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THE SHARP PU INEQUALITY FOR NON-REVERSIBLE FINSLER METRICS

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Abstract. We prove the sharp Holmes–Thompson systolic inequality for every Finsler metric on the real projective plane, with no reversibility assumption. A smooth strongly convex metric attains equality exactly when its geodesic flow is Zoll. The proof cuts along an oriented systole and relates a quadratic boundary-distance integral to that of the symmetrized boundary distance. A reversible metric built from chords and a collar realizes the latter integral in the limit while preserving the systolic lower bound. The result implies the sharp Viterbo inequality for convex bodies in $\mathbb{C}^2$ invariant under multiplication by i, with rigidity for smooth equality cases.

1. Introduction

For a Finsler metric F on $\mathbb{RP}^2$, let $\mathrm{sys}(F)$ be the infimum of the F-lengths of oriented noncontractible closed curves. The Holmes–Thompson area $\mathrm{Area}_{\mathrm{HT}}(F)$ is $1/\pi$ times the symplectic volume of the unit codisc bundle. In particular, it agrees with the usual area for Riemannian metrics. We allow continuous metrics whose fiberwise norms are asymmetric. A Zoll metric in the smooth strongly convex class is one whose geodesic flow on the unit sphere bundle has a common least positive period.

Theorem 1. *Every continuous Finsler metric F on $\mathbb{RP}^2$ satisfies*

$$\mathrm{Area}_{\mathrm{HT}}(F) \;\geq\; \frac{2}{\pi}\,\mathrm{sys}(F)^2. \tag{1.1}$$

If F is smooth and strongly convex, equality holds if and only if F is Zoll.

Pu proved (1.1) for Riemannian metrics [9]. Ivanov proved it for reversible Finsler metrics [6, Theorem 3]. Lejarza Alonso established the non-reversible inequality under a no-shortcut condition on a systolic loop [7, Theorem 5.13] and stated the general inequality in [8, Conjecture 4.1]. Chaidez and Tanny stated it together with the smooth equality conjecture [3, Conjecture 3.10].

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Corollary 2. *Let $X \subset \mathbb{C}^2$ be a convex body with $iX = X$, and let c_{EHZ} be the normalized Ekeland–Hofer–Zehnder capacity. Then*

$$c_{\mathrm{EHZ}}(X)^2 \;\le\; 2\,\mathrm{vol}_4(X). \tag{1.2}$$

For smooth X, equality holds if and only if its interior is symplectomorphic to an open Euclidean ball.

The counterexample of Haim-Kislev and Ostrover to Viterbo's volume–capacity conjecture [10, 5] raises the question of which symmetries suffice for the sharp inequality. Chaidez and Tanny established the inequality in dimension four under quaternionic symmetry and reduced the cyclic i-invariant case to the non-reversible Pu inequality [3, Theorems 8 and 10].

The main comparison is made on the cut disc. The directional boundary distance d has a fixed value on one half of the ordered boundary pairs. Its mixed derivative away from the diagonal is a positive measure carried by the other half. These two properties give the identity

$$\mathcal{B}(d) = 2\mathcal{B}(q) - \frac{2}{\pi}\,\mathrm{sys}(F)^2, \qquad q(a,b) = \frac{d(a,b)+d(b,a)}{2},$$

where $\mathcal{B}$ is the boundary integral defined in (2.1). Ivanov's cyclic maps give $\mathrm{Area}_{\mathrm{HT}}(F) \ge \mathcal{B}(d)$. We construct closed reversible comparison metrics with area tending to $\mathcal{B}(q)$ and systole at least $\mathrm{sys}(F)$.

2. The boundary integral

Let D be a smooth Finsler disc, possibly non-reversible. Parametrize its boundary by an oriented smooth circle $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$, and write $d(a,b)$ for the intrinsic directed distance from boundary point a to boundary point b. It is Lipschitz in both variables, since paths along short boundary arcs have uniformly bounded lengths in either orientation. For a Lipschitz function $h\colon \mathbb{T}_L^2 \to \mathbb{R}$ define

$$\mathcal{B}(h) = -\frac{1}{2\pi}\int_{\mathbb{T}_L^2} \partial_a h\,\partial_b h\,\mathrm{d}a\,\mathrm{d}b. \tag{2.1}$$

The derivatives here are weak derivatives.

