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THE BRÜCK CONJECTURE AND THE GROWTH OF ENTIRE SOLUTIONS OF A DIFFERENTIAL EQUATION

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Abstract. Brück conjectured that if a nonconstant entire function f of finite hyper-order not equal to a positive integer shares a finite value a counting multiplicities with its derivative, then $f' - a = c(f - a)$ for some nonzero constant c. We prove this conjecture.

1. Introduction

Two entire functions f and g are said to share a value $a \in \mathbb{C}$ counting multiplicities (CM) if neither $f - a$ nor $g - a$ is identically zero and their zeros coincide with identical multiplicities. The relation between an entire function and its derivative provides a natural setting for such questions. Rubel and Yang [18] proved that a nonconstant entire function which shares two distinct finite values CM with its derivative must coincide with that derivative. Mues and Steinmetz [16] subsequently removed the multiplicity assumption from this two-value theorem. A single shared value allows more possibilities, and the growth of the function becomes relevant.

For an entire function u and $r > 0$, put

$$M(r,u) = \max_{|z|=r} |u(z)|, \qquad T(r,u) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |u(re^{i\theta})|\, d\theta,$$

where $\log^+ t = \max\{\log t, 0\}$ for $t > 0$ and $\log^+ 0 = 0$. The order and hyper-order of a nonconstant entire function are

$$\rho(u) = \limsup_{r\to\infty} \frac{\log\log M(r,u)}{\log r}, \qquad \rho_2(u) = \limsup_{r\to\infty} \frac{\log\log T(r,u)}{\log r}.$$

The logarithms in these limits are defined for all sufficiently large r. We use $\mathbb{N} = \{1, 2, 3, \ldots\}$.

In 1996, Brück [2] proposed the following conjecture.

Conjecture 1. *Let f be a nonconstant entire function with $\rho_2(f) < \infty$ and $\rho_2(f) \notin \mathbb{N}$. If f and f' share a finite value a CM, then*

$$f' - a = c(f - a)$$

for some constant $c \neq 0$.

Brück proved the case $a = 0$ in the original paper. Gundersen and Yang [8] established the conjecture for functions of finite order. Chen and Shon [6] treated hyper-order less than $1/2$, and Cao [4] obtained the endpoint $1/2$. G. Zhang [22] reported a proof for hyper-order less than one in a preprint. Lahiri and Das [13, Theorem 5] treated polynomial sharing with higher derivatives for functions of finite hyper-order and lower hyper-order less than $1/2$. Related results for linear differential polynomials were obtained by Chen, Gao, and Zhang [5]. Dida and El Farissi [7] proved the conjecture for entire functions satisfying second-order linear differential equations with meromorphic coefficients of finite order.

Lahiri's 2025 conference survey abstract [12] describes the conjecture as still open in full generality. We prove Conjecture 1 in its stated generality.

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Theorem 1.1. *Let f be a nonconstant entire function, and suppose that f and f' share a finite value a CM. If*

$$\rho_2(f) < \infty, \qquad \rho_2(f) \notin \mathbb{N},$$

then there is a constant $c \neq 0$ such that $f' - a = c(f - a)$.

The common zero multiplicities imply that $(f'-a)/(f-a)$ extends to a zero-free entire function, and hence equals e^h for an entire function h. When $a \neq 0$, the substitution $y = (f-a)/a$ gives

$$y' = e^h y + 1. \tag{1.1}$$

The connection between (1.1) and the conjecture was studied by Yang [19, 20, 21], Li and Gao [14], and Cao [3, 4]. Lahiri's survey [11, Section 3] discusses (1.1) in the treatment of the finite-order case. Yang asked whether every entire solution of (1.1), for nonconstant entire h, must have positive integer or infinite hyper-order. He established this conclusion with at most one possible exception, as recorded in [3, Section 1]. We determine the hyper-order of each nonconstant entire solution.

