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THE MAXIMAL RANK OF AN EVEN HYPERBOLIC LATTICE WITH NONZERO EXCEPTIONAL LATTICE

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Abstract. Let L be an even hyperbolic lattice. Its exceptional lattice $E(L)$ consists of the vectors with finite orbit under the stabilizer of a chamber of the (-2)-reflection group. We prove that the largest rank for which $E(L)$ can be nonzero is 26, answering a question of Grabbel, Martin, Mezzedimi, Raitz von Frentz and Schmidt. Conway's lattice $II_{1,25}$ attains the bound.

In rank $r \geq 27$, we construct in each chamber a finite set of deficient cusps f (meaning $\dim S_f < r-1$, where S_f is the rational span of f and the roots perpendicular to f) with $\bigcap_f S_f = 0$. We use these cusps to construct a Zariski-dense subgroup of the chamber group generated by at most $r-1$ Eichler transformations. If L contains a (-2)-root, this subgroup is thin and no chamber admits a Weyl vector. For the groups obtained by this cusp construction in rank at least three, we study the principal congruence subgroups defined by the action on L. At all sufficiently large levels, the expanding eigenvalue of every systolic element has degree two or four. In rank three the systole divided by the logarithm of the level has lower limit 2 and upper limit 4. Every complex projective K3 surface with Picard lattice $U \oplus \langle -4m\rangle^{\oplus 2}$ $(m \geq 3)$ or $\langle 2\rangle \oplus \langle -4\rangle \oplus D_8(-2)$ admits a symplectic automorphism of maximal Salem degree.

1. Introduction

Let L be an even hyperbolic lattice of signature $(1, r-1)$, with $r \geq 2$. Choose a component of the positive cone in $L_{\mathbb{R}}$ and denote its stabilizer in $\mathrm{O}(L)$ by $\mathrm{O}^+(L)$. The hyperplanes perpendicular to the vectors $\delta \in L$ of square -2 divide this component into open chambers for the reflection group

$$W_2(L) = \langle s_\delta \mid \delta \in L,\ \delta^2 = -2\rangle, \qquad s_\delta(x) = x + (x,\delta)\delta.$$

For a chamber D, its stabilizer $A(L) = \mathrm{Stab}_{\mathrm{O}^+(L)}(D)$ maps isomorphically to $\mathrm{O}^+(L)/W_2(L)$ [Vin72, Nik83]. We call $A(L)$ the *chamber group*. The stabilizers of any two chambers are conjugate under $W_2(L)$.

Following Nikulin [Nik14, §4], we define the *exceptional lattice* by

$$E(L) = \{x \in L \mid \#(A(L)x) < \infty\}.$$

This sublattice is primitive, since x and mx have the same stabilizer for every nonzero integer m. When $L = \mathrm{NS}(X)$ for a complex projective K3 surface X, we take D to be the ample cone. By the global Torelli theorem the image of $\mathrm{Aut}(X)$ in $A(L)$ has finite index [Nik83, Huy16]. Thus $E(\mathrm{NS}(X))$ consists of the divisor classes with finite $\mathrm{Aut}(X)$-orbit.

Nikulin proved that for each rank $r \geq 3$, only finitely many isometry classes of even hyperbolic lattices have $E(L) \neq 0$ [Nik14, Theorem 4.4]. Grabbel, Martin, Mezzedimi, Raitz von Frentz and Schmidt proved that $E(L) = 0$ for $r \geq 46$ [GMM$^+$26, Theorem 1.1]. They asked the following question [GMM$^+$26, §1.1].

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Question 1.1. *What is the largest rank of an even hyperbolic lattice L with $E(L) \neq 0$?*

Theorem A. *Let L be an even hyperbolic lattice of rank $r \geq 27$. Then $E(L) = 0$. The largest rank of an even hyperbolic lattice with $E(L) \neq 0$ is 26.*

The example in rank 26 is due to Conway. Write $II_{1,25} = U \oplus \Lambda(-1)$, where U is the hyperbolic plane and Λ is the positive-definite Leech lattice. Conway showed that its chamber group is $\Lambda \rtimes \mathrm{Aut}(\Lambda)$ and fixes a primitive isotropic Weyl vector e [Con83, Bor87]. Every finite-index subgroup of the translation lattice Λ has common fixed space $\mathbb{Q}e$. Hence $E(II_{1,25}) = \mathbb{Z}e$.

The study of these chamber groups goes back to Vinberg, Conway, Borcherds and Nikulin [Vin72, Con83, Bor87, Bor92, Nik80, Nik83]. In rank at least 3, Brandhorst and Mezzedimi classified the 360 even hyperbolic lattices with virtually abelian chamber group. This list consists of the 118 finite cases of Nikulin and Vinberg, 194 cases of parabolic type and 48 of hyperbolic type. The rank is at most 26 in this classification and at most 19 in the finite case [BM26, Theorem 1.1].

Exceptional classes also occur when the chamber group is not virtually abelian. For Vinberg's complex K3 surface X_4 of Picard rank 20 and discriminant 4, the chamber group contains a free product of five groups of order two as a subgroup of finite index [Vin83, Theorem 2.4]. The non-symplectic involution acting as $+\mathrm{id}$ on $\mathrm{NS}(X_4)$ and $-\mathrm{id}$ on its transcendental lattice is central in $\mathrm{Aut}(X_4)$, and its fixed locus consists of ten disjoint smooth rational curves [Sie18, Proposition 5.1 and Theorem 5.4]. The automorphism group therefore permutes the ten curves. Its image has finite index in the chamber group, so their classes lie in $E(\mathrm{NS}(X_4))$. Their intersection matrix is $-2I_{10}$, so they span a negative-definite sublattice of rank 10 in $E(\mathrm{NS}(X_4))$.

To prove Theorem A, we construct cusps in the closure $\overline{D}$ of a single chamber D in $L_{\mathbb{R}}$. We identify a cusp, a rational isotropic ray in $\overline{D}$, with its primitive integral generator f and put

$$S_f = \mathrm{span}_{\mathbb{Q}}\big(\{f\} \cup \{\delta \in L \mid \delta^2 = -2,\ (f,\delta) = 0\}\big).$$

We call f *deficient* if the images of its perpendicular roots span a proper subspace of $f^{\perp}/\mathbb{Q}f$, or equivalently if $\dim S_f < r - 1$. The difference $(r-1) - \dim S_f$ is the *root-rank deficit*. A cusp is *rootless* if no root is perpendicular to it. At a deficient cusp the Eichler transformations with directions in $S_f^{\perp} \cap L$ preserve D. Their common fixed space is S_f, and every vector with finite $A(L)$-orbit lies in S_f (Lemma 2.3). It therefore suffices to find deficient cusps whose spaces S_f have zero intersection. For a positive-definite lattice P, write P_{rt} for the sublattice generated by its vectors of square 2.

Theorem B. *Let L be an even hyperbolic lattice of rank $r \geq 27$. Every chamber D of $W_2(L)$ admits a finite set F of deficient cusps such that*

$$\bigcap_{f \in F} S_f = 0.$$

More precisely, choose a maximal even overlattice $L' \cong H \oplus \Lambda(-1)$. If $H_{\mathbb{Q}}$ is anisotropic, denote by t the least positive square represented by H. In a chamber of L' one can take the following cusp rays.

(i) *Suppose $H_{\mathbb{Q}}$ is isotropic. Write $L' \cong U \oplus E_8(-1)^n \oplus M(-1)$ as in Section 3, with $n \geq 3$, and put $Q = E_8^{n-3} \oplus M$. Let e_0 be the standard cusp of U and $e = e_{123}$ the Leech cusp associated with the first three E_8 factors. If $Q_{\mathrm{rt}} = 0$, take e and e_0. If $Q_{\mathrm{rt}} \neq 0$: in rank 27 the cusps e and f_2, f_3 of denominators 2 and 3 suffice; for $r > 27$, take e, f_2, f_3 together with the cusps e_{12j} for $4 \leq j \leq n$ and $\mathrm{rk}\, M_{\mathrm{rt}} \leq 10$ cusps given by the weights in Table 1.*

(ii) *If $H_{\mathbb{Q}}$ is anisotropic and $t \geq 4$, take the two rootless cusps $(k, \pm u)$ in $H \oplus \Lambda(-1)$, where $k \in H$, $u \in \Lambda$ and $k^2 = u^2 = t$.*

(iii) *If $H_{\mathbb{Q}}$ is anisotropic and $t = 2$, take the 48 coordinate cusps $f_i^{\pm} = (2k, \pm u_i)$ in $H \oplus \Lambda(-1)$, where $k^2 = 2$. They form one orbit under the monomial subgroup $2^{12}{:}\mathrm{M}_{24} \subset \mathrm{Aut}(\Lambda)$. The $A(L')$-orbit containing them is infinite. In L' the quotient root system at each of these cusps is $23A_1$, with root-rank deficit $\operatorname{rk} H - 1$.*

The primitive generators in L of the chosen rays give F after conjugation by an element of $W_2(L)$.

From the cusps in Theorem B we choose at most $r - 1$ Eichler transformations whose common fixed space is zero. The group they generate acts irreducibly on $L_{\mathbb{R}}$ (Proposition 7.1). Recall that a subgroup of $\mathrm{SO}(L)$ is *thin* if it is finitely generated, Zariski dense in $\mathrm{SO}(L_{\mathbb{R}})$ and of infinite index in $\mathrm{SO}(L)$.

Corollary C. *Let L be an even hyperbolic lattice of rank $r \geq 27$. Then $A(L) \cap \mathrm{SO}^+(L)$ contains a Zariski-dense subgroup Γ generated by at most $r - 1$ integral Eichler transformations. If L contains a (-2)-root, then Γ is a thin subgroup of $\mathrm{SO}(L)$, and no chamber D of L admits a Weyl vector $\rho \in L_{\mathbb{R}}$ satisfying $(\rho, \delta) = 1$ for all inward simple roots δ of D.*

Barnard obtained the bound 26 for well-endowed Lorentzian lattices whose full reflection group admits a generalized Weyl vector, assuming the spectral conjecture in [Bar03, Conjecture 3.4.1 and Theorem 4.1.4]. Corollary C concerns the chambers cut out by roots of square -2.

The local criterion for splitting off the Leech lattice gives a necessary condition for $E(L) \neq 0$ in ranks 25 and 26 (Section 5).

Theorem D.

(i) *For every integer $a \geq 1$ and every positive-definite even unimodular lattice P of rank 24, the rank-25 lattice $L = \langle 2a \rangle \oplus P(-1)$ satisfies $E(L) = 0$.*

(ii) *Let L be a maximal even hyperbolic lattice of rank 25 or 26 with $E(L) \neq 0$. Then either $\operatorname{rk} L = 26$ and $L \cong II_{1,25}$, or the p-adic Witt index satisfies $i_W(L_{\mathbb{Q}_p}) = 11$ at some finite prime p. In particular, $II_{1,25}$ is the unique rank-26 maximal even lattice containing $\Lambda(-1)$ with $E(L) \neq 0$.*

In small rank, coordinate vectors in $D_n(2)$ and cosets modulo 2 give analogous cusp constructions. We apply them to Picard lattices of complex projective K3 surfaces in Section 6.

Theorem E.

(i) *For every integer $m \geq 3$, let $L_m = U \oplus \langle -4m \rangle^{\oplus 2}$. Every complex projective K3 surface X_m with $\mathrm{NS}(X_m) \cong L_m$ admits a symplectic automorphism g_m with no periodic curves. Its action on L_m is a product of two Eichler transformations at rootless cusps and has characteristic polynomial*

$$P_m(X) = X^4 - (16m^2 + 4)X^3 + (6 - 32m^2)X^2 - (16m^2 + 4)X + 1.$$

This is an irreducible quartic Salem polynomial. The lattices L_m are pairwise non-isometric. For each m, a very general member of the 16-dimensional L_m-polarized period space has Picard lattice L_m and admits such an automorphism.

(ii) *Every complex projective K3 surface X with $\mathrm{NS}(X) \cong \langle 2 \rangle \oplus \langle -4 \rangle \oplus D_8(-2)$ admits a symplectic automorphism of Salem degree 10 with no periodic curves. We give its action on $\mathrm{NS}(X)$ explicitly.*

We also give a quantitative form of Yu's criterion for maximal Salem degree [Yu18, Theorem 6.2] (see also [GMM$^+$26, Theorem 1.3]). Suppose $r \geq 3$ and there is a finite set of deficient cusps in the closure of one chamber with $\bigcap_f S_f = 0$. Let $\Gamma \subseteq A(L) \cap \mathrm{SO}^+(L)$ be the group generated by the Eichler transformations of Proposition 7.1, and let $\Gamma(N)$ be the kernel of its action on L/NL. Put $n = 2\lfloor r/2 \rfloor$. Every $\Gamma(N)$ contains elements whose expanding eigenvalue has degree n. The minimum h_N of their logarithmic spectral radii satisfies

$$n \log N - (n-2)\log 2 \;\leq\; h_N \;\leq\; 2B_r \log N + O_\Gamma(1) \qquad (N \geq 2),$$

where B_r depends only on r (Theorem 7.7). For the lower bound, if $\lambda > 1$ is an eigenvalue of $g \in \Gamma(N)$, then $(\lambda - 1)/N$ is an algebraic integer. Taking its norm gives $N^{\deg P} \mid P(1)$ for the minimal polynomial P of λ. For the rank-four examples in Theorem E(i), we obtain $h_N = 4 \log N + O_m(1)$ (Proposition 7.9).

In rank three every loxodromic element has degree two. Thus h_N equals the ordinary systole ℓ_N (the length of a shortest closed geodesic on $\mathbb{H}^2/\Gamma(N)$). For the congruence action on L/NL we prove

$$\liminf_{N\to\infty} \frac{h_N}{\log N} = 2, \qquad \limsup_{N\to\infty} \frac{h_N}{\log N} = 4.$$

Both limits persist on levels coprime to $2|\det L|$, and $h_N = 4 \log N + O_\Gamma(1)$ along odd prime powers (Proposition 7.10). In every rank $r \geq 3$, products of two Eichler transformations at distinct cusps give the upper bound $\ell_N \leq 4 \log N + O_\Gamma(1)$ for the ordinary systole ℓ_N of $\mathbb{H}^{r-1}/\Gamma(N)$. Calling a loxodromic element of $\Gamma(N)$ attaining ℓ_N a *systolic element*, it follows that at all sufficiently large levels the expanding eigenvalue of every systolic element has degree two or four. In particular, for $r \geq 6$ a systolic element cannot have maximal Salem degree (Corollary 7.11).

1.1. **Outline of the proof of the rank bound.** The first step is to split off a copy of $\Lambda(-1)$ from a maximal even overlattice L' of L. Grabbel, Martin, Mezzedimi, Raitz von Frentz and Schmidt obtain a decomposition $L' = U \oplus E_8(-1)^n \oplus M(-1)$ with $\operatorname{rk} M \leq 11$ [GMM$^+$26, Proposition 2.8]. Rank 30 guarantees three E_8 factors. Their proof in rank at least 46 uses five such factors, three to obtain $U \oplus \Lambda(-1)$ and two to treat the remaining summand [GMM$^+$26, §4].

We prove that $\Lambda(-1)$ splits off from L' if and only if the Witt index of $L'_{\mathbb{Q}_p}$ is at least 12 at every finite prime p (Proposition 2.1). The anisotropic kernel over $\mathbb{Q}_p$ has dimension at most 4, and at most 3 in odd dimension. This gives the required Witt index in every rank $r \geq 27$. In ranks 25 and 26, Witt index 11 gives an anisotropic kernel of dimension 3 and 4, respectively. This accounts for the local condition in Theorem D.

Write $L' \cong H \oplus \Lambda(-1)$. The cusps we use depend on whether $H_\mathbb{Q}$ is isotropic (Figure 1). In the isotropic case, maximality gives a summand U in H. The isometry $U \oplus \Lambda(-1) \cong U \oplus E_8(-1)^3$ then gives the decomposition of Theorem B(i), with $\operatorname{rk} M \leq 10$. At the first Leech cusp we have $S_e = \mathbb{Q}e \oplus (Q_{\mathrm{rt}})_\mathbb{Q}$. If $Q_{\mathrm{rt}} = 0$, then $S_e \cap S_{e_0} = 0$. Otherwise we use two cusps of denominators 2 and 3 with the same highest-root component $-\theta$ to eliminate the coefficient of e. For the latter cusp we use a coset modulo 3Λ whose nine shortest vectors have square 16 and span a space of dimension 8 [Mar02] (Lemmas 3.5 and 3.6). In rank 27, the only possibility with $Q_{\mathrm{rt}} \neq 0$ is $Q = A_1$. The same two pairings then eliminate the entire space S_e. In higher rank, further Leech cusps eliminate the remaining E_8 factors. For each irreducible component R of M_{rt}, we choose $\operatorname{rk} R$ weights spanning $R_\mathbb{Q}$. Table 1 gives these weights for all 15 possible types.

If $H_\mathbb{Q}$ is anisotropic, its rank is 3 or 4. Choose $k \in H$ of least positive square t. If $t \geq 4$, the Leech lattice represents t, and the two cusps $(k, \pm u)$ with $u^2 = t$

are rootless and lie in the closure of one chamber (Lemmas 4.2 and 4.1). If $t = 2$, the coordinate vectors $\pm u_i$ of square 8 in the Golay model of Λ give 48 cusps in the chamber closure containing $(k, 0)$. The perpendicular quotient root systems are $23A_1$, and the spaces $S_{f_i^\pm}$ have zero intersection (Lemma 4.3). Finally, Lemma 2.4 passes the cusps from L' to L.

$$L\ (\operatorname{rk} L \geq 27) \xrightarrow{\text{Lemma 2.4}} \text{maximal even } L' \supseteq L \xrightarrow{\text{Prop. 2.1, Cor. 2.2}} L' \cong H \oplus \Lambda(-1)$$

$$\begin{cases} H_{\mathbb{Q}} \textbf{ isotropic } (\S3): & L' \cong U \oplus E_8(-1)^n \oplus M(-1),\ n \geq 3 \\ & Q = E_8^{n-3} \oplus M, \quad e = e_{123} \\ & Q_{\mathrm{rt}} = 0: \quad e,\ e_0 \\ & Q_{\mathrm{rt}} \neq 0: \quad e,\ f_2,\ f_3 \\ & \quad \text{if } r > 27, \text{ also } e_{12j}\ (4 \leq j \leq n) \\ & \quad \text{and } \operatorname{rk} M_{\mathrm{rt}} \text{ weight cusps from Table 1} \\ H_{\mathbb{Q}} \textbf{ anisotropic } (\S4): & t = \min\{k^2 > 0 : k \in H\} \\ & \hookrightarrow t \geq 4: \text{ two rootless cusps } (k, \pm u) \text{ (Lemma 4.1)} \\ & \hookrightarrow t = 2: \text{ 48 coordinate cusps, roots } 23A_1 \text{ (Lemma 4.3)} \end{cases}$$

FIGURE 1. Choice of cusps in a maximal even overlattice L' in Theorem B.

