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Atkin–Swinnerton-Dyer congruences for Kummer twists of Hauptmoduln

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Abstract

We prove that the coefficients of $h = \sqrt{\Delta/E_4} = \sum_{n\geq 1} b(n)q^{n/2}$ satisfy the Atkin–Swinnerton-Dyer congruence

$$b(mp^r) - a_p(g)b(mp^{r-1}) + p^3 b(mp^{r-2}) \equiv 0 \pmod{p^{3r}}$$

for every prime $p \geq 5$, $m \geq 1$ prime to p, and $r \geq 1$, with $b(x) = 0$ for nonintegral x. Here $a_p(g)$ is the pth Hecke eigenvalue of the unique normalised newform g of weight four and level 24 with trivial character. Since h branches at the elliptic point $\rho = e^{2\pi i/3}$, it is not a modular form. In a local uniformising chart of the double cover $s^2 = 1/j$ above ρ, the associated differential extends holomorphically across the ramification point. It defines a parabolic cohomology class with coefficients in the symmetric square of the elliptic local system.

More generally, we study modular differentials twisted by rational powers of a Hauptmodul. When the differential is nonzero and square-integrable on the Kummer cover and the twisted parabolic cohomology consists of two lines of types $(k-1,0)$ and $(0,k-1)$, we prove three-term congruences modulo $p^{(k-1)r}$ at good primes of residue degree one. A comparison of crystalline and étale Frobenius identifies the polynomial of the recurrence with the Frobenius polynomial of a two-dimensional Galois representation. Our local criteria produce further examples, including $E_4 j^{-a}$ for every rational a with $1/3 < a < 2/3$.

1 Introduction

Let Δ be the modular discriminant and E_4 the normalised Eisenstein series of weight four. Put $q = e^{2\pi i\tau}$ and $Q = e^{\pi i\tau}$. At the cusp at infinity, the square root of their quotient has an expansion

$$h(\tau) = \sqrt{\frac{\Delta(\tau)}{E_4(\tau)}} = \sum_{n\geq 1} b(n)Q^n = Q - 132Q^3 + 22014Q^5 - 4187368Q^7 + \cdots,$$

where the leading coefficient fixes the branch. Since Δ/E_4 has a simple pole at $\rho = e^{2\pi i/3}$, it cannot be the square of a meromorphic function on the upper half-plane. Thus h is not a modular form on any finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Let $g = \sum_{n\geq 1} a_n(g)q^n$ be the unique normalised newform of weight four and level 24 with trivial character [39]. For each prime $p \geq 5$, the coefficients of h satisfy the Hecke recurrence of g modulo successively higher powers of p.

Theorem A (Atkin–Swinnerton-Dyer congruence)**.** *For every prime $p \geq 5$, every $m \geq 1$ prime to p, and every $r \geq 1$,*

$$b(mp^r) - a_p(g)b(mp^{r-1}) + p^3 b(mp^{r-2}) \equiv 0 \pmod{p^{3r}},$$

where $b(x) = 0$ for nonintegral x.

Bönisch suggested that the periods of $h = E_4/\sqrt{j}$ should be related to those of the level-24 newform g [8]. For the hypergeometric family with parameters $(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6})$, he identified the pure conifold periods, expressed as integrals involving $h(\tau)\,d\tau$, with periods of the level-72 quadratic twist $g \otimes \chi_{-3}$ [7, §6.3, Theorem 12 and p. 84] (see also [9]). Bönisch's period formula and Theorem A thus suggest that the complex periods of $h(\tau)\,d\tau$ and the congruences among the coefficients of h reflect the same modular motive, up to quadratic twist.

The same hypergeometric family gives a second p-adic comparison. Long, Tu, Yui and Zudilin [42, Theorem 1 and Table 1] prove that its truncated period at 1 is congruent modulo p^3 to $a_p(g \otimes \chi_{-3})$. Thus the case $m = r = 1$ of Theorem A gives, for $p > 5$,

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^2 (\frac{1}{6})_n (\frac{5}{6})_n}{(n!)^4} \equiv \chi_{-3}(p)\, b(p) \pmod{p^3},$$

where $\chi_{-3}(p) = (\frac{-3}{p})$, $(x)_n = x(x+1)\cdots(x+n-1)$ and $(x)_0 = 1$. For related congruences for Apéry-like sequences, see [6, 33].

The three-term recurrence for $b(m), b(mp), b(mp^2), \ldots$ in Theorem A is the analogue of the congruences that Atkin and Swinnerton-Dyer [4] observed for the Fourier coefficients of cusp forms on noncongruence subgroups. Scholl [47] used de Rham cohomology to prove a common $(2D+1)$-term congruence for a D-dimensional space of cusp forms. At good primes its coefficients are those of the Frobenius polynomial of the associated Galois representation. For three-term congruences in higher dimension and their relation to modularity, see [37, 3, 40] and the survey [38].

Kazalicki and Scholl [32] allowed poles at the cusps. Their description of de Rham cohomology by weakly holomorphic forms gives congruences under an integrality condition on the principal parts. More recently, Allen, Long and Saad [2] treated meromorphic modular forms with poles in the upper half-plane. The elliptic curves at the poles contribute additional Frobenius factors to the recurrence. Related congruences appear in work of Zhang [61] and in Bordignon's rigid cohomological treatment [11].

To see what happens to the differential at ρ, pull the double cover $s^2 = 1/j$ back to a small disc about ρ in the upper half-plane. On the normalisation of this pullback we may choose a coordinate u with $\tau - \rho = u^2$. The simple zero of $d\tau = 2u\,du$ cancels the simple pole of h in u. To vary this example, put $t = 1/j$. The identities $h = E_4\sqrt{t}$ and $q\,d\log t/dq = E_6/E_4$ give

$$h\,\frac{dq}{q} = \frac{E_4^2}{E_6}\,t^{1/2}\frac{dt}{t}.$$

The factor $t^{1/2}$ accounts for the change of sign around ρ. We may therefore vary the branching by replacing the square root with a rational power t^a of a Hauptmodul. The modular form E_4^2/E_6 is replaced by a form f of weight $k-2$. A rational factor $R(t)$ allows us to prescribe additional zeros or poles without changing the monodromy of t^a.

Definition 1.1. A *Kummer twist datum* of integer weight $k \geq 3$ is a tuple (Γ, t, f, a, R) with the following properties. The group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is commensurable with $\mathrm{PSL}_2(\mathbb{Z})$, contains a principal congruence subgroup, and has genus zero and a cusp of width one at infinity. The function t is a Hauptmodul for Γ, and f is a meromorphic modular form of weight $k-2$ on Γ with a finite-order multiplier trivial on $\Gamma \cap \Gamma(n)$ for some n. The rational number $a = j/d$ lies in $(0,1)$ and is written in lowest terms, and $R \in \mathbb{Q}(t)$ is regular at $t = 0$. There is an integer $M \geq 1$ such that, writing $A = \mathbb{Z}[1/M]$, one has

$$t \in q + q^2 A[[q]], \qquad f = F(t) \text{ with } F \in 1 + tA[[t]], \qquad R(t) \in A[[t]].$$

The normalisation $t = q + O(q^2)$ determines a preferred branch of t^a at the cusp. With $Q = q^{1/d}$, take $t^a = Q^j (t/q)^a$, where the last factor has constant term one. The coefficients in our congruences are those of the resulting differential in the cusp parameter,

$$\Phi = f\, R(t)\, t^a \frac{dt}{t} = f_\Phi(Q)\frac{dq}{q}, \qquad f_\Phi(Q) = \sum_{N \geq 1} c(N) Q^N.$$

For the datum above, $f_\Phi = h$ and $c(N) = b(N)$.

In the opening example the involution of the double cover sends the differential to its negative. To find a Frobenius relation for its coefficients, it therefore suffices to consider the part of cohomology on which this involution acts by -1. For a general exponent a, we describe the corresponding space on the t-line by first writing down the periods before twisting. On $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the invariant differential dz has periods 1 and τ along the standard cycles. Choose a local inverse $\tau = \tau(t)$ and write $F(t) = f(\tau(t))$. Evaluating $f(\tau)[dz]^{\otimes(k-2)}$ on symmetric products of these cycles gives the periods

$$F(t), \quad \tau(t)F(t), \quad \ldots, \quad \tau(t)^{k-2}F(t).$$

These functions form a basis of solutions of a linear differential equation in t of order $k-1$ with coefficients in $\mathbb{Q}(t)$. This is the Picard–Fuchs equation associated with (t, f). Analytic continuation of its solutions defines a local system $\mathbb{V}_f$ on $U = \mathbb{P}^1 \setminus S$, where S consists of the singularities of the equation together with 0 and ∞. Multiplying the solutions by a branch of t^a gives the twisted local system $\mathbb{W} = \mathbb{V}_f \otimes \mathbb{L}_a$, where $\mathbb{L}_a$ is the line of branches of t^a. Its parabolic cohomology is

$$H_a = \operatorname{im}(H^1_c(U, \mathbb{W}) \longrightarrow H^1(U, \mathbb{W})).$$

These are precisely the classes whose restrictions to punctured neighbourhoods of the points of S vanish.

When $\dim H_a = 2$, the Frobenius polynomial is quadratic, as in Theorem A. The next definition also specifies the Hodge types of the two lines and the behaviour of Φ at branch points and cusps.

Definition 1.2. A Kummer twist datum is *admissible* if Φ is nonzero and square-integrable on $s^d = t$, and

$$H_a = H_a^{k-1,0} \oplus H_a^{0,k-1}, \qquad \dim_{\mathbb{C}} H_a^{k-1,0} = \dim_{\mathbb{C}} H_a^{0,k-1} = 1.$$

In the examples, these conditions can be checked from local data. Square-integrability means that the coefficient of $d\tau$ has order greater than -1 at each interior point and the coefficient of dq_c/q_c has positive order in a parameter q_c at each cusp. Thus the order $-1/2$ at ρ in Theorem A is allowed. The local monodromies determine $\dim H_a$. When the multiplier is real and $\dim H_a = 2$, the two extreme Hodge types follow if the differentials for a and $1-a$ are both nonzero and square-integrable (Proposition 2.9).

Let L be the field generated over $\mathbb{Q}$ by the dth roots of unity and the multiplier values of f, adjoining also, for odd k, the square roots of the determinants of primitive integral matrices representing elements of Γ. We consider primes $\mathfrak{P}$ of L of residue degree one over rational primes p outside the explicit finite set Σ of Definition 2.3. At such a prime Frobenius fixes the dth roots of unity and hence the Kummer character determined by a. In Theorem A, $L = \mathbb{Q}$ and $\Sigma = \{2, 3\}$.

Theorem B. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum of weight k. For every prime $\mathfrak{P}$ as above there are coefficients $A_\mathfrak{P} \in \mathcal{O}_L$ and $B_\mathfrak{P} \in L$ such that, for all $m \geq 1$ prime to p and $r \geq 1$,*

$$c(mp^r) - A_\mathfrak{P}\, c(mp^{r-1}) + B_\mathfrak{P}\, c(mp^{r-2}) \equiv 0 \pmod{p^{(k-1)r}}.$$

Here $c(x) = 0$ for nonintegral x, and the congruence is read in $L_{\mathfrak{P}} = \mathbb{Q}_p$.

There is a system of two-dimensional representations of $\mathrm{Gal}(\overline{L}/L)$ for which $X^2 - A_{\mathfrak{P}}X + B_{\mathfrak{P}}$ is the characteristic polynomial of geometric Frobenius at $\mathfrak{P}$ in every λ-adic realisation with $\lambda \nmid p$. These realisations are unramified at $\mathfrak{P}$, and $B_{\mathfrak{P}} = \varepsilon(\mathfrak{P})p^{k-1}$ for a single algebraic Hecke character ε of L. If $L = \mathbb{Q}$, there is a normalised newform of weight k whose Hecke polynomial equals $X^2 - A_pX + B_p$ at every $p \notin \Sigma$.

Theorem A is the case $a = 1/2$ of the family $h_a = E_4 j^{-a}$, to which Theorem B applies for every rational $1/3 < a < 2/3$. The first nonquadratic case, $h_{2/5} = \Delta^{2/5}/E_4^{1/5}$, suggests a Hilbert modular analogue of Theorem A. We conjecture that the projective Galois representation attached to $h_{2/5}$ over $\mathbb{Q}(\zeta_5)$ comes by restriction from a Hilbert eigenform of weight $(4, 3)$ over $\mathbb{Q}(\sqrt{5})$. At the embeddings of $\mathbb{Q}(\zeta_5)$ above one real place of $\mathbb{Q}(\sqrt{5})$, the Hodge–Tate weights differ by 3. Above the other real place they differ by 2. These gaps predict the Hilbert weight $(4, 3)$. After restriction to $\mathbb{Q}(\zeta_5)$, the candidate's projective Frobenius invariants agree with those of the Kummer system above eighteen further primes of $\mathbb{Q}(\sqrt{5})$ that were not used to select it (Section 6.6).

Theorem 6.10 gives the full statement of Theorem B, including a point-count formula for the trace and bounds at every complex embedding. The location of the pole of t determines whether f_Φ itself is modular. Let P be the unique pole of t. If P is a cusp, or if P is an interior point whose stabilizer order is divisible by d, then f_Φ is a cusp form of weight k on a subgroup of finite index in Γ. In the remaining case, continuation around a lift of P acts nontrivially on t^a, and f_Φ has no single-valued meromorphic continuation to $\mathbb{H}$. This is what happens at the point of order three in Theorem A.

In the proof of Theorem B, the quadratic Frobenius polynomial gives a relation among three cusp expansions. To obtain the full modulus $p^{(k-1)r}$, we show that their linear combination is $(Q\,d/dQ)^{k-1}G(Q)$ for an integral power series G.

We apply Scholl's calculation at the cusp [47] to the de Rham class $x = [\omega_\Phi]$ of the twisted differential on $s^d = t$. In Theorem A this class is represented, up to normalisation, by $h(\tau)\,d\tau \otimes [dz]^{\otimes 2}$. Fix $\mathfrak{P}$ as in Theorem B and put $w = k - 1$. Sections 3 and 4 construct an integral model of the cover that includes its interior ramification points. On this model x lies in a rank-two $\mathbb{Z}_p$-lattice H with crystalline Frobenius φ (Lemma 3.12). Put $y = p^{-w}\varphi x$.

At the cusp, write ω for the invariant differential of the Tate curve. Since $dq/q = d\,dQ/Q$, dividing the pullback by d gives the expansion $f_\Phi(Q)$ relative to $\omega^{\otimes(k-2)} \otimes dQ/Q$. The Frobenius lift sends Q to Q^p and multiplies both ω and dQ/Q by p, so it sends a differential with coefficient $C(Q)$ to one with coefficient $p^w C(Q^p)$. Lemma 4.3(2) constructs an integral representative of y by dividing a cocycle for φx by p^w before passing to local cohomology. Its cusp expansion is $f_\Phi(Q^p)$. Applying the Tate-curve formula once more gives the cusp representative of φy (Lemma 3.9(2)).

$$\begin{aligned} x \longmapsto f_\Phi(Q) &= \sum_{n\geq 1} c(n)Q^n, \\ y \longmapsto f_\Phi(Q^p) &= \sum_{n\geq 1} c(n)Q^{pn}, \\ \varphi y \longmapsto p^w f_\Phi(Q^{p^2}) &= p^w \sum_{n\geq 1} c(n)Q^{p^2 n}. \end{aligned}$$

Write $T = \operatorname{tr}\varphi$. We also prove that $\det\varphi = p^w u$ with $u \in \mathbb{Z}_p^\times$. Cayley–Hamilton then gives $ux - Ty + \varphi y = 0$. The corresponding combination of expansions represents an exact differential. For the symmetric power Sym^{k-2}, the local de Rham calculation expresses its expansion as the $(k-1)$-fold application of $Q\,d/dQ$ to an integral power series (Lemma 3.8). Dividing by the unit

u and writing $A_{\mathfrak{P}} = T/u$, $B_{\mathfrak{P}} = p^w/u$, we obtain the integral identity of Theorem 4.1

$$f_\Phi(Q) - A_{\mathfrak{P}} f_\Phi(Q^p) + B_{\mathfrak{P}} f_\Phi(Q^{p^2}) = \left(Q\frac{d}{dQ}\right)^w G(Q), \qquad G(Q) \in \mathbb{Z}_p[[Q]].$$

For $p \nmid m$ and $r \geq 1$, taking the coefficient of Q^{mp^r} gives

$$c(mp^r) - A_{\mathfrak{P}} c(mp^{r-1}) + B_{\mathfrak{P}} c(mp^{r-2}) = (mp^r)^w [Q^{mp^r}] G(Q) \in p^{(k-1)r}\mathbb{Z}_p.$$

It remains to identify the coefficients $A_{\mathfrak{P}}$ and $B_{\mathfrak{P}}$ as $\mathfrak{P}$ varies. At the chosen prime they are the trace and determinant of $p^w\varphi^{-1}$. We compare crystalline and étale Frobenius traces on the same cover. At each interior fixed point both trace formulas use the same endomorphism of an elliptic curve, Frobenius composed with an isogeny, and the cusp contributions also agree. On the corresponding character components the comparison gives

$$\det(X - \rho_\lambda(\mathrm{Frob}_{\mathfrak{P}})) = X^2 - A_{\mathfrak{P}} X + B_{\mathfrak{P}} = \det\left(X - p^w\varphi^{-1}\right).$$

Here the middle polynomial is read in L_λ on the left ($\lambda \nmid p$) and in $L_{\mathfrak{P}}$ on the right. The representation ρ_λ is taken on the inverse character component, with geometric Frobenius $\mathrm{Frob}_{\mathfrak{P}}$ (Proposition 5.6). Over $\mathbb{Q}$, modularity identifies these polynomials with the Hecke polynomials of a single newform. For Theorem A, a comparison at nine primes in Section 6.5 identifies this newform with g.

This investigation began with Cooper's level-seven sequence $s_7(n)$ [14]. In Section 7 we return to its modular parametrisation. Twisting by the square root of its Hauptmodul gives a differential whose cusp coefficients $c_7(n)$ satisfy an ASD congruence modulo p^{3r} at every prime $p \neq 2, 3, 7$. Let g_{16} be the weight-four newform of level 16 obtained by twisting $\eta(2\tau)^4\eta(4\tau)^4$ by χ_{-4}. The link with Cooper's original sequence is the congruence

$$c_7(p) \equiv s_7\left(\frac{p-1}{2}\right) \equiv a_p(g_{16}) \pmod{p^2},$$

which holds for every odd prime p (Appendix B). Together with the Weil bound, this comparison identifies $a_p(g_{16})$ as the Frobenius trace in the level-seven ASD congruence.

2 Data and admissibility

We give local criteria for the admissibility condition in Theorem B. These criteria apply both to the square root in Theorem A and to the differential attached to Cooper's level-seven sequence in Section 7. The Hodge structure on parabolic cohomology comes from Zucker's theorem [62], in the form for complex variations proved in Lemma A.3.

2.1 The datum and its cohomology

Definition 2.1 (Notation for a datum)**.** Let (Γ, t, f, a, R) be a Kummer twist datum of weight k as in Definition 1.1. We use the following notation for its levels, local orders and expansion.

1. Let n_Γ be the least positive integer with $\Gamma(n_\Gamma) \subset \Gamma$. Write χ_f for the multiplier of f and n_f for the least positive integer for which χ_f is trivial on $\Gamma \cap \Gamma(n_f)$.

2. Fix $M \geq 1$ as in Definition 1.1 and write $A = \mathbb{Z}[1/M]$. Thus $t \in q + q^2A[[q]]$, $f = F(t)$ with $F \in 1 + tA[[t]]$, and $R(t) \in A[[t]]$ at $t = 0$.

3. The Hauptmodul t has a simple zero at the cusp ∞ and a simple pole at a unique point P of X_Γ. At an interior point P, let $e_P \in \{1,2,3,4,6\}$ be its stabiliser order and let $\nu_P = \operatorname{ord}_{\tau_P} f$, computed in $\tau - \tau_P$ at a lift $\tau_P \in \mathbb{H}$. At a cusp $P = \sigma\infty$, put $e_P = \infty$ and let ν_P be the least exponent of $f|_{k-2}\sigma$ in the local parameter of Γ at P.

Write $a = j/d$ in lowest terms. For the branch $t^a = Q^j(t/q)^a$ with $Q = q^{1/d}$ and $(t/q)^a$ of constant term one, the *twisted differential* Φ, its *expansion* $f_\Phi(Q)$ and its *coefficients* $c(N)$ are defined by

$$\Phi = fR(t)t^a\frac{dt}{t} = f_\Phi(Q)\frac{dq}{q}, \qquad f_\Phi(Q) = fR(t)t^a\frac{q}{t}\frac{dt}{dq} = \sum_{N\geq 1} c(N)Q^N.$$

We put $c(x) = 0$ for $x \notin \mathbb{Z}$.

Definition 2.2. Let (Γ, t, f, a, R) be a Kummer twist datum (Definition 2.1).

1. $\mathbb{V}_f$ is the local system of rank $k-1$ of solutions of the **Picard–Fuchs equation associated with** (t, f). This is the unique monic linear differential equation of order $k-1$ in d/dt whose local solutions have basis $F(t), \tau(t)F(t), \dots, \tau(t)^{k-2}F(t)$, where $\tau(t)$ is a local inverse of the Hauptmodul and $F(t) = f(\tau(t))$. The **singular set** of the datum is the set $S \subset \mathbb{P}^1(\bar{\mathbb{Q}})$ of singular points of this equation together with $t = 0$ and $t = \infty$. We write ρ_P for the smallest local exponent of this equation at $t = \infty$, in the parameter $1/t$.

2. $\mathbb{L}_a$ is the Kummer local system of t^a on $\mathbb{P}^1 \setminus \{0, \infty\}$, with monodromy $e^{2\pi i a}$ around $t = 0$.

3. The **twisted parabolic cohomology** of the datum is

$$H_a = H^1_{\mathrm{par}}(\mathbb{P}^1 \setminus S,\ \mathbb{V}_f \otimes \mathbb{L}_a) = \operatorname{im}\big(H^1_c(\mathbb{P}^1 \setminus S, \mathbb{V}_f \otimes \mathbb{L}_a) \to H^1(\mathbb{P}^1 \setminus S, \mathbb{V}_f \otimes \mathbb{L}_a)\big).$$

Definition 2.3. Let (Γ, t, f, a, R) be a Kummer twist datum (Definition 2.1), with singular set S (Definition 2.2). Its **field** L is the subfield of $\mathbb{C}$ generated over $\mathbb{Q}(\zeta_d)$ by the values of the multiplier system χ_f and, if k is odd, by the square roots of the determinants of the primitive integral matrices representing elements of Γ. Its **exceptional set** Σ is the set of primes p such that at least one of the following holds:

- $p \in \{2, 3\}$ or $p \mid dMn_\Gamma n_f$;
- $p < k$;
- p ramifies in L;
- two points of S have the same reduction at some prime of $\bar{\mathbb{Q}}$ above p.

Lemma 2.4. *Let (Γ, t, f, a, R) be a Kummer twist datum, with local systems $\mathbb{V}_f$, $\mathbb{L}_a$, singular set S and twisted parabolic cohomology H_a; put $U = \mathbb{P}^1 \setminus S$, let $o : U \hookrightarrow \mathbb{P}^1$ be the inclusion and $\mathbb{W} = \mathbb{V}_f \otimes \mathbb{L}_a$. Let $\mathcal{P}_{k-2}$ be the space of complex polynomials of degree at most $k-2$ in X, with the bilinear form $\langle\,,\rangle$, the $(k-2)$-th symmetric power of $\langle X-u, X-v\rangle = u - v$. For $\tau \in \mathbb{H}$ and $p+q = k-2$ put $\mathcal{H}^{p,q}_\tau = \mathbb{C}\,(X-\tau)^p(X-\bar\tau)^q$, and let $h(\Pi_1, \Pi_2) = (-i)^{k-2}\langle \Pi_1, \bar\Pi_2\rangle$, where $\bar\Pi$ has the conjugate coefficients.*

1. *Let $t_0 \in U$ and $\tau_0 \in \mathbb{H}$ with $t(\tau_0) = t_0$. The map $\Pi \mapsto f\Pi$, through a local inverse of t near τ_0, identifies $\mathcal{P}_{k-2}$ with the stalk $\mathbb{V}_{f,t_0}$. The subspaces and the form transported from the $\mathcal{H}^{p,q}_{\tau_0}$ and h do not depend on τ_0, and they define on $\mathbb{V}_f$ a polarised complex variation of Hodge structure of weight $k-2$, in which $F^p\mathbb{V}_{f,t_0}$ consists of the solutions vanishing to order at least p at t_0.*

2. *With $\mathbb{L}_a$ unitary of type $(0,0)$, every branch of t^a having norm one, $\mathbb{W}$ carries a polarised complex variation of Hodge structure of weight $k-2$, and its local monodromies at the points of S have roots of unity as eigenvalues. Through the identification of H_a with $H^1(\mathbb{P}^1, o_*\mathbb{W})$ of Lemma A.3(1), H_a carries the decomposition $H_a = \bigoplus_{p+q=k-1} H_a^{p,q}$ of Lemma A.3(2); its Hodge filtration is $F^pH_a = \bigoplus_{p'\geq p} H_a^{p',k-1-p'}$.*

3. *If χ_f takes real values, then $\overline{\mathbb{W}}$, with the conjugate variation, is isomorphic to $\mathbb{V}_f \otimes \mathbb{L}_{1-a}$ by an isomorphism of local systems that preserves types and Hodge metrics.*

Proof. The variation on $\mathbb{V}_f$ is the $(k-2)$-th symmetric power of the weight-one variation of the elliptic curves $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$, as in [62, §12, (12.9)], carried to $\mathbb{V}_f$ by the solutions $f\Pi$ of Lemma A.1(5).

Proof of (1). Let $\mathrm{SL}_2(\mathbb{R})$ act on $\mathcal{P}_{k-2}$ on the right by $(\Pi|\gamma)(X) = (\gamma_{21}X+\gamma_{22})^{k-2}\Pi(\gamma X)$; the form $\langle\,,\rangle$ is invariant. Put $F_\tau^p = (X-\tau)^p\mathcal{P}_{k-2-p}$. This is the $(k-2)$-th symmetric power of the weight-one variation $F_\tau^1 = \mathbb{C}(X-\tau)$: the F_τ^p vary holomorphically with $\partial_\tau F_\tau^p \subset F_\tau^{p-1}$, h is invariant under $|\gamma$ for real γ, the $\mathcal{H}_\tau^{p,q}$ are h-orthogonal, and $(-1)^q h$ is positive definite on $\mathcal{H}_\tau^{p,q}$; in particular $\|(X-\bar\tau)^{k-2}\|^2 = (2\operatorname{Im}\tau)^{k-2}$ for the Hodge metric. Let $t_0 \in U$ and $\tau_0 \in \mathbb{H}$ over t_0. By Lemma A.1(3), $e=1$ at τ_0, so t is injective on a disc B around τ_0, and $f(\tau_0)\neq 0$; by Lemma A.1(5), the solutions on a disc around t_0 inside $t(B)\setminus S$ are the $f\Pi$, $\Pi\in\mathcal{P}_{k-2}$, through $(t|_B)^{-1}$. Transport $\mathcal{H}_\tau^{p,q}$, F_τ^p and h by $\Pi \mapsto f\Pi$. Since the germ of $f\Pi$ at $\gamma\tau_0$ is that of $\chi_f(\gamma)\, f\cdot(\Pi|\gamma)$ at τ_0 for γ in the preimage $\bar\Gamma$ of Γ in $\mathrm{SL}_2(\mathbb{R})$, and $|\chi_f(\gamma)|=1$, this does not depend on τ_0. It is a polarised complex variation of weight $k-2$ on $\mathbb{V}_f$, and $F^p\mathbb{V}_{f,t_0}$ consists of the solutions vanishing to order $\geq p$ at t_0.

Proof of (2). On $\mathbb{W}$ we take the tensor product with the unitary local system $\mathbb{L}_a$ of type $(0,0)$, whose flat Hermitian form gives every germ of a branch of t^a norm 1. The local monodromies of $\mathbb{W}$ have roots of unity as eigenvalues: those of $\mathbb{V}_f$ are $e^{2\pi i\lambda}$ with $\lambda\in\mathbb{Q}$, by Lemma A.1(1), (2), and those of $\mathbb{L}_a$ are $e^{\pm 2\pi i a}$ or 1. So $\mathbb{W}$ satisfies the hypotheses of Lemma A.3 with $w=k-2$. By its item (1), H_a, which is the image of $H_c^1(U,\mathbb{W})$ in $H^1(U,\mathbb{W})$, is identified with $H^1(\mathbb{P}^1, o_*\mathbb{W})$, and its item (2) gives the decomposition, of weight $w+1=k-1$.

Proof of (3). If χ_f is real-valued, conjugation of coefficients, $f\Pi\mapsto f\bar\Pi$, is a flat antilinear involution of $\mathbb{V}_f$, since $\Pi\mapsto\chi_f(\gamma)\,\Pi|\gamma$ commutes with it; it maps $\mathcal{H}_\tau^{q,p}$ onto $\mathcal{H}_\tau^{p,q}$ and preserves the Hodge metric. So it induces a flat isomorphism from $\overline{\mathbb{V}_f}$, with the conjugate variation, onto $\mathbb{V}_f$ which preserves types and Hodge metrics. Also $\overline{\mathbb{L}_a}\cong\mathbb{L}_{1-a}$ isometrically, both being unitary of rank one and type $(0,0)$ with monodromy $e^{-2\pi i a} = e^{2\pi i(1-a)}$ around $t=0$ and trivial monodromy at the points of $S\setminus\{0,\infty\}$. Hence $\overline{\mathbb{W}}\cong\mathbb{V}_f\otimes\mathbb{L}_{1-a}$, preserving types and Hodge metrics.

□

Definition 2.5. For a Kummer twist datum with twisted differential Φ and twisted parabolic cohomology H_a, we use the following labels for the conditions of Definition 1.2.

(R) $\dim_{\mathbb{C}} H_a = 2$.

(P) $H_a = H_a^{k-1,0}\oplus H_a^{0,k-1}$, with both summands of dimension one.

(F) $\Phi\neq 0$ and Φ is square-integrable on the cover $s^d=t$: $\operatorname{ord}_{\tau_0}\Phi > -1$ at every $\tau_0\in\mathbb{H}$, where the order is taken in $\tau-\tau_0$ and may be fractional, and $\operatorname{ord}_c\Phi > 0$ at every cusp c, in the local parameter of Γ at c.

2.2 Local criteria

Local monodromy determines the rank in (R), and local orders of Φ determine square-integrability in (F). When the multiplier is real and the rank is two, the two lines in (P) follow if the differentials for a and $1-a$ are both nonzero and square-integrable. We use the local exponents computed in Lemma A.1.

Write H_Φ for the untwisted factor in the coefficient of dq/q. On a local branch of t^a we have

$$H_\Phi(\tau) = \frac{1}{2\pi i}\, f(\tau) R(t(\tau)) \frac{t'(\tau)}{t(\tau)}, \qquad f_\Phi(\tau) = H_\Phi(\tau) t(\tau)^a, \qquad \Phi = 2\pi i\, f_\Phi(\tau)\, d\tau.$$

Here $f_\Phi(\tau)$ denotes the local analytic coefficient. On the preferred branch at infinity its expansion in $Q = e^{2\pi i \tau/d}$ is the series $f_\Phi(Q)$ of Definition 2.1. The function H_Φ is meromorphic of weight k with multiplier χ_f, and $|f_\Phi| = |H_\Phi|\,|t|^a$ is independent of the branch.

Lemma 2.6. *Let (Γ, t, f, a, R) be a Kummer twist datum, with singular set S, local systems $\mathbb{V}_f$, $\mathbb{L}_a$ and twisted parabolic cohomology H_a. Put $\mathbb{W} = \mathbb{V}_f \otimes \mathbb{L}_a$ and $U = \mathbb{P}^1 \setminus S$. For $s \in S$ let T_s be the local monodromy of $\mathbb{W}$ at s, acting on the stalk at a point near s, and $\mathbb{W}^{I_s} = \ker(T_s - 1)$; define $\mathbb{V}_f^{I_s}$ in the same way.*

1. *The local monodromy of $\mathbb{V}_f$ at $t = 0$ is unipotent, and T_0 has $e^{2\pi i a} \neq 1$ as its only eigenvalue. In particular $\mathbb{W}^{I_0} = 0$ and $H^0(U, \mathbb{W}) = 0$.*
2. *$\dim_{\mathbb{C}} H_a = (|S| - 2)(k-1) - \sum_{s \in S} \dim \mathbb{W}^{I_s}$.*
3. *For $s \in S \setminus \{0, \infty\}$, $\mathbb{W}^{I_s} \cong \mathbb{V}_f^{I_s}$. If moreover $\mathbb{W}^{I_\infty} = 0$, then*

$$\dim H_a = \sum_{s \in S \setminus \{0,\infty\}} (k - 1 - \dim \mathbb{V}_f^{I_s}).$$

Proof. Item (2) is the dimension formula of Dettweiler and Wewers for parabolic cohomology on a punctured sphere [19], §1.3. It is stated there under the vanishing of the invariants only; its proof also needs the vanishing of the coinvariants, and both hold here by (1). Items (1) and (3) are local.

Proof of (1). The cusp ∞ has width one and t has a simple zero there, so q is a holomorphic function of t near $t = 0$ with $q = t\,(1 + O(t))$, and near $t = 0$

$$\tau = \frac{1}{2\pi i} \log t + \frac{1}{2\pi i} \log \frac{q}{t},$$

where the second term is holomorphic at $t = 0$. The functions $f\tau^i = F(t)\tau^i$, $0 \le i \le k-2$, are linearly independent, since a linear relation would make a nonzero polynomial in τ vanish on an open set; so they form a basis of the solutions on a slit punctured disc around $t = 0$. Continuation once around $t = 0$ in the positive sense fixes $F(t)$ and replaces τ by $\tau + 1$. So the monodromy of $\mathbb{V}_f$ at 0 is $F\tau^i \mapsto F(\tau+1)^i$, which is unipotent. Tensoring with $\mathbb{L}_a$, T_0 is $e^{2\pi i a}$ times a unipotent map, and $e^{2\pi i a} \neq 1$ as $0 < a < 1$. So $\ker(T_0 - 1) = 0$. A global flat section of $\mathbb{W}$ is fixed by T_0, hence zero.

Proof of (2). Let $m = |S|$, let $x \in U$ be a base point and $W = \mathbb{W}_x$, of dimension n. By (1), $T_0 - 1$ is invertible on W, so $\pi_1(U, x)$ has no invariants on W, and no coinvariants, since $(T_0 - 1)W = W$. By [19], §1.3, Proposition, part (ii), H_a is the parabolic cohomology $H^1(\mathbb{P}^1, o_*\mathbb{W})$, where $o : U \hookrightarrow \mathbb{P}^1$ is the inclusion; and by the Remark there, which applies since

both the invariants and the coinvariants vanish, its dimension is $(m-2)\,n - \sum_{s\in S} \dim \mathbb{W}^{I_s}$, the local monodromies being the T_s.

Proof of (3). For $s \notin \{0,\infty\}$, t is holomorphic and nonzero near s, so t^a has a single-valued branch near s and $\mathbb{L}_a$ has trivial monodromy at s. Hence T_s is the local monodromy of $\mathbb{V}_f$ at s, and $\mathbb{W}^{I_s} \cong \mathbb{V}_f^{I_s}$. If moreover $\mathbb{W}^{I_\infty} = 0$, then (2) and (1) give $\dim H_a = (m-2)n - \sum_{s\neq 0,\infty} \dim \mathbb{V}_f^{I_s} = \sum_{s\in S\setminus\{0,\infty\}} (n - \dim \mathbb{V}_f^{I_s})$.

□

Lemma 2.7. *Let (Γ, t, f, a, R) be a Kummer twist datum, with $a = j/d$ in lowest terms, and put $\mathbb{W} = \mathbb{V}_f \otimes \mathbb{L}_a$. If $R \neq 0$, let $r_\infty = \operatorname{ord}_{t=\infty} R$, in the parameter $1/t$. In (3) and (4) assume $\Phi \neq 0$, which holds if and only if $R \neq 0$.*

1. *$\rho_P = \nu_P/e_P$ if P is not a cusp, and $\rho_P = \nu_P$ if P is a cusp.*
2. *The invariants $\mathbb{W}^{I_\infty}$ of the local monodromy of $\mathbb{W}$ at $t=\infty$ have dimension $\#\{0 \le i \le k-2 : \frac{\nu_P+i}{e_P} - a \in \mathbb{Z}\}$ if P is not a cusp, which is 0 if $d \nmid e_P$; if P is a cusp, the dimension is 1 if $\nu_P - a \in \mathbb{Z}$ and 0 otherwise.*
3. *Condition (F) holds at the points over P (the points of $\Gamma\tau_P$, or the cusp P) if and only if $a < \rho_P + r_\infty$. In particular, for $R = 1$ it holds there if and only if $a < \rho_P$.*
4. *At the points of $\mathbb{H}$ not over P and at the cusps other than ∞ and P, the orders of Φ entering (F) do not depend on a, and (F) always holds at the cusp ∞. If (Γ, t, f, a, R) satisfies (F) and $1 - a < \rho_P + r_\infty$, then $(\Gamma, t, f, 1-a, R)$ satisfies (F).*

Proof. Everything is read off from the local exponents of Lemma A.1 at $t=\infty$ and from the order of t at the points over P.

Proof of (1) and (2). Apply Lemma A.1(1) with $\tau_1 = \tau_P$, so that $t_1 = \infty$, $e_1 = e_P$ and $\nu_1 = \nu_P$, or Lemma A.1(2) with $c = P$. The exponents at $t=\infty$ are $\frac{\nu_P+i}{e_P}$, $0 \le i \le k-2$, whose smallest is ν_P/e_P, respectively all equal to ν_P; this is (1). A positive loop around $1/t = 0$ is a negative loop around $t = 0$ in $\mathbb{C}^\times$, so $\mathbb{L}_a$ has monodromy $e^{-2\pi i a}$ there. If P is not a cusp, the local monodromy of $\mathbb{W}$ at ∞ is diagonal in the basis $(x^{\lambda_i} V_i) \otimes t^a$ given by Lemma A.1(1), with eigenvalues $e^{2\pi i(\frac{\nu_P+i}{e_P} - a)}$; this gives the count. If $\frac{\nu_P+i}{e_P} - a = m \in \mathbb{Z}$, then $\frac{j}{d} = \frac{\nu_P+i-me_P}{e_P}$, and since j/d is in lowest terms, $d \mid e_P$. If P is a cusp, the local monodromy is $e^{2\pi i(\nu_P - a)}$ times a unipotent map with one Jordan block, by Lemma A.1(2); its fixed space has dimension 1 if $e^{2\pi i(\nu_P-a)} = 1$ and 0 otherwise.

Proof of (3). Suppose first that P is not a cusp. Since $1/t$ is a coordinate on X_Γ vanishing simply at P, and $((\tau-\tau_P)/(\tau-\bar\tau_P))^{e_P}$ is a local coordinate of X_Γ at P, $1/t$ has order e_P at τ_P. So $t = (\tau-\tau_P)^{-e_P} u(\tau)$ with u holomorphic and $u(\tau_P) \neq 0$, and $t'/t = -e_P(\tau-\tau_P)^{-1} + u'/u$ has order -1. Moreover $R(t)$ has order $e_P r_\infty$, and $|t^a| = |\tau-\tau_P|^{-ae_P}|u|^a$. Hence

$$\operatorname{ord}_{\tau_P} \Phi = \nu_P + e_P r_\infty - 1 - a e_P,$$

and $\operatorname{ord}_{\tau_P} \Phi > -1$ if and only if $a < \nu_P/e_P + r_\infty = \rho_P + r_\infty$, by (1). For γ in the preimage $\bar\Gamma$ of Γ in $\mathrm{SL}_2(\mathbb{R})$ the order at $\gamma\tau_P$ is the same, because $|f_\Phi(\gamma\tau)| = |\gamma_{21}\tau + \gamma_{22}|^k |f_\Phi(\tau)|$, the factor $\gamma_{21}\tau + \gamma_{22}$ has no zero near τ_P, and $\gamma\tau - \gamma\tau_P$ vanishes to order one at τ_P.

Suppose now that $P = \sigma\infty$ is a cusp, with local parameter q_P. As t has a simple pole at P and q_P is a local coordinate there, $t(\sigma\tau') = q_P^{-1} E(q_P)$ with E holomorphic and $E(0) \neq 0$. The four factors of

$$f_\Phi|_k\sigma = (f|_{k-2}\sigma) \cdot R(t\circ\sigma) \cdot (t\circ\sigma)^a \cdot \frac{1}{2\pi i}\frac{d}{d\tau'} \log t(\sigma\tau')$$

have orders ν_P, r_∞, $-a$ and 0 in q_P, the last because $\frac{1}{2\pi i}\frac{d}{d\tau'}\log t(\sigma\tau') = \frac{1}{h_P}q_P\frac{d}{dq_P}\log t(\sigma\tau') = \frac{1}{h_P}(-1+O(q_P))$. So $\operatorname{ord}_P\Phi = \nu_P + r_\infty - a$, and (F) at P holds if and only if $a < \rho_P + r_\infty$, by (1).

Proof of (4). At $\tau_0 \in \mathbb{H}\setminus\Gamma\tau_P$, t is holomorphic and nonzero near τ_0, so t^a has a holomorphic branch without zeros there, and $\operatorname{ord}_{\tau_0}\Phi = \operatorname{ord}_{\tau_0}H_\Phi$ does not depend on a. At a cusp $c = \sigma\infty \notin \{\infty, P\}$, $t(c)\in\mathbb{C}^\times$, so $(t\circ\sigma)^a$ has a holomorphic branch near $q_c = 0$ which is a unit there, and the order of Φ at c is that of $H_\Phi|_k\sigma$. At the cusp ∞ write $t = q\,E(q)$ with $E(0)=1$: then $f = F(t)$ has order 0, $\frac{1}{2\pi i}t'/t = q\frac{d}{dq}\log t = 1+O(q)$, $R(t)$ has order $\operatorname{ord}_{t=0}R \geq 0$, and $t^a = Q^jE^a$ has order a in q; so $\operatorname{ord}_\infty\Phi = a + \operatorname{ord}_{t=0}R > 0$. For the datum $(\Gamma, t, f, 1-a, R)$, $\Phi\neq 0$ still holds since $R\neq 0$; its orders at the points of $\mathbb{H}$ not over P and at the cusps other than ∞, P are those of the original datum, it satisfies (F) at ∞, and by (3) it satisfies (F) at the points over P when $1-a < \rho_P + r_\infty$.

□

Lemma 2.8. *Let (Γ, t, f, a, R) be a Kummer twist datum satisfying (F). Then $H_a^{0,k-1}\neq 0$. If moreover χ_f takes real values, then $H_{1-a}^{k-1,0}\neq 0$, where H_{1-a} is the twisted parabolic cohomology of the datum $(\Gamma, t, f, 1-a, R)$.*

Proof. The twisted differential, paired with the evaluation of germs of $\mathbb{V}_f\otimes\mathbb{L}_a$, is a square-integrable holomorphic 1-form with values in the top Hodge bundle of the dual variation; by Lemma A.3(3) it is harmonic of extreme type, and conjugation carries its class into $H_a^{0,k-1}$. Throughout, $U = \mathbb{P}^1\setminus S$, $o: U\hookrightarrow\mathbb{P}^1$ is the inclusion, $\mathbb{W} = \mathbb{V}_f\otimes\mathbb{L}_a$ carries the polarised complex variation of weight $k-2$ of Lemma 2.4(2), with $\mathcal{P}_{k-2}$, $\mathcal{H}_\tau^{p,q}$ and h as there, and $H_a = H^1(\mathbb{P}^1, o_*\mathbb{W})$ carries the Hodge decomposition of that item.

The dual variation. The dual $\mathbb{W}^\vee$ carries the dual variation, of weight $2-k$: $(\mathbb{W}^\vee)^{-p,-q}$ is the annihilator of $\bigoplus_{(p',q')\neq(p,q)}\mathbb{W}^{p',q'}$, and its form $h^\vee$ is transported from h by the antilinear isomorphism $w\mapsto h(\,\cdot\,, w)$, which maps $\mathbb{W}^{p,q}$ onto $(\mathbb{W}^\vee)^{-p,-q}$. It is again polarised, its Hodge metric is the dual of that of $\mathbb{W}$, and its top piece is $F^0\mathbb{W}^\vee = (F^1\mathbb{W})^\perp = (\mathbb{W}^\vee)^{0,2-k}$.

(F) forces R to be holomorphic on U, which makes the form below holomorphic. Let $t_1\in U$ and suppose R has a pole of order $r\geq 1$ at t_1. Since the cusps map into S by Lemma A.1(3), $t_1 = t(\tau_1)$ with $\tau_1\in\mathbb{H}$, of stabiliser order e_1. Then f has order 0 at τ_1 by Lemma A.1(3), $t-t_1$ has order e_1 and t'/t order e_1-1, and t^a is a unit near τ_1; so $\operatorname{ord}_{\tau_1}\Phi = e_1(1-r)-1\leq -1$, against (F).

The form. For $t_0\in U$ let $\mathrm{ev}_{t_0}\in\mathbb{W}_{t_0}^\vee$ be evaluation at t_0 of germs of $\mathbb{W}$, which are germs of holomorphic functions $t^a y$. In a flat frame $w_1,\dots,w_n$ of $\mathbb{W}$ on a disc, $\mathrm{ev} = \sum_i w_i(t)\,w_i^\vee$ with holomorphic coefficients, so ev is a holomorphic section of $\mathbb{W}^\vee\otimes\mathcal{O}_U$. It is nonzero, and it kills $F^1\mathbb{W}_{t_0}$, whose elements vanish at t_0; so it is a section of the top Hodge bundle $F^0\mathbb{W}^\vee$. Put

$$\omega_\Phi^{\mathrm{an}} = \frac{1}{2\pi i}\,R(t)\,\frac{dt}{t}\otimes\mathrm{ev},$$

a holomorphic section of $\Omega^1_U\otimes F^0\mathbb{W}^\vee$, nonzero because $R\neq 0$ ($\Phi\neq 0$ is part of (F)). For the germ $t^a f$ of $\mathbb{W}$ near $t=0$, $\langle\omega_\Phi^{\mathrm{an}}, t^a f\rangle = \frac{1}{2\pi i}\Phi$.

Square-integrability. For $t_0 = t(\tau_0)\in U$, the orthogonal complement of $F^1\mathbb{W}_{t_0}$ is the line $\mathcal{H}^{0,k-2}$ spanned by the germ $v_0 = t^a f(\tau)(\tau-\bar\tau_0)^{k-2}$, of norm $(2\operatorname{Im}\tau_0)^{(k-2)/2}$, and $\mathrm{ev}_{t_0}(v_0) = t(\tau_0)^a f(\tau_0)(\tau_0-\bar\tau_0)^{k-2}$. Hence

$$\|\mathrm{ev}_{t_0}\|^2 = |t(\tau_0)|^{2a}|f(\tau_0)|^2(2\operatorname{Im}\tau_0)^{k-2}.$$

The L^2 norm of a $(1,0)$-form does not depend on the conformal metric on U: for $\omega = \phi\,dt$ it is $\int\|\phi\|^2\,dA$, with dA Lebesgue measure in the coordinate t. Pulling back by t, $\|\omega_\Phi^{\mathrm{an}}\|^2 dA$ becomes

$2^{k-2}|f_\Phi(\tau)|^2(\operatorname{Im}\tau)^{k-2}\,d\operatorname{Re}\tau\,d\operatorname{Im}\tau$, a Γ-invariant measure since $|f_\Phi(\gamma\tau)| = |\gamma_{21}\tau+\gamma_{22}|^k|f_\Phi(\tau)|$. As $\mathbb{P}^1$ is compact, $\|\omega_\Phi^{\mathrm{an}}\|_{L^2} < \infty$ if and only if the integral is finite near each point $t_1 \in \mathbb{P}^1$. If $t_1 = t(\tau_1)$ with $\tau_1 \in \mathbb{H}$, a small disc B around τ_1 covers a punctured neighbourhood of t_1 exactly e_1 times, and $|f_\Phi|^2 = |\tau-\tau_1|^{2\alpha}\times$(a positive continuous function) with $\alpha = \operatorname{ord}_{\tau_1}\Phi$; so the integral is finite if and only if $\alpha > -1$. If $t_1 = t(c)$ for a cusp $c = \sigma\infty$ of width h_c, then q_c is a local coordinate at c, a neighbourhood of t_1 is the image of $\{\sigma\tau' : 0 \le \operatorname{Re}\tau' < h_c,\ \operatorname{Im}\tau' > Y\}$, and the integral becomes $2^{k-2}\int|(f_\Phi|_k\sigma)(\tau')|^2(\operatorname{Im}\tau')^{k-2}d\operatorname{Re}\tau'\,d\operatorname{Im}\tau'$, with $|(f_\Phi|_k\sigma)(\tau')| = |b|e^{-2\pi\beta\operatorname{Im}\tau'/h_c}(1+o(1))$, $b\neq 0$, $\beta = \operatorname{ord}_c\Phi$; it is finite if and only if $\beta > 0$, since $k \ge 2$. So (F) gives $\omega_\Phi^{\mathrm{an}} \in L^2$.

Harmonicity. We apply Lemma A.3(3) to $\mathbb{W}^\vee$, a polarised complex variation of weight $2-k$ whose local monodromies have roots of unity as eigenvalues (the inverses of those of $\mathbb{W}$), with $p_0 = 0$, and to $\omega_\Phi^{\mathrm{an}}$, a nonzero holomorphic section of $\Omega^1_U \otimes F^0\mathbb{W}^\vee$ which is square-integrable by the previous step. So $\omega_\Phi^{\mathrm{an}}$ is harmonic of type $(0+1, 2-k+0) = (1, 2-k)$, and its class is a nonzero element of $H^1(\mathbb{P}^1, o_*\mathbb{W}^\vee)^{1,2-k}$.

Transfer to H_a. Give $\overline{\mathbb{W}}$ the conjugate variation. The map $\overline{\mathbb{W}} \to \mathbb{W}^\vee$, $\bar w \mapsto h(\,\cdot\,, w)$, is a flat $\mathbb{C}$-linear isomorphism and an isometry for the Hodge metrics, and it maps $\overline{\mathbb{W}}^{q,p} = \overline{\mathbb{W}^{p,q}}$ onto $(\mathbb{W}^\vee)^{-p,-q} = (\mathbb{W}^\vee)^{q-k+2,p-k+2}$. By Lemma A.3(4) with $n = 2-k$, $H^1(\mathbb{P}^1, o_*\overline{\mathbb{W}})^{k-1,0}$ is isomorphic to $H^1(\mathbb{P}^1, o_*\mathbb{W}^\vee)^{1,2-k} \neq 0$. By Lemma A.3(2), complex conjugation of forms maps $H_a^{0,k-1}$ onto $H^1(\mathbb{P}^1, o_*\overline{\mathbb{W}})^{k-1,0}$. Hence $H_a^{0,k-1} \neq 0$.

The second assertion. If χ_f is real-valued, then $\overline{\mathbb{W}} \cong \mathbb{V}_f \otimes \mathbb{L}_{1-a}$, preserving types and Hodge metrics, by Lemma 2.4(3). So Lemma A.3(4) with $n = 0$, together with the conjugation in Lemma A.3(2), gives $H^{k-1,0}_{1-a} \cong H^1(\mathbb{P}^1, o_*\overline{\mathbb{W}})^{k-1,0} = \overline{H_a^{0,k-1}} \neq 0$, the Hodge decomposition of H_{1-a} being that of Lemma 2.4(2) for the datum $(\Gamma, t, f, 1-a, R)$.

□

Proposition 2.9. *Let (Γ, t, f, a, R) be a Kummer twist datum such that χ_f takes real values, (R) holds, and both (Γ, t, f, a, R) and $(\Gamma, t, f, 1-a, R)$ satisfy (F). Then (P) holds: $H_a = H_a^{k-1,0} \oplus H_a^{0,k-1}$ with both summands of dimension one.*

Proof. By Lemma 2.8, applied to (Γ, t, f, a, R), which satisfies (F), $H_a^{0,k-1} \neq 0$. The datum $(\Gamma, t, f, 1-a, R)$ also satisfies (F), and χ_f is real-valued, so the second assertion of Lemma 2.8, applied to it, gives $H^{k-1,0}_{1-(1-a)} = H_a^{k-1,0} \neq 0$.

By Lemma 2.4(2), $H_a = \bigoplus_{p+q=k-1} H_a^{p,q}$. Since $k \ge 2$, the types $(k-1, 0)$ and $(0, k-1)$ are distinct, so $\dim H_a^{k-1,0} + \dim H_a^{0,k-1} \ge 2 = \dim H_a$ by (R). Hence both have dimension one and all other $H_a^{p,q}$ vanish, which is (P).

□

2.3 When the expansion is modular

Since t has no zeros in $\mathbb{H}$, the only possible interior branch points of t^a lie over its pole P. At a lift of an interior P, the order of t is $-e_P$, so the local exponent of t^a is $-je_P/d$. This gives the following criterion for the modularity of f_Φ.

Proposition 2.10. *Let (Γ, t, f, a, R) be a Kummer twist datum with $\Phi \neq 0$, write $a = j/d$ in lowest terms, and let $\bar\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be the preimage of Γ. There is $y_0 > 0$ such that $f_\Phi(Q)$ converges for $|Q| < e^{-2\pi y_0/d}$, and the preferred branch $f_\Phi(\tau)$ is holomorphic on $\mathcal{H}_{y_0} = \{\tau \in \mathbb{H} : \operatorname{Im}\tau > y_0\}$.*

1. *If P is not a cusp and $d \nmid e_P$, then for no $\lambda > 0$ is there a meromorphic function on $\mathbb{H}$ which agrees with $\tau \mapsto f_\Phi(\lambda\tau)$ on a nonempty open subset of $\lambda^{-1}\mathcal{H}_{y_0}$. In particular f_Φ is*

not the restriction to $\mathcal{H}_{y_0}$ of a single-valued meromorphic function on $\mathbb{H}$, and it is not a meromorphic modular form, of any weight and multiplier system, for any subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$.

2. *If P is a cusp or $d \mid e_P$, then f_Φ extends to a meromorphic function G on $\mathbb{H}$, and there is a surjective character $\chi_a : \bar\Gamma \to \mu_d$ with $\chi_a(-1) = 1$ and $\chi_a\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = e^{2\pi i a}$ such that*

$$G(\gamma\tau) = \chi_f(\gamma)\,\chi_a(\gamma)\,(\gamma_{21}\tau + \gamma_{22})^k\,G(\tau) \qquad (\gamma = (\gamma_{mn}) \in \bar\Gamma).$$

If moreover the datum satisfies (F), then G is a cusp form of weight k with trivial multiplier on the subgroup $\ker(\chi_f\chi_a)$ of finite index in $\bar\Gamma$; equivalently, a cusp form of weight k and multiplier system χ_f on the subgroup $\ker\chi_a$, whose image in Γ has index d.

Proof. The untwisted factor H_Φ is meromorphic on $\mathbb{H}$. It remains to determine when the preferred branch of t^a at infinity extends to $\mathbb{H}$.

The branch at infinity. Since the cusp ∞ has width one, q is a local coordinate of X_Γ at ∞; as t has a simple zero there, $t = q\,E(q)$ with E holomorphic on a disc $|q| < r$ and $E(0) = 1$. Shrinking r, we may assume that E has no zeros, that F converges on $t(\{|q| < r\})$ with $f = F(t)$ there, and that $R(t)$ is holomorphic there (R is regular at $t = 0$). Choose y_0 with $e^{-2\pi y_0} = r$. On the disc, $E^a = \exp(a \log E)$, with the branch of $\log E$ vanishing at $q = 0$, is holomorphic, and its Taylor series is the binomial series $(t/q)^a$ of Definition 2.1. On $\mathcal{H}_{y_0}$ the preferred branch is therefore $\ell_a = Q^j E(q)^a$, with $\ell_a^d = t^j$. The product $H_\Phi \ell_a$ is holomorphic there. Since $t'/(2\pi i t) = (q/t)\,dt/dq$, its expansion in Q is precisely the series $f_\Phi(Q)$, which thus converges for $|Q| < r^{1/d}$. Also $H_\Phi \not\equiv 0$, since $F(0) = 1$, $R \not\equiv 0$ and t is nonconstant.

The order of t over P. Suppose P is not a cusp, let $\tau_P \in \mathbb{H}$ lie over P, and put $w = (\tau - \tau_P)/(\tau - \bar\tau_P)$. The stabiliser of τ_P in Γ is cyclic of order e_P, acts on w by the rotations $w \mapsto \zeta w$ with $\zeta^{e_P} = 1$, and w^{e_P} is a local coordinate of X_Γ at P. As t has a simple pole at P, $1/t = \Xi(w^{e_P})$ with Ξ holomorphic near 0 and $\Xi(0) = 0 \neq \Xi'(0)$. Since w has a simple zero at τ_P,

$$\operatorname{ord}_{\tau_P} t = -e_P,$$

and, t being Γ-invariant, the same holds at every point of $\Gamma\tau_P$. Thus a small positive loop around τ_P in $\mathbb{H}$, which maps to e_P times a positive loop around P on X_Γ, multiplies every local branch of t^a by $e^{-2\pi i a e_P}$, and this factor is 1 exactly when $d \mid e_P$.

Proof of (1). Let $\lambda > 0$, and suppose that a meromorphic function G_λ on $\mathbb{H}$ agrees with $f_\Phi(\lambda\,\cdot)$ on a nonempty open set $V \subset \lambda^{-1}\mathcal{H}_{y_0}$. Then $G(\tau) = G_\lambda(\tau/\lambda)$ is meromorphic on $\mathbb{H}$ and agrees with f_Φ on the nonempty open set $\lambda V \subset \mathcal{H}_{y_0}$. The function $u = G/H_\Phi$ is meromorphic on $\mathbb{H}$ and equals ℓ_a on λV minus the discrete zero set of H_Φ. Hence $u^d - t^j$ is a meromorphic function on the connected set $\mathbb{H}$ vanishing on a nonempty open set, so $u^d = t^j$ on $\mathbb{H}$. Taking orders at τ_P gives

$$d \cdot \operatorname{ord}_{\tau_P} u = j \cdot \operatorname{ord}_{\tau_P} t = -j e_P.$$

So $d \mid j e_P$, and since $\gcd(j, d) = 1$, $d \mid e_P$, a contradiction. The case $\lambda = 1$ is the statement about the restriction to $\mathcal{H}_{y_0}$. A meromorphic modular form for a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ is in particular a meromorphic function on $\mathbb{H}$, so f_Φ is not one.

Proof of (2): a global branch of t^a. Let $Z = \Gamma\tau_P$ if P is not a cusp and $Z = \emptyset$ if P is a cusp. Then Z is closed and discrete in $\mathbb{H}$, and t is holomorphic without zeros on $\mathbb{H} \setminus Z$. For a closed piecewise smooth path β in $\mathbb{H} \setminus Z$, the argument principle in the simply connected domain $\mathbb{H}$ gives

$$\frac{1}{2\pi i}\oint_\beta \frac{t'}{t}\,d\tau = \sum_{z \in Z} n(\beta, z)\operatorname{ord}_z t = -e_P \sum_{z \in Z} n(\beta, z)$$

(a finite sum; it is 0 if $Z = \emptyset$), where $n(\beta, z)$ is the winding number. If $d \mid e_P$ then $ae_P \in \mathbb{Z}$, so $a \oint_\beta t'/t \, d\tau \in 2\pi i\mathbb{Z}$ for every such β. Fix $\tau_1 \in \mathcal{H}_{y_0}$ and a logarithm L_1 of $\ell_a(\tau_1)$. Then

$$\tilde{\ell}(\tau) = \exp\Big(L_1 + a \int_{\tau_1}^{\tau} \frac{t'(\sigma)}{t(\sigma)} \, d\sigma\Big),$$

integrated along any path in $\mathbb{H} \setminus Z$, is a well-defined holomorphic function without zeros on $\mathbb{H} \setminus Z$, with $\tilde{\ell}'/\tilde{\ell} = a\, t'/t$. On the connected set $\mathcal{H}_{y_0}$ also $\ell_a'/\ell_a = a\, t'/t$ (differentiate $\ell_a^d = t^j$), and $\tilde{\ell}(\tau_1) = \ell_a(\tau_1)$; so $\tilde{\ell} = \ell_a$ on $\mathcal{H}_{y_0}$, and $\tilde{\ell}^d = t^j$ on the connected set $\mathbb{H} \setminus Z$. Near $z \in Z$, $|\tilde{\ell}| = |t|^a$ is bounded by a constant times $|\tau - z|^{-ae_P}$ with $ae_P \in \mathbb{Z}$, so by Riemann's removable singularity theorem $\tilde{\ell}$ is meromorphic at z, with $\operatorname{ord}_z \tilde{\ell} = -ae_P$. Write $t^a := \tilde{\ell}$, a meromorphic function on $\mathbb{H}$. Then $G := H_\Phi \, t^a$ is meromorphic on $\mathbb{H}$ and $G = f_\Phi$ on $\mathcal{H}_{y_0}$.

Proof of (2): the transformation law. Let $\gamma \in \bar{\Gamma}$. Since $(t^a \circ \gamma)^d = t^j \circ \gamma = t^j = (t^a)^d$, the meromorphic function $(t^a \circ \gamma)/t^a$ takes values in μ_d on the connected set $\mathbb{H} \setminus (Z \cup \gamma^{-1}Z)$, hence is a constant $\chi_a(\gamma) \in \mu_d$. From $t^a \circ (\gamma\delta) = (t^a \circ \gamma) \circ \delta = \chi_a(\gamma)\, t^a \circ \delta = \chi_a(\gamma)\chi_a(\delta)\, t^a$, χ_a is a character; $\chi_a(-1) = 1$ because -1 acts trivially on $\mathbb{H}$. For $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\ell_a(\tau + 1) = e^{2\pi i j/d}\ell_a(\tau)$, since $Q \mapsto e^{2\pi i/d}Q$ and q is unchanged; so $\chi_a(T) = e^{2\pi i a}$ is a primitive d-th root of unity, χ_a is onto μ_d, and the image of $\ker \chi_a$ in Γ has index d. Differentiating $t(\gamma\tau) = t(\tau)$ gives $t'(\gamma\tau) = (\gamma_{21}\tau + \gamma_{22})^2 t'(\tau)$; together with the transformation laws of f, $R(t)$ and t^a this gives

$$G(\gamma\tau) = \chi_f(\gamma)\chi_a(\gamma)(\gamma_{21}\tau + \gamma_{22})^k G(\tau).$$

As χ_f has finite order, $\Gamma_\Phi := \ker(\chi_f\chi_a)$ has finite index in $\bar{\Gamma}$, and G is invariant under the weight-k slash action of Γ_Φ.

Proof of (2): cusp form under (F). At $\tau_0 \in \mathbb{H}$ the order of G is an integer; (F) makes it > -1, hence ≥ 0, so G is holomorphic on $\mathbb{H}$. Let $c = \sigma\infty$ be a cusp of Γ. The group $\sigma^{-1}\Gamma_\Phi\sigma$ has finite index in $\sigma^{-1}\bar{\Gamma}\sigma$, so it contains $\pm\left(\begin{smallmatrix} 1 & h' \\ 0 & 1 \end{smallmatrix}\right)$ for some $h' > 0$; hence $G|_k\sigma$ has period $2h'$ and a Laurent expansion in $e^{2\pi i\tau/(2h')}$ on a half-plane $\operatorname{Im}\tau > y_1$. This expansion has only finitely many negative terms: H_Φ is meromorphic at c, and $|t^a| = |t|^a$ grows at most like a power of $|e^{2\pi i\sigma^{-1}\tau/h_c}|^{-1}$ near c, because t is meromorphic at c. By (F) all exponents in this expansion are positive, so $G|_k\sigma(\tau) \to 0$ as $\operatorname{Im}\tau \to \infty$. The cusps of Γ_Φ are those of Γ, since the index is finite. Hence G is a cusp form of weight k with trivial multiplier on Γ_Φ. The equivalent formulation follows from the transformation law, since χ_a is trivial on $\ker \chi_a$.

□

3 Integral models and the Kummer lattice

To apply Frobenius to Φ we need an integral model of the cover $s^d = t$, together with the elliptic family supplying its coefficients. Elements of Γ may act on this family through isogenies. A modular model records the resulting action on the symmetric power, its normalisation with respect to the Weil pairing, and a cusp at which to recover f_Φ. We construct such a model at every residue-degree-one prime above $p \notin \Sigma$.

3.1 Modular models

Definition 3.1. Let (Γ, t, f, a, R) be a Kummer twist datum (Definition 2.1) with field L (Definition 2.3). Let $\mathfrak{P}$ be an unramified prime of L of residue degree one over p. Then $L_\mathfrak{P} = \mathbb{Q}_p$. We write $\iota_\mathfrak{P} : L \hookrightarrow \mathbb{Q}_p$ for the corresponding embedding and $\mathcal{O} = \mathcal{O}_{L,(\mathfrak{P})} \subset \mathbb{Z}_p$ for the localisation of $\mathcal{O}_L$ at $\mathfrak{P}$. The given inclusion $L \subset \mathbb{C}$ also defines $X'_\mathbb{C} = X' \otimes_\mathcal{O} \mathbb{C}$ for any $\mathcal{O}$-scheme X'. A **modular model** of the datum at $\mathfrak{P}$ is a tuple (X', Z', E, G, χ_G) with the following properties.

(M1) X' is a smooth proper curve over $\mathcal{O}$, whose fibres need not be geometrically connected; $Z' \subset X'$ is a closed subscheme étale over $\mathcal{O}$; and $E \to X'$ is a generalised elliptic curve, smooth over $Y' = X' \setminus Z'$, with Hodge bundle ω, whose Kodaira–Spencer map $\omega^{\otimes 2} \to \Omega^1_{X'/\mathcal{O}}(\log Z')$ is an isomorphism. For every finite unramified extension W of $\mathbb{Z}_p$ and every $z \in Z'(W)$ there are a formal parameter q_z of X'_W along z, an element $\delta_z \in W^\times$ and an integer $m_z \geq 1$ prime to p such that the pull-back of E to $\operatorname{Spf} W[[q_z]]$ is the Tate curve $\operatorname{Tate}(\delta_z q_z^{m_z})$. We write $H = H^1_{\mathrm{dR}}(E/X')(\log Z')$ for the canonical extension of the relative de Rham cohomology of $E|_{Y'}$, with its Gauss–Manin connection, Hodge filtration $\omega \subset H$ and structure of logarithmic F-crystal.

(M2) G is a finite group acting on X' over $\mathcal{O}$, not necessarily faithfully, and preserving Z'. For each $g \in G$ there are an isogeny $\pi_g : E|_{Y'} \to g^*E|_{Y'}$ of elliptic curves over Y', of degree prime to p, and $c_g \in L$ with $c_g^2 = (\deg \pi_g)^{-(k-2)}$, such that the maps $c_g \operatorname{Sym}^{k-2}(\pi_g^*)$ extend to $g^* \operatorname{Sym}^{k-2} H \to \operatorname{Sym}^{k-2} H$ and form an action of G on $\operatorname{Sym}^{k-2} H$ lifting the action on X'. This action is compatible with the connection, the Hodge filtration and the Frobenius, and preserves the pairing induced by the Weil pairing. The stabiliser in G of every geometric point of $X'_{\mathbb{F}_p}$ has order prime to p.

(M3) The Hauptmodul t extends to a G-invariant morphism $t : X' \to \mathbb{P}^1_{\mathcal{O}}$ that induces an isomorphism $X'_L/G \cong \mathbb{P}^1_L$, whose divisor of zeros is supported on Z', and whose reduction is non-constant on every connected component of $X'_{\mathbb{F}_p}$.

(M4) For some connected component X'_0 of $X'_{\mathbb{C}}$, with stabiliser $G_0 \subset G$, there are a torsion-free normal subgroup $\Gamma' \subset \Gamma$ of finite index, contained in $\ker \chi_f$ and in $\Gamma(n)$ for some $n \geq 3$ or in $\Gamma_1(n)$ for some $n \geq 5$, a surjection $G_0 \to \Gamma/\Gamma'$, and an isomorphism $X'_0 \cong \Gamma' \backslash \mathbb{H}^*$ carrying t to the Hauptmodul, the action of $g \in G_0$ to $\tau \mapsto \gamma\tau$ for the image $\gamma\Gamma'$ of g, and $E|_{X'_0}$ to the universal generalised elliptic curve. Moreover $\chi_G : G \to L^\times$ is a character of finite order, and there is a rational section $f\omega^{\otimes(k-2)}$ of $\omega^{\otimes(k-2)}$ over X'_L, equal on X'_0 to $f(\tau)\,(2\pi i\,dz)^{\otimes(k-2)}$, with $g^*(f\omega^{\otimes(k-2)}) = \chi_G(g)\, f\omega^{\otimes(k-2)}$ for all $g \in G$, where g^* is the action of (M2) on $\omega^{\otimes(k-2)} = \operatorname{Fil}^{k-2} \operatorname{Sym}^{k-2} H$.

(M5) There are a finite unramified extension W of $\mathbb{Z}_p$, a point $\infty' \in Z'(W)$ with $t(\infty') = 0$, a formal parameter q' of X'_W along ∞' and an integer $w \geq 1$ prime to p, such that over $\operatorname{Spf} W[[q']]$ one has $E = \operatorname{Tate}(q)$ with $q = {q'}^w$, the function t is the q-expansion of the Hauptmodul (Definition 2.1), and $f\omega^{\otimes(k-2)} = F(t)\,\omega_{\mathrm{can}}^{\otimes(k-2)}$.