Theorem 1.2. *Let h be entire, and let y be a nonconstant entire solution of (1.1). Then*

$$\rho_2(y) = \begin{cases} 0, & \text{if } h \text{ is constant,} \\ \deg h, & \text{if } h \text{ is a nonconstant polynomial,} \\ +\infty, & \text{if } h \text{ is transcendental entire.} \end{cases}$$

The polynomial case of Theorem 1.2 already follows from Li and Gao [14, Theorem 1.1], applied to $f = y + 1$ with first derivatives and with both shared polynomials equal to 1. Qi and Yang [17, Theorem 1.1] also determined the lower hyper-order in the polynomial case, where the lower hyper-order is defined by replacing lim sup with lim inf in the definition of ρ_2. For transcendental h, Cao [3, Theorem 1.2] proved infinite hyper-order when $\rho(h) < 1/2$ and later included the endpoint $\rho(h) = 1/2$ [4, Theorem 2.6]. Cao [3, Theorem 1.3] also established the conclusion under additional growth and gap conditions on the Taylor series of h. G. Zhang and Qi [23, Theorem 1.1] obtained infinite hyper-order for a particular canonical product h of order $1/2$.

Y. Zhang [24, Theorem 1.1] obtained lower bounds along certain rays for solutions of $y' = Se^P y + 1$, where S is a nonzero rational function and P is a nonconstant polynomial. Under the hypotheses of Conjecture 1, Y. Zhang [24, Theorem 4.1] showed that the exponent h is constant or transcendental entire of finite order. Theorem 1.2 excludes the latter possibility for solutions of finite hyper-order by treating arbitrary transcendental h. The following quantitative estimate gives a common proof of the lower bounds.

Theorem 1.3. *There is an absolute constant $\kappa > 0$ with the following property. For every nonconstant entire function h, there is a sequence $r_n \to \infty$, depending only on h, such that every entire solution y of (1.1) satisfies*

$$\log\log M(2r_n, y) \geq \kappa M(r_n, h) \tag{1.2}$$

for all sufficiently large n, with the threshold independent of y.

The proof uses a local inverse of h near points of maximum modulus. The local approximation we need follows from Wiman–Valiron theory and Macintyre's refinement [15, 10]; we use the formulation in Bergweiler [1]. A short additional growth estimate controls the size of the derivative of the inverse. In the coordinate $w = h(z)$, equation (1.1) becomes $Y' = q(w)(e^w Y + 1)$, where q is the derivative of the inverse and varies slowly on a disk of radius comparable to $M(r_n, h)$.

The main step is an integration along three sides of a rectangle of height 2π in the w-plane. If c is a suitable value of q, the integral on the left vertical side is $2\pi i c + o(|c|)$. Each horizontal integral is smaller than $e^{-2}|c|$. The integrating factor is exponentially small at both right endpoints, so at

least one value of the solution must be large. This argument applies to every solution and requires only a bounded contour. Figure 1 records the main dependencies.

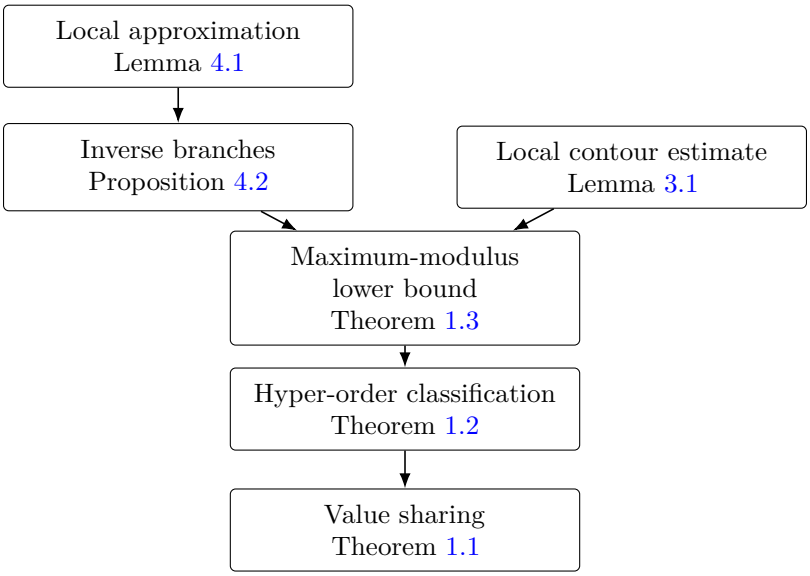


FIGURE 1. The main proof dependencies.