Sections 2–4 prove Theorems A and B. Section 5 treats the boundary ranks, minima of lattice cosets and reflections of bounded length. Section 6 gives the applications to K3 surfaces. Section 7 studies the chamber groups and their principal congruence subgroups. Appendix A gives the finite computations used in the proofs.

1.2. **Open questions.**

(1) For which ranks $21 \leq r \leq 26$ is there an even hyperbolic lattice L with $E(L) \neq 0$ and $A(L)$ not virtually abelian? Vinberg's surface X_4 gives such an example in rank 20.
(2) Which maximal even hyperbolic lattices of ranks 25 and 26 have $E(L) \neq 0$? Even without maximality, does any rank-25 lattice have $E(L) \neq 0$? For a maximal example in rank 25, Theorem D(ii) requires $i_W(L_{\mathbb{Q}_p}) = 11$ at some prime p.
(3) For the groups Γ of Theorem 7.7 in rank $r \geq 4$, what are the lower and upper limits of $h_N / \log N$? How do they depend on L and Γ? Both limits equal 4 in the rank-four examples of Proposition 7.9.

1.3. **Notation.** Throughout, a lattice is a free $\mathbb{Z}$-module of finite rank with a nondegenerate integral symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$. We assume that $x^2 = (x, x) \in 2\mathbb{Z}$ for all lattice vectors x.

- For a commutative ring F, write $L_F = L \otimes_{\mathbb{Z}} F$. For a prime p, write $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $L_{\mathbb{Q}_p} = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$, and let $i_W(L_{\mathbb{Q}_p})$ denote the Witt index of $L_{\mathbb{Q}_p}$ (the dimension of a maximal isotropic subspace).
- We write $L \cong L'$ for an integral isometry of lattices. For $m \in \mathbb{Z} \smallsetminus \{0\}$, $L(m)$ denotes the lattice with the same underlying $\mathbb{Z}$-module and form multiplied by m. The dual lattice is $L^\vee = \{x \in L_{\mathbb{Q}} : (x, L) \subseteq \mathbb{Z}\}$, and $A_L = L^\vee / L$ is the discriminant group, with exponent $\exp A_L$. We denote its minimum number of generators by $\ell(A_L)$.
- An even lattice L is *maximal even* if it admits no proper even integral overlattice in $L_{\mathbb{Q}}$.
- We follow the positive-definite convention for the root lattices A_m, D_m, E_m and the Leech lattice Λ. For any positive-definite even lattice P, $P_{\mathrm{rt}} = \langle r \in P : r^2 = 2 \rangle$ denotes its root sublattice, and $\min P = \min\{v^2 : v \in P \smallsetminus \{0\}\}$ its minimum nonzero squared norm.

- U denotes the integral hyperbolic plane with basis e_0, f_0 satisfying $e_0^2 = f_0^2 = 0$ and $(e_0, f_0) = 1$. In an orthogonal sum $U \oplus N(-1)$ with N positive definite, we write vectors as $(a, b, z) = ae_0 + bf_0 + z$ $(a, b \in \mathbb{Z},\ z \in N)$, so that $(a, b, z)^2 = 2ab - z^2$.
- For a hyperbolic lattice L of signature $(1, r-1)$, $\mathrm{O}^+(L)$ denotes the subgroup of $\mathrm{O}(L)$ preserving a chosen component of the positive cone $\{x \in L_{\mathbb{R}} : x^2 > 0\}$, and $\mathrm{SO}^+(L) = \mathrm{O}^+(L) \cap \mathrm{SO}(L)$.

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2. A Leech summand and the spaces attached to cusps

We first give a criterion for a given even unimodular lattice to occur as an orthogonal summand. We apply it to the Leech lattice.

Proposition 2.1. *Let P be a positive-definite even unimodular lattice of rank $d > 0$, and let L be a maximal even lattice of signature $(1, r-1)$ with $r \geq d+1$. There is an integral orthogonal decomposition*

$$L \cong H \oplus P(-1)$$

if and only if $i_W(L_{\mathbb{Q}_p}) \geq d/2$ for every finite prime p. In this case H is maximal even of signature $(1, r-d-1)$. The condition holds for every such L of rank $r \geq d+3$.

Proof. Since P is positive definite, even and unimodular, its rank d is divisible by 8. We claim that $P(-1)_p \cong U_p^{d/2}$ for every finite prime p. For odd p this follows from the classification of unimodular $\mathbb{Z}_p$-lattices by rank and determinant square class. At $p = 2$, the two even unimodular lattices in rank $d \in 8\mathbb{Z}$ have determinant square classes 1 and -3 in $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$. The split lattice has determinant 1, as does $P(-1)$ (and P). Thus $P(-1)_p \cong P_p \cong U_p^{d/2}$, which proves the claim and the necessity of the Witt index condition.

Conversely, suppose that $i_W(L_{\mathbb{Q}_p}) \geq d/2$ at every finite prime p. Put $q = r - d \geq 1$. We first split off $P(-1)_{\mathbb{Q}}$ from $L_{\mathbb{Q}}$. The space $L_{\mathbb{Q}} \oplus P_{\mathbb{Q}}$ has signature $(d+1, r-1)$. By Meyer's theorem we can split off $d-1$ hyperbolic planes in succession. Before the last of these splittings the signature is $(3, q+1)$ and the dimension is still at least 5. The remaining space W has signature $(2, q)$. Since $P_{\mathbb{Q}_p}$ is split, Witt cancellation gives

$$i_W(W_{\mathbb{Q}_p}) = i_W(L_{\mathbb{Q}_p}) + i_W(P_{\mathbb{Q}_p}) - (d-1) \geq 1$$

for every finite prime p. The real space $W_{\mathbb{R}}$ is also isotropic. The Hasse–Minkowski theorem gives a further rational hyperbolic plane in W [SP20, Theorems 5.31 and 6.1]. Thus $L_{\mathbb{Q}} \oplus P_{\mathbb{Q}} \cong U_{\mathbb{Q}}^d \oplus V$ for a space V of signature $(1, q-1)$. Since $U_{\mathbb{Q}}^d \cong P_{\mathbb{Q}} \oplus P(-1)_{\mathbb{Q}}$, Witt cancellation gives

$$L_{\mathbb{Q}} \cong V \oplus P(-1)_{\mathbb{Q}}.$$

We next split off $P(-1)_p$ integrally. The lattice L_p is maximal even over $\mathbb{Z}_p$. Indeed, a proper even overlattice $\widetilde{L}_p$ would give a proper even overlattice of L by taking the vectors of $L_{\mathbb{Q}}$ that lie in $\widetilde{L}_p$ at p and in L_ℓ at every prime $\ell \neq p$. For any primitive isotropic vector $e \in L_p$ we have $(e, L_p) = \mathbb{Z}_p$, since otherwise $L_p + \mathbb{Z}_p(e/p)$ would be a proper even overlattice. Choose $f \in L_p$ with $(e, f) = 1$. The vectors e and $f - (f^2/2)e$ span a unimodular hyperbolic plane. This also holds at $p = 2$

because L_2 is even. Its orthogonal complement is again maximal even. Repeating the argument $d/2$ times gives

$$L_p \cong H'_p \oplus U_p^{d/2},$$

with H'_p maximal even. By Witt cancellation we may identify H'_p with a lattice in $V_{\mathbb{Q}_p}$.

Choose a lattice $H^{(0)} \subset V$. Since L_p is unimodular for all $p \nmid \det L$, at every odd prime $p \nmid \det(L)\det(H^{(0)})$ both $H_p^{(0)}$ and H'_p are unimodular and have the determinant square class of $V_{\mathbb{Q}_p}$. They are therefore isometric, and we can choose the identifications with $V_{\mathbb{Q}_p}$ so that $H'_p = H_p^{(0)}$ at these primes. Patching the local lattices gives a lattice $H \subset V$ with $H_p \cong H'_p$ at every finite prime p [SP20, Lemma 2.11(b), Remark 2.12 and Theorem 2.19]. The local maximality of H_p implies that H is maximal even.

The lattices L and $H \oplus P(-1)$ now lie in the same genus. In particular their discriminant groups are isomorphic, so $\ell(A_L) = \ell(A_H) \le q$. Thus

$$\operatorname{rk} L - \ell(A_L) \ge r - q = d \ge 8.$$

The uniqueness theorem for indefinite even lattices in a genus gives $L \cong H \oplus P(-1)$ [Nik80, Theorem 1.14.2].

Finally, the anisotropic kernel over $\mathbb{Q}_p$ has dimension at most 4, and at most 3 in odd dimension. For odd $r \ge d+3$ this gives $i_W(L_{\mathbb{Q}_p}) \ge (r-3)/2 \ge d/2$. For even $r \ge d+3$ we have $r \ge d+4$, so $i_W(L_{\mathbb{Q}_p}) \ge (r-4)/2 \ge d/2$. □

Corollary 2.2. *Let L be a maximal even hyperbolic lattice of signature $(1, r-1)$ with $r \ge 27$. Then*

$$L \cong H \oplus \Lambda(-1)$$

for a maximal even lattice H of signature $(1, r-25)$.

Proof. Apply Proposition 2.1 with the even unimodular Leech lattice $P = \Lambda$ of rank $d = 24$. □

Fix a chamber D for $W_2(L)$ in the chosen component of the positive cone. For a cusp $f \in L \cap \overline{D}$ recall that

$$S_f = \operatorname{span}_{\mathbb{Q}}\big(\{f\} \cup \{\delta \in L \mid \delta^2 = -2,\ (f,\delta) = 0\}\big),$$

and call f *deficient* if the images of its perpendicular roots span a proper subspace of $f^\perp/\mathbb{Q}f$. Equivalently, $\dim S_f < \operatorname{rk} L - 1$. Both S_f and this inequality depend only on the ray of f. The following form of [Nik14, Theorem 4.2] and [GMM$^+$26, Proposition 3.3] identifies the Eichler transformations we use in Section 7.

Lemma 2.3. *Let $f \in L \cap \overline{D}$ be a deficient cusp. Then $E(L)_{\mathbb{Q}} \subseteq S_f$. More precisely, for every $z \in S_f^\perp \cap L$, the Eichler transformation*

$$T_{f,z}(x) = x + (x,f)z - (x,z)f - \tfrac{1}{2}z^2(x,f)f \tag{2.1}$$

belongs to $A(L)$, and the common fixed space of $\{T_{f,z} : z \in S_f^\perp \cap L\}$ in $L_{\mathbb{Q}}$ is S_f.

Proof. Formula (2.1) preserves the bilinear form and gives $T_{f,z}^n = T_{f,nz}$ for every $n \in \mathbb{Z}$, since $f^2 = (f,z) = 0$. The transformation is integral because $z^2 \in 2\mathbb{Z}$. The path $t \mapsto T_{f,tz}$, $0 \le t \le 1$, shows that $T_{f,z}$ preserves each component of the positive cone. It fixes every root perpendicular to f, since $z \in S_f^\perp$. For every other root δ, the sign of (δ, f) determines the sign of δ in D. This pairing is invariant under $T_{f,z}$, so $T_{f,z} \in A(L)$.

Since f is deficient, $S_f^\perp/\mathbb{Q}f$ is a nonzero negative-definite rational space, so we may choose $z \in S_f^\perp \cap L$ with $z^2 < 0$. If $x \in E(L)$, some positive power $T_{f,z}^n = T_{f,nz}$ $(n \ge 1)$ fixes x. Reducing $T_{f,nz}(x) = x$ modulo $\mathbb{Q}f$ gives $n(x,f)z \in \mathbb{Q}f$, whence

$(x, f) = 0$. For every $z' \in S_f^\perp \cap L$, the $T_{f,z'}$-orbit of x is then $\{x - n(x, z')f \mid n \in \mathbb{Z}\}$, whose finiteness forces $(x, z') = 0$. Since $S_f^\perp$ is spanned over $\mathbb{Q}$ by $S_f^\perp \cap L$, we obtain $x \in (S_f^\perp)^\perp = S_f$. The same calculation shows that every common fixed vector lies in S_f, and (2.1) fixes S_f pointwise. □

By Lemma 2.3, a finite set of deficient cusps in one chamber with $\bigcap_f S_f = 0$ suffices to prove $E(L) = 0$. The following descent argument is a version of [GMM$^+$26, Lemma 3.1] for such sets of cusps.

Lemma 2.4. *Every even hyperbolic lattice L has a maximal even overlattice in $L_\mathbb{Q}$. Let $L' \supseteq L$ be an even overlattice and D' a chamber of L'. Suppose $F' \subseteq L' \cap \overline{D'}$ is a finite set of deficient cusps with $\bigcap_{f' \in F'} S_{f'}(L') = 0$. Let F consist of the primitive generators in L of their rays. Then D' lies in a chamber D of L, each member of F is a deficient cusp of D, and $\bigcap_{f \in F} S_f(L) = 0$. In particular, $E(L) = 0$.*

Proof. Every even overlattice M of L satisfies $L \subseteq M \subseteq M^\vee \subseteq L^\vee$. The group $L^\vee/L$ is finite, so a maximal even overlattice L' exists. Every root of L is a root of L'. Thus D' lies in a unique chamber D of L. Given $f' \in F'$, the primitive generator f of its ray in L belongs to $\overline{D}$. Every root of L perpendicular to f is perpendicular to f' in L', so $S_f(L) \subseteq S_{f'}(L')$. The lattices have the same rank. It follows that each f is deficient and that the intersection of the $S_f(L)$ is zero. Now apply Lemma 2.3. □

3. Isotropic complements

Suppose the complement H in Corollary 2.2 is isotropic over $\mathbb{Q}$. Choose a primitive isotropic vector $e \in H$. Its divisibility $d = \operatorname{div}_H(e)$ is one, since otherwise $H + \mathbb{Z}(e/d)$ would be a proper even overlattice of H. Choose $g \in H$ with $(e, g) = 1$. Then e and $g - (g^2/2)e$ span a unimodular hyperbolic plane $U \subseteq H$, which splits off as an orthogonal summand. Choose $s \geq 0$ maximal among the decompositions

$$H \cong U \oplus E_8(-1)^s \oplus M(-1)$$

with M positive definite. We first show that $\operatorname{rk} M \leq 10$. The summand $U \oplus M(-1)$ is maximal even, since any proper even overlattice would give one of H. If $\operatorname{rk} M \geq 11$, Proposition 2.1 splits $E_8(-1)$ from $U \oplus M(-1)$ with a maximal even hyperbolic complement of rank $\operatorname{rk} M - 6 \geq 5$. By Meyer's theorem this complement is rationally isotropic, so the divisibility argument above splits off U. This gives a decomposition of H with $s + 1$ copies of $E_8(-1)$, contradicting maximality.

An E_8 component of M_{rt} would split off by unimodularity. If an irreducible component $R \subseteq M_{\mathrm{rt}}$ had rank at least 9, then $U \oplus R(-1)$ would contain $U \oplus E_8(-1)$ by [GMM$^+$26, Proposition 2.1]. This unimodular sublattice would split off from $U \oplus M(-1)$ even if R were imprimitive in M. Maximality of s thus restricts the irreducible components of M_{rt} to the 15 types

$$A_1, \ldots, A_8, \qquad D_4, \ldots, D_8, \qquad E_6, E_7. \tag{3.1}$$

The isometry $U \oplus \Lambda(-1) \cong U \oplus E_8(-1)^3$ now identifies L with

$$L_n = U \oplus E_8(-1)^n \oplus M(-1), \qquad n = s + 3 \geq 3. \tag{3.2}$$

3.1. **Standard chamber and split Leech cusps.** Fix an orthogonal decomposition $U \oplus N(-1)$ with N positive definite. In the coordinates (a, b, z) of Section 1.3, put $e_0 = (1, 0, 0)$. The *height* of a root $\delta = (A, B, u)$ at e_0 is $B = (e_0, \delta)$.

Lemma 3.1. *In $U \oplus N(-1)$, fix a system of simple roots α_i in each irreducible component R of N_{rt}, and let θ_R denote the corresponding highest root. Declare a (-2)-root $\delta = (A, B, u)$ to be positive if either $B > 0$, or $B = 0$ and $A > 0$, or $B = A = 0$ and u is a positive root of N_{rt}. The following properties hold.*

(i) *These signs define a chamber D of $U \oplus N(-1)$ whose closure contains e_0.*
(ii) *A vector $x = (a, b, -w) \in U_{\mathbb{R}} \oplus N_{\mathbb{R}}$ with $b > 0$ satisfies the height-zero chamber inequalities $((a, b, -w), \delta) \geq 0$ ($B = 0$) if and only if w lies in the scaled fundamental alcove*

$$(w, \alpha_i) \geq 0, \qquad (w, \theta_R) \leq b \tag{3.3}$$

for every irreducible component R of N_{rt}.
(iii) *At the cusp e_0 we have $S_{e_0} = \mathbb{Q}e_0 \oplus (N_{\mathrm{rt}})_{\mathbb{Q}}$.*

Proof. The roots of height zero are the vectors $(A, 0, u)$ with $u^2 = 2$. In each component R, the corresponding affine simple roots are $(0, 0, \alpha_i)$ and $(1, 0, -\theta_R)$. Their pairings with $(a, b, -w)$ give exactly (3.3). Choose $w \in N_{\mathbb{R}}$ satisfying these inequalities strictly for $b = 1$. For a root $\delta = (A, B, u)$ with $B > 0$, substituting $A = (u^2 - 2)/(2B)$ gives

$$((a, 1, -w), (A, B, u)) = \frac{\|u + Bw\|^2}{2B} + B\big(a - \tfrac{1}{2}w^2\big) - \frac{1}{B} > 0$$

whenever $a > \frac{1}{2}w^2 + 1$. Thus all the designated positive roots are positive on $(a, 1, -w)$ and define a chamber D. Since their pairings with e_0 are nonnegative, $e_0 \in \overline{D}$. This proves (i) and (ii). The description of the roots of height zero also gives (iii). □

Each triple of $E_8(-1)$ summands in (3.2) gives a cusp of the same chamber D.

Lemma 3.2. *Let D be the standard chamber of $L_n = U \oplus E_8(-1)^n \oplus M(-1)$ from Lemma 3.1. For every three-element subset $I \subset \{1, \dots, n\}$, the sublattice $U \oplus \bigoplus_{i \in I} E_8^{(i)}(-1) \cong U \oplus \Lambda(-1)$ contains a primitive isotropic vector $e_I \in \overline{D}$ whose perpendicular quotient inside $U \oplus \bigoplus_{i \in I} E_8^{(i)}(-1)$ is the rootless lattice $\Lambda(-1)$. Each e_I is a deficient cusp of D with root-rank deficit $24 + \operatorname{rk} M - \operatorname{rk} M_{\mathrm{rt}} > 0$ and*

$$S_{e_I} = \mathbb{Q}e_I \oplus \bigoplus_{j \notin I} (E_8^{(j)})_{\mathbb{Q}} \oplus (M_{\mathrm{rt}})_{\mathbb{Q}}. \tag{3.4}$$

Proof. Fix a triple I and write $L_I = U \oplus \bigoplus_{i \in I} E_8^{(i)}(-1)$ and $L_I^{\perp} = \bigoplus_{j \notin I} E_8^{(j)}(-1) \oplus M(-1)$. The isometry $L_I \cong U \oplus \Lambda(-1)$ gives a primitive future isotropic vector $y \in L_I$ with perpendicular quotient $\Lambda(-1)$, as in the proof of [GMM$^+$26, Lemma 4.2]. Let D_I be the standard chamber of L_I at e_0 for the chosen simple roots, and choose an integral vector $h \in D_I$. If $(y, \delta) < 0$ for a positive root δ of D_I, replace y by $s_\delta y$. The reflection preserves the future cone, and

$$0 < (h, s_\delta y) = (h, y) + (h, \delta)(y, \delta) < (h, y).$$

The height (h, y) is a positive integer, so these reflections terminate at a vector e_I pairing nonnegatively with every positive root of D_I. Thus $e_I \in \overline{D_I}$. It is primitive and still has perpendicular quotient $\Lambda(-1)$.