We state existence first; the three lemmas behind it supply a Galois closure of modular function fields, a smooth integral model and prime-to-p isogenies on that model.

Proposition 3.2. *Let (Γ, t, f, a, R) be a Kummer twist datum, and let $\mathfrak{P}$ be a prime of L of residue degree one over a prime $p \notin \Sigma$. Then the datum has a modular model at $\mathfrak{P}$.*

Proof. The model is cut out of the curve of Lemma 3.4 by the constants in L; the substance is (M4), where the section $f\omega^{\otimes(k-2)}$ is shown to be rational and an eigensection of all of G. From $p \notin \Sigma$ we use only: $p \neq 2, 3$, $p \nmid M n_\Gamma n_f$, and p unramified in L. Then $\mathcal{O} = \mathcal{O}_{L,(\mathfrak{P})}$ is a discrete valuation ring, unramified over $\mathbb{Z}_{(p)}$, with residue field $\mathbb{F}_p$. Fix an integer $m \geq 3$ prime to $p n_\Gamma$, and let n_1, $\mathfrak{F}_n$, $\mathcal{K}$, κ_0, Γ', n_2, $G^\flat$, $G_0^\flat$ and D be as in Lemma 3.3; X'', Z'', E'', α, X''_0, ∞'', w as in Lemma 3.4; and π_g, e_g, B_g, $\hat{G}$ as in Lemma 3.5.

The model. Let $\kappa_1 = \kappa_0 \cap L$, a subfield of $\mathbb{C}$ unramified at p, Galois over $\mathbb{Q}$. As X'' is smooth and proper with $H^0(X''_{\mathbb{Q}}, \mathcal{O}) = \kappa_0$, $H^0(X''_{\mathcal{O}}, \mathcal{O})$ is the base change to $\mathcal{O}$ of the integral closure of

$\mathbb{Z}_{(p)}$ in κ_0; it contains $\mathcal{O}_{\kappa_1,(p)} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}$, which is étale over $\mathcal{O}$ and split, as $\kappa_1 \subset L$ is unramified at p. Let e_1 be its idempotent belonging to the identity of κ_1 (the multiplication map $\kappa_1 \otimes_{\mathbb{Q}} L \to L$), and put $X' = e_1 X''_{\mathcal{O}}$, open and closed in $X''_{\mathcal{O}}$; $Z' = Z''_{\mathcal{O}} \cap X'$, $E = E''|_{X'}$. The components of $X'_{\mathbb{C}}$ are those components of $X''_{\mathbb{C}}$ whose embedding $\kappa_0 \to \mathbb{C}$ is the inclusion on κ_1; they include X''_0. Let $G \subset \hat{G}$ be the subgroup of (g,π) with g trivial on κ_1; it is the stabiliser of X', and $G_0^\flat$ acts trivially on κ_0, so G contains $G_0 = \{(g,\pi) : g \in G_0^\flat\}$.

(M1)–(M3). Smoothness, the cusp divisor, the Tate descriptions and the Kodaira–Spencer isomorphism follow from Lemma 3.4(1) by flat base change. The action preserves Z' by part (3) of that lemma. For $(g,\pi) \in G$, write $g = g_0 d$ as in Lemma 3.3(4). By Lemma 3.5(1), $\deg \pi = e_{g_0}$ is the determinant of a primitive matrix representing an element of Γ, hence is prime to p. Its normalising factor $c = e_{g_0}^{-(k-2)/2}$ lies in L by Definition 2.3 and is a unit of $\mathcal{O}$. Part (2) of the isogeny lemma gives the action, its compatibility with the connection, filtration and Frobenius, and preservation of the Weil pairing; part (3) gives tame stabilisers, since $p \geq 5$. This proves (M2).

The properties of t in (M3) follow from Lemma 3.4(3), except for the quotient description. The surjection $G^\flat \to \mathrm{Gal}(\kappa_1/\mathbb{Q})$ permutes the subschemes $e_\iota X''_L$ transitively; the stabiliser of X'_L is the image of G, whose central elements $(1, \pm\mathrm{id})$ act trivially. Hence

$$X'_L/G \cong X''_L/G^\flat \cong \mathbb{P}^1_L.$$

(M4), the analytic data. Take $X'_0 = X''_0$. Its stabiliser in G is G_0, because $G^\flat$ acts on the components of $X''_{\mathbb{C}}$ through $\mathrm{Gal}(\kappa_0/\mathbb{Q})$ simply transitively; and $G_0 \to G_0^\flat \cong \Gamma/\Gamma'$ is onto (Lemma 3.3(3)). By Lemma 3.3(1), Γ' is torsion-free, normal of finite index in Γ, and contained in $\ker \chi_f$ and in $\Gamma(m)$, $m \geq 3$; by Lemma 3.4(2) the isomorphism $X'_0 \cong \Gamma' \backslash \mathbb{H}^*$ has the required properties.

The section on X'_0 and the character on G_0. Let $s_0 = f(\tau)(2\pi i\, dz)^{\otimes(k-2)}$ on X'_0, a meromorphic, hence rational, section of $\omega^{\otimes(k-2)}$, since f is a meromorphic form of weight $k-2$ invariant under $\Gamma' \subset \ker \chi_f$ (the elements of $\Gamma' \subset \Gamma(m)$ lift to matrices $\equiv I \bmod m$). Let $(g,\pi) \in G_0$, with g the image of γ represented by the primitive $A = (A_{ij})$ of determinant e, and $\pi = \epsilon\pi_g$, $\epsilon = \pm 1$, with π_g as in Lemma 3.5(1). Then $\pi^*(dz_{\gamma\tau}) = \epsilon e\,(A_{21}\tau + A_{22})^{-1} dz_\tau$, so

$$\begin{aligned}(g,\pi)^* s_0 &= e^{-\frac{k-2}{2}} \Big(\frac{\epsilon e}{A_{21}\tau + A_{22}}\Big)^{k-2} f(\gamma\tau)\,(2\pi i\, dz)^{\otimes(k-2)} \\ &= (f|_{k-2}\tilde{\gamma})(\tau)\,(2\pi i\, dz)^{\otimes(k-2)} = \chi_f(\tilde{\gamma})\, s_0,\end{aligned}$$

where $\tilde{\gamma} = \epsilon A/\sqrt{e} \in \mathrm{SL}_2(\mathbb{R})$ and χ_f is the multiplier system of f; its values lie in L by Definition 2.3. Put $\chi(g,\pi) = \chi_f(\tilde{\gamma})$; this is the eigenvalue of (g,π) on s_0, so χ is a character of G_0.

Transport by D. For $d \in D$ let π_d be the isomorphism of Lemma 3.5(1) with $B_d = \delta_d$, where $\delta_d = \mathrm{diag}(1,u)$ is the element of $\mathrm{GL}_2(\mathbb{Z}/m)$ acting on $\mathfrak{F}_m$ as d does (Lemma 3.3, fact (c), in the normalisation of [57]); then $\hat{D} = \{(d,\pi_d)\}$ is a subgroup of $\hat{G}$, as $d \mapsto \delta_d$ is multiplicative. If $d \in D \cap G_0^\flat$, then d is the image of some $\gamma \in \Gamma(n_1)$ (Lemma 3.3(4)), represented by a matrix $\equiv I \bmod n_1$, and d fixes $\zeta_m \in \kappa_0$, so $\delta_d = I$; by uniqueness in Lemma 3.5(1), $(d,\pi_d) = (g,\pi_g)$ with $\tilde{\gamma} \equiv I \bmod n_f$, and $\chi(d,\pi_d) = 1$ because χ_f is trivial on $\Gamma \cap \Gamma(n_f)$. Hence there is a unique section s over $X''_{\mathbb{C}}$ with $s|_{X'_0} = s_0$ and $(d,\pi_d)^* s = s$ for all $d \in D$, as D permutes the components transitively.

s is defined over $\mathbb{Q}$. Near ∞'', under $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$, $z \mapsto e^{2\pi i z}$, one has $2\pi i\, dz = \omega_{\mathrm{can}}$, so $s = f(q)\,\omega_{\mathrm{can}}^{\otimes(k-2)}$ with $f(q) = F(t(q)) \in \mathbb{Q}[[q]]$. Let $\varsigma \in \mathrm{Aut}(\mathbb{C})$ and $\sigma = \varsigma|_{\mathbb{Q}(\zeta_{n_2})}$. For $h \in \mathcal{K}$, the expansion of h at $\varsigma(\infty'')$ in $\varsigma(q'')$ is ς applied to its expansion at ∞'', which is the expansion

of $d_\sigma(h) = h \circ d_\sigma^{-1}$ at ∞''; so $\varsigma(\infty'') = d_\sigma^{-1}(\infty'')$, with matching formal parameters. As E'' and π_d come from $X(m)$, where π_d is the identity on the underlying curves, the Tate curve $\mathrm{Tate}(q)$ and ω_{can} are matched too. Hence $\varsigma(s)$ and s have the same expansion $f(q)\omega_{\mathrm{can}}^{\otimes(k-2)}$ at $\varsigma(\infty'')$. Both are $\hat{D}$-invariant, the (d, π_d) being defined over $\mathbb{Q}$; so $\varsigma(s) = s$ on the component of $\varsigma(\infty'')$ and then everywhere. So s is a rational section of $\omega^{\otimes(k-2)}$ over $X''_{\mathbb{Q}}$.

The character χ_G. Let $(g, \pi) \in G_0$ and write its action as $c\,\Pi$, with $c = e_g^{-(k-2)/2}$ and $\Pi = \mathrm{Sym}^{k-2}(\pi^*) \circ g^*$ defined over $\mathbb{Q}$. Regard $X''_{\mathbb{Q}}$ as a curve over κ_0; then X'_0 is its base change along $\kappa_0 \subset \mathbb{C}$, s is κ_0-rational, and Π is κ_0-linear since g fixes κ_0. So $\lambda = c^{-1}\chi(g, \pi)$, the eigenvalue of Π on $s|_{X'_0}$, lies in κ_0, and also in L; so $\lambda \in \kappa_1$. Every component of $X'_{\mathbb{C}}$ is $\varsigma(X'_0)$ for some $\varsigma \in \mathrm{Aut}(\mathbb{C})$ trivial on κ_1, and there $\Pi(s) = \varsigma(\lambda)s = \lambda s$. So $(g, \pi)^* s = \chi(g, \pi)s$ on X'. The elements (d, π_d) with d trivial on κ_1 fix s, and together with G_0 they generate G, by $G^\flat = G_0^\flat D$. Hence $f\omega^{\otimes(k-2)} := s|_{X'_L}$ is an eigensection for all of G, and its eigenvalues define a character $\chi_G : G \to L^\times$ of finite order, extending χ. This completes (M4).

(M5). Choose the embedding $\mathbb{Z}[\zeta_{n_2}] \to W$ of Lemma 3.4(4) so that its restriction to $\mathcal{O}_{\kappa_1}$ factors through $\mathfrak{P}$. Then ∞'' is a point $\infty' \in Z'(W)$, $q' = q''$ and w are as required with $p \nmid w$, $E = \mathrm{Tate}(q)$ with $q = {q'}^w$, t is its q-expansion, and $f\omega^{\otimes(k-2)} = f(q)\omega_{\mathrm{can}}^{\otimes(k-2)} = F(t)\,\omega_{\mathrm{can}}^{\otimes(k-2)}$ by Definition 2.1.

□

Lemma 3.3. *Let (Γ, t, f, a, R) be a Kummer twist datum and p a prime with $p \nmid n_\Gamma n_f$. Fix an integer $m \geq 3$ prime to pn_Γ and put $n_1 = \mathrm{lcm}(m, n_\Gamma, n_f)$. For $n \geq 1$ let $\mathfrak{F}_n$ be the field of meromorphic functions h on $\mathbb{H}$ that are invariant under $\Gamma(n)$, meromorphic at the cusps, and have Fourier expansion at ∞ in $\mathbb{Q}(\zeta_n)((q^{1/n}))$, $\zeta_n = e^{2\pi i/n}$. Let $\mathcal{K}$ be the field generated by the functions $h \circ \gamma$ with $h \in \mathfrak{F}_{n_1}$ and $\gamma \in \Gamma$, let κ_0 be the algebraic closure of $\mathbb{Q}$ in $\mathcal{K}$, and put $\Gamma' = \bigcap_{\gamma \in \Gamma} \gamma\,\Gamma(n_1)\gamma^{-1}$.*

1. *Γ' is a torsion-free normal subgroup of finite index in Γ, contained in $\Gamma(n_1)$, hence in $\Gamma(m)$ and in $\ker \chi_f$. There is an integer n_2, prime to p and divisible by n_1, with $\Gamma(n_2) \subset \Gamma'$ and $\mathcal{K} \subset \mathfrak{F}_{n_2}$; in particular $\kappa_0 \subset \mathbb{Q}(\zeta_{n_2})$.*

2. *$\mathbb{Q}(t) \subset \mathfrak{F}_{n_1} \subset \mathcal{K}$, and $\mathcal{K}$ is a finite Galois extension of $\mathbb{Q}(t)$, namely the Galois closure of $\mathfrak{F}_{n_1}$ over $\mathbb{Q}(t)$. Put $G^\flat = \mathrm{Gal}(\mathcal{K}/\mathbb{Q}(t))$ and $G_0^\flat = \mathrm{Gal}(\mathcal{K}/\kappa_0(t))$.*

3. *$[\mathcal{K} : \kappa_0(t)] = [\Gamma : \Gamma']$, and the compositum $\mathbb{C}\mathcal{K}$ is the field $\mathcal{A}(\Gamma')$ of Γ'-invariant meromorphic functions on $\mathbb{H}$ that are meromorphic at the cusps. The rule $\gamma \mapsto (u \mapsto u \circ \gamma^{-1})$ induces an isomorphism $\Gamma/\Gamma' \cong G_0^\flat$.*

4. *For every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n_2})/\mathbb{Q})$, applying σ to Fourier coefficients maps $\mathcal{K}$ onto itself and defines an element $d_\sigma \in G^\flat$. The group $D = \{d_\sigma\}$ satisfies $G^\flat = G_0^\flat D$, and $D \cap G_0^\flat$ is contained in the image of $\Gamma(n_1)$.*

Proof. Classical facts used. For every $n \geq 1$ [57], §§6.1–6.2 (Theorem 6.6): (a) $\mathfrak{F}_n$ is a finite Galois extension of $\mathbb{Q}(j)$, $\mathbb{Q}(\zeta_n)$ is algebraically closed in $\mathfrak{F}_n$, and the compositum $\mathbb{C}\mathfrak{F}_n$ is $\mathcal{A}(\Gamma(n))$; (b) $h \circ \sigma \in \mathfrak{F}_n$ for $h \in \mathfrak{F}_n$ and $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, and applying $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ to Fourier coefficients is an automorphism $h \mapsto h^\sigma$ of $\mathfrak{F}_n$; (c) $\mathrm{Gal}(\mathfrak{F}_n/\mathbb{Q}(j)) \cong \mathrm{GL}_2(\mathbb{Z}/n)/\{\pm 1\}$, where the class of $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ acts by $h \mapsto h \circ \sigma$ (up to the inversion fixed in loc. cit.) and $\mathrm{diag}(1, s)$ by $\zeta_n \mapsto \zeta_n^s$ on coefficients; in particular this action is faithful. We also use the Galois theory of branched coverings of compact Riemann surfaces: for a normal subgroup Δ of finite index in Γ, $\mathcal{A}(\Delta)$ is Galois over $\mathcal{A}(\Gamma) = \mathbb{C}(t)$ with group Γ/Δ acting by $u \mapsto u \circ \gamma^{-1}$, and an element $g \in \mathrm{PSL}_2(\mathbb{R})$ with $u \circ g = u$ for all $u \in \mathcal{A}(\Delta)$ lies in Δ (these functions separate the points of $\Delta\backslash\mathbb{H}$, so $g\tau \in \Delta\tau$

for all τ, and by a Baire argument $g \in \Delta$). Since $\Gamma(n_\Gamma) \subset \Gamma$ and $n_\Gamma \mid n_1$, we have $\Gamma(n_1) \subset \Gamma$, of finite index.

Proof of (1). The group Γ' is normal in Γ and contained in $\Gamma(n_1)$, which is torsion-free as $n_1 \geq 3$; since $n_f \mid n_1$ and χ_f is trivial on $\Gamma \cap \Gamma(n_f)$ (Definition 2.1), $\Gamma' \subset \ker \chi_f$. We bound its level. By Lemma A.4, applied to the prime $p \nmid n_\Gamma$, every $\gamma \in \Gamma$ is the image of $A/\sqrt{e}$ with $A \in M_2(\mathbb{Z})$ primitive and $e = \det A > 0$ prime to p. If $\delta = I + n_1 eX \in \Gamma(n_1 e)$ with $X \in M_2(\mathbb{Z})$, then $\gamma^{-1}\delta\gamma = A^{-1}\delta A = I + n_1 A^{\mathrm{adj}} X A \in \Gamma(n_1)$; so

$$\Gamma(n_1 e) \subset \gamma\, \Gamma(n_1)\gamma^{-1}.$$

The subgroup $\Gamma_\mathbb{Z} = \Gamma \cap \mathrm{PSL}_2(\mathbb{Z})$ contains $\Gamma(n_\Gamma)$, hence has finite index in Γ, and it normalises $\Gamma(n_1)$. With representatives $\gamma_1, \dots, \gamma_r$ of $\Gamma/\Gamma_\mathbb{Z}$ and e_i the corresponding determinants, $\Gamma' = \bigcap_i \gamma_i \Gamma(n_1)\gamma_i^{-1}$ has finite index in Γ and contains $\Gamma(n_2)$ for $n_2 = n_1 e_1 \cdots e_r$, which is prime to p.

We show $\mathcal{K} \subset \mathfrak{F}_{n_2}$. Let $h \in \mathfrak{F}_{n_1}$ and $\gamma = \gamma_i \sigma$ with $\sigma \in \Gamma_\mathbb{Z}$. Write $A_i = \sigma_i B$ with $\sigma_i \in \mathrm{SL}_2(\mathbb{Z})$ and $B = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, $ad = e_i$ (Hermite normal form). By (b), $h_1 = h \circ \sigma_i \in \mathfrak{F}_{n_1}$, say $h_1 = \sum_k c_k q^{k/n_1}$ with $c_k \in \mathbb{Q}(\zeta_{n_1})$. Then

$$h \circ \gamma_i = h_1 \circ B = \sum_k c_k\, \zeta_{n_1 d}^{kb}\, q^{ka/(n_1 d)},$$

whose coefficients lie in $\mathbb{Q}(\zeta_{n_1 e_i})$ and whose exponents lie in $\frac{1}{n_1 e_i}\mathbb{Z}$. As above $B\Gamma(n_1 e_i)B^{-1} \subset \Gamma(n_1)$, so $h_1 \circ B$ is invariant under $\Gamma(n_1 e_i)$; it is meromorphic at the cusps, since B permutes $\mathbb{P}^1(\mathbb{Q})$. Hence $h \circ \gamma_i \in \mathfrak{F}_{n_1 e_i}$, and by (b) $h \circ \gamma = (h \circ \gamma_i) \circ \sigma \in \mathfrak{F}_{n_1 e_i} \subset \mathfrak{F}_{n_2}$. So $\mathcal{K} \subset \mathfrak{F}_{n_2}$, and $\kappa_0 \subset \mathbb{Q}(\zeta_{n_2})$ by (a).

Proof of (2). t is invariant under $\Gamma(n_\Gamma) \supset \Gamma(n_1)$, meromorphic at the cusps, and $t \in q + q^2\mathbb{Z}[\frac{1}{M}][[q]]$; so $t \in \mathfrak{F}_{n_1}$. The field $\mathfrak{F}_{n_2}$ is finite over $\mathbb{Q}(j)$ by (a), so it has transcendence degree one, j is algebraic over $\mathbb{Q}(t)$, and $\mathfrak{F}_{n_2} \supset \mathcal{K}$ is finite over $\mathbb{Q}(t)$. For $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n_1})/\mathbb{Q})$ and $\gamma \in \Gamma$, the map $\epsilon_{\sigma,\gamma} : h \mapsto h^\sigma \circ \gamma$ is a field embedding $\mathfrak{F}_{n_1} \to \mathcal{K}$ fixing t, by (b) and because $t^\sigma = t = t \circ \gamma$. These embeddings are pairwise distinct for distinct $(\sigma, \Gamma(n_1)\gamma)$: on ζ_{n_1} the embedding $\epsilon_{\sigma,\gamma}$ is σ, and if $h^\sigma \circ \gamma = h^\sigma \circ \gamma'$ for all h, then $u \circ \gamma'\gamma^{-1} = u$ for all $u \in \mathfrak{F}_{n_1}$, hence for all $u \in \mathcal{A}(\Gamma(n_1)) = \mathbb{C}\mathfrak{F}_{n_1}$, so $\gamma'\gamma^{-1} \in \Gamma(n_1)$. Their number is $\varphi(n_1)[\Gamma : \Gamma(n_1)]$. On the other hand $[\mathbb{Q}(\zeta_{n_1})(t) : \mathbb{Q}(t)] = \varphi(n_1)$ and, since $\mathbb{Q}(\zeta_{n_1})$ is algebraically closed in $\mathfrak{F}_{n_1}$ and we are in characteristic zero, $\mathfrak{F}_{n_1}$ and $\mathbb{C}$ are linearly disjoint over $\mathbb{Q}(\zeta_{n_1})$, so $[\mathfrak{F}_{n_1} : \mathbb{Q}(\zeta_{n_1})(t)] = [\mathcal{A}(\Gamma(n_1)) : \mathbb{C}(t)] = [\Gamma : \Gamma(n_1)]$. Hence the $\epsilon_{\sigma,\gamma}$ are all the $\mathbb{Q}(t)$-embeddings of $\mathfrak{F}_{n_1}$ into an algebraic closure of $\mathbb{Q}(t)$ containing $\mathcal{K}$, and the Galois closure of $\mathfrak{F}_{n_1}$ is the compositum of their images $\mathfrak{F}_{n_1}^\sigma \circ \gamma = \mathfrak{F}_{n_1} \circ \gamma$, which is $\mathcal{K}$.

Proof of (3). For $\gamma \in \Gamma$ we have $\mathbb{C}(\mathfrak{F}_{n_1} \circ \gamma) = \mathcal{A}(\gamma^{-1}\Gamma(n_1)\gamma)$, and all these groups contain the normal subgroup Γ'. In the Galois extension $\mathcal{A}(\Gamma')/\mathbb{C}(t)$ with group Γ/Γ', the compositum of the fixed fields $\mathcal{A}(\gamma^{-1}\Gamma(n_1)\gamma)$ is the fixed field of their intersection Γ'; so $\mathbb{C}\mathcal{K} = \mathcal{A}(\Gamma')$. As κ_0 is algebraically closed in $\mathcal{K}$, $[\mathcal{K} : \kappa_0(t)] = [\mathbb{C}\mathcal{K} : \mathbb{C}(t)] = [\Gamma : \Gamma']$. Each $\gamma \in \Gamma$ maps the generators $h \circ \gamma'$ of $\mathcal{K}$ to $h \circ (\gamma'\gamma^{-1})$, so $u \mapsto u \circ \gamma^{-1}$ is an automorphism of $\mathcal{K}$ fixing $\kappa_0(t)$. This gives a homomorphism $\Gamma \to G_0^\flat$, with kernel $\{\gamma : u \circ \gamma = u \text{ for all } u \in \mathcal{A}(\Gamma')\} = \Gamma'$, and it is onto because $|G_0^\flat| = [\mathcal{K} : \kappa_0(t)] = [\Gamma : \Gamma']$.

Proof of (4). By (b), $h \mapsto h^\sigma$ is an automorphism of $\mathfrak{F}_{n_2}$ fixing t; it maps $\mathcal{K}$ onto a $\mathbb{Q}(t)$-conjugate of $\mathcal{K}$ inside the algebraic extension $\mathfrak{F}_{n_2}$ of $\mathbb{Q}(t)$, which is $\mathcal{K}$ because $\mathcal{K}/\mathbb{Q}(t)$ is normal. So $d_\sigma \in G^\flat$, and D, the image of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n_2})/\mathbb{Q})$, is a subgroup. Restriction to κ_0 maps $G^\flat$ onto $\mathrm{Gal}(\kappa_0/\mathbb{Q})$ with kernel $G_0^\flat$, and it maps D onto $\mathrm{Gal}(\kappa_0/\mathbb{Q})$, as $\kappa_0 \subset \mathbb{Q}(\zeta_{n_2})$; so $G^\flat = G_0^\flat D$. Finally let $d_\sigma \in G_0^\flat$ be the image of $\gamma \in \Gamma$. Since j has rational Fourier coefficients, $j \circ \gamma^{-1} = j^\sigma = j$, so $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. On $\mathfrak{F}_{n_1}$ the element γ^{-1} of $\mathrm{SL}_2(\mathbb{Z}/n_1)$ and the element $\mathrm{diag}(1, s)$, where $\sigma(\zeta_{n_1}) = \zeta_{n_1}^s$, then act in the same way, so by (c) they agree in $\mathrm{GL}_2(\mathbb{Z}/n_1)/\{\pm 1\}$; comparing determinants, $s \equiv 1$ and $\gamma \in \Gamma(n_1)$.

□

Lemma 3.4. *Let (Γ, t, f, a, R) be a Kummer twist datum and p a prime with $p \nmid M n_\Gamma n_f$, and let m, n_2, $\mathfrak{F}_n$, $\mathcal{K}$, κ_0, Γ', $G^\flat$, $G_0^\flat$ and D be as in Lemma 3.3. Let $X(m)$ be the compactified moduli scheme over $\mathbb{Z}[\frac{1}{m}]$ of generalised elliptic curves with full level-m structure, with cuspidal subscheme $Z(m)$, $Y(m) = X(m) \setminus Z(m)$, universal generalised elliptic curve E_m and universal level structure $\alpha : (\mathbb{Z}/m)^2 \xrightarrow{\sim} E_m[m]$ over $Y(m)$. Let X'' be the normalisation of $X(m)_{\mathbb{Z}_{(p)}}$ in $\mathcal{K}$ (via $\mathfrak{F}_m \subset \mathcal{K}$), $\phi : X'' \to X(m)_{\mathbb{Z}_{(p)}}$, $Z'' = \phi^{-1}(Z(m))_{\mathrm{red}}$, $Y'' = X'' \setminus Z''$ and $E'' = \phi^* E_m$.*

1. *X'' is smooth and proper over $\mathbb{Z}_{(p)}$, ϕ is finite and étale over $Y(m)$, Z'' is étale over $\mathbb{Z}_{(p)}$, and (X'', Z'', E'') satisfies (M1) with $\mathbb{Z}_{(p)}$ in place of $\mathcal{O}$.*

2. *The connected components of $X''_{\mathbb{C}}$ correspond to the embeddings $\kappa_0 \to \mathbb{C}$. On the component X''_0 belonging to the inclusion $\kappa_0 \subset \mathbb{C}$ there is an isomorphism $X''_0 \cong \Gamma' \backslash \mathbb{H}^*$ carrying t to the Hauptmodul, (E'', α) to $\tau \mapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \alpha_\tau)$ with $\alpha_\tau(x, y) = (x\tau + y)/m$, and the action of the image of $\gamma \in \Gamma$ in $G_0^\flat$ to $\tau \mapsto \gamma\tau$; an automorphism g acts on functions by $u \mapsto u \circ g^{-1}$.*

3. *t extends to a finite morphism $t : X'' \to \mathbb{P}^1_{\mathbb{Z}_{(p)}}$ whose divisor of zeros is supported on Z'' and whose reduction is non-constant on every connected component of $X''_{\mathbb{F}_p}$. Hence X'' is the normalisation of $\mathbb{P}^1_{\mathbb{Z}_{(p)}}$ in $\mathcal{K}$, $G^\flat$ acts on X'' over $\mathbb{Z}_{(p)}$ preserving Z'', and t induces $X''_{\mathbb{Q}}/G^\flat \cong \mathbb{P}^1_{\mathbb{Q}}$. The group D acts by automorphisms lying over automorphisms of $X(m)_{\mathbb{Z}_{(p)}}$.*

4. *Let ∞'' be the image of ∞ on X''_0, w the width of Γ' at ∞, and W the ring of Witt vectors of $\mathbb{F}_p(\zeta_{n_2})$, with $\mathbb{Z}[\zeta_{n_2}] \to W$ fixed. Then $p \nmid w$, $\infty'' \in Z''(W)$, $q'' = q^{1/w}$ is a formal parameter of X''_W along ∞'', and over $\operatorname{Spf} W[[q'']]$ one has $E'' = \mathrm{Tate}(q)$ with $q = q''^w$ and t equal to its q-expansion.*

Proof. Facts on full level structures. For $n \geq 3$, $X(n)$ is smooth and proper over $\mathbb{Z}[\frac{1}{n}]$, $Y(n)$ is a fine moduli scheme, E_n is a generalised elliptic curve, the cusps are finite étale over $\mathbb{Z}[\frac{1}{n}]$, each a spectrum of a cyclotomic extension, and at a cusp the formal parameter can be taken as $q^{1/n}$ with E_n the Tate curve $\mathrm{Tate}(q)$; the Kodaira–Spencer map is an isomorphism $\omega^{\otimes 2} \cong \Omega^1(\log Z(n))$ [18], IV and VII; [30], Ch. 4, 8 and 10; [32], §2. For $m \mid n$ the forgetful map $X(n) \to X(m)$ is finite, and étale and Galois over $Y(m)$ with group $K_{n,m} = \ker(\mathrm{GL}_2(\mathbb{Z}/n) \to \mathrm{GL}_2(\mathbb{Z}/m))$ acting by $\alpha \mapsto \alpha \circ g$ (rigidity of level-m structures, $m \geq 3$; loc. cit.). Over $\mathbb{C}$, the standard component of $X(n)$ is $\Gamma(n)\backslash\mathbb{H}^*$ via $\tau \mapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \alpha_\tau)$, the function field of $X(n)_{\mathbb{Q}}$ restricted to it is $\mathfrak{F}_n$, so that $X(n)_{\mathbb{Q}}$ is the normalisation of the j-line in $\mathfrak{F}_n$ and $\mathrm{Gal}(\mathfrak{F}_n/\mathfrak{F}_m)$ acts on $X(n)$ through $K_{n,m}$; and at the cusp $\sigma\infty$, $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, of the standard component the expansion of $h \in \mathfrak{F}_n$ in $q^{1/n}$ is the Fourier expansion of $h \circ \sigma$ (loc. cit., and [57], §6.1).

Proof of (1). By Lemma 3.3(1), $\mathfrak{F}_m \subset \mathcal{K} \subset \mathfrak{F}_{n_2}$ with $m \mid n_2$ and $p \nmid n_2$. Let $\Delta = \mathrm{Gal}(\mathfrak{F}_{n_2}/\mathcal{K}) \subset K_{n_2,m}$. The normalisation of $X(m)_{\mathbb{Z}_{(p)}}$ in $\mathcal{K} = \mathfrak{F}_{n_2}^\Delta$ is the quotient $X'' = X(n_2)_{\mathbb{Z}_{(p)}}/\Delta$. As Δ acts freely on $Y(n_2)$, $Y'' = Y(n_2)/\Delta$ is finite étale over $Y(m)$, hence smooth over $\mathbb{Z}_{(p)}$. Let $\bar{c}$ be a geometric cusp of $X(n_2)$ over $\bar{\mathbb{F}}_p$ and $c \in X(n_2)(W')$, $W' = W(\bar{\mathbb{F}}_p)$, its unique lift. The stabiliser S of $\bar{c}$ in Δ fixes c and acts W'-linearly on $\widehat{\mathcal{O}}_{X(n_2)_{W'},\bar{c}} = W'[[u]]$, where $\delta u^{n_2} = q$ with $\delta \in W'^\times$; it fixes j, hence $q = 1/j + 744/j^2 + \cdots$, hence u^{n_2}, and a W'-automorphism of $W'[[u]]$ fixing u^{n_2} sends u to ζu with $\zeta \in \mu_{n_2}(W')$, because $p \nmid n_2$ and $1 + uW'[[u]]$ has no prime-to-p torsion. Elements acting trivially on $W'[[u]]$ act trivially on the component through $\bar{c}$, hence are trivial since Δ acts freely on $Y(n_2)$. So S is cyclic of order $r \mid n_2$, and the completed local ring of X'' at the image of $\bar{c}$ is $W'[[u]]^S = W'[[u^r]]$. Hence X'' is smooth, $Z'' = \{u^r = 0\}$ locally, étale over $\mathbb{Z}_{(p)}$, and $E'' = \mathrm{Tate}(q)$ with $q = \delta(u^r)^{n_2/r}$ and $p \nmid n_2/r$. Over the cusp $\phi(c)$ of $X(m)$ with parameter q_m, $\delta' q_m^m = q$, the map ϕ is $q_m \mapsto \varepsilon\,(u^r)^{n_2/(rm)}$ with $\varepsilon \in W'[[u^r]]^\times$, tamely ramified,

so $\phi^*\Omega^1_{X(m)}(\log Z(m)) = \Omega^1_{X''}(\log Z'')$, and the Kodaira–Spencer isomorphism pulls back, as in [32], §2. X'' is proper, being finite over $X(m)$. This proves (1).

Proof of (2). $X''_{\mathbb{Q}}$ is normal and connected with function field $\mathcal{K}$, so $H^0(X''_{\mathbb{Q}}, \mathcal{O}) = \kappa_0$ and the components of $X''_{\mathbb{C}}$ correspond to the embeddings of κ_0. The component for $\kappa_0 \subset \mathbb{C}$ has function field $\mathcal{K} \otimes_{\kappa_0} \mathbb{C}$, whose image in the meromorphic functions on $\mathbb{H}$ is $\mathbb{C}\mathcal{K} = \mathcal{A}(\Gamma')$ by Lemma 3.3(3); so it is $\Gamma'\backslash\mathbb{H}^*$, t being the Hauptmodul. On it, ϕ is induced by $\mathfrak{F}_m \subset \mathcal{K}$, that is $\Gamma'\backslash\mathbb{H}^* \to \Gamma(m)\backslash\mathbb{H}^*$, $\tau \mapsto \tau$, which gives (E'', α). The image of γ acts on $\mathcal{K}$ by $u \mapsto u \circ \gamma^{-1}$, so it induces $\tau \mapsto \gamma\tau$.

Proof of (4). The stabiliser of ∞ in Γ' is generated by $\tau \mapsto \tau + w$, and $\Gamma(n_2) \subset \Gamma'$ gives $w \mid n_2$, so $p \nmid w$. The standard cusp of $X(n_2)$ lies over W, with formal neighbourhood $\operatorname{Spf} W[[q^{1/n_2}]]$ and $E_{n_2} = \operatorname{Tate}(q)$; its stabiliser in Δ corresponds to $\langle \tau \mapsto \tau + w\rangle / \langle \tau \mapsto \tau + n_2\rangle$ and acts by $q^{1/n_2} \mapsto \zeta q^{1/n_2}$, so the quotient is $\operatorname{Spf} W[[q^{1/w}]]$. The expansion of t there is its Fourier expansion, by the facts above.

Proof of (3). Write $\operatorname{div}(t) = V + Z_0 - Z_\infty$ on the regular two-dimensional scheme X'', with V supported on $X''_{\mathbb{F}_p}$ and Z_0, Z_∞ the closures of the zero and pole divisors of t on $X''_{\mathbb{Q}}$. The elements $d_\sigma \in D$ of Lemma 3.3(4) map $\mathfrak{F}_m$ to itself and fix j, so they induce automorphisms of $X(m)_{\mathbb{Z}_{(p)}}$ (normalisation of the j-line in $\mathfrak{F}_m$) and of X'' over them; they fix t, and they permute the components of $X''_{\mathbb{C}}$ transitively, because D maps onto $\operatorname{Gal}(\kappa_0/\mathbb{Q})$. Since X'' is smooth and proper over $\mathbb{Z}_{(p)}$, $H^0(X'', \mathcal{O})$ is finite étale over $\mathbb{Z}_{(p)}$ and the components of the geometric special and generic fibres correspond to each other; so D also permutes the components of $X''_{\mathbb{F}_p}$ transitively, and all multiplicities in V are equal. On the component through $\infty''_{\mathbb{F}_p}$ the multiplicity is 0: by (4), $t = q''^{\,w} \cdot v$ with $v \in W[[q'']]^\times$, because $t \in q + q^2\mathbb{Z}[\frac{1}{M}][[q]]$ and $p \nmid M$. Hence $V = 0$.

The zeros of t on X''_0 are the cusps $\gamma\infty$, $\gamma \in \Gamma$, and on the other components their D-translates; so $Z_0 \subset Z''$. We show $Z_0 \cap Z_\infty = \emptyset$. Since D acts over $\mathbb{Z}_{(p)}$ and fixes t, it suffices to show that t is regular near the reduction of each cusp $c = \gamma\infty$ of X''_0. Choose $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ with $\sigma\infty = \gamma\infty$. Then $\beta = \gamma^{-1}\sigma$ fixes ∞ and is represented by $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with ad dividing the determinant of the primitive integral matrix of γ^{-1}, hence prime to p (Lemma A.4). With $t = \sum_k c_k q^k$, $c_k \in \mathbb{Z}[\frac{1}{M}]$,

$$t \circ \sigma = t \circ \beta = \sum_k c_k\, \zeta_d^{kb}\, q^{ka/d},$$

which has p-integral coefficients. By the facts above this is the expansion of t at the cusp $\sigma\infty$ of $X(n_2)$, so t is regular near its reduction on $X(n_2)$, hence on the quotient X''. So $c \notin Z_\infty$ and its reduction is not on Z_∞.

At every point of the regular scheme X'' one of t, $1/t$ is therefore regular, so t is a morphism to $\mathbb{P}^1_{\mathbb{Z}_{(p)}}$. Its reduction is non-constant on the component through $\infty''_{\mathbb{F}_p}$, as $t \equiv q''^{\,w}\bar{v}$, and hence on every component, by transitivity of D. So t is proper and quasi-finite, hence finite. Being normal and finite over $\mathbb{P}^1_{\mathbb{Z}_{(p)}}$ with function field $\mathcal{K}$, X'' is the normalisation of $\mathbb{P}^1_{\mathbb{Z}_{(p)}}$ in $\mathcal{K}$, so $G^\flat = \operatorname{Gal}(\mathcal{K}/\mathbb{Q}(t))$ acts on it over $\mathbb{P}^1_{\mathbb{Z}_{(p)}}$. It preserves Z'': $G^\flat = G^\flat_0 D$, $G^\flat_0$ acts on X''_0 through Γ, which permutes the cusps, D preserves $\phi^{-1}(Z(m))$, and Z'' is the closure of the set of cusps of the generic fibre. Finally $X''_{\mathbb{Q}}$ is the normalisation of $\mathbb{P}^1_{\mathbb{Q}}$ in $\mathcal{K}$ and $\mathcal{K}^{G^\flat} = \mathbb{Q}(t)$, so $X''_{\mathbb{Q}}/G^\flat \cong \mathbb{P}^1_{\mathbb{Q}}$. □

Lemma 3.5. *Let X'', Y'', Z'', E'' and α be as in Lemma 3.4, for a Kummer twist datum (Γ, t, f, a, R), a prime $p \nmid M n_\Gamma n_f$ and the data m, n_2, Γ', $G^\flat$, $G^\flat_0$ and D of Lemma 3.3; let H'' be the canonical extension of the relative de Rham cohomology of $E''|_{Y''}$, and for $g \in G^\flat$ put $\mathcal{H}_g = \operatorname{Hom}_{Y''}(E''|_{Y''}, g^*E''|_{Y''})$.*

1. *For every $g \in G^\flat$, $\mathcal{H}_g$ is free of rank one, generated by an isogeny π_g, unique up to sign, of*

degree e_g prime to p; there is $B_g \in \mathrm{GL}_2(\mathbb{Z}/m)$ with $\pi_g \circ \alpha = g^*\alpha \circ B_g$. If g is the image of $\gamma \in \Gamma$, represented by a primitive $A = (A_{ij}) \in M_2(\mathbb{Z})$ of determinant e, then on X_0'' one may take π_g to be $z \mapsto ez/(A_{21}\tau + A_{22})$, from $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ to $\mathbb{C}/(\mathbb{Z}\gamma\tau + \mathbb{Z})$, and $e_g = e$; for $g \in D$, π_g is an isomorphism.

2. *Let $\hat{G}$ be the set of pairs (g, π) with $g \in G^\flat$ and $\pi = \pm\pi_g$, with product $(g,\pi)(h,\pi') = (gh, \mu^{-1}\, h^*\pi \circ \pi')$, where $\mu > 0$ is the integer making $\mu^{-1}h^*\pi \circ \pi'$ a generator of $\mathcal{H}_{gh}$. Then $\hat{G}$ is a central extension of $G^\flat$ by $\{(1, \pm\mathrm{id})\}$. Each π^* extends to a map $g^*H'' \to H''$ over X'', invertible over $\mathbb{Z}_{(p)}$, compatible with the connection, the Hodge filtration and the logarithmic F-crystal structure, with $\langle \pi^*x, \pi^*y\rangle = e_g\langle x, y\rangle$ for the Weil pairing; and $(g,\pi) \mapsto e_g^{-(k-2)/2}\,\mathrm{Sym}^{k-2}(\pi^*)$, with the positive square root, satisfies the cocycle rule of an action of $\hat{G}$ on $\mathrm{Sym}^{k-2} H'' \otimes \mathbb{Z}_{(p)}[e_g^{1/2} : g \in G^\flat]$ lifting the action on X''.*

3. *Assume $p \geq 5$. For every geometric point $\bar{x}$ of $X''_{\mathbb{F}_p}$, the stabiliser of $\bar{x}$ in $\hat{G}$ has order prime to p: it divides 48 if $\bar{x} \notin Z''$, and it divides $2n_2 r$ for some r prime to p if $\bar{x} \in Z''$.*

Proof. Rank one. $Y''_{\mathbb{Q}}$ is connected with function field $\mathcal{K}$, over which E'' has transcendental j-invariant, hence no complex multiplication; so Hom over the generic point, and a fortiori $\mathcal{H}_g$, is 0 or free of rank one. Restriction from Y'' to $Y''_{\mathbb{Q}}$ is injective on $\mathcal{H}_g$, as Y'' is flat over $\mathbb{Z}_{(p)}$.

Proof of (1) for $g \in G_0^\flat$. Let g be the image of γ, with A and e as stated; $p \nmid e$ by Lemma A.4, and $\gcd(e, m) = 1$ since the primes dividing e divide n_Γ (Lemma A.4 at primes $\ell \nmid n_\Gamma$). On X_0'', with $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and $\Lambda_A = \mathbb{Z}(A_{11}\tau + A_{12}) + \mathbb{Z}(A_{21}\tau + A_{22})$, we have $\Lambda_{\gamma\tau} = (A_{21}\tau + A_{22})^{-1}\Lambda_A$ and $e\Lambda_\tau \subset \Lambda_A$ of index e, so $z \mapsto ez/(A_{21}\tau + A_{22})$ is an isogeny $E''_\tau \to E''_{\gamma\tau} = (g^*E'')_\tau$ of degree e with cyclic kernel (Lemma 3.4(2)). It depends holomorphically on τ and is compatible with Γ', as $\gamma\Gamma'\gamma^{-1} = \Gamma'$. For $v \in (\mathbb{Z}/m)^2$ (row vectors), $\pi(\alpha_\tau(v)) = \alpha_{\gamma\tau}(vA^{\mathrm{adj}})$, because $(x', y')A = (x'A_{11} + y'A_{21}, x'A_{12} + y'A_{22})$ gives $\alpha_{\gamma\tau}(v') = (A_{21}\tau + A_{22})^{-1} \cdot \frac{1}{m}\,\eta_\tau(v'A)$ with $\eta_\tau(x, y) = x\tau + y$, and $eA^{-1} = A^{\mathrm{adj}}$. So $\pi \circ \alpha = g^*\alpha \circ B_g$ with $B_g : v \mapsto vA^{\mathrm{adj}}$, invertible modulo m. The kernel of π is a union of connected components of the finite étale group $E''[e]$ over Y_0'', so it is algebraic and defined over $\bar{\mathbb{Q}}$ (connected components of finite étale covers do not change from $\bar{\mathbb{Q}}$ to $\mathbb{C}$); let C be it. The pairs $(E''/C, \bar{\pi} \circ \alpha \circ B_g^{-1})$, with $\bar{\pi} : E'' \to E''/C$ the quotient, and $(g^*E'', g^*\alpha)$ are elliptic curves with level-m structure over $Y''_{0,\bar{\mathbb{Q}}}$; their classifying morphisms to the fine moduli scheme $Y(m)$ agree on $\mathbb{C}$-points, hence agree, and the resulting unique isomorphism of pairs composed with $\bar{\pi}$ is π. So π is algebraic over $\bar{\mathbb{Q}}$. For $\varsigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\varsigma(\pi)$ is an isogeny of degree e over $\varsigma(Y_0'')$ inducing the same B_g, since E'', α and g are defined over $\mathbb{Q}$. On a connected component an isogeny of degree e inducing B_g is unique, by rank one and $-B_g \neq B_g$ ($m \geq 3$). As the components of $Y''_{\bar{\mathbb{Q}}}$ are the conjugates of Y_0'', the conjugates of π glue to a Galois-equivariant isogeny over $Y''_{\bar{\mathbb{Q}}}$, which descends to $Y''_{\mathbb{Q}}$.

For $g \in D$. By Lemma 3.4(3), g lies over an automorphism of $X(m)_{\mathbb{Z}_{(p)}}$ fixing j, that is (Lemma 3.3, fact (c), at level m) the moduli automorphism $(E, \alpha) \mapsto (E, \alpha \circ \delta)$ for some $\delta \in \mathrm{GL}_2(\mathbb{Z}/m)$. So $g^*(E'', \alpha) \cong (E'', \alpha \circ \delta)$ canonically, which gives an isomorphism π_g over Y'' with $B_g = \delta^{-1}$. For general $g = g_0 d$ with $g_0 \in G_0^\flat$, $d \in D$ (Lemma 3.3(4)), the composite $d^*\pi_{g_0} \circ \pi_d$ lies in $\mathcal{H}_g$, has cyclic kernel of order e_{g_0}, and induces $B_{g_0}B_d$ on m-torsion.

Extension over Y'' and generators. Let π be one of the isogenies above over $Y''_{\mathbb{Q}}$, of degree e prime to p. Its kernel is a union of connected components of $E''[e]_{\mathbb{Q}}$; since Y'' is normal and $E''[e]$ is finite étale over Y'', its closure $\bar{C}$ is a finite étale subgroup. The pairs $(E''/\bar{C}, \bar{\pi} \circ \alpha \circ B^{-1})$ and $(g^*E'', g^*\alpha)$ over Y'' have classifying morphisms to $Y(m)_{\mathbb{Z}_{(p)}}$ that agree on the dense open $Y''_{\mathbb{Q}}$; as Y'' is reduced and $Y(m)$ separated, they agree, and π extends to Y''. Its kernel is cyclic, so π is not a multiple of another isogeny, and it generates $\mathcal{H}_g$. This proves (1).

Proof of (2). By rank one, $h^*\pi\circ\pi' = \lambda\pi_{gh}$ with $\lambda\in\mathbb{Z}\setminus\{0\}$, $\lambda^2 e_{gh} = e_g e_h$, so $\mu = |\lambda|$ is well defined; associativity follows from that of composition, and $(1,-\mathrm{id})$ is central. Over Y'', π^* on de Rham cohomology is horizontal, preserves the Hodge filtration ω, commutes with the crystalline Frobenius by functoriality of crystalline cohomology for morphisms of abelian schemes [5], and satisfies $\langle\pi^*x,\pi^*y\rangle = \deg(\pi)\langle x,y\rangle$ since $\hat\pi\pi = [\deg\pi]$. At a cusp, over $\operatorname{Spf} W'[[u]]$ both curves are Tate curves $\mathrm{Tate}(q_1)$, $\mathrm{Tate}(q_2)$, and over the complete field $\mathrm{Frac}(W')((u))$ every homomorphism is induced by $x\mapsto x^a$ on $\mathbb{G}_m$ with $q_1^a = q_2^c$, $ac = \pm\deg\pi$ (Tate's uniformisation; [18], VII). Then $\pi^*\omega_{\mathrm{can}} = a\,\omega_{\mathrm{can}}$, and horizontality with $\nabla\omega_{\mathrm{can}} = \xi_{\mathrm{can}}\otimes dq_i/q_i$ gives $\pi^*\xi_{\mathrm{can}} = c\,\xi_{\mathrm{can}}$; as a,c divide $\deg\pi$, which is prime to p, π^* maps the lattice spanned by $\omega_{\mathrm{can}},\xi_{\mathrm{can}}$, which is the canonical extension [32], §2, isomorphically onto the corresponding lattice. So π^* extends over X'', invertibly over $\mathbb{Z}_{(p)}$, and the compatibilities extend by density of Y''. Finally, for $(g,\pi)(h,\pi') = (gh,\pi'')$ with $\pi'' = \mu^{-1}h^*\pi\circ\pi'$, the scalars satisfy $e_{gh}^{-(k-2)/2}\mu^{-(k-2)} = (e_g e_h)^{-(k-2)/2}$ because $\mu^2 e_{gh} = e_g e_h$; hence $e_{gh}^{-(k-2)/2}\operatorname{Sym}^{k-2}(\pi''^*) = e_h^{-(k-2)/2}\operatorname{Sym}^{k-2}(\pi'^*)\circ h^*(e_g^{-(k-2)/2}\operatorname{Sym}^{k-2}(\pi^*))$, the cocycle rule.

Proof of (3). Let (g,π) fix $\bar x$. The components of $X''_{\bar{\mathbb{F}}_p}$ correspond to those of $X''_{\bar{\mathbb{Q}}_p}$, on which $G^\flat$ acts through $\mathrm{Gal}(\kappa_0/\mathbb{Q})$ simply transitively; so $g\in G_0^\flat$. If $\bar x\in Z''$, its unique lift $x\in Z''(W')$ has the same stabiliser; after transport by D and an embedding $W'\to\mathbb{C}$, x is a cusp of $X_0'' = \Gamma'\backslash\mathbb{H}^*$, and its stabiliser in $G_0^\flat\cong\Gamma/\Gamma'$ is generated by the image of a parabolic $P\in\Gamma$. Write $P = \sigma\left(\begin{smallmatrix}1 & u/r\\ 0 & 1\end{smallmatrix}\right)\sigma^{-1}$ with $\sigma\in\mathrm{SL}_2(\mathbb{Z})$ and $\gcd(u,r)=1$; then r^2 is the determinant of the primitive matrix of P, prime to p, and $P^{n_2 r}\in\Gamma(n_2)\subset\Gamma'$. So the stabiliser in $\hat G$ has order dividing $2n_2 r$. If $\bar x\notin Z''$, let $\bar E = E''_{\bar x}$; then $\pi_{\bar x}$ is an isogeny $\bar E\to E''_{g\bar x} = \bar E$, and $\theta:(g,\pi)\mapsto[\pi_{\bar x}]$ is a homomorphism from the stabiliser to $B^\times/\mathbb{Q}^\times$, $B = \mathrm{End}(\bar E)\otimes\mathbb{Q}$. Every element of $B^\times/\mathbb{Q}^\times$ of finite order r comes from $x\in B^\times$ generating a quadratic field F, and $x/\bar x$ is then a root of unity of order r in F, so $r\in\{1,2,3,4,6\}$. As B is $\mathbb{Q}$, an imaginary quadratic field or a definite quaternion algebra, $B^\times/\mathbb{Q}^\times$ embeds in $(B\otimes\mathbb{R})^\times/\mathbb{R}^\times\subset\mathrm{SO}(3)$, whose finite subgroups are cyclic, dihedral, A_4, S_4 or A_5; with element orders in $\{1,2,3,4,6\}$ the order divides 24. If $\theta(g,\pi)=1$, then $\pi_{\bar x} = [\lambda]$ with $\lambda\in\mathbb{Z}$, and $\lambda\alpha_{\bar x} = \alpha_{\bar x}B_g$, so $B_g = \lambda$. Over $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{X''_{W'},\bar x}$ the kernel of π is the unique finite étale subgroup lifting $\bar E[\lambda]$, namely $E''[\lambda]$, so $\pi = u\circ[\lambda]$ for an isomorphism $u:E''\to g^*E''$ reducing to the identity, and $u\circ\alpha = g^*\alpha$. As ϕ is étale and $Y(m)$ is a fine moduli scheme, $\widehat{\mathcal{O}}$ pro-represents deformations of $(\bar E,\alpha_{\bar x})$, and the deformations (E'',α) and $g^*(E'',\alpha)$ are isomorphic; so g induces the identity on $\widehat{\mathcal{O}}$, hence on the component of $X''_{W'}$ through $\bar x$. As $G_0^\flat\cong\Gamma/\Gamma'$ acts faithfully on each component, $g=1$, and the kernel of θ is $\{(1,\pm\mathrm{id})\}$. So the stabiliser has order dividing 48.

□

3.2 The cover and the eigenlattice

On the Kummer cover the twist becomes single-valued. Its differential transforms under the deck group by $\zeta\mapsto\zeta^j$. Together with the multiplier of f, this action singles out the integral lattice used in the Frobenius calculation. We fix one modular model for Sections 3–5.

Definition 3.6. Let (X',Z',E,G,χ_G) be a modular model, at a prime $\mathfrak{P}$ of L of residue degree one over $p\notin\Sigma$, of a Kummer twist datum (Γ,t,f,a,R) with $a = j/d$, with $\mathcal{O} = \mathcal{O}_{L,(\mathfrak{P})}\subset\mathbb{Z}_p$, Hodge bundle ω, H as in (M1) and section $f\omega^{\otimes(k-2)}$ as in (M4).

1. The **Kummer cover** of the model is the normalisation $\rho:C\to X'$ of X' in the étale $L(X')$-algebra $L(X')[s]/(s^d-t)$, where $L(X')$ is the ring of rational functions on X'_L. We put $D_C = \rho^{-1}(Z')_{\mathrm{red}}$, $Z_\infty = t^{-1}(\infty)_{\mathrm{red}}$, $P_C = \rho^{-1}(Z_\infty\setminus Z')_{\mathrm{red}}$ and $C_{\mathbb{Z}_p} = C\otimes_{\mathcal{O}}\mathbb{Z}_p$.

2. The group $\tilde{G} = G \times \mu_d(L)$ acts on C: an element $g \in G$ through its action on X' and fixing s, and $\zeta \in \mu_d(L)$ by $s \mapsto \zeta s$ over X'. The **twist character** is $\chi_\Phi : \tilde{G} \to L^\times$, $\chi_\Phi(g, \zeta) = \chi_G(g)\,\zeta^j$; through $\mathcal{O} \subset \mathbb{Z}_p$ it takes values in $\mathbb{Z}_p^\times$.

3. $M_k = \operatorname{Sym}^{k-2} \rho^* H$, with the connection, the Hodge filtration $\operatorname{Fil}^i M_k = (\rho^*\omega)^i \cdot \operatorname{Sym}^{k-2-i} \rho^* H$ and the logarithmic F-crystal structure induced from H. The **twisted crystal** of the model is

$$H_{\mathbb{Z}} = \mathbb{H}^1\big(C_{\mathbb{Z}_p},\, M_k \xrightarrow{\nabla} M_k \otimes \Omega^1_{C/\mathbb{Z}_p}(\log D_C)\big).$$

4. The **twisted differential on** C is the rational section

$$\omega_\Phi = \rho^*\big(f\omega^{\otimes(k-2)}\big) \otimes R(t)\, s^{j-1}\, ds$$

of $\rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_{C/\mathcal{O}}$. The normalisation $s^{j-1}\,ds = d^{-1}s^j\,dt/t$ makes its coefficient relative to dQ/Q equal to $f_\Phi(Q)$, since $dq/q = d\,dQ/Q$.

From here to the end of Section 5, fix a Kummer twist datum (Γ, t, f, a, R), a prime $\mathfrak{P}$ of L of residue degree one over $p \notin \Sigma$, and a modular model (X', Z', E, G, χ_G) at $\mathfrak{P}$. Put $\mathcal{O} = \mathcal{O}_{L,(\mathfrak{P})}$ and let $\iota_{\mathfrak{P}} : L \to \mathbb{Q}_p$ be the embedding defined by $\mathfrak{P}$. Let $\rho : C \to X'$ be its Kummer cover, with D_C, P_C, M_k, $H_{\mathbb{Z}}$, $\widetilde{G}$ and χ_Φ as in its definition. Admissibility is assumed only where stated.

Lemma 3.7. *1. Z_∞ is finite and étale over $\mathcal{O}_{L,(\mathfrak{P})}$, and every branch point of $X'_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ lies over S.*

2. C is smooth and proper over $\mathcal{O}_{L,(\mathfrak{P})}$, and $D_C \cup P_C$ is étale over $\mathcal{O}_{L,(\mathfrak{P})}$. The ramification index of ρ at every point of $D_C \cup P_C$ divides d, and ρ is étale elsewhere.

Proof. Item (1) rests on the tame stabilisers of (M2) and the collision clause of Σ (distinct points of S have distinct reductions at every prime above p); item (2) then follows from a local computation of the Kummer extension, which is tame as $p \nmid d$. We use (M1)–(M4) of Definition 3.1, and from $p \notin \Sigma$ only $p \neq 2, 3$, $p \nmid d$ and the collision clause.

Branch points lie over S. Since $X'_L/G \cong \mathbb{P}^1_L$ is geometrically connected, G permutes the connected components of $X'_{\mathbb{C}}$ transitively, so the branch points of $X'_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ are those of $X'_0 \to X'_0/G_0$. By (M4) this map is $\Gamma'\backslash\mathbb{H}^* \to \Gamma\backslash\mathbb{H}^*$, t being the coordinate on the target. As Γ' is torsion-free, a point of $\Gamma'\backslash\mathbb{H}$ has nontrivial stabiliser in Γ/Γ' only if it lies over an elliptic point of Γ; so the branch points lie over elliptic points and cusps of Γ. We show that their t-values are in S. By Definition 2.2, $\mathbb{V}_f$ is spanned by $f, \tau f, \dots, \tau^{k-2} f$, and continuation along the loop around $t(\tau_1)$ attached to $\gamma = \pm(\gamma_{ij}) \in \Gamma$ fixing τ_1 sends $\tau^i f$ to $\chi_f(\gamma)(\gamma_{11}\tau + \gamma_{12})^i(\gamma_{21}\tau + \gamma_{22})^{k-2-i} f$; so the local monodromy is $\chi_f(\gamma)\operatorname{Sym}^{k-2}$ of a matrix representing γ, up to transposition. If τ_1 is elliptic of order $e \geq 2$, the matrix has eigenvalues $\varepsilon^{\pm 1}$ with $\varepsilon^2 \neq 1$, and Sym^{k-2} has the distinct eigenvalues ε^{k-2}, ε^{k-4} (as $k \geq 3$), so the monodromy is not the identity. At a cusp the matrix is unipotent and $\neq \pm 1$, so Sym^{k-2} of it is a nontrivial unipotent ($k \geq 3$) and its product with a scalar is not the identity. Hence these points are singular points of $\mathbb{V}_f$ and their t-values lie in S.

Proof of (1). The reduction of t is non-constant on every component of $X'_{\mathbb{F}_p}$ (M3), so t is a unit at the generic points of $X'_{\mathbb{F}_p}$, its divisor has no vertical component, and Z_∞ is the closure in X' of the finite set $t^{-1}(\infty)_L$. Thus Z_∞ is finite and flat over $\mathcal{O}_{L,(\mathfrak{P})}$, with reduced generic fibre, and it is étale if and only if no two distinct points of $Z_\infty(\bar{\mathbb{Q}}_p)$ have the same reduction in $X'(\bar{\mathbb{F}}_p)$ (reduction is defined because X' is proper). Suppose $x_1 \neq x_2$ in $Z_\infty(\bar{\mathbb{Q}}_p)$ reduce to the same point $\bar{x}$. As $X'_{\bar{\mathbb{Q}}_p} \to \mathbb{P}^1_{\bar{\mathbb{Q}}_p}$ is the quotient by G, there is $g \in G$ with $gx_1 = x_2$. Reduction commutes

with G, which acts over $\mathcal{O}_{L,(\mathfrak{P})}$, so $g\bar{x} = \bar{x}$. Let $\bar{S}$ be the stabiliser of $\bar{x}$; then $\mathrm{Stab}(x_1) \subsetneq \bar{S}$, since $g \in \bar{S}$, and $|\bar{S}|$ is prime to p by (M2).

Next, $\bar{x}$ lifts to a point fixed by $\bar{S}$. Let $W = W(\bar{\mathbb{F}}_p)$. The completed local ring of X'_W at $\bar{x}$ is $W[[\pi]]$, on which $\bar{S}$ acts W-linearly. It acts on the line $\mathfrak{m}/(\mathfrak{m}^2 + p)$ through a character, which lifts to $\chi : \bar{S} \to W^\times$ by Teichmüller lifting; as $|\bar{S}|$ is invertible, $\pi' = |\bar{S}|^{-1} \sum_{h \in \bar{S}} \chi(h)^{-1} h(\pi)$ is again a parameter, $W[[\pi]] = W[[\pi']]$, and $h(\pi') = \chi(h)\pi'$. The ideal (π') is $\bar{S}$-stable, so it defines $y \in X'(W)$ with reduction $\bar{x}$ and $\bar{S} \subset \mathrm{Stab}(y)$; conversely $\mathrm{Stab}(y) \subset \bar{S}$, so $\mathrm{Stab}(y) = \bar{S}$. Let $K = W[1/p]$. The points y and x_1 reduce to the same point $\bar{x}$ of the smooth special fibre, so they lie on the same connected component X'_1 of $X'_{\bar{K}}$. Let G_1 be the stabiliser of X'_1 in G and N_1 the kernel of its action on X'_1; since $\mathrm{Stab}(y)$ and $\mathrm{Stab}(x_1)$ lie in G_1, the ramification indices of $X'_1 \to X'_1/G_1 = \mathbb{P}^1_{\bar{K}}$ at y and at x_1 are $|\bar{S}|/|N_1|$ and $|\mathrm{Stab}(x_1)|/|N_1|$, and the first is the larger. So y is a ramification point of $X'_K \to \mathbb{P}^1_K$; the ramification points are defined over $\bar{\mathbb{Q}}$, so $t(y) \in \bar{\mathbb{Q}}$, and by the first step $t(y) \in S$. Moreover $t(y) \neq \infty$: the points over ∞ form one G-orbit, whose stabilisers are conjugate to $\mathrm{Stab}(x_1)$ and so have fewer than $|\bar{S}|$ elements. But $t(y)$ and ∞ have the same reduction $t(\bar{x})$. So two points of S have the same reduction at a prime above p, and $p \in \Sigma$, a contradiction. Hence Z_∞ is étale.

Proof of (2). Let $Z_0 = t^{-1}(0)_{\mathrm{red}}$. By (M3) and the absence of vertical components, Z_0 is a union of connected components of Z', hence étale, and $Z_0 \cap Z_\infty = \emptyset$ because t is a morphism. So $B = Z_0 \cup Z_\infty$ is étale over $\mathcal{O}_{L,(\mathfrak{P})}$. Let $\bar{x}$ be a closed point of $X'_{\mathbb{F}_p}$ and $\mathcal{O}$ its local ring, regular of dimension 2. If $\bar{x} \notin B$, then $t \in \mathcal{O}^\times$, $p \nmid d$, and $\mathcal{O}[s]/(s^d - t)$ is étale over $\mathcal{O}$, so C is étale over X' above $\bar{x}$. If $\bar{x} \in B$, the ideal of B at $\bar{x}$ is principal, $= (\pi)$, since $\mathcal{O}$ is factorial and B is a reduced divisor; $\mathcal{O}/\pi$ is a discrete valuation ring étale over $\mathcal{O}_{L,(\mathfrak{P})}$, so (π, p) is a regular system of parameters, and $t = \pi^e v$ with $e \in \mathbb{Z} \setminus \{0\}$ and $v \in \mathcal{O}^\times$. Put $g = \gcd(d, e)$, $d = gd'$, $e = ge'$, and choose integers α, β with $\alpha d' + \beta e' = 1$. After the étale extension $\mathcal{O}'$ of $\mathcal{O}$ obtained by adjoining $v^{1/d}$ and μ_g and localising at a point over $\bar{x}$ (étale as $p \nmid d$), the $\mathrm{Frac}(\mathcal{O}')$-algebra $\mathrm{Frac}(\mathcal{O}')[s]/(s^d - t)$ is the product over $\zeta \in \mu_g$ of the fields $\mathrm{Frac}(\mathcal{O}')[y]/(y^{d'} - \zeta\pi^{e'})$, $y = s\, v^{-1/d}$. In each factor, $w = y^\beta \pi^\alpha$ satisfies $w^{d'} = \zeta^\beta \pi$ and $y = \zeta^\alpha w^{e'}$. The ring $\mathcal{O}'[w]/(w^{d'} - \zeta^\beta\pi)$ is finite over $\mathcal{O}'$, local with maximal ideal (w, p) (as $\pi = \zeta^{-\beta} w^{d'}$), hence regular of dimension 2, so it is the normalisation of the factor. Modulo p it is an Eisenstein extension of the discrete valuation ring $\mathcal{O}'/p$, hence regular. Normalisation commutes with étale base change, so C is regular with regular special fibre above $\bar{x}$; as C is flat over $\mathcal{O}_{L,(\mathfrak{P})}$, it is smooth there. Its ramification index over $\pi = 0$ is d', which divides d, and $\rho^{-1}(B)$ is $\{w = 0\}$, étale over $\mathcal{O}/\pi$ and hence over $\mathcal{O}_{L,(\mathfrak{P})}$. So C is smooth, and proper as it is finite over X'. Finally, $Z' \cup Z_\infty$ is étale. If P is a cusp, the points of X'_L over $t = \infty$ are cusps, so $Z_\infty \subset Z'$. If P is not a cusp, $Z' \cap Z_\infty = \emptyset$: a common point $\bar{x}$ would be the reduction of a point of Z_∞ and of a point c of Z', which lies over a cusp of Γ by (M4), so $t(c) \neq \infty$ and $t(c) \in S$ by the first step; then $t(c)$ and ∞ are distinct points of S with the same reduction $t(\bar{x})$, and $p \in \Sigma$. In both cases $Z' \cup Z_\infty$ is étale, and $D_C \cup P_C = \rho^{-1}(Z' \cup Z_\infty)_{\mathrm{red}}$. Over $Z' \setminus B$ the map ρ is étale, and over B we have seen that $\rho^{-1}(B)$ is étale; so $D_C \cup P_C$ is étale over $\mathcal{O}_{L,(\mathfrak{P})}$.

□

The local Tate calculation gives the expansion map and its Frobenius action. These are local cusp statements, so they remain available when the cover is ramified in the interior; the corresponding global statement in [47, Theorem 3.6] assumes an étale cover over the open modular curve.

Lemma 3.8. *Let p be an odd prime, $k \geq 2$ an integer with $p > k - 2$, and $m \geq 1$ an integer prime to p. Let W be the ring of Witt vectors of a perfect field of characteristic p, with Frobenius automorphism σ. Let $\mathcal{A} = W((u))$ be the p-adic completion of $W[[u]][u^{-1}]$ and $\theta = u\, d/du$. Fix*

$\delta \in W^\times$, put $q = \delta u^m$, and let $\gamma \in 1 + pW$ be the unique element with $\gamma^m = \delta^p/\sigma(\delta)$. Let $\omega = \omega_{\mathrm{can}}$ be the canonical differential of the Tate curve $\mathrm{Tate}(q)$ over $\mathcal{A}$ and $\xi = \nabla_{q\,d/dq}\,\omega$, so that $\nabla_\theta \omega = m\,\xi$ and $\nabla_\theta \xi = 0$. Let $M_{\mathcal{A}} = \mathrm{Sym}^{k-2} H^1_{\mathrm{dR}}(\mathrm{Tate}(q)/\mathcal{A})$, free with basis $e_i = \omega^{k-2-i}\xi^i$ $(0 \le i \le k-2)$, and let H^1_{loc} be the cokernel of $\nabla_\theta : M_{\mathcal{A}} \to M_{\mathcal{A}}$, whose elements we write as classes of forms $x \otimes du/u$.

1. *The map $g \mapsto [g\,e_0]$ induces an isomorphism $\iota : \mathcal{A}/\theta^{k-1}\mathcal{A} \to H^1_{\mathrm{loc}}$, with inverse*

$$\psi\Big[\sum_{i=0}^{k-2} a_i e_i\Big] = \sum_{i=0}^{k-2} (-1)^i\,\frac{(k-2-i)!}{(k-2)!\,m^i}\,\theta^i a_i.$$

2. *Let $V : \mathcal{A} \to \mathcal{A}$ be the σ-semilinear ring endomorphism with $V(u) = \gamma u^p$, that is $V(\sum_N a_N u^N) = \sum_N \sigma(a_N)\gamma^N u^{pN}$; it satisfies $V(q) = q^p$. Let $F : V^*M_{\mathcal{A}} \to M_{\mathcal{A}}$ be the map induced on de Rham cohomology by the isogeny $\mathrm{Tate}(q) \to V^*\mathrm{Tate}(q) = \mathrm{Tate}(q^p)$, $x \mapsto x^p$. Then*

$$F\big(V^*(g\,e_0)\big) \otimes V^*\frac{du}{u} = p^{k-1}\,V(g)\,e_0 \otimes \frac{du}{u} \qquad (g \in \mathcal{A}),$$

and $x \otimes \frac{du}{u} \mapsto F(V^*x) \otimes V^*\frac{du}{u}$ induces $p^{k-1}V$ on $\mathcal{A}/\theta^{k-1}\mathcal{A}$ through ι.

Proof. For power series and cusp forms, (1) is [47, Theorems 2.7(ii) and 2.12(ii)] and (2) is [47, (3.2.1), (3.3.1), (3.3.2)] (the weight $k+2$ there is our k), in the normalisation of [32, §4], for a cusp of width m.