Section 2 records the growth comparisons used below. Section 3 proves the local contour estimate. Section 4 constructs the inverse branches and establishes their quantitative bounds. Section 5 proves Theorems 1.3, 1.2, and 1.1.

Formalization. Theorem 1.1 has been formalized in Lean 4 using Mathlib. The formalization covers all finite shared values, including zero, and verifies that the formal definitions of CM sharing and hyper-order agree with their classical counterparts. The main theorem, `Bruck.bruck_conjecture`, is proved without `sorry` or additional axioms beyond Lean's standard logical axioms. The source code and verification instructions are available at `https://github.com/QuanyuTang/bruck-conjecture-formalization`.

Statement on AI usage. The authors acknowledge the use of AI tools for exploratory discussions and language refinement during the preparation of this work. ChatGPT assisted in developing certain technical arguments in the proof of the main theorem. OpenAI's Codex assisted with the Lean 4 formalization of the main theorem.

2. Preliminary growth estimates

We recall a few standard facts about entire functions; see Hayman [9]. For every entire function u, the Poisson inequality for the subharmonic function $\log^+ |u|$ gives

$$T(r,u) \le \log^+ M(r,u) \le 3T(2r,u). \tag{2.1}$$

Indeed, the Poisson kernel for the disk of radius $2r$, evaluated at a point of modulus at most r, is bounded above by 3. For nonconstant u, comparison at r and $2r$ in (2.1) yields

$$\rho_2(u) = \limsup_{r\to\infty} \frac{\log\log\log M(r,u)}{\log r}. \tag{2.2}$$

If $\alpha \neq 0$ and β are constants, the elementary inequalities in both directions between u and $\alpha u + \beta$ give $T(r, \alpha u + \beta) = T(r,u) + O(1)$. Thus hyper-order is unchanged by these affine transformations.

We will also use the identity

$$\rho_2(e^v) = \rho(v) \tag{2.3}$$

for every nonconstant entire function v. To verify it, first note that $T(r, e^v) \le M(r, v)$. The Poisson inequality for $\operatorname{Re} v$ and the Borel–Carathéodory inequality give

$$M(r, v) \le 2 \max_{|z|\le 2r} \operatorname{Re} v(z) + 3|v(0)| \le 6T(4r, e^v) + 3|v(0)|. \tag{2.4}$$

Taking logarithms in these two bounds proves (2.3).

Finally, a zero-free entire function of finite order is the exponential of a polynomial. This is the zero-free case of Hadamard factorization, and it also follows directly from (2.4). Write the function as e^v with v entire. Finite order gives $\log^+ M(r, e^v) \le r^K$ for some $K > 0$ and all sufficiently large r. Equations (2.1) and (2.4) imply $M(r, v) = O(r^K)$. Cauchy's coefficient estimates then show that v is a polynomial.

3. A uniform contour estimate

For $w_0 \in \mathbb{C}$ and $s > 0$, write $D(w_0, s) = \{w \in \mathbb{C} : |w - w_0| < s\}$. The following estimate is the main analytic step.