We show that $e_I \in \overline{D}$. Write a positive root of L_n as $\delta = \delta_I + u_{I^c}$ with $\delta_I = (A, B, u_I) \in L_I$ and $u_{I^c} \in L_I^{\perp}$. Then $(e_I, \delta) = (e_I, \delta_I)$.

(1) Suppose $B > 0$ and $u_{I^c} \neq 0$. The lattice $L_I^{\perp}$ is even and negative definite, so $u_{I^c}^2 \leq -2$ and $\delta_I^2 = -2 - u_{I^c}^2 \geq 0$. Since $(e_0, \delta_I) = B > 0$, the vectors δ_I and e_I belong to the same closed future cone. Hence $(e_I, \delta_I) \geq 0$.
(2) If $B > 0$ and $u_{I^c} = 0$, then $\delta = \delta_I$ is a positive-height root of L_I, so $(e_I, \delta) \geq 0$ because $e_I \in \overline{D_I}$.
(3) Suppose $B = 0$. The negative-definite component of δ has square -2, so it lies entirely in one of the two orthogonal even summands. If it lies in L_I, then δ is a positive root of L_I and $(e_I, \delta) \geq 0$. Otherwise $\delta_I = Ae_0$ with $A \geq 0$, and $(e_I, \delta) = A(e_I, e_0) \geq 0$.

Thus $e_I \in \overline{D}$. The same three cases, with any point of D_I in place of e_I, give $D_I \subseteq \overline{D}$.

Now suppose that $(e_I, \delta) = 0$. The image of δ_I in $(e_I^\perp \cap L_I)/\mathbb{Z}e_I \cong \Lambda(-1)$ is either zero or has square at most -4. The latter is impossible since $\delta_I^2 + u_{I^c}^2 = -2$ and $u_{I^c}^2 \le 0$. Thus the roots perpendicular to e_I are exactly the vectors $me_I + r$ with $m \in \mathbb{Z}$ and r a root of $L_I^\perp$. This gives (3.4). □

For $I_0 = \{1, 2, 3\}$, put

$$Q = E_8^{n-3} \oplus M,$$

and choose coordinates in which

$$L_n = U \oplus \Lambda(-1) \oplus Q(-1), \qquad e = e_{123} = (1, 0, 0, 0). \tag{3.5}$$

By Lemma 3.2, the roots of height zero at e are the vectors $ae + r$ with r a root of $Q(-1)$. The cusps e and e_0 are distinct future rays, so $(e_0, e) > 0$. Thus $ae + r$ is positive in D if and only if $a > 0$ or $a = 0$ and r is positive in Q_{rt}. Since $e \in \overline{D}$, every root of positive e-height is positive in D. In the coordinates (3.5), the height-zero inequalities at e are therefore the alcove inequalities (3.3) in Q.

We use minima in cosets of $\Lambda \oplus Q$ to construct further cusps of D and bound the rank of their perpendicular roots.

Lemma 3.3. *Let $N = \Lambda \oplus Q$, let $b \ge 2$, and write $z = (v, -w) \in N$ with $v \in \Lambda$ and $w \in Q$. Suppose that $a = z^2/(2b)$ is an integer, w satisfies the alcove inequalities* (3.3) *in Q, and*

$$\min\{y^2 \mid y \in jz + bN\} \ge 2b^2 \qquad (1 \le j < b). \tag{3.6}$$

Then $f = (a, b, z)$ is isotropic and lies in $\overline{D}$. Reduce each root perpendicular to f by a multiple of f so that its e-height lies in $\{0, \dots, b-1\}$. Suppose the Leech components of the reduced roots and v lie in a rational subspace $W \subseteq \Lambda_{\mathbb{Q}}$ of dimension d. Then the images of the perpendicular roots in $f^\perp/\mathbb{Q}f$ have rank at most $\operatorname{rk} Q + d$. In particular, the ray of f is a deficient cusp if $d < 24$.

Proof. For any root $\delta = (A, B, u)$ of e-height $B > 0$, substituting $A = (u^2 - 2)/(2B)$ and $a = z^2/(2b)$ and completing the square gives

$$(f, \delta) = \frac{b}{2B}\left(\left\|u - \frac{B}{b}z\right\|^2 - 2\right) = \frac{1}{2bB}\big(\|bu - Bz\|^2 - 2b^2\big). \tag{3.7}$$

If $B \equiv j \pmod b$ with $1 \le j < b$, then $bu - Bz \in -jz + bN$. Condition (3.6) gives $(f, \delta) \ge 0$. If $B = tb$, then $u - tz \in N$ is nonzero, since $u = tz$ would give $A = ta - 1/B \notin \mathbb{Z}$. Thus $(u - tz)^2 \ge 2$, and again $(f, \delta) \ge 0$. At height zero the assertion follows from (3.3). Hence $f \in \overline{D}$, and $f^2 = 2ab - z^2 = 0$.

Translating a perpendicular root by a multiple of f preserves its class in $f^\perp/\mathbb{Q}f$. For a reduced root $\delta = (A, B, u)$ write u_Λ for the projection of u to Λ. At $B = 0$ we have $u^2 = 2$, hence $u_\Lambda = 0$ since $\min \Lambda = 4$. For $0 < B < b$, equations (3.6) and (3.7) show that $bu - Bz$ is shortest in $-Bz + bN$, of square $2b^2$. Since $N = \Lambda \oplus Q$ is an orthogonal sum, its projection $\xi = (bu - Bz)_\Lambda$ is shortest in $-Bv + b\Lambda$. We have

$$u_\Lambda = \frac{\xi + Bv}{b}.$$

Thus a subspace containing v and these shortest Leech vectors also contains the Leech projections of all reduced perpendicular roots.

By hypothesis, f and the reduced roots lie in $U_{\mathbb{Q}} \oplus W \oplus Q_{\mathbb{Q}}$. This space has dimension $2 + d + \operatorname{rk} Q$ and contains e, with $(e, f) = b \ne 0$. Intersecting with $f^\perp$ and passing to the quotient by $\mathbb{Q}f$ gives a space of dimension $d + \operatorname{rk} Q$ containing all the root classes. The full quotient has dimension $24 + \operatorname{rk} Q$. □

Lemma 3.4. *Let P be a positive-definite even lattice, $z \in P$, and $b \geq 2$. Suppose $|(z, r)| \leq b$ for every root $r \in P$ of square 2. Then z has minimum square in its coset $z + bP$ whenever*

$$z^2 \leq 8 \quad (b = 2), \qquad z^2 \leq b^2 + b \quad (b \geq 3).$$

Proof. For any $x \in P \setminus \{0\}$, the norm change $(z + bx)^2 - z^2 = b^2x^2 + 2b(z, x)$ is a multiple of $2b$. If $x^2 = 2$, the hypothesis $|(z, x)| \leq b$ gives $b^2x^2 + 2b(z, x) \geq 2b^2 - 2b^2 = 0$. If $x^2 \geq 4$ and $z^2 \leq b^2 + b < (b + 1/2)^2$, Cauchy–Schwarz gives

$$b^2x^2 + 2b(z, x) \geq b^2x^2 - 2b\sqrt{x^2z^2} \geq 4b^2 - 4b\sqrt{z^2} > -2b,$$

since $t \mapsto b^2t^2 - 2bt\sqrt{z^2}$ is increasing for $t = \sqrt{x^2} \geq 2 > \sqrt{z^2}/b$. Because the norm change lies in $2b\mathbb{Z}$, it is nonnegative.

It remains to check $b = 2$ and $z^2 = 8$. Any shorter vector $v \in z + 2P$ satisfies $v^2 \equiv z^2 \equiv 0 \pmod 4$, so $v^2 = 0$ or $v^2 = 4$. If $v = 0$, then $z/2 \in P$ is a root of square 2 with $(z, z/2) = 4 > 2$, contradicting the hypothesis. If $v^2 = 4$, then $u_\pm = (z \pm v)/2 \in P$ are nonzero vectors satisfying $u_+^2 + u_-^2 = (z^2 + v^2)/2 = 6$ and $(u_+, u_-) = (z^2 - v^2)/4 = 1$. Since P is even, one of $u_\pm$ has square 2 and the other has square 4, so the root among them pairs with $z = u_+ + u_-$ to $2 + 1 = 3 > 2$, again a contradiction. □

3.2. **Leech cosets and cusps of heights two and three.** We use the standard Golay coordinates for Λ at scale $1/\sqrt{8}$ [Wil09, §5.4.1]. A vector $v = (a_0, \dots, a_{22}, a_\infty)/\sqrt{8}$ with $a_i \in \mathbb{Z}$ belongs to Λ if and only if all a_i have the same parity $p \in \{0, 1\}$, $\sum_i a_i \equiv 4p \pmod 8$, and the binary vector $\big(((a_i - p)/2) \bmod 2\big)_i$ belongs to the extended binary Golay code $\mathcal{G}_{24}$. On the coordinates $\{0, \dots, 22, \infty\}$ we generate $\mathcal{G}_{24}$ from the cyclic Golay code of length 23 with generator polynomial

$$g(X) = 1 + X + X^5 + X^6 + X^7 + X^9 + X^{11} \in \mathbb{F}_2[X]/(X^{23} - 1),$$

reciprocal to the generator in [BBC+26, §5.4-1], together with the overall parity check at ∞. The 8-element subset $O = \{0, 1, 2, 3, 4, 15, 17, 20\}$ is an octad of $\mathcal{G}_{24}$ because

$$\sum_{i \in O} X^i = g(X)\big(1 + X^2 + X^4 + X^6 + X^7 + X^9\big) \quad \text{in } \mathbb{F}_2[X].$$

We need one coset of minimum 16 in $\Lambda/3\Lambda$. Martinet classifies all the cosets [Mar02, Theorem 4.1]. Let e_j denote the standard coordinate vectors of $\mathbb{Z}^{24}$. Put

$$v_{16} = \frac{1}{\sqrt{8}}(8, 4, 4, 4, 4, 0^{19}) \in \Lambda,$$

and define eight norm-4 vectors $x_1, \dots, x_8 \in \Lambda$ by

$$\frac{4(e_0 + e_j)}{\sqrt{8}} \ (1 \leq j \leq 4), \quad \frac{2}{\sqrt{8}}\left(\sum_{j=0}^{4} e_j + \epsilon_{15}e_{15} + \epsilon_{17}e_{17} + \epsilon_{20}e_{20}\right), \ \epsilon_{15}\epsilon_{17}\epsilon_{20} = 1,$$

where $\epsilon_j \in \{\pm 1\}$. The coordinate-sum condition and the octad support O confirm that $v_{16}, x_1, \dots, x_8 \in \Lambda$. Direct inspection gives $x_i^2 = 4$, $(x_i, x_j) = 2$ for $i \neq j$, $\sum_{i=1}^{8} x_i = 3v_{16}$, and $(v_{16}, x_i) = 6$.

Lemma 3.5. *The shortest vectors in the coset $v_{16} + 3\Lambda$ have square 16 and are precisely the nine vectors v_{16} and $v_{16} - 3x_i$ $(1 \leq i \leq 8)$. Their rational span has dimension 8.*

Proof. For any $x \in \Lambda$ of square $q = x^2$, we have $(v_{16} - 3x)^2 - v_{16}^2 = 9q - 6(v_{16}, x)$. If $q = 4$ and $x \notin \{x_1, \dots, x_8\}$, then $x - x_i \neq 0$ implies $(x - x_i)^2 \geq 4$, so $(x, x_i) \leq 2$ for each i; summing over $i = 1, \dots, 8$ yields $3(v_{16}, x) = \sum_i (x, x_i) \leq 16$, so $(v_{16}, x) \leq 5$ and $9q - 6(v_{16}, x) \geq 36 - 30 = 6 > 0$. If $q = 6$, the same inequality $(x - x_i)^2 \geq 4$

gives $(x, x_i) \le 3$ for each i, so $3(v_{16}, x) \le 24$, $(v_{16}, x) \le 8$, and $9q - 6(v_{16}, x) \ge 54 - 48 = 6 > 0$. If $q \ge 8$, Cauchy–Schwarz gives $9q - 6(v_{16}, x) \ge 9q - 24\sqrt{q} > 0$. Thus $(v_{16} - 3x)^2 \le 16$ holds if and only if $x = 0$ or $x = x_i$ $(1 \le i \le 8)$, which produces the nine stated vectors of square 16. Their Gram matrix is $18I_9 - 2J_9$, where every entry of J_9 is one. The kernel of the Gram matrix is spanned by $(1, \dots, 1)^{\mathsf{T}}$, so the nine vectors span a space of dimension 8. □

If $Q_{\mathrm{rt}} \neq 0$, we construct two cusps with the same Q-component and different heights at e. Orthogonality to both forces the coefficient of e in $S_e = \mathbb{Q}e \oplus (Q_{\mathrm{rt}})_{\mathbb{Q}}$ to vanish.

Lemma 3.6. *Assume $Q_{\mathrm{rt}} \neq 0$, let θ be the highest root of an irreducible component of Q_{rt}, let $v_6 \in \Lambda$ be a vector of square 6, and define*

$$f_2 = (2, 2, v_6, -\theta), \qquad f_3 = (3, 3, v_{16}, -\theta) \tag{3.8}$$

in the coordinates (3.5)*. The following properties hold.*

(i) *The vectors f_2 and f_3 are isotropic and lie in $\overline{D}$.*
(ii) *Both cusps are deficient. The dimension d in Lemma 3.3 can be taken at most 1 for f_2 and at most 8 for f_3.*
(iii) *Any vector $x = \alpha e + m \in S_e$ (with $\alpha \in \mathbb{Q}$ and $m \in (Q_{\mathrm{rt}})_{\mathbb{Q}}$) belonging to $S_{f_2} \cap S_{f_3}$ satisfies $\alpha = 0$.*

Proof. We verify the hypotheses of Lemma 3.3 for f_2 $(b = 2)$ and f_3 $(b = 3)$. Since θ is the highest root of a component R of Q_{rt} and is orthogonal to all other components, we have $(\theta, \alpha_i) \ge 0$ for all simple roots, $(\theta, \theta) = 2 \le b$, and $|(\theta, r)| \le 2 \le b$ for every root $r \in Q_{\mathrm{rt}}$.

For f_2 we have $b = 2$, $z = (v_6, -\theta)$ and $a = 2$. The squares in $v_6 + 2\Lambda$ and in $-\theta + 2Q$ are each congruent to 2 (mod 4), since $v_6^2 = 6$ and $\theta^2 = 2$. The squares in $z + 2N$ are therefore congruent to 0 (mod 4). As $\min \Lambda = 4$, the first coset has minimum 6, attained by v_6, and the second has minimum 2, attained by $-\theta$. Thus $\min(z + 2N) = 6 + 2 = 8 = 2b^2$. Only $\pm v_6$ attain the minimum in $v_6 + 2\Lambda$. Indeed, if $u \in v_6 + 2\Lambda$ had $u^2 = 6$ with $u \neq \pm v_6$, then $(u \pm v_6)/2 \in \Lambda$ would be two nonzero Leech vectors whose squares sum to $(u^2 + v_6^2)/2 = 6$, contradicting $\min \Lambda = 4$. For a reduced perpendicular root of height $B = 1$, the projection formula in the proof of Lemma 3.3 gives $u_\Lambda = (\xi + v_6)/2$ with $\xi = \pm v_6$. Hence $u_\Lambda \in \mathbb{Q}v_6$ and $d \le 1$.

For f_3 ($b = 3$, $z = (v_{16}, -\theta)$, $z^2 = 18$, $a = 3$), Lemma 3.5 shows that $\min(\pm v_{16} + 3\Lambda) = 16$ with shortest vectors spanning an 8-dimensional subspace of $\Lambda_{\mathbb{Q}}$, while Lemma 3.4 applied to $\theta \in Q$ gives $\min(\mp\theta + 3Q) = 2$. Thus $\min(jz + 3N) = 16 + 2 = 18 = 2b^2$ for $j \in \{1, 2\}$. The span of the shortest Leech representatives contains v_{16}. The same projection formula therefore gives $d \le 8$. Lemma 3.3 now gives (i) and (ii).

For (iii), let $x = \alpha e + m \in S_e \cap S_{f_2} \cap S_{f_3}$ with $m \in (Q_{\mathrm{rt}})_{\mathbb{Q}}$. Since $S_{f_b} \subseteq f_b^{\perp}$, we have $(x, f_b) = 0$ for $b = 2, 3$. In the form $2ab - v^2 - w^2$, these pairings are

$$0 = (x, f_2) = 2\alpha + (\theta, m), \qquad 0 = (x, f_3) = 3\alpha + (\theta, m),$$

and subtracting the two equations yields $\alpha = 0$. □

3.3. **Spanning the residual root lattice.** It remains to impose independent conditions on $(M_{\mathrm{rt}})_{\mathbb{Q}}$. For each irreducible component R of M_{rt} we construct cusps from dominant weights $w \in R$ with $(w, \theta_R) \le b$.

Lemma 3.7. *Fix a denominator b and a pair of squared norms (w^2, v^2) from the following table.*

b	(w^2, v^2)
2	$(2,6),\ (6,6),\ (4,4),\ (8,4)$
3	$(6,12),\ (8,10),\ (10,8),\ (12,6)$
4	$(12,20),\ (14,18)$

(3.9)

The Leech lattice contains a vector v of the indicated square. If $w \in Q$ satisfies the alcove inequalities (3.3) for b, then $f = (a, b, v, -w)$, where $a = (v^2 + w^2)/(2b) \in \mathbb{Z}$, spans a deficient cusp in $\overline{D}$. The dimension d of Lemma 3.3 is at most 1, 2 or 3 for $b = 2$, 3 or 4, respectively.

Proof. In the Golay coordinates of Λ, the numerators $(4,4)$, (8), $(8,4,4)$ and $(12,4)$, extended by zeros, give vectors of squares $4, 8, 12, 20$ respectively. Placing seven entries 2 and one entry -2 on the octad O, and placing 4 on $1, 3$, or 7 coordinates outside O, produces vectors in Λ of squares $6, 10$, and 18.

We next check the coset minimum condition (3.6) for $z = (v, -w)$. Since w satisfies (3.3), it satisfies $|(w, r)| \le b$ for every root $r \in Q_{\mathrm{rt}}$.