Proof of (1). From $\nabla_\theta\omega = m\xi$, $\nabla_\theta\xi = 0$ and the Leibniz rule, $\nabla_\theta e_i = (k-2-i)\,m\,e_{i+1}$, with $e_{k-1} := 0$. Hence, for $a_0, \dots, a_{k-2} \in \mathcal{A}$ and $a_{-1} := 0$,

$$\nabla_\theta\Big(\sum_{i=0}^{k-2} a_i e_i\Big) = \sum_{i=0}^{k-2} \big(\theta a_i + (k-1-i)\,m\,a_{i-1}\big)e_i. \tag{$*$}$$

The map ψ vanishes on coboundaries modulo $\theta^{k-1}\mathcal{A}$. Put $c_i = (k-2-i)!/((k-2)!\,m^i)$. Since $p > k-2$ and $p \nmid m$, each c_i is a unit of $\mathbb{Z}_{(p)}$, and $c_i\,(k-1-i)\,m = c_{i-1}$ for $1 \le i \le k-2$. Applying ψ to $(*)$ and shifting the index in the second sum,

$$\begin{aligned}
&\sum_{i=0}^{k-2} (-1)^i c_i \theta^{i+1} a_i + \sum_{i=1}^{k-2} (-1)^i c_i (k-1-i) m\,\theta^i a_{i-1} \\
&\qquad = \sum_{i=0}^{k-2} (-1)^i c_i \theta^{i+1} a_i - \sum_{i=0}^{k-3} (-1)^i c_i \theta^{i+1} a_i \\
&\qquad = (-1)^{k-2} c_{k-2}\,\theta^{k-1} a_{k-2}.
\end{aligned}$$

So ψ induces a map $H^1_{\mathrm{loc}} \to \mathcal{A}/\theta^{k-1}\mathcal{A}$.

That $\theta^{k-1}c \cdot e_0$ is a coboundary for every $c \in \mathcal{A}$, and that every class has a representative $a\,e_0$, is the computation in the proof of [47, Theorem 2.7(ii)], pp. 58–59, which uses only $(*)$ and the invertibility of $(k-2)!\,m^{k-2}$, and holds verbatim over $\mathcal{A}$. Since $c_0 = 1$, $\psi \circ \iota$ is the identity; hence ι is bijective, with inverse ψ.

Proof of (2). The isogeny $x \mapsto x^p$ of multiplicative uniformisations pulls back dx/x to $p\,dx/x$, so $F(V^*\omega) = p\,\omega$ [47, (3.2.1)]. Since $\gamma \in W$ is a constant, $V^*\frac{du}{u} = \frac{d(\gamma u^p)}{\gamma u^p} = p\,\frac{du}{u}$. Hence

$$F\big(V^*(g\,e_0)\big) \otimes V^*\frac{du}{u} = V(g)\,(p\,\omega)^{k-2} \otimes p\,\frac{du}{u} = p^{k-1}\,V(g)\,e_0 \otimes \frac{du}{u}.$$

The map F is induced by a morphism of abelian schemes lying over V, so it is horizontal: $\nabla_\theta F(V^*x) = p\,F(V^*\nabla_\theta x)$ for $x \in M_{\mathcal{A}}$, the factor p coming from $V^*\frac{du}{u} = p\frac{du}{u}$. Therefore $F(V^*\nabla_\theta x)\otimes V^*\frac{du}{u} = \nabla_\theta F(V^*x)\otimes\frac{du}{u}$, so the map on forms carries coboundaries to coboundaries and induces an endomorphism of H^1_{loc}. By (1) every class is $\iota(g)$, and by the display this endomorphism sends $\iota(g)$ to $\iota(p^{k-1}V(g))$. It is well defined on $\mathcal{A}/\theta^{k-1}\mathcal{A}$ because $\theta\circ V = p\,V\circ\theta$, so $p^{k-1}V(\theta^{k-1}c) = \theta^{k-1}V(c)$.

□

Lemma 3.9. *Let p be an odd prime, $k\geq 2$ an integer with $p > k-2$, and W the ring of Witt vectors of a perfect field of characteristic p, with Frobenius σ. Let X' be a smooth proper curve over W, $Z'\subset X'$ a divisor étale over W, and $E\to X'$ a generalised elliptic curve, smooth over $X'\setminus Z'$; let H be its logarithmic de Rham cohomology, with Gauss–Manin connection, Hodge filtration and structure of logarithmic F-crystal. Let $\rho : C\to X'$ be a finite morphism of smooth proper curves over W such that $D_C = \rho^{-1}(Z')_{\mathrm{red}}$ is étale over W. Put $M_k = \mathrm{Sym}^{k-2}\rho^*H$ and*

$$H_{\mathbb{Z}} = \mathbb{H}^1\big(C,\ M_k \xrightarrow{\nabla} M_k\otimes\Omega^1_{C/W}(\log D_C)\big),$$

with its crystalline Frobenius φ, which is σ-semilinear. Let $x_\infty\in D_C(W)$, let u be a formal parameter of C along x_∞, and assume that the pull-back of E to $\mathrm{Spf}\,W[[u]]$ is the Tate curve $\mathrm{Tate}(q)$ with $q=\delta u^m$, $\delta\in W^\times$ and $p\nmid m$. Let $\mathcal{A}$, θ, γ, V, ι, ψ be as in Lemma 3.8, and let σ_{can} be the Frobenius lift of $W[[u]]$ with $\sigma_{\mathrm{can}}(u)=\gamma u^p$.

1. *Let $x\in H_{\mathbb{Z}}$ be represented by a Čech–de Rham cocycle on an affine open cover of C, and let η be its component on a member U of the cover containing x_∞. Then $\mathcal{E}(x) = \psi[\eta|_{\mathrm{Spf}\,\mathcal{A}}]\in\mathcal{A}/\theta^{k-1}\mathcal{A}$ depends only on x, and $\mathcal{E}: H_{\mathbb{Z}}\to\mathcal{A}/\theta^{k-1}\mathcal{A}$ is W-linear.*
2. *$\mathcal{E}\circ\varphi = p^{k-1}V\circ\mathcal{E}$.*
3. *Let σ_U be a Frobenius lift on the p-adic completion of U, logarithmic along D_C, and F_{σ_U}, $F_{\sigma_{\mathrm{can}}}$ the induced Frobenius maps on $M_k\otimes\Omega^1_C(\log D_C)$. Put $\delta_U = \log\big(\sigma_U(u)/\sigma_{\mathrm{can}}(u)\big)$. Then $\delta_U\in pW[[u]]$, and for every section η of $M_k\otimes\Omega^1_C(\log D_C)$ over U, written $\eta|_{W[[u]]} = y\otimes\frac{du}{u}$ with $y\in M_k\otimes W[[u]]$,*

$$F_{\sigma_U}(\eta)\big|_{W[[u]]} = F_{\sigma_{\mathrm{can}}}(\eta|_{W[[u]]}) + \nabla h(y),$$
$$h(y) = \sum_{n\geq 1}\frac{\delta_U^n}{n!}\,F_{\sigma_{\mathrm{can}}}(\sigma_{\mathrm{can}}^*\nabla_\theta^{n-1}y)\in M_k\otimes W[[u]].$$

Proof. For (1) and (2) we follow [47, Theorem 3.6], in the form of [32, §4]; only the formal disc at x_∞ enters.

The Frobenius φ is computed as follows. By the comparison between logarithmic crystalline and logarithmic de Rham cohomology [29], Thm. 6.4, $H_{\mathbb{Z}}$ is the logarithmic crystalline cohomology of the special fibre of (C, D_C) with coefficients in the logarithmic F-crystal $M_k = \mathrm{Sym}^{k-2}\rho^*H$, and φ is the Frobenius it induces. It is computed on Čech–de Rham cocycles as in [48, 2.3]. Cover C by finitely many affine opens U_i such that each U_i meeting D_C has an étale coordinate u_i with $D_C\cap U_i = \{u_i = 0\}$. On the p-adic completion $\widehat{U}_i$ the lift $u_i\mapsto u_i^p$ of $\mathrm{Spf}\,W\langle u_i\rangle$ (semilinear over σ) extends uniquely, by étaleness, to a Frobenius lift σ_i, which is logarithmic along D_C; on the U_i not meeting D_C any Frobenius lift will do. Each σ_i induces F_{σ_i} on the de Rham complex of M_k over $\widehat{U}_i$, with $F_{\sigma_i}(y\otimes\eta_0) = F_{\sigma_i}(\sigma_i^*y)\otimes\sigma_i^*\eta_0$. These maps make M_k an F-crystal with logarithmic singularities along D_C in the sense of [48, 1.5]: over $C\setminus D_C$ they are the Frobenius of the crystalline cohomology of the elliptic curve ρ^*E, an F-crystal in the usual sense, and they

extend across D_C, which suffices by [48, 1.6]. So on overlaps F_{σ_i} and F_{σ_j} are homotopic through the homotopies of [48, 2.1, Proposition 2.2]. If x is the class of a cocycle (η_i, m_{ij}), then $\varphi(x)$ is the class of a cocycle whose component on U_i is $F_{\sigma_i}(\eta_i)$. This is the only feature of the Čech formula used below. The cohomology of the scheme C and of its p-adic completion agree, as C is proper over W.

Proof of (3). This is [48, Proposition 2.2(ii)] on the formal disc $\operatorname{Spf} W[[u]]$ at x_∞, a pair of type 1.2(ii) there, for the lifts σ_{can} and σ', the lift of $W[[u]]$ induced by σ_U. Both are admissible, that is, logarithmic along D_C, σ_{can} because $\gamma \in W^\times$. By (2.1.2) loc. cit., with the parameter u, $h(y)$ is the value at $y \otimes \frac{du}{u}$ of the homotopy of 2.1 loc. cit. between the two lifts, and $\delta_U \in pW[[u]]$ by (1.4.3) loc. cit.

Proof of (1). Changing the cocycle by the coboundary of (g_i), $g_i \in M_k(U_i)$, replaces η by $\eta + \nabla g$ with g regular at x_∞, whose restriction $\nabla_\theta(g|_{W[[u]]}) \otimes \frac{du}{u}$ is zero in H^1_{loc}. If U' is another member of the cover containing x_∞, with component η', the cocycle condition gives $\eta' - \eta = \nabla y$ with $y \in M_k(U \cap U')$, and $U \cap U'$ contains x_∞, so again the difference restricts to a coboundary. Two covers have a common refinement, and passing to a refinement replaces η by its restriction to a smaller open containing x_∞. Hence $\mathcal{E}$ is well defined, and it is W-linear because ψ is.

Proof of (2). Let x be the class of (η_i, m_{ij}) and let $U = U_{i_0}$ contain x_∞, with $\sigma_U = \sigma_{i_0}$ and $\eta = \eta_{i_0}$. The component of $\varphi(x)$ on U is $F_{\sigma_U}(\eta)$. By (3) its restriction to $\operatorname{Spf} W[[u]]$ is $F_{\sigma_{\mathrm{can}}}(\eta|_{W[[u]]})$ plus the coboundary $\nabla h(y)$, so $\mathcal{E}(\varphi x) = \psi[F_{\sigma_{\mathrm{can}}}(\eta|_{W[[u]]})]$. Since $\sigma_{\mathrm{can}}(q) = \sigma(\delta)\gamma^m u^{pm} = q^p$, the Frobenius of the restriction of $\rho^* H$ to $\operatorname{Spf} W[[u]]$ with respect to σ_{can} is the map induced by the isogeny $\mathrm{Tate}(q) \to \mathrm{Tate}(q^p)$, $x \mapsto x^p$, which lifts the relative Frobenius [47, Proposition 3.2, (3.2.1)]. Hence $F_{\sigma_{\mathrm{can}}}$ on $M_k \otimes \frac{du}{u}$ is the map F of Lemma 3.8(2), and that lemma gives $\psi[F_{\sigma_{\mathrm{can}}}(\eta|_{W[[u]]})] = p^{k-1} V\big(\psi[\eta|_{W[[u]]}]\big) = p^{k-1} V \mathcal{E}(x)$.

□

The eigenlattice is obtained by imposing both the character condition and vanishing residues at the cusps. The latter condition is needed when the twisted local system has cusp invariants. The differential then lies in its top Hodge line.

Lemma 3.10. *For $z \in D_C(W)$, $W = W(\bar{\mathbb{F}}_p)$, let $\operatorname{Res}_z(x) \in W$ be the constant term of the expansion of $x \in H_{\mathbb{Z}}$ at z (Lemma 3.9(1)).*

1. *$H_{\mathbb{Z}}$ is a free $\mathbb{Z}_p$-module of finite rank, with a $\mathbb{Z}_p$-linear crystalline Frobenius φ, the Hodge filtration $\operatorname{Fil}^i H_{\mathbb{Z}} = \operatorname{im}(\mathbb{H}^1(\operatorname{Fil}^i K) \to \mathbb{H}^1(K))$, where $K = [M_k \to M_k \otimes \Omega^1(\log D_C)]$ and $\operatorname{Fil}^i K = [\operatorname{Fil}^i M_k \to \operatorname{Fil}^{i-1} M_k \otimes \Omega^1(\log D_C)]$, and an action $x \mapsto g^* x$ of $\tilde{G}$ commuting with φ and preserving $\operatorname{Fil}^\bullet$.*

2. *The submodule*

$$H_{a,p} = \{x \in H_{\mathbb{Z}} : \ \operatorname{Res}_z(x) = 0 \text{ for all } z \in D_C(W), \quad g^* x = \chi_\Phi(g)\, x \text{ for all } g \in \tilde{G}\}$$

   is saturated in $H_{\mathbb{Z}}$ and stable under φ; in particular it is free. We call $H_{a,p}$ the eigenlattice *of the model.*

3. *Let $\iota : \mathbb{Q}_p \to \mathbb{C}$ be a field embedding that is the identity on $L \subset \mathbb{C}$. Then $H_{a,p} \otimes_{\mathbb{Z}_p, \iota} \mathbb{C} \cong H^1_{\mathrm{par}}(\mathbb{P}^1 \setminus S, \mathbb{W}_\Phi)$, compatibly with Hodge filtrations, where $\mathbb{W}_\Phi$ is the χ_Φ-eigensheaf of $(t \circ \rho)_* \operatorname{Sym}^{k-2} R^1\pi_* \mathbb{C}$ on $\mathbb{P}^1 \setminus S$, with π the pull-back of the elliptic curve to $C \setminus D_C$. The variation $\mathbb{W}_\Phi$ has weight $k-2$, and there are isomorphisms of complex variations of Hodge structure*

$$\mathbb{V}_f \otimes \mathbb{L}_a \ \cong \ \mathbb{W}_\Phi^\vee(2-k) \ \cong \ \overline{\mathbb{W}_\Phi},$$

where the left-hand side carries the variation of Lemma 2.4. Consequently $H_{a,p}\otimes_\iota\mathbb{C}$ is the complex conjugate of H_a, with the two Hodge indices exchanged. In particular, if the datum is admissible, then $H_{a,p}$ is free of rank 2 and $\mathrm{Fil}^1(H_{a,p}\otimes\mathbb{Q}_p)=\mathrm{Fil}^{k-1}(H_{a,p}\otimes\mathbb{Q}_p)$ is a line.

Proof. Proof of (1). The complex $K=[M_k\to M_k\otimes\Omega^1(\log D_C)]$ consists of locally free sheaves on the proper smooth $\mathbb{Z}_p$-scheme $C_{\mathbb{Z}_p}$ (Lemma 3.7), with a $\mathbb{Z}_p$-linear differential, so $H_\mathbb{Z}$ is finitely generated, lying between $H^0(M_k\otimes\Omega^1(\log D_C))$ and $H^1(M_k)$. By the sequence $0\to K\xrightarrow{p}K\to K/p\to 0$, the p-torsion of $H_\mathbb{Z}$ is the image of $H^0(C_{\mathbb{F}_p},K/p)$, the horizontal global sections of $M_k\otimes\mathbb{F}_p$; we show there are none. Let m be one. By (M1) of Definition 3.1, $\omega^{\otimes 2}\cong\Omega^1_{X'}(\log Z')$, whose degree on each component of $X'_{\mathbb{F}_p}$ equals its degree on the corresponding component of $X'_\mathbb{C}$, namely $[\mathrm{PSL}_2(\mathbb{Z}):\Gamma']/6>0$ by (M4) and the transitivity of G on components; as ρ is finite and surjective on components, $\rho^*\omega$ has positive degree on every component of $C_{\mathbb{F}_p}$. So the image of m in $\mathrm{gr}^0 M_k=\rho^*\omega^{\otimes(2-k)}$ vanishes, and $m\in\mathrm{Fil}^1$. For $1\le i\le k-2$, if $m\in\mathrm{Fil}^i$, then $\mathrm{gr}^i\nabla=i\,(\kappa\otimes\mathrm{id})$ kills the image $\bar m_i$ of m in $\mathrm{gr}^i M_k$, where $\kappa:\rho^*\omega^{\otimes 2}\to\Omega^1_C(\log D_C)$ is the pull-back of the Kodaira–Spencer isomorphism followed by $\rho^*\Omega^1_{X'}(\log Z')\to\Omega^1_C(\log D_C)$ [32], §2. The map κ is an injective map of line bundles, since $\rho_{\mathbb{F}_p}$ is generically étale ($p\nmid d$), and i is a unit as $i<p$; so $\bar m_i=0$ and $m\in\mathrm{Fil}^{i+1}$. As $\mathrm{Fil}^{k-1}M_k=0$, $m=0$. Hence $H_\mathbb{Z}$ is torsion-free, so free. By [29], Thm. 6.4, $H_\mathbb{Z}$ is the logarithmic crystalline cohomology of $(C_{\mathbb{F}_p},D_C)$ with coefficients in the logarithmic F-crystal M_k (the pull-back of $\mathrm{Sym}^{k-2}H$), which gives φ, linear over $\mathbb{Z}_p$. The Hodge filtration is $\mathrm{Fil}^i H_\mathbb{Z}=\mathrm{im}\,H^1(\mathrm{Fil}^i K)$ with $\mathrm{Fil}^i K=[\mathrm{Fil}^i M_k\to\mathrm{Fil}^{i-1}M_k\otimes\Omega^1(\log D_C)]$. The group G acts on $(X',Z',\mathrm{Sym}^{k-2}H)$ by (M2) and on C fixing s, as t is G-invariant; $\mu_d(L)$ acts on C over X'. Both preserve D_C and act on $(M_k,\nabla,\mathrm{Fil}^\bullet)$ compatibly with the F-crystal structure, so $\tilde G$ acts on $H_\mathbb{Z}$ commuting with φ and preserving $\mathrm{Fil}^\bullet$.

Proof of (2). Let $z\in D_C(W)$. By Lemma 3.7 and (M1), and as p is odd and $p\ge k$ ($p\notin\Sigma$), the hypotheses of Lemma 3.9 hold for $C_W\to X'_W$ at z: ρ has ramification index $r\mid d$ at z, so $q_{\rho(z)}=\varepsilon u^r$ for a suitable formal parameter u at z and $\varepsilon\in W^\times$ (adjust u by an r-th root of a unit, $p\nmid r$), and E pulls back to $\mathrm{Tate}(\delta u^m)$ with $m=m_{\rho(z)}r$ prime to p. So Res_z is defined, as $\theta^{k-1}\mathcal{A}$ has no constant term. By Lemma 3.9(2), applied to $H_\mathbb{Z}\otimes W$ with Frobenius $\varphi\otimes\sigma$, $\mathcal{E}_z(\varphi x)=p^{k-1}V\mathcal{E}_z(x)$; the constant term of Vg is $\sigma(g_0)$, so $\mathrm{Res}_z(\varphi x)=p^{k-1}\sigma(\mathrm{Res}_z x)$. Hence the residue conditions are preserved by φ, and so is the eigen-condition, $\tilde G$ commuting with φ. If $px\in H_{a,p}$ then $p\,\mathrm{Res}_z(x)=0$ in W and $p(g^*x-\chi_\Phi(g)x)=0$ in the torsion-free $H_\mathbb{Z}$, so $x\in H_{a,p}$. A saturated submodule of a finitely generated free $\mathbb{Z}_p$-module is a direct summand, hence free.

Proof of (3). We compare with $\mathbb{C}$, where the residue conditions cut out parabolic cohomology and the χ_Φ-eigenspace descends to a local system $\mathbb{W}_\Phi$ on $\mathbb{P}^1\setminus S$, which is then identified with the dual of $\mathbb{V}_f\otimes\mathbb{L}_a$. *Reduction to* $\mathbb{C}$. Over a field $K\supset\mathbb{Q}$ and for $z\in D_C(\bar K)$ let $\mathrm{Res}_z^{\mathrm{alg}}$ send a class to the residue at z of its component near z, in $\mathrm{coker}(\mathrm{Res}_\nabla:M_k(z)\to M_k(z))$; in the Tate basis e_i of Lemma 3.8 one has $\mathrm{Res}_\nabla e_i=(k-2-i)m\,e_{i+1}$, so this cokernel is spanned by e_0 and $\mathrm{Res}_z^{\mathrm{alg}}$ is the constant term of the e_0-coefficient, that is, Res_z. Let $H_L=H^1_{\mathrm{dR}}(C_L/L,M_k(\log D_C))$ with its actions of G and $\mu_d(L)$, and $H_L^{\mathrm{par},\chi}$ the intersection of $\ker\mathrm{Res}^{\mathrm{alg}}$ with the χ_Φ-eigenspace. Coherent cohomology, kernels and eigenspaces commute with flat base change, and $\mathfrak{P}$ identifies $\mu_d(L)$ with $\mu_d(\mathbb{Z}_p)$; hence $H_{a,p}\otimes\mathbb{Q}_p=H_L^{\mathrm{par},\chi}\otimes_{L,\mathfrak{P}}\mathbb{Q}_p$ and, as $\iota|_L=\mathrm{id}$, $H_{a,p}\otimes_\iota\mathbb{C}=H_\mathbb{C}^{\mathrm{par},\chi}$, compatibly with the Hodge filtrations.

Topology. Let $\mathbb{V}=R^1\pi_*\mathbb{C}$ for the pull-back of E to $C^{\mathrm{an}}\setminus D_C$. The residues of $\rho^*\nabla$ along D_C are nilpotent, so M_k is the canonical extension of $\mathrm{Sym}^{k-2}\mathbb{V}\otimes\mathcal{O}$, and $H_\mathbb{C}\cong H^1(C^{\mathrm{an}}\setminus D_C,\mathrm{Sym}^{k-2}\mathbb{V})$, $\tilde G$-equivariantly [15], II, §5 (canonical extension) and §6 (comparison). On a small disc Δ_z the logarithmic de Rham complex computes $H^1(\Delta_z^*,\mathrm{Sym}^{k-2}\mathbb{V})=\mathrm{coker}\,\mathrm{Res}_\nabla$, and restriction to Δ_z^* is $\mathrm{Res}_z^{\mathrm{alg}}$ (loc. cit., II, §6). So $\ker\mathrm{Res}^{\mathrm{alg}}=\ker(H^1(C\setminus D_C)\to\bigoplus_z H^1(\Delta_z^*))=H^1_{\mathrm{par}}=H^1(\bar C,o_*\mathrm{Sym}^{k-2}\mathbb{V})$, with o the inclusion of the complement of D_C.

Descent. Put $U = \mathbb{P}^1 \setminus S$ and $\rho_t = t \circ \rho$. The group $\widetilde{G}$ permutes the components of $C_{\mathbb{C}}$ transitively, since G does so on $X'_{\mathbb{C}}$ and μ_d does so over each component, and $C_{\mathbb{C}}/\widetilde{G} = \mathbb{P}^1$. Choose a component C_0 and let $\widetilde{G}_0$ be its stabiliser. Restriction identifies the global χ_Φ-eigensheaf with the $\chi_\Phi|_{\widetilde{G}_0}$-eigensheaf on C_0: an eigensection on one component determines those on all the others, and the stabiliser condition is exactly the compatibility required. By Lemma 3.7(1), and as ρ is branched only over $t \in \{0, \infty\} \subset S$, $C_0 \cap \rho_t^{-1}(U) \to U$ is finite étale Galois, with group the image of $\tilde{G}_0$ in $\mathrm{Aut}(C_0)$. Let $\mathbb{W}_\Phi$ be the subsheaf of $\rho_{t*}\operatorname{Sym}^{k-2}\mathbb{V}$ on U of sections s with $g^*s = \chi_\Phi(g)s$ for $g \in \tilde{G}_0$, a sub-variation of weight $k-2$. The kernel K of $\tilde{G}_0 \to \mathrm{Aut}(C_0)$ acts on $\operatorname{Sym}^{k-2}\mathbb{V}|_{C_0}$ by horizontal automorphisms, hence by scalars, since this local system is irreducible (its monodromy contains a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, Zariski dense in SL_2); as the section ψ below is a nonzero χ_Φ-eigensection, these scalars equal $\chi_\Phi|_K$, so $\mathbb{W}_\Phi$ is a local system of rank $k-1$. By Shapiro's lemma and exactness of isotypic projection in characteristic 0, $H^1(\bar{C}, o_*\operatorname{Sym}^{k-2}\mathbb{V})^\chi = H^1(\mathbb{P}^1, (\rho_{t*}o_*\operatorname{Sym}^{k-2}\mathbb{V})^\chi)$, and the stalk of the latter sheaf at $s \in S$ is $(o_*\operatorname{Sym}^{k-2}\mathbb{V})_x^{\chi|\mathrm{Stab}(x)} = \Gamma(\Delta_x^*, \operatorname{Sym}^{k-2}\mathbb{V})^{\chi|\mathrm{Stab}(x)} = \mathbb{W}_\Phi^{I_s}$ for $x \in C_0$ over s. So it is $o_*\mathbb{W}_\Phi$, and $H_{\mathbb{C}}^{\mathrm{par},\chi} \cong H^1_{\mathrm{par}}(U, \mathbb{W}_\Phi)$.

Identification of the variations. Put $m = k-2$. The Weil pairing on $\operatorname{Sym}^m\mathbb{V}$ takes values in $\mathbb{C}(-m)$ and is $\tilde{G}_0$-invariant by (M2) and the action of μ_d on the base. It pairs the χ- and χ^{-1}-parts perfectly. The action commutes with the real structure on $R^1\pi_*\mathbb{R}$, since each π_g^* is defined over $\mathbb{Z}$ and each c_g is real. As $\bar{\chi} = \chi^{-1}$, the pairing identifies $\overline{\mathbb{W}_\Phi}$ with $\mathbb{W}_\Phi^\vee(-m)$ as variations of weight m.

The section $\psi = \rho^*(f\omega^{\otimes m})s^j$ transforms by χ_Φ by (M4) and $[\zeta]^*s^j = \zeta^j s^j$. On a disc in U, choose a lift to the cover, a branch of τ, and a real horizontal frame e_1, e_2 for $\mathbb{V}$ dual to cycles with periods $1, \tau$. Write $v(\tau) = e_1 + \tau e_2$, so that $2\pi i\, dz = 2\pi i\, v(\tau)$. Pairing ψ with a flat functional λ gives, up to the fixed nonzero scalar $(2\pi i)^m$, the solution

$$f(\tau)t^a\, P_\lambda(\tau), \qquad P_\lambda(X) = \lambda((e_1 + Xe_2)^m).$$

Every polynomial of degree at most m occurs for a unique λ. Since ψ is an eigensection, these local period maps glue to the identification of the underlying local system of $\mathbb{W}_\Phi^\vee(-m)$ with $\mathbb{V}_f \otimes \mathbb{L}_a$.

Fix τ_0 on the disc and put $v_0 = v(\tau_0)$. On U, t is a local coordinate and $f\,t^a$ is a unit by Lemma A.1(3). For $1 \le i \le m$, the first i Taylor coefficients of $v(\tau)^m$ at τ_0 span

$$\langle v_0^{m-r}e_2^r : 0 \le r < i\rangle = v_0^{m-i+1}\operatorname{Sym}^{i-1}\mathbb{V}_{\tau_0} = F^{m-i+1}\operatorname{Sym}^m\mathbb{V}_{\tau_0}.$$

Thus the solution vanishes to order at least i exactly when λ annihilates F^{m-i+1}, the defining condition for F^i of the shifted dual. This is the filtration of Lemma 2.4(1). The Hodge types agree as well. In the basis $v_0, \bar{v}_0$, the functional dual to $v_0^{m-i}\bar{v}_0^i$ has shifted type $(i, m-i)$, and its polynomial is a nonzero multiple of $(X-\tau_0)^i(X-\bar{\tau}_0)^{m-i}$. This is precisely the $(i, m-i)$ line of that lemma for every $0 \le i \le m$.

Hodge structures. By [62, main theorem, p. 416, and (7.12)], $H^1_{\mathrm{par}} = H^1(\bar{C}, o_*\operatorname{Sym}^{k-2}\mathbb{V})$ carries a pure Hodge structure of weight $k-1$; it is the lowest-weight part of the mixed Hodge structure on $H^1(C^{\mathrm{an}} \setminus D_C, \operatorname{Sym}^{k-2}\mathbb{V})$, whose Hodge filtration is induced by the filtered logarithmic de Rham complex of the canonical extension [59], §§4–5. The inclusion is a morphism of mixed Hodge structures, hence strict for F, so the Hodge filtration of H^1_{par} is the one induced by the logarithmic de Rham filtration; and $\tilde{G}$ acts by morphisms of Hodge structures; so $H_{\mathbb{C}}^{\mathrm{par},\chi} \cong H^1_{\mathrm{par}}(U, \mathbb{W}_\Phi)$ as Hodge structures. The identification of variations above and Lemma A.3(2) give $H_a \cong H^1_{\mathrm{par}}(U, \overline{\mathbb{W}_\Phi}) \cong \overline{H^1_{\mathrm{par}}(U, \mathbb{W}_\Phi)}$, with the two Hodge indices exchanged. Under (R) and (P) we have $h^{k-1,0}(H_a) = h^{0,k-1}(H_a) = 1$. Hence $H_{a,p} \otimes_\iota \mathbb{C}$ has dimension 2 and

type $\{(k-1,0),(0,k-1)\}$ with one-dimensional summands, so $\mathrm{Fil}^1 = \mathrm{Fil}^{k-1}$ is a line; filtration dimensions are unchanged by $\otimes_\iota \mathbb{C}$.

□

3.3 The differential

At an interior ramification point, the zero of the pulled-back differential of a local parameter can cancel a fractional pole of Φ. At a cusp the relevant order is instead the order of the coefficient of the logarithmic differential. The next lemma gives both orders explicitly.

Lemma 3.11. *Let (Γ, t, f, a, R) be a Kummer twist datum, with $a = j/d$ and twisted differential Φ. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index, contained in $\Gamma(n)$ for some $n \geq 3$ or in $\Gamma_1(n)$ for some $n \geq 5$, on which χ_f is trivial. Let $X' = \Gamma' \backslash \mathbb{H}^*$, let $E \to X'$ be the universal generalised elliptic curve, ω its Hodge bundle, and $f\,\omega^{\otimes(k-2)}$ the meromorphic section of $\omega^{\otimes(k-2)}$ given by $f(\tau)\,(2\pi i\,dz)^{\otimes(k-2)}$ over $\mathbb{H}$. Let $\rho : C \to X'$ be the normalisation of X' in the étale $\mathbb{C}(X')$-algebra $\mathbb{C}(X')[s]/(s^d - t)$, $D_C = \rho^{-1}(X' \setminus Y')$ the points over the cusps, and*

$$\omega_\Phi = \rho^*\big(f\,\omega^{\otimes(k-2)}\big) \otimes R(t)\, s^{j-1}\, ds,$$

a meromorphic section of $\rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C(\log D_C)$. Write $\Phi = \psi\, d\tau$ with $\psi = f\,R(t)\,t^a\,t'/t$, $t' = dt/d\tau$. For $\tau_0 \in \mathbb{H}$ put $\mathrm{ord}_{\tau_0}\Phi = \mathrm{ord}_{\tau_0} f + \mathrm{ord}_{\tau_0} R(t) + a\,\mathrm{ord}_{\tau_0} t + \mathrm{ord}_{\tau_0}(t'/t)$, orders in $\tau - \tau_0$; for a cusp $c = \sigma_c\infty$ of Γ with $\sigma_c \in \mathrm{SL}_2(\mathbb{Z})$ and local parameter q_c of Γ at c, let $\mathrm{ord}_c\,\Phi$ be the order in q_c of $\psi|_k\sigma_c$, the exponents in q_c being allowed to be fractional.

1. *If $x \in C \setminus D_C$ lies over the image of $\tau_0 \in \mathbb{H}$ and ρ has ramification index r_x at x, then $\mathrm{ord}_x\,\omega_\Phi = r_x\big(\mathrm{ord}_{\tau_0}\Phi + 1\big) - 1$.*
2. *If $x \in D_C$ lies over the cusp c and the composite $C \to X' \to X_\Gamma$ has ramification index r_x at x, then the order of ω_Φ at x, as a section of $\rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C(\log D_C)$, is $r_x\,\mathrm{ord}_c\,\Phi$.*
3. *Consequently the datum satisfies (F) if and only if ω_Φ is a nonzero holomorphic section of $\rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C$, that is, a section of $\rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C(\log D_C)$ that is holomorphic on C and vanishes along D_C.*

Proof. Since Γ' lies in $\Gamma(n)$ with $n \geq 3$ or in $\Gamma_1(n)$ with $n \geq 5$, it is torsion-free, it lifts to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ not containing -1, and all its cusps are regular; so the universal generalised elliptic curve over X' exists [18], and at a point of Y' over τ_0 the function $\tau - \tau_0$ is a local coordinate. Since $s^{j-1}ds = \frac{1}{d}\,s^j\,\frac{dt}{t}$ and $s^d = t$, we may write $\omega_\Phi = \frac{1}{d}\,\rho^*\big(f\,\omega^{\otimes(k-2)}\big) \otimes R(t)\,s^j\,\frac{dt}{t}$, where s^j is a branch of t^a and $\mathrm{ord}_x(s^j) = a\,\mathrm{ord}_x(t)$ at every point x of C.

Proof of (1). Let v be a local coordinate at x. As ρ has ramification index r_x at x and $\tau - \tau_0$ is a local coordinate on X' at $\rho(x)$, we have $\tau - \tau_0 = v^{r_x} \cdot (\text{unit})$. Hence every meromorphic function h of τ near τ_0 satisfies $\mathrm{ord}_v(h \circ \rho) = r_x\,\mathrm{ord}_{\tau_0} h$, and $\mathrm{ord}_v(d\tau) = r_x - 1$. Over $\mathbb{H}$ the section $2\pi i\,dz$ of ω does not vanish, so $\mathrm{ord}_x\,\rho^*(f\,\omega^{\otimes(k-2)}) = r_x\,\mathrm{ord}_{\tau_0} f$. Writing $dt/t = (t'/t)\,d\tau$, we obtain

$$\mathrm{ord}_x\,\omega_\Phi = r_x\big(\mathrm{ord}_{\tau_0} f + \mathrm{ord}_{\tau_0} R(t) + a\,\mathrm{ord}_{\tau_0} t + \mathrm{ord}_{\tau_0}(t'/t)\big) + (r_x - 1) = r_x\big(\mathrm{ord}_{\tau_0}\Phi + 1\big) - 1.$$

Proof of (2). Let $c' = \rho(x)$ and let e be the ramification index of $X' \to X_\Gamma$ at c'. The local parameter of X' at c' is $q_{c'}$ with $q_{c'}^e = q_c$, and near c' the universal curve is the Tate curve in the variable $\tau' = \sigma_c^{-1}\tau$, so that $f\,\omega^{\otimes(k-2)} = (f|_{k-2}\sigma_c)(\tau')\,\omega_{\mathrm{can}}^{\otimes(k-2)}$ with ω_{can} a generator of ω at c' [32], §2. Since χ_f is trivial on Γ', $f|_{k-2}\sigma_c$ is a Laurent series in $q_{c'}$. Let v be a local coordinate at x; then

$q_c = v^{r_x}\cdot(\text{unit})$, so $\frac{dq_c}{q_c}$ is a unit multiple of $\frac{dv}{v}$, and $\operatorname{ord}_v(h) = r_x \operatorname{ord}_{q_c}(h)$ for every meromorphic function h of $q_c^{1/m}$, $m \geq 1$, near c. By the chain rule $(t'/t)|_2\sigma_c = \frac{d}{d\tau'}\log t(\sigma_c\tau') = \frac{2\pi i}{h_c}\, q_c \frac{d}{dq_c}\log t$, where h_c is the width of Γ at c; hence

$$\psi|_k\sigma_c = \frac{2\pi i}{h_c}\,(f|_{k-2}\sigma_c)\,R(t)\,t^a\,q_c\frac{d\log t}{dq_c}, \qquad \omega_\Phi = \frac{h_c}{2\pi i\, d}\,(\psi|_k\sigma_c)\,\omega_{\mathrm{can}}^{\otimes(k-2)}\otimes\frac{dq_c}{q_c},$$

with $t^a = s^j$. As $\omega_{\mathrm{can}}^{\otimes(k-2)}\otimes\frac{dv}{v}$ generates $\rho^*\omega^{\otimes(k-2)}\otimes\Omega^1_C(\log D_C)$ at x, the order of ω_Φ there is $r_x \operatorname{ord}_{q_c}(\psi|_k\sigma_c) = r_x \operatorname{ord}_c\Phi$.

Proof of (3). By (1), $r_x \operatorname{ord}_{\tau_0}\Phi$ is an integer, so $\operatorname{ord}_{\tau_0}\Phi \in \frac{1}{r_x}\mathbb{Z}$. Hence $\operatorname{ord}_x\omega_\Phi \geq 0$, that is $\operatorname{ord}_{\tau_0}\Phi \geq \frac{1}{r_x} - 1$, holds if and only if $\operatorname{ord}_{\tau_0}\Phi > -1$. By (2), $r_x \operatorname{ord}_c\Phi$ is an integer, so the logarithmic order at $x \in D_C$ is at least 1 if and only if $\operatorname{ord}_c\Phi > 0$; and a section of $\rho^*\omega^{\otimes(k-2)}\otimes\Omega^1_C(\log D_C)$ of logarithmic order at least 1 along D_C is a section of $\rho^*\omega^{\otimes(k-2)}\otimes\Omega^1_C$. Every $\tau_0 \in \mathbb{H}$ and every cusp of Γ lies under some point of C, as $C \to X' \to X_\Gamma$ is surjective, and $\omega_\Phi \neq 0$ if and only if $\Phi \neq 0$. This proves (3).

□

Lemma 3.12. *Assume that the datum is admissible. Let $H_{a,p}$ be the eigenlattice of the model and ω_Φ the twisted differential on C.*

1. *ω_Φ is a global section over C of $\rho^*\omega^{\otimes(k-2)}\otimes\Omega^1_{C/\mathcal{O}_{L,(\mathfrak{P})}} \subset \mathrm{Fil}^{k-2} M_k\otimes\Omega^1_C(\log D_C)$, and $g^*\omega_\Phi = \chi_\Phi(g)\,\omega_\Phi$ for every $g \in \tilde{G}$.*
2. *Let ∞_C be a point of C over the cusp ∞' of (M5), defined over a finite unramified extension W of $\mathbb{Z}_p$, and let $Q = s\,(t/q)^{-1/d}$ in the completed local ring of C_W at ∞_C, so that $Q^d = q$. The restriction of ω_Φ to the formal neighbourhood of ∞_C is $f_\Phi(Q)\,\omega_{\mathrm{can}}^{\otimes(k-2)}\otimes\frac{dQ}{Q}$, and $f_\Phi \in Q^j\,\mathbb{Z}_{(p)}[[Q^d]]$.*
3. *The class $[\omega_\Phi]$ lies in $H_{a,p}$ and spans $\mathrm{Fil}^{k-1}(H_{a,p}\otimes\mathbb{Q}_p) = \mathrm{Fil}^1(H_{a,p}\otimes\mathbb{Q}_p)$.*

Proof. Rationality and the eigen-property. By (M4) of Definition 3.1, $f\omega^{\otimes(k-2)}$ is a rational section of $\omega^{\otimes(k-2)}$ over X'_L, and $R(t)\,s^{j-1}ds$ is a rational differential on C_L; so ω_Φ is a rational section of the line bundle $\mathcal{L} = \rho^*\omega^{\otimes(k-2)}\otimes\Omega^1_{C/\mathcal{O}_{L,(\mathfrak{P})}}$ on the smooth $\mathcal{O}_{L,(\mathfrak{P})}$-scheme C (Lemma 3.7). Let $g = (h,\zeta) \in \tilde{G}$. Since $\rho\circ h = h\circ\rho$, (M4) gives $h^*\rho^*(f\omega^{\otimes(k-2)}) = \chi_G(h)\,\rho^*(f\omega^{\otimes(k-2)})$, and h fixes t and s; the automorphism $[\zeta]$ satisfies $\rho\circ[\zeta] = \rho$ and $[\zeta]^*(s^{j-1}ds) = \zeta^j s^{j-1}ds$. Hence $g^*\omega_\Phi = \chi_\Phi(g)\,\omega_\Phi$.

Holomorphy on the generic fibre. By (M4), the component X'_0 of $X'_{\mathbb{C}}$ is $\Gamma'\backslash\mathbb{H}^*$ with Γ' as in Lemma 3.11, E restricts to the universal curve, and $f\omega^{\otimes(k-2)}$ restricts to $f(\tau)(2\pi i\,dz)^{\otimes(k-2)}$; so on the part of $C_{\mathbb{C}}$ over X'_0, ω_Φ is the section of Lemma 3.11. As the datum satisfies (F), Lemma 3.11(3) shows that ω_Φ is a holomorphic section of $\mathcal{L}$ there. The group $\tilde{G}$ permutes the components of $C_{\mathbb{C}}$ transitively (G those of $X'_{\mathbb{C}}$, as $X'_L/G \cong \mathbb{P}^1_L$ is geometrically connected, and μ_d those over a given one) and acts on $\mathcal{L}$ by bundle automorphisms, and ω_Φ is an eigensection; so ω_Φ is holomorphic on all of $C_{\mathbb{C}}$, and the horizontal part of its divisor on C is effective.

Proof of (2). Over $\operatorname{Spf} W[[q']]$, (M5) gives $E = \mathrm{Tate}(q)$, t equal to the q-expansion of the Hauptmodul, and $f\omega^{\otimes(k-2)} = F(t)\,\omega_{\mathrm{can}}^{\otimes(k-2)} = f(q)\,\omega_{\mathrm{can}}^{\otimes(k-2)}$. By Definition 2.1, $t/q \in 1+q\,\mathbb{Z}_{(p)}[[q]]$, so $(t/q)^{-1/d}$ is given by the binomial series, which has coefficients in $\mathbb{Z}[\frac{1}{d}] \subset \mathbb{Z}_{(p)}$ as $p \nmid d$; thus $Q = s\,(t/q)^{-1/d}$ lies in the completed local ring of C_W at ∞_C and $Q^d = s^d\,q/t = q$. Using $s^{j-1}ds = \frac{1}{d}\,s^j\frac{dt}{t}$, $s^j = Q^j(t/q)^a$, $\frac{dt}{t} = \frac{q}{t}\frac{dt}{dq}\frac{dq}{q}$ and $\frac{dq}{q} = d\,\frac{dQ}{Q}$,

$$\omega_\Phi = f\,R(t)\,Q^j\,(t/q)^a\,\frac{q}{t}\frac{dt}{dq}\,\omega_{\mathrm{can}}^{\otimes(k-2)}\otimes\frac{dQ}{Q} = f_\Phi(Q)\,\omega_{\mathrm{can}}^{\otimes(k-2)}\otimes\frac{dQ}{Q},$$

the last equality being the definition of the coefficients $c(N)$ in Definition 2.1. The same definition gives $f_\Phi = Q^j \cdot$ (a series in $Q^d = q$) with coefficients in $\mathbb{Z}[\frac{1}{dM}] \subset \mathbb{Z}_{(p)}$, as $p \nmid dM$. This proves (2).

For (1) it remains to show that the vertical part of the divisor of ω_Φ is effective; it has one multiplicity for each connected component of $C \otimes W(\bar{\mathbb{F}}_p)$. On the component containing ∞_C the multiplicity is ≥ 0: by (2), on the formal neighbourhood of ∞_C the section is $f_\Phi(Q)\, \omega_{\mathrm{can}}^{\otimes(k-2)} \otimes \frac{dQ}{Q}$ with integral coefficients, where $\frac{dQ}{Q} = w' \frac{du}{u}$ for a formal parameter u, and f_Φ is divisible by Q^j, hence by u, so the section is integral there. The group $\tilde{G}$ acts on $C \otimes \mathbb{Z}_p$ and on $\mathcal{L}$ over $\mathbb{Z}_p$, permutes these components transitively (they correspond to the components of the generic fibre, C being smooth and proper), and ω_Φ is an eigensection with eigenvalues $\chi_\Phi(g) \in \mathbb{Z}_p^\times$; so all vertical multiplicities agree and are ≥ 0. Hence $\omega_\Phi \in H^0(C, \mathcal{L})$. Since $\Omega^1_C \subset \Omega^1_C(\log D_C)$ and $\rho^*\omega^{\otimes(k-2)} = \mathrm{Fil}^{k-2} M_k$, this proves (1).

Proof of (3). The section ω_Φ of $\mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$ defines a class in $\mathrm{Fil}^{k-1} H_\mathbb{Z}$, represented on any affine cover by the cocycle $(\omega_\Phi|_{U_i}, 0)$. It is a χ_Φ-eigenvector by (1), and its residue at every point of D_C vanishes because ω_Φ is a section of $\rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C$; by the proof of Lemma 3.10(3), Res_z is the constant term of the e_0-coefficient of the residue. So $[\omega_\Phi] \in H_{a,p}$. We show $[\omega_\Phi] \neq 0$ in $H_\mathbb{Z} \otimes \mathbb{Q}_p$. If it were zero, the cocycle $(\omega_\Phi|_{U_i}, 0)$ would be the coboundary of sections m_i with $m_j - m_i = 0$, that is, $\omega_\Phi = \nabla m$ for a global section m of M_k over $C_{\mathbb{Q}_p}$. As in the proof of Lemma 3.10(1): the image of m in $\mathrm{gr}^0 M_k = \rho^*\omega^{\otimes(2-k)}$ vanishes, $\rho^*\omega$ having positive degree on every component; and if $m \in \mathrm{Fil}^i$ with $1 \leq i \leq k-2$, the component of $\nabla m = \omega_\Phi$ in $\mathrm{gr}^{i-1} M_k \otimes \Omega^1_C(\log D_C)$ is $i\,\kappa(\bar{m}_i)$, while that of $\omega_\Phi \in \mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$ is zero since $i - 1 < k - 2$; as κ is injective, $\bar{m}_i = 0$. So $m \in \mathrm{Fil}^{k-1} M_k = 0$, and $\omega_\Phi = 0$, contradicting $\Phi \neq 0$ in (F). Finally, by Lemma 3.10(3) and admissibility, $\mathrm{Fil}^{k-1}(H_{a,p} \otimes \mathbb{Q}_p) = \mathrm{Fil}^1(H_{a,p} \otimes \mathbb{Q}_p)$ is a line, the filtration on $H_{a,p} \otimes \mathbb{Q}_p$ being induced from $H_\mathbb{Z} \otimes \mathbb{Q}_p$; it contains the nonzero class $[\omega_\Phi]$, which therefore spans it.

□

4 The crystalline congruence

Two integral facts allow us to read the characteristic equation of crystalline Frobenius in the Q-expansion. First, the class $p^{1-k}\varphi[\omega_\Phi]$ has an integral representative with cusp expansion $f_\Phi(Q^p)$ relative to dQ/Q (Lemma 4.3). Second, $p^{1-k} \det \varphi$ is a unit (Proposition 4.8). The latter permits normalisation of the coefficient of $f_\Phi(Q)$ to one.

Theorem 4.1. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum, with coefficients $c(N)$ and expansion f_Φ, let $\mathfrak{P}$ be a prime of L of residue degree one over a prime $p \notin \Sigma$, and let $H_{a,p}$ be the eigenlattice, with crystalline Frobenius φ, of a modular model of the datum at $\mathfrak{P}$. Then the following hold.*

1. *$v_p\big(\det(\varphi \mid H_{a,p})\big) = k - 1$; in particular φ is invertible on $H_{a,p} \otimes \mathbb{Q}_p$.*
2. *Let $X^2 - A^\vee X + B^\vee$ be the characteristic polynomial of $p^{k-1}\varphi^{-1}$ on $H_{a,p} \otimes \mathbb{Q}_p$, that is,*

$$A^\vee = \frac{p^{k-1} \operatorname{Tr}(\varphi \mid H_{a,p})}{\det(\varphi \mid H_{a,p})}, \qquad B^\vee = \frac{p^{2(k-1)}}{\det(\varphi \mid H_{a,p})}.$$

   Then $A^\vee \in \mathbb{Z}_p$ and $B^\vee \in p^{k-1}\mathbb{Z}_p^\times$.

3. *Let V be the ring endomorphism of $\mathbb{Z}_p[[Q]]$ with $V(Q) = Q^p$, and $\theta = Q\, d/dQ$. Then*

$$(1 - A^\vee V + B^\vee V^2)\, f_\Phi \in \theta^{k-1}\, \mathbb{Z}_p[[Q]];$$

equivalently, $c(mp^r) - A^\vee c(mp^{r-1}) + B^\vee c(mp^{r-2}) \equiv 0 \pmod{p^{(k-1)r}}$ *for all* $m \geq 1$ *with* $p \nmid m$ *and all* $r \geq 1$.

Proof. We apply Scholl's argument [47, 5.7(a)] in the expansion form of [32, §4]. Write $w = k-1$ and $x = [\omega_\Phi]$. By Lemma 3.10(3), $H_{a,p}$ is free of rank two. Proposition 4.8 gives (1), so we may write

$$T = \mathrm{Tr}(\varphi \mid H_{a,p}), \qquad \det(\varphi \mid H_{a,p}) = p^w \delta, \qquad \delta \in \mathbb{Z}_p^\times.$$

In dimension two, the trace of the inverse is the trace divided by the determinant. Thus $A^\vee = T/\delta$ and $B^\vee = p^w/\delta$, proving (2).

By Lemmas 3.12(3) and 4.3(2), both x and $y = p^{-w}\varphi x$ lie in $H_{a,p}$. Substituting $\varphi x = p^w y$ in the characteristic equation $(\varphi^2 - T\varphi + p^w\delta)x = 0$ and using torsion-freeness gives

$$\delta x - Ty + \varphi y = 0 \qquad \text{in } H_{a,p}.$$

To expand this identity at ∞_C, take W, the parameter u, the ring $\mathcal{A}$, and the expansion map $\mathcal{E}$ of Lemma 4.3(2). There $Q = \beta u^{w'}$ for $\beta \in W^\times$ and $p \nmid w'$. The σ-semilinear Frobenius lift of $\mathcal{A}$ in Lemma 4.3(2) sends Q to Q^p; as σ fixes $\mathbb{Z}_p$, it restricts on $\mathbb{Z}_p[[Q]] \subset W[[u]]$ to the endomorphism V of the statement. Put $\theta_u = u\, d/du$. In $\mathcal{A}/\theta_u^w\mathcal{A}$ the three classes have expansions

$$\mathcal{E}(x) = w' f_\Phi(Q), \qquad \mathcal{E}(y) = w' f_\Phi(Q^p), \qquad \mathcal{E}(\varphi y) = w' p^w f_\Phi(Q^{p^2}).$$

The first two identities are Lemma 4.3(2), and the third follows by applying Lemma 3.9(2) to y. The common factor w' comes from $dQ/Q = w'\, du/u$. In particular

$$w' S(Q) \in \theta_u^w \mathcal{A}, \qquad S(Q) = (\delta - TV + p^w V^2) f_\Phi(Q) \in Q\mathbb{Z}_p[[Q]],$$

where integrality and vanishing of the constant term follow from Lemma 3.12(2).

Write $S(Q) = \sum_{N \geq 1} S_N Q^N$. The coefficient of $u^{w'N}$ in $w'S(Q)$ is $w'\beta^N S_N$, whereas the coefficient of u^n in any element of $\theta_u^w\mathcal{A}$ belongs to $n^w W$. As w' and β are units and W is unramified over $\mathbb{Z}_p$, we obtain $S_N \in N^w\mathbb{Z}_p$. Consequently

$$G(Q) = \sum_{N \geq 1} \frac{S_N}{\delta N^w} Q^N \in \mathbb{Z}_p[[Q]], \qquad (1 - A^\vee V + B^\vee V^2) f_\Phi = \theta^w G.$$

For $N = mp^r$ with $p \nmid m$ and $r \geq 1$, comparison of the coefficients of Q^N gives

$$c(mp^r) - A^\vee c(mp^{r-1}) + B^\vee c(mp^{r-2}) = (mp^r)^w [Q^{mp^r}]G \in p^{wr}\mathbb{Z}_p.$$

Conversely, the congruences imply that the N-th coefficient of the series on the left of the power-series identity is divisible in $\mathbb{Z}_p$ by N^w for each N divisible by p. For $p \nmid N$ this divisibility is automatic. Since the constant term vanishes, dividing the N-th coefficient by N^w for each $N \geq 1$ gives an integral primitive of order w. This proves the equivalence in (3). □

4.1 Frobenius divisibility

The quotient of cusp expansions by θ^{k-1} can have p-torsion, so the identity $\mathcal{E}\varphi = p^{k-1}V\mathcal{E}$ alone does not determine the expansion of $p^{1-k}\varphi[\omega_\Phi]$. We divide the Frobenius cocycle on the cover before applying $\mathcal{E}$. This requires divisibility by p^{k-1} for the local Frobenius maps and for the homotopies comparing two lifts, including at the interior ramification points.

Lemma 4.2. *Let $W = W(\kappa)$ for a finite field κ of characteristic p, and let $U = \operatorname{Spf} B$ be a* chart *of $C_W = C \otimes W$: either an affine open of the p-adic completion $\widehat{C}_W$ with $v \in B$ such that $U \cap D_C = \emptyset$ and $\Omega^1_{U/W}$ is free on $\eta_0 = dv$, or $U \cap D_C$ is the divisor of v and $\Omega^1_{U/W}(\log D_C)$ is free on $\eta_0 = dv/v$; or the formal disc $\operatorname{Spf} W[[v]]$ at a W-point of D_C, with $\eta_0 = dv/v$. Let ∂ be the derivation dual to η_0, and let σ, σ' be Frobenius lifts on U, semilinear over the Frobenius of W and logarithmic along D_C, that is, $\sigma(v), \sigma'(v) \in v^p B^\times$ when $\eta_0 = dv/v$. For $m \in M_k(U)$ write $F_\sigma(m \otimes \eta_0) = F_\sigma(\sigma^* m) \otimes \sigma^* \eta_0$, and let $h_{\sigma,\sigma'}$ be the homotopy between the Frobenius maps of the two lifts [48, 2.1], that is,*

$$h_{\sigma,\sigma'}(m) = \sum_{n \geq 1} \frac{\delta^n}{n!} F_\sigma(\sigma^* \nabla_\partial^{n-1} m), \qquad \delta = \sigma'(v) - \sigma(v) \quad \text{or} \quad \delta = \log \frac{\sigma'(v)}{\sigma(v)}$$

according as $\eta_0 = dv$ or $\eta_0 = dv/v$. Then:

1. *$F_\sigma(\sigma^* \operatorname{Fil}^i M_k) \subset p^i M_k$ for $0 \leq i \leq k-2$, and $\sigma^* \eta_0 \in pB\,\eta_0$. Hence $F_\sigma(\operatorname{Fil}^{k-2} M_k \otimes \eta_0) \subset p^{k-1} M_k \otimes \eta_0$.*
2. *$h_{\sigma,\sigma'}(m) \in p^{k-1} M_k(U)$ for every $m \in \operatorname{Fil}^{k-2} M_k(U)$.*

Proof. The inclusions are local, and we prove them on the charts of the Kummer cover; (2) is where $p \geq k$ is used.

As $p \notin \Sigma$, p is odd and $p \geq k$. By Lemma 3.7(2), C_W is smooth over W and D_C is étale over W. The formal scheme U is the disjoint union of the open formal subschemes over the connected components of $U_0 = U \otimes_W \kappa$, so we may assume U_0 connected. It is then a smooth connected curve over κ, or $\operatorname{Spec} \kappa[[v]]$, hence integral, and v is a non-zero-divisor in B/pB when $\eta_0 = dv/v$. In that case write $\sigma(v) = v^p u$ with $u \in B^\times$; as $\sigma(v) \equiv v^p \pmod p$ and v is a non-zero-divisor modulo p, $u \in 1 + pB$.

Frobenius on Fil^1. We show that $F_\sigma(\sigma^* \operatorname{Fil}^1 \rho^* H) \subset p\,\rho^* H$. Let $U^\circ = \operatorname{Spf} B^\circ$ be $U \setminus D_C$: $B^\circ = B$ if $U \cap D_C = \emptyset$, and otherwise B° is the p-adic completion of $B[1/v]$. Over U° the curve $E_{U^\circ} = E \times_{X'} U^\circ$ is an elliptic curve, $\rho^* H(U^\circ) = H^1_{\mathrm{dR}}(E_{U^\circ}/U^\circ)$, and $\operatorname{Fil}^1 \rho^* H(U^\circ)$ is the image of $H^0(E_{U^\circ}, \Omega^1_{E_{U^\circ}/U^\circ})$; both are locally free and commute with base change. By construction of the crystalline Frobenius, F_σ is the composite of the base-change isomorphism $\sigma^* H^1_{\mathrm{cris}}(E_0/U^\circ) \cong H^1_{\mathrm{cris}}(E_0^{(p)}/U^\circ)$, where $E_0^{(p)} = E_0 \times_{U_0^\circ, \mathrm{Frob}} U_0^\circ$, with the map induced by the relative Frobenius $F_{E_0/U_0^\circ} : E_0 \to E_0^{(p)}$ [5, §7]. By the comparison of crystalline with de Rham cohomology and its compatibility with base change [5, §7], the reduction of F_σ modulo p is the pull-back $F^*_{E_0/U_0^\circ} : H^1_{\mathrm{dR}}(E_0^{(p)}/U_0^\circ) \to H^1_{\mathrm{dR}}(E_0/U_0^\circ)$. This pull-back is induced by a map of relative de Rham complexes that is zero on 1-forms, since $F^*_{E_0/U_0^\circ}\, d(x \otimes 1) = d(x^p) = 0$. So it kills the image of $H^0(\Omega^1_{E_0^{(p)}/U_0^\circ})$, which is Fil^1 modulo p, and $F_\sigma(\sigma^* \operatorname{Fil}^1 \rho^* H(U^\circ)) \subset p\,\rho^* H(U^\circ)$. If $U \cap D_C \neq \emptyset$, let $e \in \operatorname{Fil}^1 \rho^* H(U)$ and $y = F_\sigma(\sigma^* e)$. Then $y \in p\,\rho^* H(U^\circ)$. The map $B/pB \to B^\circ/pB^\circ = (B/pB)[1/v]$ is injective, so $B \cap pB^\circ = pB$, and since $\rho^* H(U)$ is locally free over B, $y \in p\,\rho^* H(U)$.

Proof of (1). The Frobenius of $M_k = \operatorname{Sym}^{k-2} \rho^* H$ is the symmetric power of that of $\rho^* H$, and $\operatorname{Fil}^i M_k$ is spanned by products of i sections of $\operatorname{Fil}^1 \rho^* H$ and $k-2-i$ sections of $\rho^* H$. So $F_\sigma(\sigma^* \operatorname{Fil}^i M_k) \subset p^i M_k$. If $\eta_0 = dv$, write $\sigma(v) = v^p + pb$; then $\sigma^* dv = p v^{p-1} dv + p\,db \in pB\,dv$. If $\eta_0 = dv/v$, then $\sigma^* \frac{dv}{v} = \frac{d(v^p u)}{v^p u} = p\frac{dv}{v} + \frac{du}{u}$, and $\frac{du}{u} \in pB\,\eta_0$ since $u \in 1 + pB$. Hence $F_\sigma(\operatorname{Fil}^{k-2} M_k \otimes \eta_0) \subset p^{k-2} \cdot p\, M_k \otimes \eta_0$.

Proof of (2). Let $m \in \operatorname{Fil}^{k-2} M_k(U)$. Both lifts reduce to the p-th power map, so $\delta \in pB$ if $\eta_0 = dv$; if $\eta_0 = dv/v$, then $\delta \in pB$ by [48, (1.4.3)]. Griffiths transversality, $\nabla_\partial \operatorname{Fil}^i \subset \operatorname{Fil}^{i-1}$, gives

$\nabla_\partial^{n-1} m \in \mathrm{Fil}^{\max(k-1-n,0)} M_k$. By (1) and $\delta \in pB$, the n-th term of $h_{\sigma,\sigma'}(m)$ lies in $p^{e(n)} M_k(U)$ with

$$e(n) = n - v_p(n!) + \max(k-1-n, 0).$$

For $1 \le n \le k-1$ we have $n < p$, so $v_p(n!) = 0$ and $e(n) = k-1$. For $n \ge k$, Legendre's formula gives $v_p(n!) \le \frac{n-1}{p-1}$, so $e(n) \ge \frac{n(p-2)+1}{p-1}$, and $n(p-2)+1-(k-1)(p-1) \ge k(p-2)+1-(k-1)(p-1) = p-k \ge 0$. Hence $e(n) \ge k-1$ for all $n \ge 1$, and $h_{\sigma,\sigma'}(m) \in p^{k-1} M_k(U)$.

□

Lemma 4.3. *Assume that the datum is admissible.*

1. *For every global section ω of $\mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$ over $C_{\mathbb{Z}_p}$, $\varphi([\omega]) \in p^{k-1} H_{\mathbb{Z}}$.*
2. *Let ∞_C, W and Q be as in Lemma 3.12(2), and let u be a formal parameter of C_W along ∞_C such that the pull-back of E to $\mathrm{Spf}\, W[[u]]$ is $\mathrm{Tate}(\delta u^m)$ with $\delta \in W^\times$ and $p \nmid m$. Let $\mathcal{A}$, θ, γ and V be as in Lemma 3.8 for (W, m, δ), and let $\mathcal{E} : H_{\mathbb{Z}} \otimes W \to \mathcal{A}/\theta^{k-1}\mathcal{A}$ be the expansion map of Lemma 3.9 for $(x_\infty, u) = (\infty_C, u)$. Then $m = dw'$ for an integer w' prime to p, $Q = \beta u^{w'}$ for some $\beta \in W^\times$, and $V(Q) = Q^p$. For every global section ω as in (1), write $\omega|_{W[[u]]} = g_\omega\, \omega_{\mathrm{can}}^{\otimes(k-2)} \otimes \frac{du}{u}$ with $g_\omega \in W[[u]]$; then, in $\mathcal{A}/\theta^{k-1}\mathcal{A}$,*

$$\mathcal{E}([\omega]) = g_\omega, \qquad \mathcal{E}\big(p^{1-k}\varphi([\omega])\big) = V(g_\omega).$$

In particular $\varphi([\omega_\Phi]) \in p^{k-1} H_{a,p}$, $g_{\omega_\Phi} = w' f_\Phi(Q)$, and $\mathcal{E}\big(p^{1-k}\varphi([\omega_\Phi])\big) = w' f_\Phi(Q^p)$.

Proof. For covers étale over $Y(N)$ and cusp forms, item (1) is [47, Proposition 3.4], p. 64, and item (2) is the reading of Frobenius at a cusp of [47, Theorem 3.6], in the form of [32, §4]. The proof of Proposition 3.4 uses only the local inclusions of Lemma 4.2, so it applies here; we recall it over a finite unramified extension W of $\mathbb{Z}_p$, possibly $\mathbb{Z}_p$ itself, as (2) needs the cocycle. Item (2) is proved at a cusp of width m, with the division by p^{k-1} made on cochains.

Proof of (1). Cover the p-adic completion of C_W by finitely many charts U_i in the sense of Lemma 4.2, with Frobenius lifts σ_i, logarithmic along D_C, such that each $U_{ij} = U_i \cap U_j$ is a chart for the coordinate of U_i. The Frobenius of $H_{\mathbb{Z}} \otimes W$ is $\varphi \otimes \sigma$. A class of $H_{\mathbb{Z}} \otimes W$ is represented by a cocycle $((\omega_i), (m_{ij}))$ with ω_i a section of $M_k \otimes \Omega^1_C(\log D_C)$ on U_i, $m_{ij} \in M_k(U_{ij})$, $\omega_j - \omega_i = \nabla m_{ij}$ and (m_{ij}) a Čech cocycle, and the image of this class under $\varphi \otimes \sigma$ is represented by

$$\Big((F_{\sigma_i}(\omega_i)),\ (F_{\sigma_i}(\sigma_i^* m_{ij}) + h_{\sigma_i,\sigma_j}(\omega_j))\Big)$$

[48, 2.3], with [5, §7] and [29, Thm. 6.4] for the comparison with the crystalline Frobenius, where $h_{\sigma_i,\sigma_j}(m \otimes \eta_0) = h_{\sigma_i,\sigma_j}(m)$ is the homotopy of Lemma 4.2.

A global section ω of $\mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$ over C_W has the cocycle $((\omega|_{U_i}), (0))$, so $(\varphi \otimes \sigma)([\omega])$ is represented by $\big((F_{\sigma_i}(\omega)), (h_{\sigma_i,\sigma_j}(\omega))\big)$. Only the next step differs from the source: on each chart $\Omega^1_C(\log D_C)$ is free on the form η_0 of the chart, so by Lemma 4.2(1) on U_i and Lemma 4.2(2) on U_{ij}, each component is p^{k-1} times an integral cochain. The cochain groups are p-torsion-free, so

$$c = \Big((p^{1-k} F_{\sigma_i}(\omega)),\ (p^{1-k} h_{\sigma_i,\sigma_j}(\omega))\Big)$$

is an integral cocycle with $p^{k-1}[c] = (\varphi \otimes \sigma)([\omega])$. Hence $(\varphi \otimes \sigma)([\omega]) \in p^{k-1}(H_{\mathbb{Z}} \otimes W)$; for $W = \mathbb{Z}_p$ this is (1). Since $H_{\mathbb{Z}}$ is torsion-free (Lemma 3.10(1)), $[c] = p^{1-k}(\varphi \otimes \sigma)([\omega])$.

Proof of (2): Q in terms of u. Such a parameter u exists (proof of Lemma 3.10(2)). By Lemma 3.12(2), Q lies in the completed local ring $W[[u]]$ of C_W at ∞_C, and $Q^d = q = \delta u^m$. Write

$Q = u^n\beta(u)$ with $n \geq 0$ and $\beta(u) \in W[[u]]$ not divisible by u. Since $W[[u]]$ is factorial and u is prime, $u^{nd}\beta(u)^d = \delta u^m$ gives $m = nd$ and $\beta(u)^d = \delta$. So $\beta(u)$ is a unit with constant d-th power, and as $p \nmid d$ it is constant, $\beta(u) = \beta \in W^\times$. Put $w' = n$; it is prime to p because $p \nmid m$. Next, $V(Q) = \sigma(\beta)\gamma^{w'}u^{pw'}$, and $\big(\sigma(\beta)\gamma^{w'}\beta^{-p}\big)^d = \sigma(\delta)\gamma^m\delta^{-p} = 1$ by the definition of γ in Lemma 3.8. As $\sigma(\beta) \equiv \beta^p$ and $\gamma \equiv 1 \pmod p$, the element $\sigma(\beta)\gamma^{w'}\beta^{-p}$ lies in $(1+pW) \cap \mu_d(W) = \{1\}$, p being odd. Hence $V(Q) = \beta^p u^{pw'} = Q^p$.

Proof of (2): the expansions. Choose the cover above, over W, with $\infty_C \in U_{i_0}$, and let $e_0 = \omega_{\mathrm{can}}^{\otimes(k-2)}$, so that $\omega|_{W[[u]]} = g_\omega\, e_0 \otimes \frac{du}{u}$. The class $[\omega]$ is represented by $((\omega|_{U_i}), (0))$, so by the definitions of $\mathcal{E}$ in Lemma 3.9(1) and of ψ in Lemma 3.8(1), $\mathcal{E}([\omega]) = \psi[g_\omega e_0] = g_\omega$. By the proof of (1), $p^{1-k}(\varphi \otimes \sigma)([\omega])$ is represented by a cocycle whose component on U_{i_0} is $p^{1-k}F_{\sigma_{i_0}}(\omega)$. Let σ_{can} be the lift $u \mapsto \gamma u^p$ of $W[[u]]$. By Lemma 3.9(3) with $\sigma_U = \sigma_{i_0}$, that is, by [48, Proposition 2.2(ii)] on the disc $\operatorname{Spf} W[[u]]$ for the logarithmic lifts σ_{can} and σ_{i_0} ,

$$F_{\sigma_{i_0}}(\omega)\big|_{W[[u]]} = F_{\sigma_{\mathrm{can}}}(g_\omega e_0 \otimes \tfrac{du}{u}) + \nabla h(g_\omega e_0), \qquad h = h_{\sigma_{\mathrm{can}},\sigma_{i_0}}.$$

Lemma 4.2(2) applies to $g_\omega e_0 \in \mathrm{Fil}^{k-2} M_k \otimes W[[u]]$ and gives $h(g_\omega e_0) \in p^{k-1}M_k \otimes W[[u]]$. Hence $p^{1-k}F_{\sigma_{i_0}}(\omega)|_{W[[u]]}$ and $p^{1-k}F_{\sigma_{\mathrm{can}}}(g_\omega e_0 \otimes \frac{du}{u})$ differ by ∇ of an integral section, and they have the same class in H^1_{loc}. The Frobenius $F_{\sigma_{\mathrm{can}}}$ is the map F of Lemma 3.8(2) (proof of Lemma 3.9(2)), so by the display in Lemma 3.8(2),

$$p^{1-k}F_{\sigma_{\mathrm{can}}}(g_\omega e_0 \otimes \tfrac{du}{u}) = V(g_\omega)\, e_0 \otimes \tfrac{du}{u}$$

as forms. Applying ψ gives $\mathcal{E}(p^{1-k}\varphi([\omega])) = V(g_\omega)$. The division by p^{k-1} was made on cochains; this is needed because $\mathcal{A}/\theta^{k-1}\mathcal{A}$ has p-torsion, so the identity does not follow from $\mathcal{E} \circ \varphi = p^{k-1}V \circ \mathcal{E}$.