Lemma 3. *The boundary distance of D satisfies $\mathrm{Area}_{\mathrm{HT}}(D) \ge \mathcal{B}(d)$.*

Proof. We use the cyclic-map estimate [6, Example 3.2 and Proposition 3.7]. For any positively cyclically ordered boundary points $a_1, \dots, a_N$, the distance functions $f_j(x) = d_D(a_j, x)$ satisfy

$$\mathrm{Area}_{\mathrm{HT}}(D) \;\ge\; \frac{1}{2\pi}\sum_{j=1}^{N}\int_D \mathrm{d}f_j \wedge \mathrm{d}f_{j+1} = \frac{1}{2\pi}\sum_{j=1}^{N}\int_{\mathbb{T}_L} d(a_j,b)\,\partial_b d(a_{j+1},b)\,\mathrm{d}b.$$

Indices in the sum are cyclic. The last equality is Stokes' formula for Lipschitz functions. The cyclic-map estimate holds for asymmetric norms and for arbitrary smooth disc boundaries.

Put $\delta = L/N$ and take $a_j = x + (j-1)\delta$. Average over $0 \le x < \delta$. The right-hand side becomes

$$\frac{1}{2\pi\delta} \int_{\mathbb{T}^2_L} d(a,b)\, \partial_b d(a+\delta, b)\, \mathrm{d}a\, \mathrm{d}b.$$

We may subtract $d(a,b)$ from $d(a+\delta,b)$ under the b-derivative, because $\int d(a,b)\partial_b d(a,b)\, \mathrm{d}b = 0$. Integration by parts in b and convergence of Lipschitz difference quotients in L^2 now give (2.1). □

Lemma 4. *Let $h\colon \mathbb{T}^2_L \to \mathbb{R}$ be Lipschitz, with $h(a,a) = 0$. Suppose that $\partial_a\partial_b h$ is a nonnegative distribution off the diagonal. There is a finite nonnegative measure μ_h off the diagonal with two equal, absolutely continuous marginals α_h such that*

$$\partial_a\partial_b h = \mu_h - \iota_*\alpha_h, \qquad \iota(a) = (a,a). \tag{2.2}$$

In this case

$$\mathcal{B}(h) = \frac{1}{2\pi} \int h\, \mathrm{d}\mu_h. \tag{2.3}$$

The hypothesis holds for the directed boundary distance of a Finsler disc.

Proof. For a.e. a the bounded function $\partial_a h(a,\cdot)$ has a nondecreasing representative on the circle cut at a. Indeed, on each rectangle away from the diagonal, testing the mixed derivative with nonnegative product test functions gives this monotonicity on almost every vertical slice. A countable cover by such rectangles gives the assertion on the whole cut circle for a.e. a. Its positive derivative in the cut circle has total mass at most $2\|\partial_a h\|_\infty$. Integrating these derivatives over a gives μ_h, with first marginal bounded by $2\|\partial_a h\|_\infty\, \mathrm{d}a$. The periodic distributional derivative in b has one further term, the jump at $b = a$, with mass equal to the negative of the off-diagonal mass on that slice. Hence it has the form (2.2) with the first marginal of μ_h. Repeating the argument with a, b interchanged identifies the second marginal with the first.

For (2.3), mollify h on the torus. Its derivatives converge in L^2, the functions converge uniformly, and the mollified mixed derivative converges weakly as a signed measure with bounded total variation. Thus integration by parts gives $2\pi\mathcal{B}(h) = \int h\, \mathrm{d}(\partial_a\partial_b h)$. The diagonal term vanishes. The same mollification argument shows

$$\int h_1\, \mathrm{d}(\partial_a\partial_b h_2) = \int h_2\, \mathrm{d}(\partial_a\partial_b h_1) \tag{2.4}$$

whenever the h_j are Lipschitz and their mixed derivatives are finite signed measures.

For a boundary distance, take four distinct boundary points in cyclic order a_1, a_2, b_1, b_2. Any path from a_1 to b_1 in a disc meets every path from a_2 to b_2, by the alternation of their endpoints. Switching the outgoing parts at a

meeting point, using paths of lengths arbitrarily close to the two distances, gives

$$d(a_1, b_1) + d(a_2, b_2) \geq d(a_1, b_2) + d(a_2, b_1).$$

Thus the mixed distributional derivative is nonnegative on rectangles with separated source and target arcs. These rectangles cover the complement of the diagonal. □

3. A reversible metric with the averaged boundary distance

The averaged boundary distance admits a Crofton representation. This is a continuous form of the circular split representation for finite metrics [4].