- For $b = 2$, Lemma 3.4 shows that w is shortest in $w + 2Q$. In $v + 2\Lambda$, norm congruence modulo 4 and $\min \Lambda = 4$ give minimum 6 when $v^2 = 6$ and minimum 4 when $v^2 = 4$. The argument for v_6 in Lemma 3.6 shows that in both cases only $\pm v$ attain the minimum. Thus $\min(z + 2N) = w^2 + v^2 \ge 8 = 2b^2$, and the Leech projection dimension is $d \le 1$.
- For $b = 3$, both $w^2 \le 12 = b^2 + b$ and $v^2 \le 12 = b^2 + b$, so Lemma 3.4 shows that each of $w + 3Q$ and $v + 3\Lambda$ attains its minimum at the given representative. The opposite cosets have the same minima. Thus $\min(jz + 3N) = w^2 + v^2 = 18 = 2b^2$ for $j \in \{1, 2\}$.
- For $b = 4$, both $w^2 \le 14 \le b^2 + b$ and $v^2 \le 20 = b^2 + b$, so Lemma 3.4 gives $\min(jz + 4N) = w^2 + v^2 = 32 = 2b^2$ for $j \in \{1, 3\}$. For $j = 2$, every vector in $2z + 4N = 2(z + 2N)$ is twice a vector of $z + 2N$. When $(w^2, v^2) = (12, 20)$, neither $w/2$ nor $v/2$ lies in the lattice because $(w/2)^2 = 3$ and $(v/2)^2 = 5$ are odd. Thus $\min(w + 2Q) \ge 4$ and $\min(v + 2\Lambda) \ge 4$, giving $\min(2z + 4N) \ge 4(4 + 4) = 32$. When $(w^2, v^2) = (14, 18)$, congruence modulo 4 gives $\min(w + 2Q) \ge 2$ and $\min(v + 2\Lambda) \ge 6$, again giving $\min(2z + 4N) \ge 4(2 + 6) = 32$.

Finally, we bound the Leech projection dimension d for $b = 3$ and $b = 4$. Any two distinct vectors v_1, v_2 of square $t = v^2$ in the same coset of $\Lambda / b\Lambda$ satisfy $(v_1 - v_2)/b \in \Lambda \setminus \{0\}$, so $(v_1 - v_2)^2 \ge 4b^2$ and $(v_1, v_2) \le t - 2b^2$. If there are h such shortest vectors $v_1, \dots, v_h$ in a coset, then

$$0 \le \left\| \sum_{i=1}^{h} v_i \right\|^2 \le h\big(ht - 2b^2(h-1)\big).$$

For $b = 3$ and $t \le 12$ we obtain $h \le 3$. If $h = 3$, then $t = 12$ and $v_1 + v_2 + v_3 = 0$. The shortest vectors in $v + 3\Lambda$ and their negatives in the opposite coset therefore span a space of dimension at most two containing v. For $b = 4$ and $t \le 20$, the same bound gives $h \le 2$ in $v + 4\Lambda$. For a reduced perpendicular root at height $B = 2$, the preceding bounds force $\xi = (4u - 2z)_\Lambda = 2s$, where $s \in v + 2\Lambda$ has square 4 or 6 according to the norm pair in (3.9). In either case s is unique up to sign by the argument for $b = 2$. Since $u_\Lambda = (s + v)/2$, these roots add at most one dimension. We obtain $d \le 2$ for $b = 3$ and $d \le 3$ for $b = 4$. Lemma 3.3 gives deficiency in every case. □

For each type R in (3.1), Table 1 gives $\operatorname{rk} R$ pairs (b, w) whose weights span $R_{\mathbb{Q}}$. A string $i_1 \cdots i_k$ denotes $\omega_{i_1} + \cdots + \omega_{i_k}$. For A_m we number the chain from 1 to m. For D_m the nodes $m - 1$ and m meet the chain $1, \dots, m - 2$ at $m - 2$. For E_6 and

E_7 the main chain is $1, 3, 4, 5, 6$ (extended to 7 in E_7), with node 2 attached to 4. The last column gives the absolute determinant of the matrix of fundamental-weight coordinates.

TABLE 1. Spanning weights for the 15 possible irreducible components of M_{rt}. There are 79 weights in all. The entry $i_1 \cdots i_k$ denotes $\omega_{i_1} + \cdots + \omega_{i_k}$.

R	$b = 2$	$b = 3$	$b = 4$	$\lvert\det(c_{ij})\rvert$
A_1	11			2
A_2	12	222		3
A_3	22, 13	233		4
A_4	23, 14	334, 244		5
A_5	33, 24, 15	444, 345		6
A_6	34, 25, 16	455, 446, 356		7
A_7	44, 35, 26, 17	466, 457, 367		8
A_8	45, 36, 27, 18	558, 477, 468, 378		9
D_4	44, 33, 2, 11			8
D_5	45, 2, 11	155, 13		4
D_6	66, 55, 4, 2, 11	13		8
D_7	67, 4, 2, 11	177, 15, 13		4
D_8	88, 77, 6, 4, 2, 11	15, 13		8
E_6	2, 16	666, 56, 4, 13		3
E_7	77, 6, 1	3, 27	57, 4	2

Let C be the Cartan matrix of R, $c \in \mathbb{Z}_{\geq 0}^{\operatorname{rk} R}$ the column vector of fundamental-weight coefficients of w, and m the column vector of highest-root marks. Then the simple-root coordinates of w are $C^{-1}c$, its squared norm is $w^2 = c^{\mathsf{T}}C^{-1}c$, and its level is $(w, \theta_R) = m^{\mathsf{T}}c$. For A_m, D_m, E_6 and E_7, the highest-root marks are respectively

$$(1, \ldots, 1), \quad (1, 2, \ldots, 2, 1, 1), \quad (1, 2, 2, 3, 2, 1), \quad (2, 2, 3, 4, 3, 2, 1).$$

Applying these formulas to the 79 entries of Table 1 gives the following bounds (see Appendix A and `anc/verify_rank27.py`). For each entry, $C^{-1}c \in \mathbb{Z}^{\operatorname{rk} R}$ (so $w \in R$), $m^{\mathsf{T}}c \leq b$, and w^2 belongs to $\{2, 4, 6, 8\}$ for $b = 2$, to $\{6, 8, 10, 12\}$ for $b = 3$, and to $\{12, 14\}$ for $b = 4$. (Whenever $b = 2$ and $w^2 = 8$, $C^{-1}c$ also has an odd entry, giving an independent direct check that $w/2 \notin Q$; this already follows from the level inequality $m^{\mathsf{T}}c \leq 2$, as in Lemma 3.4.) Because roots in distinct irreducible components of Q_{rt} are mutually orthogonal, these alcove inequalities hold in Q even when Q_{rt} is not saturated in Q.

Proposition 3.8. *If $L_n = U \oplus E_8(-1)^n \oplus M(-1)$ in* (3.2) *has rank $r \geq 27$ and every irreducible component of M_{rt} belongs to* (3.1)*, then the standard chamber D admits a finite set F of deficient cusps with $\bigcap_{f \in F} S_f = 0$.*

Proof. In the first-triple coordinates (3.5), Lemma 3.2 gives the deficient cusp $e = e_{123}$ with $S_e = \mathbb{Q}e \oplus (Q_{\mathrm{rt}})_{\mathbb{Q}}$. If $Q_{\mathrm{rt}} = 0$, then $n = 3$, $M_{\mathrm{rt}} = 0$, and $\operatorname{rk} M = r - 26 \geq 1$. Take $F = \{e, e_0\}$. By Lemma 3.1(iii), $\dim S_{e_0} = 1 + 24 < \operatorname{rk} L_n - 1 = 25 + \operatorname{rk} M$, so e_0 is deficient. Since $S_{e_0} \subseteq e_0^{\perp}$, $S_e = \mathbb{Q}e$, and $(e_0, e) > 0$, we have $S_{e_0} \cap S_e = 0$.

Now suppose $Q_{\mathrm{rt}} \neq 0$, and include e and the cusps f_2, f_3 of Lemma 3.6 in F. If $r = 27$, then $r = 2 + 8n + \operatorname{rk} M$ and $n \geq 3$ give $n = 3$ and $\operatorname{rk} M = 1$. The positive root θ has square two and is primitive in the even lattice M. Hence $Q = M = \mathbb{Z}\theta \cong A_1$.

Write $x \in S_e \cap S_{f_2} \cap S_{f_3}$ as $\alpha e + \beta\theta$. Pairing with f_2 and f_3 gives

$$2\alpha + 2\beta = 0, \qquad 3\alpha + 2\beta = 0.$$

Thus $\alpha = \beta = 0$, so $F = \{e, f_2, f_3\}$ suffices in this case.

Now let $r > 27$, and also include in F the split Leech cusps e_{12j} for $4 \le j \le n$ from Lemma 3.2. By Lemma 3.6(iii), the intersection $S_e \cap S_{f_2} \cap S_{f_3}$ lies in $(Q_{\mathrm{rt}})_{\mathbb{Q}} = \bigoplus_{j=4}^{n}(E_8^{(j)})_{\mathbb{Q}} \oplus (M_{\mathrm{rt}})_{\mathbb{Q}}$. Let x lie in this intersection and in all $S_{e_{12j}}$ for $4 \le j \le n$. In the original decomposition (3.2), x has zero U-component. By (3.4), the only term of $S_{e_{12j}}$ with nonzero projection to U is the line $\mathbb{Q}e_{12j}$ (since e_{12j} is nonzero and isotropic while $\bigoplus_{i\in\{1,2,j\}} E_8^{(i)}(-1)$ is negative definite). Its coefficient in x must vanish. The remaining summands of $S_{e_{12j}}$ have zero projection to $E_8^{(j)}$. Applying this for every j gives $x \in (M_{\mathrm{rt}})_{\mathbb{Q}}$.

Finally, for each irreducible component R of M_{rt}, include in F the $\operatorname{rk} R$ weight cusps $f = (a, b, v, -w)$ from Lemma 3.7 and Table 1, extending each $w \in R$ by zero on the remaining components. For any $x \in (M_{\mathrm{rt}})_{\mathbb{Q}} \cap S_f$, orthogonality $(x, f) = 0$ gives $(w, x) = 0$ in the positive-definite inner product on M_{rt}. Since the chosen weights w span $R_{\mathbb{Q}}$ for every component R of M_{rt}, we conclude that $x = 0$. □

4. Anisotropic complements and the sharp bound

Write K for the hyperbolic complement in $K \oplus \Lambda(-1)$ and suppose that K is anisotropic over $\mathbb{Q}$. By Meyer's theorem, $q = \operatorname{rk} K$ is at most 4. We treat all $2 \le q \le 4$, corresponding to total ranks 26 through 28.

The construction depends on the least positive square represented by K.

Lemma 4.1. *Let K be an even hyperbolic lattice of rank $q \ge 2$, and let $t = \min\{x^2 > 0 : x \in K\}$ be its least positive square. Assume $t \ge 4$. Let N be a positive-definite even lattice with $\min N \ge 4$, and suppose there exist $k \in K$ and $u \in N$ with $k^2 = u^2 = t$. Then in $K \oplus N(-1)$ the isotropic vectors $f_\pm = (k, \pm u)$ are primitive, lie in the closure of the same chamber, have no perpendicular (-2)-roots, and are deficient cusps whenever $\operatorname{rk}(K \oplus N) \ge 3$. Consequently, $S_{f_+} \cap S_{f_-} = 0$ and $E(K \oplus N(-1)) = 0$.*

Proof. By minimality of t, the vector k is primitive in K, so $f_\pm = (k, \pm u)$ are primitive in $K \oplus N(-1)$. No (-2)-root $x \in K$ can be orthogonal to k, for otherwise $(k + x)^2 = t - 2$ would be a positive square in K strictly smaller than t. For any root $\delta = (x, v) \in K \oplus N(-1)$ with $v \ne 0$, put $s = v^2 \ge 4$ and $y = (k, x)$. Then $x^2 = s - 2 \ge 2 > 0$, so the reversed Cauchy–Schwarz inequality in the hyperbolic space $K_{\mathbb{R}}$ gives $y^2 \ge tx^2 > 0$. It follows that $(k, 0)$ lies on no root mirror of $K \oplus N(-1)$ and therefore belongs to a unique chamber D, whose positive roots are characterized by $y = (k, x) > 0$.

Let $\delta = (x, v)$ be a positive root. If $v = 0$, we have $(f_\pm, \delta) = y > 0$. If $v \ne 0$, minimality of t in K forces $x^2 \ge t$, whence $s = x^2 + 2 \ge t + 2$. By the hyperbolic Cauchy–Schwarz inequality on K and the Euclidean Cauchy–Schwarz inequality on N,

$$(f_\pm, \delta) = y \mp (u, v) \ge \sqrt{t(s-2)} - \sqrt{ts} = \frac{-2\sqrt{t}}{\sqrt{s-2} + \sqrt{s}} \ge \frac{-2\sqrt{t}}{\sqrt{t} + \sqrt{t+2}} > -1.$$

Since $(f_\pm, \delta)$ is an integer, $(f_\pm, \delta) \ge 0$, so $f_\pm \in \overline{D}$.

Suppose $\delta = (x, v)$ were a root perpendicular to $f_\pm$. Translating δ by an integer multiple of $f_\pm$ shifts $y = (k, x)$ by a multiple of t, so we may arrange $|y| \le t/2$. If the translated root had $v \ne 0$, then $x^2 = v^2 - 2 \ge t$ would force $y^2 \ge tx^2 \ge t^2$, contradicting $|y| \le t/2$. If $v = 0$, then x is a root in K perpendicular to k, which

we excluded above. Thus f_+ and f_- have no perpendicular roots, so $S_{f_\pm} = \mathbb{Q}f_\pm$ are two distinct lines with root-rank deficit $\mathrm{rk}(K \oplus N) - 2 > 0$. □

Lemma 4.1 applies to the Leech summand for every even $t \geq 4$.

Lemma 4.2. *The Leech lattice Λ represents every even integer $t \geq 4$.*

Proof. In the Golay model, $4a/\sqrt{8}$ lies in Λ for every $a \in \mathbb{Z}^{24}$ of even coordinate sum, and its square is $2\sum a_i^2$. For $m \geq 0$, write $2m = a_1^2 + \cdots + a_4^2$ by Lagrange's four-square theorem. The sum $a_1 + \cdots + a_4$ is even, so placing $4a_1, \ldots, 4a_4$ on four coordinates gives a vector of square $4m$. The vector of square 6 in Lemma 3.7 has support on nine coordinates. Adding our vector on four of the remaining coordinates gives square $6 + 4m$. These two families contain every even integer $t \geq 4$. □

For $t = 2$, choose $k \in K$ with $k^2 = 2$. Relabeling the 24 Golay coordinates by $1, \ldots, 24$, we use the coordinate frame of square 8 in the Golay model of Λ [Wil09, §5.4.2]. Put

$$u_i = \frac{8e_i}{\sqrt{8}} \in \Lambda, \qquad f_i^\epsilon = (2k, \epsilon u_i) \in K \oplus \Lambda(-1) \qquad (1 \leq i \leq 24,\ \epsilon \in \{\pm 1\}). \quad (4.1)$$

Each f_i^ϵ is isotropic since $(2k)^2 = u_i^2 = 8$. A norm-eight vector in an even lattice can have a nontrivial integral divisor only if it is divisible by two. Here $u_i/2 = 4e_i/\sqrt{8}$ fails the coordinate-sum condition modulo 8, so u_i and f_i^ϵ are primitive.

Lemma 4.3. *Let K be a rationally anisotropic even hyperbolic lattice of rank $q \geq 2$, and suppose $k \in K$ has square 2. Then $(k, 0)$ lies in a unique chamber D of $L = K \oplus \Lambda(-1)$. Its positive roots $\delta = (x, v)$ satisfy $(k, x) > 0$. The vectors in* (4.1) *have the following properties.*

(i) *All 48 rays lie in $\overline{D}$.*

(ii) *For each f_i^ϵ, the perpendicular roots modulo $\mathbb{Z}f_i^\epsilon$ have the following 46 representatives*

$$\delta_{ij}^\pm = \left(k, \frac{4(\epsilon e_i \pm e_j)}{\sqrt{8}}\right) \qquad (j \neq i).$$

Since $\delta_{ij}^+ + \delta_{ij}^- = f_i^\epsilon$, the 46 classes form 23 opposite pairs. In $(f_i^\epsilon)^\perp/\mathbb{Q}f_i^\epsilon$, the induced root system is $23A_1$, and the root-rank deficit is $(q + 22) - 23 = q - 1 > 0$.

(iii) *Writing $V_i = \{(0, v) \in L_\mathbb{Q} : v \in \Lambda_\mathbb{Q},\ (e_i, v) = 0\}$, we have*

$$S_{f_i^\epsilon} = \mathbb{Q}f_i^\epsilon \oplus V_i, \qquad S_{f_i^+} \cap S_{f_i^-} = V_i, \qquad \bigcap_{i=1}^{24} (S_{f_i^+} \cap S_{f_i^-}) = 0.$$

Proof. The vector k is primitive since $k^2 = 2$. A (-2)-root $x \in K$ perpendicular to k would give a nonzero isotropic vector $k + x$, contrary to anisotropy. As in the proof of Lemma 4.1, $(k, 0)$ lies on no root mirror and defines a chamber D whose positive roots $\delta = (x, v)$ have $y = (k, x) > 0$.

To prove (i), write $v = (a_1, \ldots, a_{24})/\sqrt{8} \in \Lambda$ in Golay coordinates and set $s = v^2$. If $v = 0$, then $(f_i^\epsilon, \delta) = 2y > 0$. If $v \neq 0$, then $s \geq 4$ implies $x^2 = s - 2 \geq 2$, so the hyperbolic Cauchy–Schwarz inequality $y^2 \geq k^2x^2 = 2(s - 2)$ gives

$$y \geq 2, \qquad s \leq \tfrac{1}{2}y^2 + 2.$$

We claim that $|a_i| \leq 2y$. Suppose otherwise. If all a_j are even, then $|a_i| \geq 2y + 2$ and $s \geq (2y + 2)^2/8 > y^2/2 + 2$, since $y \geq 2$. If all a_j are odd, then $|a_i| \geq 2y + 1$. The other 23 entries have square at least one, so $s \geq ((2y + 1)^2 + 23)/8 > y^2/2 + 2$. Both cases contradict the upper bound on s. Thus

$$(f_i^\epsilon, \delta) = 2y - \epsilon a_i \geq 0,$$

proving $f_i^\epsilon \in \overline{D}$.

For (ii), translate a root $\delta = (x, v)$ perpendicular to f_i^ϵ by a multiple of f_i^ϵ. This changes $y = (k, x)$ by a multiple of 4, so we can take $|y| \le 2$. We have $v \neq 0$, since no root of K is perpendicular to k. Thus $x^2 = v^2 - 2 \ge 2$. The inequalities $4 \ge y^2 \ge 2x^2 \ge 4$ are equalities. It follows that $x^2 = 2$, $v^2 = 4$, and x is proportional to k. If $x = -k$, adding f_i^ϵ gives a representative of the same class with $x = k$ and $y = 2$.