Lemma 3.1. *Fix $\delta, C, K > 0$. There are $\varepsilon > 0$, depending only on δ, C, and $R_0 > 0$, depending only on δ, C, K, such that the following holds. Let $R \ge R_0$, and let q be holomorphic and zero-free on $D = D(-iR, \delta R)$. Suppose that*

$$\left|\frac{q'(w)}{q(w)}\right| \le \frac{C}{R} \quad (w \in D), \qquad R^{-K} \le |q(-iR)| \le R^K. \tag{3.1}$$

There are points $w_0, w_1 \in D$, depending only on q and R, such that every holomorphic solution of

$$Y' = q(w)(e^w Y + 1) \tag{3.2}$$

on D satisfies

$$\log\log \max\{|Y(w_0)|, |Y(w_1)|\} \ge \frac{\varepsilon R}{2}.$$

Proof. Put $c = q(-iR)$. Choose $b \in \mathbb{R}$ such that

$$|b + R| \le \pi, \qquad ce^{ib} = |c|.$$

Fix

$$0 < \varepsilon < \min\left\{\frac{\delta}{4}, \frac{1}{100(C+1)}\right\}, \qquad x = \log\frac{4}{|c|}, \qquad X = \varepsilon R.$$

By (3.1), $x = O(\log R)$. For sufficiently large R, we have $x < X$, and the closed rectangle

$$\mathcal{R} = \{u + iv : x \le u \le X,\ b \le v \le b + 2\pi\}$$

lies in D. Denote its left vertices by $\xi_j = x + i(b + 2\pi j)$ and its right vertices by $w_j = X + i(b + 2\pi j)$, where $j = 0, 1$. Let V be the upward segment from ξ_0 to ξ_1. The contour used below is shown in Figure 2.

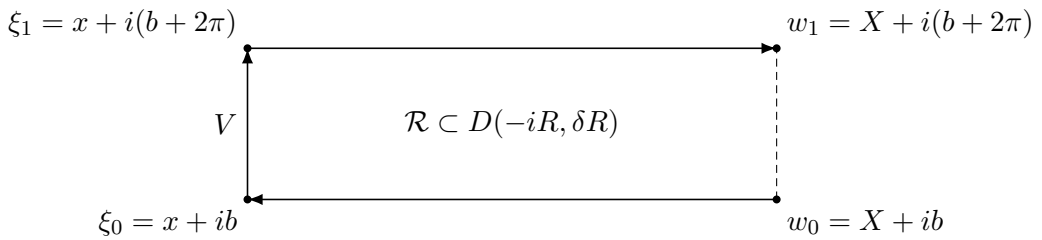


FIGURE 2. Integration from w_0 to w_1 along the three solid sides. The right vertical side is not part of the contour. The drawing is not to scale.

Since q has a holomorphic logarithm on D, integration of q'/q along straight segments starting at $-iR$ gives

$$|q(w)/c-1|<\frac{1}{10},\qquad |q(w)|\le 2|c|\quad (w\in\mathcal{R}) \tag{3.3}$$

for all sufficiently large R. The distance from $-iR$ to any point of V is $O(\log R)$, so the same argument gives the sharper bound

$$\eta_R:=\sup_{w\in V}|q(w)/c-1|=O(\log R/R). \tag{3.4}$$

All constants and thresholds in this proof are independent of Y.

Let S be the holomorphic primitive on D specified by $S'=qe^w$ and $S(\xi_0)=4$. Since $ce^{\xi_0}=4$, we have, for $w\in V$,

$$S(w)-ce^w=\int_{\xi_0}^{w}e^{\zeta}(q(\zeta)-c)\,d\zeta.$$

Using $|c|e^x=4$ and the length 2π of V, we obtain

$$\sup_{w\in V}|S(w)-ce^w|\le 8\pi\eta_R. \tag{3.5}$$

On this segment, (3.4) and (3.5) imply

$$\int_V e^{-S(w)}q(w)\,dw=c\int_V e^{-ce^w}\,dw+O(|c|\eta_R)=2\pi ic+o(|c|). \tag{3.6}$$

For the last equality, expand e^{-ce^w} in its uniformly convergent exponential series on V. Its constant term integrates to $2\pi i$, whereas every term involving e^{mw} with $m\ge 1$ has integral zero. The period identity for the model integrand also appears in Y. Zhang [24, Eq. (1.7)].