Let E be the closed Euclidean unit disc with its boundary parametrized by

$$x(a) = \big(\cos(2\pi a/L), \sin(2\pi a/L)\big).$$

Write $r(a, b)$ for the shorter arclength distance on $\mathbb{T}_L$. We first treat a symmetric boundary function q satisfying the hypotheses of Lemma 4. In the application it is a metric. Denote by $\mathcal{L}$ the set of unoriented lines that meet the open disc. For $a \neq b$, let $\lambda(a, b)$ be the line through $x(a), x(b)$, and put

$$\nu = \tfrac{1}{4}\lambda_* \mu_q. \tag{3.1}$$

Lemma 5. *The mass of lines separating $x(a)$ and $x(b)$, counted with ν, equals $q(a, b)$. There are smooth strongly convex reversible Finsler metrics G_j on E whose boundary distances q_j converge uniformly to q, with $q_j(a, b) \leq Cr(a, b)$ for a fixed C, and*

$$\lim_{j\to\infty} \mathrm{Area}_{\mathrm{HT}}(E, G_j) = \mathcal{B}(q). \tag{3.2}$$

Proof. For an arc A from a to b, rectangle integration of (2.2) gives

$$-2q(a, b) = (\partial_a \partial_b q)(A \times A) = -\mu_q(A \times A^c).$$

For the first equality one can mollify and pass to the limit. The marginals are atomless, so the rectangle boundary has zero mass for the mixed derivative. By symmetry of μ_q, the total μ_q-mass of pairs with exactly one endpoint in A is $4q(a, b)$. Except for the zero-mass case of a shared endpoint, these pairs represent exactly the chords separating $x(a)$ and $x(b)$. This proves the first assertion.

We describe a smooth approximation of the line measure that controls the boundary distances. Restrict the endpoint measure first to $r(a, b) \geq \varepsilon_j$, where $\varepsilon_j \downarrow 0$. Then average its pushforwards by small simultaneous orientation-preserving circle diffeomorphisms of the two endpoints. For example, use smooth compactly supported probability densities in the parameters $z = (z_0, z_1, z_2)$, with support shrinking to zero, for the maps

$$\phi_z(t) = t + z_0 + z_1 \sin(2\pi t/L) + z_2 \cos(2\pi t/L) \pmod{L}.$$

The endpoint measures after averaging have smooth densities supported away from the diagonal. To see this, for a fixed distinct pair the parameter

map to its two images has a two-by-three matrix of rank two. The rank follows from the distinctness of the sine and cosine pairs and the identical first entry 1. On the compact set of retained pairs the smaller singular value has a positive lower bound. Smooth parameter integration along the fibers thus gives a smooth density with uniform bounds on each derivative for this fixed cutoff. One may equivalently use the coarea formula on finitely many charts in the parameter space.

Pass to unoriented lines using λ and the factor $1/4$ in (3.1). This yields smooth line densities supported on lines a positive distance from tangency. Add $\beta_j/2$ times the measure $\mathrm{d}\theta\,\mathrm{d}p$ on lines meeting E°, with $\beta_j > 0$ tending to zero. Here $\theta \in [0,\pi)$ is the angle of a choice of unit normal n_θ, and the line equation is $\langle n_\theta, x\rangle = p$. Let ν_j be the resulting line measure, with density $w_j(\theta,p)$. The associated Crofton norm [2] is

$$G_j(x,v) = \int_0^\pi |\langle n_\theta, v\rangle|\, w_j(\theta, \langle n_\theta, x\rangle)\,\mathrm{d}\theta \tag{3.3}$$

gives a reversible Finsler metric on E. The added line measure contributes $\beta_j|v|$ to G_j and can be represented by a constant line density on a neighborhood of E. Smoothness in $v \neq 0$ follows by shifting the angular variable to put the nonsmooth factor at a fixed angle. The positively weighted Euclidean norm gives a positive second derivative transverse to the radial direction in each fiber, which makes G_j strongly convex.