The equation $(f_i^\epsilon, \delta) = 2y - \epsilon a_i = 0$ gives $a_i = 4\epsilon$. All coordinates are even, and the squares of the remaining 23 numerators sum to 16. They therefore have either one nonzero entry ± 4 or four nonzero entries ± 2. The second case would give a word of weight 4 in the extended binary Golay code, whose minimum nonzero weight is 8. Conversely, each displayed $\delta_{ij}^\pm$ belongs to L, has square -2, and is perpendicular to f_i^ϵ. For each $j \neq i$, their sum is f_i^ϵ. The 23 vectors δ_{ij}^+ have Gram matrix $-2I_{23}$, so their images and their negatives form $23A_1$. The quotient has dimension $q + 22$, giving deficit $q - 1$.

For (iii), $\delta_{ij}^+ - \delta_{ij}^- = (0, u_j)$ for every $j \neq i$. Together with $\delta_{ij}^+ + \delta_{ij}^- = f_i^\epsilon$, this gives $S_{f_i^\epsilon} = \mathbb{Q} f_i^\epsilon \oplus V_i$. In $L_\mathbb{Q}/V_i$, the images of f_i^+ and f_i^- are linearly independent, as one sees from their K-components $2k$ and their ith coordinate components $\pm u_i$. Hence $S_{f_i^+} \cap S_{f_i^-} = V_i$. Intersecting the 24 coordinate hyperplanes V_i gives zero. □

Proposition 4.4. *Let K be a rationally anisotropic even hyperbolic lattice of rank $q \ge 2$. Then $L = K \oplus \Lambda(-1)$ admits a finite set F of deficient cusps in one chamber with $\bigcap_{f \in F} S_f = 0$.*

Proof. Let $t \ge 2$ be the least positive square in K and choose $k \in K$ with $k^2 = t$. If $t \ge 4$, Lemma 4.2 provides $u \in \Lambda$ with $u^2 = t$, and Lemma 4.1 gives two rootless deficient cusps $f_\pm = (k, \pm u)$ in one chamber with $S_{f_+} \cap S_{f_-} = 0$. If $t = 2$, Lemma 4.3 gives 48 deficient cusps $f_i^\pm$ in one chamber with $\bigcap_{i,\epsilon} S_{f_i^\epsilon} = 0$. □

Remark 4.5. The monomial subgroup $2^{12}{:}\mathrm{M}_{24} \subset \mathrm{Aut}(\Lambda)$ fixes $(k, 0)$ and hence preserves D. The Mathieu group M_{24} acts transitively on the 24 coordinates, and the sign change along an octad containing i sends u_i to $-u_i$. Thus the 48 cusps $f_i^\pm$ form one orbit of this finite subgroup. Their common $A(L)$-orbit is infinite, since Proposition 4.4 and Lemma 2.3 give $E(L) = 0$. At each cusp the perpendicular roots give 23 mutually orthogonal affine systems of type $\widetilde{A}_1$, one for each pair with $\delta_{ij}^+ + \delta_{ij}^- = f_i^\epsilon$. Near the cusp the horospherical chamber slice is a 23-dimensional rectangular box times $\mathbb{R}^{q-1}$.

Proof of Theorems A and B. Let L be an even hyperbolic lattice of rank $r \ge 27$. By Lemma 2.4, we may choose a maximal even overlattice $L' \supseteq L$, and Corollary 2.2 decomposes L' as $H \oplus \Lambda(-1)$ where $\operatorname{rk} H = r - 24 \ge 3$. If $H_\mathbb{Q}$ is isotropic, Proposition 3.8 gives a finite set F' of deficient cusps of a chamber D' of L' with $\bigcap_{f' \in F'} S_{f'}(L') = 0$: for $Q_{\mathrm{rt}} = 0$ we take $F' = \{e, e_0\}$; for $Q_{\mathrm{rt}} \neq 0$ we take e, f_2, f_3 (which suffice in rank 27), together with the split cusps e_{12j} $(4 \le j \le n)$ and $\operatorname{rk} M_{\mathrm{rt}} \le \operatorname{rk} M \le 10$ cusps from Table 1 when $r > 27$. If $H_\mathbb{Q}$ is anisotropic, Proposition 4.4 gives two rootless cusps when $t \ge 4$, and the 48 coordinate cusps when $t = 2$.

By Lemma 2.4, the primitive generators in L of the rays in F' form a set F of deficient cusps of the chamber of L containing D', and $\bigcap_{f \in F} S_f(L) = 0$. Acting by $W_2(L)$ carries this chamber to any prescribed chamber of L. This proves Theorem B, and Lemma 2.3 yields $E(L) = 0$.

In rank 26, consider Conway's chamber in $II_{1,25} = U \oplus \Lambda(-1)$. Its Weyl vector $e = (1, 0, 0)$ is primitive and isotropic and has no perpendicular roots. Hence

$S_e = \mathbb{Q}e$, and Lemma 2.3 gives $E(II_{1,25}) \subseteq \mathbb{Q}e \cap II_{1,25} = \mathbb{Z}e$. Conway's description of $A(II_{1,25}) \cong \Lambda \rtimes \mathrm{Aut}(\Lambda)$ shows that the whole group fixes e [Con83, Bor87]. Thus $E(II_{1,25}) = \mathbb{Z}e$, and the maximal rank with nonzero exceptional lattice is 26. □

5. Boundary ranks and integral reflections

5.1. **Ranks 25 and 26.** Let P and P' be positive-definite even unimodular lattices of the same rank $d > 0$. They are isometric at every finite prime by the local argument in Proposition 2.1. If K is a nondegenerate even lattice with a positive direction, then $K \oplus P(-1)$ and $K \oplus P'(-1)$ are indefinite lattices in the same genus. Both have discriminant group A_K, and their common rank exceeds $\ell(A_K)$ by at least $d \geq 8$. Nikulin's uniqueness theorem [Nik80, Theorem 1.14.2] gives

$$K \oplus P(-1) \cong K \oplus P'(-1). \tag{5.1}$$

In rank 25, this allows us to replace any Niemeier summand by the Leech lattice.

Proposition 5.1. *Let $a \geq 1$ be an integer and let P be a positive-definite even unimodular lattice of rank 24. Then*

$$E(\langle 2a \rangle \oplus P(-1)) = 0.$$

Proof. By (5.1), we may take $L = \mathbb{Z}k \oplus \Lambda(-1)$ with $k^2 = 2a$. Since $k^\perp = \Lambda(-1)$ has no roots, k lies in a unique chamber D. The group $\mathrm{Aut}(\Lambda)$ fixes k and therefore preserves D.

We first show that $\mathrm{Aut}(\Lambda)$ acts irreducibly on $\Lambda_{\mathbb{Q}}$. In Golay coordinates, the subgroup $2^{12}{:}\mathrm{M}_{24}$ acts by sign changes on the supports of codewords in $\mathcal{G}_{24}$ and by the 5-transitive Mathieu group on the coordinates [Wil09, §§5.2.3 and 5.4.1]. The coordinate axes afford distinct characters of the sign-change group. Indeed, equality of two characters would give a vector of weight 2 in $\mathcal{G}_{24}^\perp = \mathcal{G}_{24}$, whose minimum nonzero weight is 8. Every subspace invariant under the sign changes is thus a sum of coordinate axes. Transitivity on the axes gives the assertion.

Since $\mathrm{id}_{\mathbb{Z}k} \oplus (-\mathrm{id}_\Lambda)$ belongs to $A(L)$, its eigenspaces give the decomposition

$$E(L)_{\mathbb{Q}} = \big(E(L)_{\mathbb{Q}} \cap \mathbb{Q}k\big) \oplus \big(E(L)_{\mathbb{Q}} \cap \Lambda_{\mathbb{Q}}\big),$$

where the second summand is either 0 or $\Lambda_{\mathbb{Q}}$ by irreducibility. Suppose $E(L)_{\mathbb{Q}}$ is a nonzero proper subspace. It is then either $\mathbb{Q}k$ or $(k^\perp)_{\mathbb{Q}}$, so $A(L)$ preserves $\mathbb{Q}k$. Since k is primitive and $A(L)$ preserves the positive cone, $A(L)$ fixes k. The stabilizer of k in $\mathrm{O}(L)$ acts faithfully on the negative-definite lattice $k^\perp$ and is therefore finite. If $E(L)_{\mathbb{Q}} = L_{\mathbb{Q}}$, then the vectors of a basis of L have finite orbits, which again implies that $A(L)$ is finite. By the Nikulin–Vinberg classification, an even hyperbolic lattice with finite chamber group has rank at most 19 [Nik83], [BM26, Theorem 1.1]. Here $\mathrm{rk}\, L = 25$. Hence $E(L) = 0$. □

Corollary 5.2. *Let L be a maximal even hyperbolic lattice with $E(L) \neq 0$.*

(i) *If $\mathrm{rk}\, L = 25$, then $i_W(L_{\mathbb{Q}_p}) = 11$ at some finite prime p.*
(ii) *If $\mathrm{rk}\, L = 26$, then either $L \cong II_{1,25}$ or $i_W(L_{\mathbb{Q}_p}) = 11$ at some finite prime p. In particular, among rank-26 maximal even lattices admitting a $\Lambda(-1)$ summand, $II_{1,25}$ is the unique lattice with nonzero exceptional lattice.*

Proof. In dimension 25 or 26 over $\mathbb{Q}_p$, the anisotropic kernel has dimension at most 4. Thus $i_W(L_{\mathbb{Q}_p}) \geq 11$. Suppose that $i_W(L_{\mathbb{Q}_p}) \geq 12$ at every finite prime. By Proposition 2.1, we may write $L = H \oplus \Lambda(-1)$ with H maximal even. In rank 25 we have $H = \langle 2a \rangle$ for some $a \geq 1$, so $E(L) = 0$ by Proposition 5.1.

In rank 26, the complement H is binary. If $H_{\mathbb{Q}}$ is anisotropic, Proposition 4.4 gives $E(L) = 0$. If $H_{\mathbb{Q}}$ is isotropic, maximality gives $H \cong U$ and hence $L \cong II_{1,25}$. The same argument applies whenever a Leech summand is given. □

This proves Theorem D.

5.2. **Sharp norm thresholds for coset minima.** Lemma 3.4 gives a sufficient condition for $z \in P$ to be shortest in $z + bP$. Under the inequalities $|(z, r)| \le b$ for every root r, it suffices that $z^2 \le 8$ if $b = 2$, or $z^2 \le b^2 + b$ if $b \ge 3$. The next even square gives a counterexample for every b.

Proposition 5.3. *For each integer $b \ge 2$, there is a positive-definite even binary lattice P and a vector $z \in P$ satisfying $|(z, r)| \le b$ for every root $r \in P_{\mathrm{rt}}$, such that $z + bP$ contains a vector shorter than z. One can take $z^2 = 10$ for $b = 2$ and $z^2 = b^2 + b + 2$ for $b \ge 3$.*

Proof. For $b \ge 3$, let $P = \mathbb{Z}e \oplus \mathbb{Z}x$ be the binary lattice with Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$. Since $(ue + vx)^2/2 = (u + v/2)^2 + 7v^2/4$, its only roots are $\pm e$. Take $z = e + kx$ if $b = 2k$ and $z = (k + 1)x - e$ if $b = 2k + 1$. In both cases

$$z^2 = b^2 + b + 2, \qquad |(z, e)| \le b, \qquad (z, x) = 2b + 1, \qquad (z - bx)^2 = z^2 - 2b < z^2.$$

For $b = 2$, let $P = \mathbb{Z}e \oplus \mathbb{Z}x$ have Gram matrix $\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)$. Its quadratic form $(ue + vx)^2 = 3u^2 + 3v^2 + (u + v)^2$ has minimum 4, so P has no roots. Take $z = e + x$. Then $z^2 = 10$ and $(z - 2e)^2 = (x - e)^2 = 6$. □

Remark 5.4. More generally, let $b \ge 1$ and let $s \ge 4$ be even. Suppose that $z^2 \le b^2 s/4 + b$ and $|(z, x)| \le bx^2/2$ for all $x \in P$ with $0 < x^2 < s$. Then z is shortest in $z + bP$. Indeed, for $0 < x^2 < s$, the change in square $(z + bx)^2 - z^2 = b^2x^2 + 2b(z, x)$ is nonnegative by hypothesis. For $q = x^2 \ge s$, the Cauchy–Schwarz inequality bounds this change below by $b^2q - 2b\sqrt{z^2 q}$. This expression is increasing for $q \ge s$. Its value at s is greater than $-2b$, since $z^2 \le b^2 s/4 + b < (bs + 2)^2/(4s)$. The change belongs to $2b\mathbb{Z}$, so it is nonnegative in this case as well.

5.3. **Reflections of bounded length.** We now allow the integral reflections of other lengths that occur in Vinberg's reflection group [Vin72]. A primitive vector $\delta \in L$ of square $-2d$, with $d \ge 1$, defines the reflection

$$s_\delta(x) = x - \frac{2(x, \delta)}{\delta^2}\delta = x + \frac{(x, \delta)}{d}\delta,$$

which preserves L precisely when $d \mid \mathrm{div}_L(\delta)$. In that case $\delta/d \in L^\vee$, and its class in $A_L = L^\vee/L$ has order d by primitivity. In particular, $d \mid \exp A_L$.

For an integer $M \ge 1$, let $W_{\le M}(L) \subseteq \mathrm{O}^+(L)$ be the group generated by the integral reflections in primitive vectors of square $-2d$ with $1 \le d \le M$. For $M = \exp A_L$ this is the full reflection group $W(L)$. Fix an open chamber $D_{\le M}$, write $A_{\le M}(L)$ for its stabilizer in $\mathrm{O}^+(L)$, and let $E_{\le M}(L)$ be the sublattice of vectors with finite $A_{\le M}(L)$-orbit. The following proposition extends Lemma 4.1.

Proposition 5.5. *Suppose $L = K \oplus P(-1)$, where K is an even hyperbolic lattice of rank at least 2, P is a positive-definite even lattice, and $t = \min\{x^2 > 0 : x \in K\}$ is the least positive square in K. Assume $t > 2M$, $\min P > 2M$, and P represents t. Then $E_{\le M}(L) = 0$.*

Proof. Choose $k \in K$ and $u \in P$ with $k^2 = u^2 = t$. Let $\delta = (x, v) \in L$ be a primitive reflective vector of square $\delta^2 = -2d$ with $1 \le d \le M$ and $d \mid (\delta, L)$. If $v = 0$, then $x \in K$ cannot be orthogonal to k, for otherwise $(k + x)^2 = t - 2d$ would lie in $(0, t)$, contradicting the minimality of t. If $v \ne 0$, then $x^2 = v^2 - 2d \ge \min P - 2M > 0$, so the hyperbolic Cauchy–Schwarz inequality gives $(k, x)^2 \ge tx^2 > 0$. Thus $(k, 0)$ lies on no mirror of $W_{\le M}(L)$ and belongs to a unique chamber $D_{\le M}$, whose positive roots are characterized by $(k, x) > 0$.

Consider the isotropic vectors $f_\pm = (k, \pm u)$. If $\delta = (x, 0)$ is a positive reflective vector, then $(f_\pm, \delta) = (k, x) > 0$. If $v \neq 0$, put $s = v^2$. Since $x^2 = s - 2d > 0$, minimality gives $s - 2d \geq t$. Applying Cauchy–Schwarz in K and in P, we obtain

$$(f_\pm, \delta) \geq \sqrt{t(s-2d)} - \sqrt{ts} = -\frac{2d\sqrt{t}}{\sqrt{s} + \sqrt{s-2d}} > -d.$$

As $(f_\pm, \delta) \in d\mathbb{Z}$, this gives $(f_\pm, \delta) \geq 0$ and hence $f_\pm \in \overline{D_{\leq M}}$.

Suppose a reflective vector δ of square $-2d$, $1 \leq d \leq M$, is perpendicular to $f_\pm$. After adding a multiple of $f_\pm$, we obtain (x, v) of the same square with $|(k, x)| \leq t/2$. If $v \neq 0$, we have $x^2 \geq t$ and $(k, x)^2 \geq t^2$, a contradiction. If $v = 0$, perpendicularity gives $(k, x) = 0$, so $(k + x)^2 = t - 2d \in (0, t)$, again a contradiction.

Thus every positive reflective vector pairs strictly positively with $f_\pm$. Each integral isometry $T_{f_\pm, z}$ with $z \in f_\pm^\perp \cap L$ fixes $f_\pm$ and permutes the primitive reflective vectors of each length and divisibility. It therefore preserves their signs and belongs to $A_{\leq M}(L)$. The fixed-space argument of Lemma 2.3 gives $E_{\leq M}(L)_\mathbb{Q} \subseteq \mathbb{Q}f_+ \cap \mathbb{Q}f_- = 0$. □

Remark 5.6. Suppose that $e \in K$ is primitive isotropic of divisibility d'. Choose $y \in K$ with $(e, y) = d'$. Since $(y + je)^2 = y^2 + 2jd'$, some translate has positive square at most $2d'$. The class of e/d' has order d' in A_K, so $t \leq 2d' \leq 2 \exp A_K$. In particular, the hypothesis $t > 2M$ requires $M < \operatorname{div}_K(e)$ for every primitive isotropic $e \in K$. For anisotropic K there is no such restriction.

In the following example Proposition 5.5 applies at $M = 2$. A mirror of square -6 separates the two cusps at $M = 3$.

Example 5.7. Let $K = \langle 6 \rangle \oplus \langle -10 \rangle$ and $P = \langle 6 \rangle$. Then $L = K \oplus P(-1)$ has $\exp A_L = \operatorname{lcm}(6, 10, 6) = 30$. The congruence $3a^2 - 5b^2 \equiv 1 \pmod 5$ excludes square 2 in K, and $3a^2 - 5b^2 \equiv 2 \pmod 3$ excludes square 4. Thus $t = 6$. Similarly $3a^2 - 5b^2 - 3c^2 \equiv -1 \pmod 3$ has no solution, so $W_2(L) = \{1\}$. Since $t = \min P = 6 > 4$, Proposition 5.5 gives $E_{\leq 2}(L) = 0$.

In the diagonal basis, $\delta = (1, 1, 0)$ has square -4 and divisibility 2. Write $R = s_\delta$. The cusps $f_s = (1, 0, s)$ for $s = \pm 1$ have no perpendicular mirrors at $M = 2$. With $z = (0, 1, 0)$, the isometries $T_s = T_{f_s, z}$ belong to $A_{\leq 2}(L)$. Their matrices and the reflection matrix are

$$R = \begin{pmatrix} 4 & -5 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_s = \begin{pmatrix} 31 & 10 & -30s \\ 6 & 1 & -6s \\ 30s & 10s & -29 \end{pmatrix}.$$

The product T_+T_- has characteristic polynomial $(X - 1)(X^2 - 3842X + 1)$ and hence is loxodromic. Its fixed line is $\mathbb{Q}(0, 3, -1)$. Each T_s is a nontrivial unipotent with fixed line $\mathbb{Q}f_s$, and every nonzero power has the same fixed line. A vector with finite orbit under both transformations must therefore lie in $\mathbb{Q}f_+ \cap \mathbb{Q}f_- = 0$. At $M = 3$, the third basis vector $v = (0, 0, 1)$ defines an integral reflection because $v^2 = -6$ and $\operatorname{div}_L(v) = 6$. The point $(1, 0, 0)$ lies on its mirror, and $(f_s, v) = -6s$ places the two cusps on opposite sides.