For $v=b$ or $v=b+2\pi$, equation (3.3) gives

$$\operatorname{Re}(e^{u+iv}q(u+iv))\ge\frac{9}{10}|c|e^u.$$

At both left endpoints, $\operatorname{Re}S(\xi_j)\ge 4-8\pi\eta_R$, by (3.5). Integration along the horizontal sides therefore gives

$$\operatorname{Re}S(u+iv)\ge 4-8\pi\eta_R+\frac{9}{10}|c|(e^u-e^x)=\frac{9}{10}|c|e^u+\frac{2}{5}-8\pi\eta_R.$$

In particular, for all sufficiently large R,

$$\operatorname{Re}S(u+iv)\ge\frac{1}{2}|c|e^u\quad (x\le u\le X,\ v=b,b+2\pi). \tag{3.7}$$

Each horizontal integral is consequently bounded by

$$\begin{aligned}\left|\int_{x+iv}^{X+iv}e^{-S(w)}q(w)\,dw\right|&\le 2|c|\int_x^X e^{-|c|e^u/2}\,du\\&\le 2|c|\int_2^\infty\frac{e^{-s}}{s}\,ds\le e^{-2}|c|.\end{aligned} \tag{3.8}$$

Equation (3.2) gives the exact identity $(e^{-S}Y)'=e^{-S}q$. Integrating along the contour of Figure 2 and using (3.6) and (3.8), we find

$$|e^{-S(w_1)}Y(w_1)-e^{-S(w_0)}Y(w_0)|\ge(2\pi-2e^{-2}-o(1))|c|\ge\pi|c|.$$

At least one of the two weighted endpoint values has modulus at least $\pi|c|/2$. Since $\operatorname{Re}S(w_j)\ge|c|e^{\varepsilon R}/2$ by (3.7),

$$\max_{j=0,1}|Y(w_j)|\ge\frac{\pi}{2}|c|\exp\left(\frac{1}{2}|c|e^{\varepsilon R}\right).$$

The bounds $R^{-K} \le |c| \le R^K$ now give

$$\log\log \max_{j=0,1} |Y(w_j)| \ge \varepsilon R - K\log R - O(1) \ge \frac{\varepsilon R}{2}$$

for sufficiently large R. □

4. Local inverse branches

We first record the precise consequence of classical local growth theory that we need. In the sequential estimates below, $o(1)$ denotes a quantity tending to zero as $n \to \infty$, uniformly on the specified sets.

Lemma 4.1. *Let h be a nonconstant entire function. There are points z_n, radii $r_n = |z_n| \to \infty$, and positive numbers ν_n such that*

$$R_n := |h(z_n)| = M(r_n, h), \qquad \frac{1}{2} \le \nu_n \le 2(\log R_n)^2, \tag{4.1}$$

and

$$\frac{h(z_n e^{\zeta/\nu_n})}{h(z_n)e^{\zeta}} \longrightarrow 1 \quad \textit{uniformly for } |\zeta| \le 10. \tag{4.2}$$

Proof. If h is a polynomial of degree $d \ge 1$, take $\nu_n = d$ and any sequence $r_n \to \infty$, with z_n a point of maximum modulus on $|z| = r_n$. The leading term of h proves (4.2) uniformly for bounded ζ, and (4.1) holds after omitting finitely many terms.

Suppose that h is transcendental. Put $\Phi(t) = \log M(e^t, h)$ and $\alpha(r) = \Phi'_+(\log r)$, where Φ'_+ is the right derivative. The three-circles theorem makes Φ convex. Cauchy's coefficient estimates imply $\Phi(t)/t \to \infty$, so $\alpha(r) \to \infty$.