The twisted differential. By Lemma 3.12(1),(3), ω_Φ is a global section of $\mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$ with $[\omega_\Phi] \in H_{a,p}$. By (1), $\varphi([\omega_\Phi]) \in p^{k-1}H_{\mathbb{Z}}$, and $\varphi([\omega_\Phi]) \in H_{a,p}$, which is φ-stable and saturated in $H_{\mathbb{Z}}$ (Lemma 3.10(2)); so $\varphi([\omega_\Phi]) \in p^{k-1}H_{a,p}$. By Lemma 3.12(2), $\omega_\Phi|_{W[[u]]} = f_\Phi(Q)\, e_0 \otimes \frac{dQ}{Q}$, and $\frac{dQ}{Q} = w'\frac{du}{u}$ because β is constant; so $g_{\omega_\Phi} = w' f_\Phi(Q)$. The coefficients of f_Φ lie in $\mathbb{Z}_{(p)}$ and are fixed by σ, and V is a continuous σ-semilinear ring endomorphism with $V(Q) = Q^p$; hence $V(g_{\omega_\Phi}) = w' f_\Phi(Q^p)$.

□

Lemma 4.4. *Assume that the datum is admissible. Write* $\mathcal{L}_{\log} = \mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$.

1. *If* $\omega \in H^0(C_{\mathbb{Z}_p}, \mathcal{L}_{\log})$ *is not divisible by* p *in* $H^0(C_{\mathbb{Z}_p}, \mathcal{L}_{\log})$, *then* $[\omega] \notin pH_{\mathbb{Z}}$.
2. *There are* $c \geq 0$ *and* $x_1 \in H_{a,p} \backslash pH_{a,p}$ *with* $p^c x_1 = [\omega_\Phi]$ *and* $\varphi(x_1) \in p^{k-1}H_{a,p}$. *Consequently* $\det(\varphi \mid H_{a,p}) \in p^{k-1}\mathbb{Z}_p$.

Proof. Proof of (1). We reduce to a statement about global sections modulo p. Let K be the complex $M_k \xrightarrow{\nabla} M_k \otimes \Omega^1_C(\log D_C)$ on $C_{\mathbb{Z}_p}$, so that $H_{\mathbb{Z}} = \mathbb{H}^1(K)$. Its terms are locally free, so $0 \to K \xrightarrow{p} K \to K \otimes \mathbb{F}_p \to 0$ is exact, and the long exact sequence shows that $H_{\mathbb{Z}}/pH_{\mathbb{Z}} \to \mathbb{H}^1(C_{\mathbb{F}_p}, K \otimes \mathbb{F}_p)$ is injective. The image of $[\omega]$ is the class of the Čech–de Rham cocycle $((\bar\omega|_{U_i}), (0))$, where $\bar\omega$ is the reduction of ω. Suppose $[\omega] \in pH_{\mathbb{Z}}$. Then this cocycle is the coboundary of some (m_i), that is, $\nabla m_i = \bar\omega$ and $m_j - m_i = 0$; so the m_i glue to a global section m of $M_k \otimes \mathbb{F}_p$ with $\nabla m = \bar\omega$. We show that this forces $\bar\omega = 0$.

If $k = 2$, then $M_k = \mathcal{O}$, $\nabla = d$, and m is locally constant on the proper reduced curve $C_{\mathbb{F}_p}$, so $\bar\omega = dm = 0$. Let $k \geq 3$. We collect three facts on $C_{\mathbb{F}_p}$.

- $\operatorname{gr}^j M_k \cong \rho^*\omega^{\otimes(2j+2-k)}$ for $0 \le j \le k-2$, since $H/\operatorname{Fil}^1 H \cong \omega^{-1}$.
- ∇ induces $\mathcal{O}$-linear maps $\operatorname{gr}^j M_k \to \operatorname{gr}^{j-1} M_k \otimes \Omega^1_C(\log D_C)$ for $1 \le j \le k-2$, equal to $j\kappa$, where $\kappa : \rho^*\omega^{\otimes 2} \to \Omega^1_C(\log D_C)$ is ρ^* of the Kodaira–Spencer map followed by $\rho^*\Omega^1_{X'}(\log) \to \Omega^1_C(\log D_C)$. The Kodaira–Spencer map is an isomorphism $\omega^{\otimes 2} \cong \Omega^1_{X'}(\log Z')$ over $\mathcal{O}_{L,(\mathfrak{P})}$ by (M1) of Definition 3.1 (for modular curves see [18], [30]). By Lemma 3.7(2), ρ is étale outside $D_C \cup P_C$, which is finite and étale over $\mathcal{O}_{L,(\mathfrak{P})}$, so $\rho_{\mathbb{F}_p}$ is étale outside finitely many points, and the second map is nonzero on each component of $C_{\mathbb{F}_p}$. So κ is a nonzero map of line bundles on each component of the smooth curve $C_{\mathbb{F}_p}$, hence injective, and so is $j\kappa$, because $j \le k-2 < p$.
- $H^0(C_{\mathbb{F}_p}, \operatorname{gr}^0 M_k) = H^0(C_{\mathbb{F}_p}, \rho^*\omega^{\otimes(2-k)}) = 0$. Indeed $k \ge 3$, and $\rho^*\omega$ has positive degree on every component of $C_{\mathbb{F}_p}$, as shown in the proof of Lemma 3.10(1): by (M1), $\omega^{\otimes 2} \cong \Omega^1_{X'}(\log Z')$, whose degree on each component of $X'_{\mathbb{F}_p}$ equals its degree $[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma']/6 > 0$ on the corresponding component of $X'_{\mathbb{C}}$ (M4); and ρ is finite and surjective on components.

Now suppose $m \ne 0$, and let $j \le k-2$ be maximal with $m \in H^0(\operatorname{Fil}^j M_k \otimes \mathbb{F}_p)$, with nonzero image $\bar m \in H^0(\operatorname{gr}^j)$. If $j = 0$ this contradicts the third fact. If $j \ge 1$, then $\nabla m = \bar\omega \in \operatorname{Fil}^{k-2} \otimes \Omega^1 \subset \operatorname{Fil}^j \otimes \Omega^1$, so the image of ∇m in $\operatorname{gr}^{j-1} \otimes \Omega^1$, which is $j\kappa(\bar m)$, vanishes; by the second fact $\bar m = 0$, a contradiction. Hence $m = 0$ and $\bar\omega = \nabla m = 0$. Since $\mathcal{L}_{\log}$ is flat over $\mathbb{Z}_p$, $\bar\omega = 0$ means $\omega \in pH^0(C_{\mathbb{Z}_p}, \mathcal{L}_{\log})$, contrary to the hypothesis. Hence $[\omega] \notin pH_{\mathbb{Z}}$.

Proof of (2). The module $H^0(C_{\mathbb{Z}_p}, \mathcal{L}_{\log})$ is finitely generated and torsion-free, as C is proper, and ω_Φ is a nonzero element of it by Lemma 3.12(1),(3). So $\omega_\Phi = p^c\omega_1$ with $c \ge 0$ and ω_1 not divisible by p. Put $x_1 := [\omega_1]$. Then $p^c x_1 = [\omega_\Phi] \in H_{a,p}$ (Lemma 3.12(3)), and $x_1 \in H_{a,p}$ because $H_{a,p}$ is saturated in $H_{\mathbb{Z}}$ (Lemma 3.10(2)). By (1), $x_1 \notin pH_{\mathbb{Z}}$, so $x_1 \notin pH_{a,p}$. By Lemma 4.3(1), $\varphi(x_1) \in p^{k-1}H_{\mathbb{Z}}$, and $\varphi(x_1) \in H_{a,p}$ by Lemma 3.10(2). Since $H_{a,p}$ is saturated in $H_{\mathbb{Z}}$, $p^{k-1}H_{\mathbb{Z}} \cap H_{a,p} = p^{k-1}H_{a,p}$. As $H_{a,p}$ is free of rank 2 (Lemma 3.10(3)) and x_1 is not divisible by p, there is y with $\{x_1, y\}$ a basis of $H_{a,p}$. The first column of the matrix of φ in this basis is divisible by p^{k-1}, so $\det(\varphi \mid H_{a,p}) \in p^{k-1}\mathbb{Z}_p$.

□

4.2 The determinant

To determine the valuation of the determinant, we construct a primitive generator x_1 of the top Hodge line and extend it to a basis x_1, y of $H_{a,p}$. The other extreme Hodge type identifies the kernel of the relevant mod-p projection with the line spanned by $\bar x_1$. This allows us to show that $p^{1-k}\varphi(x_1)$ and $\varphi(y)$ are independent modulo p.

Lemma 4.5. *Assume that the datum is admissible. Put $\chi = \chi_\Phi$. Let $\operatorname{gr}^0 M_k = M_k/\operatorname{Fil}^1 M_k$, with the action of $\tilde G$ induced from M_k, and let $\mathcal{L}^* = (\operatorname{gr}^0 M_k)^\vee \otimes \Omega^1_C$, with the dual action on $(\operatorname{gr}^0 M_k)^\vee$ and the natural action on Ω^1_C. For a character ψ of $\tilde G$ and a $\tilde G$-module V write $V^\psi = \{v \in V :\ g^*v = \psi(g)\, v \text{ for all } g \in \tilde G\}$. Then*

$$H^1(C_{\mathbb{Q}_p}, \operatorname{gr}^0 M_k)^\chi \ne 0 \qquad \text{and} \qquad H^0(C_{\mathbb{Q}_p}, \mathcal{L}^*)^{\chi^{-1}} \ne 0.$$

Proof. The first assertion, from a class of $H_{a,p}$ outside Fil^1. Let $K = [M_k \xrightarrow{\nabla} M_k \otimes \Omega^1_C(\log D_C)]$ on $C_{\mathbb{Q}_p}$, and $\operatorname{Fil}^1 K = [\operatorname{Fil}^1 M_k \to M_k \otimes \Omega^1_C(\log D_C)]$, so that $K/\operatorname{Fil}^1 K = \operatorname{gr}^0 M_k[0]$. By flat base change, $H_{\mathbb{Z}} \otimes \mathbb{Q}_p = \mathbb{H}^1(C_{\mathbb{Q}_p}, K)$, and by Lemma 3.10(1) its Hodge filtration has $\operatorname{Fil}^1 = \operatorname{im}\big(\mathbb{H}^1(C_{\mathbb{Q}_p}, \operatorname{Fil}^1 K) \to \mathbb{H}^1(C_{\mathbb{Q}_p}, K)\big)$. The long exact sequence of $0 \to \operatorname{Fil}^1 K \to K \to \operatorname{gr}^0 M_k[0] \to 0$ contains the exact segment

$$\mathbb{H}^1(C_{\mathbb{Q}_p}, \operatorname{Fil}^1 K) \longrightarrow \mathbb{H}^1(C_{\mathbb{Q}_p}, K) \xrightarrow{r} H^1(C_{\mathbb{Q}_p}, \operatorname{gr}^0 M_k),$$

so $\ker r = \mathrm{Fil}^1(H_{\mathbb{Z}} \otimes \mathbb{Q}_p)$. Since $\tilde{G}$ acts on $(M_k, \nabla, \mathrm{Fil}^\bullet)$ (Lemma 3.10(1)), the map r is $\tilde{G}$-equivariant. As the datum is admissible, Lemma 3.10(3) shows that $H_{a,p} \otimes \mathbb{Q}_p$ has dimension 2 and that $\mathrm{Fil}^1(H_{a,p} \otimes \mathbb{Q}_p)$, the intersection of $H_{a,p} \otimes \mathbb{Q}_p$ with $\mathrm{Fil}^1(H_{\mathbb{Z}} \otimes \mathbb{Q}_p)$, is a line; this is where (R) and (P) enter. Choose $x \in H_{a,p} \otimes \mathbb{Q}_p$ outside this line. Then $r(x) \neq 0$, and $g^* r(x) = r(g^* x) = \chi(g)\, r(x)$ for all $g \in \tilde{G}$, by the definition of $H_{a,p}$ in Lemma 3.10(2). Hence $0 \neq r(x) \in H^1(C_{\mathbb{Q}_p}, \mathrm{gr}^0 M_k)^{\chi}$.

The second assertion, by Serre duality. Cohomology and eigenspaces commute with the flat base change $\mathbb{Q}_p \to \bar{\mathbb{Q}}_p$, so it suffices to show that $H^0(C_{\bar{\mathbb{Q}}_p}, \mathcal{L}^*)^{\chi^{-1}} \neq 0$; the characters $\chi^{\pm 1}$ take values in $\mathbb{Z}_p^\times$. The curve $C_{\bar{\mathbb{Q}}_p}$ is a finite disjoint union of smooth projective connected curves. For the line bundle $\mathcal{F} = \mathrm{gr}^0 M_k$, Serre duality [28, III, §7] gives a perfect pairing

$$\langle\ ,\ \rangle :\ H^1(C_{\bar{\mathbb{Q}}_p}, \mathcal{F}) \times H^0(C_{\bar{\mathbb{Q}}_p}, \mathcal{F}^\vee \otimes \Omega^1_C) \longrightarrow H^1(C_{\bar{\mathbb{Q}}_p}, \Omega^1_C) \xrightarrow{\mathrm{tr}} \bar{\mathbb{Q}}_p,$$

the cup product followed by the trace map, which on each component is the sum of residues [52, Ch. II]. Residues of differentials are preserved by isomorphisms of curves, so $\mathrm{tr} \circ g^* = \mathrm{tr}$ for every automorphism g of $C_{\bar{\mathbb{Q}}_p}$; and as the action of $\tilde{G}$ on $\mathcal{L}^* = \mathcal{F}^\vee \otimes \Omega^1_C$ is dual to that on $\mathcal{F}$, the cup product is $\tilde{G}$-equivariant. Hence $\langle g^*\xi, g^*\eta\rangle = \langle \xi, \eta\rangle$ for all $g \in \tilde{G}$.

It remains to pass to eigenspaces. For a character ψ of $\tilde{G}$ with values in $\mathbb{Z}_p^\times$ put $e_\psi = |\tilde{G}|^{-1} \sum_{g \in \tilde{G}} \psi(g)^{-1} g^*$, the projector onto the ψ-eigenspace. By the invariance of the pairing, $\langle e_\chi \xi, \eta\rangle = \langle \xi, e_{\chi^{-1}}\eta\rangle$. Let ξ be the image of $r(x)$ in $H^1(C_{\bar{\mathbb{Q}}_p}, \mathcal{F})$, a nonzero χ-eigenvector, so $e_\chi \xi = \xi$, and choose η with $\langle \xi, \eta\rangle \neq 0$. Then $\langle \xi, e_{\chi^{-1}}\eta\rangle = \langle e_\chi \xi, \eta\rangle \neq 0$, so $e_{\chi^{-1}}\eta$ is a nonzero element of $H^0(C_{\bar{\mathbb{Q}}_p}, \mathcal{L}^*)^{\chi^{-1}}$.

□

Lemma 4.6. *Assume that the datum is admissible. Write $\bar{M} = M_k \otimes \mathbb{F}_p$ on $C_{\mathbb{F}_p}$ and $\mathrm{gr}^0 \bar{M} = \bar{M}/\mathrm{Fil}^1 \bar{M}$. For $x \in H_{\mathbb{Z}}$ represented by a Čech–de Rham cocycle $((\omega_i), (m_{ij}))$ on a finite cover of the p-adic completion of $C_{\mathbb{Z}_p}$ by charts as in Lemma 4.2, let $\pi(x) \in H^1(C_{\mathbb{F}_p}, \mathrm{gr}^0 \bar{M})$ be the class of the Čech cocycle $(m_{ij} \bmod (p, \mathrm{Fil}^1))$. Then $\pi : H_{\mathbb{Z}} \to H^1(C_{\mathbb{F}_p}, \mathrm{gr}^0 \bar{M})$ is a well-defined homomorphism, and*

$$\pi(x) \neq 0 \implies \varphi(x) \notin pH_{\mathbb{Z}}.$$

Proof. Modulo p, Frobenius sees only the gr^0-part of a class, through a map Λ that is a form of the Cartier isomorphism; the proof shows that Λ is a bijection onto the horizontal sections.

The map π. A coboundary $((\nabla n_i), (n_j - n_i))$ changes (m_{ij}) by a Čech coboundary, and refining the cover does not change the class, so π is well defined.

The reduced Frobenius. On a chart U with lift σ put $\bar{F}(m) = F_\sigma(\sigma^* \tilde{m}) \bmod p$ for $m \in \bar{M}(U)$ and any lift $\tilde{m}$; it does not depend on $\tilde{m}$. By the Taylor formula of the crystal M_k [5, §2, §7], [29, §6], with $\delta \in pB$ (both lifts reduce to the p-th power map, and [48, (1.4.3)] when $\eta_0 = dv/v$) , $F_{\sigma'}(\sigma'^* \tilde{m}) - F_\sigma(\sigma^* \tilde{m}) = \sum_{n \geq 1} \frac{\delta^n}{n!} F_\sigma(\sigma^* \nabla^n_\partial \tilde{m}) \in pM_k$, so $\bar{F}$ does not depend on σ either, and it glues to an additive map of sheaves $\bar{F} : \bar{M} \to \bar{M}$ with $\bar{F}(fm) = f^p \bar{F}(m)$. By Lemma 4.2(1), $\bar{F}$ kills $\mathrm{Fil}^1 \bar{M}$, so it factors through a map $\Lambda : \mathrm{gr}^0 \bar{M} \to \bar{M}$. Its values are horizontal, since $\nabla F_\sigma(\sigma^* \tilde{m}) = F_\sigma(\sigma^* \nabla_\partial \tilde{m}) \otimes \sigma^* \eta_0$ and $\sigma^* \eta_0 \in p\,\Omega^1$. So Λ maps $\mathrm{gr}^0 \bar{M}$ to the sheaf $\bar{M}^\nabla$ of horizontal sections. The same construction with $\rho^* H$ in place of M_k gives $\bar{F}$ on $\bar{H} = \rho^* H \otimes \mathbb{F}_p$, and $\bar{F}$ on $\bar{M} = \mathrm{Sym}^{k-2} \bar{H}$ is its symmetric power.

The reduction of $\varphi(x)$. By the description of Frobenius on Čech–de Rham cocycles [48, 2.3], with [5, §7] and [29, Thm. 6.4], written out in the proof of Lemma 4.3, $\varphi(x)$ is represented by $((F_{\sigma_i}(\omega_i)), (F_{\sigma_i}(\sigma_i^* m_{ij}) + h_{\sigma_i, \sigma_j}(\omega_j)))$. Here $F_{\sigma_i}(\omega_i) \equiv 0 \pmod p$ because $\sigma_i^* \eta_0 \in p\,\Omega^1$, and $h_{\sigma_i, \sigma_j}(\omega_j) \equiv 0 \pmod p$ because each term contains $\delta^n/n! \in pB$. So modulo p, $\varphi(x)$ is represented by $((0), (\Lambda(\mu_{ij})))$ with $\mu_{ij} = m_{ij} \bmod (p, \mathrm{Fil}^1)$.

Λ *is a bijection onto* $\bar{M}^{\nabla}$. We prove three facts.

(a) $u = \bar{F}(\xi)$ *is nowhere zero, for* ξ *a local generator of* $\bar{H}/\operatorname{Fil}^1 \bar{H}$. Let $y \notin D_C$ be a closed point with residue field κ_y. By the proof of Lemma 4.2, $\bar{F}$ on $\bar{H}$ is the pull-back by the relative Frobenius of E_0 composed with base change along the absolute Frobenius; both commute with restriction to y, so the fibre of $\bar{F}$ at y is $F^* \circ \mathrm{Frob}^*_{\kappa_y}$ on $H^1_{\mathrm{dR}}(E_y/\kappa_y)$, where $F : E_y \to E'_y = E_y^{(p)}$ is the relative Frobenius. The map of complexes $F^* : \Omega^\bullet_{E'_y} \to F_*\Omega^\bullet_{E_y}$ is zero in degree 1 and maps $\mathcal{O}_{E'_y}$ isomorphically onto $\mathcal{Z} = \ker(d : F_*\mathcal{O}_{E_y} \to F_*\Omega^1_{E_y})$, since a function with $df = 0$ is a p-th power (κ_y is perfect). So on H^1 it is the projection $H^1_{\mathrm{dR}}(E'_y) \to H^1(E'_y, \mathcal{O})$, followed by the isomorphism $H^1(\mathcal{O}) \cong H^1(\mathcal{Z})$ and by the map induced by $\mathcal{Z}[0] \to F_*\Omega^\bullet_{E_y}$, which is injective on H^1 because the quotient complex has no $\mathcal{H}^0$. Hence F^* is injective on $H^1_{\mathrm{dR}}/\operatorname{Fil}^1$, and $u(y) \neq 0$. At a point $z \in D_C$, defined over $W = W(\kappa_z)$, choose a formal parameter v at z with $q = \delta v^m$, $\delta \in W^\times$ and $p \nmid m$ (proof of Lemma 3.10(2)), and compute $\bar{F}$ on the disc with the logarithmic lift $\sigma_{\mathrm{can}}(v) = \gamma v^p$ of Lemma 3.8, which satisfies $\sigma_{\mathrm{can}}(q) = q^p$; this is allowed, as $\bar{F}$ does not depend on the lift, by the Taylor formula on the disc, as above. The Frobenius F for σ_{can} is induced by the isogeny $\mathrm{Tate}(q) \to \mathrm{Tate}(q^p)$, $x \mapsto x^p$ (proof of Lemma 3.9(2)), so $F(\sigma^*_{\mathrm{can}}\omega_{\mathrm{can}}) = p\,\omega_{\mathrm{can}}$ (proof of Lemma 3.8(2)). Since F is horizontal, $\nabla\omega_{\mathrm{can}} = \xi_{\mathrm{can}} \otimes \frac{dq}{q}$ and $\sigma^*_{\mathrm{can}}\frac{dq}{q} = p\frac{dq}{q}$, applying ∇ to this identity gives $p\,F(\sigma^*_{\mathrm{can}}\xi_{\mathrm{can}}) \otimes \frac{dq}{q} = p\,\xi_{\mathrm{can}} \otimes \frac{dq}{q}$, so $F(\sigma^*_{\mathrm{can}}\xi_{\mathrm{can}}) = \xi_{\mathrm{can}}$, as in [47, Proposition 3.2, (3.2.1)]. Hence u is a unit multiple of ξ_{can} near z.

(b) *The* p*-curvature* ψ *of* $\bar{H}$ *is nonzero at the generic point of every component of* $C_{\mathbb{F}_p}$. Every component of $C_{\mathbb{F}_p}$ contains a point $x \in D_C$: by (M3) of Definition 3.1, t is non-constant, hence surjective onto $\mathbb{P}^1_{\mathbb{F}_p}$, on every component of $X'_{\mathbb{F}_p}$, and its zeros lie in Z'; and ρ is finite and surjective on components. Near x, with coordinate v and $q = v^e \cdot (\text{unit})$, the derivation $D = v\,d/dv$ satisfies $D^p = D$, and the Gauss–Manin connection of the Tate curve (Lemma 3.8) gives $\nabla_D\omega_{\mathrm{can}} = c\,\xi_{\mathrm{can}}$, $\nabla_D\xi_{\mathrm{can}} = 0$, with $c = D(q)/q = \sum_n c_n v^n$ and $c_0 = e$. Hence $\psi(D)\omega_{\mathrm{can}} = (\nabla_D^p - \nabla_D)\omega_{\mathrm{can}} = (D^{p-1}(c) - c)\,\xi_{\mathrm{can}}$, and $D^{p-1}(c) - c = -\sum_{p\mid n} c_n v^n$ has constant term $-e \neq 0$.

(c) *Every local horizontal section of* $\bar{M}$ *is* $b^p u^{k-2}$ *for a unique local section* b. Let K be the function field of a component and $D \neq 0$ a derivation of K; then $\ker D = K^p$. Near the generic point choose w with $u \wedge w = 1$; this uses $\wedge^2\bar{H} \cong \mathcal{O}$ with the trivial connection (the Weil pairing is horizontal). Then $\nabla_D w = \alpha u + \beta w$, and $\beta = 0$ because $u \wedge \nabla_D w = \nabla_D(u \wedge w) = 0$. If $\alpha = D(a)$ with $a \in K$, then u and $w - au$ are independent horizontal sections and $\psi = 0$ at the generic point, contrary to (b); so $\alpha \notin D(K)$. Let $s = \sum_{i=0}^{k-2} c_i u^{k-2-i} w^i$ be horizontal over K. Then $D(c_i) + (i+1)\alpha c_{i+1} = 0$ for $0 \le i \le k-2$, with $c_{k-1} = 0$. If $c_{i+1} \neq 0$ for some $i \ge 0$, take i largest; then $D(c_{i+1}) = 0$, so $c_{i+1} \in K^p$, and $\alpha = D(-c_i/((i+1)c_{i+1})) \in D(K)$, using $i+1 \le k-2 < p$; this is impossible. So $s = c_0 u^{k-2}$ with $c_0 \in K^p$. Since u is nowhere zero by (a), $\mathcal{O}\,u^{k-2}$ is a subbundle of $\bar{M}$, so near any point y we get $c_0 \in \mathcal{O}_y \cap K^p = \mathcal{O}_y^p$, as $\mathcal{O}_y$ is normal. Finally $\Lambda(b\,\xi^{k-2}) = b^p u^{k-2}$, so Λ is injective and its image is $\bar{M}^{\nabla}$. For $k = 2$ the statement is $\ker d = \mathcal{O}^p$.

Conclusion. Suppose $\varphi(x) \in pH_{\mathbb{Z}}$. Since $H_{\mathbb{Z}}/p \to \mathbb{H}^1(C_{\mathbb{F}_p}, K \otimes \mathbb{F}_p)$ is injective (proof of Lemma 4.4(1)), the cocycle $((0), (\Lambda(\mu_{ij})))$ is a coboundary: there are $n_i \in \bar{M}(U_i)$ with $\nabla n_i = 0$ and $\Lambda(\mu_{ij}) = n_j - n_i$. Write $n_i = \Lambda(\nu_i)$. Then $\Lambda(\mu_{ij}) = \Lambda(\nu_j - \nu_i)$, so $\mu_{ij} = \nu_j - \nu_i$ and $\pi(x) = 0$. □

Lemma 4.7. *Assume that the datum is admissible. Put* $\chi = \chi_\Phi$*, and for a* $\tilde{G}$*-module* V *put* $V^\chi = \{v :\ g^*v = \chi(g)v \text{ for all } g \in \tilde{G}\}$. *Let* $\mathcal{L} = \rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C$ *and* $\mathcal{L}_{\log} = \mathcal{L}(D_C) = \operatorname{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$. *Then* $H^0(C_{\mathbb{Z}_p}, \mathcal{L})^\chi = \mathbb{Z}_p\,\omega_1$ *is free of rank one,* $H^0(C_{\mathbb{F}_p}, \bar{\mathcal{L}})^\chi = \mathbb{F}_p\,\bar{\omega}_1$, *for every geometric point* x *of* $C_{\mathbb{F}_p}$ *the germ of* $\bar{\omega}_1$ *at* x *is nonzero, and*

$$\{\bar{x} \in H_{a,p}/pH_{a,p} :\ \pi(x) = 0\} = \mathbb{F}_p\,[\bar{\omega}_1],$$

with π as in Lemma 4.6.

Proof. The proof counts the χ-parts of the graded pieces of the Hodge filtration over $\mathbb{Q}_p$, where the cusp form on the conjugate component of Lemma 4.5 leaves no room for intermediate Hodge pieces in the χ-part (Step 1); tame descent to $\mathbb{P}^1$ carries this to the special fibre (Step 2); and the kernel of π is then read off along the filtration (Step 3).

Let $K = [M_k \xrightarrow{\nabla} M_k \otimes \Omega^1_C(\log D_C)]$ with the filtration $\mathrm{Fil}^j K = [\mathrm{Fil}^j M_k \to \mathrm{Fil}^{j-1} M_k \otimes \Omega^1_C(\log D_C)]$, $0 \le j \le k-1$, where $\mathrm{Fil}^{-1} M_k = M_k$. Its graded pieces are $\mathrm{gr}^0 K = \mathrm{gr}^0 M_k[0]$ with $\mathrm{gr}^0 M_k \cong \rho^*\omega^{\otimes(2-k)}$, $\mathrm{gr}^{k-1} K = \mathcal{L}_{\log}[-1]$, and, for $1 \le j \le k-2$, $\mathrm{gr}^j K = [\mathrm{gr}^j M_k \xrightarrow{j\kappa} \mathrm{gr}^{j-1} M_k \otimes \Omega^1_C(\log D_C)] \simeq \mathcal{Q}_j[-1]$, where $\kappa : \rho^*\omega^{\otimes 2} \to \Omega^1_C(\log D_C)$ is the pull-back of the Kodaira–Spencer isomorphism of (M1) and $\mathcal{Q}_j = \mathrm{coker}(j\kappa)$. Since $j\kappa$ is injective on both fibres (proof of Lemma 4.4(1)), $\mathcal{Q}_j$ is flat over $\mathbb{Z}_p$, supported on the ramification points of ρ outside D_C. Let $\mathcal{L}^* = (\mathrm{gr}^0 M_k)^\vee \otimes \Omega^1_C$, with the action of $\tilde{G}$ dual to that on $\mathrm{gr}^0 M_k$, as in Lemma 4.5. Write $h^i_\chi(\mathcal{F}) = \dim_{\mathbb{Q}_p} H^i(C_{\mathbb{Q}_p}, \mathcal{F})^\chi$, and similarly for χ^{-1}.

Step 1: a count over $\mathbb{Q}_p$. We collect the terms.

- $h^1_\chi(K) = \mathrm{rank}\, H^\chi_{\mathbb{Z}} = 2 + b$, where b is the rank of $\mathrm{res}(H^\chi_{\mathbb{Z}})$ and 2 that of its kernel $H_{a,p}$ (Lemma 3.10(2),(3)). Next, $h^0(K) = 0$: a horizontal global section of M_k maps to $H^0(\mathrm{gr}^0 M_k) = 0$, and walking down the filtration with the injective maps $j\kappa$, as in Lemma 4.4(1), shows it is zero. And $\mathbb{H}^2(K) = 0$, because $\mathbb{H}^2(\mathrm{gr}^j K) = 0$ for every j: for $j = 0$ it is H^2 of a coherent sheaf on a curve, for $1 \le j \le k-2$ it is H^1 of the skyscraper sheaf $\mathcal{Q}_j$, and for $j = k-1$ it is $H^1(\mathcal{L}_{\log}) = 0$ (third item of this list).
- $h^0(\mathrm{gr}^0 M_k) = 0$, and $h^1_\chi(\mathrm{gr}^0 M_k) = h^0_{\chi^{-1}}(\mathcal{L}^*)$ by $\tilde{G}$-equivariant Serre duality.
- $h^1(\mathcal{L}_{\log}) = 0$, since $\mathcal{L}^\vee_{\log} \otimes \Omega^1_C = \rho^*\omega^{\otimes(2-k)}(-D_C)$ has negative degree on every component.
- $h^0_\chi(\mathcal{L}_{\log}) = h^0_\chi(\mathcal{L}) + b$. Indeed, in $0 \to \mathcal{L} \to \mathcal{L}_{\log} \to \mathcal{L}_{\log}|_{D_C} \to 0$, $H^1(\mathcal{L})$ is dual to $H^0(\mathrm{gr}^0 M_k) = 0$, so $H^0(\mathcal{L}_{\log})^\chi \to H^0(\mathcal{L}_{\log}|_{D_C})^\chi$ is onto. At $y \in D_C$ the residue N_y of ∇ maps $\omega^i_{\mathrm{can}}\xi^{k-2-i}_{\mathrm{can}}$ to a nonzero multiple of $\omega^{i-1}_{\mathrm{can}}\xi^{k-1-i}_{\mathrm{can}}$ for $1 \le i \le k-2$ (Lemma 3.8, where $\nabla_\theta e_i = (k-2-i)\, m\, e_{i+1}$ for $e_i = \omega^{k-2-i}_{\mathrm{can}}\xi^i_{\mathrm{can}}$), so $\mathrm{coker}\, N_y$ is the line $\mathrm{Fil}^{k-2} M_{k,y} = \mathcal{L}_{\log,y}$, and the residue of the class of a log form is its residue. So res maps $H^\chi_{\mathbb{Z}} \otimes \mathbb{Q}_p$ onto $H^0(\mathcal{L}_{\log}|_{D_C})^\chi$, which has dimension b.

The Euler characteristic of χ-parts is additive along the filtration, so

$$-(2+b) = \sum_{j=0}^{k-1} \chi_\chi(\mathrm{gr}^j K) = -h^0_{\chi^{-1}}(\mathcal{L}^*) - \sum_{j=1}^{k-2} h^0_\chi(\mathcal{Q}_j) - h^0_\chi(\mathcal{L}) - b.$$

Here $h^0_\chi(\mathcal{L}) \ge 1$, because ω_Φ is a nonzero χ-eigensection of $\mathcal{L}$ by Lemma 3.12(1),(3); this is where (F) enters. And $h^0_{\chi^{-1}}(\mathcal{L}^*) \ge 1$ by Lemma 4.5; this is where (P) enters. Hence $h^0_\chi(\mathcal{Q}_j) = 0$ for $1 \le j \le k-2$, and $h^0_\chi(\mathcal{L}) = 1$.

Step 2: tame descent. Let $\pi_Y : C_{\mathbb{Z}_p} \to Y = C_{\mathbb{Z}_p}/\tilde{G}$. Since $C_{\mathbb{Z}_p}/\mu_d = X'_{\mathbb{Z}_p}$, $Y = X'_{\mathbb{Z}_p}/G$, which is normal and finite and birational over $\mathbb{P}^1_{\mathbb{Z}_p}$ by (M3) of Definition 3.1; so $Y \cong \mathbb{P}^1_{\mathbb{Z}_p}$. By (M2) and $p \nmid d$, every geometric point x of C has a stabiliser $S_x \subset \tilde{G}$ of order prime to p. On the completed stalk at $\pi_Y(x)$, $\pi_{Y*}\mathcal{F}$ is induced from S_x, and its χ-part is $\widehat{\mathcal{F}}_x^{\chi|S_x}$ (Frobenius reciprocity), cut out by the idempotent of S_x. So $\mathcal{F} \mapsto (\pi_{Y*}\mathcal{F})^\chi$ is exact on $\tilde{G}$-equivariant coherent sheaves and commutes with reduction modulo p, and $H^0(Y, (\pi_{Y*}\mathcal{F})^\chi) = H^0(C, \mathcal{F})^\chi$.

- For $\mathcal{Q}_j$, $(\pi_{Y*}\mathcal{Q}_j)^\chi$ is a skyscraper sheaf flat over $\mathbb{Z}_p$ whose generic fibre has no sections by Step 1; so it is zero, and $H^0(C_{\mathbb{F}_p}, \bar{\mathcal{Q}}_j)^\chi = 0$.

- For $\mathcal{L}$, $\mathcal{M} = (\pi_{Y*}\mathcal{L})^\chi$ is a line bundle on Y. Let x be a geometric point of $C_{\mathbb{F}_p}$, $W = W(\bar{\mathbb{F}}_p)$, $W[[v]]$ the completed local ring of C_W at x, and K_x the kernel of the action of S_x on $W[[v]]$. The group S_x/K_x acts faithfully and W-linearly on $W[[v]]$ and has order e prime to p; averaging a parameter as in the proof of Lemma 3.7(1), we may take v with $h(v) = \zeta(h)v$, where ζ identifies S_x/K_x with $\mu_e(W)$. Similarly $\widehat{\mathcal{L}}_x = W[[v]]\,\ell$ with $h^*\ell = \lambda(h)\ell$ for a character λ of S_x. Every $h \in K_x$ fixes the connected component of C_W through x pointwise, so it acts on $\widehat{\mathcal{L}}_x$ by the constant $\lambda(h)$. The section ω_Φ is a χ-eigensection that is nonzero on every connected component of C_W (Lemma 3.12(1), and $\tilde{G}$ permutes the components of $C_{\mathbb{C}}$ transitively, proof of Lemma 3.10(3)); comparing $h^*\omega_\Phi = \chi(h)\omega_\Phi$ with $h^*\omega_\Phi = \lambda(h)\omega_\Phi$ on its nonzero germ at x gives $\lambda = \chi$ on K_x. So $\chi\lambda^{-1} = \zeta^{n_0}$ for a unique $0 \le n_0 < e$, and the χ-part of $\widehat{\mathcal{L}}_x$ is $v^{n_0}W[[v^e]]\,\ell$, free of rank one over $W[[v^e]] = W[[v]]^{S_x}$, the completed local ring of Y_W at the image of x. By Step 1, $h^0(\mathcal{M}_{\mathbb{Q}_p}) = 1$; as every line bundle on $Y \cong \mathbb{P}^1_{\mathbb{Z}_p}$ is some $\mathcal{O}(n)$, $\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}$. Hence $H^0(C_{\mathbb{Z}_p}, \mathcal{L})^\chi = \mathbb{Z}_p\,\omega_1$ and $H^0(C_{\mathbb{F}_p}, \bar{\mathcal{L}})^\chi = H^0(\mathbb{P}^1_{\mathbb{F}_p}, \bar{\mathcal{M}}) = \mathbb{F}_p\,\bar{\omega}_1$, where $\bar{\omega}_1$ is a nowhere vanishing section of $\bar{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_p}}$. At a geometric point x of $C_{\mathbb{F}_p}$, the completed stalk of $\bar{\mathcal{M}}$ at the image of x is the χ-part of $\widehat{\bar{\mathcal{L}}}_x$, and the image of $\bar{\omega}_1$ there is its germ at x; this germ is therefore a generator, in particular nonzero.

Step 3: the kernel of π. For $0 \le j \le k-2$ we have $H^0(\mathrm{gr}^j \bar{K}) = 0$: for $j = 0$ it is $H^0(C_{\mathbb{F}_p}, \rho^*\omega^{\otimes(2-k)}) = 0$, and for $j \ge 1$, $\mathrm{gr}^j \bar{K} \simeq \bar{\mathcal{Q}}_j[-1]$. So the maps $\mathbb{H}^1(\mathrm{Fil}^{j+1} \bar{K}) \to \mathbb{H}^1(\mathrm{Fil}^j \bar{K})$ are injective, and they are $\tilde{G}$-equivariant. Let $\bar{x} \in H_{a,p}/p$ with $\pi(x) = 0$. As π is the map $\mathbb{H}^1(\bar{K}) \to \mathbb{H}^1(\mathrm{gr}^0 \bar{K})$, $\bar{x}$ has a unique preimage $z_1 \in \mathbb{H}^1(\mathrm{Fil}^1 \bar{K})$, which is a χ-eigenvector by uniqueness. Its image in $\mathbb{H}^1(\mathrm{gr}^1 \bar{K}) = H^0(\bar{\mathcal{Q}}_1)$ is a χ-eigenvector, hence 0 by Step 2, so z_1 comes from a unique χ-eigenvector $z_2 \in \mathbb{H}^1(\mathrm{Fil}^2 \bar{K})$. Continuing, $\bar{x} = [\bar{\omega}]$ with $\bar{\omega} \in H^0(C_{\mathbb{F}_p}, \bar{\mathcal{L}}_{\log})^\chi = \mathbb{H}^1(\mathrm{Fil}^{k-1} \bar{K})^\chi$. Since $\mathrm{res}(x) = 0$, the residues of $\bar{\omega}$ vanish in $\mathrm{coker}\, \bar{N}_y = \bar{\mathcal{L}}_{\log,y}$ (Step 1, valid modulo p as $i < p$) , so $\bar{\omega} \in H^0(\bar{\mathcal{L}})^\chi = \mathbb{F}_p\,\bar{\omega}_1$. Conversely $[\omega_1] \in H_{a,p}$, since ω_1 is a cusp form, and $\pi([\omega_1]) = 0$. Finally $[\bar{\omega}_1] \ne 0$: ω_1 is not divisible by p in $H^0(C_{\mathbb{Z}_p}, \mathcal{L}_{\log})$ either, because $\mathcal{L}_{\log}/\mathcal{L}$ is supported on D_C, which is étale over $\mathcal{O}_{L,(\mathfrak{P})}$, and so is flat over $\mathbb{Z}_p$; hence Lemma 4.4(1) applies. □

Proposition 4.8. *Assume that the datum is admissible. Let $x_1 = [\omega_1]$, where ω_1 is a generator of the free $\mathbb{Z}_p$-module of rank one $H^0(C_{\mathbb{Z}_p}, \rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C)^{\chi_\Phi}$ of Lemma 4.7. Then x_1 extends to a basis $\{x_1, y\}$ of $H_{a,p}$, and for any such y, $\{p^{1-k}\varphi(x_1), \varphi(y)\}$ is a basis of $H_{a,p}$. In particular*

$$v_p(\det(\varphi \mid H_{a,p})) = k - 1.$$

Proof. To prove independence modulo p, we detect the class of $p^{1-k}\varphi(x_1)$ by its expansion at the cusp and the class of $\varphi(y)$ by the reduced Frobenius of Lemma 4.6.

The class x_1. The section ω_1 is a χ_Φ-eigensection of $\mathcal{L} = \rho^*\omega^{\otimes(k-2)} \otimes \Omega^1_C$, so its class is a χ_Φ-eigenvector with zero residues, that is, $x_1 \in H_{a,p}$ (Lemma 3.10(2); the residue of the class of a section of $\mathcal{L}$ vanishes, as in the proof of Lemma 3.12(3)). By Lemma 4.7, the reduction $\bar{\omega}_1$ is nonzero, so ω_1 is not divisible by p in $H^0(C_{\mathbb{Z}_p}, \mathcal{L})$; as $\mathcal{L}_{\log}/\mathcal{L}$ is supported on D_C, which is étale over $\mathcal{O}_{L,(\mathfrak{P})}$, it is not divisible by p in $H^0(C_{\mathbb{Z}_p}, \mathcal{L}_{\log})$ either, where $\mathcal{L}_{\log} = \mathrm{Fil}^{k-2} M_k \otimes \Omega^1_C(\log D_C)$. So $x_1 \notin pH_{\mathbb{Z}}$ by Lemma 4.4(1), hence $x_1 \notin pH_{a,p}$, and as $H_{a,p}$ is free of rank 2 (Lemma 3.10(3)), x_1 extends to a basis $\{x_1, y\}$. By Lemma 4.3(1), $\varphi(x_1) \in p^{k-1}H_{\mathbb{Z}}$; as $\varphi(x_1) \in H_{a,p}$ and $H_{a,p}$ is saturated in $H_{\mathbb{Z}}$ (Lemma 3.10(2)), $\omega_1' := p^{1-k}\varphi(x_1) \in H_{a,p}$.

The expansion of ω_1 is not divisible by p. Let ∞_C, W, u, γ, $\mathcal{A}$, θ, V and $\mathcal{E}$ be as in Lemma 4.3(2), and write $\omega_1|_{W[[u]]} = g_1\,\omega_{\mathrm{can}}^{\otimes(k-2)} \otimes \frac{du}{u}$ with $g_1 = \sum_{N\ge 0} g_{1,N}u^N \in W[[u]]$. By

Lemma 4.7, the germ of $\bar{\omega}_1$ at the reduction of ∞_C is nonzero; since $\omega_{\mathrm{can}}^{\otimes(k-2)} \otimes \frac{du}{u}$ is a basis of $\mathcal{L}_{\log}$ on the formal disc at ∞_C, $g_1 \not\equiv 0 \pmod{p}$. By Lemma 4.3(2), $\mathcal{E}(\omega_1') = V(g_1)$.

$\omega_1' \notin pH_{a,p} + \varphi(H_{a,p})$. Suppose that $\omega_1' = pz + \varphi(w)$ with $z, w \in H_{a,p}$, and let $h_z, h_w \in \mathcal{A}$ represent $\mathcal{E}(z)$ and $\mathcal{E}(w)$. By Lemma 3.9(2), $\mathcal{E}(\varphi(w))$ is represented by $p^{k-1}V(h_w)$, so

$$V(g_1) - p\,h_z - p^{k-1}V(h_w) \in \theta^{k-1}\mathcal{A}.$$

Let $N \geq 0$. The coefficient of u^{Np} in $V(g_1)$ is $\sigma(g_{1,N})\gamma^N$; those in $p\,h_z$ and in $p^{k-1}V(h_w)$ lie in pW; and that of an element of $\theta^{k-1}\mathcal{A}$ lies in $(Np)^{k-1}W \subset pW$. Hence $\sigma(g_{1,N})\gamma^N \in pW$, and since $\gamma \in W^\times$ and σ is an automorphism of W preserving pW, $g_{1,N} \in pW$. This holds for every N, so $g_1 \equiv 0 \pmod{p}$, a contradiction.

$\varphi(y) \notin pH_{a,p}$. By Lemma 4.7, the $\bar{x} \in H_{a,p}/pH_{a,p}$ with $\pi(x) = 0$ form the line $\mathbb{F}_p\bar{x}_1$, and $\bar{y}$ is not on it; so $\pi(y) \neq 0$, and $\varphi(y) \notin pH_{\mathbb{Z}}$ by Lemma 4.6.

Conclusion. Since $\varphi(x_1) = p^{k-1}\omega_1' \in pH_{a,p}$, the image of $\varphi(H_{a,p})$ in $H_{a,p}/pH_{a,p}$ is the line spanned by the class of $\varphi(y)$, which is nonzero by the previous step; and the class of ω_1' is not on this line, by the one before it. So the classes of ω_1' and $\varphi(y)$ form a basis of $H_{a,p}/pH_{a,p}$, and by Nakayama's lemma $\{\omega_1', \varphi(y)\}$ is a basis of $H_{a,p}$. In the basis $\{x_1, y\}$ the columns of the matrix of φ are $\varphi(x_1) = p^{k-1}\omega_1'$ and $\varphi(y)$; hence $\det\varphi = p^{k-1}\nu$, where ν is the determinant of the change of basis from $\{x_1, y\}$ to $\{\omega_1', \varphi(y)\}$, a unit of $\mathbb{Z}_p$.

□

5 The trace comparison

The characteristic polynomial in the cusp congruence of Theorem 4.1 belongs to the normalised inverse $p^{k-1}\varphi^{-1}$ on the crystalline summand containing the class of Φ. We identify this polynomial with that of geometric Frobenius on the étale summand for the inverse character χ_Φ^{-1}. The two summands are related by duality on the Kummer cover, which accounts for both the inverse and the factor p^{k-1}. Throughout, ℓ-adic Frobenius is geometric, and $\mathbb{Q}_p(-1)$ has Hodge–Tate weight one.

5.1 Étale eigensheaves

Lemma 5.1. *Let (X', Z', E, G, χ_G) be a modular model, at a prime $\mathfrak{P}$ of L of residue degree one over $p \notin \Sigma$, of a Kummer twist datum (Γ, t, f, a, R) with singular set S, and let $\bar{S}$ be the closure of S in $\mathbb{P}^1_{\mathcal{O}}$, where $\mathcal{O} = \mathcal{O}_{L,(\mathfrak{P})}$. Then $t(Z') \subset \bar{S}$. Consequently $t^{-1}(\mathbb{P}^1_{\mathcal{O}} \setminus \bar{S})$ is contained in $Y' = X' \setminus Z'$, over which E is an elliptic curve and the isogenies π_g of (M2) are defined.*

Proof. As Z' is étale over $\mathcal{O}$ (M1), it is flat over $\mathcal{O}$, hence the closure of its generic fibre Z_L'; as t is continuous, $t(Z')$ lies in the closure of $t(Z_L')$. It therefore suffices to show that t maps the geometric points of Z_L' into S. On each connected component X_0' of $X_{\mathbb{C}}'$, E is the universal generalised elliptic curve (M4), which is singular exactly at the cusps; as E is smooth over Y', the cusps lie in Z', and as the Kodaira–Spencer map is an isomorphism $\omega^{\otimes 2} \cong \Omega^1_{X'/\mathcal{O}}(\log Z')$ (M1), $Z' \cap X_0'$ is exactly the set of cusps. Their images under t are values of the Hauptmodul at cusps of Γ, which lie in S by Lemma A.1(3). Hence $t(Z') \subset \bar{S}$. The last assertion follows, as E is smooth over Y' (M1) and the π_g are isogenies over Y' (M2).

□

Definition 5.2. Let (X', Z', E, G, χ_G) be a modular model, at a prime $\mathfrak{P}$ of L of residue degree one over $p \notin \Sigma$, of a Kummer twist datum (Γ, t, f, a, R) with $a = j/d$ and singular set S, with

isogenies π_g and constants c_g as in (M2), Kummer cover $\rho : C \to X'$, group $\tilde{G}$ and twist character χ_Φ, and put $\mathcal{O} = \mathcal{O}_{L,(\mathfrak{P})}$. Let $\bar{S}$ be the closure of S in $\mathbb{P}^1_{\mathcal{O}}$, $U = \mathbb{P}^1_{\mathcal{O}} \setminus \bar{S}$ with inclusion $o : U \hookrightarrow \mathbb{P}^1_{\mathcal{O}}$, $\varpi' : X'_U \to U$ and $\varpi : C_U \to U$ the restrictions of t and $t \circ \rho$ over U, and $[d] : \mathbb{G}_{m,\mathcal{O}} \to \mathbb{G}_{m,\mathcal{O}}$ the map $s \mapsto s^d$. Let ℓ be a prime and λ a prime of L above ℓ; put $\mathcal{U} = U$ if $\ell \neq p$ and $\mathcal{U} = U_L$ if $\ell = p$. Over $\mathcal{U}$ let $\mathcal{H} = R^1\pi_{E*}\mathbb{Q}_\ell$, for the structure map $\pi_E : E \to X'$, and $\mathcal{M} = \varpi_*\rho^* \operatorname{Sym}^{k-2} \mathcal{H} \otimes_{\mathbb{Q}_\ell} L_\lambda$.

1. For $g \in G$ let $g^\sharp = c_g \operatorname{Sym}^{k-2}(\pi_g^*) \circ g^*$, and for $\zeta \in \mu_d(L)$ let $\zeta^\sharp = [\zeta]^*$, as endomorphisms of $\mathcal{M}$; the same formulas define $g^\sharp$ on $\varpi'_* \operatorname{Sym}^{k-2} \mathcal{H} \otimes_{\mathbb{Q}_\ell} L_\lambda$ and $\zeta^\sharp$ on $[d]_* L_\lambda$.

2. The **Kummer eigensheaves** of the model are the subsheaves $\mathcal{F}_a$ and $\mathcal{F}_{\bar{a}}$ of $\mathcal{M}$ on which $\zeta^\sharp g^\sharp$ acts by $\chi_\Phi(g,\zeta)$, respectively by $\chi_\Phi(g,\zeta)^{-1}$, for all $(g,\zeta) \in \tilde{G}$, and the subsheaves $\mathbb{V}_{\chi_G^{\pm 1}}$ of $\varpi'_* \operatorname{Sym}^{k-2} \mathcal{H} \otimes_{\mathbb{Q}_\ell} L_\lambda$ on which every $g^\sharp$ acts by $\chi_G(g)^{\pm 1}$. The **Kummer sheaves** are the subsheaves $\mathcal{L}_{\pm j}$ of $[d]_* L_\lambda$ on which every $\zeta^\sharp$ acts by $\zeta^{\pm j}$.

3. The **λ-adic realisations** are the representations of $\operatorname{Gal}(\bar{L}/L)$

$$V_{a,\lambda} = H^1\big(\mathbb{P}^1_{\bar{L}}, o_*\mathcal{F}_a\big), \qquad V_{\bar{a},\lambda} = H^1\big(\mathbb{P}^1_{\bar{L}}, o_*\mathcal{F}_{\bar{a}}\big),$$

   with $\mathcal{F}_a$, $\mathcal{F}_{\bar{a}}$ restricted to U_L. For $\ell \neq p$ we put

$$V_a = H^1\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_*\mathcal{F}_a\big), \qquad V_{\bar{a}} = H^1\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_*\mathcal{F}_{\bar{a}}\big),$$

   with $\mathcal{F}_a$, $\mathcal{F}_{\bar{a}}$ restricted to $U_{\mathbb{F}_p}$, and with the geometric Frobenius F relative to $\mathbb{F}_p$.

4. For $x \in \mathbb{F}_p^\times$, the **$d$-th power residue symbol** $(x/\mathfrak{P})_d$ is the element of $\mu_d(L)$ congruent to $x^{(p-1)/d}$ modulo $\mathfrak{P}$.

Lemma 5.3. *Let (X', Z', E, G, χ_G) be a modular model, at a prime $\mathfrak{P}$ of L of residue degree one over $p \notin \Sigma$, of a Kummer twist datum (Γ, t, f, a, R), with isogenies π_g and constants c_g as in (M2), and let ℓ be a prime and λ a prime of L above ℓ.*

1. Over $Y'[1/\ell]$, where $Y' = X' \setminus Z'$, the cup product $\langle\,,\rangle : R^1\pi_{E*}\mathbb{Q}_\ell \otimes R^1\pi_{E*}\mathbb{Q}_\ell \to R^2\pi_{E*}\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-1)$ is a perfect alternating pairing, and

$$\langle x_1 \cdots x_{k-2}, y_1 \cdots y_{k-2} \rangle = \frac{1}{(k-2)!} \sum_{\sigma \in S_{k-2}} \prod_{i=1}^{k-2} \langle x_i, y_{\sigma(i)} \rangle$$

is a perfect pairing of $\operatorname{Sym}^{k-2} R^1\pi_{E*}\mathbb{Q}_\ell$ with itself into $\mathbb{Q}_\ell(-(k-2))$. An isogeny ψ of degree δ satisfies $\langle \psi^* x, \psi^* y \rangle = \delta^{k-2}\langle x, y \rangle$ on Sym^{k-2}.

2. Suppose that ϖ is finite étale and that $(g,\zeta) \mapsto \zeta^\sharp g^\sharp$ is a right action of $\tilde{G}$ on $\mathcal{M}$. Then the pairing of (1), composed with the trace $\varpi_* L_\lambda(-(k-2)) \to L_\lambda(-(k-2))$, is a perfect $\tilde{G}$-invariant L_λ-bilinear pairing $\mathcal{M} \otimes \mathcal{M} \to L_\lambda(-(k-2))$ on $\mathcal{U}$, and it restricts to a perfect pairing $\mathcal{F}_a \otimes \mathcal{F}_{\bar{a}} \to L_\lambda(-(k-2))$ of the Kummer eigensheaves.

Proof. Proof of (1). Over Y', E is an elliptic curve (M1), so over $Y'[1/\ell]$ the sheaf $R^1\pi_{E*}\mathbb{Q}_\ell$ is lisse of rank two and the cup product into $R^2\pi_{E*}\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-1)$ is a perfect alternating pairing; it induces the perfect pairing of (1) on Sym^{k-2}. An isogeny ψ of degree δ acts on R^2 by δ, so $\langle \psi^* x, \psi^* y \rangle = \delta^{k-2}\langle x, y \rangle$ on Sym^{k-2}.

Proof of (2). By Lemma 5.1, $X'_{\mathcal{U}} \subset Y'$, and ℓ is invertible on $\mathcal{U}$; so (1) applies to $\mathcal{H}$ on $X'_{\mathcal{U}}$. As $c_g^2 = (\deg \pi_g)^{-(k-2)}$, each $g^\sharp$ preserves the pairing by (1), and so does each $\zeta^\sharp = [\zeta]^*$, as $\rho^* \operatorname{Sym}^{k-2} \mathcal{H}$ is pulled back from X'. Composing with the trace of the finite étale map ϖ

gives a perfect $\tilde{G}$-invariant L_λ-bilinear pairing on $\mathcal{M}$, the orthogonal sum over the points of each geometric fibre. If x lies in the χ-part and y in the χ'-part, for characters χ, χ' of $\tilde{G}$ with values in $L^\times$, then $\langle x, y\rangle = \langle \gamma^\sharp x, \gamma^\sharp y\rangle = \chi(\gamma)\chi'(\gamma)\langle x, y\rangle$ for all $\gamma \in \tilde{G}$, so these parts are orthogonal unless $\chi\chi' = 1$. As $\mathcal{M}$ is the direct sum of its isotypic parts for the action of $\tilde{G}$, the pairing restricts to a perfect pairing $\mathcal{F}_a \otimes \mathcal{F}_{\bar{a}} \to L_\lambda(-(k-2))$.

□

Lemma 5.4. *Let ℓ be a prime and λ a prime of L above ℓ, and let $\mathcal{F}_a$, $\mathcal{F}_{\bar{a}}$, $\mathbb{V}_{\chi_G^{\pm 1}}$ be the Kummer eigensheaves of the fixed model and $\mathcal{L}_{\pm j}$ its Kummer sheaves.*

1. *S is stable under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$; $\bar{S}$ is finite and étale over $\mathcal{O}$; ϖ' and ϖ are finite étale; and $C_U = X'_U \times_U [d]^{-1}(U)$, with $\varpi = \varpi' \times_U [d]$.*

2. *$(g, \zeta) \mapsto \zeta^\sharp g^\sharp$ is a right action of $\tilde{G}$ on $\mathcal{M}$ by L_λ-linear automorphisms. The sheaves $\mathcal{F}_a$, $\mathcal{F}_{\bar{a}}$, $\mathbb{V}_{\chi_G^{\pm 1}}$ and $\mathcal{L}_{\pm j}$ are lisse direct summands, $\mathcal{L}_{\pm j}$ has rank one, and on $\mathcal{U}$*

$$\mathcal{F}_a \cong \mathbb{V}_{\chi_G} \otimes \mathcal{L}_j, \qquad \mathcal{F}_{\bar{a}} \cong \mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j}.$$

   If $\ell \neq p$ and $x \in \mathbb{F}_p^\times$, the geometric Frobenius Frob_x acts on the stalk of $\mathcal{L}_{\pm j}$ at $\bar{x}$ by $(x/\mathfrak{P})_d^{\pm j}$.

3. *There is a perfect L_λ-bilinear pairing of lisse sheaves $\mathcal{F}_a \otimes \mathcal{F}_{\bar{a}} \to L_\lambda(-(k-2))$ on $\mathcal{U}$.*

4. *Fix an embedding $\bar{L} \subset \mathbb{C}$ extending $L \subset \mathbb{C}$. There is a local system $\mathcal{V}$ of L-vector spaces on $U(\mathbb{C})$ with $\mathcal{V} \otimes_L \mathbb{C} \cong \mathbb{V}_f \otimes \mathbb{L}_a$ such that Artin's comparison isomorphism identifies $\mathcal{F}_{\bar{a}}|_{U(\mathbb{C})}$ with $\mathcal{V} \otimes_L L_\lambda$. Consequently $\mathcal{F}_a$, $\mathcal{F}_{\bar{a}}$ and $\mathbb{V}_{\chi_G^{\pm 1}}$ have rank $k-1$, $\mathbb{V}_{\chi_G^{-1}}|_{U(\mathbb{C})}$ corresponds in the same way to an L-form of $\mathbb{V}_f$, and*

$$H^1\big(\mathbb{P}^1_{\mathbb{C}}, o_*\mathcal{F}_{\bar{a}}\big) \cong H^1\big(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}\big) \otimes_L L_\lambda, \qquad H^1\big(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}\big) \otimes_L \mathbb{C} \cong H_a.$$

5. *Let $\ell \neq p$ and let x_0 be a geometric point of $\mathbb{P}^1_{\mathbb{F}_p}$ over $t = 0$, with inertia group I_{x_0}. Then I_{x_0} has no nonzero invariants and no nonzero coinvariants on the generic stalks of $\mathcal{F}_a|_{U_{\mathbb{F}_p}}$ and of $\mathcal{F}_{\bar{a}}|_{U_{\mathbb{F}_p}}$.*

Proof. The main steps are 3, 5 and 9: tame stabilisers turn two points with the same reduction into a ramification point over S; de Rham cohomology fixes the sign in $g^\sharp h^\sharp = \pm(hg)^\sharp$; and inertia at $t = 0$ is computed on the Tate curve. Write $W = W(\bar{\mathbb{F}}_p)$ and $K = W[\frac{1}{p}]$; as p is unramified in L and $\mathfrak{P}$ has degree one, $\mathcal{O} \subset \mathbb{Z}_p \subset W$.

Step 1: S and $\bar{S}$. We first show that S is Galois-stable. Let $T \in t + t^2\mathbb{Z}[\frac{1}{M}][[t]]$ be the inverse of the series $t(q)$ of Definition 2.1, and put $y_i = F(t)\,(\log T(t))^i$, $0 \le i \le k-2$, elements of $\mathbb{Q}((t))[\log t]$ since $\log T(t) = \log t + \log(T(t)/t)$ with $\log(T(t)/t) \in t\mathbb{Q}[[t]]$. As $2\pi i\tau = \log q$, they span the solutions near $t = 0$ of the Picard–Fuchs equation D associated with (t, f) (Definition 2.2), which we take monic in $\partial = d/dt$. Its coefficients are $\pm \mathrm{Wr}_m/\mathrm{Wr}$, with Wr the Wronskian of the y_i and Wr_m minors of the Wronskian matrix. The substitution $\log t \mapsto \log t + c$, $c \in \mathbb{Q}$, acts on the span of the y_i by a unipotent matrix and so fixes every coefficient; a rational function of $\log t$ over $\mathbb{Q}((t))$, in which $\log t$ is transcendental, that is fixed by all these translations has no poles (they would form a finite set stable under infinitely many translations) and is therefore constant, so the coefficients lie in $\mathbb{Q}((t))$. By Lemma A.1(1),(2) every point of $\mathbb{P}^1(\mathbb{C})$ is a regular or regular singular point of D, so its coefficients are meromorphic on $\mathbb{P}^1$, hence rational; and a rational function with Laurent coefficients in $\mathbb{Q}$ lies in $\mathbb{Q}(t)$ (numerator and denominator

solve a linear system with rational coefficients). So the singular set of D, and with it S, is stable under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Hence the geometric points of $\bar{S}_L$ are the $r = |S|$ points of S, and $\bar{S}$, closed in $\mathbb{P}^1_{\mathcal{O}}$ and the closure of its generic fibre, is finite and flat of degree r over $\mathcal{O}$. Fix the embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ extending $\mathcal{O} \subset \mathbb{Z}_p$. As $p \notin \Sigma$, the reductions of the points of S are pairwise distinct; every geometric point of the special fibre of $\bar{S}$ is such a reduction (flatness), so the special fibre has r geometric points and degree r, hence is étale, and $\bar{S}$ is finite étale over $\mathcal{O}$. So U is smooth over $\mathcal{O}$ and irreducible, and $U \subset \mathbb{G}_{m,\mathcal{O}}$ as $0, \infty \in S$.

Step 2: t is finite and flat. The morphism $t : X'_{\mathcal{O}} \to \mathbb{P}^1_{\mathcal{O}}$ is proper, and its fibres are finite because t is non-constant on every connected component of every fibre of X' (M3); so t is finite [21, (4.4.2)]. Both schemes are regular of dimension two and the fibres of t have dimension zero, so t is flat [43, Theorem 23.1]. Let n be its degree.

Step 3: ϖ' is étale. By Lemma 3.7(1), ϖ' is étale over U_L. Let $\bar{u} \in U(\bar{\mathbb{F}}_p)$; as U is smooth over $\mathcal{O}$, $\bar{u}$ lifts to $u \in U(W)$. The fibre Y_u of t over u is finite flat of degree n over W, with étale generic fibre since $u_K \notin S$; so $Y_u(\bar{K})$ has n points, which reduce to points of $X'(\bar{\mathbb{F}}_p)$ as X' is proper. Suppose two of them, $x_1 \neq x_2$, reduce to the same point $\bar{x}$. By (M3), $X'_L/G \cong \mathbb{P}^1_L$, and as invariants commute with flat base change, $X'_{\bar{K}}/G = \mathbb{P}^1_{\bar{K}}$, so G acts transitively on $Y_u(\bar{K})$ [12, Ch. V, §2, no. 2, Th. 2]: $x_2 = gx_1$ with $g \in G$, and $g\bar{x} = \bar{x}$ since G acts over $\mathcal{O}$. Let $G_{\bar{x}}$ be the stabiliser of $\bar{x}$; $|G_{\bar{x}}|$ is prime to p by (M2). The completed local ring of X'_W at $\bar{x}$ is $W[[v]]$, and $G_{\bar{x}}$ acts on the line $\mathfrak{m}/(\mathfrak{m}^2 + p)$ by a character, which lifts by Teichmüller to $\chi : G_{\bar{x}} \to W^\times$; then $v' = |G_{\bar{x}}|^{-1} \sum_{h \in G_{\bar{x}}} \chi(h)^{-1} h(v)$ is a parameter with $h(v') = \chi(h)v'$, and $v' = 0$ defines $y \in X'(W)$, fixed by $G_{\bar{x}}$, with reduction $\bar{x}$. The points y and x_1 reduce to the same point of the smooth special fibre, so they lie on the same connected component X'_1 of $X'_{\bar{K}}$. Let G_1 be its stabiliser and N_1 the kernel of $G_1 \to \mathrm{Aut}(X'_1)$; as G permutes the components of $X'_{\bar{K}}$ transitively, $\bar{K}(X'_1)/\bar{K}(t)$ is Galois with group G_1/N_1. Now $G_{\bar{x}} \subset G_1$, and the image of g in G_1/N_1 is nontrivial ($gx_1 \neq x_1$) and fixes y. In a Galois covering of smooth curves over an algebraically closed field of characteristic zero, a point with nontrivial stabiliser is a ramification point; so $t(y) \in S$ by Lemma 3.7(1). But $t(y) \in \mathbb{P}^1(W)$ reduces to $\bar{u} \notin \bar{S}(\bar{\mathbb{F}}_p)$, a contradiction. Hence the n points have distinct reductions, the special fibre of Y_u has n geometric points and is étale, and the flat morphism t_W is étale at every point over $\bar{u}$. So ϖ' is étale.

Step 4: C_U and ϖ. On X'_U the function t is a unit and $d \in \mathcal{O}^\times$, so $\mathcal{O}_{X'_U}[s]/(s^d - t)$ is finite étale over X'_U, hence regular and normal; as C is a normalisation (Lemma 3.7), $C_U = X'_U \times_U [d]^{-1}(U)$ and $\varpi = \varpi' \times_U [d]$. As $[d]$ is finite étale over $U \subset \mathbb{G}_m$, ϖ is finite étale. This proves (1). Moreover $X'_U \subset Y'$ by Lemma 5.1, so E is an elliptic curve over X'_U and $\mathcal{H}$ is lisse.

Step 5: the action. We first show that $\mathrm{End}(E_\eta) = \mathbb{Z}$ at the generic point η of every connected component $\mathcal{Y}$ of X'_U; η lies in characteristic zero. Let $\alpha \in \mathrm{End}(E_\eta)$, let $H_\eta = H^1_{\mathrm{dR}}(E_\eta/\eta)$ with Hodge line ω_η and Gauss–Manin connection ∇. Then α^* preserves ω_η, is horizontal, and acts on ω_η and on H_η/ω_η by scalars $\beta_1, \beta_2 \in \kappa(\eta)$. For $x \in \omega_\eta$, projecting $\nabla(\alpha^* x) = d\beta_1 \otimes x + \beta_1 \nabla x$ and $\nabla(\alpha^* x) = (\alpha^* \otimes 1)\nabla x$ to $(H_\eta/\omega_\eta) \otimes \Omega^1$ gives $\beta_1 \,\mathrm{KS}(x) = \beta_2 \,\mathrm{KS}(x)$, where the Kodaira–Spencer map KS is nonzero by (M1); so $\beta_1 = \beta_2$, and α^* has the single eigenvalue β_1 on H_η. Comparing with Betti cohomology after embedding a field of definition of α in $\mathbb{C}$, the characteristic polynomial of α^* is $X^2 - \mathrm{tr}(\alpha)X + \deg \alpha \in \mathbb{Z}[X]$; so $2\beta_1 \in \mathbb{Z}$ and $\beta_1^2 \in \mathbb{Z}$, whence $\beta_1 \in \mathbb{Z}$. Then $\alpha - [\beta_1]$ kills ω_η; a nonzero endomorphism in characteristic zero is étale and injective on invariant differentials, so $\alpha = [\beta_1]$. Consequently, for $h \in G$ and isogenies $\psi, \psi' : E \to h^*E$ over $\mathcal{Y}$, $\psi'^\vee \circ \psi$ is an integer and $\psi = r\psi'$ in $\mathrm{Hom}(E, h^*E) \otimes \mathbb{Q}$ with $r \in \mathbb{Q}^\times$ (homomorphisms are determined at η, as E is flat and separated over the integral scheme $\mathcal{Y}$).