6. Lattices of small rank and automorphisms of K3 surfaces

In small rank, the coordinate vectors of $D_n(2)$ play the role of the Leech vectors in Section 4. We use them and certain cosets modulo 2 to construct cusps with $E(L) = 0$. For lattices of rank at most 10, these constructions give genus-one fibrations and symplectic automorphisms of complex projective K3 surfaces.

6.1. **Cusps from $D_n(2)$.** Give $D_n = \{a \in \mathbb{Z}^n : \sum_{i=1}^n a_i \in 2\mathbb{Z}\}$ its Euclidean inner product. We include $D_2 \cong A_1 \oplus A_1$ and $D_3 \cong A_3$. The following construction is the analogue of Lemma 4.3 for $D_n(2)$.

Theorem 6.1. *Let K be an even hyperbolic lattice of rank $q \geq 2$. Suppose $k \in K$ has square 2 and $k^\perp \cap K$ contains no (-2)-root. Let $P = 0$ or let P be a positive-definite even lattice with $\min P \geq 6$. For each $n \geq 2$, the lattice*

$$L = K \oplus D_n(-2) \oplus P(-1)$$

has $2n$ deficient cusps in the closure of one chamber. At each cusp the quotient root system is $(n-1)A_1$ and the root-rank deficit is $q + \operatorname{rk} P - 1$. The spaces S_f of these cusps have trivial intersection. In particular, $E(L) = 0$.

Proof. Let $e_1, \dots, e_n$ be the coordinate vectors of $\mathbb{Z}^n$. The vector k is primitive, and $2e_i \in D_n$ has square 8 in $D_n(2)$, whereas $e_i \notin D_n$. Thus the vectors

$$f_i^\epsilon = (2k, 2\epsilon e_i, 0) \in L \qquad (1 \leq i \leq n,\ \epsilon \in \{\pm 1\})$$

are primitive and isotropic. Let $\delta = (x, a, w)$ be a (-2)-root and write $y = (k, x)$. If $(a, w) \neq 0$, then $x^2 = 2\sum_j a_j^2 + w^2 - 2 \geq 2$, so $y^2 \geq 2x^2 \geq 4$. If $(a, w) = 0$, then $y \neq 0$ by the hypothesis on $k^\perp \cap K$. Hence $(k, 0, 0)$ lies in a chamber D whose positive roots have $y > 0$.

For a positive root with $(a, w) = 0$, we have $(f_i^\epsilon, \delta) = 2y > 0$. Otherwise $y \geq 2$. If $(f_i^\epsilon, \delta) = 2y - 4\epsilon a_i < 0$, then $2\epsilon a_i \geq y + 1$ by integrality. It follows that

$$2\sum_{j=1}^n a_j^2 + w^2 \geq 2a_i^2 \geq \tfrac{1}{2}(y+1)^2 > \tfrac{1}{2}y^2 + 2 \geq x^2 + 2,$$

contradicting $\delta^2 = x^2 - 2\sum a_j^2 - w^2 = -2$. Thus $f_i^\epsilon \in \overline{D}$ for all i, ϵ.

For a root perpendicular to f_i^ϵ, we have $y = 2\epsilon a_i$. Adding a multiple of f_i^ϵ gives a representative with $|y| \leq 2$. For this representative $(a, w) \neq 0$, since otherwise x would be a root in $k^\perp \cap K$. Thus $4 \geq y^2 \geq 2x^2 \geq 4$. Equality gives $x = \pm k$ and $2\sum_j a_j^2 + w^2 = 4$. Hence $w = 0$ and $\sum_j a_j^2 = 2$. If $x = -k$, adding f_i^ϵ replaces x by k. The $2(n-1)$ classes of perpendicular roots modulo $\mathbb{Z}f_i^\epsilon$ are therefore represented by

$$\delta_{ij}^\pm = (k, \epsilon e_i \pm e_j, 0) \qquad (j \neq i).$$

For each $j \neq i$, the two representatives sum to f_i^ϵ and give opposite roots in the quotient. The classes of the δ_{ij}^+ have Gram matrix $-2I_{n-1}$. Thus the quotient root system is $(n-1)A_1$, of deficit $(q + n + \operatorname{rk} P - 2) - (n-1) = q + \operatorname{rk} P - 1 > 0$. Put $V_i = \{(0, a, 0) \in L_\mathbb{Q} : a_i = 0,\ a \in (D_n)_\mathbb{Q}\}$. Then

$$S_{f_i^\epsilon} = \mathbb{Q}f_i^\epsilon \oplus V_i, \qquad S_{f_i^+} \cap S_{f_i^-} = V_i, \qquad \bigcap_{i=1}^n V_i = 0,$$

and Lemma 2.3 gives $E(L) = 0$. □

Corollary 6.2. *If K is a rationally anisotropic even hyperbolic lattice of rank $q \geq 2$, then*

$$E\big(K \oplus D_4(-2) \oplus \langle -6 \rangle\big) = 0.$$

The resulting lattices have ranks $7, 8$, and 9 according as $q = 2, 3, 4$.

Proof. The lattice $N = D_4(2) \oplus \langle 6 \rangle$ has minimum 4 and represents every even integer $t \geq 4$. Indeed, by the four-square theorem we can write $2a = \sum_{i=1}^4 a_i^2$ for each $a \geq 0$. The coordinate sum is even, so $(a_1, \dots, a_4) \in D_4$ has square $4a$ in $D_4(2)$. Adding the generator of $\langle 6 \rangle$ gives square $6 + 4a$.

Let t be the least positive square in K. If $t \geq 4$, Lemma 4.1 gives $E(K \oplus N(-1)) = 0$. If $t = 2$, take $k \in K$ with $k^2 = 2$. A perpendicular root r would give the isotropic

vector $k + r$, contrary to anisotropy. Theorem 6.1 now applies with $n = 4$ and $P = \langle 6 \rangle$. Meyer's theorem gives $q \leq 4$. □

6.2. **A scaling bound.** We write $(a, b, u)^2 = 2ab - u^2$ in $U \oplus N(-1)$.

Lemma 6.3. *Let N be a positive-definite even lattice with $\min N \geq 4$. Suppose $w \in N \smallsetminus 2N$ satisfies $w^2 \in 4\mathbb{Z}$ and $\min(w + 2N) > 8$. Then $e = (1, 0, 0)$ and $f = (w^2/4, 2, w)$ are rootless cusps in the closure of a common chamber of $U \oplus N(-1)$.*

Proof. A root $\delta = (A, B, v)$ satisfies $2AB = v^2 - 2$. If $v \neq 0$, then $v^2 \geq 4$ and A, B have the same nonzero sign. If $v = 0$, the only root with $B > 0$ is $(-1, 1, 0)$, whose pairing with $(2, 1, 0)$ is 1. Thus $(2, 1, 0)$ lies in a chamber D in which the positive roots are exactly those with $B > 0$. Since $(e, \delta) = B$, the cusp e lies in $\overline{D}$ and has no perpendicular roots. For f and any positive root we have

$$(f, \delta) = \frac{1}{B}\big(\|v - Bw/2\|^2 - 2\big) = \frac{1}{4B}\big(\|2v - Bw\|^2 - 8\big).$$

If B is odd, then $2v - Bw \in w + 2N$, so $(f, \delta) > 0$ by the bound on this coset. If $B = 2j$ is even, then $v = jw$ would give $A = jw^2/4 - 1/B \notin \mathbb{Z}$. Hence $v - jw$ is a nonzero vector in N and $\|v - jw\|^2 \geq 4$, which again gives $(f, \delta) > 0$. Finally, f is primitive because $w \notin 2N$. Thus f is a second rootless cusp in $\overline{D}$. □

Theorem 6.4. *Let P be a positive-definite even lattice of rank at least 3. For every integer $c \geq 3$, we have $E(U \oplus P(-c)) = 0$. The hypothesis $c \geq 3$ cannot be weakened to $c \geq 2$.*

Proof. The quadratic form $q(\bar{v}) = v^2/2 \bmod 2$ on $P/2P$ has a nonzero isotropic vector by Chevalley–Warning, since $\operatorname{rk} P \geq 3$. Indeed, if q took the value 1 on every nonzero vector, its polar form B would satisfy $B(a, b) = 1$ on every independent pair. Three independent vectors a, b, d would then give $1 = B(a, b + d) = 1 + 1 = 0$. Choose $w \in P \smallsetminus 2P$ lifting a nonzero isotropic vector. Every vector in $w + 2P$ has positive square divisible by 4. Thus in $N = P(c)$ we have $\min N \geq 2c$, $w^2 \in 4\mathbb{Z}$, and $\min(w + 2N) \geq 4c > 8$. Lemma 6.3 gives two rootless cusps e, f in the same chamber. As $S_e \cap S_f = \mathbb{Q}e \cap \mathbb{Q}f = 0$, Lemma 2.3 proves the assertion.

For $c = 2$, take $P = D_4$. To compute its covering radius, round the coordinates of $x \in \mathbb{R}^4$ to nearest integers and let $t \leq 1/2$ be the largest absolute rounding error. If the rounded vector has even coordinate sum, its squared distance from x is at most $4t^2 \leq 1$. Otherwise, replace an entry attaining t by the nearest integer of the opposite parity. This gives a vector of D_4 at squared distance at most $3t^2 + (1 - t)^2 = 1 - 2t(1 - 2t) \leq 1$. The point e_1 has distance 1 from D_4, so the covering radius is 1. Hence $D_4(2)$ is rootless of covering radius $\sqrt{2}$. By the covering-radius criterion of Brandhorst and Mezzedimi [BM23, Proposition 4.5 and Theorem 1.3], $A(U \oplus D_4(-2))$ fixes a cusp. In particular, $E(U \oplus D_4(-2)) \neq 0$. □

Corollary 6.5. *Let P be a positive-definite even lattice. If $\operatorname{rk} P \geq 2$, then $E(U \oplus P(-4)) = 0$. If $\operatorname{rk} P \geq 1$, then $E(U \oplus P(-c)) = 0$ for every even integer $c \geq 6$.*

Proof. For even $c \geq 6$, every norm in $N = P(c)$ is divisible by 4 and $\min N \geq 2c \geq 12 > 8$, so Lemma 6.3 applies to any $w \in P \smallsetminus 2P$. The same argument applies when $c = 4$ and $\min P \geq 4$.

Suppose $c = 4$, $\operatorname{rk} P \geq 2$, and P contains a vector p of square 2. The only vectors of square 2 in $p + 2P$ are $\pm p$. Indeed, any other such v would give two nonzero vectors $(p \pm v)/2 \in P$ whose squares sum to 2.

Take $e = (1, 0, 0)$ and $f = (2, 2, p)$ in $U \oplus P(-4)$. In the chamber of Lemma 6.3, e is rootless. The pairing formula in its proof gives $(f, \delta) > 0$ for even $B > 0$ and $(f, \delta) \geq 0$ for odd $B = 2j + 1 > 0$. Equality occurs precisely when $2v - Bp = \pm p$ in $P(4)$. Subtracting jf gives one of the roots $(-1, 1, 0)$ and $(3, 1, p)$, whose sum is

f. The quotient root system is therefore A_1. Since $1 < \operatorname{rk} P = \operatorname{rk} L - 2$, the cusp f is deficient. As $S_f \subseteq f^\perp$ and $(e, f) = 2$, we have $S_e \cap S_f = 0$. Lemma 2.3 gives $E(L) = 0$. □

For $\operatorname{rk} P = 1$ the conclusion of Corollary 6.5 at $c = 4$ can fail: for example, $A(U \oplus \langle -8 \rangle)$ fixes a cusp when $P = \langle 2 \rangle$ [BM23, Proposition 4.9].

6.3. **Realization on K3 surfaces.** By Morrison's theorem [Mor84, Corollary 2.9(i)], every even hyperbolic lattice L of rank at most 10 occurs as the Néron–Severi lattice of a complex projective K3 surface X. After composing the marking with an element of $W_2(\mathrm{NS}(X))$, we may identify the chosen chamber D with the ample cone of X.

Let $f, z \in L$ satisfy $f^2 = (f, z) = 0$. The Eichler formula (2.1) defines an integral isometry $T_{f,z}$ of $H^2(X, \mathbb{Z})$. It fixes $L^\perp$ pointwise and hence acts trivially on $H^{2,0}(X)$. For $x \in L^\vee$, both (x, f) and (x, z) are integers, so $(T_{f,z} - I)x \in L$. In particular, $T_{f,z}$ acts trivially on A_L. If $T_{f,z} \in A(L)$, it preserves the ample cone. Global Torelli [Huy16, Chapter 7, Theorem 5.3] then realizes $T_{f,z}$ as the pullback of a unique symplectic automorphism of X.

More generally, the kernel of $A(L) \to \mathrm{O}(A_L)$ extends by the identity on $L^\perp$ and preserves the ample cone. Thus the image of $\mathrm{Aut}(X)$ has finite index in $A(L)$. If $E(L) = 0$, every nonzero divisor class on X has infinite $\mathrm{Aut}(X)$-orbit, and the same holds for every curve.

A primitive nef isotropic class $f \in \mathrm{NS}(X)$ defines a genus-one fibration $X \to \mathbb{P}^1$. Its index, the least positive degree of a multisection, is $\mathrm{div}_L(f)$. By the Shioda–Tate formula, the Mordell–Weil rank of its Jacobian equals the root-rank deficit at f [Huy16, Chapter 11, §§3–4]. For instance, take $K = \langle 2 \rangle \oplus \langle -2d \rangle$ with $d \geq 2$ and $P = 0$ in Theorem 6.1. For $2 \leq n \leq 8$ the corresponding K3 surface has Picard rank $n + 2$ and carries $2n$ genus-one fibrations of index 4. Each has reducible fibers of total root type $(n-1)A_1$ and Jacobian of Mordell–Weil rank 1.

McMullen [McM02, McM11, McM16], Esnault, Oguiso and Yu [EOY16], and Yu [Yu18] constructed K3 automorphisms with small Salem numbers or with maximal Salem degree. Their constructions use lattice gluing or Mordell–Weil translations. Bhargava, Ho and Kumar computed quartic Salem polynomials for automorphisms of K3 surfaces arising from penteracts [BHK16, §7.4]. For a Salem polynomial $X^4 - aX^3 - bX^2 - aX + 1$ with a, b even, Lee realized the eighth power of its Salem root as the dynamical degree of a symplectic automorphism in Picard rank four [Lee23, Theorem 1(2)].

The Salem numbers λ_m of Theorem 6.6 also occur in two earlier constructions. Since $P_m(1) = -(8m)^2$, Reschke's torus construction and its Kummer quotient realize λ_m on a K3 surface [Res12, Theorem 1.1(2) and §6]. In Picard rank four, Hashimoto and Oda realize the Gaussian principal congruence group of level two on a K3 surface with rootless Picard lattice $U(2) \oplus \langle -4 \rangle^{\oplus 2}$ [HO26, Theorem 3.17]. Take the product of the upper and lower unipotent matrices with off-diagonal entries 2 and $2mi$, respectively. Its trace is $t = 2 + 4mi$. On Hermitian matrices the characteristic polynomial has coefficients $-|t|^2$ in degrees three and one and $t^2 + \bar{t}^2 - 2$ in degree two, which gives P_m. We realize this polynomial on every K3 surface with Picard lattice $L_m = U \oplus \langle -4m \rangle^{\oplus 2}$. Each L_m contains roots, and its determinant distinguishes it from the other lattices in the family.

Theorem 6.6. *Let $m \geq 3$ be an integer. Every complex projective K3 surface X_m with Néron–Severi lattice $L_m = U \oplus \langle -4m \rangle^{\oplus 2}$ admits a symplectic automorphism $g_m \in \mathrm{Aut}(X_m)$ whose action on L_m has characteristic polynomial*

$$P_m(X) = X^4 - (16m^2 + 4)X^3 + (6 - 32m^2)X^2 - (16m^2 + 4)X + 1. \tag{6.1}$$

The polynomial $P_m(X)$ is irreducible over $\mathbb{Q}$, its largest real root λ_m is a Salem number of degree 4, and g_m has topological entropy $\log \lambda_m$ and no periodic curves.

Proof. In the coordinates (a, b, c_1, c_2) of $L_m = U \oplus \langle -4m \rangle^{\oplus 2}$ with quadratic form $2ab - 4m(c_1^2 + c_2^2)$, take

$$e = (1, 0, 0, 0), \quad f = (m, 2, 1, 0), \quad u = (2m, 0, 1, 0), \quad v = (0, 0, 0, 1).$$

In $N = \langle 4m \rangle^{\oplus 2}$, the vector $w = (1, 0)$ has square $4m$, and $\min N = 4m \geq 12$. Hence Lemma 6.3 puts the rootless cusps e and f in the closure of one chamber D. We have $u, v \in e^\perp \cap f^\perp$, $u^2 = v^2 = -4m$, and $(u, v) = 0$. By Lemma 2.3, the Eichler transformations $T = T_{e,u}$ and $S = T_{f,v}$ preserve D. On column vectors they act by

$$T = \begin{pmatrix} 1 & 2m & 4m & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 4m^2+1 & 2m^3 & -8m^3 & 4m^2 \\ 8m & 4m^2+1 & -16m^2 & 8m \\ 4m & 2m^2 & 1-8m^2 & 4m \\ 2 & m & -4m & 1 \end{pmatrix}.$$

Direct multiplication gives $\operatorname{tr}(TS) = 16m^2 + 4$ and $\det(XI - TS) = P_m(X)$. Under the reciprocal substitution $Y = X + X^{-1}$, $X^{-2}P_m(X)$ becomes $Y^2 - (16m^2+4)Y + 4 - 32m^2$, whose roots are

$$Y_\pm = 8m^2 + 2 \pm 8m\sqrt{m^2+1}, \qquad Y_- = 2 - \frac{8m}{\sqrt{m^2+1} + m} \in (-2, 2).$$

Since $Y_+ > 2$ and $-2 < Y_- < 2$, the roots of P_m are a positive real pair $\lambda_m, \lambda_m^{-1}$ with $\lambda_m > 1$ and a conjugate pair on the unit circle. The only possible rational roots are ± 1, and $P_m(1) = -64m^2$, $P_m(-1) = 16$. In a factorization into rational quadratics the two nonreal roots would belong to the same factor. Their sum Y_- is irrational, since $m^2 + 1$ is not a square for $m \geq 1$. Hence P_m is an irreducible quartic Salem polynomial.

The product TS extends by the identity on $L_m^\perp$ and preserves the ample cone. Global Torelli gives a symplectic automorphism g_m with $g_m^*|_{L_m} = TS$. By the Gromov–Yomdin theorem [Gro03, Yom87, McM02], $h_{\mathrm{top}}(g_m) = \log \lambda_m$. A periodic curve would give a nonzero divisor class fixed by a positive power of TS. This is impossible since P_m has no root of unity. □

The class e defines an elliptic fibration with a section, and f defines a genus-one fibration of index $\gcd(m, 2)$. Both Jacobians have Mordell–Weil rank 2. The lattices L_m are pairwise non-isometric since $|\det L_m| = 16m^2$. For each $m \geq 3$, the period space of L_m-polarized K3 surfaces has dimension $20 - \operatorname{rk} L_m = 16$. At a very general point the Néron–Severi lattice equals L_m, so the corresponding surface admits a symplectic automorphism of Salem degree 4.