Bergweiler's theorem [1, Theorem 1.1 and Remark 1], with $\psi(s) = s^{3/2}$, gives a set E_1 of finite logarithmic measure such that, if $r \notin E_1$ and $|z_r| = r$, $|h(z_r)| = M(r, h)$, then

$$h(z) = (1 + o(1))h(z_r)\left(\frac{z}{z_r}\right)^{\alpha(r)} \quad \text{uniformly for } |z - z_r| \le \frac{r}{\alpha(r)^{3/4}}. \tag{4.3}$$

The power uses the local logarithm of z/z_r which is zero at $z = z_r$. Finite logarithmic measure means $\int_{E_1 \cap [1,\infty)} dr/r < \infty$.

We also need a bound on $\alpha(r)$ in terms of $M(r, h)$. Choose $r_0 > 1$ with $\log M(r_0, h) > 0$, and let

$$E_2 = \{r \ge r_0 : \alpha(r) > (\log M(r, h))^2\}.$$

Convexity gives local absolute continuity of Φ, and hence

$$\int_{E_2} \frac{dr}{r} \le \int_{r_0}^{\infty} \frac{\alpha(r)}{(\log M(r, h))^2}\frac{dr}{r} = \frac{1}{\log M(r_0, h)} < \infty.$$

This is also the elementary growth estimate in [1, Remark 1].

Choose $r_n \to \infty$ outside $E_1 \cup E_2$, choose z_n as above, and set $\nu_n = \alpha(r_n)$. Then (4.1) holds for large n. For $|\zeta| \le 10$,

$$|z_n e^{\zeta/\nu_n} - z_n| \le \frac{10 r_n}{\nu_n} e^{10/\nu_n} \le \frac{20 r_n}{\nu_n} < \frac{r_n}{\nu_n^{3/4}}$$

eventually. Since $\nu_n \to \infty$, the local logarithm satisfies $\log(e^{\zeta/\nu_n}) = \zeta/\nu_n$ on this set. Substitution in (4.3) proves (4.2). □

The next proposition converts this approximation into an inverse branch with the bounds required by Lemma 3.1.

Proposition 4.2. *For every nonconstant entire function h, there are radii $r_n \to \infty$, with $R_n = M(r_n, h)$, and holomorphic maps*

$$\chi_n : D(-iR_n, R_n/10) \longrightarrow D(0, 2r_n)$$

such that $h \circ \chi_n(w) = w$. For all sufficiently large n, the derivatives $q_n = \chi_n'$ are zero-free and satisfy

$$\left|\frac{q_n'(w)}{q_n(w)}\right| \le \frac{10}{R_n} \quad (w \in D(-iR_n, R_n/10)), \qquad R_n^{-2} \le |q_n(-iR_n)| \le R_n^2. \tag{4.4}$$

Proof. Use Lemma 4.1 and suppress the index n. Write $z_0 = z_n$, $r = r_n$, $\nu = \nu_n$, $h_0 = h(z_0)$, and $R = |h_0|$. By (4.2), the functions

$$J(\zeta) = \log \frac{h(z_0 e^{\zeta/\nu})}{h_0 e^{\zeta}}, \qquad A(\zeta) = \zeta + J(\zeta)$$

are well defined on $D(0, 10)$ for all sufficiently large indices, using the logarithm near 1. Moreover, $J = o(1)$ uniformly there. Cauchy's estimates give $J', J'' = o(1)$ on $D(0, 9)$. In particular, $|A' - 1| < 1/2$ on that disk eventually. Integration along a straight segment shows that

$$|A(\zeta_1) - A(\zeta_2) - (\zeta_1 - \zeta_2)| \le \frac{1}{2}|\zeta_1 - \zeta_2|,$$

so A is injective on $D(0, 9)$. Rouché's theorem on $|\zeta| = 9$ shows that its image contains $D(0, 8)$. Let B be the inverse on $D(0, 8)$. The identity $A(B(\lambda)) = \lambda$ gives

$$B(\lambda) = \lambda - J(B(\lambda)) = \lambda + o(1) \qquad (|\lambda| < 8), \tag{4.5}$$

uniformly.