We next show that $g \mapsto g^\sharp$ is a right action. For $g, h \in G$ the isogenies $\psi = g^*\pi_h \circ \pi_g$ and

$\psi' = \pi_{hg}$ map E to $g^*h^*E = (hg)^*E$, and, in étale as in de Rham cohomology,

$$g^\sharp h^\sharp = c_g c_h \operatorname{Sym}^{k-2}(\psi^*) \circ (hg)^*, \qquad (hg)^\sharp = c_{hg} \operatorname{Sym}^{k-2}(\psi'^*) \circ (hg)^*.$$

On a component $\mathcal{Y}$ write $\psi = r\psi'$. Then $\operatorname{Sym}^{k-2}(\psi^*) = r^{k-2} \operatorname{Sym}^{k-2}(\psi'^*)$ in both theories, $\deg \psi = r^2 \deg \psi'$, and

$$(c_g c_h r^{k-2})^2 = (\deg \pi_g \deg \pi_h)^{-(k-2)} r^{2(k-2)} = (\deg \psi')^{-(k-2)} = c_{hg}^2,$$

so $c_g c_h r^{k-2} = \pm c_{hg}$, and $g^\sharp h^\sharp = \pm (hg)^\sharp$ over $\mathcal{Y}$, with the same sign in both theories. In de Rham cohomology (M2) gives $g^\sharp h^\sharp = (hg)^\sharp$, and $\operatorname{Sym}^{k-2}(\psi'^*)$ is injective there; so the sign is $+$, and $g^\sharp h^\sharp = (hg)^\sharp$ on $\mathcal{M}$. By Step 4, G acts on the factor X'_U of C_U and μ_d on the factor $[d]^{-1}(U)$, and $\rho^* \operatorname{Sym}^{k-2} \mathcal{H}$ is pulled back from the first factor; so the $\zeta^\sharp$ commute with the $g^\sharp$, and $(g, \zeta) \mapsto \zeta^\sharp g^\sharp$ is a right action of $\tilde{G}$. It is L_λ-linear because $c_g \in L$.

The values of χ_Φ lie in $L^\times$ ((M4) and $\mu_d \subset L$). For a character χ of $\tilde{G}$ with values in $L^\times$, $e_\chi = |\tilde{G}|^{-1} \sum_{\gamma \in \tilde{G}} \chi(\gamma)^{-1} \gamma^\sharp$ is an idempotent endomorphism of $\mathcal{M}$ with image the χ-part. As ϖ is finite étale, $\mathcal{M}$ is lisse, so $\mathcal{F}_a = e_{\chi_\Phi} \mathcal{M}$ and $\mathcal{F}_{\bar{a}} = e_{\chi_\Phi^{-1}} \mathcal{M}$ are lisse direct summands; likewise for $\mathbb{V}_{\chi_G^{\pm 1}}$ and $\mathcal{L}_{\pm j}$.

Step 6: the Kummer sheaves. By Step 4 and the projection formula for the finite étale maps ϖ' and $[d]$, $\mathcal{M} = \varpi'_* \operatorname{Sym}^{k-2} \mathcal{H} \otimes [d]_* L_\lambda$ on $\mathcal{U}$, with $g^\sharp$ acting on the first factor and $\zeta^\sharp$ on the second. The $(\chi_G^{\pm 1} \boxtimes \zeta^{\pm j})$-isotypic part of such a tensor product is the tensor product of the isotypic parts, which gives $\mathcal{F}_a \cong \mathbb{V}_{\chi_G} \otimes \mathcal{L}_j$ and $\mathcal{F}_{\bar{a}} \cong \mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j}$. The stalk of $[d]_* L_\lambda$ at a geometric point $\bar{x}$ is the space of L_λ-valued functions ϕ on $[d]^{-1}(\bar{x})$, with $(\zeta^\sharp \phi)(s) = \phi(\zeta s)$; as μ_d acts simply transitively on $[d]^{-1}(\bar{x})$, each $\mathcal{L}_{\pm j}$ has rank one. Let $\ell \neq p$ and $x \in \mathbb{F}_p^\times$; then $\mu_d \subset \mathbb{F}_p^\times$, as $\mathfrak{P}$ has degree one and $\mu_d \subset L$. The geometric Frobenius acts on the stalk by $(\mathrm{Frob}_x \phi)(s) = \phi(s^p)$ [16, (1.15)]. If $\phi(\zeta s) = \zeta^{\pm j} \phi(s)$ for all ζ, then $(\mathrm{Frob}_x \phi)(s) = \phi(\zeta' s)$ with $\zeta' = s^{p-1} = x^{(p-1)/d} \in \mu_d(\mathbb{F}_p)$, so $\mathrm{Frob}_x \phi = (x/\mathfrak{P})_d^{\pm j} \phi$, the root of unity ζ' being identified with $(x/\mathfrak{P})_d \in \mu_d(L)$ by reduction modulo $\mathfrak{P}$. This proves (2).

Step 7: duality. By (1) and Step 5, ϖ is finite étale and $(g, \zeta) \mapsto \zeta^\sharp g^\sharp$ is a right action of $\tilde{G}$ on $\mathcal{M}$. So Lemma 5.3(2) applies: the pairing of Lemma 5.3(1) on $\operatorname{Sym}^{k-2} \mathcal{H}$, composed with the trace of ϖ, is a perfect $\tilde{G}$-invariant pairing on $\mathcal{M}$ and restricts to a perfect pairing $\mathcal{F}_a \otimes \mathcal{F}_{\bar{a}} \to L_\lambda(-(k-2))$. This proves (3). The proof of Lemma 5.3 applies verbatim in Betti cohomology.

Step 8: comparison with the Betti eigensheaf. Let $V_\mathbb{Z} = R^1 \pi_{E*}^{\mathrm{an}} \mathbb{Z}$ on $X'_U(\mathbb{C})$ and put

$$\mathcal{M}_B = \varpi_*^{\mathrm{an}} \rho^* \operatorname{Sym}^{k-2} V_\mathbb{Z} \otimes L.$$

The actions of Step 5 and the invariant pairing of Step 7 are defined on $\mathcal{M}_B$ as well. Artin's comparison theorem [55, Exp. XVI, Theorem 4.1], applied first with coefficients $\mathbb{Z}/\ell^n$ and then by passage to the limit, gives

$$\mathcal{M}|_{U(\mathbb{C})} \cong \mathcal{M}_B \otimes_L L_\lambda$$

equivariantly for $\widetilde{G}$. This equivariance follows from functoriality for the isogenies π_g and the covering automorphisms.

Let $\mathcal{V} = (\mathcal{M}_B)^{\chi_\Phi^{-1}}$. Its complexification is the dual of the Betti eigensheaf $\mathbb{W}_\Phi$ of Lemma 3.10(3), by the pairing of Step 7. That lemma identifies $\mathbb{W}_\Phi^\vee$ with $\mathbb{V}_f \otimes \mathbb{L}_a$ and gives its rank $k-1$. Hence

$$\mathcal{F}_{\bar{a}}|_{U(\mathbb{C})} \cong \mathcal{V} \otimes_L L_\lambda, \qquad \mathcal{V} \otimes_L \mathbb{C} \cong \mathbb{V}_f \otimes \mathbb{L}_a.$$

Duality gives the same rank for $\mathcal{F}_a$. The sheaf $\mathcal{L}_{-j}$ corresponds to $\mathbb{L}_a$: both have the same monodromy about 0, with the conventions for the Kummer cover in Step 6. The tensor-product description in Step 6 therefore identifies the Betti counterpart of $\mathbb{V}_{\chi_G^{-1}}$ with an L-form of $\mathbb{V}_f$; in particular the two $\mathbb{V}_{\chi_G^{\pm 1}}$ also have rank $k-1$.

Apply the same comparison theorem to the constructible sheaf $o_*\mathcal{F}_{\bar{a}}$ on $\mathbb{P}^1_{\mathbb{C}}$. It gives

$$H^1(\mathbb{P}^1_{\mathbb{C}}, o_*\mathcal{F}_{\bar{a}}) \cong H^1(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}) \otimes_L L_\lambda.$$

After extension to $\mathbb{C}$, the group on the right is H_a, since $H^1(\mathbb{P}^1, o_*\mathbb{W})$ is parabolic cohomology by Lemma A.3(1). This proves (4).

Step 9: inertia at $t = 0$. Let $\ell \neq p$. Let $\infty' \in Z'(W)$, q' and w be as in (M5), $\bar{x}$ the reduction of ∞', and $G_{\bar{x}}$ the stabiliser of $\bar{x}$ in G. As Z' is étale over $\mathcal{O}$, a W-point of Z' is determined by its reduction, so $G_{\bar{x}}$ is also the stabiliser of ∞'. By (M3) the zeros of t lie in Z'. The sheaf $(t_*\mathcal{O}_{X'})^G$ is $\mathcal{O}_{\mathbb{P}^1}$: it is torsion-free and finite over the normal sheaf $\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}}}$, and equal to it after inverting p, by (M3). So G acts transitively on the points of $X'(\bar{\mathbb{F}}_p)$ over x_0 [12, Ch. V, §2, no. 2, Th. 2], and these points are the $g\bar{x}$.

Each $g \in G_{\bar{x}}$ acts on E near ∞' as a multiplication. Let $T \in t + t^2\mathbb{Z}[\frac{1}{M}][[t]]$ be the inverse of the q-expansion $t(q)$ (Definition 2.1), so $q = T(t)$. By (M5) and [21, (5.4.1)], $E \times_{X'} \operatorname{Spec} W[[q']]$ is $\operatorname{Tate}(q) \otimes_{\mathbb{Z}[[q]]} W[[q']]$ with $q = q'^w$; since $q = T(t)$ with $t \in W[[q']]$, this is the base change of the generalised elliptic curve $E_0 = \operatorname{Tate}(T(t))$ over $W[[t]]$. For $g \in G_{\bar{x}}$, the automorphism $\hat{g}$ of $W[[q']]$ induced by g is $W[[t]]$-linear ($t \circ g = t$), so $\hat{g}^*(E_0 \otimes W[[q']]) = E_0 \otimes W[[q']]$ canonically, and π_g, defined over Y', restricts to an endomorphism α_g of the elliptic curve $E_0 \otimes W[[q']][\frac{1}{q'}]$. Extend scalars to $\mathcal{A} = W((u))^\wedge$, $u = q'$, and use the basis ω, ξ of $H^1_{\mathrm{dR}}(\operatorname{Tate}(q)/\mathcal{A})$ given by Lemma 3.8 with $k = 3$, $\delta = 1$, $m = w$ (its hypotheses hold, as p is odd for $p \notin \Sigma$, and $p \nmid w$ by (M5)): $\nabla_\theta \omega = w\xi$, $\nabla_\theta \xi = 0$, and $\langle \omega, \xi \rangle \in \mathcal{A}^\times$, as the cup product is a perfect alternating pairing on this free module of rank two. Write $\alpha_g^*\omega = \beta_1\omega$ and $\alpha_g^*\xi = \beta_2\xi + \beta_3\omega$. Horizontality gives $\theta(\beta_1)\omega + \beta_1 w\xi = w(\beta_2\xi + \beta_3\omega)$, so $\beta_1 = \beta_2$ and $\theta(\beta_1) = w\beta_3$; the cup product gives $\beta_1\beta_2 = \deg \pi_g$, so β_1^2 is a constant, $\theta(\beta_1) = 0$ ($\mathcal{A}$ is a domain), $\beta_3 = 0$ and $\alpha_g^* = \beta_1 \cdot \mathrm{id}$. As in Step 5, $\beta_1 \in \mathbb{Z}$ and $\alpha_g - [\beta_1]$ vanishes over the characteristic-zero field $\operatorname{Frac}\mathcal{A}$, hence over $W[[q']][\frac{1}{q'}]$, and so also after reduction modulo p, over $\bar{\mathbb{F}}_p((q'))$. So $\alpha_g = [m_g]$ with $m_g \in \mathbb{Z}$, and $\deg \pi_g = m_g^2$.

Next, $\chi_G(g) = c_g m_g^{k-2}$ for $g \in G_{\bar{x}}$. Indeed, on $\operatorname{Spf} W[[q']]$ we have $f\omega^{\otimes(k-2)} = F(t)\,\omega_{\mathrm{can}}^{\otimes(k-2)}$ (M5), and $g^\sharp = c_g \operatorname{Sym}^{k-2}([m_g]^*) \circ \hat{g}^* = c_g m_g^{k-2}\hat{g}^*$ there, where $\hat{g}^*$ fixes $F(t)$ and the canonical differential ω_{can} of E_0; as $F(t) \neq 0$, (M4) gives the claim. In particular $\chi_G(g)^2 = c_g^2 m_g^{2(k-2)} = 1$ on $G_{\bar{x}}$.

Let K_0 be the fraction field of the strict henselisation of $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$ at x_0, so $I_{x_0} = \operatorname{Gal}(K_0^{\mathrm{sep}}/K_0)$, and $\hat{K}_0 = \bar{\mathbb{F}}_p((t))$ its completion. The valuation of K_0 extends uniquely to every finite separable extension L' (K_0 is henselian, [45, Ch. II, (6.6)]), so the minimal polynomial of a generator of L' stays irreducible over $\hat{K}_0$ [45, Ch. II, (8.2)]. Hence $\operatorname{Gal}(\hat{K}_0^{\mathrm{sep}}/\hat{K}_0) \to I_{x_0}$ is surjective, and it suffices to treat the restrictions of $\mathcal{F}_a$ and $\mathcal{F}_{\bar{a}}$ to $\operatorname{Spec}\hat{K}_0$. By [45, Ch. II, (8.3)], applied to the function field of $X'_{\bar{\mathbb{F}}_p}$, which is separable over $\bar{\mathbb{F}}_p(t)$ by Step 3, $X'_U \times_U \operatorname{Spec}\hat{K}_0$ is the disjoint union of the $\operatorname{Spec}\hat{K}_z$, $\hat{K}_z$ the completed function field at the points $z = g\bar{x}$, permuted transitively by G, with $G_{\bar{x}}$ the stabiliser of $z = \bar{x}$; and $\hat{K}_{\bar{x}} = \bar{\mathbb{F}}_p((q'))$. Let $\nu : \operatorname{Spec}\hat{K}_{\bar{x}} \to \operatorname{Spec}\hat{K}_0$ and $\mathcal{S}_0 = \operatorname{Sym}^{k-2} R^1(E_0)_*\mathbb{Q}_\ell$ on $\operatorname{Spec}\hat{K}_0$. By the base change above (reduced modulo p), $\operatorname{Sym}^{k-2}\mathcal{H}|_{\hat{K}_{\bar{x}}} = \nu^*\mathcal{S}_0$, and for $h \in G_{\bar{x}}$, $h^\sharp = c_h m_h^{k-2}(\mathrm{id} \otimes \hat{h}^*) = \chi_G(h)(\mathrm{id} \otimes \hat{h}^*)$ on $\nu_*\nu^*\mathcal{S}_0 = \mathcal{S}_0 \otimes \nu_*\mathbb{Q}_\ell$. Projection to the summand at $\bar{x}$ identifies the $\chi_G^{\pm 1}$-part of $(\varpi'_* \operatorname{Sym}^{k-2}\mathcal{H})|_{\hat{K}_0}$ with the $\chi_G^{\pm 1}|_{G_{\bar{x}}}$-part of $\mathcal{S}_0 \otimes \nu_*\mathbb{Q}_\ell$, which is $\mathcal{S}_0 \otimes (\nu_*\mathbb{Q}_\ell)^{G_{\bar{x}}}$ because $\chi_G = \chi_G^{-1}$ on $G_{\bar{x}}$. It contains $\mathcal{S}_0$ (the constants) and has rank $k-1 = \operatorname{rank}\mathcal{S}_0$ by Step 8; so $\mathbb{V}_{\chi_G^{\pm 1}}|_{\hat{K}_0} \cong \mathcal{S}_0 \otimes L_\lambda$.

The smooth locus of the generalised elliptic curve $E_0 \otimes \bar{\mathbb{F}}_p[[t]]$ is a separated smooth group scheme of finite type extending $E_{0,\hat{K}_0}$, with special fibre $\mathbb{G}_m$ ($T(t)$ has a simple zero, so the closed fibre is a Néron 1-gon), of unipotent rank zero. So $E_{0,\hat{K}_0}$ has semistable reduction [56, Exp. IX, Prop. 3.2 and Déf. 3.4], and $\mathrm{Gal}(\hat{K}_0^{\mathrm{sep}}/\hat{K}_0)$, which is the inertia group as the residue field is algebraically closed, acts unipotently on $T_\ell(E_0)$ [56, Exp. IX, Prop. 3.5], hence on $\mathcal{S}_0$. Since $X^d - t$ is Eisenstein, $\hat{K}_0(t^{1/d})/\hat{K}_0$ has degree d and group μ_d; choose γ with $\gamma(t^{1/d}) = \zeta_1 t^{1/d}$, ζ_1 a primitive d-th root of unity. Then γ acts on the stalk of $\mathcal{L}_{\pm j}$ by a primitive d-th root of unity, as $\gcd(j,d) = 1$, and this is $\neq 1$ as $d \geq 2$. Hence all eigenvalues of γ on $\mathcal{F}_a = \mathbb{V}_{\chi_G} \otimes \mathcal{L}_j$ and on $\mathcal{F}_{\bar{a}} = \mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j}$ differ from 1, so $\gamma - 1$ is invertible on both stalks, and both the invariants and the coinvariants vanish. This proves (5).

□

Lemma 5.5. *Let ℓ be a prime, λ a prime of L above ℓ, and Ω an algebraically closed field with a homomorphism $\mathcal{O}_{L,(\mathfrak{P})} \to \Omega$, of characteristic zero if $\ell = p$. Let $M = \rho^* \operatorname{Sym}^{k-2} R^1\pi_{E*}\mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell} L_\lambda$ on C_Ω°, for the structure map $\pi_E : E \to X'$, with the action of $\tilde{G}$ by the maps $\zeta^\sharp g^\sharp$; let $o' : C_\Omega^\circ \hookrightarrow C_\Omega$ be the inclusion; and for a character $\psi : \tilde{G} \to L^\times$ put $e_\psi = |\tilde{G}|^{-1} \sum_{\gamma \in \tilde{G}} \psi(\gamma)^{-1}\gamma^*$.*

1. *The natural map $H^1(C_\Omega, o'_*M) \to H^1(C_\Omega^\circ, M)$ is injective, and its image is the image of $H^1_c(C_\Omega^\circ, M)$.*
2. *$e_{\chi_\Phi} H^1(C_\Omega, o'_*M) = H^1(\mathbb{P}^1_\Omega, o_*\mathcal{F}_a)$ and $e_{\chi_\Phi^{-1}} H^1(C_\Omega, o'_*M) = H^1(\mathbb{P}^1_\Omega, o_*\mathcal{F}_{\bar{a}})$, compatibly with the automorphisms of Ω over $\mathcal{O}_{L,(\mathfrak{P})}$ and, for $\Omega = \bar{\mathbb{F}}_p$, with the geometric Frobenius.*

Proof. By Lemma 3.7(2), C_Ω is a proper smooth curve and $D_{C,\Omega}$ is finite; M is lisse on C_Ω°, as E is an elliptic curve over Y' and $\rho^{-1}(Y') = C^\circ$.

(1) The exact sequence $0 \to o'_!M \to o'_*M \to Q \to 0$, with Q supported on the finite set $D_{C,\Omega}$, gives $H^1(C_\Omega, Q) = 0$, so $H^1_c(C_\Omega^\circ, M) = H^1(C_\Omega, o'_!M) \to H^1(C_\Omega, o'_*M)$ is surjective. The Leray spectral sequence of o' makes $H^1(C_\Omega, o'_*M) \to H^1(C_\Omega^\circ, M)$ injective. The composite of the two maps is the natural map.

(2) Let $\bar{\varpi} : C_\Omega \to \mathbb{P}^1_\Omega$ be the finite map extending ϖ (Lemma 5.4(1)) and $o'' : C_{U,\Omega} \hookrightarrow C_\Omega$ the inclusion; $C_U \subset C^\circ$ by Lemma 5.1. Both $o_*\varpi_*$ and $\bar{\varpi}_* o''_*$ are the push-forward from $C_{U,\Omega}$ to $\mathbb{P}^1_\Omega$; $\bar{\varpi}_*$ is exact, as $\bar{\varpi}$ is finite; and $o''_*(M|_{C_{U,\Omega}}) = o'_*M$, as M is lisse on the normal curve $C_\Omega^\circ \supset C_{U,\Omega}$. Hence $H^1(\mathbb{P}^1_\Omega, o_*\varpi_*(M|_{C_U})) = H^1(C_\Omega, o'_*M)$, compatibly with the automorphisms of Ω over $\mathcal{O}_{L,(\mathfrak{P})}$, with the geometric Frobenius when $\Omega = \bar{\mathbb{F}}_p$, and with the actions of $\tilde{G}$, which on both sides are induced by the pairs $(\gamma, \gamma^\sharp)$, all defined over $\mathcal{O}_{L,(\mathfrak{P})}$. Now $\varpi_*(M|_{C_U})$ is the sheaf $\mathcal{M}$, whose χ_Φ- and χ_Φ^{-1}-eigensheaves are $\mathcal{F}_a$ and $\mathcal{F}_{\bar{a}}$, and e_ψ is an idempotent commuting with o_* and with cohomology. Applying $e_{\chi_\Phi^{\pm 1}}$ gives (2).

□

5.2 Twisted Lefschetz formulas

We compute the two traces on the same cover by applying the Lefschetz formula to Frobenius composed with each group element. At an interior fixed point the crystalline and étale terms come from the same endomorphism of an elliptic curve [31, Theorem 2(2)]. At the cusps the Tate curve gives the local terms. Averaging with the character χ_Φ then isolates the two summands in Proposition 5.6.

Proposition 5.6. *Assume that the datum is admissible. Let $H_{a,p}$ be its eigenlattice, with crystalline Frobenius φ, and $\iota_{\mathfrak{P}} : L \to \mathbb{Q}_p$ the embedding defined by $\mathfrak{P}$. Let $\ell \neq p$ be a prime and*

λ a prime of L above ℓ, and let V_a, $V_{\bar a}$ be the cohomology over $\bar{\mathbb{F}}_p$ of the Kummer eigensheaves of the model, with the geometric Frobenius F. Let $S(\mathbb{F}_p)$ be the set of $\mathbb{F}_p$-rational points of the reduction of S through $\mathfrak{P}$, and $o : U \hookrightarrow \mathbb{P}^1$ the inclusion of the complement U of the closure of S.

1. *For every $n \geq 1$, $\operatorname{Tr}(F^n \mid V_{\bar a}) \in L$ and*

$$\iota_{\mathfrak{P}} \operatorname{Tr}(F^n \mid V_{\bar a}) = \operatorname{Tr}\big((p^{k-1}\varphi^{-1})^n \mid H_{a,p} \otimes \mathbb{Q}_p\big).$$

2. *$\dim V_a = \dim V_{\bar a} = 2$, and there are $A_{\mathfrak{P}} \in \mathcal{O}_L$ and $\varepsilon_{\mathfrak{P}} \in L^\times$ with*

$$\det(X - F \mid V_{\bar a}) = X^2 - A_{\mathfrak{P}} X + \varepsilon_{\mathfrak{P}}\, p^{k-1},$$

   whose image under $\iota_{\mathfrak{P}}$ is the characteristic polynomial of $p^{k-1}\varphi^{-1}$ on $H_{a,p}$. The characteristic polynomial of F on V_a is $X^2 - (A_{\mathfrak{P}}/\varepsilon_{\mathfrak{P}})X + \varepsilon_{\mathfrak{P}}^{-1} p^{k-1}$, and its image under $\iota_{\mathfrak{P}}$ is the characteristic polynomial of φ on $H_{a,p}$.

3. *With the d-th power residue symbol $(x/\mathfrak{P})_d$,*

$$A_{\mathfrak{P}} = -\sum_{x \in \mathbb{F}_p \setminus S(\mathbb{F}_p)} \Big(\frac{x}{\mathfrak{P}}\Big)_d^{-j} \operatorname{Tr}\big(\mathrm{Frob}_x \mid \mathbb{V}_{\chi_G^{-1},\bar x}\big) - \sum_{x \in S(\mathbb{F}_p)} \operatorname{Tr}\big(\mathrm{Frob}_x \mid (o_*\mathcal{F}_{\bar a})_{\bar x}\big).$$

4. *The eigenvalues of F on $V_{\bar a}$ have absolute value $p^{(k-1)/2}$ under every complex embedding; hence $|\sigma(A_{\mathfrak{P}})| \leq 2p^{(k-1)/2}$ and $|\sigma(\varepsilon_{\mathfrak{P}})| = 1$ for every embedding $\sigma : L \to \mathbb{C}$.*

5. *Let $x \in S(\mathbb{F}_p)$, let $\bar\eta$ be a geometric generic point of $U_{\bar{\mathbb{F}}_p}$, and let $I_x \subset \pi_1(U_{\bar{\mathbb{F}}_p}, \bar\eta)$ be an inertia group at x. Then $(o_*\mathcal{F}_{\bar a})_{\bar x}$ is the space of invariants $(\mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j})_{\bar\eta}^{I_x}$, on which Frob_x acts through a Frobenius element of a decomposition group at x.*

6. *Let a bar denote complex conjugation on $L \subset \mathbb{C}$. Then $\overline{A_{\mathfrak{P}}} = A_{\mathfrak{P}}/\varepsilon_{\mathfrak{P}}$ and $\overline{\varepsilon_{\mathfrak{P}}} = \varepsilon_{\mathfrak{P}}^{-1}$; so the characteristic polynomial of φ on $H_{a,p}$ is the image under $\iota_{\mathfrak{P}}$ of $X^2 - \overline{A_{\mathfrak{P}}}\, X + \overline{\varepsilon_{\mathfrak{P}}}\, p^{k-1}$.*

Proof. We first establish the equality of traces in (1) using the twisted Lefschetz formulas of Lemma 5.7 and the local comparison of Lemma 5.8. We then recover the common polynomial, the point count and the bounds on its roots.

Let $\chi = \chi_\Phi$, valued in $L^\times$, and for a character ψ of $\tilde G$ let $e_\psi = |\tilde G|^{-1} \sum_{g \in \tilde G} \psi(g)^{-1} g^*$, which projects onto the ψ-eigenspace of a right action. We use the notation H_{cris}, M_ℓ, H_ℓ, $\mathrm{Fix}_q(g)$, Ψ of Lemma 5.7.

Step 1. Both sides are eigenspaces in the cohomology of the cover. By definition (Lemma 3.10(2)), $H_{a,p} \otimes \mathbb{Q}_p = e_\chi H_{\mathrm{cris}}$, χ acting through $\iota_{\mathfrak{P}}$. On the ℓ-adic side, Lemma 5.5(2) with $\Omega = \bar{\mathbb{F}}_p$ gives $V_a = e_\chi H_\ell$ and $V_{\bar a} = e_{\chi^{-1}} H_\ell$, compatibly with F.

Step 2. The twisted traces agree. Let $n \geq 1$, $q = p^n$ and $g \in \tilde G$. By Lemma 5.7(2) and Lemma 5.8,

$$\begin{aligned}
\operatorname{Tr}\big(g^*\varphi^{-n} \mid H_{\mathrm{cris}}\big) &= -q^{-1} \sum_{y \in \mathrm{Fix}_q(g)} \operatorname{Tr}\big((\Psi^{\mathrm{cris}}_{g,y})^{-1}\big) \\
&= -q^{-(k-1)}\, \iota_{\mathfrak{P}} \sum_{y \in \mathrm{Fix}_q(g)} \operatorname{Tr}\big(\Psi^{\ell}_{g,y}\big) \\
&= q^{-(k-1)}\, \iota_{\mathfrak{P}} \operatorname{Tr}\big(F^n (g^{-1})^* \mid H_\ell\big),
\end{aligned}$$

and $\operatorname{Tr}(F^n(g^{-1})^* \mid H_\ell) \in L$.

Step 3. Averaging over $\tilde{G}$ gives (1). As e_χ is an idempotent commuting with φ, and $e_{\chi^{-1}}$ one commuting with F,

$$\begin{aligned}\operatorname{Tr}(\varphi^{-n} \mid H_{a,p} \otimes \mathbb{Q}_p) &= \frac{1}{|\tilde{G}|} \sum_g \iota_{\mathfrak{P}} \chi(g)^{-1} \operatorname{Tr}(g^* \varphi^{-n} \mid H_{\mathrm{cris}}) \\ &= \frac{q^{-(k-1)}}{|\tilde{G}|} \iota_{\mathfrak{P}} \sum_g \chi(g) \operatorname{Tr}(F^n g^* \mid H_\ell) \\ &= q^{-(k-1)} \iota_{\mathfrak{P}} \operatorname{Tr}(F^n \mid V_{\bar{a}}),\end{aligned}$$

where the middle step substitutes $g \mapsto g^{-1}$ in Step 2, and the last uses $e_{\chi^{-1}} = |\tilde{G}|^{-1} \sum_g \chi(g) g^*$ and Step 1. Multiplying by $q^{k-1} = p^{n(k-1)}$ gives (1); $\operatorname{Tr}(F^n \mid V_{\bar{a}}) \in L$ by Step 2.

Step 4. The power sums determine the characteristic polynomials, which gives (2). By Lemma 3.10(3) and admissibility, $H_{a,p} \otimes \mathbb{Q}_p$ has dimension 2, and φ is bijective on it ([48], Theorem 5.2(ii), with Lemma 5.7(1)); F is bijective on $V_{\bar{a}}$. For an invertible endomorphism u of a finite-dimensional vector space over a field of characteristic zero,

$$\det(1 - Tu) = \exp\Big(-\sum_{m \geq 1} \operatorname{Tr}(u^m) T^m / m\Big)$$

is a polynomial of degree the dimension. By (1) the power sums of F on $V_{\bar{a}}$ lie in L, so $\det(1 - TF \mid V_{\bar{a}}) \in L[T]$, and its image under $\iota_{\mathfrak{P}}$ is $\det(1 - T\, p^{k-1}\varphi^{-1} \mid H_{a,p} \otimes \mathbb{Q}_p)$. Comparing degrees, $\dim V_{\bar{a}} = 2$, and $\det(X - F \mid V_{\bar{a}}) = X^2 - A_{\mathfrak{P}} X + \varepsilon_{\mathfrak{P}} p^{k-1}$ with $A_{\mathfrak{P}} \in L$ and $\varepsilon_{\mathfrak{P}} \in L^\times$ (the constant term is nonzero as F is bijective), whose image under $\iota_{\mathfrak{P}}$ is the characteristic polynomial of $p^{k-1}\varphi^{-1}$. By Lemma 5.4(3) and the duality theorem [16], Théorème (2.12), $V_a \cong V_{\bar{a}}^\vee(-(k-1))$ compatibly with F , so $\dim V_a = 2$ and the eigenvalues of F on V_a are p^{k-1}/α', p^{k-1}/β', where α', β' are those on $V_{\bar{a}}$; their sum is $p^{k-1} A_{\mathfrak{P}} / (\varepsilon_{\mathfrak{P}} p^{k-1}) = A_{\mathfrak{P}} / \varepsilon_{\mathfrak{P}}$ and their product $p^{2(k-1)}/(\varepsilon_{\mathfrak{P}} p^{k-1}) = \varepsilon_{\mathfrak{P}}^{-1} p^{k-1}$. The eigenvalues of φ are likewise p^{k-1} divided by those of $p^{k-1}\varphi^{-1}$, so $\iota_{\mathfrak{P}}$ maps the characteristic polynomial of F on V_a to that of φ.

Step 5. Grothendieck's trace formula on the line gives (3), once $H^i(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_* \mathcal{F}_{\bar{a}}) = 0$ for $i \neq 1$. For $i > 2$ this holds because $\mathbb{P}^1$ is a curve [16], (1.10). Let $\bar{\eta}$ be a geometric generic point of $U_{\bar{\mathbb{F}}_p}$ and x_0 a geometric point of $\mathbb{P}^1_{\mathbb{F}_p}$ over $t = 0$, with inertia group I_{x_0}; by Lemma 5.4(5), I_{x_0} has neither nonzero invariants nor nonzero coinvariants on $\mathcal{F}_{\bar{a},\bar{\eta}}$. By [17, (1.4.1.1), (1.4.1.3)], $H^0(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_* \mathcal{F}_{\bar{a}})$ is the space of invariants and $H^2(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_* \mathcal{F}_{\bar{a}})$ the Tate twist of the space of coinvariants of $\pi_1(U_{\bar{\mathbb{F}}_p}, \bar{\eta})$ on $\mathcal{F}_{\bar{a},\bar{\eta}}$; both vanish, as the invariants and the coinvariants of I_{x_0} do. Now Grothendieck's trace formula [17, (1.4.5.1)], in its trace form over $\mathbb{F}_p$, applied to the constructible L_λ-sheaf $o_* \mathcal{F}_{\bar{a}}$ on the proper curve $\mathbb{P}^1_{\mathbb{F}_p}$, gives

$$A_{\mathfrak{P}} = \operatorname{Tr}(F \mid V_{\bar{a}}) = -\sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} \operatorname{Tr}(\mathrm{Frob}_x \mid (o_* \mathcal{F}_{\bar{a}})_{\bar{x}}).$$

As $0, \infty \in S$, $U(\mathbb{F}_p) = \mathbb{F}_p \setminus S(\mathbb{F}_p)$, and by Lemma 5.4(2) the stalk of $\mathcal{F}_{\bar{a}} = \mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j}$ at $x \in U(\mathbb{F}_p)$ has trace $(x/\mathfrak{P})_d^{-j} \operatorname{Tr}(\mathrm{Frob}_x \mid \mathbb{V}_{\chi_G^{-1}, \bar{x}})$. Splitting the sum into $x \in U(\mathbb{F}_p)$ and $x \in S(\mathbb{F}_p)$ gives (3).

Step 6. $A_{\mathfrak{P}}$ is integral. By Step 5, $A_{\mathfrak{P}} = -\sum_{x \in \mathbb{P}^1(\mathbb{F}_p)} \operatorname{Tr}(\mathrm{Frob}_x \mid (o_* \mathcal{F}_{\bar{a}})_{\bar{x}})$, and $A_{\mathfrak{P}} \in L$ by Step 4; so it suffices that each eigenvalue of each Frob_x on $(o_* \mathcal{F}_{\bar{a}})_{\bar{x}}$ be an algebraic integer. This stalk is a Frob_x-stable subspace of $(o_* \varpi_* M_\ell)_{\bar{x}} = (\bar{\varpi}_* o_* M_\ell)_{\bar{x}} = \bigoplus_{y \in \bar{\varpi}^{-1}(\bar{x})} (o_* M_\ell)_{\bar{y}}$ (proper base change for the finite $\bar{\varpi}$), on which Frob_x permutes the summands. If y lies in an orbit of length r, Frob_x^r acts on $(o_* M_\ell)_{\bar{y}}$ as the geometric Frobenius of the $\mathbb{F}_{p^r}$-rational point y: for

$y \notin D_C$ as Sym^{k-2} of the Frobenius of the elliptic curve $E_{\rho(y)}$ over $\mathbb{F}_{p^r}$, whose eigenvalues are algebraic integers of absolute value $p^{r/2}$ [58], Theorem V.1.1 and V.2.3.1; for $y \in D_C$ trivially, the Galois group acting trivially on the invariant line (proof of Lemma 5.8, at the cusps). So every eigenvalue of Frob_x on the direct sum is an r-th root of an algebraic integer, hence an algebraic integer.

Step 7. Purity gives the weights in (4). The same description shows that for every $r \geq 1$ and $x \in U(\mathbb{F}_{p^r})$ every eigenvalue of Frob_x on $\mathcal{F}_{\bar{a},\bar{x}}$ has absolute value $p^{r(k-2)/2}$ under every complex embedding: $\mathcal{F}_{\bar{a}}$ is pointwise pure of weight $k-2$. By [17], Théorème (3.2.3), applied for every isomorphism $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$, the eigenvalues α', β' of F on $V_{\bar{a}} = H^1(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_*\mathcal{F}_{\bar{a}})$ have absolute value $p^{(k-1)/2}$ under every complex embedding of $\mathbb{Q}(\alpha', \beta')$. Extending σ to $L(\alpha', \beta')$, $|\sigma(A_{\mathfrak{P}})| = |\sigma\alpha' + \sigma\beta'| \leq 2p^{(k-1)/2}$ and $|\sigma(\varepsilon_{\mathfrak{P}})| = |\sigma(\alpha'\beta')|/p^{k-1} = 1$.

Step 8. The stalks at the points of S, (5). Let D be the reduction of $\bar{S}$, a finite set of closed points of $\mathbb{P}^1_{\mathbb{F}_p}$ because $\bar{S}$ is finite and étale over $\mathcal{O}$ (Lemma 5.4(1)), so that $U_{\mathbb{F}_p} = \mathbb{P}^1_{\mathbb{F}_p} \setminus D$, and let $x \in S(\mathbb{F}_p)$, a point of D. The stalk of $o_*\mathcal{F}_{\bar{a}}$ at $\bar{x}$ is the space of invariants of a local monodromy group [16], (2.11): the strict henselisation of $\mathbb{P}^1_{\bar{\mathbb{F}}_p}$ at x is a henselian discrete valuation ring whose fraction field has absolute Galois group I_x, so $(o_*\mathcal{F}_{\bar{a}})_{\bar{x}} = \mathcal{F}^{I_x}_{\bar{a},\bar{\eta}}$; as x is $\mathbb{F}_p$-rational, Frob_x acts on it through a Frobenius element of a decomposition group at x, which normalises I_x. The isomorphism $\mathcal{F}_{\bar{a}} \cong \mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j}$ of Lemma 5.4(2) on $U_{\mathbb{F}_p}$ identifies this space, compatibly with Frob_x, with $(\mathbb{V}_{\chi_G^{-1}} \otimes \mathcal{L}_{-j})^{I_x}_{\bar{\eta}}$. This proves (5).

Step 9. Complex conjugation, (6). The field L is generated over $\mathbb{Q}$ by roots of unity and square roots of integers (Definition 2.3), so it is abelian over $\mathbb{Q}$. Let $\sigma : L \to \mathbb{C}$ be an embedding. As $L \subset \mathbb{C}$ is Galois over $\mathbb{Q}$, σ is the restriction of some $\tau \in \mathrm{Gal}(L/\mathbb{Q})$, and complex conjugation restricts to an automorphism $\xi \mapsto \bar{\xi}$ of L; since $\mathrm{Gal}(L/\mathbb{Q})$ is abelian,

$$\overline{\sigma(\xi)} = \overline{\tau(\xi)} = \tau(\bar{\xi}) = \sigma(\bar{\xi}) \qquad (\xi \in L).$$

Let α', β' be the roots of $X^2 - A_{\mathfrak{P}}X + \varepsilon_{\mathfrak{P}}p^{k-1}$, the eigenvalues of F on $V_{\bar{a}}$, and extend σ to an embedding $\tilde{\sigma}$ of $L(\alpha', \beta')$ in $\mathbb{C}$. By (4), $|\tilde{\sigma}\alpha'|^2 = |\tilde{\sigma}\beta'|^2 = p^{k-1}$, so $\overline{\tilde{\sigma}\alpha'} = p^{k-1}/\tilde{\sigma}\alpha'$ and $\overline{\tilde{\sigma}\beta'} = p^{k-1}/\tilde{\sigma}\beta'$. As $\tilde{\sigma}\alpha' + \tilde{\sigma}\beta' = \sigma(A_{\mathfrak{P}})$ and $\tilde{\sigma}\alpha'\,\tilde{\sigma}\beta' = \sigma(\varepsilon_{\mathfrak{P}})p^{k-1}$,

$$\sigma\big(\overline{A_{\mathfrak{P}}}\big) = \frac{p^{k-1}}{\tilde{\sigma}\alpha'} + \frac{p^{k-1}}{\tilde{\sigma}\beta'} = \frac{\sigma(A_{\mathfrak{P}})}{\sigma(\varepsilon_{\mathfrak{P}})}, \qquad \sigma\big(\overline{\varepsilon_{\mathfrak{P}}}\big) = \frac{\overline{\tilde{\sigma}\alpha'}\;\overline{\tilde{\sigma}\beta'}}{p^{k-1}} = \frac{1}{\sigma(\varepsilon_{\mathfrak{P}})}.$$

As σ is injective, $\overline{A_{\mathfrak{P}}} = A_{\mathfrak{P}}/\varepsilon_{\mathfrak{P}}$ and $\overline{\varepsilon_{\mathfrak{P}}} = \varepsilon_{\mathfrak{P}}^{-1}$. The last assertion of (6) is then the second sentence of (2).

□

Lemma 5.7. *Let $H_{\mathrm{cris}} \subset H_{\mathbb{Z}} \otimes \mathbb{Q}_p$ be the subspace of classes all of whose residues* Res_z *vanish, with the action $x \mapsto g^*x$ of $\tilde{G}$ of Lemma 3.10(1). Let $\widehat{C}$ be the p-adic completion of $C \otimes \mathbb{Z}_p$, regard M_k as an F-crystal with logarithmic singularities on $(\widehat{C}, \widehat{D}_C)$ in the sense of [48], 1.5, and let $\Omega^\bullet(M_k)$ be the complex of loc. cit., 2.5. Put $C_0 = C \otimes \mathbb{F}_p$, $C_0^\circ = C_0 \setminus D_C$ and $o : C_0^\circ \hookrightarrow C_0$. Let $\ell \neq p$ be a prime, λ a prime of L above ℓ, $M_\ell = \rho^* \mathrm{Sym}^{k-2} R^1\pi_{E*}\mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell} L_\lambda$ on C_0°, for the structure map $\pi_E : E \to X'$, with the action of $\tilde{G}$ by the maps $\zeta^\sharp g^\sharp$, where $g^\sharp = c_g \mathrm{Sym}^{k-2}(\pi_g^*) \circ g^*$ and $\zeta^\sharp = [\zeta]^*$, and $H_\ell = H^1(C_{0,\bar{\mathbb{F}}_p}, o_*M_\ell)$, with the geometric Frobenius F and the induced right action $g \mapsto g^*$. For $n \geq 1$ put $q = p^n$, let Fr_q be the q-power Frobenius endomorphism of $C_{\bar{\mathbb{F}}_p}$, and for $g \in \tilde{G}$ put*

$$\mathrm{Fix}_q(g) = \{y \in C(\bar{\mathbb{F}}_p) : \ \mathrm{Fr}_q(y) = g(y)\}.$$

1. *$H^i(\widehat{C}, \Omega^\bullet(M_k)) \otimes \mathbb{Q}_p = 0$ and $H^i(C_{0,\bar{\mathbb{F}}_p}, o_*M_\ell) = 0$ for $i \neq 1$, and $H^1(\widehat{C}, \Omega^\bullet(M_k)) \otimes \mathbb{Q}_p = H_{\mathrm{cris}}$.*

2. *Let $n \geq 1$, $q = p^n$ and $g \in \tilde{G}$. There are a smooth projective curve $C^{(g)}$ over $W(\mathbb{F}_q)$ with a divisor $D^{(g)}$ finite étale over $W(\mathbb{F}_q)$, an F-crystal with logarithmic singularities $M^{(g)}$ on $(\widehat{C^{(g)}}, \widehat{D^{(g)}})$ satisfying the conditions 1.7 of loc. cit., and a lisse L_λ-sheaf $M_\ell^{(g)}$ on $C^{(g)}_{\mathbb{F}_q} \setminus D^{(g)}$, which become (C, D_C, M_k) and (C_0°, M_ℓ) after base change to $W(\bar{\mathbb{F}}_p)$, respectively $\bar{\mathbb{F}}_p$, in such a way that the q-power Frobenius endomorphism of $C^{(g)}_{\mathbb{F}_q}$ becomes $g^{-1} \circ \mathrm{Fr}_q$, so that $C^{(g)}(\mathbb{F}_q) = \mathrm{Fix}_q(g)$, and the q-linear Frobenius on the cohomology of the twisted objects becomes $\varphi^n(g^{-1})^*$ on H_{cris} and $F^n(g^{-1})^*$ on H_ℓ. For $y \in \mathrm{Fix}_q(g)$ let $\Psi^{\mathrm{cris}}_{g,y}$ be the q-linear Frobenius at y of $M^{(g)}$ on $B^{\mathrm{cris}}_y = \mathrm{coker}(R_y) \otimes \mathbb{Q}$, where R_y is the residue map of $M^{(g)}$ at y ($R_y = 0$ if $y \notin D_C$), and $\Psi^\ell_{g,y}$ the geometric Frobenius at y on $B^\ell_y = (o_* M_\ell^{(g)})_{\bar{y}}$. Then $\mathrm{Fix}_q(g)$ is finite and*

$$\mathrm{Tr}(g^* \varphi^{-n} \mid H_{\mathrm{cris}}) = -q^{-1} \sum_{y \in \mathrm{Fix}_q(g)} \mathrm{Tr}((\Psi^{\mathrm{cris}}_{g,y})^{-1} \mid B^{\mathrm{cris}}_y),$$
$$\mathrm{Tr}(F^n(g^{-1})^* \mid H_\ell) = - \sum_{y \in \mathrm{Fix}_q(g)} \mathrm{Tr}(\Psi^\ell_{g,y} \mid B^\ell_y).$$

3. *Let $y \in \mathrm{Fix}_q(g) \setminus D_C$, $g = (h, \zeta)$, and $x = \rho(y)$, so that $\mathrm{Fr}_q(x) = h(x)$. After extending scalars to $W(\kappa(x))$, $\Psi^{\mathrm{cris}}_{g,y}$ becomes $c_h \,\mathrm{Sym}^{k-2}(u^*_{g,y})$ on $\mathrm{Sym}^{k-2} H^1_{\mathrm{cris}}(E_x/W(\kappa(x))) \otimes \mathbb{Q}$, and $\Psi^\ell_{g,y}$ is $c_h \,\mathrm{Sym}^{k-2}(u^*_{g,y})$ on $\mathrm{Sym}^{k-2} H^1(E_{\bar{x}}, \mathbb{Q}_\ell) \otimes L_\lambda$, where*

$$u_{g,y} = \hat{\pi}_{h,x} \circ F_{x,q} \in \mathrm{End}_{\kappa(x)}(E_x),$$

$F_{x,q} : E_x \to E_x^{(q)} = E_{h(x)}$ is the relative q-power Frobenius, $\hat{\pi}_{h,x} : E_{h(x)} \to E_x$ is the dual of the fibre at x of the isogeny $\pi_h : E \to h^* E$ of (M2), and c_h is the constant of (M2).

Proof. The formulas of (2) are Scholl's and Grothendieck's trace formulas, applied to the forms of (C, D_C, M_k) and of M_ℓ twisted by g, on a curve whose $\mathbb{F}_q$-points are $\mathrm{Fix}_q(g)$; this twisting (Step 3, after Scholl) is the idea of the proof. Write K for the log de Rham complex $M_k \to M_k \otimes \Omega^1_{C/\mathbb{Z}_p}(\log D_C)$ on $C_{\mathbb{Z}_p}$, so that $H_{\mathbb{Z}} = \mathbb{H}^1(C_{\mathbb{Z}_p}, K)$ is the twisted crystal.

Step 1: the crystalline side of (1). The terms of $\Omega^\bullet(M_k)$ and of K are coherent on the proper $\mathbb{Z}_p$-scheme $C_{\mathbb{Z}_p}$, so by [21], (4.1.5), and the hypercohomology spectral sequences, $\mathbb{H}^i(\widehat{C}, \widehat{K}) = \mathbb{H}^i(C_{\mathbb{Z}_p}, K)$, and likewise for $\Omega^\bullet(M_k)$. The residues of ∇ along D_C are nilpotent (Lemma 3.8, and (M1)), M_k is locally free, and its Frobenius is an isogeny, so M_k satisfies the conditions 1.7 of [48]. The exact sequence $0 \to \Omega^\bullet(M_k) \to \widehat{K} \to \mathrm{coker}(R_{D_C})[-1] \to 0$ of loc. cit., 2.5 and (5.6.1), gives, after tensoring with $\mathbb{Q}$,

$$0 \to H^0(\Omega^\bullet) \to H^0(K) \to 0,$$
$$0 \to H^1(\Omega^\bullet) \to H^1(K) \xrightarrow{r} \mathrm{coker}(R_{D_C}) \to H^2(\Omega^\bullet) \to H^2(K) \to 0,$$

where r is induced by $\mathrm{id} \otimes \mathrm{res}_{D_C}$. At $z \in D_C$ the cokernel of the residue is spanned by e_0 and r is Res_z (proof of Lemma 3.10(3), Reduction); so $H^1(\Omega^\bullet) \otimes \mathbb{Q} = H_{\mathrm{cris}}$. Scholl's endomorphism F of these groups (loc. cit., 2.3) is the crystalline Frobenius φ: both are computed on a Čech–de Rham complex of an affine open cover of $\widehat{C}$ with admissible (that is, logarithmic) local Frobenius lifts, glued by the homotopies $L(\phi', \phi)$ of loc. cit., 2.1, which is the description of the logarithmic crystalline Frobenius through a log-smooth lifting [29], §6, used in Lemma 3.9(1),(3). Likewise g^* is the action of loc. cit., 7.2, for the pairs $(g, g^\sharp)$.

For H^0, the proof of Lemma 3.10(1) shows that $M_k \otimes \mathbb{F}_p$ has no nonzero horizontal global sections, that is $H^0(C_{\mathbb{F}_p}, K/p) = 0$. As $H^0(K)$ is a finitely generated $\mathbb{Z}_p$-module and $H^0(K)/p \hookrightarrow H^0(K/p)$, Nakayama's lemma gives $H^0(K) = 0$, hence $H^0(\Omega^\bullet) = 0$.

For H^2 we pass to $\mathbb{C}$. Fix a field embedding $\iota : \mathbb{Q}_p \to \mathbb{C}$ that is the identity on L. Coherent cohomology commutes with flat base change, so it suffices to show that $\mathbb{H}^2(C_{\mathbb{C}}, K) = 0$ and that $r \otimes_\iota \mathbb{C}$ is surjective. By [15], II, §§5–6, as in the proof of Lemma 3.10(3) (Topology), $\mathbb{H}^i(C_{\mathbb{C}}, K) = H^i(C^{\mathrm{an}} \setminus D_C, \mathrm{Sym}^{k-2}\mathbb{V})$ with $\mathbb{V} = R^1\pi_{E*}\mathbb{C}$, and $r \otimes \mathbb{C}$ is the restriction to the punctured discs Δ_z^* around the points of D_C. Every connected component of $C_{\mathbb{C}}$ meets D_C: $t \circ \rho$ is non-constant on it (M3) and its zeros lie in $\rho^{-1}(Z')$. So each component of $C^{\mathrm{an}} \setminus D_C$ is a non-compact Riemann surface, on which H^2 of a local system vanishes. The cokernel of the restriction $H^1(C^{\mathrm{an}} \setminus D_C, \mathrm{Sym}^{k-2}\mathbb{V}) \to \bigoplus_z H^1(\Delta_z^*, \mathrm{Sym}^{k-2}\mathbb{V})$ injects into $H^2_c(C^{\mathrm{an}} \setminus D_C, \mathrm{Sym}^{k-2}\mathbb{V})$, which is dual to $H^0(C^{\mathrm{an}} \setminus D_C, \mathrm{Sym}^{k-2}\mathbb{V}^\vee)$ by Poincaré duality. This vanishes: $\mathbb{V}^\vee \cong \mathbb{V}$ by the Weil pairing, and on each component the monodromy of $\mathrm{Sym}^{k-2}\mathbb{V}$ contains Sym^{k-2} of a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$, Zariski dense in SL_2, which has no nonzero invariants as $k \geq 3$ (proof of Lemma 3.10(3), Descent). So $H^2(\Omega^\bullet) \otimes \mathbb{Q} = 0$.

Step 2: the ℓ-adic side of (1). We first show $H^0(C_{0,\bar{\mathbb{F}}_p}, o_*M_\ell) = H^0(C^\circ_{0,\bar{\mathbb{F}}_p}, M_\ell) = 0$. Let V be the strict henselisation of $\mathcal{O}_{L,(\mathfrak{P})}$ and $\bar\eta$ a geometric generic point of $\operatorname{Spec} V$. The sheaf M_ℓ is the restriction to the special fibre of the sheaf $\rho^* \mathrm{Sym}^{k-2} R^1\pi_{E*}\mathbb{Q}_\ell \otimes L_\lambda$ on $C^\circ_V = (C \setminus D_C)_V$, which has the lattice $\rho^* \mathrm{Sym}^{k-2} R^1\pi_{E*}\mathbb{Z}_\ell \otimes \mathcal{O}_\lambda$; let M_n be its reduction modulo λ^n, a locally constant constructible sheaf. The morphism $C^\circ_V \to \operatorname{Spec} V$ is smooth, hence locally acyclic relative to M_n [55], Exp. XV. The M_n are tamely ramified along the relative normal crossings divisor $D_{C,V}$, finite étale over V: near D_C they are trivialised by adjoining ℓ^n-th roots of the Tate parameter of (M1), a Kummer covering of degree prime to p. So by [54], Th. finitude, Appendice, 1.3.3, the formation of Ro_*M_n commutes with base change, and by the proper base change theorem so does the formation of $R\Gamma(C^\circ, M_n)$; by loc. cit., 2.4, specialisation and cospecialisation are then inverse isomorphisms between the cohomology of the special and of the generic geometric fibre. Passing to the limit, $H^0(C^\circ_{0,\bar{\mathbb{F}}_p}, M_\ell) \cong H^0(C^\circ_{\bar\eta}, M_\ell)$. The latter vanishes: after an extension of algebraically closed fields of characteristic zero, which does not change étale cohomology, Artin's comparison theorem [55], Exp. XVI, Th. 4.1, identifies it with the space of global monodromy invariants of $\mathrm{Sym}^{k-2}\mathbb{V} \otimes L_\lambda$ on $C^{\mathrm{an}} \setminus D_C$, which is zero as in Step 1. The dual of M_ℓ is $M_\ell(k-2)$ (the pairing of Lemma 5.3(1), pulled back by ρ), so by Poincaré duality $H^2(C_{0,\bar{\mathbb{F}}_p}, o_*M_\ell)$ is dual to $H^0(C_{0,\bar{\mathbb{F}}_p}, o_*(M_\ell^\vee)(1)) = H^0(C_{0,\bar{\mathbb{F}}_p}, o_*M_\ell)(k-1) = 0$. Groups H^i with $i > 2$ vanish on a curve. This proves (1).

Step 3: twisted forms. We construct the objects of (2) as in [47], proof of Proposition 4.4, (4.4.3)–(4.4.4). Let m be the order of g in $\tilde G$, $R_1 = W(\mathbb{F}_q)$, $R_m = W(\mathbb{F}_{q^m})$, and $\tau \in \mathrm{Gal}(R_m/R_1)$ the q-power Frobenius. The group $\tilde G$ acts on $C \otimes \mathbb{Z}_p$ over $\mathbb{Z}_p$ (G over $\mathcal{O}_{L,(\mathfrak{P})} \subset \mathbb{Z}_p$, and $\mu_d(\mathbb{Z}_p)$ through $s \mapsto \zeta s$), preserving D_C. Let γ be the τ-semilinear automorphism $g \otimes \tau$ of $C_{R_m} = C \otimes R_m$; as g commutes with the $\mathbb{Z}_p$-structure, $\gamma^m = 1$, and $\langle\gamma\rangle$ acts on $\operatorname{Spec} R_m$ through $\mathrm{Gal}(R_m/R_1)$, freely. Since C is projective, Galois descent [10], §6.2, Example B, gives a smooth projective R_1-scheme $C^{(g)}$ with $C^{(g)} \otimes_{R_1} R_m = C_{R_m}$ on which the descent datum is γ, and likewise $D^{(g)}$, finite étale over R_1. The isomorphisms $\gamma^* M_k \cong M_k$ given by $g^\sharp$ and the base change along τ are compatible with ∇, the Hodge filtration and the Frobenius structures (M2), and satisfy the cocycle condition because $g \mapsto g^\sharp$ is an action (proof of Lemma 5.4, Step 5). So, by faithfully flat descent of modules with connection, the morphisms F_ϕ descending as they commute with the descent datum [48], 7.2, M_k descends to an F-crystal with logarithmic singularities $M^{(g)}$ on $(\widehat{C^{(g)}}, \widehat{D^{(g)}})$, which satisfies the conditions 1.7 of loc. cit. because they can be checked after the faithfully flat base change to R_m. In the same way, étale descent along $C_{\mathbb{F}_{q^m}} \to C^{(g)}_{\mathbb{F}_q}$ gives $M_\ell^{(g)}$ from M_ℓ and the maps $g^\sharp$.

The arithmetic Frobenius of the twisted $\mathbb{F}_q$-structure is γ; as the absolute q-power Frobenius is the identity on closed points, the q-power Frobenius endomorphism of $C^{(g)}_{\mathbb{F}_q}$ acts on $C(\bar{\mathbb{F}}_p)$ as

$\mathrm{Fr}_q \circ g^{-1} = g^{-1} \circ \mathrm{Fr}_q$. Hence $C^{(g)}(\mathbb{F}_q) = \mathrm{Fix}_q(g)$, a finite set. On cohomology, which commutes with the base changes $R_1 \to R_m$ and $\mathbb{F}_q \to \bar{\mathbb{F}}_p$, the q-linear Frobenius of the twisted objects is the pull-back along $g^{-1} \circ \mathrm{Fr}_q$, that is $\varphi^n(g^{-1})^*$ on $H^\bullet(\widehat{C}, \Omega^\bullet(M_k)) \otimes R_m$ (compare (4.4.4) of [47]) and $F^n(g^{-1})^*$ on $H^\bullet(C_{0,\bar{\mathbb{F}}_p}, o_* M_\ell)$, the formation of o_* commuting with the extension $\mathbb{F}_q \to \bar{\mathbb{F}}_p$. At $y \in \mathrm{Fix}_q(g)$ the fibre of the twisted objects is the fibre at y, and the Frobenius at y is the composite of $(g^\sharp)^{-1}$, from the fibre at y to the fibre at $g(y) = \mathrm{Fr}_q(y)$, with the q-power Frobenius of the crystal, respectively of the sheaf, from the fibre at $\mathrm{Fr}_q(y)$ to that at y.

Step 4: the two trace formulas. We apply [48], Theorem 5.2(ii), to $(C^{(g)}, D^{(g)}, M^{(g)})$ over $R = R_1$, whose absolute ramification index 1 is less than $p-1$, with residue field $\mathbb{F}_q$, $q = p^n$ odd, and with the integer s of loc. cit. equal to 1:

$$\sum_i (-1)^i \operatorname{Tr}\bigl(F'^{-n} \mid H^i(\widehat{C^{(g)}}, \Omega^\bullet(M^{(g)}))_{\mathbb{Q}}\bigr) = q^{-1} \sum_{y \in C^{(g)}(\mathbb{F}_q)} \operatorname{Tr}\bigl((\Psi^{\mathrm{cris}}_{g,y})^{-1} \mid \operatorname{coker}(R_y)_{\mathbb{Q}}\bigr),$$

where F' is the σ-linear Frobenius of the twisted crystal, so that F'^n is R_1-linear. By Step 3 and (1), after extending scalars to R_m only $i = 1$ contributes, and F'^{-n} becomes $(\varphi^n(g^{-1})^*)^{-1} = g^*\varphi^{-n}$ on $H_{\mathrm{cris}} \otimes R_m$; traces are unchanged by extension of scalars. This is the first formula of (2). For the second, Grothendieck's trace formula [17, (1.4.5.1)], in its trace form over $\mathbb{F}_q$, applied to the constructible L_λ-sheaf $o_* M_\ell^{(g)}$ on the proper curve $C^{(g)}_{\mathbb{F}_q}$, gives $\sum_i (-1)^i \operatorname{Tr}(F^n (g^{-1})^* \mid H^i(C_{0,\bar{\mathbb{F}}_p}, o_* M_\ell)) = \sum_{y \in \mathrm{Fix}_q(g)} \operatorname{Tr}(\Psi^\ell_{g,y} \mid B^\ell_y)$, and (1) removes $i \neq 1$.

Step 5: the local operators off the cusps, (3). For $g = (h, \zeta)$ the factor ζ acts on C over X' and trivially on the fibres of $M_k = \rho^* \operatorname{Sym}^{k-2} H$ and of M_ℓ; so $\rho(g(y)) = h(x)$, and as ρ commutes with Fr_q, $\mathrm{Fr}_q(x) = h(x)$. The fibre of H at (a lift of) x is $H^1_{\mathrm{cris}}(E_x/W(\kappa(x)))$, and the Frobenius structure of the crystal H restricts on fibres to the map induced by the relative Frobenius of E over X' [5], §7; so the q-power Frobenius from the fibre at $h(x) = \mathrm{Fr}_q(x)$ to the fibre at x is $\operatorname{Sym}^{k-2}(F^*_{x,q})$, and likewise for $R^1\pi_{E*}\mathbb{Q}_\ell$. On fibres, $g^\sharp = c_h \operatorname{Sym}^{k-2}(\pi^*_{h,x})$ from the fibre at $h(x)$ to that at x. As $\pi^*_{h,x}\hat{\pi}^*_{h,x} = (\hat{\pi}_{h,x}\pi_{h,x})^* = \deg \pi_h$ on H^1 and $c_h^2 (\deg \pi_h)^{k-2} = 1$, its inverse is

$$(g^\sharp)^{-1} = c_h^{-1} (\deg \pi_h)^{-(k-2)} \operatorname{Sym}^{k-2}(\hat{\pi}^*_{h,x}) = c_h \operatorname{Sym}^{k-2}(\hat{\pi}^*_{h,x}).$$

So $\Psi_{g,y} = \operatorname{Sym}^{k-2}(F^*_{x,q}) \circ c_h \operatorname{Sym}^{k-2}(\hat{\pi}^*_{h,x}) = c_h \operatorname{Sym}^{k-2}((\hat{\pi}_{h,x} \circ F_{x,q})^*)$ in both theories. Both $F_{x,q}$ and $\hat{\pi}_{h,x}$ are defined over $\kappa(x)$, which contains the residue field of $h(x) = \mathrm{Fr}_q(x)$.

□

Lemma 5.8. *Let $\ell \neq p$ be a prime, λ a prime of L above ℓ, $n \geq 1$, $q = p^n$ and $g = (h, \zeta) \in \tilde{G}$, and let $\mathrm{Fix}_q(g)$, the local Frobenius operators $\Psi^{\mathrm{cris}}_{g,y}$ on B^{cris}_y and $\Psi^\ell_{g,y}$ on B^ℓ_y, and the endomorphisms $u_{g,y}$ be those of Lemma 5.7(2),(3). Let $y \in \mathrm{Fix}_q(g)$. Then $\operatorname{Tr}(\Psi^\ell_{g,y} \mid B^\ell_y)$ lies in L, and*

$$q^{-1} \operatorname{Tr}\bigl((\Psi^{\mathrm{cris}}_{g,y})^{-1} \mid B^{\mathrm{cris}}_y\bigr) = q^{-(k-1)}\, \iota_{\mathfrak{P}} \operatorname{Tr}(\Psi^\ell_{g,y} \mid B^\ell_y).$$

More precisely:

1. *If $y \notin D_C$, let $x = \rho(y)$ and $u = u_{g,y}$, and let α, β be the roots of the characteristic polynomial $P_u(X) = X^2 - \operatorname{tr}(u)X + \deg u \in \mathbb{Z}[X]$ of u^* on $H^1(E_{\bar{x}}, \mathbb{Q}_\ell)$. Then P_u is also the characteristic polynomial of u^* on $H^1_{\mathrm{cris}}(E_x/W(\kappa(x))) \otimes \mathbb{Q}$, and*

$$\operatorname{Tr}(\Psi^{\mathrm{cris}}_{g,y}) = \iota_{\mathfrak{P}} \operatorname{Tr}(\Psi^\ell_{g,y}) = c_h \sum_{i=0}^{k-2} \alpha^i \beta^{k-2-i}, \qquad \operatorname{Tr}\bigl((\Psi^{\mathrm{cris}}_{g,y})^{-1}\bigr) = q^{-(k-2)} \operatorname{Tr}(\Psi^{\mathrm{cris}}_{g,y}).$$

2. *If $y \in D_C$, then B_y^{cris} and B_y^{ℓ} are lines, and there is a nonzero integer r_y such that $\Psi_{g,y}^{\mathrm{cris}}$ acts by $c_h(qr_y)^{k-2}$ and $\Psi_{g,y}^{\ell}$ by $c_h(\deg \pi_h / r_y)^{k-2}$; r_y is the degree on the tori of the Tate parametrisations of the restriction of $\hat{\pi}_h$ to the punctured formal neighbourhood of the cusp.*

Proof. **Off the cusps.** Let $y \notin D_C$. By Lemma 5.7(3) both local operators are $c_h \operatorname{Sym}^{k-2}(u^*)$, with $u = u_{g,y}$ an endomorphism of the elliptic curve E_x over the finite field $\kappa(x)$, of degree $\deg u = q \deg \pi_h$. The graph of u is a cycle of dimension one with integer coefficients on $E_x \times E_x$, and E_x is projective, smooth and absolutely irreducible; by [31], Theorem 2(2), its characteristic polynomial on H^1 lies in $\mathbb{Z}[X]$ and is the same for ℓ-adic cohomology and for crystalline cohomology tensored with the fraction field of $W(\bar{\mathbb{F}}_p)$, the latter being a Weil cohomology by [27]. On ℓ-adic H^1 this polynomial is $X^2 - \operatorname{tr}(u)X + \deg u$ [58], Proposition III.8.6 (on the Tate module, whose dual is H^1). So in both theories $\operatorname{Sym}^{k-2}(u^*)$ has eigenvalues $\alpha^i \beta^{k-2-i}$, $0 \le i \le k-2$, and its trace $\sum_i \alpha^i \beta^{k-2-i}$, a symmetric polynomial in α, β with integer coefficients, is an integer; as $c_h \in L$ (M2) acts on crystalline cohomology through $\iota_{\mathfrak{P}}$, $\operatorname{Tr}(\Psi_{g,y}^{\ell}) \in L$ and $\operatorname{Tr}(\Psi_{g,y}^{\mathrm{cris}}) = \iota_{\mathfrak{P}} \operatorname{Tr}(\Psi_{g,y}^{\ell})$. For the inverse, $\alpha\beta = q \deg \pi_h$ and $c_h^{-1}(\deg \pi_h)^{-(k-2)} = c_h$, which is the normalisation $c_h^2 = (\deg \pi_h)^{-(k-2)}$ of (M2), give

$$c_h^{-1} \alpha^{-i} \beta^{-(k-2-i)} = c_h^{-1} (\alpha\beta)^{-(k-2)} \alpha^{k-2-i} \beta^i = q^{-(k-2)}\, c_h\, \alpha^{k-2-i} \beta^i,$$

so the eigenvalues of $(\Psi_{g,y}^{\mathrm{cris}})^{-1}$ are those of $q^{-(k-2)} \Psi_{g,y}^{\mathrm{cris}}$. This proves (1).

At the cusps. Let $y \in D_C$, $z = \rho(y) \in Z'(\bar{\mathbb{F}}_p)$ and $W = W(\bar{\mathbb{F}}_p)$; then $h(z) = \mathrm{Fr}_q(z)$ (Step 5 of the proof of Lemma 5.7). As Z' and D_C are étale over $\mathcal{O}_{L,(\mathfrak{P})}$, y, $g(y)$, z, $h(z)$ have unique lifts $\tilde{y}, \widetilde{g(y)} \in D_C(W)$, $\tilde{z}, \widetilde{h(z)} \in Z'(W)$, and $\widetilde{h(z)} = \sigma^n(\tilde{z})$ for the Frobenius σ of W, as X' and Z' are defined over $\mathcal{O}_{L,(\mathfrak{P})} \subset \mathbb{Z}_p$. Fix data (q_z, δ_z, m_z) as in (M1) at $\tilde{z}$, take the σ^n-conjugate data at $\widetilde{h(z)}$, and transport both to the cusps of C above them as in the proof of Lemma 3.10(2): with a formal parameter v of C_W along $\tilde{y}$, the pull-back of E to $\operatorname{Spf} W[[v]]$ is $\mathrm{Tate}(Q)$, $Q = \delta v^m$, $\delta \in W^\times$, m prime to p; likewise along $\widetilde{g(y)}$.

Both spaces are lines. In the basis $e_i = \omega^{k-2-i} \xi^i$ of Lemma 3.8, $\omega = \omega_{\mathrm{can}}$, the residue R_y maps e_i to $(k-2-i)m\, e_{i+1}$, so B_y^{cris} is the line spanned by the image of $e_0 = \omega^{\otimes(k-2)}$. Let K_y be the fraction field of the strict henselisation of $C_{0,\bar{\mathbb{F}}_p}$ at y, with inertia group I. Over K_y, E is the Tate curve $\mathrm{Tate}(Q)$ with Q of valuation $m \ge 1$, and $\mathrm{Tate}(Q)[\ell^\nu]$ is an extension of $\mathbb{Z}/\ell^\nu$ by μ_{ℓ^ν}, the quotient being generated by the class of an ℓ^ν-th root of Q [18], VII, §1. So I acts on $T_\ell(E)$ through a nontrivial unipotent group, trivially on $T_\ell(\mu)$ and on $T_\ell(E)/T_\ell(\mu) \cong \mathbb{Z}_\ell$, and $H^1(E, \mathbb{Q}_\ell)^I$ is the line dual to $T_\ell(E)/T_\ell(\mu)$, on which the whole Galois group of K_y acts trivially. As Sym^{k-2} of a nontrivial unipotent automorphism of a plane has one-dimensional invariants, $B_y^{\ell} = (o_* M_\ell)_{\bar{y}} = \operatorname{Sym}^{k-2}(H^1(E, \mathbb{Q}_\ell)^I) \otimes L_\lambda$ is a line.