Coarea along a path shows that its G_j-length is the ν_j-weighted number of line crossings, counted with multiplicity. A straight segment crosses each separating line once, so it realizes the distance between its endpoints. In particular q_j is the mass of the separating lines. The endpoint cutoff removes a total mass tending to zero, and the circular order is preserved by each ϕ_z. Before adding the Euclidean term, the new boundary distance is the average of the truncated boundary distance evaluated at $\phi_z^{-1}a, \phi_z^{-1}b$. It converges uniformly to q by uniform continuity and the cutoff mass bound. It is bounded by a constant times $r(a,b)$ since the truncated distances are bounded above by q, $q(a,a) = 0$, and the inverse diffeomorphisms have uniformly bounded Lipschitz constants. The added Euclidean distances tend uniformly to zero and satisfy the same type of bound.

For the area formula, the unit codisc at x in (3.3) is the Minkowski integral of the segments $[-n_\theta, n_\theta]$ with the indicated density. Its area is twice the double integral of $|\det(n_\theta, n_{\theta'})|$ against these two segment weights. This follows by approximating the integral by finite sums of segments and using mixed area. Integrating over E and changing variables from x to the two normal offsets of lines with these angles gives

$$\mathrm{Area}_{\mathrm{HT}}(E, G_j) = \frac{2}{\pi} \iint \mathbf{1}_{\{\ell \text{ and } \ell' \text{ cross in } E^\circ\}}\,\mathrm{d}\nu_j(\ell)\,\mathrm{d}\nu_j(\ell'). \tag{3.4}$$

The determinant is the Jacobian of the change of variables. Parallel directions give zero in the determinant integral.

The line measures converge weakly in chord endpoint coordinates, by the vanishing cutoff mass and the shrinking supports of the averaging parameters. For a pair of chords having four distinct endpoints, the crossing indicator is locally constant as a function of these endpoints. The exceptional set has $\nu \otimes \nu$-measure zero by the atomless endpoint marginals. Thus the right-hand side of (3.4) converges to the corresponding integral with ν. By the separation formula just proved, the inner integral for $\ell = \lambda(a, b)$ is $q(a, b)$. The limit is therefore $(1/(2\pi)) \int q \, \mathrm{d}\mu_q = \mathcal{B}(q)$, as asserted. □

Lemma 6. *Suppose D is the disc obtained by cutting an oriented noncontractible simple closed curve in a smooth Finsler $\mathbb{RP}^2$, and write s for the systole of the original metric. Suppose further that the boundary identification is $a \sim a + L/2$ in the chosen parameter. For the intrinsic directed boundary distance d of D and $q = (d + d^{\mathsf{T}})/2$,*

$$\mathcal{B}(q) \;\geq\; \frac{2}{\pi} s^2. \tag{3.5}$$

Proof. The function q is a symmetric metric, and it satisfies Lemma 4 because d and d^{T} do. Use the smooth reversible comparison metrics G_j from Lemma 5. Write $\eta_j = \sup_{a,b} |q_j(a, b) - q(a, b)|$. Choose $H > 0$ so that $q_j(a, b), q(a, b) \leq Hr(a, b)$ for all j, a, b. Attach the collar $[0, 1] \times \mathbb{T}_L$ to E along the boundary at $u = 0$. Its length norm is

$$\tau_j |\dot{u}| + H|\dot{a}|, \qquad \tau_j = \tfrac{1}{2}(\eta_j + j^{-1}).$$

Any path between two points a, b of the outer boundary has length at least $q(a, b)$ in this piecewise metric. If it stays in the collar, this follows from the tangential cost. Otherwise let a', b' be its first and last contacts with the inner boundary. The initial and final portions cost at least $Hr(a, a') + \tau_j$ and $Hr(b', b) + \tau_j$, respectively. The middle portion costs at least $q_j(a', b')$, since any excursion into the collar between inner-boundary points costs at least their q_j-distance and can be replaced by a segment in E. Consequently the total cost is at least

$$Hr(a, a') + q(a', b') + Hr(b', b) \;\geq\; q(a, b).$$

For this argument it suffices to use paths transverse to the inner circle, since we will test smooth loops transverse to both circles after making a smooth majorant.