Choose $\Delta \in [-\pi, \pi]$ such that $h_0 e^{i\Delta} = -iR$. For $w \in D(-iR, R/10)$, define

$$\Lambda(w) = i\Delta + \log \frac{w}{-iR}, \qquad \chi(w) = z_0 \exp\left(\frac{B(\Lambda(w))}{\nu}\right),$$

where the logarithm is the branch satisfying $\log 1 = 0$ on $D(1, 1/10)$. Its power series gives $|\Lambda(w)| \le \pi - \log(9/10) < 5$. Thus the definition is valid, and

$$h(\chi(w)) = h_0 e^{A(B(\Lambda(w)))} = h_0 e^{\Lambda(w)} = w.$$

Furthermore, by (4.5),

$$\operatorname{Re} B(\Lambda(w)) = \log|w/(-iR)| + o(1) \le \log(11/10) + o(1).$$

Since $\nu \ge 1/2$, we have $|\chi(w)| < 2r$ for all sufficiently large indices. This estimate uses no assumption that $\nu \to \infty$.

Put $\zeta = B(\Lambda(w))$ and $q = \chi'$. Differentiation gives the exact identities

$$q(w) = \frac{\chi(w)}{\nu w A'(\zeta)} \tag{4.6}$$

and

$$\frac{q'(w)}{q(w)} = \frac{1}{\nu w A'(\zeta)} - \frac{1}{w} - \frac{A''(\zeta)}{w(A'(\zeta))^2}. \tag{4.7}$$

The first identity shows that q has no zeros. For large indices, $|\zeta| < 6$, $|A'(\zeta)| \ge 1/2$, $|A''(\zeta)| \le 1$, $\nu \ge 1/2$, and $|w| \ge 9R/10$. Equation (4.7) therefore gives $|q'/q| \le 10/R$.

At $w = -iR$, equation (4.5) gives $\operatorname{Re} B(i\Delta) = o(1)$. Thus (4.6) yields

$$|q(-iR)| = (1 + o(1))\frac{r}{\nu R}.$$

Using $r \geq 1$ and the upper bound on ν in (4.1), we obtain

$$|q(-iR)| \geq \frac{1}{4R(\log R)^2} \geq R^{-2}$$

eventually. Write

$$h(z) = \sum_{m=0}^{\infty} h_m z^m.$$

Since h is nonconstant, we may choose a nonzero coefficient h_m with $m \geq 1$. Cauchy's estimate gives $R \geq |h_m| r^m$, so r/R is bounded for $r \geq 1$. The lower bound $\nu \geq 1/2$ then gives $|q(-iR)| \leq 4r/R \leq R^2$ eventually. This proves (4.4). □

The factor $1/w$ in each term of (4.7) is useful here. It supplies the bound of order $1/R$ without requiring a rate for the $o(1)$ error in (4.2). The estimate of order $\log R/R$ on the vertical segment in Lemma 3.1 follows from this logarithmic derivative bound.

5. Proofs of the main theorems

We first prove the maximum-modulus estimate of Theorem 1.3.

Proof of Theorem 1.3. Take the maps χ_n from Proposition 4.2 and put $Y_n = y \circ \chi_n$. Since $h \circ \chi_n(w) = w$, the chain rule and (1.1) give

$$Y_n' = q_n(w)(e^w Y_n + 1).$$

By (4.4), Lemma 3.1 applies with $\delta = 1/10$, $C = 10$, and $K = 2$. Its two endpoint values are values of y at points of modulus less than $2r_n$. The maximum-modulus principle gives (1.2), with $\kappa = \varepsilon/2$. The parameters in the local lemma are absolute, and the inverse branches depend only on h. Both κ and the threshold are therefore independent of the solution y. □

We next prove Theorem 1.2.