Over $B = \operatorname{Spec} W[[v]][\frac{1}{v}]$, which maps to Y', the isogeny $\hat{\pi}_h : h^*E \to E$ restricts to an isogeny $\psi : \mathrm{Tate}(Q') \to \mathrm{Tate}(Q)$ of degree $\deg \pi_h$, where $\mathrm{Tate}(Q')$ is the pull-back along g of the Tate curve along $\widetilde{g(y)}$. Let R_0 be the local ring of C_W at the generic point of the divisor $\tilde{y}$, a discrete valuation ring with residue field $\kappa_0 = \operatorname{Frac} W$ of characteristic zero and completion $\kappa_0[[v]]$, the ideal of $\tilde{y}$ in $W[[v]]$ being (v); its fraction field is the function field of C_W, to which the generic point of B maps. Over $\operatorname{Frac} R_0$ both curves have split multiplicative reduction, and the special fibre of the identity component of their Néron models is the torus $\mathbb{G}_{m,\kappa_0}$ of the Tate parametrisation, on which ω_{can} restricts to dx/x and whose ℓ^ν-torsion is the reduction of the subgroup scheme $\mu_{\ell^\nu} \subset \mathrm{Tate}(\cdot)[\ell^\nu]$ [18], VII, §1, [58], Proposition VII.5.1. By the Néron mapping property [10], §1.2, Definition 1, ψ extends to the Néron models, and on the special fibres of the identity components it is a homomorphism of tori, $x \mapsto x^{r_y}$ with $r_y \in \mathbb{Z} \setminus \{0\}$, as ψ is an isogeny. Hence the function $\psi^* \omega_{\mathrm{can}} / \omega_{\mathrm{can}}$, which lies in $W[[v]]$ because $c_h \operatorname{Sym}^{k-2}(\hat{\pi}_h^*)$ extends

to the logarithmic crystal (M2), takes the value r_y at $v=0$. Moreover ψ maps μ_{ℓ^ν} by $x \mapsto x^{r_y}$ over R_0. Indeed, both are morphisms from μ_{ℓ^ν,R_0}, a disjoint union of copies of the local scheme $\operatorname{Spec} R_0$ because W contains μ_{ℓ^ν}, to an étale separated R_0-scheme, and they agree on the closed point of each copy; the locus where two such morphisms agree is open and closed, the diagonal of an étale separated morphism being an open and closed immersion. The same argument over B, the components of $\mu_{\ell^\nu,B}$ being copies of the integral scheme B, whose generic point maps to $\operatorname{Spec}\operatorname{Frac} R_0$, shows that ψ restricts to $x \mapsto x^{r_y}$ on μ_{ℓ^ν} over B, hence over K_y. So over K_y, ψ acts by r_y on $T_\ell(\mu)$ and by $r'_y = \deg \pi_h / r_y$ on $T_\ell(E)/T_\ell(\mu)$, the determinant of ψ on T_ℓ being $\deg \psi$ [58], Proposition III.8.6.

On the crystalline side $(g^\sharp)^{-1} = c_h \operatorname{Sym}^{k-2}(\hat\pi_h^*)$ (Step 5 of the proof of Lemma 5.7), extended to the logarithmic crystal by (M2), maps the image of e_0 in the fibre at the cusp $\tilde y$ (where $v=0$) to $c_h r_y^{k-2}$ times the image of e_0 at $\widetilde{g(y)}$, as $\psi^*\omega_{\mathrm{can}}/\omega_{\mathrm{can}}$ takes the value r_y at $v=0$, and the q-power Frobenius, induced on Tate curves with conjugate data by $x \mapsto x^q$ (iterate Lemma 3.8(2)), maps ω_{can} at $\widetilde{g(y)} = \mathrm{Fr}_q(\tilde y)$ to $q\,\omega_{\mathrm{can}}$ at $\tilde y$; so $\Psi^{\mathrm{cris}}_{g,y} = c_h (q r_y)^{k-2}$. On the ℓ-adic side, $(g^\sharp)^{-1}$ acts on the invariant lines by $c_h (r'_y)^{k-2}$, and the q-power Frobenius carries the generator dual to the class of the roots of the Tate parameter at $\mathrm{Fr}_q(y)$ to the corresponding generator at y, the data being conjugate and Galois acting trivially on $T_\ell(E)/T_\ell(\mu)$; so $\Psi^{\ell}_{g,y} = c_h (r'_y)^{k-2}$. This proves (2).

The identity. Off the cusps, (1) gives

$$\begin{aligned} q^{-1}\operatorname{Tr}((\Psi^{\mathrm{cris}})^{-1}) &= q^{-1} q^{-(k-2)} \operatorname{Tr}(\Psi^{\mathrm{cris}}) \\ &= q^{-(k-1)} \iota_{\mathfrak P} \operatorname{Tr}(\Psi^{\ell}). \end{aligned}$$

At a cusp, (2) and $c_h^{-1} = c_h (\deg \pi_h)^{k-2} = c_h (r_y r'_y)^{k-2}$, again by (M2), give $(\Psi^{\mathrm{cris}})^{-1} = c_h^{-1}(q r_y)^{-(k-2)} = q^{-(k-2)} c_h (r'_y)^{k-2}$, and $c_h (r'_y)^{k-2} \in L$.

□

5.3 Specialisation

The computation above takes place over $\overline{\mathbb{F}}_p$. Tame specialisation identifies these cohomology groups, with their Frobenius action, with the restrictions of the generic-fibre representations to a decomposition group at $\mathfrak P$.

Lemma 5.9. *Assume that the datum is admissible. Let $\ell \neq p$ be a prime, λ a prime of L above ℓ, $V_{a,\lambda}$, $V_{\bar a,\lambda}$ the λ-adic realisations of the model, and V_a, $V_{\bar a}$ the cohomology over $\bar{\mathbb{F}}_p$ of its Kummer eigensheaves, with the geometric Frobenius F. Fix a prime of $\bar L$ above $\mathfrak P$, and let $D \supset I$ be its decomposition and inertia groups. Then I acts trivially on $V_{a,\lambda}$ and on $V_{\bar a,\lambda}$, and there are isomorphisms $V_{a,\lambda} \cong V_a$ and $V_{\bar a,\lambda} \cong V_{\bar a}$, compatible with the action of D, which acts on V_a and $V_{\bar a}$ through $D/I = \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$, so that a geometric Frobenius element $\mathrm{Frob}_{\mathfrak P} \in D$ acts as F. In particular $V_{a,\lambda}$ and $V_{\bar a,\lambda}$ are two-dimensional and unramified at $\mathfrak P$, and*

$$\det(X - \mathrm{Frob}_{\mathfrak P} \mid V_{\bar a,\lambda}) = X^2 - A_{\mathfrak P} X + \varepsilon_{\mathfrak P}\, p^{k-1}$$

with $A_{\mathfrak P}$, $\varepsilon_{\mathfrak P}$ the numbers of Proposition 5.6(2).

Proof. We use the notation of Lemma 5.7 and of Lemma 5.5, with $\chi = \chi_\Phi$; the index $*$ stands for a or $\bar a$, with e_χ for $* = a$ and $e_{\chi^{-1}}$ for $* = \bar a$. Let $\mathcal O = \mathcal O_{L,(\mathfrak P)}$, let $\mathcal O^{\mathrm{sh}}$ be the strict henselisation of $\mathcal O$ at $\mathfrak P$, with closed point s, and $\bar\eta$ a geometric generic point of $\operatorname{Spec}\mathcal O^{\mathrm{sh}}$ over the fixed prime of $\bar L$. Let M be the lisse sheaf $\rho^* \operatorname{Sym}^{k-2} R^1\pi_{E*}\mathbb{Q}_\ell \otimes L_\lambda$ on $C^\circ_{\mathrm{sh}} = (C \setminus D_C)_{\mathcal O^{\mathrm{sh}}}$, with the lattice $M^0 = \rho^* \operatorname{Sym}^{k-2} R^1\pi_{E*}\mathbb{Z}_\ell \otimes \mathcal O_\lambda$ and $M_n = M^0/\lambda^n$, and $o : C^\circ_{\mathrm{sh}} \hookrightarrow C_{\mathrm{sh}}$.

Step 1: specialisation on C°. By Lemma 3.7(2), C_{sh} is smooth and proper over $\mathcal{O}^{\mathrm{sh}}$ and $D_{C,\mathrm{sh}}$ is finite étale over $\mathcal{O}^{\mathrm{sh}}$, hence a divisor with normal crossings relative to $\mathcal{O}^{\mathrm{sh}}$. Each M_n is locally constant and tamely ramified along $D_{C,\mathrm{sh}}$: near a point of D_C the curve E is a Tate curve $\mathrm{Tate}(Q)$ (M1), $Q = \delta v^m$ with $p \nmid m$, whose ℓ^n-torsion, and hence M_n, is trivialised by adjoining an ℓ^n-th root of v, a Kummer covering of degree prime to p [18], VII, §1. So by [54], Th. finitude, Appendice, 1.3.3(i), the formation of Ro_*M_n commutes with every base change, and by the proper base change theorem [55], Exp. XII, Th. 5.1, for $C_{\mathrm{sh}} \to \operatorname{Spec}\mathcal{O}^{\mathrm{sh}}$, the formation of $R\Gamma(C^\circ, M_n)$ and of $R\Gamma_c(C^\circ, M_n)$ commutes with every base change. The specialisation maps on both are therefore defined [54], Th. finitude, Appendice, 2.1, and compatible with the forget-supports maps and with the action of $\tilde{G}$. The smooth morphism $C^\circ_{\mathrm{sh}} \to \operatorname{Spec}\mathcal{O}^{\mathrm{sh}}$ is locally acyclic relative to the locally constant M_n [55], Exp. XV, so by [54], Th. finitude, Appendice, 2.4, $\mathrm{sp} : H^1(C^\circ_{\bar{s}}, M_n) \to H^1(C^\circ_{\bar{\eta}}, M_n)$ is an isomorphism. Passing to the limit and tensoring with $\mathbb{Q}$, the same holds for M.

Step 2: the parabolic parts. On each geometric fibre, $H^1(C, o_*M)$ is the image of $H^1_c(C^\circ, M) \to H^1(C^\circ, M)$ and injects into $H^1(C^\circ, M)$, by Lemma 5.5(1). Hence sp maps $H^1(C_{\bar{s}}, o_*M) = H_\ell$ injectively and $\tilde{G}$-equivariantly into $H^1(C_{\bar{\eta}}, o_*M)$. By Lemma 5.5(2), applied to the geometric generic fibre, $e_{\chi^{\pm 1}}H^1(C_{\bar{\eta}}, o_*M) = H^1(\mathbb{P}^1_{\bar{\eta}}, o_*\mathcal{F}_*)$, and this is $V_{*,\lambda}$, as H^1 of a proper scheme is unchanged by the extension $\bar{L} \subset \bar{\eta}$ of algebraically closed fields (proper base change). So sp maps $V_* = e_{\chi^{\pm 1}}H_\ell$ injectively into $V_{*,\lambda}$. Both have dimension 2: V_* by Proposition 5.6(2); $V_{a,\lambda}$ by the duality of Lemma 5.4(3) once $V_{\bar{a},\lambda}$ is known; and $V_{\bar{a},\lambda}$ because it is an L_λ-form of H_a, of dimension 2 by (R). To see the last point, fix an embedding $\bar{L} \subset \mathbb{C}$ extending $L \subset \mathbb{C}$. Over $\bar{L}$, which has characteristic zero, S is a divisor of $\mathbb{P}^1_{\bar{L}}$ étale over $\bar{L}$, and the reductions of an integral lattice of $\mathcal{F}_{\bar{a}}$ are locally constant on $U_{\bar{L}}$ and tamely ramified along S. Hence the formation of their direct images under o commutes with the base change to $\mathbb{C}$ [54], Th. finitude, Appendice, 1.3.3(i), and the proper base change theorem [55], Exp. XII, Th. 5.1, applied to $\mathbb{P}^1_{\bar{L}} \to \operatorname{Spec}\bar{L}$, gives $V_{\bar{a},\lambda} \cong H^1(\mathbb{P}^1_{\mathbb{C}}, o_*\mathcal{F}_{\bar{a}})$. By Lemma 5.4(4) the right side is $H^1(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}) \otimes_L L_\lambda$ for a local system $\mathcal{V}$ of L-vector spaces with $H^1(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}) \otimes_L \mathbb{C} \cong H_a$. So $\mathrm{sp} : V_* \to V_{*,\lambda}$ is an isomorphism.

Step 3: Galois action. The decomposition group D is the Galois group of the henselisation of L at $\mathfrak{P}$ (for the fixed prime), it acts on $\mathcal{O}^{\mathrm{sh}}$ over the henselisation of $\mathcal{O}$, and sp is D-equivariant, D acting on the special fibre through $D/I = \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. So I acts trivially on $V_{*,\lambda}$. An element of D inducing the arithmetic Frobenius of $\bar{\mathbb{F}}_p$ acts on $H^1(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, o_*\mathcal{F}_*)$ as the inverse of the geometric Frobenius F [16], (1.15); so $\mathrm{Frob}_{\mathfrak{P}}$ acts as F. The last assertion is then Proposition 5.6(2).

□

6 The global system and the general theorem

6.1 Independence of the model

The modular model used in Section 5 was chosen at a single prime $\mathfrak{P}$. To show that the polynomials obtained at different primes come from one system over L, we compare the eigensheaves on their common generic base. Geometric irreducibility leaves a possible rank-one arithmetic twist. The invariant line at the Tate cusp determines that twist and shows that it is trivial.

Proposition 6.1. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum with field L, exceptional set Σ and singular set S. Let ℓ be a prime and λ a prime of L above ℓ.*

1. *There are lisse L_λ-sheaves $\mathcal{F}_{\bar{a},\lambda}$ and $\mathcal{F}_{a,\lambda}$ on $U_L = \mathbb{P}^1_L \setminus S$, unique up to isomorphism, such that $\mathcal{F}_{\bar{a},\lambda} \cong \mathcal{F}_{\bar{a}}|_{U_L}$ and $\mathcal{F}_{a,\lambda} \cong \mathcal{F}_a|_{U_L}$ for the Kummer eigensheaves of every modular model*

of the datum at every prime of L of residue degree one over a prime not in Σ. In particular the λ-adic realisations $V_{\bar{a},\lambda}$ and $V_{a,\lambda}$ do not depend, up to isomorphism, on the model or the prime; they are two-dimensional over L_λ. Moreover, for an embedding $\bar{L} \subset \mathbb{C}$ extending $L \subset \mathbb{C}$, there is a local system $\mathcal{V}$ of L-vector spaces on $U(\mathbb{C})$ with

$$V_{\bar{a},\lambda} \cong H^1\big(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}\big) \otimes_L L_\lambda, \qquad H^1\big(\mathbb{P}^1(\mathbb{C}), o_*\mathcal{V}\big) \otimes_L \mathbb{C} \cong H_a.$$

2. *Let $\mathfrak{P}$ be a prime of L of residue degree one over a prime $p \notin \Sigma$, with $\mathfrak{P} \nmid \ell$. Then $V_{\bar{a},\lambda}$ and $V_{a,\lambda}$ are unramified at $\mathfrak{P}$, and, for a geometric Frobenius* $\mathrm{Frob}_\mathfrak{P}$*,*

$$\det(X - \mathrm{Frob}_\mathfrak{P} \mid V_{\bar{a},\lambda}) = X^2 - A_\mathfrak{P} X + \varepsilon_\mathfrak{P}\, p^{k-1},$$

where $A_\mathfrak{P} \in \mathcal{O}_L$ and $\varepsilon_\mathfrak{P} \in L^\times$ are the numbers of Proposition 5.6(2) for any modular model of the datum at $\mathfrak{P}$; in particular $A_\mathfrak{P}$ and $\varepsilon_\mathfrak{P}$ depend neither on the model nor on ℓ and λ.

3. *If $L = \mathbb{Q}$, then $(V_{\bar{a},\ell})_\ell$ is a weakly compatible system of two-dimensional ℓ-adic representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*, unramified outside Σ: for every prime $p \notin \Sigma$ and every prime $\ell \neq p$, $V_{\bar{a},\ell}$ is unramified at p, and a geometric Frobenius at p has characteristic polynomial $X^2 - A_p X + \varepsilon_p\, p^{k-1}$, independent of ℓ, with $A_p \in \mathbb{Z}$, $\varepsilon_p \in \{\pm 1\}$, and roots of absolute value $p^{(k-1)/2}$ under every complex embedding.*

Proof. Proof of (1). The set Σ is finite, and L is generated over $\mathbb{Q}$ by roots of unity and square roots of integers, so it is abelian over $\mathbb{Q}$ and infinitely many primes split completely in it; hence there is a prime $\mathfrak{P}_0$ of L of residue degree one over some $p_0 \notin \Sigma$, and the datum has a modular model at $\mathfrak{P}_0$ by Proposition 3.2. Let $\mathcal{F}_{\bar{a},\lambda}$ and $\mathcal{F}_{a,\lambda}$ be the restrictions to U_L of the Kummer eigensheaves of this model. By Proposition 6.2, the Kummer eigensheaves of any other modular model at any prime of residue degree one over a prime not in Σ are isomorphic to them on U_L; uniqueness up to isomorphism is then clear. Applying Lemma 5.9 to a modular model at a prime $\mathfrak{P}$ of residue degree one over $p \notin \Sigma$ with $\mathfrak{P} \nmid \ell$ (such primes exist, as above), $V_{\bar{a},\lambda}$ and $V_{a,\lambda}$ are two-dimensional. For the last assertion, compute $V_{\bar{a},\lambda}$ with the Kummer eigensheaf $\mathcal{F}_{\bar{a}}$ of one model: as o_* commutes with the base change from $\bar{L}$ to $\mathbb{C}$ for the tamely ramified sheaf $\mathcal{F}_{\bar{a}}$ [54], Th. finitude, Appendice, 1.3.3(i), the proper base change theorem [55], Exp. XII, Th. 5.1, gives $V_{\bar{a},\lambda} \cong H^1(\mathbb{P}^1_\mathbb{C}, o_*\mathcal{F}_{\bar{a}})$, and Lemma 5.4(4) gives the displayed isomorphisms.

Proof of (2). By Proposition 3.2 the datum has a modular model at $\mathfrak{P}$; by (1), $V_{\bar{a},\lambda}$ and $V_{a,\lambda}$ are, as representations of $\mathrm{Gal}(\bar{L}/L)$, the λ-adic realisations of this model. As $\ell \neq p$, Lemma 5.9 shows that they are unramified at $\mathfrak{P}$ and that the characteristic polynomial of $\mathrm{Frob}_\mathfrak{P}$ on $V_{\bar{a},\lambda}$ is $X^2 - A_\mathfrak{P} X + \varepsilon_\mathfrak{P} p^{k-1}$, with $A_\mathfrak{P}$, $\varepsilon_\mathfrak{P}$ those of Proposition 5.6(2) for this model. The left-hand side does not depend on the model, by (1); since $A_\mathfrak{P}$ is the trace and $\varepsilon_\mathfrak{P} p^{k-1}$ the determinant, neither do $A_\mathfrak{P}$ and $\varepsilon_\mathfrak{P}$. They do not depend on ℓ and λ either: by Proposition 5.6(2) the polynomial lies in $L[X]$ and its image under the injective map $\iota_\mathfrak{P}$ is the characteristic polynomial of $p^{k-1}\varphi^{-1}$ on $H_{a,p}$, in which ℓ and λ do not occur.

Proof of (3). If $L = \mathbb{Q}$, every prime p is the unique prime of L above itself, of residue degree one, and $\lambda = \ell$. For $p \notin \Sigma$ and $\ell \neq p$, (2) gives unramifiedness and the characteristic polynomial, independent of ℓ; $A_p \in \mathcal{O}_\mathbb{Q} = \mathbb{Z}$; and by Proposition 5.6(4) the roots have absolute value $p^{(k-1)/2}$ and $|\varepsilon_p| = 1$, so $\varepsilon_p = \pm 1$ as $\varepsilon_p \in \mathbb{Q}^\times$.

□

Proposition 6.2. *Let (Γ, t, f, a, R) be a Kummer twist datum with field L, exceptional set Σ and singular set S. For $i = 1, 2$ let $p_i \notin \Sigma$ be primes (possibly equal), $\mathfrak{P}_i$ a prime of L of residue degree one over p_i, and $(X_i', Z_i', E_i, G_i, \chi_{G_i})$ a modular model of the datum at $\mathfrak{P}_i$. Let ℓ be a prime, λ*

a prime of L above ℓ, and $\mathcal{F}_a^{(i)}$, $\mathcal{F}_{\bar{a}}^{(i)}$ the Kummer eigensheaves of the i-th model, restricted to $U_L = \mathbb{P}^1_L \setminus S$. Then

$$\mathcal{F}_{\bar{a}}^{(1)} \cong \mathcal{F}_{\bar{a}}^{(2)} \qquad \text{and} \qquad \mathcal{F}_a^{(1)} \cong \mathcal{F}_a^{(2)}$$

as lisse L_λ-sheaves on U_L. In particular $H^1(\mathbb{P}^1_{\bar{L}}, o_*\mathcal{F}_{\bar{a}}^{(1)}) \cong H^1(\mathbb{P}^1_{\bar{L}}, o_*\mathcal{F}_{\bar{a}}^{(2)})$ and $H^1(\mathbb{P}^1_{\bar{L}}, o_*\mathcal{F}_a^{(1)}) \cong H^1(\mathbb{P}^1_{\bar{L}}, o_*\mathcal{F}_a^{(2)})$ as representations of $\mathrm{Gal}(\bar{L}/L)$.

Proof. The local system $\mathbb{V}_f \otimes \mathbb{L}_a$ is irreducible. Indeed, put $n = \mathrm{lcm}(n_\Gamma, n_f)$. By Lemma A.1(3),(5), its monodromy restricted to $\Gamma(n)$ is the $(k-2)$-th symmetric power of the standard representation of SL_2, tensored with a rank-one character. The group $\Gamma(n)$ is Zariski-dense in SL_2: its upper and lower unipotents with entries in $n\mathbb{Z}$ are Zariski-dense in the corresponding one-parameter subgroups, which generate SL_2. In characteristic zero the symmetric power of the standard representation is irreducible. Tensoring with a rank-one character preserves irreducibility. Its endomorphisms are therefore scalars.

The space $\mathcal{N} = \mathrm{Hom}_{U_{\bar{L}}}(\mathcal{F}_{\bar{a}}^{(1)}, \mathcal{F}_{\bar{a}}^{(2)})$ is a line over L_λ spanned by an isomorphism. Fix an embedding $\bar{L} \subset \mathbb{C}$ extending $L \subset \mathbb{C}$. By Lemma 5.4(4), Artin's comparison theorem [55], Exp. XVI, Th. 4.1, identifies $\mathcal{F}_{\bar{a}}^{(i)}|_{U(\mathbb{C})}$ with $\mathcal{V}^{(i)} \otimes_L L_\lambda$, where $\mathcal{V}^{(i)}$ is a local system of L-vector spaces with $\mathcal{V}^{(i)} \otimes_L \mathbb{C} \cong \mathbb{V}_f \otimes \mathbb{L}_a$. By the first step, $\mathrm{Hom}(\mathcal{V}^{(1)}, \mathcal{V}^{(2)}) \otimes_L \mathbb{C}$ is a line and each $\mathcal{V}^{(i)}$ is absolutely irreducible; as invariants of a linear action commute with extension of scalars, $\mathrm{Hom}(\mathcal{V}^{(1)} \otimes_L L_\lambda, \mathcal{V}^{(2)} \otimes_L L_\lambda)$ is a line, and each $\mathcal{V}^{(i)} \otimes_L L_\lambda$ is irreducible, so a nonzero homomorphism between them is an isomorphism. Now $\mathcal{N}$ is the space of global sections over $U_{\bar{L}}$ of the lisse sheaf $\mathcal{H}om(\mathcal{F}_{\bar{a}}^{(1)}, \mathcal{F}_{\bar{a}}^{(2)})$; each of its reductions is trivialised by a connected finite étale Galois cover of $U_{\bar{L}}$, which stays connected, with the same group, over the algebraically closed field $\mathbb{C}$, so global sections do not change from $U_{\bar{L}}$ to $U_{\mathbb{C}}$, and by Artin's comparison they are the homomorphisms of the associated local systems on $U(\mathbb{C})$. Hence $\mathcal{N}$ is a line, and a nonzero $\phi \in \mathcal{N}$ is an isomorphism, being one on a geometric stalk.

Consequently the two sheaves differ by a character of $\mathrm{Gal}(\bar{L}/L)$. The group $\pi_1(U_L)$ acts on $\mathcal{N}$ by conjugation, through its quotient $\pi_1(U_L)/\pi_1(U_{\bar{L}}) = \mathrm{Gal}(\bar{L}/L)$, hence by a continuous character ν. Evaluation $\mathcal{F}_{\bar{a}}^{(1)} \otimes_{L_\lambda} \mathcal{N} \to \mathcal{F}_{\bar{a}}^{(2)}$ is equivariant, so it is a morphism of lisse sheaves on U_L, with $\mathcal{N}$ the rank-one sheaf pulled back from $\mathrm{Spec}\, L$; it is an isomorphism by the previous step. Thus $\mathcal{F}_{\bar{a}}^{(2)} \cong \mathcal{F}_{\bar{a}}^{(1)} \otimes \nu$.

It remains to show that $\nu = 1$. Let K, $\bar{\xi}$, G_K, I, E_0 and S_0 be as in Lemma 6.3, and let ψ be the character of G_K on the stalk at $\bar{\xi}$ of the Kummer sheaf $\mathcal{L}_{-j}$, of rank one by Lemma 5.4(2), which is built from $[d] : s \mapsto s^d$ alone and so is the same for both models. By Lemma 5.4(2), $\mathcal{F}_{\bar{a}}^{(i)} \cong \mathbb{V}^{(i)}_{\chi_{G_i}^{-1}} \otimes \mathcal{L}_{-j}$, and by Lemma 6.3(2) the stalk of $\mathbb{V}^{(i)}_{\chi_{G_i}^{-1}}$ at $\bar{\xi}$ is S_0 for both i. So, as representations of G_K,

$$S_0 \otimes \psi \cong S_0 \otimes \psi \otimes \nu|_{G_K}, \qquad \text{hence} \qquad S_0 \cong S_0 \otimes \nu|_{G_K},$$

where $\nu|_{G_K}$ is ν composed with restriction $G_K \to \mathrm{Gal}(\bar{L}/L)$, which is trivial on I. Taking I-invariants gives $S_0^I \cong S_0^I \otimes \nu$ as representations of $G_K/I = \mathrm{Gal}(\bar{L}/L)$ (Lemma 6.3(3)). As S_0^I is a line, $\nu = 1$. So $\mathcal{F}_{\bar{a}}^{(1)} \cong \mathcal{F}_{\bar{a}}^{(2)}$.

By Lemma 5.4(3), $\mathcal{F}_a^{(i)} \cong \mathcal{H}om(\mathcal{F}_{\bar{a}}^{(i)}, L_\lambda(-(k-2)))$, so $\mathcal{F}_a^{(1)} \cong \mathcal{F}_a^{(2)}$. The last assertion follows by functoriality of o_* and of $H^1(\mathbb{P}^1_{\bar{L}}, -)$.

□

Lemma 6.3. *Let (X', Z', E, G, χ_G) be a modular model, at a prime $\mathfrak{P}$ of L of residue degree one over $p \notin \Sigma$, of a Kummer twist datum (Γ, t, f, a, R) with singular set S. Let ℓ be a prime, λ*

a prime of L above ℓ, and $\mathbb{V}_{\chi_G}$, $\mathbb{V}_{\chi_G^{-1}}$ its Kummer eigensheaves, restricted to $U_L = \mathbb{P}^1_L \setminus S$. Let $T \in t + t^2\mathbb{Z}[\frac{1}{M}][[t]]$ be the compositional inverse of the q-expansion $t(q)$ of the Hauptmodul, so that $q = T(t)$. Let $K = L((t))$, fix an algebraic closure $\bar{K}$, let $\bar{L}$ be the algebraic closure of L in $\bar{K}$, put $G_K = \mathrm{Gal}(\bar{K}/K)$ and $I = \mathrm{Gal}(\bar{K}/\bar{L}K)$, and let $\bar{\xi} : \operatorname{Spec}\bar{K} \to U_L$ be the geometric point defined by $L(t) \subset K \subset \bar{K}$. Let $E_0 = \mathrm{Tate}(T(t))$ be the Tate curve over K with parameter $T(t)$, and

$$S_0 = \operatorname{Sym}^{k-2} H^1(E_{0,\bar{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} L_\lambda,$$

with the natural action of G_K. Then:

1. $\mathrm{End}_{\bar{K}}(E_{0,\bar{K}}) = \mathbb{Z}$.
2. *The stalks $(\mathbb{V}_{\chi_G})_{\bar{\xi}}$ and $(\mathbb{V}_{\chi_G^{-1}})_{\bar{\xi}}$, with their actions of G_K, are both isomorphic to S_0.*
3. *S_0^I is a line, and restriction to $\bar{L}$ induces an isomorphism $G_K/I \cong \mathrm{Gal}(\bar{L}/L)$.*

Proof. Since $T(t)$ has t-adic valuation one, E_0 is the Tate curve with split multiplicative reduction, and for every n the group $E_0[\ell^n](\bar{K})$ is the extension of $\mathbb{Z}/\ell^n$ by μ_{ℓ^n} formed by the classes of $\zeta\, T(t)^{b/\ell^n}$, $\zeta \in \mu_{\ell^n}$, $b \in \mathbb{Z}$ [18], VII, §1. The group I fixes $\mu_{\ell^\infty} \subset \bar{L}$, and it acts transitively on the ℓ^n-th roots of $T(t)$, because $T(t)$ is a uniformiser of the discretely valued field $\bar{L}K = \bigcup_{L'} L'((t))$ (L' finite over L) and $X^{\ell^n} - T(t)$ is Eisenstein. Hence, in a suitable basis e, e' of $V = H^1(E_{0,\bar{K}}, \mathbb{Q}_\ell)$, every element of I acts by a matrix $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, the line $\mathbb{Q}_\ell e$ being the space of forms on $T_\ell E_0$ that vanish on $T_\ell\mu$, and the image of every open subgroup of I contains a nontrivial unipotent.

Proof of (1). Let $\alpha \in \mathrm{End}_{\bar{K}}(E_{0,\bar{K}})$. It is defined over a finite extension $K_1 \subset \bar{K}$ of K, so α^* commutes on V with $I \cap \mathrm{Gal}(\bar{K}/K_1)$, which is open in I and so contains an element acting by a unipotent $u \neq 1$. The commutant of u in $\mathrm{End}(V)$ is $\mathbb{Q}_\ell \oplus \mathbb{Q}_\ell N$ with $N = u - 1$, $N^2 = 0$; write $\alpha^* = a + bN$. The characteristic polynomial of α^* on V is $(X - a)^2$, and it equals $X^2 - \mathrm{tr}(\alpha)X + \deg\alpha$ with integer coefficients [58], Proposition III.8.6 (stated for T_ℓ; V is its dual). So $2a \in \mathbb{Z}$ and $a^2 \in \mathbb{Z}$, whence $a \in \mathbb{Z}$. Then $\beta = \alpha - [a]$ acts on V by the nilpotent bN, so $\deg\beta = \det(\beta^*) = 0$ and $\beta = 0$.

Proof of (3). In characteristic zero $\operatorname{Sym}^{k-2}(u)$ of a nontrivial unipotent $u \in \mathrm{GL}(V)$ is a single unipotent Jordan block of size $k-1$: its logarithm is the derivation of $\operatorname{Sym}^\bullet V$ induced by N, which maps $e^i e'^{k-2-i}$ to a nonzero multiple of $e^{i+1}e'^{k-3-i}$. So S_0^I has dimension at most one, and it contains e^{k-2}. As L is algebraically closed in K, restriction $G_K \to \mathrm{Gal}(\bar{L}/L)$ is surjective, and its kernel is I.

Let B be the set of K-morphisms $y : \operatorname{Spec}\bar{K} \to X'_U$ with $\varpi' \circ y = \bar{\xi}$; put $E_y = y^*E$, an elliptic curve over $\bar{K}$, and $S_y = \operatorname{Sym}^{k-2} H^1(E_y, \mathbb{Q}_\ell) \otimes L_\lambda$. Then $(\varpi'_* \operatorname{Sym}^{k-2}\mathcal{H} \otimes L_\lambda)_{\bar{\xi}} = \bigoplus_{y\in B} S_y$. The group G acts on B by $y \mapsto g \circ y$, transitively, since $X'_L/G \cong \mathbb{P}^1_L$ (M3) [12], Ch. V, §2, no. 2, Th. 2; and $g^\sharp$ maps $v = (v_y)$ to $(c_g \pi_{g,y}^* v_{g\circ y})_y$, where $\pi_{g,y} : E_y \to E_{g\circ y}$ is the fibre of π_g. The group G_K acts on B by $\sigma(y) = y \circ \operatorname{Spec}(\sigma)$, and on the stalk by transport of structure: the projection $f_{\sigma,y} : E_{\sigma(y)} = E_y \times_{\operatorname{Spec}\bar{K}, \operatorname{Spec}\sigma} \operatorname{Spec}\bar{K} \to E_y$ is an isomorphism of schemes over $\operatorname{Spec}\sigma$, and σ maps S_y to $S_{\sigma(y)}$ by $f_{\sigma,y}^*$. The two actions commute. In the same way σ acts on S_0 by $(f_\sigma^0)^*$, where $f_\sigma^0 = \mathrm{id} \times \operatorname{Spec}\sigma$ is the transport of $E_{0,\bar{K}} = E_0 \times_K \operatorname{Spec}\bar{K}$. Finally, $f\omega^{\otimes(k-2)}$ is a rational section over X'_L (M4), regular and nonzero at the generic points of X'_L (on X'_0 it is $f(\tau)(2\pi i\, dz)^{\otimes(k-2)}$, and G permutes the components); its fibres $\phi_y \in H^0(E_y, \Omega^1)^{\otimes(k-2)}$ satisfy $f_{\sigma,y}^*\phi_y = \phi_{\sigma(y)}$, and (M4) gives

$$c_g\, \pi_{g,y}^*\, \phi_{g\circ y} = \chi_G(g)\, \phi_y \qquad (g \in G,\ y \in B). \tag{6.1}$$

We next produce a point $y_0 \in B$ with a normalised isomorphism $\iota_0 : E_{y_0} \to E_{0,\bar{K}}$. Let W, $\infty' \in Z'(W)$, q' and w be as in (M5), and let $x_0 \in X'_L$ be the image of the generic point of ∞',

a closed point with $t(x_0)=0$. The completed local ring A of X'_L at x_0 is a discrete valuation ring finite over $L[[t]]$ (t is finite, (M3)), so $F=\operatorname{Frac}A$ is finite over K. The generic point of ∞' is a $\operatorname{Frac}(W)$-point of $X'_L\otimes_L\operatorname{Frac}(W)$ over x_0, at which q' restricts to a uniformiser; so the completed local ring there is $\operatorname{Frac}(W)[[q']]$, the induced map $A\to\operatorname{Frac}(W)[[q']]$ is a local injection, and F embeds over K into $F^+=\operatorname{Frac}(W)((q'))$, with $t\mapsto t(q)$, $q=q'^w$. By (M5) and [21], (5.4.1), $E\times_{X'}\operatorname{Spec}W[[q']]$ is $\mathrm{Tate}(q)\otimes W[[q']]$, and $f\omega^{\otimes(k-2)}=F(t)\,\omega_{\mathrm{can}}^{\otimes(k-2)}$ there; as $T(t)=q$ in F^+, we get an isomorphism $\iota^+:E_{F^+}\to E_{0,F^+}$ with $\iota^{+*}(F(t)\omega_{\mathrm{can}}^{\otimes(k-2)})=f\omega^{\otimes(k-2)}$. Since $j(E_0)=j(T(t))$ has a pole at $t=0$, it is neither 0 nor 1728, so E_F and $E_{0,F}$, which have the same j-invariant, are quadratic twists of each other, and their isomorphisms over any extension of F are $\pm$ one of them [58], Theorem III.10.1 and Proposition X.5.4. Hence ι^+ is the base change of an isomorphism ι' defined over $F'=F(\sqrt{\delta})\subset F^+$ for some $\delta\in F^\times$, and $\iota'^*(F(t)\omega_{\mathrm{can}}^{\otimes(k-2)})=f\omega^{\otimes(k-2)}$, as this identity holds in F^+. Choose a K-embedding $F'\to\bar K$. It defines $y_0\in B$ (the image of $\operatorname{Spec}F\to X'_L$ is a generic point, so y_0 lies over U) and $\iota_0=\iota'\otimes\bar K$, with $\iota_0^*(F(t)\omega_{\mathrm{can}}^{\otimes(k-2)})=\phi_{y_0}$.

Let $\sigma\in G_K$ and $g\in G$ with $g\circ\sigma^{-1}(y_0)=y_0$; put $y=\sigma^{-1}(y_0)$ and $\Pi=\pi_{g,y}\circ f_{\sigma,y}:E_{y_0}\to E_y\to E_{y_0}$, an isogeny of group schemes over $\operatorname{Spec}\sigma$ of degree $\deg\pi_g$. Then $(f_\sigma^0)^{-1}\circ\iota_0\circ\Pi\circ\iota_0^{-1}$ is a $\bar K$-linear isogeny of $E_{0,\bar K}$ of degree $\deg\pi_g$, so by (1) it is $[m]$ with $m\in\mathbb{Z}$, $m^2=\deg\pi_g$:

$$\iota_0\circ\Pi\circ\iota_0^{-1}=[m]\circ f_\sigma^0. \tag{6.2}$$

Apply both sides to differentials. Pull-back along Π is σ-semilinear and, by the identity $c_g\pi_{g,y}^*\phi_{y_0}=\chi_G(g)\phi_y$ (the (M4) display with $g\circ y=y_0$) and $f_{\sigma,y}^*\phi_y=\phi_{y_0}$, maps ϕ_{y_0} to $c_g^{-1}\chi_G(g)\phi_{y_0}$ (as $c_g,\chi_G(g)\in L\subset K$). On the other side, $(f_\sigma^0)^*$ fixes the K-rational ω_{can} and $F(t)\in K$, and $[m]^*\omega_{\mathrm{can}}=m\,\omega_{\mathrm{can}}$; with $\iota_0^*(F(t)\omega_{\mathrm{can}}^{\otimes(k-2)})=\phi_{y_0}$ this gives $\Pi^*\phi_{y_0}=m^{k-2}\phi_{y_0}$. Since $\phi_{y_0}\neq0$,

$$\chi_G(g)=c_g\,m^{k-2},\qquad \chi_G(g)^2=c_g^2m^{2(k-2)}=(\deg\pi_g)^{-(k-2)}(\deg\pi_g)^{k-2}=1. \tag{6.3}$$

Proof of (2). Let $\chi\in\{\chi_G,\chi_G^{-1}\}$ and let p_0 be the projection of the stalk onto S_{y_0}. On the χ-part p_0 is injective: if $g\circ y=y_0$, then $\chi(g)v_y=c_g\pi_{g,y}^*v_{y_0}$. It is surjective: for $w\in S_{y_0}$ put $v_y=\chi(g)^{-1}c_g\pi_{g,y}^*w$ for any g with $g\circ y=y_0$; this does not depend on g, and $v=(v_y)$ lies in the χ-part, by the composition rule $g^\sharp h^\sharp=(hg)^\sharp$ of Lemma 5.4(2) and because, for h fixing y_0, the case $\sigma=1$ above gives $\pi_{h,y_0}=[m_h]$ (by (1), through ι_0) and $c_h\operatorname{Sym}^{k-2}(\pi_{h,y_0}^*)=c_hm_h^{k-2}=\chi_G(h)=\chi(h)$. Transport the action of G_K to S_{y_0} through p_0. For $\sigma\in G_K$ and v in the χ-part, with y and g as above, $(\sigma v)_{y_0}=f_{\sigma,y}^*v_y=\chi(g)^{-1}c_g\,f_{\sigma,y}^*\pi_{g,y}^*v_{y_0}$, that is, σ acts on S_{y_0} by $\chi(g)^{-1}c_g\operatorname{Sym}^{k-2}(\Pi^*)$. By the displayed identity for Π, $\Pi^*=\iota_0^*\circ m\,(f_\sigma^0)^*\circ(\iota_0^*)^{-1}$ on H^1, so σ acts by

$$\chi(g)^{-1}c_gm^{k-2}\cdot\iota_0^*\circ\operatorname{Sym}^{k-2}\big((f_\sigma^0)^*\big)\circ(\iota_0^*)^{-1}=\chi(g)^{-1}\chi_G(g)\cdot\iota_0^*\circ\sigma_{S_0}\circ(\iota_0^*)^{-1},$$

where $\sigma_{S_0}=\operatorname{Sym}^{k-2}((f_\sigma^0)^*)$ is the natural action of σ on S_0, and $\chi(g)^{-1}\chi_G(g)=1$ because $\chi_G(g)=\pm1$. So $(\iota_0^*)^{-1}\circ p_0$ identifies $(\mathbb{V}_\chi)_{\bar\xi}$ with S_0, G_K-equivariantly.

□

Crystalline comparison is needed at auxiliary split primes. Its range $p>k$ therefore imposes no extra exclusion on the prime in the final congruence, where $p\geq k$ suffices. The proof applies Faltings' logarithmic comparison [22, Theorems 5.3 and 6.2] to the Kummer cover and checks its isogeny eigenspaces and residue kernels.

Proposition 6.4. *Let (Γ,t,f,a,R) be an admissible Kummer twist datum with field L and exceptional set Σ. Let $p\notin\Sigma$ with $p>k$, let $\mathfrak{P}$ be a prime of L of residue degree one over*

p, fix an embedding $\bar{L} \subset \bar{\mathbb{Q}}_p$ inducing on L the embedding $\iota_{\mathfrak{P}} : L \to \mathbb{Q}_p$ defined by $\mathfrak{P}$, and let $D_{\mathfrak{P}} \subset \mathrm{Gal}(\bar{L}/L)$ be the corresponding decomposition group, identified with $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $V_{a,\lambda}$, $V_{\bar{a},\lambda}$ be the λ-adic realisations.

1. *For every prime λ of L above p, the restrictions of $V_{a,\lambda}$ and $V_{\bar{a},\lambda}$ to $D_{\mathfrak{P}}$ are crystalline.*
2. *Let $\lambda = \mathfrak{P}$, so that $L_\lambda = \mathbb{Q}_p$. For every modular model of the datum at $\mathfrak{P}$, with eigenlattice $H_{a,p}$ and crystalline Frobenius φ,*

$$D_{\mathrm{cris}}(V_{a,\mathfrak{P}}) = \left(V_{a,\mathfrak{P}} \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\right)^{D_{\mathfrak{P}}} \cong H_{a,p} \otimes \mathbb{Q}_p$$

as filtered φ-modules. Consequently $V_{a,\mathfrak{P}}$ and $V_{\bar{a},\mathfrak{P}}$ are two-dimensional with Hodge–Tate weights 0 and $k-1$, each of multiplicity one, in the normalisation in which $\mathbb{Q}_p(-1)$ has weight 1.

Proof. Let (X', Z', E, G, χ_G) be a modular model at $\mathfrak{P}$: one exists by Proposition 3.2, and in (2) we take the given one; the argument uses only (M1)–(M5). By Proposition 6.1(1) we may compute $V_{a,\lambda}$ and $V_{\bar{a},\lambda}$ with the sheaves of this model. Let $C^\circ = C \setminus D_C$ and $\mathbb{T} = \rho^* R^1\pi_* \mathbb{Z}_p$, a lisse sheaf on $C^\circ_{\mathbb{Q}_p}$.

Let $\lambda \mid p$, let $M_\lambda = \mathrm{Sym}^{k-2} \mathbb{T} \otimes_{\mathbb{Z}_p} L_\lambda$ on $C^\circ_{\bar{\mathbb{Q}}_p}$, with $\tilde{G}$ acting by the maps $g^\sharp$ (constants c_g and values of χ taken in L_λ), and let $\mathcal{W}_\lambda = H^1(C_{\bar{\mathbb{Q}}_p}, o_* M_\lambda)$, the image of $H^1_c(C^\circ_{\bar{\mathbb{Q}}_p}, M_\lambda) \to H^1(C^\circ_{\bar{\mathbb{Q}}_p}, M_\lambda)$ (Lemma 5.5(1)). The cohomology of the proper $\mathbb{P}^1$ does not change under the extension $\bar{L} \subset \bar{\mathbb{Q}}_p$ of algebraically closed fields, and Lemma 5.5(2), with $\Omega = \bar{\mathbb{Q}}_p$ and $\ell = p$, gives

$$V_{a,\lambda}|_{D_{\mathfrak{P}}} = e_\chi \mathcal{W}_\lambda, \qquad V_{\bar{a},\lambda}|_{D_{\mathfrak{P}}} = e_{\chi^{-1}} \mathcal{W}_\lambda,$$

with $e_{\chi^{\pm 1}}$ the idempotents attached to $\chi^{\pm 1}$. The automorphisms in $\tilde{G}$ and the isogenies π_g are defined over $\mathcal{O}_{L,(\mathfrak{P})} \subset \mathbb{Z}_p$, so the $g^\sharp$, and the $e_{\chi^{\pm 1}}$, commute with $D_{\mathfrak{P}} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. As a representation over $\mathbb{Q}_p$, $\mathcal{W}_\lambda = \mathcal{W} \otimes_{\mathbb{Q}_p} L_\lambda$ with $\mathcal{W} = H^1(C_{\bar{\mathbb{Q}}_p}, o_* \mathrm{Sym}^{k-2} \mathbb{T} \otimes \mathbb{Q}_p)$.

Following Faltings, we show that $\mathcal{W}$ is crystalline and describe $D_{\mathrm{cris}}(\mathcal{W})$. The hypotheses of the theory hold: $C_{\mathbb{Z}_p}$ is smooth and proper over $\mathbb{Z}_p$ and D_C is finite étale, a divisor with normal crossings relative to $\mathbb{Z}_p$ (Lemma 3.7(2)). By (M1), $E \to X'$ is a proper generalised elliptic curve which near each point of Z' is a Tate curve $\mathrm{Tate}(\delta q^m)$, with nodes of local equation $xy = \delta q^m$, so it is logarithmically smooth over (X', Z') for the logarithmic structures defined by Z' and its inverse image [29], Theorem 3.5, as for modular curves in [20], §2.2. By [22], Theorem 6.2, applied to $E \to X'$ as in [23], §1, p. 4, and [20], §2.2, H with its Hodge filtration, logarithmic Gauss–Manin connection and Frobenius is an object of $\mathcal{MF}^\nabla_{[0,1]}(X'_{\mathbb{Z}_p})$ associated to $R^1\pi_*\mathbb{Z}_p$. Pulling back along the morphism of logarithmic schemes $\rho : (C, D_C) \to (X', Z')$, $\rho^* H$ is an object of $\mathcal{MF}^\nabla_{[0,1]}(C_{\mathbb{Z}_p})$ associated to $\mathbb{T}$ (compatibility with pull-back, [22], V a), as used in [20], §3.3). Alternatively, Theorem 6.2 applies directly to $E \times_{X'} C \to C$, a proper generalised elliptic curve with nodes of local equation $xy = \delta' v^{em}$, v a parameter at a point of D_C and $e \mid d$ the ramification index of ρ, logarithmically smooth over (C, D_C) by the same criterion. As the association respects tensor products ([22], II h), [20], §2.2) and $k - 2 \le p - 2$, M_k is an object of $\mathcal{MF}^\nabla_{[0,k-2]}(C_{\mathbb{Z}_p})$ associated to $\mathrm{Sym}^{k-2}\mathbb{T}$. Since $1 + (k-2) \le p - 2$, that is $p > k$, the comparison theorem [22], Theorem 5.3 (in the form of [44], published version, Astérisque 280 (2002), §6.1, Theorem 7, with $i = 1$ and $a = k - 2$), identifies $H^1(C^\circ_{\bar{\mathbb{Q}}_p}, \mathrm{Sym}^{k-2}\mathbb{T})$, naturally and functorially, with the Galois module associated to the logarithmic crystalline cohomology H^1_{crys} of (C, D_C) with coefficients M_k, an object of $\mathcal{MF}_{[0,k-1]}(\mathbb{Z}_p)$; and likewise H^1_c with $H^1_{c,\mathrm{crys}}$, whose underlying filtered module is the hypercohomology of the logarithmic de Rham complex of

$M_k(-D_C)$, compatibly with the forget-supports maps ([23], §1, p. 4; [20], §§2.2 and 2.5). By the theory of Fontaine and Laffaille [25], as recalled in [20], §1.1, the Galois module associated to an object A of $\mathcal{MF}_{[0,p-2]}(\mathbb{Z}_p)$ spans a crystalline representation with $D_{\mathrm{cris}} \cong A \otimes \mathbb{Q}_p$ as filtered φ-modules, functorially. As D_{cris} is exact on crystalline representations and $\mathcal{W}$ is the image of $H^1_c \otimes \mathbb{Q}_p \to H^1 \otimes \mathbb{Q}_p$, $\mathcal{W}$ is crystalline and

$$D_{\mathrm{cris}}(\mathcal{W}) = \mathrm{Im}(H^1_{c,\mathrm{crys}} \to H^1_{\mathrm{crys}}) \otimes \mathbb{Q}_p.$$

The underlying filtered module of H^1_{crys} is the hypercohomology $H_{\mathbb{Z}}$ of the logarithmic de Rham complex of M_k, and its Frobenius is the crystalline Frobenius φ of Lemma 3.10(1), both being the logarithmic crystalline cohomology of $(C_{\mathbb{F}_p}, D_C)$ with coefficients M_k [29], Theorem 6.4.

Proof of (1). For $\lambda \mid p$, $V_{a,\lambda}|_{D_{\mathfrak{P}}}$ and $V_{\bar{a},\lambda}|_{D_{\mathfrak{P}}}$ are direct summands of $\mathcal{W} \otimes_{\mathbb{Q}_p} L_\lambda$, a direct sum of copies of the crystalline $\mathcal{W}$. A direct summand of a crystalline representation is crystalline, since the comparison map $D_{\mathrm{cris}}(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \to X \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$ is additive in X, and X is crystalline exactly when it is an isomorphism [24], §1.

The comparison commutes with $\tilde{G}$. On the étale side $(g, \zeta) \in \tilde{G}$ acts by the maps $g^\sharp = c_g \,\mathrm{Sym}^{k-2}(\pi_g^*) \circ g^*$ and $[\zeta]^*$; on the crystalline side by the action $x \mapsto g^* x$ of Lemma 3.10(1), given by the same formulas, where $c_g \,\mathrm{Sym}^{k-2}(\pi_g^*)$ extends over Z' and respects all structures by (M2). The morphisms of $\mathcal{MF}^\nabla$ and of étale sheaves to be compared are determined by their restrictions to the open part, the functor of [22], Theorem 2.6, being fully faithful ([20], §2.2), and there they come from the isogeny π_g of elliptic curves. Both are induced by the automorphism (g, ζ) of (C, D_C) together with the isogeny π_g of degree prime to p. The comparison isomorphisms are natural for such data: by [22], 4 c), and the proof of its Theorem 6.2, an isogeny of the families induces a morphism of the associated objects of $\mathcal{MF}^\nabla$ which corresponds to the induced morphism of étale sheaves (loc. cit., V a)), exactly as for the isogenies defining Hecke operators in [20], §3.3, and [23], proof of Theorem 1.2(1); and the isomorphism of Theorem 5.3 is functorial ([44], published version, §6.1, Theorem 7). So D_{cris} carries e_χ on $\mathcal{W}$ to e_χ on $D_{\mathrm{cris}}(\mathcal{W})$, the constants acting through $\iota_{\mathfrak{P}}$ on both sides.

Proof of (2). Let $\lambda = \mathfrak{P}$. By the comparison above and its equivariance, $D_{\mathrm{cris}}(V_{a,\mathfrak{P}}) = e_\chi \,\mathrm{Im}(H^1_{c,\mathrm{crys}} \to H^1_{\mathrm{crys}}) \otimes \mathbb{Q}_p \subset e_\chi H_{\mathbb{Z}} \otimes \mathbb{Q}_p$. Classes in the image of $H^1_{c,\mathrm{crys}}$ have vanishing residues: on an affine neighbourhood of $z \in D_C$ such a class is represented by a logarithmic 1-form with coefficients in $M_k(-D_C)$, and Res_z is the constant term at z of the e_0-coefficient of the component near z (proof of Lemma 3.10(3)), which vanishes. Hence $D_{\mathrm{cris}}(V_{a,\mathfrak{P}}) \subset H_{a,p} \otimes \mathbb{Q}_p$, the residue-free χ-part (Lemma 3.10(2)). Both have dimension two: $V_{a,\mathfrak{P}}$ by Proposition 6.1(1), and it is crystalline by (1); $H_{a,p}$ by Lemma 3.10(3) and (R). So they are equal, as φ-modules, and as filtered modules, the filtration of $D_{\mathrm{cris}}(V_{a,\mathfrak{P}})$ being induced from $D_{\mathrm{cris}}(\mathcal{W})$ (D_{cris} is strict) and that of $H_{a,p} \otimes \mathbb{Q}_p$ from $H_{\mathbb{Z}} \otimes \mathbb{Q}_p$.

It remains to compute the Hodge–Tate weights. Over $\mathbb{Q}_p$, $D_{\mathrm{dR}} = D_{\mathrm{cris}}$ with its filtration, and the Hodge–Tate weights are the jumps of the filtration, the jump of $\mathbb{Q}_p(-1)$ being at 1. On $H_{a,p} \otimes \mathbb{Q}_p$, F^0 is everything, $F^1 = F^{k-1}$ is a line by Lemma 3.10(3) and (P), and $F^k = 0$, the filtration of H^1 with coefficients in $\mathcal{MF}_{[0,k-2]}$ lying in $[0, k-1]$. So the weights of $V_{a,\mathfrak{P}}$ are 0 and $k-1$. By Lemma 5.4(3) and Poincaré duality on $\mathbb{P}^1_{\bar{\mathbb{Q}}_p}$ [16], (2.12) (p is invertible on $\bar{\mathbb{Q}}_p$), $V_{\bar{a},\mathfrak{P}} \cong \mathrm{Hom}(V_{a,\mathfrak{P}}, \mathbb{Q}_p(-(k-1)))$, $D_{\mathfrak{P}}$-equivariantly; it is crystalline by (1), with weights $(k-1)-0$ and $(k-1)-(k-1)$, that is $k-1$ and 0.

□

6.2 The determinant as a Hecke character

We now identify the normalised determinants $\varepsilon_{\mathfrak{P}}$ as values of one algebraic Hecke character of L. At an auxiliary split prime the determinant of the global representation is Hodge–Tate, so Serre's theory gives an algebraic Hecke character. A second auxiliary prime allows us to remove the first from the modulus. We use the degree-one-prime formulations in Lemmas A.5 and A.6.

Definition 6.5. Let K be a number field, Galois over $\mathbb{Q}$, and $\mathfrak{m}$ a nonzero integral ideal of K; let $I^{\mathfrak{m}}$ be the group of fractional ideals of K prime to $\mathfrak{m}$. An **algebraic Hecke character of K with values in K and modulus $\mathfrak{m}$** is a homomorphism $\chi : I^{\mathfrak{m}} \to K^{\times}$ for which there are integers $(a_\sigma)_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}$, its **infinity type**, such that

$$\chi((\alpha)) = \prod_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})} \sigma(\alpha)^{a_\sigma}$$

for every totally positive $\alpha \in K^{\times}$ with $\alpha \equiv 1 \bmod {}^{\times}\mathfrak{m}$.

Proposition 6.6. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum with field L and exceptional set Σ. There is an algebraic Hecke character ε of L with values in L, whose modulus is prime to every prime of L of residue degree one over a prime not in Σ, such that*

$$\varepsilon(\mathfrak{P}) = \varepsilon_{\mathfrak{P}}$$

for every prime $\mathfrak{P}$ of L of residue degree one over a prime $p \notin \Sigma$, where $\varepsilon_{\mathfrak{P}}$ is the number of Proposition 6.1(2).

Proof. Put

$$\delta(\mathfrak{P}) = \varepsilon_{\mathfrak{P}} p^{k-1}$$

for a degree-one prime $\mathfrak{P}$ above $p \notin \Sigma$, with the coefficients of Proposition 6.1(2). We construct a Hecke character with these values and then divide by the $(k-1)$st power of the norm character.

The field L is abelian over $\mathbb{Q}$, since it is generated by roots of unity and square roots of integers. Choose distinct primes $\ell_1, \ell_2 > k$, outside Σ, that split completely in L, by Chebotarev's theorem [53, Ch. I, §2.2], and primes $\lambda_i \mid \ell_i$. Write $\mathcal{P}$ for the degree-one primes of L above rational primes outside Σ.

Apply Lemma A.5 to

$$\psi_i = \det V_{\bar{a},\lambda_i} : G_L \longrightarrow L_{\lambda_i}^{\times} = \mathbb{Q}_{\ell_i}^{\times}.$$

Its Frobenius condition is Proposition 6.1(2). At every $w \mid \ell_i$, the representation $V_{\bar{a},\lambda_i}$ is crystalline by Proposition 6.4(1), since ℓ_i splits completely and $\ell_i > k$. Its determinant is therefore crystalline, hence Hodge–Tate [24, 1.5.2, 3.8, 5.1.4 and 5.1.7]. The lemma gives an algebraic Hecke character χ_i, with values in L, such that

$$\chi_i(\mathfrak{P}) = \delta(\mathfrak{P}) \qquad (\mathfrak{P} \in \mathcal{P},\ \mathfrak{P} \nmid \ell_i).$$

Its modulus is supported on the primes above ℓ_i and on a finite set T_i disjoint from $\mathcal{P}$.

Choose an ideal $\mathfrak{n}$, supported on $T_1 \cup T_2$, and ideals $\mathfrak{r}_i$, supported on the primes above ℓ_i, with exponents large enough that $\mathfrak{n}\mathfrak{r}_i$ is a modulus for χ_i. These three supports are pairwise disjoint: every prime above either ℓ_i belongs to $\mathcal{P}$. The characters agree on the degree-one primes prime to both moduli, so Lemma A.6(2) gives an algebraic Hecke character χ of modulus $\mathfrak{n}$ restricting to both. Every $\mathfrak{P} \in \mathcal{P}$ is prime to $\mathfrak{n}$ and to at least one ℓ_i; using that i shows $\chi(\mathfrak{P}) = \delta(\mathfrak{P})$.

The absolute norm N is an algebraic Hecke character of modulus (1) and infinity type $(1)_\sigma$: for totally positive α,

$$\mathrm{N}((\alpha)) = |N_{L/\mathbb{Q}}(\alpha)| = \prod_\sigma \sigma(\alpha).$$

Here L is totally real or totally imaginary, so the norm on such α is positive. Thus

$$\varepsilon = \chi\,\mathrm{N}^{1-k}$$

is an algebraic Hecke character with values in L, whose modulus is prime to $\mathcal{P}$. Since $\mathrm{N}\mathfrak{P} = p$ for $\mathfrak{P} \in \mathcal{P}$ above p, it has the required value $\varepsilon(\mathfrak{P}) = \varepsilon_{\mathfrak{P}}$.

□

6.3 The rational-field newform

When $L = \mathbb{Q}$, the Kummer character is quadratic and the multiplier is real. In this case the two character summands coincide, and Poincaré duality gives a pairing on the global representation itself. This determines the determinant and gives oddness. Lemma A.9 then produces a weight-k newform from the Frobenius polynomials at good primes and the crystalline comparison at an auxiliary prime.

Lemma 6.7. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum with field $L = \mathbb{Q}$ and exceptional set Σ. For a prime ℓ let $V_{a,\ell}$ and $V_{\bar{a},\ell}$ be its ℓ-adic realisations, representations of $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then $V_{a,\ell} \cong V_{\bar{a},\ell}$, and there is a perfect $\mathbb{Q}_\ell$-bilinear pairing*

$$B_\ell : V_{\bar{a},\ell} \times V_{\bar{a},\ell} \to \mathbb{Q}_\ell(1-k)$$

such that $B_\ell(\sigma x, \sigma y) = \sigma B_\ell(x, y)$ for $\sigma \in G_\mathbb{Q}$ and

$$B_\ell(y, x) = (-1)^{k-1} B_\ell(x, y) \qquad (x, y \in V_{\bar{a},\ell}).$$

Thus B_ℓ is alternating if k is even and symmetric if k is odd.

Proof. We first identify the two sheaves. Fix a prime $p \notin \Sigma$ with $p \neq \ell$, and a modular model of the datum at p (Definition 3.1), which exists by Proposition 3.2; then $\mathcal{U} = U$ is smooth over $\mathbb{Z}_{(p)}$ with generic fibre $U_\mathbb{Q} = \mathbb{P}^1_\mathbb{Q} \setminus S$. The character $\chi_\Phi(g, \zeta) = \chi_G(g)\zeta^j$ of $\tilde{G}$ has finite order and values in $L^\times = \mathbb{Q}^\times$ ((M4) and $\mu_2 \subset \mathbb{Q}$), hence values ± 1, so $\chi_\Phi^{-1} = \chi_\Phi$ and the χ_Φ- and χ_Φ^{-1}-parts of $\mathcal{M}$ coincide: $\mathcal{F}_a = \mathcal{F}_{\bar{a}}$ by their definition (Definition 5.2(2)). Put $\mathcal{F} = \mathcal{F}_a|_{U_\mathbb{Q}}$. By Proposition 6.1(1), $V_{a,\ell}$ and $V_{\bar{a},\ell}$ are both isomorphic, as representations of $G_\mathbb{Q}$, to $V = H^1(\mathbb{P}^1_{\bar{\mathbb{Q}}}, o_*\mathcal{F})$, where $o : U_{\bar{\mathbb{Q}}} \hookrightarrow \mathbb{P}^1_{\bar{\mathbb{Q}}}$. It suffices to construct B on V.

We next show that $\mathcal{F}$ carries a perfect pairing $\beta : \mathcal{F} \otimes \mathcal{F} \to \mathbb{Q}_\ell(2-k)$ of lisse sheaves on $U_\mathbb{Q}$ with $\beta(y \otimes x) = (-1)^k \beta(x \otimes y)$. By Lemma 5.3(1), the perfect alternating cup-product pairing $\langle\,,\rangle : \mathcal{H} \otimes \mathcal{H} \to \mathbb{Q}_\ell(-1)$ induces the pairing

$$\langle x_1 \cdots x_{k-2}, y_1 \cdots y_{k-2} \rangle = \frac{1}{(k-2)!} \sum_{\sigma \in S_{k-2}} \prod_{i=1}^{k-2} \langle x_i, y_{\sigma(i)} \rangle$$

on $\mathrm{Sym}^{k-2}\mathcal{H}$; composed with the trace of the finite étale map ϖ it gives a pairing on $\mathcal{M}$ whose restriction to $\mathcal{F}_a \otimes \mathcal{F}_{\bar{a}}$ is perfect on U, by Lemma 5.3(2), whose hypotheses hold by Lemma 5.4(1), (2). Let β be its restriction to $U_\mathbb{Q}$. Since $\langle y_i, x_{\sigma(i)} \rangle = -\langle x_{\sigma(i)}, y_i \rangle$, substituting σ^{-1} for σ gives $\langle y, x \rangle = (-1)^{k-2}\langle x, y \rangle$ on $\mathrm{Sym}^{k-2}\mathcal{H}$; this sign survives the trace and the restriction, so $\beta(y \otimes x) = (-1)^k \beta(x \otimes y)$.

By β, $\mathcal{F} \cong \mathcal{F}^\vee(2-k)$, so $\mathcal{F}^\vee(1) \cong \mathcal{F}(k-1)$. By the duality theorem [16], Théorème (2.12), applied to the projective smooth connected curve $X = \mathbb{P}^1_{\bar{\mathbb{Q}}}$ over the algebraically closed field $\bar{\mathbb{Q}}$, the dense open set $U_{\bar{\mathbb{Q}}}$ and the lisse sheaf $\mathcal{F}|_{U_{\bar{\mathbb{Q}}}}$, the pairing $\mathrm{Tr}(x \cup y)$ between $H^1(X, o_*\mathcal{F})$ and $H^1(X, o_*\mathcal{F}^\vee(1))$ is perfect. Transported along $\mathcal{F}^\vee(1) \cong \mathcal{F}(k-1)$ it becomes

$$B(x,y) = \mathrm{Tr}\big(\beta_*(x \cup y)\big) \in \mathbb{Q}_\ell(1-k) \qquad (x, y \in V),$$

where $x \cup y \in H^2(X, o_*\mathcal{F} \otimes o_*\mathcal{F})$, β_* is induced by $o_*\mathcal{F} \otimes o_*\mathcal{F} \to o_*(\mathcal{F} \otimes \mathcal{F}) \xrightarrow{\beta} o_*\mathbb{Q}_\ell(2-k) = \mathbb{Q}_\ell(2-k)$, and $\mathrm{Tr} : H^2(X, \mathbb{Q}_\ell(2-k)) \to \mathbb{Q}_\ell(1-k)$ is the trace map. So B is perfect.

The pairing B is $G_{\mathbb{Q}}$-equivariant. The sheaf $\mathcal{F}$ and the pairing β are defined on $U_{\mathbb{Q}}$, and $G_{\mathbb{Q}}$ acts on the cohomology of $X = \mathbb{P}^1_{\mathbb{Q}} \otimes \bar{\mathbb{Q}}$ through the second factor. The cup product and β_* commute with this action by functoriality, and so does the trace map: it is the stalk at $\mathrm{Spec}\,\bar{\mathbb{Q}}$ of the trace morphism $R^2h_*\mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell$ of the smooth proper map $h : \mathbb{P}^1_{\mathbb{Q}} \to \mathrm{Spec}\,\mathbb{Q}$, a morphism of sheaves on $\mathrm{Spec}\,\mathbb{Q}$ [55], Exp. XVIII, Théorème 2.9. Hence $B(\sigma x, \sigma y) = \sigma B(x,y)$, where σ acts on $\mathbb{Q}_\ell(1-k)$ through χ_ℓ^{1-k}, χ_ℓ the ℓ-adic cyclotomic character.

Finally we compute the symmetry of B. For classes x, y of degree one, $y \cup x = -s_*(x \cup y)$, where s is the symmetry of $o_*\mathcal{F} \otimes o_*\mathcal{F}$: this is the Koszul sign rule for cup products [54], Cycle, 1.2–1.3, used in the same way in loc. cit., Sommes trig., 3.6 c). As $\beta \circ s = (-1)^k\beta$, we get $\beta_* \circ s_* = (-1)^k\beta_*$, and so

$$B(y,x) = \mathrm{Tr}\big(\beta_*(y \cup x)\big) = -(-1)^k\,\mathrm{Tr}\big(\beta_*(x \cup y)\big) = (-1)^{k-1}B(x,y).$$

□

Proposition 6.8. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum with field $L = \mathbb{Q}$ and exceptional set Σ. For a prime ℓ let $V_\ell = V_{\bar{a},\ell}$ be its ℓ-adic realisation, a representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and for $p \notin \Sigma$ let $A_p \in \mathbb{Z}$ and $\varepsilon_p \in \{\pm 1\}$ be as in Proposition 6.1(3). Let χ_ℓ be the ℓ-adic cyclotomic character and $c \in G_{\mathbb{Q}}$ a complex conjugation.*

1. *There is a continuous character $\psi : G_{\mathbb{Q}} \to \{\pm 1\}$, unramified outside Σ, such that $\det V_\ell = \psi\,\chi_\ell^{1-k}$ for every prime ℓ, and $\psi(\mathrm{Frob}_p) = \varepsilon_p$ for every prime $p \notin \Sigma$.*
2. *If k is even, then $\psi = 1$.*
3. *If k is odd, then $\psi \neq 1$ and $\ker\psi = G_K$ for an imaginary quadratic field K unramified outside Σ; moreover $A_p = 0$ for every prime $p \notin \Sigma$ inert in K.*
4. *$\det V_\ell(c) = -1$ for every prime ℓ.*
5. *There is a prime $p_0 \notin \Sigma$ such that $X^2 - A_{p_0}X + \varepsilon_{p_0}p_0^{k-1}$ is irreducible over $\mathbb{Q}$.*

Proof. The crux is that K in (3) is imaginary: otherwise the Hodge–Tate weights 0 and $k-1$ at a prime split in K would make a unit of infinite order a root of unity.

Proof of (1). For a prime ℓ put $\psi_\ell = \det V_\ell \cdot \chi_\ell^{k-1}$, a continuous character $G_{\mathbb{Q}} \to \mathbb{Q}_\ell^\times$, unramified outside $\Sigma \cup \{\ell\}$. For $p \notin \Sigma \cup \{\ell\}$, $\psi_\ell(\mathrm{Frob}_p) = \varepsilon_p p^{k-1} \cdot p^{1-k} = \varepsilon_p \in \{\pm 1\}$. So ψ_ℓ^2 is trivial on a dense subset, hence trivial, and ψ_ℓ takes values in $\{\pm 1\}$. For two primes ℓ, ℓ', ψ_ℓ and $\psi_{\ell'}$ agree at Frob_p for all $p \notin \Sigma \cup \{\ell, \ell'\}$, hence everywhere. Their common value ψ is unramified outside $\Sigma \cup \{\ell\}$ for every ℓ, hence outside Σ, and $\psi(\mathrm{Frob}_p) = \varepsilon_p$ for $p \notin \Sigma$ (take $\ell \neq p$). This proves (1).

Proof of (2). Let ℓ be a prime. By Lemma 6.7, B_ℓ is a perfect alternating pairing on the two-dimensional space V_ℓ with $B_\ell(\sigma x, \sigma y) = \chi_\ell(\sigma)^{1-k}B_\ell(x,y)$ for $\sigma \in G_{\mathbb{Q}}$. For a basis e_1, e_2 of

V_ℓ we have $B_\ell(\sigma e_1, \sigma e_2) = \det(\sigma)\, B_\ell(e_1, e_2)$ and $B_\ell(e_1, e_2) \neq 0$; so $\det V_\ell = \chi_\ell^{1-k}$, that is, $\psi_\ell = 1$, and $\psi = 1$. This proves (2).