Proposition 6.7. *Every complex projective K3 surface X with Néron–Severi lattice*

$$L = \langle 2 \rangle \oplus \langle -4 \rangle \oplus D_8(-2)$$

admits a symplectic automorphism $g \in \operatorname{Aut}(X)$ of maximal Salem degree 10 with no periodic curves.

Proof. Write vectors as (a, b, t) with $a, b \in \mathbb{Z}$ and $t = (t_0, \ldots, t_7) \in D_8$, indexing the D_8 coordinates by $0, \ldots, 7$. Apply Theorem 6.1 with $K = \langle 2 \rangle \oplus \langle -4 \rangle$, $k = (1, 0)$, $n = 8$, and $P = 0$. The 16 cusps $f_i^\epsilon = (2, 0, 2\epsilon e_i)$ for $0 \leq i \leq 7$ and $\epsilon = \pm 1$ belong to the closure of one chamber D. The vector $z = (0, 1, 0)$ is perpendicular to every $S_{f_i^\epsilon}$. Thus $T_i^\epsilon = T_{f_i^\epsilon, z}$ belongs to $A(L)$. It fixes all coordinates except (a, b, t_i), on which it acts by

$$\begin{pmatrix} 17 & 8 & -16\epsilon \\ 4 & 1 & -4\epsilon \\ 16\epsilon & 8\epsilon & -15 \end{pmatrix}.$$

Since t_i changes by an even integer, T_i^ϵ preserves the parity condition on D_8. The Eichler formula gives $(T_i^\epsilon - I)L^\vee \subseteq L$, so T_i^ϵ acts trivially on A_L. Consider the product

$$g = T_0^+ T_0^- T_1^+ T_2^+ \cdots T_7^+ \in A(L).$$

Multiplying the matrices gives a reciprocal characteristic polynomial $P(X) = \sum_{j=0}^{10} a_j X^j$ with $a_{10-j} = a_j$ and

$$(a_0, a_1, a_2) = (1,\ -531713972298,\ 1239885808173),$$
$$(a_3, a_4, a_5) = (-2475101071224,\ 3369426357970,\ -3754750059132).$$

Let $\bar{P}$ be the reduction of P modulo 13, and let R_j be the remainder of X^{13^j} modulo $\bar{P}$. Computing successive 13th powers modulo $\bar{P}$, we obtain the coefficient vectors in ascending order of degree

$$R_2 = (6, 0, 10, 8, 0, 5, 2, 3, 2, 10),$$
$$R_5 = (2, 1, 2, 3, 7, 3, 2, 1, 2, 12), \qquad R_{10} = X,$$

where $\gcd(\bar{P}, R_2 - X) = \gcd(\bar{P}, R_5 - X) = 1$ in $\mathbb{F}_{13}[X]$. Since the quotients of 10 by its prime divisors are 2 and 5, Rabin's criterion gives irreducibility over $\mathbb{F}_{13}$ and hence over $\mathbb{Q}$. The trace is $531713972298 > 10$. A Lorentzian isometry preserving the positive cone has at most one pair of eigenvalues off the unit circle, both real and positive [EOY16, Proposition 3.1]. Hence P is a Salem polynomial of degree 10. As in the proof of Theorem 6.6, global Torelli gives a symplectic automorphism, and irreducibility excludes periodic curves. The matrix and polynomial calculations are also recorded in Appendix A. □

Theorem 6.6 and Proposition 6.7 establish Theorem E.

Remark 6.8. Let g be a symplectic automorphism of a K3 surface S as in Theorem 6.6 or Proposition 6.7. Write P for its characteristic polynomial on $\mathrm{NS}(S)$, $\lambda > 1$ for its Salem root, and $\rho \in \{4, 10\}$ for the Picard number. For every $n \geq 2$, the induced automorphism $g^{[n]}$ of the Hilbert scheme $S^{[n]}$ is regular and symplectic. Under the decomposition $H^2(S^{[n]}, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$, with $\delta^2 = -2(n-1)$, its pullback is $g^* \oplus \mathrm{id}$ [Bea83, Boi12]. Its characteristic polynomial on $\mathrm{NS}(S^{[n]})$ is $(X-1)P(X)$. The Salem degree $\rho = 2\lfloor(\rho+1)/2\rfloor$ is maximal in Picard rank $\rho + 1$. By Oguiso's theorem [Ogu09, Theorem 1.1], the dynamical degrees are $d_k(g^{[n]}) = \lambda^k$ for $0 \leq k \leq n$, and the topological entropy is $n \log \lambda$.

An Eichler transformation involving δ need not be induced by a birational map of $S^{[n]}$. Let f be one of the primitive nef isotropic classes above. Take the pullback H of an ample class on S, so $(H, f) > 0$. Let $0 < \varepsilon < (H, f)$ be small enough that $x = H - \varepsilon\delta$ is ample on $S^{[n]}$. Then $(T_{f,\delta} x, \delta) = \delta^2((H, f) - \varepsilon) < 0$. The exceptional divisor 2δ of the Hilbert–Chow morphism is prime. By the description of the closed movable cone in [Mar13, §1.5], every class in this cone pairs nonnegatively with 2δ. Hence $T_{f,\delta} x$ lies outside the closed movable cone.

7. Chamber groups and congruence subgroups

Let L be an even hyperbolic lattice of rank $r \geq 3$, and fix a chamber D for $W_2(L)$. Suppose that $\overline{D}$ contains a finite set F of cusps such that

$$\dim S_f < r - 1 \quad \text{for every } f \in F, \qquad \bigcap_{f \in F} S_f = 0. \tag{7.1}$$

Theorem B gives such a set in every rank $r \geq 27$. Section 6 gives examples in smaller ranks. In the K3 setting, Yu constructed an irreducible action on the span of the classes of elliptic fibrations with infinite stabilizer [Yu18, Theorem 5.1]. Under (7.1) we choose at most $r - 1$ Eichler transformations and obtain irreducibility even

after replacing each generator by a nonzero power. Recall that a finitely generated subgroup of $\mathrm{SO}(L)$ is *thin* if it is Zariski dense in $\mathrm{SO}(L_{\mathbb{R}})$ and has infinite index in $\mathrm{SO}(L)$.

Proposition 7.1. *Assume* (7.1). *There are $k \leq r-1$ integral Eichler transformations $T_1, \dots, T_k \in A(L) \cap \mathrm{SO}^+(L)$ with the following property. For any nonzero integers $a_1, \dots, a_k$, the group $\langle T_1^{a_1}, \dots, T_k^{a_k} \rangle$ acts irreducibly on $L_{\mathbb{R}}$ and is Zariski dense in $\mathrm{SO}(L_{\mathbb{R}})$. If L contains a (-2)-root, this group is thin in $\mathrm{SO}(L)$.*

Proof. For each $f \in F$, choose $z_1, \dots, z_d \in L \cap S_f^{\perp}$ whose images form a basis of $S_f^{\perp}/\mathbb{Q}f$. Each z_j has negative square. By Lemma 2.3, the transformations T_{f,z_j} belong to $A(L) \cap \mathrm{SO}^+(L)$. Since f and z_j are linearly independent, the fixed space of T_{f,z_j} is $f^{\perp} \cap z_j^{\perp}$ and has codimension two. For a given f, these fixed spaces intersect in $S_f \otimes_{\mathbb{Q}} \mathbb{R}$. Their intersection over all $f \in F$ is therefore zero by (7.1).

Choose one transformation T_1. Its fixed space has dimension $r-2$. As long as the common fixed space is nonzero, we can choose a further transformation that decreases its dimension. After at most $r-1$ choices the common fixed space is zero. Taking a nonzero power preserves each fixed space, since $T_{f,z}^a = T_{f,az}$.

Write $H = \langle T_1^{a_1}, \dots, T_k^{a_k} \rangle$, and suppose that H preserves a nonzero proper subspace $V \subseteq L_{\mathbb{R}}$. If V is degenerate, its radical is an invariant isotropic line. Each generator is unipotent and fixes this line pointwise. If V is nondegenerate, either V or $V^{\perp}$ is negative definite. Every unipotent isometry of a definite space is the identity, so again the generators have a nonzero common fixed vector. Both cases contradict our choice of the T_i. Hence H acts irreducibly on $L_{\mathbb{R}}$. By Benoist and de la Harpe [BdlH04, Proposition 1], it is Zariski dense in $\mathrm{SO}(L_{\mathbb{R}})$.

Suppose that L contains a (-2)-root δ. If its H-orbit were finite, some positive power of each $T_i^{a_i}$ would fix δ, contradicting the common fixed space calculation. The orbit thus gives infinitely many reflection hyperplanes, so $W_2(L)$ is infinite. Since $\mathrm{O}^+(L) = W_2(L) \rtimes A(L)$ and $[\mathrm{O}^+(L) : \mathrm{SO}^+(L)] \leq 2$, the group $A(L) \cap \mathrm{SO}^+(L)$ has infinite index in $\mathrm{SO}(L)$. The same holds for H, which is therefore thin. The larger group $A(L) \cap \mathrm{SO}^+(L)$ is also thin, since $A(L)$ is a quotient of the finitely generated arithmetic group $\mathrm{O}^+(L)$ and $A(L) \cap \mathrm{SO}^+(L)$ has index at most 2 in $A(L)$. □

Corollary 7.2. *Let L be an even hyperbolic lattice of rank $r \geq 27$ containing a (-2)-root. Then no chamber D of L admits a Weyl vector $\rho \in L_{\mathbb{R}}$ satisfying $(\rho, \delta) = 1$ for every inward-pointing simple root δ of D.*

Proof. By Theorem B and Proposition 7.1, a subgroup $\Gamma \subseteq A(L) \cap \mathrm{SO}^+(L)$ acts irreducibly on $L_{\mathbb{R}}$ with zero fixed space. The set $\Delta(D)$ of inward-pointing simple roots is nonempty because L contains a (-2)-root. Its span is a nonzero Γ-invariant subspace, hence equals $L_{\mathbb{R}}$. A solution of $(\rho, \delta) = 1$ for all $\delta \in \Delta(D)$ would therefore be unique. Since Γ permutes $\Delta(D)$, it would fix this nonzero vector. □

Proposition 7.1 and Corollary 7.2 complete the proof of Corollary C.

Remark 7.3. Let $L' \supseteq L$ be an even overlattice, let D' be a chamber of L', and let D be the unique chamber of L containing D'. Suppose that an Eichler transformation $T_{f',z'}$ with $f', z' \in L'$ preserves D', as do the cusp generators of Lemma 2.3. Choose $m \geq 1$ with $mL' \subseteq L$. Then $mf', mz' \in L$, so

$$T_{f',z'}^{m^2} = T_{mf',mz'} \in \mathrm{O}^+(L).$$

This power preserves D'. It therefore preserves D, and hence belongs to $A(L)$.

Remark 7.4. Let $\Gamma = \langle T_1, \dots, T_k \rangle$, let $S = \{T_1^{\pm 1}, \dots, T_k^{\pm 1}\}$, and denote reduction modulo q by π_q. By Proposition 7.1, the Zariski closure of Γ over $\mathbb{Q}$ is the special

orthogonal group of $L_{\mathbb{Q}}$. Since $r \geq 3$, this is a connected semisimple algebraic group and hence is perfect. By Salehi Golsefidy and Varjú [SGV12, Theorem 1], the Cayley graphs of $\pi_q(\Gamma)$ with generating sets $\pi_q(S)$ form a family of expanders for squarefree q coprime to a fixed integer q_0.

The associated random walks mix exponentially on every Γ-orbit in L/qL, with a uniform rate. Indeed, the same theorem applies to the group generated by the T_i^2, by Proposition 7.1. We may enlarge q_0 to include its exceptional primes and 2. Since $T_i^q \equiv I \pmod q$, each $\pi_q(T_i)$ has odd order, so the squares generate $\pi_q(\Gamma)$. The two-step walk assigns each $\pi_q(T_i^{\pm 2})$ probability at least $|S|^{-2}$. Thus its Dirichlet form dominates a fixed positive multiple of that of the walk on the squares, which has a uniform spectral gap. This gives exponential decay on mean-zero functions on $\pi_q(\Gamma)$, and hence on each Γ-orbit in L/qL by pulling back functions along the orbit map.

7.1. **Minimal entropy in maximal Salem degree.** Fix the group $\Gamma = \langle T_1, \dots, T_k \rangle$ of Proposition 7.1, and fix a $\mathbb{Z}$-basis of L to write elements of $\mathrm{SO}(L)$ as $r \times r$ integer matrices. For $N \geq 1$, its principal congruence subgroup of level N is

$$\Gamma(N) = \{g \in \Gamma \mid g \equiv I \pmod N\},$$

where $g \equiv I \pmod N$ means $(g - I)L \subseteq NL$. Write

$$n = 2\lfloor r/2 \rfloor, \qquad \varepsilon = r - n \in \{0, 1\}.$$

For $g \in \mathrm{SO}^+(L)$, the characteristic polynomial $\chi_g(X) = \det(XI - g)$ is a product of cyclotomic polynomials and at most one irreducible noncyclotomic reciprocal factor [EOY16, Proposition 3.1]. We say that g has *maximal Salem degree* if

$$\chi_g(X) = (X - 1)^{\varepsilon} P_g(X),$$

where P_g is an irreducible Salem polynomial of degree n for $r \geq 4$. In rank three we allow an irreducible reciprocal quadratic polynomial with real roots $\lambda_g > 1 > \lambda_g^{-1}$. For any loxodromic g, we write $\lambda_g = \rho(g) > 1$ for its spectral radius.

Esnault, Oguiso and Yu characterize maximal Salem degree by the nonvanishing of a regular function on $\mathrm{SO}(L_{\mathbb{R}})$ [EOY16, §4]. The escape lemma of Eskin, Mozes and Oh [EMO05, Proposition 3.2] then bounds the length of a word of maximal Salem degree. We give an explicit bound depending only on r.

Lemma 7.5. *Let S' be a finite symmetric generating set of a subgroup of $\mathrm{SO}^+(L)$ that is Zariski dense in $\mathrm{SO}(L_{\mathbb{R}})$. Set*

$$\mathcal{C}_n = \{j \geq 1 \mid \varphi(j) \leq n\}, \qquad D_r = r \sum_{j \in \mathcal{C}_n} \varphi(j), \qquad B_r = \binom{r^2 + D_r}{D_r} - 1.$$

Some word in S' of length at most B_r has maximal Salem degree.

Proof. For $g \in \mathrm{SO}(L_{\mathbb{R}})$, the spectrum is invariant under inversion. In odd rank 1 is an eigenvalue since $\det(g) = 1$. Thus $\beta_g(X) = \chi_g(X)/(X-1)^{\varepsilon}$ has degree n, and its coefficients are polynomials of degree at most r in the entries of g. The inequality $\varphi(j) \geq \sqrt{j/2}$ shows that $\mathcal{C}_n$ is finite. Write Φ_j for the jth cyclotomic polynomial. The product

$$F(g) = \prod_{j \in \mathcal{C}_n} \mathrm{Res}_X(\Phi_j, \beta_g)$$

has degree at most D_r in the entries of g. To see that F does not vanish identically on $\mathrm{SO}(L_{\mathbb{R}})$, choose a real isometry with eigenvalues λ, λ^{-1}, $\lambda > 1$, on a hyperbolic plane. On its orthogonal complement take irrational rotations (by angles in $\mathbb{R} \smallsetminus \pi\mathbb{Q}$) on $(n-2)/2$ two-planes and the identity on the remaining ε-dimensional space. Then $F(g) \neq 0$. For integral $g \in \mathrm{SO}^+(L)$, the condition $F(g) \neq 0$ excludes every

cyclotomic factor of β_g. The factorization above then gives $\beta_g = P_g$ and g has maximal Salem degree.

Let $v(g)$ be the vector of the $B_r + 1$ monomials of degree at most D_r in the entries of g. Let V_j be the span of $v(g)$ for words of length at most j in S'. Right multiplication by $s \in S'$ induces a linear map on these vectors. Hence $V_j = V_{j+1}$ implies that V_j contains $v(g)$ for every $g \in \langle S' \rangle$. Since $\dim V_0 = 1$, the chain stabilizes by $j = B_r$. Now $F(g)$ is a linear functional of $v(g)$. If it vanished on all words of length at most B_r, it would vanish on $\langle S' \rangle$, contrary to Zariski density. □

Remark 7.6. The bound B_r grows superexponentially with r. For fixed r, Theorem 7.7 uses it to bound the minimum logarithmic spectral radius in maximal Salem degree by a linear function of $\log N$.

Theorem 7.7. *Let Γ be the group of Proposition 7.1, and let $N \geq 1$. For a loxodromic element $g \in \Gamma(N)$, let P be the minimal polynomial of λ_g and put $d = \deg P$. Then*

$$N^d \mid P(1), \qquad \log \lambda_g \geq d \log N - (d-2) \log 2. \tag{7.2}$$

The minimum

$$h_N = \min\{\log \lambda_g \mid g \in \Gamma(N),\ g \text{ has maximal Salem degree}\}$$

exists for every $N \geq 1$. There is a constant $C \geq 1$ depending only on $T_1, \dots, T_k$ such that, for every $N \geq 2$,

$$n \log N - (n-2) \log 2 \ \leq\ h_N \ \leq\ 2B_r \log N + B_r \log C. \tag{7.3}$$

Proof. Write $g = I + NH$ with $H \in \mathrm{End}(L)$. The eigenvalue $(\lambda_g - 1)/N$ of H is an algebraic integer. Its norm from $\mathbb{Q}(\lambda_g)$ to $\mathbb{Q}$ is the nonzero integer $(-1)^d P(1)/N^d$. This proves the divisibility. The roots of P are $\lambda_g, \lambda_g^{-1}$ and $d-2$ conjugates on the unit circle. It follows that

$$N^d \leq |P(1)| \leq 2^{d-2}(\lambda_g - 1)(1 - \lambda_g^{-1}) = 2^{d-2}(\lambda_g + \lambda_g^{-1} - 2) < 2^{d-2}\lambda_g.$$

Taking logarithms gives the inequality in (7.2). For $d = n$ this is the lower bound in (7.3).