Proof of Theorem 1.2. Suppose first that h is transcendental. For each nonzero coefficient h_m in the Taylor expansion of h, Cauchy's estimate gives $M(r, h) \geq |h_m| r^m$. Such indices m are arbitrarily large, and hence

$$\frac{\log M(r, h)}{\log r} \longrightarrow +\infty.$$

Taking one further logarithm in (1.2) yields

$$\log\log\log M(2r_n, y) \geq \log\kappa + \log M(r_n, h).$$

Divide by $\log(2r_n)$ and use (2.2). It follows that $\rho_2(y) = +\infty$.

If h is a polynomial of degree $d \geq 1$, then $M(r, h) \asymp r^d$, meaning that it is bounded above and below by positive constant multiples of r^d for all large r. The same argument gives $\rho_2(y) \geq d$. For the reverse inequality, integrate (1.1) along each radial segment and apply Gronwall's inequality:

$$M(r, y) \leq (|y(0)| + r)\exp\{rM(r, e^h)\}.$$

For some constant $C_1 > 0$ and all sufficiently large r, $M(r, e^h) \leq \exp(C_1 r^d)$, so

$$\log M(r, y) \leq \log(|y(0)| + r) + r\exp(C_1 r^d).$$

Taking two more logarithms and using (2.2) gives $\rho_2(y) \leq d$.

Finally, if $e^h = \lambda \neq 0$ is constant, then $y(z) = Ce^{\lambda z} - 1/\lambda$. The nonconstant solutions have $C \neq 0$ and hyper-order zero. □

We finish with Theorem 1.1.

Proof of Theorem 1.1. By the sharing assumption,

$$Q(z) = \frac{f'(z) - a}{f(z) - a}$$

extends to a zero-free entire function. At a common zero, removability and the nonzero extended value follow from equality of the multiplicities. Neither numerator nor denominator is identically zero, by the definition of sharing. Since $\mathbb{C}$ is simply connected, $Q = e^h$ for some entire h.

If $a \neq 0$, put $y = (f - a)/a$. Then y is nonconstant entire, satisfies (1.1), and has the same hyper-order as f. Theorem 1.2 forces h to be constant. Thus $f' - a = c(f - a)$ with $c = e^h \neq 0$.

For completeness, we include the classical argument for $a = 0$. If f had a zero of multiplicity $m \geq 1$, then f' would have multiplicity $m - 1$ there, contradicting the sharing assumption. Hence both f and f' are zero-free. Write $f = e^g$ with g nonconstant entire. Then g' is zero-free, and (2.3) gives $\rho(g) = \rho_2(f) < \infty$. If g' is constant, it is nonzero and $f' = g'f$ proves the claim. We may therefore assume that g' is nonconstant. Cauchy's derivative estimate shows that g' has finite order. The factorization recalled in Section 2 therefore gives $g' = e^P$ for a polynomial P.

If $\deg P = d \geq 1$, then $\rho(g') = d$. The estimates

$$M(r, g') \leq \frac{M(2r, g)}{r}, \qquad M(r, g) \leq |g(0)| + rM(r, g')$$

give $\rho(g) = \rho(g') = d$. This contradicts $\rho_2(f) \notin \mathbb{N}$. Thus P is constant, and $f' = cf$ with $c = e^P \neq 0$. □

Solving the equation in Theorem 1.1 gives

$$f(z) = Ce^{cz} + a - \frac{a}{c}, \qquad Cc \neq 0.$$

The excluded hyper-orders can all occur. Indeed, if H is the entire primitive of e^h with $H(0) = 0$, then all solutions of (1.1) are

$$y(z) = e^{H(z)} \left(C + \int_0^z e^{-H(\zeta)}\, d\zeta \right).$$

The integral is path independent. For nonconstant h these solutions are nonconstant, and $f = 1 + y$ shares 1 CM with f' while $(f' - 1)/(f - 1) = e^h$ is nonconstant. Taking $h(z) = z^d$ for $d \in \mathbb{N}$, or $h(z) = e^z$, recovers the positive-integer and infinite hyper-orders appearing in Brück's examples [2].

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