Proof of (3). Until the proof of (4), k is odd. We first show that $\psi \neq 1$. The family $(V_\ell)_\ell$ satisfies the hypotheses of Lemma A.7(2) with $T = \Sigma$: (a) is Proposition 6.1(3), and (b) is Proposition 6.4(2) with $\mathfrak{P} = \ell$, which applies to every decomposition group at ℓ because for $L = \mathbb{Q}$ every embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell$ induces $\iota_{\mathfrak{P}}$. Fix a prime $\ell_0 \notin \Sigma$ with $\ell_0 > k$; then $V = V_{\ell_0}$ is absolutely irreducible. Let B be the pairing of Lemma 6.7 on V; it is symmetric, as k is odd. If $\psi = 1$, then $\wedge^2 V = \det V \cong \mathbb{Q}_{\ell_0}(1-k)$, and $D(x, y) = x \wedge y$ is a perfect alternating $G_{\mathbb{Q}}$-equivariant pairing $V \times V \to \mathbb{Q}_{\ell_0}(1-k)$. Both B and D define isomorphisms $V \to V^\vee(1-k)$ of absolutely irreducible representations, so by Schur's lemma $B = \gamma D$ with $\gamma \in \mathbb{Q}_{\ell_0}^\times$; but a nonzero bilinear form in characteristic zero is not both symmetric and alternating. Hence $\psi \neq 1$, and $\ker \psi = G_K$ for a quadratic field K, unramified outside Σ as ψ is.

Let ℓ be any prime and $B = B_\ell$ the symmetric pairing of Lemma 6.7. As B is nondegenerate and $\bar{\mathbb{Q}}_\ell$ is algebraically closed of characteristic zero, there is a basis e_1, e_2 of $V_\ell \otimes \bar{\mathbb{Q}}_\ell$ with $B(e_1, e_1) = B(e_2, e_2) = 0$ and $B(e_1, e_2) = 1$. Then $B(x_1 e_1 + x_2 e_2, x_1 e_1 + x_2 e_2) = 2x_1 x_2$, so the isotropic lines are exactly $\bar{\mathbb{Q}}_\ell e_1$ and $\bar{\mathbb{Q}}_\ell e_2$. Every $\sigma \in G_{\mathbb{Q}}$ satisfies $B(\sigma x, \sigma y) = \chi_\ell(\sigma)^{1-k} B(x, y)$, so it permutes these two lines.

- If σ fixes both, $\sigma e_1 = \alpha e_1$ and $\sigma e_2 = \beta e_2$ with $\alpha\beta = B(\sigma e_1, \sigma e_2) = \chi_\ell(\sigma)^{1-k}$, so $\det \sigma = \chi_\ell(\sigma)^{1-k}$.
- If σ exchanges them, $\sigma e_1 = \beta e_2$ and $\sigma e_2 = \alpha e_1$ with $\alpha\beta = \chi_\ell(\sigma)^{1-k}$, so $\det \sigma = -\chi_\ell(\sigma)^{1-k}$ and $\operatorname{tr} \sigma = 0$.

By (1), σ fixes both lines if and only if $\psi(\sigma) = 1$, that is, $\sigma \in G_K$. So G_K acts on $\bar{\mathbb{Q}}_\ell e_i$ through a character $\xi_i : G_K \to \bar{\mathbb{Q}}_\ell^\times$, and for $\tau \in G_{\mathbb{Q}} \setminus G_K$ and $\delta \in G_K$,

$$\xi_1(\tau\delta\tau^{-1}) = \xi_2(\delta).$$

If $p \notin \Sigma$ is inert in K, take $\ell \neq p$: then $\mathrm{Frob}_p \notin G_K$, so $A_p = \operatorname{tr} V_\ell(\mathrm{Frob}_p) = 0$.

We next compute Hodge–Tate weights at a prime split in K. Let $\ell \notin \Sigma$ be a prime with $\ell > k$ that splits in K; there are infinitely many. Let $\iota_w : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell$ be an embedding, with decomposition group D_w, let $\tau \in G_{\mathbb{Q}} \setminus G_K$, and put $\iota_{w'} = \iota_w \circ \tau^{-1}$, with decomposition group $D_{w'} = \tau D_w \tau^{-1}$. Identify D_w and $D_{w'}$ with $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ through ι_w and $\iota_{w'}$; then $\delta \mapsto \tau\delta\tau^{-1}$ becomes the identity of $G_{\mathbb{Q}_\ell}$. As ℓ splits in K, $D_w, D_{w'} \subset G_K$, and $\iota_1 = \iota_w|_K$, $\iota_2 = \iota_{w'}|_K$ are the two embeddings $K \to \mathbb{Q}_\ell$, with $\iota_2(x) = \iota_1(\bar{x})$, where $x \mapsto \bar{x}$ is the nontrivial automorphism of K (the restriction of τ^{-1}). Let $E \subset \bar{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$ containing the coordinates of e_1, e_2 in a $\mathbb{Q}_\ell$-basis of V_ℓ; then the ξ_i take values in $E^\times$, and $(V_\ell \otimes_{\mathbb{Q}_\ell} E)|_{G_K} = E(\xi_1) \oplus E(\xi_2)$. By Proposition 6.4(2), $V_\ell|_{D_w}$ is crystalline with Hodge–Tate weights 0 and $k-1$. As crystalline representations are stable under subquotients and D_{HT} is exact on Hodge–Tate representations [24], §1, each $\xi_i|_{D_w}$ is crystalline with a single Hodge–Tate weight h_i, and $\{h_1, h_2\} = \{0, k-1\}$. By the display above and the identification of $\delta \mapsto \tau\delta\tau^{-1}$ with the identity, the character of $G_{\mathbb{Q}_\ell}$ given by $\xi_1|_{D_{w'}}$ is the one given by $\xi_2|_{D_w}$; so $\xi_1|_{D_{w'}}$ has Hodge–Tate weight h_2. As $0 \leq h_i \leq k-1 \leq \ell-2$, Lemma A.7(1) gives $\xi_1|_{D_w} = \mu_1 \chi_\ell^{-h_1}$ and $\xi_1|_{D_{w'}} = \mu_2 \chi_\ell^{-h_2}$ as characters of $G_{\mathbb{Q}_\ell}$, with μ_1, μ_2 unramified.

Suppose now that K is real, and let $\eta = \xi_1 \circ \mathrm{Art}_K$, where $\mathrm{Art}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ is the reciprocity map; η is a continuous character, trivial on $K^\times$, and its restriction η_v to $K_v^\times$ is ξ_1 composed with the local reciprocity map of K_v and the decomposition map [45], Ch. VI, §5. We evaluate η_v on units.

- If v is finite and lies over a prime not in $\Sigma \cup \{\ell\}$, then ξ_1 is unramified at v, since V_ℓ is unramified at the prime below v, and the local reciprocity map sends $\mathcal{O}_v^\times$ into the inertia group [45], Ch. V, §1; so $\eta_v(\mathcal{O}_v^\times) = 1$.
- If v lies over $q \in \Sigma$, then $q \neq \ell$, and η_v is trivial on an open subgroup U_v of $\mathcal{O}_v^\times$: $\mathcal{O}_v^\times$ is a finite group times a pro-q group, $\mathcal{O}_E^\times$ has an open pro-ℓ subgroup, and a pro-q group has no nontrivial continuous pro-ℓ quotient.
- If v is real, η_v is trivial on the connected group $\mathbb{R}_{>0}$, as $E^\times$ is totally disconnected.
- If $v = v_i$ is the place of ι_i, so that $K_{v_i} = \mathbb{Q}_\ell$, then for $x \in \mathbb{Z}_\ell^\times$, $\eta_{v_i}(x) = \chi_\ell(\mathrm{Art}_{\mathbb{Q}_\ell}(x))^{-h_i} = x^{h_i}$, since μ_i is unramified and $\chi_\ell(\mathrm{Art}_{\mathbb{Q}_\ell}(x)) = x^{-1}$ in the normalisation in which uniformisers map to arithmetic Frobenius elements [45], Ch. V, §2; the other normalisation replaces x^{h_i} by x^{-h_i}, which changes nothing below.

By Dirichlet's unit theorem [45], Ch. I, (7.4), $\mathcal{O}_K^\times$ has rank one. Its subgroup of totally positive units lying in U_v for every v over Σ has finite index, so it contains a unit u of infinite order, and $u\bar{u} = N_{K/\mathbb{Q}}(u) = 1$ because u is totally positive. Then

$$1 = \eta(u) = \prod_v \eta_v(u) = \iota_1(u)^{h_1}\,\iota_2(u)^{h_2} = \iota_1(u)^{h_1}\,\iota_1(\bar{u})^{h_2} = \iota_1(u)^{h_1-h_2}.$$

As $h_1 - h_2 = \pm(k-1) \neq 0$, $\iota_1(u)$ is a root of unity, hence so is u, a contradiction. So K is imaginary, and (3) is proved.

Proof of (4). If k is even, $\psi = 1$ by (2), and $\det V_\ell(c) = \chi_\ell(c)^{1-k} = (-1)^{k-1} = -1$. If k is odd, $\chi_\ell(c)^{1-k} = 1$, and c acts nontrivially on the imaginary quadratic field K, so $\det V_\ell(c) = \psi(c) = -1$.

Proof of (5). If k is even, every $p \notin \Sigma$ will do: a rational root r of Q_p is real with $|r| = p^{(k-1)/2}$, so $r^2 = p^{k-1}$ with $k-1$ odd, which is impossible. Let k be odd, and fix ℓ and w as in the computation of Hodge–Tate weights. As $\xi_1|_{D_w}$ and $\xi_2|_{D_w}$ have the distinct Hodge–Tate weights $h_1 \neq h_2$, $\xi_1 \neq \xi_2$, so $\{\delta \in G_K : \xi_1(\delta) \neq \xi_2(\delta)\}$ is open, nonempty and stable under conjugation. By Chebotarev it contains a Frobenius element Frob_v at a place v of K of degree one over a prime $p_0 \notin \Sigma \cup \{\ell\}$. Then p_0 splits in K, so $\varepsilon_{p_0} = \psi(\mathrm{Frob}_{p_0}) = 1$, and Frob_v is a Frobenius element at p_0 in $G_\mathbb{Q}$. The roots of Q_{p_0} are therefore $\xi_1(\mathrm{Frob}_v) \neq \xi_2(\mathrm{Frob}_v)$, with product p_0^{k-1}. If Q_{p_0} were reducible over $\mathbb{Q}$, its roots would be real of absolute value $p_0^{(k-1)/2}$ and distinct, hence $\pm p_0^{(k-1)/2}$, with product $-p_0^{k-1}$, a contradiction.

□

Proposition 6.9. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum with field $L = \mathbb{Q}$ and exceptional set Σ, and for a prime $p \notin \Sigma$ let $A_p \in \mathbb{Z}$ and $\varepsilon_p \in \{\pm 1\}$ be the numbers of Proposition 6.1(2), (3).*

1. *There is a newform g of weight k and nebentypus ψ, of level divisible only by primes in Σ, such that $A_p = a_p(g)$ and $\varepsilon_p = \psi(p)$ for every prime $p \notin \Sigma$.*
2. *If k is odd, there is an imaginary quadratic field K, unramified outside Σ, with quadratic character χ_K, such that $\psi(p) = \chi_K(p)$ and $a_p(g) = \chi_K(p)\,a_p(g)$ for every prime $p \notin \Sigma$; that is, g has complex multiplication by K.*

Proof. We check the hypotheses of Lemma A.9 for the family $(V_\ell)_\ell$, with $T = \Sigma$ and $Q_p(X) = X^2 - A_pX + \varepsilon_p p^{k-1}$.

- (a): by Proposition 6.1(1), (3), each V_ℓ is two-dimensional, and for $p \notin \Sigma$ and $\ell \neq p$ it is unramified at p; Q_p lies in $\mathbb{Z}[X]$ ($A_p \in \mathbb{Z}$, $\varepsilon_p = \pm 1$), does not depend on ℓ, and has roots of absolute value $p^{(k-1)/2}$ under every complex embedding.
- (b): for $\ell \notin \Sigma$ with $\ell > k$, Proposition 6.4(2), with $\mathfrak{P} = \ell$ and $\lambda = \mathfrak{P}$, shows that V_ℓ restricted to a decomposition group at ℓ is crystalline with Hodge–Tate weights 0 and $k-1$, in the normalisation in which $\mathbb{Q}_\ell(-1)$ has weight 1.
- (c): $\det V_\ell(c) = -1$ for every prime ℓ, by Proposition 6.8(4).
- (d): by Proposition 6.8(5) there is $p_0 \notin \Sigma$ with Q_{p_0} irreducible over $\mathbb{Q}$.

By Lemma A.9 there is a newform g of weight k and nebentypus ψ, of level divisible only by primes in Σ, with $X^2 - A_p X + \varepsilon_p p^{k-1} = X^2 - a_p(g)X + \psi(p)p^{k-1}$ for every prime $p \notin \Sigma$. Comparing coefficients, $A_p = a_p(g)$ and $\varepsilon_p = \psi(p)$ for every prime $p \notin \Sigma$. This proves (1).

For (2), let k be odd, let ψ' be the character of $G_\mathbb{Q}$ of Proposition 6.8(1), and let K be the imaginary quadratic field of Proposition 6.8(3), with $\ker \psi' = G_K$, unramified outside Σ. A prime $p \notin \Sigma$ is unramified in K, and $\chi_K(p) = \psi'(\mathrm{Frob}_p) = \varepsilon_p = \psi(p)$, by Proposition 6.8(1) and item (1) above. If p is inert in K, then $a_p(g) = A_p = 0$ by Proposition 6.8(3); otherwise $\chi_K(p) = 1$. In both cases $a_p(g) = \chi_K(p)a_p(g)$.

□

6.4 The general theorem

We combine the integral cusp relation with the common Frobenius polynomial to prove Theorem B. The full statement records the point-count formula, the bounds on the trace and the criterion for the cusp expansion to extend to a modular form.

Theorem 6.10. *Let (Γ, t, f, a, R) be an admissible Kummer twist datum, with $a = j/d$, coefficients $c(N)$, field L and exceptional set Σ. Let $\mathfrak{P}$ be a prime of L of residue degree one over a prime $p \notin \Sigma$ (so $p \equiv 1 \pmod d$), and let $\iota_\mathfrak{P} : L \to \mathbb{Q}_p$ be the embedding defined by $\mathfrak{P}$, which maps $\mathcal{O}_L[\frac{1}{dM}]$ into $\mathbb{Z}_p$. Read congruences through $\iota_\mathfrak{P}$. There are $A_\mathfrak{P} \in \mathcal{O}_L$ and $\varepsilon_\mathfrak{P} \in L^\times$ with the following properties. Write $B_\mathfrak{P} = \varepsilon_\mathfrak{P} p^{k-1}$ for the constant term of the polynomial.*

1. *For all integers $m \geq 1$ prime to p and $r \geq 1$,*

$$c(mp^r) - A_\mathfrak{P}\, c(mp^{r-1}) + B_\mathfrak{P}\, c(mp^{r-2}) \equiv 0 \pmod{p^{(k-1)r}},$$

   where $c(x) = 0$ for $x \notin \mathbb{Z}$.

2. *For every prime λ of L not above p, the global λ-adic realisation $V_{\bar a,\lambda}$ of H_a is unramified at $\mathfrak{P}$, and $X^2 - A_\mathfrak{P} X + B_\mathfrak{P}$ is the characteristic polynomial of geometric Frobenius there. The trace $A_\mathfrak{P}$ is given by the twisted point-count formula of Proposition 5.6(3). Moreover $|\sigma(A_\mathfrak{P})| \leq 2p^{(k-1)/2}$ and $|\sigma(\varepsilon_\mathfrak{P})| = 1$ for every embedding $\sigma : L \to \mathbb{C}$, and there is an algebraic Hecke character ε of L with values in L such that $\varepsilon(\mathfrak{P}') = \varepsilon_{\mathfrak{P}'}$ for every prime $\mathfrak{P}'$ of L of residue degree one over a prime not in Σ.*

3. *If $L = \mathbb{Q}$ (so that $d = 2$ and χ_f takes the values ± 1), there is a normalised newform g of weight k and nebentypus ψ, of level supported on Σ, such that $A_{p'} = a_{p'}(g)$ and $\varepsilon_{p'} = \psi(p')$ for every prime $p' \notin \Sigma$, where $A_{p'}$ and $\varepsilon_{p'}$ are the numbers attached to $\mathfrak{P} = p'\mathbb{Z}$.*

4. *Let P be the pole of t on X_Γ and, if P is an interior point, let e_P be its stabiliser order. If P is not a cusp and $d \nmid e_P$, then $\tau \mapsto f_\Phi(e^{2\pi i\tau/d})$ is not the restriction to a half-plane $\operatorname{Im}\tau > y_0$, on which it converges, of a single-valued meromorphic function on $\mathbb{H}$; in particular it is not a meromorphic modular form, of any weight and multiplier system, for any subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. If P is a cusp or $d \mid e_P$, it is a cusp form of weight k on a subgroup of finite index in Γ.*

Proof. The main inputs are Theorem 4.1 and Proposition 5.6. Choose a modular model at $\mathfrak{P}$ by Proposition 3.2, with eigenlattice $H_{a,p}$ and Frobenius φ. Proposition 5.6(2) supplies $A_\mathfrak{P} \in \mathcal{O}_L$ and $\varepsilon_\mathfrak{P} \in L^\times$ such that

$$X^2 - \iota_\mathfrak{P}(A_\mathfrak{P})X + \iota_\mathfrak{P}(\varepsilon_\mathfrak{P})p^{k-1}$$

is the characteristic polynomial of $p^{k-1}\varphi^{-1}$. By Proposition 6.1(1),(2),the single global system $V_{\bar{a},\lambda}$ is unramified at $\mathfrak{P}$ for every λ not above p, and these are the coefficients of its geometric Frobenius polynomial there; they are independent of the modular model and of λ.

Apply Theorem 4.1(2),(3) to this model. Its coefficients $A^\vee$ and $B^\vee$ are precisely $\iota_\mathfrak{P}(A_\mathfrak{P})$ and $\iota_\mathfrak{P}(B_\mathfrak{P})$, so it gives (1). The required integrality of the expansion is Lemma 3.12(2).

The point-count expression and the bounds in (2) are Proposition 5.6(3),(4). The algebraic Hecke character with these normalised determinant values is supplied by Proposition 6.6; its values agree at every degree-one prime outside Σ with those of the same global system. This proves (2).

If $L = \mathbb{Q}$, then $\mu_d \subset L$ and $0 < a < 1$ in lowest terms force $d = 2$, and the finite-order multiplier has values ± 1. Every rational prime outside Σ has degree one. Applying Proposition 6.9(1) to the global system gives (3), with one newform for all these primes.

Finally, Proposition 2.10 applies because admissibility includes (F). It gives the failure of single-valued meromorphic continuation when P is not a cusp and $d \nmid e_P$, and the cusp-form conclusion in the other cases. A meromorphic modular form on a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ is single-valued on $\mathbb{H}$, which gives the stated consequence in (4).

□

6.5 The level-one specialisation

Proof of Theorem A. For $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ take

$$(t, f, a, R) = \left(\frac{1}{j}, \frac{E_4^2}{E_6}, \frac{1}{2}, 1\right).$$

Since $q\,d\log t/dq = E_6/E_4$ and $j = E_4^3/\Delta$, the coefficient of dq/q in Φ is $E_4\sqrt{t} = h$. The normalisations at the cusp hold with $M = 1$ since $t \in q + q^2\mathbb{Z}[[q]]$, and E_4^2/E_6 has integral coefficients and constant term one. The compositional inverse $q(t)$ is integral as well. The multiplier is trivial and the singular set is $\{0, 1/1728, \infty\}$, corresponding to the cusp, i and ρ, respectively. Indeed, the only interior zeros and poles of f occur at the two elliptic points. Their t-values and the cusp value are distinct modulo every $p \geq 5$. Thus $L = \mathbb{Q}$ and $\Sigma = \{2, 3\}$ by Definition 2.3.

The untwisted local system is the symmetric square of the standard elliptic local system. At the cusp its monodromy is unipotent, and the quadratic twist multiplies it by -1. At i the twist is unramified and the eigenvalues are $1, -1, -1$. At ρ the untwisted eigenvalues are the three cube roots of unity and the twist multiplies them by -1. The local invariant dimensions are therefore $0, 1, 0$, so Lemma 2.6(2) gives

$$\dim H_{1/2} = (3 - 2) \cdot 3 - (0 + 1 + 0) = 2.$$

The coefficient h has order $-1/2$ at ρ, is holomorphic away from its translates in $\mathbb{H}$, and has order $1/2$ at the cusp in the parameter q. These orders give condition (F). As $a = 1 - a$ and the multiplier is trivial, Proposition 2.9 gives the two extreme Hodge types. The datum is admissible.

Let A_p be its Frobenius trace for $p \geq 5$. The weight is even, so Proposition 6.8(2) gives determinant p^3. Applying Theorem 6.10 at $m = r = 1$ gives $A_p \equiv b(p) \pmod{p^3}$, since $b(1) = 1$. Proposition 6.1(3) gives $A_p \in \mathbb{Z}$ and $|A_p| \leq 2p^{3/2}$. The interval has length $4p^{3/2} < p^3$, so this congruence determines A_p. For the primes needed below it gives

p	5	7	11	13	17	19	23	31	37
A_p	14	-24	-28	-74	82	92	8	80	30

These values follow by solving $E_4(q)B(q)^2 = \prod_{n\geq 1}(1-q^n)^{24}$ modulo q^{19} with $B(0) = 1$, and taking $b(p) = [q^{(p-1)/2}]B(q)$.

The Hecke eigenvalues $a_p(g)$ of the newform `24.4.a.a` agree with all nine entries [39]. To deduce equality at every $p \geq 5$, compare $V_{\bar{a},2}$ with the two-dimensional representation attached to g, in the convention that geometric Frobenius has trace $a_p(g)$. Both representations are defined over $\mathbb{Q}_2$, are unramified outside $\{2, 3\}$, and have determinant $\mathbb{Q}_2(-3)$. For $V_{\bar{a},2}$ these assertions follow from Propositions 6.1 and 6.8. For g they follow from its rational coefficients, level 24 and trivial character. Their common trace at 11 is -28, hence even. The nine trace equalities now imply that the semisimplifications are isomorphic by Schütt's comparison criterion [49, Proposition 1]. Thus $A_p = a_p(g)$ for every $p \geq 5$, and the asserted congruences follow from Theorem 6.10. □

6.6 The quintic twist and a Hilbert modular conjecture

Keeping $t = 1/j$, $f = E_4^2/E_6$ and $R = 1$ in the preceding example, let the exponent a vary. The coefficient of dq/q is now $h_a = E_4 t^a$. Its orders at the cusp and at ρ are a and $1 - 3a$, respectively. For $0 < a < 1$ the cusp condition holds for both a and $1 - a$, and neither twist has any other interior singularity up to the modular group. Thus the two square-integrability conditions at ρ are

$$1 - 3a > -1, \qquad 1 - 3(1 - a) > -1.$$

They give exactly $1/3 < a < 2/3$. For rational a in this interval the Kummer character at ρ has order not dividing three, so none of the three monodromy eigenvalues there is 1 after twisting. The local invariant dimensions at the cusp, i and ρ are again $0, 1, 0$. Lemma 2.6 and Proposition 2.9 give admissibility throughout the interval.

The smallest denominator after two is five. The exponents $2/5$ and $3/5$ give inverse characters on the cover $s^5 = t$. Take $a = 2/5$ and write

$$h_{2/5} = \frac{\Delta^{2/5}}{E_4^{1/5}} = \sum_{n\geq 1} c(n)Q^n, \qquad Q = q^{1/5}, \quad c(2) = 1.$$

Put $K = \mathbb{Q}(\zeta_5)$ and let ρ_λ be the representation on the inverse character component as in Theorem B. For every prime $p \equiv 1 \pmod 5$ and $\mathfrak{P} \mid p$ in K, let $X^2 - A_{\mathfrak{P}}X + B_{\mathfrak{P}}$ be its geometric Frobenius polynomial. The theorem gives

$$c(mp^r) - A_{\mathfrak{P}}c(mp^{r-1}) + B_{\mathfrak{P}}c(mp^{r-2}) \equiv 0 \pmod{p^{3r}} \qquad (p \nmid m,\ m, r \geq 1)$$

in $K_{\mathfrak{P}}$, with $c(x) = 0$ for nonintegral x.

To determine the possible Hilbert weight, consider the four embeddings of K. They fall into two pairs above the real places of $F = \mathbb{Q}(\sqrt{5})$. At the two embeddings corresponding to the

characters with exponents $2/5$ and $3/5$, admissibility gives Hodge–Tate weights $\{0,3\}$, in the convention that $\mathbb{Q}_\ell(-1)$ has weight one. At the remaining two embeddings the pairs are $\{1,3\}$ and $\{0,2\}$. To see this, observe that $E_4 t^{1/5}$ is still square-integrable. It provides a line in the top filtration of one character component on the cover, and its complex conjugate provides a bottom line in the inverse component (Lemma 2.8). The determinant distinguishes the remaining weights.[1] Thus the difference of the weights is 3 above one real place and 2 above the other. A classical newform over $\mathbb{Q}$ would give the same difference at all four embeddings, even after a scalar twist. For a Hilbert eigenform of weight (k_1, k_2) the corresponding differences are $k_1 - 1$ and $k_2 - 1$, which predicts the mixed-parity weight $(4,3)$.

Using the definite-quaternion method of Dembélé, Loeffler and Pacetti [63], we find a candidate of this weight and conductor $(2)^2(3)^2(\sqrt{5})$, of absolute norm 6480. To compare the projective representations we use the scalar-invariant quantity $(\operatorname{tr} M)^2/\det M$ on both sides. In the 14-dimensional quaternionic space with the chosen cuspidal types at (2) and (3) and principal series type at $(\sqrt{5})$, precisely two eigenlines give the prescribed values at the eight primes of F above $11, 31, 41, 61$. The quadratic character of K/F exchanges them, so they determine the same projective Hecke eigensystem. This eigensystem also agrees at both primes above each of $19, 29, 71, 101, 131, 151, 181, 191, 211$, which were not used to select it. At primes inert in K/F the comparison uses the square of the Hilbert Frobenius parameter. The finite comparisons and the characters defining these local types are recorded in the accompanying computational files.

Conjecture 6.11. *Let π be either of these Hilbert eigenforms of weight $(4,3)$ over F. For every λ, after extending coefficients, the projectivisation of ρ_λ is isomorphic to the restriction to $\mathrm{Gal}(\overline{K}/K)$ of the projective Galois representation attached to π.*

7 The level-seven congruence

We now turn to Cooper's modular parametrisation $z = F(y)$ of the generating series for his level-seven sequence. The square root of the Hauptmodul y branches above an elliptic point of order three, as in Theorem A. Here the congruences for the cusp coefficients are governed by a newform of level 16. We identify its Hecke eigenvalues by comparing the coefficient at Q^p with a half-index value of Cooper's sequence. Appendix B gives the comparison modulo p^2, and the Weil bound then determines the trace.

Definition 7.1. The **Cooper sequence** $s_7(n)$ is the sequence in $\mathbb{Q}$ indexed by integers $n \geq -1$ determined by $s_7(-1) = 0$, $s_7(0) = 1$, and, for every integer $n \geq 0$,

$$(n+1)^3 s_7(n+1) = (2n+1)(13n^2+13n+4)s_7(n) + 3n(9n^2-1)s_7(n-1).$$

Its **generating series** is $F(Y) = \sum_{n=0}^{\infty} s_7(n) Y^n \in \mathbb{Q}[[Y]]$. The sequence and its recurrence are due to Cooper [14, (20), (22)].

Definition 7.2. For $\tau \in \mathbb{H}$ put $q = e^{2\pi i \tau}$. Define

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \qquad \sigma_1(n) = \sum_{d \mid n} d, \qquad E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

[1] For instance, fix $z = \zeta_5$ and the prime $z \equiv 4 \bmod 11$. The trace formula of Proposition 5.6 over $\mathbb{F}_{11}$ and $\mathbb{F}_{11^2}$ gives $B_{\mathfrak{P}} = 11^2(-7 - 9z + 3z^2 - 3z^3)$ by $2\det M = (\operatorname{tr} M)^2 - \operatorname{tr}(M^2)$. Under the coefficient embeddings $z \equiv 4, 5, 9, 3 \bmod 11$ its valuations are $3, 4, 2, 3$. By weak admissibility these are the sums of the filtration weights. The two extreme pairs already account for both sums equal to three.

The **level-seven functions** x, D, y, w, z are

$$x(\tau) = \frac{\eta(\tau)^4}{\eta(7\tau)^4}, \qquad D(\tau) = x(\tau)^2 + 13x(\tau) + 49,$$

$$y(\tau) = \frac{x(\tau)}{D(\tau)}, \qquad w(\tau) = \frac{x(\tau)^2 - 49}{D(\tau)}, \qquad z(\tau) = \frac{7E_2(7\tau) - E_2(\tau)}{6}.$$

Let $W_7 = \frac{1}{\sqrt{7}}\begin{pmatrix} 0 & -1 \\ 7 & 0 \end{pmatrix}$ and $\Gamma_0(7)^+ = \Gamma_0(7) \cup W_7\Gamma_0(7) \subset \mathrm{PSL}_2(\mathbb{R})$, with $\Gamma_0(7)$ identified with its image in $\mathrm{PSL}_2(\mathbb{R})$. The **Cooper datum** is the tuple $(\Gamma_0(7)^+, y, z, \frac{1}{2}, 1)$.

Definition 7.3. Let Q be a formal variable, let y be the level-seven function of Definition 7.2, regarded as a power series in q, and let F be the generating series of Definition 7.1. The **square-root coordinate** $s(Q)$ is the unique series in $Q\mathbb{Q}[[Q^2]]$ such that

$$s(Q)^2 = y(Q^2), \qquad [Q]s(Q) = 1,$$

where $[Q^N]$ denotes coefficient extraction. The **coefficients** $c_7(N)$, for integers N, are defined by

$$C(Q) := QF(y(Q^2))\frac{ds(Q)}{dQ} = \sum_{N=1}^{\infty} c_7(N)Q^N,$$

and $c_7(N) = 0$ for $N \le 0$. The derivative is formal.

Definition 7.4. For $q = e^{2\pi i\tau}$ define

$$f_8(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\ge1} a_8(n)q^n, \qquad g_{16}(\tau) = \frac{\eta(4\tau)^{16}}{\eta(2\tau)^4\eta(8\tau)^4} = \sum_{n\ge1} a_{16}(n)q^n.$$

The character χ_{-4} is 0 on even integers and $(-1)^{(n-1)/2}$ on odd integers n. For each odd prime p put $A_p^\eta = a_{16}(p)$.

Proposition 7.5. *The forms f_8 and g_{16} are normalised newforms of weight 4, trivial character and levels 8 and 16, respectively. Both have integral Fourier coefficients. They satisfy*

$$a_{16}(n) = \chi_{-4}(n)a_8(n) \qquad (n \ge 1).$$

For every odd prime p and every integer $n \ge 1$,

$$a_{16}(pn) + p^3 a_{16}(n/p) = A_p^\eta a_{16}(n),$$

where the coefficient with nonintegral index is interpreted as zero.

Proof. The eta-quotient criteria for modularity and for the orders at the cusps give $f_8 \in S_4(\Gamma_0(8))$ and $g_{16} \in S_4(\Gamma_0(16))$, both with trivial character [46]. The curve $X_0(8)$ has genus zero, four cusps and no elliptic points, so $\dim S_4(\Gamma_0(8)) = 1$. The same dimension formula gives $S_4(\Gamma_0(4)) = 0$, since $X_0(4)$ has genus zero, three cusps and no elliptic points [57, Proposition 1.43 and Theorem 2.23]. Thus f_8, whose leading coefficient is 1, is a normalised newform. Its expansion is supported on odd powers of q, so $U_2 f_8 = 0$.

To prove the twist identity directly, put $P_m(q) = \prod_{n\ge1}(1 - q^{mn})$. Then $f_8 = qP_2^4P_4^4$ and

$$\frac{f_8(iq)}{i} = q\left(\prod_{n\ge1}(1 - (-1)^n q^{2n})\right)^4 P_4^4 = q\frac{P_4^{16}}{P_2^4P_8^4} = g_{16}.$$

Here the middle identity follows by separating even and odd n; the odd-factor product is $P_4^2/(P_2P_8)$ and the even-factor product is P_4. Since f_8 has only odd powers, substitution multiplies its coefficient of q^n by $i^{n-1} = \chi_{-4}(n)$. This proves the asserted coefficient identity and integrality of both forms.

For odd p, the Hecke relation for f_8 and multiplicativity of χ_{-4} give

$$a_{16}(pn) + p^3 a_{16}(n/p) = \chi_{-4}(p)a_8(p)a_{16}(n).$$

This includes even n, for which every term is zero. The eigenvalue is $a_{16}(p) = A_p^\eta$. The odd support of g_{16} also gives $U_2 g_{16} = 0$. To see that it is new at level 16, observe that the oldspace is spanned by $f_8(\tau)$ and $f_8(2\tau)$. Both have T_3-eigenvalue $a_8(3) = -4$, whereas $a_{16}(3) = 4$. Self-adjointness of T_3 therefore puts g_{16} in the newspace.

□

We next verify the hypotheses of Theorem B for the Cooper datum. The local calculations use the following identities on $X_0(7)$ and its Fricke quotient.

Lemma 7.6. *1. x is a modular function on $\Gamma_0(7)$ with no zeros or poles on $\mathbb{H}$, a simple pole at the cusp ∞ and a simple zero at the cusp 0; it is a coordinate on $X_0(7)$.*

2. $z \in M_2(\Gamma_0(7))$ and $q\frac{d}{dq}\log x = -z$.

3. The two roots r of $D = x^2 + 13x + 49$ are the images of the two elliptic points of order three of $\Gamma_0(7)$. At a point $\tau_0 \in \mathbb{H}$ above such a root, $x - r$ has analytic order 3 and z has analytic order exactly 2.

4. $q\frac{d}{dq}\log y = z\,w$.

Proof. The coordinate $49/x$ and its elliptic values occur in the table of canonical Hauptmoduln in [41]. We include the computation of the orders on $\mathbb{H}$, which enter the admissibility criterion.

Proof of (1). By the eta-quotient criterion, $x = \eta(\tau)^4/\eta(7\tau)^4$ is a modular function on $\Gamma_0(7)$, holomorphic and nonvanishing on $\mathbb{H}$, with $x = q^{-1} + O(1)$ at ∞. The transformation $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$ gives $x(-1/(7\tau)) = 49/x(\tau)$, so x has a simple zero at the cusp 0. Hence x has degree one on $X_0(7)$ and is a coordinate.

Proof of (2). The form $z = \frac{7E_2(7\tau)-E_2(\tau)}{6}$ is the standard holomorphic weight-two form on $\Gamma_0(7)$. Logarithmic differentiation of the eta product, with $q\frac{d}{dq}\log\eta(\tau) = E_2(\tau)/24$, gives $q\frac{d}{dq}\log x = \frac{E_2(\tau)-7E_2(7\tau)}{6} = -z$.

Proof of (3). Both sides of

$$j = \frac{(x^2 + 13x + 49)(x^2 + 245x + 2401)^3}{x^7}$$

are modular functions on $X_0(7)$ with poles only at the two cusps, of order at most one at ∞ and at most seven at 0. Their q-expansions, computed directly, agree through q^7. The difference therefore has a zero of order at least eight at ∞ and a pole of order at most seven elsewhere, so it vanishes identically. Let r be a root of D. The roots of D are distinct, nonzero, irrational and different from ± 7, and $x^2 + 245x + 2401$ does not vanish at r (subtracting D leaves $232x + 2352$, whose root is rational). So j has a simple zero at the point $x = r$ of $X_0(7)$. Zeros of j on $\mathbb{H}$ have analytic order three, so at τ_0 above $x = r$ the function $x - r$ has analytic order 3; thus τ_0 is an elliptic point of order three, and $\Gamma_0(7)$ has exactly two such points, matching the two roots. Since $x(\tau_0) \neq 0$ and $q\frac{d}{dq}x = -zx$ has order 2 at τ_0, z has order exactly 2 there.

Proof of (4). Logarithmic differentiation of $y = x/D$ gives

$$\frac{d\log y}{d\log x} = 1 - \frac{x(2x+13)}{D} = \frac{49-x^2}{D} = -w.$$

The assertion follows from (2).

□

Proposition 7.7. *One has*

$$w^2 = 1 - 26y - 27y^2, \qquad D^2 - 26xD - 27x^2 = (x^2-49)^2.$$

For $\xi = x/7$ the rational parametrisation is

$$y = \frac{\xi}{7\xi^2+13\xi+7}, \qquad w = \frac{7(\xi^2-1)}{7\xi^2+13\xi+7}.$$

The involution $x \mapsto 49/x$ fixes y and sends w to $-w$; for the chosen eta-quotient x this is the action of the Fricke involution on $X_0(7)$.

Proof. These are elementary identities of the level-seven theory; the conic relation is [14, (43)], p. 175, with $u = 1/x$. Expanding $D = x^2 + 13x + 49$ gives the stated polynomial identity. Division by D^2 proves the conic relation, and substitution $x = 7\xi$ proves the parametrisation. Direct substitution $x \mapsto 49/x$ into D, y and w gives the asserted signs. The Fricke action on $x = \eta(\tau)^4/\eta(7\tau)^4$ follows from the Dedekind eta transformation $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, applied at τ and 7τ.

□

Lemma 7.8. *$y \in q + q^2\mathbb{Z}[[q]]$ and $z, w \in 1 + q\mathbb{Z}[[q]]$.*

Proof. We have $x = q^{-1}\prod_{n\geq 1}(1-q^n)^4(1-q^{7n})^{-4}$, so $x^{-1} \in q + q^2\mathbb{Z}[[q]]$. Hence

$$y = \frac{x^{-1}}{1+13x^{-1}+49x^{-2}} \in q + q^2\mathbb{Z}[[q]], \qquad w = \frac{1-49x^{-2}}{1+13x^{-1}+49x^{-2}} \in 1 + q\mathbb{Z}[[q]].$$

Also $z = 1 + 4\sum_{n\geq 1}(\sigma_1(n) - 7\sigma_1(n/7))q^n$, with $\sigma_1(n/7) = 0$ for $7 \nmid n$.

□

Definition 7.9. For integers $n \geq 0$ and $0 \leq k \leq n$, define

$$L_{n,k} = \binom{n}{k}\binom{n+k}{k}, \qquad L_n(X) = \sum_{k=0}^{n} L_{n,k}X^k, \qquad \mathrm{Ap}(n) = \sum_{k=0}^{n} L_{n,k}^2.$$

Lemma 7.10. *For every integer $n \geq 0$,*

$$s_7(n) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}\binom{n+k}{k} = \sum_{k=0}^{n} L_{n,k}L_{n,n-k}.$$

In particular $s_7(n) \in \mathbb{Z}$.

Proof. The first equality is Zudilin's formula [14, p. 171], shown there to satisfy the recurrence (20) of the Cooper sequence by creative telescoping. Integrality is immediate from it. For the second equality we realise $s_7(n)$ as the middle coefficient of $L_n(X)^2$ and expand that square by a terminating Clausen identity,

$$L_n(X)^2 = \sum_{k=0}^{n} L_{n,k} \binom{2k}{k} \big(X(1+X)\big)^k.$$

Put $v = X(1+X)$ and $N = n(n+1)$. The coefficient formula for L_n gives $vL_n'' + v'L_n' - NL_n = 0$. Consequently $W = L_n^2$ satisfies

$$v^2W''' + 3vv'W'' + \big((v')^2 + vv'' - 4Nv\big)W' - 2Nv'W = 0.$$

Writing $W = R(v)$ is legitimate formally because $v = X + O(X^2)$. Since $(v')^2 = 1 + 4v$ and $v'' = 2$, this equation becomes

$$v^2(1+4v)R''' + (3v + 18v^2)R'' + \big(1 + (12 - 4N)v\big)R' - 2NR = 0.$$

The solution with $R(0) = 1$ is unique: its coefficients d_k satisfy

$$(k+1)^3 d_{k+1} = 2(2k+1)\big(N - k(k+1)\big)d_k.$$

These are exactly $d_k = L_{n,k}\binom{2k}{k}$, with $d_k = 0$ for $k > n$, which proves the Clausen identity.

Now take the coefficient of X^n on both sides. On the left it is $\sum_k L_{n,k}L_{n,n-k}$. On the right it is $\sum_k L_{n,k}\binom{2k}{k}\binom{k}{n-k}$, and

$$\binom{2k}{k}\binom{k}{n-k} = \binom{n}{k}\binom{2k}{n},$$

both sides vanishing when $2k < n$. Since $L_{n,k}\binom{n}{k} = \binom{n}{k}^2\binom{n+k}{k}$, this is the binomial formula, proving the second equality.

□

Lemma 7.11. *The Cooper datum $(\Gamma_0(7)^+, y, z, \frac{1}{2}, 1)$ has the following properties.*

1. *It is a Kummer twist datum with $n_\Gamma = 7$, $M = 1$, $k = 4$, $n_f = 1$, $F = \sum_{n\geq 0} s_7(n)Y^n$ and $e_P = 3$, where P is the image of the two elliptic points of order three of $\Gamma_0(7)$. Its multiplier system χ_f is trivial on $\Gamma_0(7)$ and equal to -1 on $W_7\Gamma_0(7)$.*
2. *Its coefficients are $c(N) = c_7(N)$ for all N; that is, $f_\Phi = \sum_{N\geq 1} c_7(N)Q^N$. Moreover $f_\Phi \in Q(1 + Q^2\,\mathbb{Z}[\frac{1}{2}][[Q^2]])$, so $c_7(1) = 1$ and every $c_7(N)$ lies in $\mathbb{Z}[\frac{1}{2}]$.*
3. *Its singular set is $S = \{0, -1, \frac{1}{27}, \infty\}$.*
4. *Its field is $L = \mathbb{Q}$, and its exceptional set is $\Sigma = \{2, 3, 7\}$.*
5. *It is admissible: it satisfies (R), (P) and (F).*

Proof. Items (1), (2) and (4) are checks against the definitions; (3) and (5) follow from the local exponents of Lemma A.1 and the rank, integrability and Hodge criteria of Lemma 2.6, Lemma 2.7 and Proposition 2.9.

The group and the Hauptmodul. The set Γ is a group containing $\Gamma_0(7)$ with index two, since W_7 normalises $\Gamma_0(7)$ and $W_7^2 = -1$; it is commensurable with $\mathrm{PSL}_2(\mathbb{Z})$, and $\Gamma \cap \mathrm{PSL}_2(\mathbb{Z}) = \Gamma_0(7)$,

because no element of $W_7\Gamma_0(7)$ has integral entries. As $\Gamma(n)$ contains the image of $\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$, it follows that $\Gamma(n) \subset \Gamma$ if and only if $\Gamma(n) \subset \Gamma_0(7)$, that is, if and only if $7 \mid n$; so Γ contains $\Gamma(7)$ and $n_\Gamma = 7$.

The elements of $W_7\Gamma_0(7)$ act on $X_0(7)$ as w_7, which exchanges the cusps $x = \infty$ and $x = 0$. Hence the stabiliser of ∞ in Γ is its stabiliser in $\Gamma_0(7)$, generated by $\tau \mapsto \tau + 1$, so the cusp ∞ has width one, and $X_\Gamma = X_0(7)/\langle w_7\rangle$ has a single cusp. The function $y = x/D$ is a rational function of degree two in x, as $D(0) \neq 0$, and it is invariant under $x \mapsto 49/x$; so it is the pull-back of a function of degree one on X_Γ, which therefore has genus zero, with Hauptmodul y. By Lemma 7.8, $y \in q + q^2\mathbb{Z}[[q]]$. On $X_0(7)$, y vanishes exactly at the two cusps $x = 0, \infty$, so on X_Γ it vanishes only at the cusp ∞. Its poles on $X_0(7)$ are the roots r_1, r_2 of D, which are the images of the two elliptic points of order three of $\Gamma_0(7)$ (Lemma 7.6(3)); since $r_1 r_2 = 49$, w_7 exchanges them, and y has a single simple pole on X_Γ, at their common image P. A point $\tau_P \in \mathbb{H}$ over r_1 has a stabiliser of order three in $\Gamma_0(7)$, and no element of $W_7\Gamma_0(7)$ fixes it, since $w_7(r_1) = r_2 \neq r_1$; so P is not a cusp and $e_P = 3$.

The form. By Lemma 7.6(2), $z \in M_2(\Gamma_0(7))$ and $q\frac{d}{dq}\log x = -z$; with $x \circ W_7 = 49/x$ this gives $z|_2 W_7 = -z$. Hence $z(\gamma\tau) = \chi_f(\gamma)(\gamma_{21}\tau + \gamma_{22})^2 z(\tau)$ for γ in the preimage of Γ in $\mathrm{SL}_2(\mathbb{R})$, where χ_f is the character of order two that is trivial on the preimage of $\Gamma_0(7)$ and equal to -1 on that of $W_7\Gamma_0(7)$. It is trivial on $\Gamma \cap \Gamma(1) = \Gamma_0(7)$, so $n_f = 1$; the weight is $2 = k - 2$, so $k = 4$. By Cooper's identity [14, Theorem 3.1], $z = F(y)$ near ∞, and $F \in 1 + Y\mathbb{Z}[[Y]]$ by Lemma 7.10. Finally $a = \frac{1}{2}$ is in lowest terms, with $j = 1$, $d = 2$, and $R = 1$. This proves (1), with $M = 1$.

The coefficients. In Definition 2.1, $Q = q^{1/2}$ and $t^a = Q\,(y/q)^{1/2}$, where $(y/q)^{1/2} \in 1 + q\,\mathbb{Z}[\frac{1}{2}][[q]]$ is the binomial series. The series $s = Q\,(y(Q^2)/Q^2)^{1/2} \in Q\,\mathbb{Q}[[Q^2]]$ satisfies $s^2 = y(Q^2)$ and $[Q]s = 1$, so it is the square-root coordinate of Definition 7.3, and $t^a = s$. Differentiating $s^2 = y(Q^2)$ gives $s\,\frac{ds}{dQ} = Q\,y'(Q^2)$, with $y' = dy/dq$. With $z = F(y)$ and $q = Q^2$,

$$f\,R(t)\,t^a\,\frac{q}{t}\frac{dt}{dq} = \frac{z\,s\,q\,y'}{y} = \frac{z\,q\,y'}{s} = Q\,F\big(y(Q^2)\big)\frac{ds}{dQ} = C(Q),$$

so $c(N) = c_7(N)$ for all N. Moreover $\frac{q}{y}\,y' = zw$ by Lemma 7.6(4), so $f_\Phi = Q\,z^2 w\,(y/q)^{1/2}$ with $q = Q^2$; since $z, w \in 1 + q\,\mathbb{Z}[[q]]$ (Lemma 7.8), $f_\Phi \in Q(1 + Q^2\,\mathbb{Z}[\frac{1}{2}][[Q^2]])$. This proves (2).

The singular set. Let $\tau_0 \in \mathbb{H}$, and let e_0 be the order of its stabiliser in $\Gamma_0(7)$. As x is a coordinate on $X_0(7)$, and the local coordinate of $X_0(7)$ at the image of τ_0 has order e_0 in $\tau - \tau_0$, the function $x - x(\tau_0)$ has order e_0 at τ_0. As moreover x has no zeros on $\mathbb{H}$, the identity $z = -\frac{1}{2\pi i}\frac{dx/d\tau}{x}$ of Lemma 7.6(2) shows that $z(\tau_0) = 0$ if and only if $e_0 > 1$, that is, if and only if τ_0 lies over one of the elliptic points of $\Gamma_0(7)$; these lie over P, since none has order two.

The points of $\mathbb{H}$ with nontrivial stabiliser in Γ are those over P and those over $x = \pm 7$. Indeed, a nontrivial element of $\Gamma_0(7)$ fixing τ_0 makes τ_0 an elliptic point of $\Gamma_0(7)$, and an element of $W_7\Gamma_0(7)$ fixing τ_0 acts on $X_0(7)$ as w_7 and fixes the image of τ_0, so that $x(\tau_0) = 49/x(\tau_0)$. Conversely, let $\tau_\pm \in \mathbb{H}$ lie over $x = \pm 7$, which are not cusps. As w_7 fixes the image of $\tau_\pm$, there is $\gamma \in \Gamma_0(7)$ with $W_7\tau_\pm = \gamma\tau_\pm$, and $\gamma^{-1}W_7 \in \Gamma \setminus \Gamma_0(7)$ fixes $\tau_\pm$. Since $D(\pm 7) \neq 0$, $\tau_\pm$ is not over P; so its stabiliser in Γ meets $\Gamma_0(7)$ trivially and has order two, and $z(\tau_\pm) \neq 0$. Moreover $y(\tau_+) = \frac{7}{189} = \frac{1}{27}$ and $y(\tau_-) = \frac{-7}{7} = -1$.

Now let $t_1 \in \mathbb{P}^1$. The point $t_1 = 0$, the image of the only cusp, and the point $t_1 = \infty$, the image of P, lie in S by definition. For $t_1 = y(\tau_\pm)$, Lemma A.1(1) applies with $e_1 = 2$, so $t_1 \in S$ by Lemma A.1(3). As y is a coordinate on X_Γ, every other t_1 is $y(\tau_1)$ for a point $\tau_1 \in \mathbb{H}$ over neither P nor $x = \pm 7$; then $e_1 = 1$ and $\nu_1 = \mathrm{ord}_{\tau_1} z = 0$, and Lemma A.1(1) gives a basis of solutions of the Picard–Fuchs equation near t_1 of the form $(t - t_1)^i V_i$, $0 \leq i \leq 2$, with V_i

holomorphic at t_1 and $V_i(t_1) \neq 0$. These solutions are holomorphic at t_1, and their Wronskian $\det(\frac{d^r}{dt^r}((t-t_1)^i V_i))_{0\leq r,i\leq 2}$ takes the value $2V_0(t_1)V_1(t_1)V_2(t_1) \neq 0$ at t_1. The coefficients of the Picard–Fuchs equation, normalised to leading coefficient one, are quotients of minors of this Wronskian matrix by the Wronskian, hence holomorphic at t_1, and t_1 is not a singular point. Hence $S = \{0, -1, \frac{1}{27}, \infty\}$. This proves (3).

The field and the exceptional set. The field L is generated over $\mathbb{Q}(\zeta_2) = \mathbb{Q}$ by the values ± 1 of χ_f, and k is even; so $L = \mathbb{Q}$, in which no prime ramifies. The primes 2, 3 and those dividing $dMn_\Gamma n_f = 14$ are $2, 3, 7$, and the primes below $k = 4$ are $2, 3$. The reductions of $0, \infty, -1, \frac{1}{27}$ in $\mathbb{P}^1(\mathbb{F}_p)$ are pairwise distinct unless $p \in \{2, 3, 7\}$. Hence $\Sigma = \{2, 3, 7\}$. This proves (4).

Admissibility. For (R): at $s = -1$ and $s = \frac{1}{27}$, that is, at $y(\tau_\pm)$, Lemma A.1(1) with $e_1 = 2$ and $\nu_1 = 0$ gives the local exponents $0, \frac{1}{2}, 1$ and a semisimple local monodromy of $\mathbb{V}_f$ with eigenvalues $1, -1, 1$, so $\dim \mathbb{V}_f^{I_s} = 2$. At $t = \infty$ the twist leaves no invariants, since $d = 2$ does not divide $e_P = 3$ (Lemma 2.7(2)). By Lemma 2.6(3), $\dim H_{1/2} = (3-2) + (3-2) = 2$. For (F): $\Phi \neq 0$ as $R = 1$. By Lemma 7.6(3), $\nu_P = \operatorname{ord}_{\tau_P} z = 2$, so $\rho_P = \nu_P / e_P = \frac{2}{3}$ by Lemma 2.7(1), (3), and since $a = \frac{1}{2} < \frac{2}{3}$, (F) holds at the points over P by the same item. It holds at the cusp ∞, the only cusp of Γ, as for every Kummer twist datum (Lemma 2.7(4)). At $\tau_0 \in \mathbb{H}$ not over P, y is holomorphic and nonzero, as its only zero on X_Γ is the cusp and its only pole is P; so $y^{1/2}$ has a holomorphic branch near τ_0, the coefficient $z\, y^{1/2} y^{-1} \frac{dy}{d\tau}$ of $d\tau$ in Φ is holomorphic there, and $\operatorname{ord}_{\tau_0} \Phi \geq 0 > -1$. For (P), the multiplier is real-valued and (R) is established. Since $a = 1 - a = \frac{1}{2}$, the datum with exponent $1 - a$ is the same datum, for which (F) has just been proved. Both square-integrability hypotheses of Proposition 2.9 therefore hold, and that proposition gives (P). This proves (5).

□

Corollary 7.12. *For every prime $p \notin \{2, 3, 7\}$, every integer $m \geq 1$ prime to p and every $r \geq 1$,*

$$c_7(mp^r) - a_{16}(p) c_7(mp^{r-1}) + p^3 c_7(mp^{r-2}) \equiv 0 \pmod{p^{3r}},$$

where $c_7(x) = 0$ for $x \notin \mathbb{Z}$ and $a_{16}(p) = A_p^\eta = \chi_{-4}(p) a_8(p)$. The series $C(Q)$ has no single-valued meromorphic continuation to $\mathbb{H}$ as a function of τ with $Q = e^{\pi i \tau}$. Moreover, for every modular model of the Cooper datum at p, the characteristic polynomials of φ and of $p^3\varphi^{-1}$ on its eigenlattice $H_{1/2,p}$, and that of the geometric Frobenius F on the cohomology $V_{\bar{a}} = V_a$ over $\bar{\mathbb{F}}_p$ of its Kummer eigensheaves, with $a = \frac{1}{2}$, for every prime $\ell \neq p$, are all equal to $X^2 - A_p^\eta X + p^3$.

Proof. By Lemma 7.11, the Cooper datum is admissible with $k = 4$, $L = \mathbb{Q}$, $\Sigma = \{2, 3, 7\}$ and coefficients $c_7(N) \in \mathbb{Z}[\frac{1}{2}]$, with $c_7(1) = 1$. The pole of y has stabiliser order 3. Since $2 \nmid 3$, the assertion on continuation follows from Theorem 6.10(4). Fix $p \notin \Sigma$ and any modular model at p, which exists by Proposition 3.2. The trace identity Proposition 5.6(2),(4) and the global system Proposition 6.1(2),(3) identify the characteristic polynomial of $p^3\varphi^{-1}$ with

$$X^2 - A_p X + \varepsilon_p p^3, \qquad A_p \in \mathbb{Z}, \quad |A_p| \leq 2p^{3/2}.$$

Since $k = 4$ is even and $L = \mathbb{Q}$, Proposition 6.8(1),(2) give $\varepsilon_p = 1$. Thus Theorem 4.1(3) gives

$$(1 - A_p V + p^3 V^2) \sum_{N \geq 1} c_7(N) Q^N \in \theta^3 \mathbb{Z}_p[[Q]].$$

Its coefficient of Q^p is $c_7(p) - A_p$, so $c_7(p) \equiv A_p \pmod{p^3}$. On the other hand, Theorem B.1 gives $c_7(p) \equiv A_p^\eta \pmod{p^2}$. Both traces are integers, and consequently

$$A_p - A_p^\eta \in p^2 \mathbb{Z}.$$

By Proposition 7.5, $A_p^\eta = a_{16}(p)$ is the p-th coefficient of a weight-four newform, so Deligne's bound gives $|A_p^\eta| \le 2p^{3/2}$ [16, Théorème 8.2]. For $p \ge 17$ the difference of the two traces has absolute value at most $4p^{3/2} < p^2$, hence is zero. At $p = 5, 11, 13$, respectively, $A_p^\eta = -2, 44, 22$. A different integer in the same residue class modulo p^2 has absolute value at least $23, 77, 147$, respectively, while $2p^{3/2} < 22.4, 73, 93.8$. Thus $A_p = A_p^\eta$ in these cases as well.

Taking the coefficient of Q^{mp^r} in the power-series relation gives the claimed congruence, since $\theta^3 Q^{mp^r} = (mp^r)^3 Q^{mp^r}$. The characteristic polynomial of $p^3\varphi^{-1}$ is $X^2 - A_p^\eta X + p^3$, so $\det\varphi = p^6/p^3 = p^3$. Hence $p^3\varphi^{-1}$ is the adjugate of φ and has its trace; the two operators have the same characteristic polynomial. The trace identity gives this polynomial also for geometric Frobenius on $V_{\bar a}$. Finally χ_Φ has values ± 1, since $L = \mathbb{Q}$, so $\chi_\Phi^{-1} = \chi_\Phi$ and $V_{\bar a} = V_a$ by Definition 5.2(2). The model was arbitrary.

□

A Variants of results from the literature

We collect the local and global results needed in the proof of Theorem B. The first subsection computes the exponents of the Picard–Fuchs equation, applies Zucker's theorem to the complex Hodge structure on parabolic cohomology, and treats the determinants of elements of Γ. The second shows how the degree-one Frobenius data determine a Hecke character and establishes the modularity criterion used over $\mathbb{Q}$.

A.1 Picard–Fuchs equations and Hodge theory

Lemma A.1. *Let (Γ, t, f, a, R) be a Kummer twist datum, with local system $\mathbb{V}_f$ and singular set S, and let $\bar\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be the preimage of Γ. For $t_1 \in \mathbb{P}^1$ let x be the local parameter $t - t_1$, or $1/t$ if $t_1 = \infty$. Local exponents are taken with respect to x.*

1. *Let $\tau_1 \in \mathbb{H}$, let e_1 be the order of its stabiliser in Γ, $\nu_1 = \mathrm{ord}_{\tau_1} f$ in $\tau - \tau_1$, and $t_1 = t(\tau_1)$. Then t_1 is a regular or regular singular point of the Picard–Fuchs equation associated with (t, f), with local exponents $\lambda_i = \frac{\nu_1 + i}{e_1}$, $0 \le i \le k-2$, and there is a basis of solutions near t_1 of the form $x^{\lambda_i} V_i(x)$ with V_i holomorphic at 0 and $V_i(0) \ne 0$. In particular the local monodromy of $\mathbb{V}_f$ at t_1 is semisimple with eigenvalues $e^{2\pi i \lambda_i}$.*

2. *Let $c = \sigma\infty$ ($\sigma \in \mathrm{SL}_2(\mathbb{R})$) be a cusp of Γ of width h_c, $q_c = e^{2\pi i \sigma^{-1}\tau/h_c}$, ν_c the least exponent in the expansion of $f|_{k-2}\sigma$ in powers of q_c, and $t_c = t(c)$. Then $\nu_c \in \mathbb{Q}$, t_c is a regular or regular singular point of the Picard–Fuchs equation, all of whose $k-1$ local exponents equal ν_c, and the local monodromy of $\mathbb{V}_f$ at t_c is $e^{2\pi i \nu_c}$ times a unipotent map with a single Jordan block.*

3. *If $t_1 \notin S$ in (1), then $\nu_1 = 0$ and $e_1 = 1$. In (2), $t_c \in S$.*

4. *In (1), the stabiliser of τ_1 in $\bar\Gamma$ is cyclic of order $2e_1$; if γ_1 generates it and $\mu = \gamma_{1,21}\tau_1 + \gamma_{1,22}$, then μ is a primitive $2e_1$-th root of unity and*

$$\chi_f(\gamma_1) = \mu^{-(2\nu_1 + k - 2)}.$$

In particular, if k is even and χ_f is trivial on the stabiliser of τ_1 in $\bar\Gamma$, then $\nu_1 \equiv 1 - \frac{k}{2} \pmod{e_1}$.

5. *Let $B \subset \mathbb{H}$ be a connected open set on which t is injective, and B' a simply connected open subset of $t(B) \setminus S$. Then the solutions of the Picard–Fuchs equation on B' are exactly the functions $(f\Pi) \circ (t|_B)^{-1}$, with Π a polynomial of degree $\le k-2$.*

Proof. That $f, \tau f, \dots, \tau^{k-2} f$, regarded as functions of t, are linearly independent solutions of a linear differential equation of order $k-1$ whose coefficients are algebraic functions of t is [60, Theorem 1], p. 5, for meromorphic forms with a multiplier system on groups commensurable with $\mathrm{SL}_2(\mathbb{Z})$; the weight k there is our $k-2$.

Proof of (5). By [60, Theorem 1], the Picard–Fuchs operator, written with $d/dt = t'(\tau)^{-1} d/d\tau$, annihilates $f\Pi$ on $\mathbb{H}$ for every polynomial Π of degree $\le k-2$. So every $(f\Pi) \circ (t|_B)^{-1}$ is a solution on B'. These functions form a space of dimension $n = k-1$, since $f \not\equiv 0$. The equation has no singular point in the simply connected set B', so its solution space there also has dimension n, and these functions are all the solutions.

Proof of (1) and (4). The stabiliser of τ_1 in $\mathrm{SL}_2(\mathbb{R})$ is a conjugate of $\mathrm{SO}(2)$, so its intersection with the discrete group $\bar\Gamma$ is finite cyclic; it contains -1 and maps onto the stabiliser in Γ, so it has order $2e_1$. Let γ_1 generate it. Since $\gamma_1 \binom{\tau_1}{1} = \mu \binom{\tau_1}{1}$ and $\gamma_1 \binom{\bar\tau_1}{1} = \bar\mu \binom{\bar\tau_1}{1}$, μ and $\bar\mu = \mu^{-1} = j(\gamma_1, \bar\tau_1)$ are the eigenvalues of γ_1, so μ is a primitive $2e_1$-th root of unity. Put $w = (\tau - \tau_1)/(\tau - \bar\tau_1)$; then $\tau - \bar\tau_1 = (\tau_1 - \bar\tau_1)/(1-w)$ and $\tau - \tau_1 = (\tau_1 - \bar\tau_1) w/(1-w)$. The transformation γ_1 fixes $w = 0$ and $w = \infty$, so in the coordinate w it is $w \mapsto \zeta w$ with $\zeta = \gamma_1'(\tau_1) = \mu^{-2}$, a primitive e_1-th root of unity. Moreover w^{e_1} is a local coordinate of X_Γ at the image of τ_1 [57, §1.5].

Put $\phi(w) = f(\tau)(1-w)^{2-k}$, a meromorphic function of w near 0 with $\operatorname{ord}_0 \phi = \nu_1$. From $\gamma z - \gamma z' = (z - z')/(j(\gamma, z) j(\gamma, z'))$ with $z' = \bar\tau_1$ we get $\gamma_1 \tau - \bar\tau_1 = \mu(\tau - \bar\tau_1)/j(\gamma_1, \tau)$, that is, $j(\gamma_1, \tau)(1-w) = \mu(1 - \zeta w)$. Hence

$$\begin{aligned} \phi(\zeta w) &= f(\gamma_1 \tau)(1 - \zeta w)^{2-k} \\ &= \chi_f(\gamma_1) j(\gamma_1, \tau)^{k-2} f(\tau)\, \mu^{k-2} \big(j(\gamma_1, \tau)(1-w)\big)^{2-k} = \chi_f(\gamma_1)\, \mu^{k-2}\, \phi(w). \end{aligned}$$

Write $\phi(w) = \sum_{m \ge \nu_1} \phi_m w^m$ with $\phi_{\nu_1} \ne 0$. Then $\zeta^m \phi_m = \chi_f(\gamma_1) \mu^{k-2} \phi_m$ for all m. For $m = \nu_1$ this gives $\chi_f(\gamma_1) = \mu^{-2\nu_1 - (k-2)}$, which is (4); if k is even and $\chi_f(\gamma_1) = 1$, then $2e_1 \mid 2\nu_1 + k - 2$, that is, $\nu_1 \equiv -\frac{k-2}{2} = 1 - \frac{k}{2} \pmod{e_1}$. For general m it gives $\phi_m = 0$ unless $\zeta^m = \zeta^{\nu_1}$, that is, unless $m \equiv \nu_1 \pmod{e_1}$. Hence $\phi(w) = w^{\nu_1} \psi(w^{e_1})$ with ψ holomorphic near 0 and $\psi(0) \ne 0$.

Since t is a coordinate on X_Γ, $x = \Xi(w^{e_1})$ with Ξ holomorphic near 0 and $\Xi(0) = 0 \ne \Xi'(0)$; inverting, $w^{e_1} = x\, U(x)$ with U holomorphic and $U(0) \ne 0$. Near τ_1 (and $\tau \ne \tau_1$) t is locally invertible, and on a slit disc around $x = 0$ the local inverse has $w = x^{1/e_1} U(x)^{1/e_1}$ for suitable branches. The polynomials $(X - \tau_1)^i (X - \bar\tau_1)^{k-2-i}$, $0 \le i \le k-2$, form a basis of the polynomials of degree $\le k-2$. So by (5), applied to a slit punctured disc B' around t_1 that avoids $S \setminus \{t_1\}$ and to its image B under such a branch of the local inverse, the solutions on B' have the basis

$$y_i = f(\tau)(\tau - \tau_1)^i (\tau - \bar\tau_1)^{k-2-i} = (\tau_1 - \bar\tau_1)^{k-2}\, w^{\nu_1 + i} \psi(w^{e_1}) = x^{\lambda_i} V_i(x),$$

with $V_i(x) = (\tau_1 - \bar\tau_1)^{k-2} U(x)^{\lambda_i} \psi(xU(x))$ holomorphic at 0 and $V_i(0) \ne 0$.

Put $\theta = x\, d/dx$ and let $L = \theta^n + b_1 \theta^{n-1} + \cdots + b_n$ be the monic operator with solutions $y_0, \dots, y_{k-2}$. After a factor x^{λ_i} is taken out of each column, the θ-Wronskian has constant term $\prod_i V_i(0)$ times the Vandermonde determinant of the distinct λ_i. The Wronskian formulas therefore give b_m holomorphic at 0. The Picard–Fuchs equation is regular or regular singular there, with indicial polynomial $I(X) = X^n + b_1(0) X^{n-1} + \cdots + b_n(0)$. At $x = 0$ the identity $x^{-\lambda_i} L(y_i) = 0$ gives $I(\lambda_i) V_i(0) = 0$, so the λ_i are precisely its roots. Continuation once around $x = 0$ multiplies y_i by $e^{2\pi i \lambda_i}$.

Proof of (2). Write $\tau = \sigma\tau'$. Since $f(\sigma\tau')\Pi(\sigma\tau') = (f|_{k-2}\sigma)(\tau')\tilde\Pi(\tau')$ with $\tilde\Pi(\tau') = j(\sigma, \tau')^{k-2} \Pi(\sigma\tau')$, and $\Pi \mapsto \tilde\Pi$ is a bijection of the polynomials of degree $\le k-2$, (5) shows

that the solutions on a slit punctured disc around t_c avoiding $S \setminus \{t_c\}$ are the $(f|_{k-2}\sigma)(\tau')\tilde{\Pi}(\tau')$, through a branch of the local inverse of $\tau' \mapsto t(\sigma\tau')$. The stabiliser of ∞ in $\sigma^{-1}\bar{\Gamma}\sigma$ is generated by -1 and $\pm\left(\begin{smallmatrix}1 & h_c\\ 0 & 1\end{smallmatrix}\right)$, so $f|_{k-2}\sigma(\tau'+h_c) = \varepsilon\, f|_{k-2}\sigma(\tau')$ for a root of unity ε, as χ_f has finite order; as f is meromorphic at c, $f|_{k-2}\sigma = q_c^{\nu_c}B(q_c)$ with B holomorphic at 0 and $B(0) \neq 0$, and $e^{2\pi i\nu_c} = \varepsilon$, so $\nu_c \in \mathbb{Q}$. Since q_c is a local coordinate of X_Γ at c, $q_c = x\,U(x)$ with U holomorphic and $U(0) \neq 0$, and $\tau' = \frac{h_c}{2\pi i}\log q_c$. So the solutions near t_c are $x^{\nu_c}C(x)\,Z$ with $C = U^{\nu_c}B(xU)$ a holomorphic unit and Z ranging over the polynomials of degree $\leq k-2$ in $\log q_c$, which are the solutions of $\vartheta^{k-1}Z = 0$ for $\vartheta = q_c\,d/dq_c = \epsilon_1\theta$, $\epsilon_1 = U/(U + xU')$, $\epsilon_1(0) = 1$. Hence the Picard–Fuchs equation has the same solutions as $M = x^{\nu_c}C \circ \vartheta^{k-1} \circ C^{-1}x^{-\nu_c}$. Using $x^\nu\theta x^{-\nu} = \theta - \nu$ and $C\theta C^{-1} = \theta - \theta(C)/C$, M is a polynomial in θ with coefficients holomorphic at 0 and leading coefficient ϵ_1^{k-1}, a unit; and $M(x^\lambda V) = x^\lambda((\lambda-\nu_c)^{k-1}V(0) + O(x))$ for V holomorphic. So $\epsilon_1^{1-k}M$ is the normalised operator, it is regular singular, and its indicial polynomial is $(\lambda-\nu_c)^{k-1}$. A positive loop around $x = 0$ is a positive loop around $q_c = 0$, that is $\tau' \mapsto \tau' + h_c$; it multiplies $f|_{k-2}\sigma$ by $e^{2\pi i\nu_c}$ and acts on the $\tilde{\Pi}$ by $\tilde{\Pi}(\tau') \mapsto \tilde{\Pi}(\tau'+h_c)$, which is unipotent with a single Jordan block.

Proof of (3). If $t_1 \notin S$, then t_1 is a regular point of the Picard–Fuchs equation, so its exponents are $0, 1, \ldots, k-2$. Comparing with (1), $\nu_1/e_1 = 0$ and, as $k-2 \geq 1$, $(\nu_1+1)/e_1 = 1$; so $\nu_1 = 0$ and $e_1 = 1$. If $t_c \notin S$, the exponents $0, \ldots, k-2$ would all equal ν_c by (2), which is impossible as $k-2 \geq 1$.