For the upper bound, the Eichler formula (2.1) gives $T_i^{\pm N} = T_{f_i, \pm N z_i} = I \pm NU_i + N^2 V_i$ for fixed integral matrices U_i, V_i. In particular, $T_i^{\pm N} \in \Gamma(N)$. With $C = 1 + \max_i(\|U_i\|_\infty + \|V_i\|_\infty)$, their ℓ_∞ operator norms are at most CN^2. By Proposition 7.1, the group generated by the T_i^N is Zariski dense in $\mathrm{SO}(L_\mathbb{R})$. Lemma 7.5 gives a word g of maximal Salem degree and length at most B_r in the $T_i^{\pm N}$. Thus $\lambda_g \leq \|g\|_\infty \leq (CN^2)^{B_r}$. For bounded λ_g, the coefficients of the monic integral polynomial P_g are bounded, so only finitely many λ_g occur. This proves existence of the minimum and the upper bound. □

Corollary 7.8. *Let X be a complex projective K3 surface with Néron–Severi lattice $L = \mathrm{NS}(X)$ of rank $r \geq 3$. Suppose its ample chamber satisfies* (7.1)*. There is a finitely generated group of symplectic automorphisms of X with the following property. For each $N \geq 1$, the group contains an element that acts trivially on $H^2(X, \mathbb{Z}/N\mathbb{Z})$ and has maximal Salem degree on L. For $N \geq 2$, the minimum topological entropy of such elements in the group satisfies* (7.3)*.*

Proof. By Section 6, each generator $T_i = T_{f_i, z_i}$ extends by the identity on $L^\perp$ to an integral Hodge isometry of $H^2(X, \mathbb{Z})$ induced by a symplectic automorphism of X. For $x \in H^2(X, \mathbb{Z})$ we have

$$(T_i^N - I)x = N(x, f_i)z_i - N(x, z_i)f_i - \tfrac{1}{2}N^2 z_i^2 (x, f_i) f_i \in NH^2(X, \mathbb{Z}).$$

Thus every word in the $T_i^{\pm N}$ acts trivially on $H^2(X, \mathbb{Z}/N\mathbb{Z})$. In particular, the word used for the upper bound in Theorem 7.7 has this property. Conversely, L

is primitive in $H^2(X,\mathbb{Z})$, so $L \cap NH^2(X,\mathbb{Z}) = NL$. An element acting trivially on cohomology modulo N therefore acts on L through $\Gamma(N)$. If it has maximal Salem degree, Theorem 7.7 gives the lower bound. Its topological entropy is $\log \lambda_g$ by the Gromov–Yomdin theorem [Gro03, Yom87]. □

The rank-four family of Theorem 6.6 attains the coefficient n in the lower bound of (7.3).

Proposition 7.9. *Fix an integer $m \geq 3$ and let $\Gamma = \langle T, S\rangle$ be the group of Theorem 6.6 for $L_m = U \oplus \langle -4m\rangle^{\oplus 2}$. For each integer $N \geq 1$, the element $g_N = T^N S^N \in \Gamma(N)$ has characteristic polynomial*

$$\begin{aligned} P_{mN^2}(X) = X^4 - (16m^2N^4+4)X^3 + (6 - 32m^2N^4)X^2 \\ - (16m^2N^4+4)X + 1, \end{aligned} \tag{7.4}$$

which is an irreducible quartic Salem polynomial. For this group,

$$h_N = 4\log N + O_m(1) = n \log N + O_m(1).$$

Proof. We have $T^N = T_{e,Nu}$ and $S^N = T_{f,Nv}$. In the rational basis (e, f, Nu, Nv), the first two vectors are isotropic with $(e,f) = 2$. The last two are orthogonal to the first two and to each other, and both have square $-4mN^2$. This is the same Gram matrix as that of (e,f,u,v) in Theorem 6.6, with m replaced by mN^2. Since the matrices of $T_{e,Nu}$ and $T_{f,Nv}$ in this rational basis depend only on the Gram matrix, substituting mN^2 for m in (6.1) gives (7.4). Since $(mN^2)^2 + 1$ is not a square, the proof of that theorem gives irreducibility and shows that the largest root is $\lambda_{g_N} = 16m^2N^4 + O(1)$. Thus $\log \lambda_{g_N} = 4\log N + \log(16m^2) + o(1)$. The lower bound $h_N \geq 4\log N - 2\log 2$ now gives the assertion. □

The finite calculation in Appendix A also gives (7.4). The entries of $T^N S^N$ have degree at most four in N, so the characteristic coefficients have degree at most sixteen. Evaluation at $N = 0, \dots, 16$ over $\mathbb{Z}[m]$ determines all of them.

7.2. **Ordinary systoles.** Give $\mathbb{H}^{r-1}$ the metric of sectional curvature -1. The translation length of a loxodromic element g is $\log \lambda_g$. Thus the length ℓ_N of a shortest closed geodesic in $\mathbb{H}^{r-1}/\Gamma(N)$ is

$$\ell_N = \min\{\log \lambda_g \mid g \in \Gamma(N) \text{ is loxodromic}\}.$$

We call a loxodromic element $g \in \Gamma(N)$ attaining $\log \lambda_g = \ell_N$ a *systolic element*. The set is nonempty by Theorem 7.7. As in its proof, a bound on λ_g leaves only finitely many possibilities for its monic integral minimal polynomial, so the minimum is attained.

Choose two generators $T_{f,z}, T_{h,w}$ with distinct cusp rays. Such a pair exists because a common cusp would give a nonzero vector fixed by all the generators. We have $(f,h) > 0$. By the Eichler formula (2.1),

$$\operatorname{tr}(T_{f,Nz}T_{h,Nw}) = r + 2\big((f,w)(h,z) - (f,h)(z,w)\big)N^2 + \tfrac{1}{4}(f,h)^2 z^2 w^2 N^4. \tag{7.5}$$

Indeed, the linear term of each transformation is skew-adjoint, and the quadratic term is a multiple of $f \otimes f^\flat$ or $h \otimes h^\flat$, where $v^\flat = (v, \cdot\,)$. Hence the cubic terms have trace zero. Since $z^2, w^2 < 0$, the coefficient of N^4 is positive. For large N the trace exceeds r, so $T_{f,Nz}T_{h,Nw} \in \Gamma(N)$ is loxodromic. Its expanding eigenvalue differs from the trace by a bounded amount, since each of the other eigenvalues has absolute value at most one. Thus

$$\ell_N \leq 4\log N + O_\Gamma(1). \tag{7.6}$$

Katz, Schaps and Vishne obtained logarithmic systole bounds for arithmetic surfaces from trace estimates in quaternion orders [KSV07]. For the arithmetic

groups considered in his thesis, Murillo obtained a lower bound of $2\log \mathrm{N}(I) - O(1)$ in the orthogonal representation [Mur17, §6.3, (6.9)]. For spin congruence subgroups he proved a lower bound of $4\log \mathrm{N}(I) - O(1)$ using the Clifford algebra [Mur19, §§2.4 and 4 of the arXiv version]. We use the action on L/NL throughout. In rank three both constants occur for the same group Γ, as the level varies.

Proposition 7.10. *Suppose $r = 3$. Then $h_N = \ell_N$ and*

$$\liminf_{N\to\infty} \frac{h_N}{\log N} = 2, \qquad \limsup_{N\to\infty} \frac{h_N}{\log N} = 4.$$

The same formulas hold when N runs over levels coprime to $2|\det L|$. For odd prime powers N one has $h_N = 4\log N + O_\Gamma(1)$.

Proof. In rank three a loxodromic element has eigenvalues $1, \lambda, \lambda^{-1}$ with $\lambda > 1$. The last two are conjugate quadratic units, since a rational algebraic unit is ± 1. Thus every loxodromic element has maximal Salem degree and $h_N = \ell_N$. The bounds (7.2) and (7.6) give

$$2\log N \le h_N \le 4\log N + O_\Gamma(1).$$

For the lower limit we use a recurrence that Bonechi and De Biévre used to compute periods of integral hyperbolic torus automorphisms modulo selected levels [BD00, Proposition 3.4 and (3.8)]. Fix a loxodromic $g \in \Gamma$ with expanding eigenvalue λ. Put $t = \mathrm{tr}(g) - 1 = \lambda + \lambda^{-1} \in \mathbb{Z}$. Define $u_0 = 0$, $u_1 = 1$ and $u_{j+1} = tu_j - u_{j-1}$. Then

$$u_j = \frac{\lambda^j - \lambda^{-j}}{\lambda - \lambda^{-1}}, \qquad g^{2j} - I = u_j g^j (g - g^{-1}).$$

The second identity holds on each of the three eigenspaces of g. It gives $g^{2j} \in \Gamma(u_j)$ for $j \ge 1$, and hence

$$h_{u_j} \le 2j\log\lambda = 2\log u_j + O_g(1).$$

Since $u_j \to \infty$, these levels realize the lower limit. The same holds with the coprimality restriction. Indeed, put $B = 2|\det L|$. Since the transition matrix $\left(\begin{smallmatrix} t & -1 \\ 1 & 0 \end{smallmatrix}\right)$ has determinant 1, the recurrence permutes $(\mathbb{Z}/B\mathbb{Z})^2$, so (u_j, u_{j-1}) returns to $(1, 0)$ infinitely often modulo B. Along this subsequence u_j is coprime to B.

For the upper limit we use the split ternary model. The space $L_\mathbb{Q}$ contains an isotropic line, so it splits as $U_\mathbb{Q} \oplus \langle c\rangle_\mathbb{Q}$ for some $c \in \mathbb{Q}^\times$; rescaling the form (which does not change $\mathrm{SO}(L_\mathbb{Q})$) identifies it with the space $\mathfrak{sl}_2(\mathbb{Q})$ of traceless 2×2 matrices equipped with det. Conjugation $M \mapsto AMA^{-1}$ for $A \in \mathrm{GL}_2(\mathbb{Q})$ preserves det and has determinant 1 on $\mathfrak{sl}_2(\mathbb{Q})$ (as is seen over $\overline{\mathbb{Q}}$ after scaling A into $\mathrm{SL}_2(\overline{\mathbb{Q}})$), with kernel the scalar matrices. This yields an injective homomorphism $\mathrm{Ad}\colon \mathrm{PGL}_2(\mathbb{Q}) \to \mathrm{SO}(L_\mathbb{Q})$. To see that Ad is surjective, identify the set of isotropic lines in $\mathfrak{sl}_2(\mathbb{Q})$ with $\mathbb{P}^1(\mathbb{Q})$ by sending a line spanned by a nonzero nilpotent matrix to its kernel; conjugation acts by the standard action of $\mathrm{PGL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$, which is 3-transitive. Given any $g \in \mathrm{SO}(L_\mathbb{Q})$, after multiplying by an element of $\mathrm{Ad}(\mathrm{PGL}_2(\mathbb{Q}))$ we may assume g fixes three distinct isotropic lines $\mathbb{Q}v_1, \mathbb{Q}v_2, \mathbb{Q}v_3$. In a nondegenerate ternary space any two distinct isotropic lines are nonorthogonal, so v_1, v_2, v_3 form a basis with $(v_i, v_j) \ne 0$ for $i \ne j$. Writing $gv_i = \lambda_i v_i$, orthogonality gives $\lambda_i\lambda_j = 1$ for $i \ne j$ and $\det g = \lambda_1\lambda_2\lambda_3 = 1$, whence $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $g = I$. Thus Ad identifies $\mathrm{PGL}_2(\mathbb{Q})$ with $\mathrm{SO}(L_\mathbb{Q})$.

Each Eichler generator has a lift to $\mathrm{SL}_2(\mathbb{Q})$. Indeed, if a, b are the eigenvalues of a rational matrix representing its projective class, conjugation on traceless matrices has eigenvalues $1, a/b, b/a$. The Eichler transformation is unipotent, so $a = b$. Their common value is half the trace of the representative, hence is rational and nonzero. Dividing the representative by this value gives a rational lift of determinant one.

Thus every $g \in \Gamma$ lifts to some $A \in \mathrm{SL}_2(\mathbb{Q})$. Put $s = \operatorname{tr} A$. The trace of conjugation on traceless matrices is $s^2 - 1 = \operatorname{tr}(g)$. Hence $s^2 \in \mathbb{Z}$, and rationality gives $s \in \mathbb{Z}$.

Take $N = p^a$ with p an odd prime and $a \geq 1$. For loxodromic $g \in \Gamma(N)$ the quadratic factor is $P(X) = X^2 - (s^2 - 2)X + 1$ with $|s| > 2$. By (7.2),

$$N^2 \mid P(1) = 4 - s^2.$$

The two factors $s - 2$ and $s + 2$ are nonzero, and their greatest common divisor divides 4. Since p is odd, one factor is divisible by N^2. Thus $|s| \geq N^2 - 2$. Since $N \geq 3$ and $\lambda^{-1} < 1$, substituting into $\lambda + \lambda^{-1} = s^2 - 2$ gives

$$\lambda > s^2 - 3 \geq (N^2 - 2)^2 - 3 = N^4 - 4N^2 + 1 > \tfrac{1}{2}N^4,$$

whence $\log \lambda > 4 \log N - \log 2$. Taking the minimum and using (7.6) gives the asserted estimate along odd prime powers. The powers of any odd prime $p \nmid B$ give the same upper limit among levels coprime to B. □

Dória, Freire and Murillo constructed congruence covers of compact arithmetic hyperbolic manifolds in which a totally geodesic surface contains a systole [DFM24, Theorem C]. For the groups considered here we can bound the degree of every systolic element at a given large level.

Corollary 7.11. *Let Γ be as in Proposition 7.1. For all sufficiently large N, the expanding eigenvalue of every systolic element in $\Gamma(N)$ has degree two or four. In particular, for $r \geq 6$ no systolic element at such a level has maximal Salem degree.*

Proof. Let $g \in \Gamma(N)$ be systolic, let P be the minimal polynomial of λ_g, and put $d = \deg P = [\mathbb{Q}(\lambda_g) : \mathbb{Q}]$. Since P divides χ_g and $\chi_g(0) = \pm 1$, we have $P(0) = \pm 1$, so the moduli of the roots of P multiply to 1. Because λ_g and λ_g^{-1} are the only roots of χ_g off the unit circle, λ_g^{-1} is also a root of P, and the remaining $d - 2$ roots form nonreal complex conjugate pairs on the unit circle. Hence d is even and $2 \leq d \leq r$. For $d \geq 6$, inequality (7.2) gives

$$\ell_N = \log \lambda_g \geq 6 \log N - (r - 2) \log 2.$$

For large N this exceeds the upper bound (7.6). □

Appendix A. Finite computations

A.1. **Roots perpendicular to a cusp.** Let $f = (a, b, z) \in U \oplus P(-1)$ be integral and isotropic, with P positive definite and $b \geq 2$. To compute S_f, consider a (-2)-root $\delta = (A, B, u)$ perpendicular to f. Adding a multiple of f preserves its square and arranges $0 \leq B < b$. For $0 < B < b$, equation (3.7) gives the conditions

$$w = bu - Bz \in -Bz + bP, \qquad w^2 = 2b^2, \qquad u = \frac{w + Bz}{b}, \qquad A = \frac{u^2 - 2}{2B} \in \mathbb{Z}.$$

For $B = 0$, take the roots $u \in P$ with $u^2 = 2$ and $A = (z, u)/b \in \mathbb{Z}$. Since P is positive definite, each of these sets is finite. The resulting roots together with f span S_f.

One can also test membership in the standard closed chamber at $e_0 = (1, 0, 0)$. A root of height $B > 0$ with $(f, \delta) < 0$ satisfies $(f, \delta) \leq -1$. Equation (3.7) then gives

$$1 \leq B \leq b - 1, \qquad (bu - Bz)^2 \leq 2b(b - B),$$

where $B = b$ is excluded because $bu - bz = 0$ would give $A = a - 1/b \notin \mathbb{Z}$. Thus f lies in the chamber closure if and only if the height-zero inequalities hold and none of the vectors $w \in -Bz + bP$ within these bounds gives an integral value of $A = (u^2 - 2)/(2B)$, where $u = (w + Bz)/b$. In our applications we use the sufficient criterion of Lemma 3.3. It suffices that the projection of $-z$ onto $(P_{\mathrm{rt}})_{\mathbb{R}}$ satisfy the alcove inequalities (3.3) and that $\min(jz + bP) \geq 2b^2$ for $1 \leq j < b$.

A.2. **Weights, coordinate cusps and K3 automorphisms.** The following computations use integer and rational arithmetic. The three programs in `anc/` require only the Python 3 standard library.

For Table 1, `anc/verify_rank27.py` reads the 79 weights in `anc/weight_certificates.json` for the 15 systems $A_1, \dots, A_8$, $D_4, \dots, D_8$, E_6, and E_7. For each weight it checks the integrality of $C^{-1}c$, the highest-root inequality $m^{\mathsf{T}}c \le b$, and that $w^2 = c^{\mathsf{T}}C^{-1}c$ is one of the allowed squares at denominator b in (3.9). When $(b, w^2) = (2, 8)$, it also checks that $C^{-1}c$ has an odd entry, giving an independent direct check that $w/2 \notin Q$. For each system R it computes the nonzero determinant of the rk R selected weights.

The same program constructs the extended binary Golay code $\mathcal{G}_{24}$, checks the octad polynomial factorization, and verifies the Leech vectors of squares $4, 6, 8, 10, 12, 16, 18, 20$ used in Section 3. The nine vectors of square 16 in Lemma 3.5 have Gram matrix $18I_9 - 2J_9$ of rank 8. The lemma proves that they exhaust the shortest vectors in $v_{16} + 3\Lambda$.

For Lemma 4.3, the program also runs `anc/verify_coordinate_cusps.py` and compares the result with `anc/coordinate_cusp_certificates.json`. Each of the 24 vectors $u_i = 8e_i/\sqrt{8}$ belongs to Λ and has square 8, so the 48 vectors $f_i^{\pm} = (2k, \pm u_i)$ are isotropic. For each f_i^{ϵ}, the program checks that all 46 representatives $\delta_{ij}^{\pm}$ belong to L, have square -2, and are perpendicular to f_i^{ϵ}. It checks the pair sums $\delta_{ij}^{+} + \delta_{ij}^{-} = f_i^{\epsilon}$ and the Gram matrix $-2I_{23}$ for one root from each pair in the quotient. The computation thus covers $48 \cdot 46 = 2208$ representatives. Lemma 4.3 shows that these give the full quotient root system $23A_1$ at each cusp. Finally, the 48 hyperplanes $S_{f_i^{\pm}}$ in $\mathbb{Q}k \oplus \Lambda_{\mathbb{Q}}$ have trivial common intersection.

For Theorem 6.6, `anc/verify_k3_applications.py` computes $T = T_{e,u}$ and $S = T_{f,v}$ over $\mathbb{Z}[m]$ for $L_m = U \oplus \langle -4m \rangle^{\oplus 2}$. It checks that both matrices preserve the form and that $\det(XI - TS) = P_m(X)$. To verify the identity

$$\det(XI - T^N S^N) = P_{mN^2}(X)$$

of Proposition 7.9, it evaluates both sides over $\mathbb{Z}[m]$ at $N = 0, \dots, 16$. The entries of $T^N = T_{e,Nu}$ and $S^N = T_{f,Nv}$ have degree at most 2 in N. The entries of their product therefore have degree at most 4, so its characteristic coefficients have degree at most 16. The same bound holds on the right. The 17 evaluations thus prove the identity in $\mathbb{Z}[m, N][X]$.

For Proposition 6.7, the same program forms the product $g = T_0^+ T_0^- T_1^+ T_2^+ \cdots T_7^+$ on $L = \langle 2 \rangle \oplus \langle -4 \rangle \oplus D_8(-2)$. Using an integral basis of L, it verifies that g preserves L and acts trivially on A_L. It then computes the characteristic polynomial and checks the irreducibility criterion modulo 13 given in the proof.

Statements and Declarations

Data and code availability. Appendix A describes the finite computations and identifies the verification programs and certificate files by their paths in the companion directory `anc/`.

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