□

Definition A.2. Let $S \subset \mathbb{P}^1(\mathbb{C})$ be a finite nonempty set, $U = \mathbb{P}^1 \setminus S$, and $\mathbb{W}$ a local system of finite-dimensional complex vector spaces on U, with flat connection ∇ on $\mathbb{W} \otimes \mathcal{O}_U$.

A **polarised complex variation of Hodge structure of weight** w on $\mathbb{W}$ is a decomposition $\mathbb{W} \otimes \mathcal{C}^\infty_U = \bigoplus_{p+q=w} \mathbb{W}^{p,q}$ into C^∞ subbundles such that every $F^p = \bigoplus_{p' \geq p} \mathbb{W}^{p',w-p'}$ is a holomorphic subbundle with $\nabla F^p \subset \Omega^1_U \otimes F^{p-1}$, together with a nondegenerate flat Hermitian form h on $\mathbb{W}$ for which the $\mathbb{W}^{p,q}$ are mutually orthogonal and $(-1)^q h$ is positive definite on $\mathbb{W}^{p,q}$. The **Hodge metric** is the orthogonal sum of the forms $(-1)^q h$ on the $\mathbb{W}^{p,q}$. A $\mathbb{W}$-valued differential form of degree (r,s) with values in $\mathbb{W}^{p,q}$ has **type** $(p+r, q+s)$. The **conjugate variation** on the conjugate local system $\overline{\mathbb{W}}$ is given by $\overline{\mathbb{W}}^{p,q} = \overline{\mathbb{W}^{q,p}}$ and the form $(-1)^w\bar{h}$.

Equip U with a Kähler metric as in [62, (3.2)]. A $\mathbb{W}$-valued form is **square-integrable** if its pointwise norm, for this metric and the Hodge metric, is square-integrable on U, and **harmonic** if it is smooth, square-integrable and annihilated by $\Box_{D''} = D''\mathfrak{d}'' + \mathfrak{d}''D''$, where D'' is the component of type $(0,1)$ of the flat differential and $\mathfrak{d}''$ its formal adjoint [62, (1.9) and §2].

Lemma A.3. *Let $S \subset \mathbb{P}^1(\mathbb{C})$ be a finite nonempty set, $U = \mathbb{P}^1 \setminus S$, $o : U \hookrightarrow \mathbb{P}^1$ the inclusion, and $\mathbb{W}$ a local system of finite-dimensional complex vector spaces on U, with flat connection ∇ on $\mathbb{W} \otimes \mathcal{O}_U$, such that every eigenvalue of every local monodromy of $\mathbb{W}$ at a point of S is a root of unity. Assume that $\mathbb{W}$ carries a polarised complex variation of Hodge structure of weight w, with Hermitian form h; square-integrable and harmonic forms are taken for a Kähler metric on U as in [62, (3.2)].*

1. *The edge map $H^1(\mathbb{P}^1, o_*\mathbb{W}) \to H^1(U, \mathbb{W})$ of the Leray spectral sequence of o is injective, and its image is the image of the forgetful map $H^1_c(U, \mathbb{W}) \to H^1(U, \mathbb{W})$. Through it we identify $\operatorname{im}(H^1_c(U, \mathbb{W}) \to H^1(U, \mathbb{W}))$ with $H^1(\mathbb{P}^1, o_*\mathbb{W})$.*

2. *Every class in $H^1(\mathbb{P}^1, o_*\mathbb{W})$ is represented by a unique harmonic $\mathbb{W}$-valued 1-form on U, and*
$$H^1(\mathbb{P}^1, o_*\mathbb{W}) = \bigoplus_{P+Q=w+1} H^1(\mathbb{P}^1, o_*\mathbb{W})^{P,Q},$$

where $H^1(\mathbb{P}^1, o_*\mathbb{W})^{P,Q}$ is the space of classes whose harmonic representative has type (P,Q). This decomposition depends neither on the Kähler metric nor on h. Complex conjugation of forms maps $H^1(\mathbb{P}^1, o_*\mathbb{W})^{P,Q}$ onto $H^1(\mathbb{P}^1, o_*\overline{\mathbb{W}})^{Q,P}$.

3. *Let p_0 be the largest p with $\mathbb{W}^{p,w-p} \neq 0$. Every square-integrable holomorphic section ω of $\Omega^1_U \otimes F^{p_0}$ is harmonic of type $(p_0+1, w-p_0)$, and if $\omega \neq 0$ its class is a nonzero element of $H^1(\mathbb{P}^1, o_*\mathbb{W})^{p_0+1,w-p_0}$.*

4. *Let $\mathbb{W}_1$ and $\mathbb{W}_2$ be local systems on U satisfying the hypotheses above, and $\phi : \mathbb{W}_1 \to \mathbb{W}_2$ an isomorphism of local systems which maps each $\mathbb{W}_1^{p,q}$ onto $\mathbb{W}_2^{p+n,q+n}$, for a fixed integer n, and is an isometry for the Hodge metrics. Then ϕ maps $H^1(\mathbb{P}^1, o_*\mathbb{W}_1)^{P,Q}$ onto $H^1(\mathbb{P}^1, o_*\mathbb{W}_2)^{P+n,Q+n}$ for all P, Q.*

Proof. We reduce to Zucker's theorem for real variations [62, (7.12)] by realification, retaining the complex summand throughout.

Let $\mathbb{V}_\mathbb{R}$ be the underlying real local system of $\mathbb{W}$. Its complexification is $\mathbb{V} = \mathbb{W} \oplus \overline{\mathbb{W}}$, with conjugation $(x, \bar{y}) \mapsto (y, \bar{x})$. Define

$$\mathbb{V}^{p,q} = \mathbb{W}^{p,q} \oplus \overline{\mathbb{W}^{q,p}}, \qquad Q((x_1,\bar{y}_1),(x_2,\bar{y}_2)) = i^{-w}\big(h(x_1,y_2) + (-1)^w h(x_2,y_1)\big).$$

This is a real polarisation: Q is flat and $(-1)^w$-symmetric, pairs only opposite Hodge components, and for $u = (x,\bar{y}) \in \mathbb{V}^{p,q}$,

$$i^{p-q}Q(u,\bar{u}) = (-1)^q h(x,x) + (-1)^p h(y,y) = \|x\|^2 + \|y\|^2.$$

The Hodge filtration is $F^p\mathbb{W} \oplus \overline{\bar{F}^p}$, where $\bar{F}^p$ is the h-orthogonal complement of $F^{w-p+1}\mathbb{W}$. Flatness of h makes the second summand holomorphic in the conjugate bundle and gives Griffiths transversality. Thus $\mathbb{V}_\mathbb{R}$ is a polarised real variation of weight w. Its monodromy is quasi-unipotent, because the eigenvalues for $\mathbb{W}$ and their conjugates are roots of unity.

The projection $\pi : \mathbb{V} \to \mathbb{W}$ is flat, preserves Hodge types and is orthogonal for the Hodge metric. Consequently it commutes with D, D'', their formal adjoints and $\Box_{D''}$, and preserves square-integrability. We may therefore take the π-summand in Zucker's harmonic-representative and Hodge-decomposition theorems. Here the harmonic space is exactly the space of smooth square-integrable forms killed by $\Box_{D''}$: regularity is [62, (7.11)], and the converse follows from essential self-adjointness on the complete metric of [62, (3.2),(3.4), p. 445]. These facts apply to each Hodge type and to its orthogonal direct summands.

For (1), the cokernel of $o_!\mathbb{W} \to o_*\mathbb{W}$ is supported on S, so

$$H^1_c(U, \mathbb{W}) \longrightarrow H^1(\mathbb{P}^1, o_*\mathbb{W})$$

is surjective [62, (14.3)]. The Leray edge map from the latter group to $H^1(U, \mathbb{W})$ is injective [62, proof of (12.5)]. Their composite is the forgetful map, proving the asserted identification.

For (2), apply [62, (6.15)(i),(7.6),(7.10),(7.12)] to $\mathbb{V}_\mathbb{R}$ and take the π-summand. A class in $H^1(\mathbb{P}^1, o_*\mathbb{W})$ has a unique harmonic representative: its representative in $\mathbb{V}$ equals its projection by uniqueness, and the same holds for every type component. The resulting decomposition is the intersection with Zucker's Hodge decomposition for $\mathbb{V}$, which is independent of the metric and polarisation [62, p. 447, after (7.13)]. The decomposition of $\mathbb{V}$ used here depends only on the original Hodge components, so the same independence holds for the summand. The real conjugation on $\mathbb{V}$ exchanges $\mathbb{W}$ and $\overline{\mathbb{W}}$ and interchanges the two Hodge indices, giving the final assertion.

For (3), put $q_0 = w - p_0$. The holomorphic top-bundle form ω has type (p_0+1, q_0). In Zucker's decomposition [62, (1.9),(1.10)], the degree-$(2,0)$ part of $D''\omega$ vanishes on a curve, and its remaining part is the $\bar\partial$ of a holomorphic section. Thus $D''\omega = 0$. Its adjoint has type (p_0+1, q_0-1), which cannot occur because every $\mathbb{W}$-valued form has second type index at least q_0. Hence $\Box_{D''}\omega = 0$. The remaining components of $D\omega$ vanish as well: one has degree $(2,0)$ and the other takes values in the absent Hodge component $\mathbb{W}^{p_0+1,q_0-1}$. Thus ω represents a parabolic class; by (2), a nonzero harmonic representative cannot represent zero. This proves (3).

Finally, the flat isometry ϕ in (4) intertwines D and shifts both type indices by n. It therefore intertwines D'', its adjoint and the Laplacian, and maps the square-integrable de Rham complexes isometrically onto one another. It carries the harmonic representative of each class to the harmonic representative of its image, with the stated type shift. Applying the same argument to ϕ^{-1} proves equality of the two type spaces.

□

Lemma A.4. *Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ and contain $\Gamma(n)$ for some $n \geq 1$, and let p be a prime not dividing n. Then every $\gamma \in \Gamma$ is the image of $A/\sqrt{\det A}$ for a primitive matrix $A \in M_2(\mathbb{Z})$ with $\det A > 0$ and $p \nmid \det A$.*

Proof. The first step is the classical description of groups commensurable with $\mathrm{PSL}_2(\mathbb{Z})$: their elements are rational matrices up to scalars. What is added here is that the determinants can be taken prime to p, which comes from a fixed point of the closure of Γ on the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Elements of Γ are rational up to scalars. Let $\tilde\gamma \in \mathrm{SL}_2(\mathbb{R})$ lift γ. Since $\Gamma(n)$ and $\tilde\gamma\Gamma(n)\tilde\gamma^{-1}$ both have finite index in Γ, the group $\Delta = \Gamma(n) \cap \tilde\gamma\Gamma(n)\tilde\gamma^{-1}$ has finite index in both. The $\mathbb{Q}$-linear span of a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is $M_2(\mathbb{Q})$, as the subgroup contains $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ h & 1\end{smallmatrix}\right)$ for some $h \geq 1$. So conjugation by $\tilde\gamma$ maps the span $M_2(\mathbb{Q})$ of $\tilde\gamma^{-1}\Delta\tilde\gamma$ onto the span $M_2(\mathbb{Q})$ of Δ, and is a $\mathbb{Q}$-algebra automorphism of $M_2(\mathbb{Q})$. By the Skolem–Noether theorem it is conjugation by some $A \in \mathrm{GL}_2(\mathbb{Q})$; then $A^{-1}\tilde\gamma$ commutes with $M_2(\mathbb{Q})$, hence with $M_2(\mathbb{R})$, and is a real scalar. Scaling A to be primitive in $M_2(\mathbb{Z})$, $\det\tilde\gamma = 1$ gives $\det A > 0$ and $\tilde\gamma = \pm A/\sqrt{\det A}$.

The closure of Γ in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is compact. By the first step Γ embeds in $\mathrm{PGL}_2(\mathbb{Q}) \subset \mathrm{PGL}_2(\mathbb{Q}_p)$. Let $K \subset \mathrm{PGL}_2(\mathbb{Q}_p)$ be the image of $\mathrm{GL}_2(\mathbb{Z}_p)$, compact and open. The subgroup $\Gamma \cap \mathrm{PSL}_2(\mathbb{Z})$ contains $\Gamma(n)$, so it has finite index in Γ; with coset representatives $\gamma_1, \dots, \gamma_r$ we get $\Gamma \subset \bigcup_i \gamma_i K$, a compact set. Hence the closure $\overline{\Gamma}$ is a compact subgroup. It contains the image of $\mathrm{SL}_2(\mathbb{Z}_p)$: for every s, $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/np^s)$ is surjective, so by the Chinese remainder theorem $\Gamma(n) \to \mathrm{SL}_2(\mathbb{Z}/p^s)$ is surjective, and $\Gamma(n)$ is dense in $\mathrm{SL}_2(\mathbb{Z}_p)$.

Fixed points on the tree. The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on the Bruhat–Tits tree, whose vertices are the homothety classes of lattices in $\mathbb{Q}_p^2$, and the stabiliser of the vertex $v_0 = [\mathbb{Z}_p^2]$ is K [50], Ch. II, §1. As $\overline{\Gamma}$ is compact and K open, the orbit $\overline{\Gamma}v_0$ is finite, so $\overline{\Gamma}$ fixes a vertex or the midpoint of an edge (loc. cit., Ch. I, §6). We show that the image of $\mathrm{SL}_2(\mathbb{Z}_p)$ fixes no point other than v_0. It fixes v_0 and acts on the $p+1$ neighbours of v_0, which correspond to the lines of $\mathbb{F}_p^2$, through $\mathrm{SL}_2(\mathbb{F}_p)$, transitively. Its fixed set is a subtree; if it contained a point $x \neq v_0$, it would contain the first edge $[v_0, v_1]$ of the geodesic from v_0 to x up to a point other than v_0, and, since $\mathrm{SL}_2(\mathbb{Q}_p)$ acts without inversion, also v_1, contradicting transitivity on the neighbours. Hence the fixed point of $\overline{\Gamma}$ is v_0, and $\overline{\Gamma} \subset K$.

Conclusion. Let $\gamma \in \Gamma$ and A as in the first step. As $\gamma \in K$, $A = \lambda B$ with $\lambda \in \mathbb{Q}_p^\times$ and $B \in \mathrm{GL}_2(\mathbb{Z}_p)$. Both A and B have an entry that is a p-adic unit and no entry of negative valuation, so $v_p(\lambda) = 0$ and $\det A = \lambda^2 \det B \in \mathbb{Z}_p^\times$.

□

A.2 Hecke characters and modularity

In the Hecke-character arguments, rational Frobenius values are known at the degree-one primes, a set of density one. The proofs below explain why this is sufficient for the uses of [53, Ch. II, §2.4, Proposition 2(iii); Ch. III, §2.3, Theorem 2, with §2.4].

Lemma A.5. *Let K be a number field, Galois over $\mathbb{Q}$, let S_0 be a finite set of rational primes, and let $\mathcal{P}$ be the set of primes of K of residue degree one over a rational prime not in S_0. Let ℓ be a prime that splits completely in K, let λ be a prime of K above ℓ and $\iota_\lambda : K \to K_\lambda = \mathbb{Q}_\ell$ the completion map, let $\delta : \mathcal{P} \to K^\times$ be a map, and let $\psi : \mathrm{Gal}(\bar{K}/K) \to \mathbb{Q}_\ell^\times$ be a continuous character such that:*

(a) *for every $\mathfrak{P} \in \mathcal{P}$ not above ℓ, ψ is unramified at $\mathfrak{P}$ and $\psi(\mathrm{Frob}_\mathfrak{P}) = \iota_\lambda(\delta(\mathfrak{P}))$, where $\mathrm{Frob}_\mathfrak{P}$ is a geometric Frobenius element at $\mathfrak{P}$;*

(b) *for every place w of K above ℓ, the restriction of ψ to a decomposition group at w is Hodge–Tate.*

Let T be the set of primes of K not above ℓ at which ψ is ramified. Then T is finite and disjoint from $\mathcal{P}$, and there are a nonzero integral ideal $\mathfrak{m}$ of K whose prime divisors are the primes in T and the primes above ℓ, and an algebraic Hecke character χ of K with values in K and modulus $\mathfrak{m}$, such that:

1. *ψ is unramified at every prime $v \nmid \mathfrak{m}$, and $\psi(\mathrm{Frob}_v) = \iota_\lambda(\chi(v))$;*
2. *$\chi(\mathfrak{P}) = \delta(\mathfrak{P})$ for every $\mathfrak{P} \in \mathcal{P}$ not above ℓ.*

Proof. For a rational representation, that is, one whose Frobenius traces lie in K at almost all primes, this is the one-dimensional case of Serre's theorem that a locally algebraic abelian representation comes from an algebraic Hecke character [53], Ch. III, §2.3, Theorem 2, in its form with coefficients in a number field (loc. cit., §2.4); passing from Hodge–Tate to locally algebraic characters follows [26], §6. We check the hypotheses and give the step that differs.

Let $w \mid \ell$. As ℓ splits completely in K, $K_w = \mathbb{Q}_\ell$, and the restriction of ψ to a decomposition group at w is Hodge–Tate by (b) and, being one-dimensional, semisimple on inertia; so it is locally algebraic by Tate's theorem (loc. cit., Ch. III, §1.2; proved in the Appendix to Ch. III as Theorem 3, p. III-52). Hence ψ is locally algebraic (loc. cit., Ch. III, §2.1) and has a modulus of definition (loc. cit., §2.2, Proposition 2, p. III-11), which can be taken of the form $\mathfrak{m} = \prod_{v \in T} v^{e_v} \prod_{w \mid \ell} w^{e_w}$ with all $e_v, e_w \geq 1$; in particular T is finite. By (a), ψ is unramified at every prime of $\mathcal{P}$ not above ℓ, so $T \cap \mathcal{P} = \emptyset$.

The step that differs is the last: up to it, the proof of Theorem 2 uses only local algebraicity and the modulus of definition, and yields a character ϕ of the group $S_\mathfrak{m}$ (loc. cit., Ch. II, §2.2) over $K_\lambda = \mathbb{Q}_\ell$ with the same Frobenius values as ψ at the primes $v \nmid \mathfrak{m}\ell$. Rationality enters only to show that ϕ is defined over K, by loc. cit., Ch. II, §2.4, Proposition 2, whose criterion (iii) asks only that $\mathrm{Tr}\,\phi(F_v) \in K$ for v in a set of places of density one, F_v denoting the arithmetic Frobenius. The primes of $\mathcal{P}$ prime to $\mathfrak{m}\ell$ form such a set: the primes of residue degree one over $\mathbb{Q}$ have density one (loc. cit., Ch. I, §2.2, p. I-7), and S_0 and the support of $\mathfrak{m}\ell$ are finite. For such a prime $\mathfrak{P}$, (a) gives $\mathrm{Tr}\,\phi(F_\mathfrak{P}) = \psi(\mathrm{Frob}_\mathfrak{P})^{-1} = \iota_\lambda(\delta(\mathfrak{P}))^{-1} \in \iota_\lambda(K)$. So ϕ is defined over K. The characters of $S_\mathfrak{m}$ defined over K are the algebraic Hecke characters of K with values in K and modulus $\mathfrak{m}$ (loc. cit., Ch. II, §2.4, Exercise, and §2.7); this gives χ with $\psi(\mathrm{Frob}_v) = \iota_\lambda(\chi(v))$ for every prime $v \nmid \mathfrak{m}$, which is (1). Finally (2) is (1) at the primes $\mathfrak{P} \in \mathcal{P}$ not above ℓ, which do not divide $\mathfrak{m}$ as $T \cap \mathcal{P} = \emptyset$, together with (a) and the injectivity of ι_λ.

□

Lemma A.6. *Let K be a number field, Galois over $\mathbb{Q}$, let S_0 be a finite set of rational primes, and let $\mathcal{P}$ be the set of primes of K of residue degree one over a rational prime not in S_0. For $i = 1, 2$ let χ_i be an algebraic Hecke character of K with values in K, modulus $\mathfrak{m}_i$ and infinity type $(a_{i,\sigma})_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})}$, and suppose that $\chi_1(\mathfrak{P}) = \chi_2(\mathfrak{P})$ for all but finitely many $\mathfrak{P} \in \mathcal{P}$ prime to $\mathfrak{m}_1\mathfrak{m}_2$.*

1. *Then $a_{1,\sigma} = a_{2,\sigma}$ for every $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, and $\chi_1 = \chi_2$ on $I^{\mathfrak{m}_1\mathfrak{m}_2}$.*
2. *Suppose moreover that $\mathfrak{m}_1 = \mathfrak{n}\mathfrak{r}_1$ and $\mathfrak{m}_2 = \mathfrak{n}\mathfrak{r}_2$, where $\mathfrak{n}$, $\mathfrak{r}_1$, $\mathfrak{r}_2$ are integral ideals with pairwise disjoint sets of prime divisors. Then there is a unique algebraic Hecke character χ of K with values in K and modulus $\mathfrak{n}$ whose restriction to $I^{\mathfrak{m}_i}$ is χ_i for $i = 1, 2$; its infinity type is $(a_{1,\sigma})_\sigma$.*

Proof. Part (1) is the analogue, for algebraic Hecke characters, of the uniqueness theorem for compatible rational representations [53], Ch. I, §2.3, Theorem, and rests, as its proof does, on the density of the Frobenius elements of a set of places of density one (loc. cit., Ch. I, §2.2, Corollary 2(b)).

Put $\mathfrak{M} = \mathfrak{m}_1\mathfrak{m}_2$, $c_\sigma = a_{1,\sigma} - a_{2,\sigma}$, and $\xi = \chi_1/\chi_2 : I^{\mathfrak{M}} \to K^\times$. A totally positive $\beta \in K^\times$ with $\beta \equiv 1 \bmod {}^\times\mathfrak{M}$ satisfies $\beta \equiv 1 \bmod {}^\times\mathfrak{m}_i$ for $i = 1, 2$, so $\xi((\beta)) = \prod_\sigma \sigma(\beta)^{c_\sigma}$; and $\xi(\mathfrak{P}) = 1$ for every $\mathfrak{P} \in \mathcal{P}$ prime to $\mathfrak{M}$ outside a finite set F_0. Let $P^+_{\mathfrak{M}}$ be the group of principal ideals with a totally positive generator $\equiv 1 \bmod {}^\times\mathfrak{M}$. By class field theory, $I^{\mathfrak{M}}/P^+_{\mathfrak{M}}$ is the Galois group of a finite abelian extension $K_{\mathfrak{M}}/K$ in which every prime $v \nmid \mathfrak{M}$ is unramified with Frobenius element corresponding to its class (loc. cit., Ch. II, §2.1, Remark, and §2.3).

The infinity types agree. Let K' be the Galois closure of $K_{\mathfrak{M}}$ over $\mathbb{Q}$. By Chebotarev's density theorem (loc. cit., Ch. I, §2.2) infinitely many rational primes split completely in K'; choose one, p, with $p \notin S_0$ and such that no prime of K above p divides $\mathfrak{M}$ or lies in F_0. Let $\mathfrak{P}$ be a prime of K above p. Since p splits completely in K, $\mathfrak{P} \in \mathcal{P}$, and $\mathrm{Gal}(K/\mathbb{Q})$ acts simply transitively on the primes of K above p. Since $\mathfrak{P}$ splits completely in $K_{\mathfrak{M}}$, its Frobenius element is trivial, so $\mathfrak{P} = (\beta)$ with β totally positive and $\beta \equiv 1 \bmod {}^\times\mathfrak{M}$; hence $\prod_\sigma \sigma(\beta)^{c_\sigma} = \xi(\mathfrak{P}) = 1$. For $\sigma_0 \in \mathrm{Gal}(K/\mathbb{Q})$, the ideal $(\sigma(\beta)) = \sigma(\mathfrak{P})$ equals $\sigma_0(\mathfrak{P})$ only for $\sigma = \sigma_0$, and otherwise is prime to it. Taking the $\sigma_0(\mathfrak{P})$-adic valuation,

$$0 = \sum_\sigma c_\sigma \, v_{\sigma_0(\mathfrak{P})}\big(\sigma(\beta)\big) = c_{\sigma_0}.$$

The characters agree. By the first step ξ is trivial on $P^+_{\mathfrak{M}}$, so it is a character of the finite group $I^{\mathfrak{M}}/P^+_{\mathfrak{M}} = \mathrm{Gal}(K_{\mathfrak{M}}/K)$, trivial on the Frobenius elements of the primes of $\mathcal{P}$ prime to $\mathfrak{M}$ and not in F_0. These primes form a set of density one, as $\mathcal{P}$ has density one (loc. cit., p. I-7), so by loc. cit., Ch. I, §2.2, Corollary 2(b), their Frobenius elements meet every class. So $\xi = 1$, which completes the proof of (1).

Proof of (2). Every $\mathfrak{a} \in I^{\mathfrak{n}}$ factors uniquely as $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$ with $\mathfrak{b}$ divisible only by primes dividing $\mathfrak{r}_1$ and $\mathfrak{c}$ prime to $\mathfrak{r}_1$. Then $\mathfrak{b} \in I^{\mathfrak{m}_2}$, as no prime dividing $\mathfrak{r}_1$ divides $\mathfrak{n}\mathfrak{r}_2$, and $\mathfrak{c} \in I^{\mathfrak{m}_1}$. Put $\chi(\mathfrak{a}) = \chi_2(\mathfrak{b})\chi_1(\mathfrak{c})$; this is a homomorphism $I^{\mathfrak{n}} \to K^\times$. On $I^{\mathfrak{m}_1}$ it equals χ_1, since there $\mathfrak{b} = (1)$. On $I^{\mathfrak{m}_2}$ it equals χ_2: for $\mathfrak{a} \in I^{\mathfrak{m}_2}$, $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ lies in $I^{\mathfrak{m}_1} \cap I^{\mathfrak{m}_2} = I^{\mathfrak{M}}$, where $\chi_1 = \chi_2$ by (1). It is unique, because $I^{\mathfrak{n}}$ is generated by $I^{\mathfrak{m}_1}$ and $I^{\mathfrak{m}_2}$. For the infinity type, let $\alpha \in K^\times$ be totally positive with $\alpha \equiv 1 \bmod {}^\times\mathfrak{n}$. By the approximation theorem [45], Ch. II, (3.4), there is $\gamma \in K^\times$, positive at every real place, with $\gamma \equiv 1 \bmod {}^\times\mathfrak{n}$, $\gamma/\alpha \equiv 1 \bmod {}^\times\mathfrak{r}_1$ and $\gamma \equiv 1 \bmod {}^\times\mathfrak{r}_2$. Then α/γ is totally positive with $\alpha/\gamma \equiv 1 \bmod {}^\times\mathfrak{m}_1$, and γ is totally positive with $\gamma \equiv 1 \bmod {}^\times\mathfrak{m}_2$, so

$$\chi((\alpha)) = \chi_1((\alpha/\gamma))\,\chi_2((\gamma)) = \prod_\sigma \sigma(\alpha/\gamma)^{a_\sigma} \prod_\sigma \sigma(\gamma)^{a_\sigma} = \prod_\sigma \sigma(\alpha)^{a_\sigma}, \qquad a_\sigma = a_{1,\sigma} = a_{2,\sigma}.$$

□

Lemma A.7. *1. Let ℓ be a prime, E a finite extension of $\mathbb{Q}_\ell$, and $\xi : \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to E^\times$ a crystalline character with Hodge–Tate weight h, $0 \le h \le \ell - 2$, in the normalisation in which $\mathbb{Q}_\ell(-1)$ has weight 1. Then $\xi\chi_\ell^h$ is unramified, where χ_ℓ is the ℓ-adic cyclotomic character.*

2. Let $k \ge 3$ be an integer, T a finite set of primes, and, for every prime ℓ, V_ℓ a two-dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Assume:

- **(a)** *for every prime $p \notin T$ and every prime $\ell \neq p$, V_ℓ is unramified at p, and the characteristic polynomial $Q_p(X)$ of a geometric Frobenius at p on V_ℓ lies in $\mathbb{Z}[X]$, does not depend on ℓ, and has roots of absolute value $p^{(k-1)/2}$ under every complex embedding;*
- **(b)** *for every prime $\ell \notin T$ with $\ell > k$, the restriction of V_ℓ to a decomposition group at ℓ is crystalline with Hodge–Tate weights 0 and $k-1$, in the normalisation in which $\mathbb{Q}_\ell(-1)$ has weight 1.*

Then V_ℓ is absolutely irreducible for every prime $\ell \notin T$ with $\ell > k$.

Proof. Proof of (1). The character $\xi\chi_\ell^h$ is crystalline with Hodge–Tate weight 0. As $0 \le h \le \ell - 2$, the rank-one case of the theory of Fontaine and Laffaille [25], Théorème 8.4, applies: the filtered φ-module of $\xi\chi_\ell^h$ is weakly admissible of rank one with trivial filtration, so its Frobenius is a unit, and the corresponding representation is unramified.

Proof of (2). Fix a prime $\ell \notin T$ with $\ell > k$, and suppose that $V_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell$ is reducible. A $G_\mathbb{Q}$-stable line is spanned by a vector v whose coordinates in a $\mathbb{Q}_\ell$-basis of V_ℓ lie in a finite extension E of $\mathbb{Q}_\ell$; for $g \in G_\mathbb{Q}$ both v and gv lie in $V_\ell \otimes E$, so $gv \in E^\times v$. Hence there are continuous characters $\chi_1, \chi_2 : G_\mathbb{Q} \to E^\times$ and an exact sequence of $E[G_\mathbb{Q}]$-modules

$$0 \to E(\chi_1) \to V_\ell \otimes_{\mathbb{Q}_\ell} E \to E(\chi_2) \to 0.$$

The images of χ_1, χ_2 are compact, so they lie in $\mathcal{O}_E^\times$.

Let $D \subset G_\mathbb{Q}$ be a decomposition group at ℓ, identified with $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. As a $\mathbb{Q}_\ell[D]$-module, $V_\ell \otimes_{\mathbb{Q}_\ell} E$ is a direct sum of copies of $V_\ell|_D$, which is crystalline with Hodge–Tate weights 0 and $k-1$ by (b). Crystalline representations are stable under subquotients, and D_{HT} is exact on Hodge–Tate representations [24], §1. So each $\chi_i|_D$ is crystalline; its D_{HT} is a graded E-vector space of dimension one, so $\chi_i|_D$ has a single Hodge–Tate weight h_i, and $\{h_1, h_2\} = \{0, k-1\}$. As $0 \le h_i \le k-1 \le \ell - 2$, (1) shows that $\chi_i|_D$ is $\chi_\ell^{-h_i}$ times an unramified character.

Next, $\epsilon_i = \chi_i\chi_\ell^{h_i}$ has finite order. It is unramified at ℓ by the previous step, and at every prime $p \notin T \cup \{\ell\}$, because V_ℓ is unramified there by (a) and so is χ_ℓ. By the Kronecker–Weber theorem, ϵ_i therefore factors through $\mathrm{Gal}(\mathbb{Q}(\mu_{N^\infty})/\mathbb{Q}) \cong \prod_{q \in T} \mathbb{Z}_q^\times$, where $N = \prod_{q \in T} q$. Each $\mathbb{Z}_q^\times$ is a finite group times a pro-q group. The group $\mathcal{O}_E^\times$ contains an open pro-ℓ subgroup U, and the preimage of U in a pro-q group P with $q \neq \ell$ is an open subgroup of P mapped to U trivially, a pro-q group having no nontrivial continuous pro-ℓ quotient. As $\ell \notin T$, ϵ_i has finite image.

Finally we compare absolute values. Let $p \notin T \cup \{\ell\}$ and let Frob_p be a geometric Frobenius at p. The roots of Q_p are $\chi_1(\mathrm{Frob}_p)$ and $\chi_2(\mathrm{Frob}_p)$, and $\chi_i(\mathrm{Frob}_p) = \epsilon_i(\mathrm{Frob}_p)\, p^{h_i}$ with $\epsilon_i(\mathrm{Frob}_p)$ a root of unity. These roots are algebraic numbers, and under every complex embedding $\chi_i(\mathrm{Frob}_p)$ has absolute value p^{h_i}; by (a) this equals $p^{(k-1)/2}$. Hence $h_1 = h_2 = \frac{k-1}{2}$, contradicting $\{h_1, h_2\} = \{0, k-1\}$ with $k - 1 \neq 0$. So V_ℓ is absolutely irreducible.

□

The modularity argument needs the Serre weight, including the orientation of a reducible residual extension. It uses the lifting theorems at one auxiliary prime: the hypotheses of [34, Theorem 10.1(i)], which require Weil–Deligne compatibility at every prime, have not been established for our system.

Lemma A.8. *Let ℓ be a prime and k an integer with $3 \le k \le \ell - 1$. Let V be a two-dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, crystalline with Hodge–Tate weights 0 and $k-1$ in the normalisation in which $\mathbb{Q}_\ell(-1)$ has weight 1. Let $T \subset V$ be a $G_{\mathbb{Q}_\ell}$-stable $\mathbb{Z}_\ell$-lattice and $\bar{W} = \mathrm{Hom}_{\mathbb{F}_\ell}(T/\ell T, \mathbb{F}_\ell)$. Let $I \subset G_{\mathbb{Q}_\ell}$ be the inertia subgroup, ω the mod ℓ cyclotomic character and ω_2 a fundamental character of level two. Then $(\bar{W} \otimes \bar{\mathbb{F}}_\ell)|_I$ is either $\omega_2^{k-1} \oplus \omega_2^{\ell(k-1)}$ or an extension of the trivial character by ω^{k-1}. Consequently, if $\bar{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$, $\mathbb{F}$ a finite field of characteristic ℓ, restricts on a decomposition group at ℓ to $\bar{W} \otimes_{\mathbb{F}_\ell} \mathbb{F}$, then the weight $k(\bar{r})$ of [51], §2, equals k.*

Proof. The action of inertia on the reductions of Fontaine–Laffaille lattices is [25], Théorème 5.3.

The Hodge–Tate weights of V lie in $[0, k-1] \subset [0, \ell - 2]$, so $\mathrm{Fil}^0 D_{\mathrm{cris}}(V) = D_{\mathrm{cris}}(V)$ and $\mathrm{Fil}^\ell D_{\mathrm{cris}}(V) = 0$, and no nonzero quotient V' of V has $V'(\ell - 1)$ unramified, since V' would have Hodge–Tate weight $\ell - 1$. By [20], §1.1 (from [25], 6.1 and 8.4), $T \cong \mathbb{V}(A)$ for an object A of $\mathcal{MF}^0$, and the filtration of A has jumps 0 and $k-1$, each of rank one. As $\mathbb{V}$ is exact, $T/\ell T \cong \mathbb{V}(N)$ with $N = A/\ell A$, and as N is killed by ℓ, $\bar{W} \cong \underline{U}_S(N)$.

Extending scalars from $W(\mathbb{F}_\ell)$ to $W(\bar{\mathbb{F}}_\ell)$ restricts the Galois action to I [25], 3.11: $\bar{W}|_I \cong \underline{U}_S(\bar{N})$ for the object $\bar{N} = N \otimes \bar{\mathbb{F}}_\ell$ of the category $\tilde{\mathcal{M}}$ of loc. cit., 4.1, with filtration jumps 0 and $k-1$. Let $L = \mathrm{Fil}^{k-1} \bar{N}$, a line, and let $\varphi_{\bar{N}} : \mathrm{gr}\, \bar{N} = \bar{N}/L \oplus L \to \bar{N}$ be the bijective semilinear map of loc. cit., 4.2. Put $I_0 = \varphi_{\bar{N}}(\bar{N}/L)$ and $I_1 = \varphi_{\bar{N}}(L)$, lines with $\bar{N} = I_0 \oplus I_1$. By loc. cit., Lemme 4.7, a line of $\bar{N}$ is a sub-object if and only if it is stable under the map f of loc. cit., 4.6, which sends $\bar{N} \setminus L$ into $I_0 \setminus \{0\}$ and $L \setminus \{0\}$ into $I_1 \setminus \{0\}$. So the sub-objects of rank one are L, when $I_1 = L$, and I_0, when $I_0 \ne L$.

- If $I_0 = L$, then $I_1 \ne L$, $\bar{N}$ has no sub-object of rank one and is simple, and by loc. cit., Proposition 4.4, $\bar{N} \cong M(2; i)$ with $\{i_0, i_1\} = \{0, k-1\}$. By loc. cit., Théorème 5.3, applicable as $0 \le i_n \le \ell - 1$, $\underline{U}_S(\bar{N})$ is a line over $\mathbb{F}_{\ell^2}$ on which I acts through $\chi_2^{i_0 + \ell i_1}$, χ_2 the fundamental character of level two (loc. cit., 5.1). Over $\bar{\mathbb{F}}_\ell$ this is $\omega_2^{k-1} \oplus \omega_2^{\ell(k-1)}$.
- If $I_0 \ne L$, then I_0 is a sub-object with filtration jump 0, isomorphic to $M(1; 0)$, and $\bar{N}/I_0$ has jump $k-1$ and is isomorphic to $M(1; k-1)$. As $\underline{U}_S$ is exact and contravariant, $\underline{U}_S(\bar{N})$ is an extension of $\underline{U}_S(I_0)$ by $\underline{U}_S(\bar{N}/I_0)$. By loc. cit., Théorème 5.3(iii), I acts trivially on the first and through χ_1^{k-1} on the second, and $\chi_1 = \omega$ on I.

Finally we compute Serre's weight, which depends only on the restriction to I [51], §2. In the first case $\bar{r}|_I \otimes \bar{\mathbb{F}}_\ell \cong \omega_2^{a + \ell b} \oplus \omega_2^{b + \ell a}$ with $a = 0$, $b = k-1$ and $0 \le a < b \le \ell - 1$, so $k(\bar{r}) = 1 + \ell a + b = k$. In the second, $\bar{r}|_I \otimes \bar{\mathbb{F}}_\ell$ is an extension of the trivial character by ω^{k-1} with $2 \le k - 1 \le \ell - 2$, so $k(\bar{r}) = 1 + (k-1) = k$, whether or not the extension splits; the exponent 1, for which the weight depends on whether the extension is peu or très ramifiée, does not occur. □

Lemma A.9. *Let $k \ge 3$, T and $(V_\ell)_\ell$ be as in Lemma A.7(2), satisfying (a) and (b) there, and write $Q_p(X) = X^2 - a_p X + b_p$ for $p \notin T$. Assume moreover:*

(c) $\det V_\ell(c) = -1$ *for every prime $\ell \notin T$, where $c \in G_{\mathbb{Q}}$ is a complex conjugation;*

(d) *there is a prime $p_0 \notin T$ such that Q_{p_0} is irreducible over $\mathbb{Q}$.*

Then there is a newform g of weight k and nebentypus ψ, of level N_g divisible only by primes in T, such that

$$Q_p(X) = X^2 - a_p(g)\,X + \psi(p)\,p^{k-1} \qquad \text{for every prime } p \notin T.$$

Moreover $V_\ell \otimes_{\mathbb{Q}_\ell} \bar{\mathbb{Q}}_\ell \cong \rho_{g,\lambda} \otimes \bar{\mathbb{Q}}_\ell$ for every prime $\ell \notin T$ with $\ell > k$ and every prime λ above ℓ of the coefficient field of g, where $\rho_{g,\lambda}$ is the λ-adic representation attached to g, normalised so that a geometric Frobenius at $p \nmid N_g\ell$ has characteristic polynomial $X^2 - a_p(g)X + \psi(p)p^{k-1}$.

Proof. For systems that are compatible in the sense of [34], §5, this is [34], Theorem 10.1(i). We apply instead the two theorems behind it, Serre's modularity conjecture and the modularity lifting theorem of Khare and Wintenberger, at a single auxiliary prime ℓ, chosen so that two Frobenius elements force the residual hypotheses.

We first choose the auxiliary prime ℓ. There is an odd prime $p_1 \notin T$ with $a_{p_1} \neq 0$: otherwise, for a fixed prime ℓ_1, $\operatorname{tr} V_{\ell_1}(\mathrm{Frob}_p) = 0$ for all odd primes $p \notin T \cup \{\ell_1\}$, so the continuous function $\operatorname{tr} V_{\ell_1}$ would vanish on a dense subset of $G_{\mathbb{Q}}$, hence everywhere, while its value at 1 is 2. Let $\Delta = a_{p_0}^2 - 4b_{p_0} \in \mathbb{Z}$, not a square by (d), and $F = \mathbb{Q}(\sqrt{\Delta}, \sqrt{p_1})$. Some element of $\mathrm{Gal}(F/\mathbb{Q})$ acts nontrivially on both $\sqrt{\Delta}$ and $\sqrt{p_1}$: the element $(-1,-1)$ of $\mathrm{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2)^2$ if $\mathbb{Q}(\sqrt{\Delta}) \neq \mathbb{Q}(\sqrt{p_1})$, and the nontrivial element otherwise. By Chebotarev's theorem infinitely many primes have this Frobenius in F, that is, are inert both in $\mathbb{Q}(\sqrt{\Delta})$ and in $\mathbb{Q}(\sqrt{p_1})$. Fix such a prime ℓ with $\ell \notin T \cup \{2, p_0, p_1\}$, $\ell > k$ and $\ell \nmid \Delta a_{p_1}$.

We next show that the residual representation is large. Choose a $G_{\mathbb{Q}}$-stable $\mathbb{Z}_\ell$-lattice in V_ℓ, which exists as $G_{\mathbb{Q}}$ is compact, let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ be the resulting representation and $\bar\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell)$ its reduction.

- $\bar\rho$ is irreducible over $\mathbb{F}_\ell$. By (a), $\bar\rho(\mathrm{Frob}_{p_0})$ has characteristic polynomial $Q_{p_0} \bmod \ell$, whose discriminant Δ is a non-square in $\mathbb{F}_\ell^\times$, as ℓ is odd, $\ell \nmid \Delta$ and ℓ is inert in $\mathbb{Q}(\sqrt{\Delta})$. So $\bar\rho(\mathrm{Frob}_{p_0})$ has no eigenvalue in $\mathbb{F}_\ell$, and no $\mathbb{F}_\ell$-line is stable.

- $\bar\rho$ is absolutely irreducible. Otherwise a stable line L of $\bar\rho \otimes \bar{\mathbb{F}}_\ell$ is defined over a finite field $\mathbb{F} \supset \mathbb{F}_\ell$. Its image $\phi(L)$ under the Frobenius ϕ of $\mathbb{F}/\mathbb{F}_\ell$ is stable, and $\phi(L) \neq L$, since a ϕ-stable line has its slope in $\mathbb{P}^1(\mathbb{F}_\ell)$ and would be an $\mathbb{F}_\ell$-line. So $\bar\rho \otimes \mathbb{F} = L \oplus \phi(L)$, with $G_{\mathbb{Q}}$ acting by characters θ and $\phi \circ \theta$. As $c^2 = 1$, $\theta(c) = \pm 1 \in \mathbb{F}_\ell$, so $\phi(\theta(c)) = \theta(c)$ and $\det \bar\rho(c) = \theta(c)^2 = 1$, whereas $\det \bar\rho(c) = -1$ by (c), and $-1 \neq 1$ in $\mathbb{F}_\ell$.

- $\bar\rho|_{G_{\mathbb{Q}(\zeta_\ell)}}$ is absolutely irreducible. Suppose not, and put $H = G_{\mathbb{Q}(\zeta_\ell)}$, a normal subgroup with cyclic quotient. By Clifford's theorem $(\bar\rho \otimes \bar{\mathbb{F}}_\ell)|_H$ is semisimple, hence a sum of two characters θ_1, θ_2 of H. If $\theta_1 = \theta_2$, then $\bar\rho(H)$ consists of scalars, $\bar\rho(G_{\mathbb{Q}})$ is generated by scalars and one element, so it is abelian and $\bar\rho \otimes \bar{\mathbb{F}}_\ell$ is reducible, which is excluded. So $\theta_1 \neq \theta_2$, $G_{\mathbb{Q}}$ permutes the two isotypic lines, and the stabiliser G' of one of them has index 2 and contains H; every $g \notin G'$ exchanges the lines, so $\operatorname{tr} \bar\rho(g) = 0$. Now $G' = G_{\mathbb{Q}(\sqrt{\ell^*})}$, $\ell^* = (-1)^{(\ell-1)/2}\ell$, the quadratic subfield of $\mathbb{Q}(\zeta_\ell)$. By quadratic reciprocity $\left(\frac{\ell^*}{p_1}\right) = \left(\frac{p_1}{\ell}\right) = -1$, the last because ℓ is inert in $\mathbb{Q}(\sqrt{p_1})$; so p_1 is inert in $\mathbb{Q}(\sqrt{\ell^*})$, $\mathrm{Frob}_{p_1} \notin G'$, and $a_{p_1} \equiv \operatorname{tr} \bar\rho(\mathrm{Frob}_{p_1}) = 0 \pmod{\ell}$, contradicting $\ell \nmid a_{p_1}$.

We next show that V_ℓ is modular. Put $W = \rho^\vee$, a lift of $\bar\rho^\vee$; by the previous step $\bar\rho^\vee$ is odd, absolutely irreducible, and absolutely irreducible on $G_{\mathbb{Q}(\zeta_\ell)}$, these properties passing to the dual. By Serre's modularity conjecture, proved by Khare and Wintenberger [34], Theorem 9.1, together with Kisin's proof of their Hypothesis (H) [36], Theorem 0.1, $\bar\rho^\vee$ arises from a newform. By (a), W is unramified outside $T \cup \{\ell\}$, and $b_p = \pm p^{k-1}$ (an integer of absolute value p^{k-1}), so

$\det V_\ell \cdot \chi_\ell^{k-1}$ takes the values ± 1 at all Frob_p, $p \notin T \cup \{\ell\}$, and is a character of order at most two; thus $\det W$ is a finite-order character times χ_ℓ^{k-1}. By (b), W is crystalline at ℓ, duals of crystalline representations being crystalline [24], §1, with Hodge–Tate weights 0 and $k-1$ in the normalisation in which χ_ℓ has weight 1. So $W \otimes \mathbb{C}_\ell \cong \mathbb{C}_\ell \oplus \mathbb{C}_\ell(k-1)$, that is, W is crystalline of weight k in the sense of [34], §4, and [35], Definition 3.4, with $2 \le k \le \ell+1$. By Lemma A.8, applied to V_ℓ restricted to a decomposition group at ℓ and to the lattice of ρ, the Serre weight of $\bar\rho^\vee$ is k; so $\bar\rho^\vee$ lies in the setting of [35], §1, and of the paragraph preceding [34], Theorem 4.1 (Serre weight in $[2, \ell+1]$), in which [34], Theorem 4.1, is proved. By [34], Theorem 4.1(2)(i), proved in [35], W arises from a newform. For $k \le \ell - 1$, the case used here, this lifting theorem is due to Diamond, Flach and Guo [20], the theorem of the subsection "Consequences" of the section on the Taylor–Wiles construction (see [35], §10.2). Hence there are a newform f and a prime λ of K_f above ℓ such that $V_\ell \otimes \bar{\mathbb{Q}}_\ell$ is isomorphic to $\rho_{f,\lambda} \otimes \bar{\mathbb{Q}}_\ell$ or to its contragredient. Let $p \notin T \cup \{\ell\}$ with $p \nmid N_f$. The geometric Frobenius at p has determinant b_p on V_ℓ, of absolute value p^{k-1} by (a), and determinant $\psi_f(p)^{\pm 1} p^{\pm(k_f-1)}$ on $\rho_{f,\lambda}$, respectively on its contragredient. As $k \ge 3$ and $k_f \ge 2$, only $V_\ell \otimes \bar{\mathbb{Q}}_\ell \cong \rho_{f,\lambda} \otimes \bar{\mathbb{Q}}_\ell$ is possible, and $k_f = k$. Comparing characteristic polynomials, $a_p(f) = a_p$ and $\psi_f(p) p^{k-1} = b_p$ for all such p, the identities holding in K_f because the embedding $K_f \to \bar{\mathbb{Q}}_\ell$ is injective and $a_p, b_p \in \mathbb{Z}$.

We next compare f with the whole system. Let $\ell' \notin T$ be a prime with $\ell' > k$ and λ' a prime of K_f above ℓ'. For every prime $p \notin T \cup \{\ell, \ell'\}$ with $p \nmid N_f$, both $V_{\ell'}$ and $\rho_{f,\lambda'}$ are unramified at p, and a geometric Frobenius has characteristic polynomial Q_p on both, by (a) and the previous step. These Frobenius elements are dense in $G_{\mathbb{Q}}$ and characteristic polynomials vary continuously, so $V_{\ell'} \otimes \bar{\mathbb{Q}}_{\ell'}$ and $\rho_{f,\lambda'} \otimes \bar{\mathbb{Q}}_{\ell'}$ have the same characteristic polynomial at every element of $G_{\mathbb{Q}}$, and by the Brauer–Nesbitt theorem isomorphic semisimplifications. As $V_{\ell'}$ is absolutely irreducible by Lemma A.7(2), so is $\rho_{f,\lambda'}$, and $V_{\ell'} \otimes \bar{\mathbb{Q}}_{\ell'} \cong \rho_{f,\lambda'} \otimes \bar{\mathbb{Q}}_{\ell'}$.

It remains to determine the level. Let $p \notin T$ and choose a prime $\ell' \notin T \cup \{p\}$ with $\ell' > k$. By (a), $V_{\ell'}$ is unramified at p, hence so is $\rho_{f,\lambda'}$. By local–global compatibility at $p \ne \ell'$ [13], Théorème (A), the component at p of the automorphic representation of f is then unramified, so $p \nmid N_f$. Therefore N_f is divisible only by primes in T, and for every $p \notin T$

$$Q_p(X) = \det(X - \mathrm{Frob}_p \mid \rho_{f,\lambda'}) = X^2 - a_p(f) X + \psi_f(p) p^{k-1}.$$

The lemma holds with $g = f$ and $\psi = \psi_f$.

□

B A level-seven supercongruence

The trace in Corollary 7.12 is identified by comparing $c_7(p)$ with Cooper's sequence at $(p-1)/2$, and then with the Apéry number at the same index. We first state the two congruences together. Their proofs use the modular parametrisation of Section 7 and the classical Apéry supercongruence of Ahlgren and Ono [1].

Theorem B.1. *For every odd prime p,*

$$c_7(p) \equiv s_7\left(\frac{p-1}{2}\right) \equiv A_p^{\eta} \pmod{p^2}.$$

Proof. Put $h = (p-1)/2$. By Lemma B.10, the first congruence amounts to $d_h \equiv 0 \pmod p$. By Lemma B.9, d_h is the harmonic Cooper sum e_h of Definition B.4, and we show that every summand of e_h vanishes modulo p, by the symmetry $m \mapsto p-1-m$ of the harmonic numbers. For $0 \le m \le p-1$, the harmonic numbers in Definition B.2 are p-integral and satisfy

$$\mathfrak{h}_{p-1-m} \equiv \mathfrak{h}_m \pmod p.$$

Indeed, pairing j with $p-j$ proves $\mathfrak{h}_{p-1} \equiv 0 \pmod p$, and subtracting the last m terms gives the displayed congruence. For each $0 \le k \le h$, the equalities $h+k = p-1-(h-k)$ and $2h-k = p-1-k$ therefore yield

$$\mathfrak{h}_{h+k} + \mathfrak{h}_{2h-k} - \mathfrak{h}_k - \mathfrak{h}_{h-k} \equiv 0 \pmod p.$$

All factorial arguments in the coefficient

$$\frac{(h+k)!(2h-k)!}{k!^3(h-k)!^3}$$

are between 0 and $p-1$, so this coefficient is p-integral. The factor $3/2$ is also p-integral, including at $p=3$. Hence $e_h \in p\mathbb{Z}_p$ term by term, and $d_h = e_h \in p\mathbb{Z}_p$. Applying Lemma B.10 gives

$$c_7(p) - s_7(h) \equiv -\frac{p}{2} d_h \equiv 0 \pmod{p^2}.$$

The second congruence $s_7(h) \equiv A_p^{\eta} \pmod{p^2}$ is Proposition B.12.

□

Definition B.2. In the notation of Definition 7.1, let $G(Y) = (1+Y)(1-27Y)$, use the formal branch $\sqrt{G(Y)} \in 1 + Y\mathbb{Q}[[Y]]$ and define

$$\rho(Y) = \frac{1}{F(Y)\sqrt{G(Y)}}, \qquad \ell(Y) = \int_0^Y \frac{\rho(T)-1}{T}\,dT, \qquad \mathcal{H}(Y) = F(Y)\ell(Y) = \sum_{n\ge 0} d_n Y^n.$$

The integral denotes termwise formal integration, with zero constant term. The series $\mathcal{H}$ is the **logarithmic Cooper companion**. The **harmonic numbers** are $\mathfrak{h}_0 = 0$ and $\mathfrak{h}_m = \sum_{j=1}^{m} 1/j$ for integers $m \ge 1$.

Definition B.3. With $\theta = Y\,d/dY$, the **Cooper operators** are

$$\mathcal{L} = \theta^3 - Y(2\theta+1)(13\theta^2+13\theta+4) - 3Y^2(\theta+1)(3\theta+2)(3\theta+4),$$

$$\mathcal{D} = (1+Y)(1-27Y)\,\theta^2 - (13Y+27Y^2)\,\theta - (2Y+6Y^2),$$

acting on $\mathbb{Q}[[Y]]$ and on holomorphic functions on open subsets of $\mathbb{C}^{\times}$. Up to the factor Y they are Cooper's: $\mathcal{D}/Y$ and $\mathcal{L}/Y$ are the second- and third-order operators of [14, p. 176].

Definition B.4. The **Cooper recurrence polynomials** are

$$A(v) = (2v-1)(13v^2-13v+4), \qquad B(v) = 3(v-1)\big(9(v-1)^2-1\big),$$

with derivatives A', B', so that the recurrence of Definition 7.1 reads $n^3 s_7(n) = A(n)s_7(n-1) + B(n)s_7(n-2)$ for $n \ge 1$. A sequence $(u_n)_{n\ge 0}$ satisfies the **deformed Cooper recurrence** if $u_0 = 0$, $u_1 = 9$ and

$$n^3 u_n - A(n)u_{n-1} - B(n)u_{n-2} = -3n^2 s_7(n) + A'(n)s_7(n-1) + B'(n)s_7(n-2) \qquad (n \ge 2).$$

For integers $0 \le k \le n$ put

$$T(n,k) = \frac{(n+k)!\,(2n-k)!}{k!^3\,(n-k)!^3}.$$

With the harmonic numbers $\mathfrak{h}_m$ of Definition B.2, the **harmonic Cooper sums** are, for $n \ge 0$,

$$e_n = \frac{3}{2}\sum_{k=0}^{n} T(n,k)\big(\mathfrak{h}_{n+k} + \mathfrak{h}_{2n-k} - \mathfrak{h}_k - \mathfrak{h}_{n-k}\big).$$

Lemma B.5. *Put*

$$\mathcal{T}(v,u)=\frac{\Gamma(v+u+1)\Gamma(2v-u+1)}{\Gamma(u+1)^3\Gamma(v-u+1)^3},\qquad Z(v,u)=\frac{P(v,u)\Gamma(v+u-1)\Gamma(2v-u-2)}{\Gamma(u)^3\Gamma(v-u+1)^3},$$

where

$$\begin{aligned}P(v,u)&=(42v^2-57v+20)u^3+(-175v^3+300v^2-165v+28)u^2\\&\quad+v(244v^3-488v^2+321v-70)u-v^2(v-1)(115v^2-144v+46).\end{aligned}$$

Then $\mathcal{T}(n,k)=T(n,k)$ for integers $0\le k\le n$, and, as meromorphic functions of $(v,u)\in\mathbb{C}^2$,

$$v^3\mathcal{T}(v,u)-A(v)\mathcal{T}(v-1,u)-B(v)\mathcal{T}(v-2,u)=Z(v,u+1)-Z(v,u).$$

Consequently, for every integer $n\ge 0$ and as meromorphic functions of (v,x),

$$\sum_{k=0}^{n}\Big(v^3\mathcal{T}(v,x+k)-A(v)\mathcal{T}(v-1,x+k)-B(v)\mathcal{T}(v-2,x+k)\Big)=Z(v,x+n+1)-Z(v,x).$$

Proof. For integers $0\le k\le n$ all four gamma arguments are positive integers, and $\Gamma(m+1)=m!$ gives $\mathcal{T}(n,k)=T(n,k)$.

For the certificate identity, set

$$D=(v+u)(v+u-1)(2v-u)(2v-u-1)(2v-u-2)(2v-u-3).$$

Apply $\Gamma(z+1)=z\Gamma(z)$ to write each of the five terms as a polynomial in v,u times $\beta=\Gamma(v+u-1)\Gamma(2v-u-3)/(\Gamma(u+1)^3\Gamma(v-u+1)^3)=\mathcal{T}(v,u)/D$, and divide by β. The identity becomes the polynomial equality

$$\begin{aligned}&(v-u)^3(v+u-1)P(v,u+1)-u^3(2v-u-3)P(v,u)\\&\quad=v^3D-A(v)(v-u)^3(v+u-1)(2v-u-2)(2v-u-3)\\&\qquad-B(v)\big((v-u)(v-u-1)\big)^3,\end{aligned}$$

which holds by expansion. Both sides of the original identity are meromorphic on $\mathbb{C}^2$ and agree on the open set where the division is legitimate, so they agree identically.

Finally, substitute $u=x+k$ for $k=0,\dots,n$ and sum; the right-hand side telescopes to $Z(v,x+n+1)-Z(v,x)$.

□

Lemma B.6. *The harmonic Cooper sums $(e_n)_{n\ge0}$ satisfy the deformed Cooper recurrence.*

Proof. The proof is by creative telescoping. We deform the Cooper summand by a continuous parameter r. The certificate of Lemma B.5 shows that the deformed sums satisfy the Cooper recurrence up to $O(r^2)$, and differentiating at $r=0$ produces the harmonic sums together with the stated inhomogeneity. We use the notation $\mathcal{T}$, Z of Lemma B.5.

The deformed sums. For $n\ge0$ define, analytically near $r=0$,

$$a_n(r)=\sum_{k=0}^{n}\mathcal{T}(n+r,k+r/2).$$

By Lemma B.5 and Lemma 7.10, $a_n(0)=\sum_k T(n,k)=\sum_k L_{n,k}L_{n,n-k}=s_7(n)$. Logarithmic differentiation of the gamma factors, using $\psi(m+1)=\mathfrak{h}_m-\gamma$, gives $a_n'(0)=e_n$; the Euler constants cancel. In particular $e_0=0$ and $e_1=9$.

Telescoping. For $n \geq 2$, apply the telescoped identity of Lemma B.5 with $v = n+r$ and $x = r/2$. The boundary terms are $O(r^2)$. Indeed, $Z(n+r, r/2) = O(r^3)$ because $1/\Gamma(r/2)^3 = O(r^3)$ and both numerator gamma factors are regular. At the upper boundary, $Z(n+r, n+1+r/2)$ also contains $1/\Gamma(r/2)^3$; its only possibly singular numerator factor is $\Gamma(n-3+3r/2)$, with at most a simple pole. Hence this boundary is $O(r^2)$, including $n = 2, 3$. The sum of $\mathcal{T}(n-1+r, k+r/2)$ differs from $a_{n-1}(r)$ only by its term $k = n$, and that of $\mathcal{T}(n-2+r, k+r/2)$ differs from $a_{n-2}(r)$ only by its terms $k = n-1, n$. Every such extra term is $O(r^2)$: the inverse cube of a gamma function at a nonpositive integer supplies a zero of order three, while at most one numerator gamma factor has a simple pole. Consequently

$$(n+r)^3 a_n(r) - A(n+r)a_{n-1}(r) - B(n+r)a_{n-2}(r) = O(r^2).$$

Differentiating at $r = 0$ and substituting $a_m(0) = s_7(m)$, $a'_m(0) = e_m$ gives the assertion.

□

Lemma B.7. *1.* $\mathcal{L}F = 0$.

2. If u, v are analytic solutions of $\mathcal{D}u = \mathcal{D}v = 0$ on a connected open subset of $\mathbb{C} \setminus \{0, -1, \frac{1}{27}\}$, then $\mathcal{L}(uv) = 0$ there.

3. $\mathcal{D}$ has a unique solution $\phi \in 1 + Y\mathbb{Q}[[Y]]$, and $\phi^2 = F$.

Proof. Item (1) is the recurrence of the Cooper numbers in operator form; (2) and (3) apply the classical formula for the square of a solution of a second-order equation.

Proof of (1). Extracting the coefficient of Y^n in $\mathcal{L}F$ gives

$$n^3 s_7(n) - (2n-1)(13n^2 - 13n + 4)s_7(n-1) - 3(n-1)(3n-4)(3n-2)s_7(n-2) = 0.$$

For $n \geq 2$ this is the defining recurrence shifted by one, and for $n = 1$ it is the recurrence at index 0, that is, $s_7(1) = 4s_7(0)$; the equation at $n = 0$ follows from the initial values.

Proof of (2). Put $G = (1+Y)(1-27Y)$. In ordinary derivative form, $\mathcal{D}u = 0$ is $u'' + au' + bu = 0$ with

$$a = \frac{1}{Y} + \frac{G'}{2G}, \qquad b = -\frac{2(1+3Y)}{YG}.$$

By [14, Lemma 6.1], p. 175, the square of a solution satisfies

$$g''' + 3ag'' + (a' + 4b + 2a^2)g' + (2b' + 4ab)g = 0,$$

and substitution of a, b identifies this operator with $\mathcal{L}/(Y^3 G)$; note that the coefficient of θ^3 in $\mathcal{L}$ is G. As $uv = \frac{1}{4}((u+v)^2 - (u-v)^2)$ and $\mathcal{L}$ is linear, $\mathcal{L}(uv) = 0$.

Proof of (3). The formula for the square used in (2) is a differential-algebraic identity, valid for formal solutions in $\mathbb{Q}[[Y]]$, with $a, b \in \mathbb{Q}((Y))$. At $Y = 0$ both $\mathcal{D}$ and $\mathcal{L}$ have indicial polynomial a power of θ, so their coefficient recurrences have leading coefficients n^2 and n^3, respectively. Hence each has a unique power-series solution with constant term 1; denote that of $\mathcal{D}$ by ϕ. Then $\phi^2 \in 1 + Y\mathbb{Q}[[Y]]$ solves $\mathcal{L}$, and by (1) and uniqueness $\phi^2 = F$.

□

Lemma B.8. *The coefficients $(d_n)_{n \geq 0}$ of the logarithmic Cooper companion satisfy the deformed Cooper recurrence.*

Proof. The series $F\log Y+\mathcal{H}$ is the product of the two Frobenius solutions of $\mathcal{D}$, hence a logarithmic solution of $\mathcal{L}$, and the recurrence is the coefficient form of this fact.

The logarithmic solution. Let ϕ be as in Lemma B.7(3), so $\phi^2=F$; all series below converge near $Y=0$. In ordinary derivative form the coefficient of ϕ' in $\mathcal{D}$ is $a=1/Y+G'/(2G)$, so $\exp(-\int a)=1/(Y\sqrt{G})$. Reduction of order gives the second solution

$$\phi_1=\phi\big(\log Y+\ell(Y)\big),\qquad\text{since}\qquad \frac{d}{dY}\big(\log Y+\ell(Y)\big)=\frac{\rho(Y)}{Y}=\frac{1}{Y\,\phi^2\sqrt{G}},$$

on a slit neighbourhood of 0. By Lemma B.7(2), $\phi\phi_1=F\log Y+\mathcal{H}$ solves $\mathcal{L}$.

The coefficient form. Write $\mathcal{L}=\theta^3-YP_1(\theta)-Y^2P_2(\theta)$ with $P_1(\theta)=(2\theta+1)(13\theta^2+13\theta+4)$ and $P_2(\theta)=3(\theta+1)(3\theta+2)(3\theta+4)$, so $P_1(n-1)=A(n)$ and $P_2(n-2)=B(n)$. Since $\theta^j(g\log Y)=(\theta^j g)\log Y+j\theta^{j-1}g$, one has $\mathcal{L}(F\log Y)=(\mathcal{L}F)\log Y+(\partial_\theta\mathcal{L})F$, where $\partial_\theta\mathcal{L}$ is the formal θ-derivative of $\mathcal{L}$. As $\mathcal{L}F=0$ (Lemma B.7(1)) and $\mathcal{L}(F\log Y+\mathcal{H})=0$,

$$\mathcal{L}\mathcal{H}=-(\partial_\theta\mathcal{L})F.$$

Taking the coefficient of Y^n, with $d_m=s_7(m)=0$ for $m<0$, gives the relation of the deformed Cooper recurrence for every $n\geq 0$, in particular for $n\geq 2$, since $\partial_\theta P_1(\theta)|_{\theta=n-1}=A'(n)$ and $\partial_\theta P_2(\theta)|_{\theta=n-2}=B'(n)$.

Initial values. $d_0=0$ because ℓ has zero constant term. Since $F=1+4Y+O(Y^2)$ and $\sqrt{G}=1-13Y+O(Y^2)$, $\rho=1+9Y+O(Y^2)$, so $[Y]\ell=9$ and $d_1=9$.

□

Lemma B.9. *For every integer $n\geq 0$, $d_n=e_n$.*

Proof. By Lemma B.6 and Lemma B.8, both (e_n) and (d_n) satisfy the deformed Cooper recurrence, so $e_0=d_0=0$ and $e_1=d_1=9$. Since $n^3\neq 0$, the relation of that recurrence determines its n-th term from the two preceding ones for $n\geq 2$. Induction on n gives $d_n=e_n$ for all $n\geq 0$.

□

Lemma B.10. *For every integer $n\geq 0$,*

$$c_7(2n+1)=[Y^n]F(Y)\exp\!\left(-\frac{2n+1}{2}\ell(Y)\right).$$

For every odd prime p this implies

$$c_7(p)-s_7\!\left(\frac{p-1}{2}\right)\equiv-\frac{p}{2}\,d_{(p-1)/2}\pmod{p^2}.$$

Proof. The exact formula. Cooper's identity $z=F(y)$ [14, Theorem 3.1], the identity $q\frac{d}{dq}\log y=zw$ of Lemma 7.6(4) and the conic identity $w^2=G(y)$ of Proposition 7.7 give

$$q\frac{d\log y(q)}{dq}=F(y(q))\sqrt{G(y(q))}.$$

Both square roots have constant term one. Let $q=\psi(Y)$ be the compositional inverse of $y(q)=q+O(q^2)$. Formal differentiation gives

$$Y\frac{d\log\psi(Y)}{dY}=\rho(Y),\qquad \psi(Y)=Y\exp(\ell(Y)).$$

Thus the inverse of $t = s(Q)$ is

$$Q(t) = t \exp\!\left(\frac{1}{2}\ell(t^2)\right).$$

For $m = 2n+1$, invariance of the formal residue under this unit-leading change of variable and the defining differential identity give

$$\begin{aligned} c_7(m) &= \operatorname{Res}_{Q=0} C(Q) Q^{-m} \frac{dQ}{Q} \\ &= \operatorname{Res}_{t=0} F(t^2) Q(t)^{-m}\, dt \\ &= [t^{m-1}] F(t^2) \exp\!\left(-\frac{m}{2}\ell(t^2)\right) \\ &= [Y^n] F(Y) \exp\!\left(-\frac{m}{2}\ell(Y)\right). \end{aligned}$$

This proves the exact formula.

The congruence. Fix an odd prime p and put $h = (p-1)/2$. The integrality of $s_7(n)$ is Lemma 7.10; $\sqrt{G}$ and its inverse are integral at every odd prime by recursive coefficient comparison. Hence $\rho \in \mathbb{Z}_p[[Y]]$. For $1 \le j \le h$, the coefficient $[Y^j]\ell = ([Y^j]\rho)/j$ is p-integral. Since $h < p$, every factorial appearing in the exponential through degree h is a p-adic unit. Therefore, in $\mathbb{Z}_p[[Y]]/(Y^{h+1})$,

$$\exp\!\left(-\frac{p}{2}\ell(Y)\right) \equiv 1 - \frac{p}{2}\ell(Y) \pmod{p^2}.$$

Taking the degree-h coefficient after multiplying by F proves the congruence.

□

Lemma B.11. *For every odd prime p, with $h = (p-1)/2$,*

$$s_7(h) \equiv (-1)^h \operatorname{Ap}(h) \pmod{p^2}.$$

Proof. Put $\epsilon = (-1)^h$. By Lemma 7.10, $s_7(h) = \sum_{k=0}^{h} L_{h,k} L_{h,h-k}$. For $0 \le k \le h$ the factorial denominators are prime to p, and $h \equiv -h-1 \pmod{p}$ gives

$$L_{h,k} \equiv (-1)^k \binom{h}{k}^2 \pmod{p}.$$

By symmetry of the binomial coefficients, $L_{h,h-k} \equiv \epsilon L_{h,k} \pmod{p}$. The following identity is exact over $\mathbb{Q}$:

$$s_7(h) - \epsilon \operatorname{Ap}(h) = -\frac{\epsilon}{2} \sum_{k=0}^{h} \bigl(L_{h,k} - \epsilon L_{h,h-k}\bigr)^2.$$

Every squared term lies in $p^2\mathbb{Z}$, and 2 is a p-adic unit.

□

Proposition B.12. *For every odd prime p,*

$$s_7\left(\frac{p-1}{2}\right) \equiv (-1)^{(p-1)/2} \operatorname{Ap}\left(\frac{p-1}{2}\right) \equiv A_p^{\eta} \pmod{p^2}.$$

Proof. Put $h = (p-1)/2$ and $\epsilon = (-1)^h$. The first congruence is Lemma B.11.

The second congruence. The Apéry congruence theorem of [1] states, for every odd prime p, that $\operatorname{Ap}((p-1)/2) \equiv a_8(p) \pmod{p^2}$, where $\sum_{m\ge1} a_8(m) q^m = \eta(2\tau)^4 \eta(4\tau)^4$. By the twist identity in Proposition 7.5, $A_p^{\eta} = \epsilon\, a_8(p)$. Multiplying the cited congruence by ϵ completes the proof.

□

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