The collar has Holmes–Thompson area $4LH\tau_j/\pi$, which tends to zero. Glue the outer boundary by $a \sim a + L/2$. The piecewise norm has smooth strongly convex reversible majorants on the quotient with area arbitrarily close to the sum of the disc and collar areas. Here is an explicit smoothing procedure. In the collar, replace each absolute-value term by a Euclidean norm of that term and a small positive multiple of the other coordinate. Their sum majorizes the collar norm and converges uniformly to it on unit vectors. The resulting norm is constant in the collar coordinates and is invariant under the change $(a, u - 1) \mapsto (a + L/2, 1 - u)$ at the outer seam.

Near the inner seam use a large smooth Riemannian norm dominating both of the adjacent norms on a fixed small neighborhood, and interpolate to the adjacent norms with a smooth cutoff in a thinner strip. The smoothing is a pointwise majorant on both sides. For each fixed j, the area cost of this interpolation tends to zero with the strip width. The regularized collar area tends to the stated collar area by uniform convergence on its compact tangent unit sphere. We thus obtain smooth reversible metrics on $\mathbb{RP}^2$ with areas at most $\mathrm{Area}_{\mathrm{HT}}(E, G_j) + 4LH\tau_j/\pi + j^{-1}$.

Consider a noncontractible loop for such a majorant. By approximation we may take it smooth and transverse to both seams, with finitely many crossings. It must cross the outer seam, since the seam complement is an open disc. Cutting the loop at that seam gives portions with ordered outer-boundary endpoint pairs (a_k, b_k). Their lengths sum to at least $\sum_k q(a_k, b_k)$. Replace those portions in the original cut disc D by directed paths from a_k to b_k of lengths approaching $d(a_k, b_k)$. The projected loop has the same nontrivial homotopy class. Indeed, one may identify the two discs by a boundary-preserving homeomorphism and then homotope each portion with fixed endpoints in the disc. It follows that $\sum_k d(a_k, b_k) \geq s$. Reversing the ordered circuit and making the same replacements shows $\sum_k d(b_k, a_k) \geq s$. Thus $\sum_k q(a_k, b_k) \geq s$, and the smooth comparison metric has systole at least s.

Apply the reversible Pu inequality [6, Theorem 3] to these smooth reversible metrics. Their areas are at least $2s^2/\pi$. Taking the limit using (3.2) proves (3.5). □

4. The sharp inequality and equality

We first prove the sharp bound for a smooth strongly convex F. Set $s = \mathrm{sys}(F)$. A shortest noncontractible oriented loop exists by compactness and is a smooth closed geodesic [3, Theorem 3.5]. It is embedded, because a self-intersection would split it into two strictly shorter oriented loops of which at least one is noncontractible. Cutting along this one-sided curve gives a disc D. The boundary goes twice around the loop. Orient and parametrize it by forward arclength, so $L = 2s$ and the gluing is $a \sim a + s$. Let d be the intrinsic Finsler distance of boundary points in D, with the order of the arguments indicating the direction.

If $b = a + t \pmod{L}$ and $0 \leq t \leq s$, then

$$d(a, b) = t. \tag{4.1}$$

The forward boundary arc gives the upper bound. A shorter directed path from a to b, followed by the forward boundary arc from b to $a + s$, would project to a noncontractible loop of length less than s. To see the homotopy class, the concatenated path in the disc is homotopic relative to its endpoints to the boundary half-circle from a to $a + s$.

Set $q = (d + d^{\mathsf{T}})/2$. By (4.1), μ_d has no mass on the open region of forward separation $0 < t < s$. Thus $d(b, a) = r(a, b)$ for μ_d-almost every

(a, b). This also holds on the antipodal locus $b = a + s$. Since $\partial_a \partial_b r$ is twice Lebesgue measure on the antipodal graph minus twice Lebesgue measure on the diagonal, (2.4) gives

$$\int r \, \mathrm{d}\mu_d = \int d \, \mathrm{d}(\partial_a \partial_b r) = 2 \int_{\mathbb{T}_L} d(a, a + s) \, \mathrm{d}a = 4s^2.$$

Here r vanishes on the diagonal. By symmetry of q and $\mu_q = (\mu_d + \mu_{d^\intercal})/2$, we have $\int q \, \mathrm{d}\mu_d = \int q \, \mathrm{d}\mu_q$. It follows from (2.3) that

$$\mathcal{B}(d) = 2\mathcal{B}(q) - \frac{2}{\pi} s^2. \tag{4.2}$$

Lemmas 3 and 6 now imply

$$\mathrm{Area}_{\mathrm{HT}}(F) = \mathrm{Area}_{\mathrm{HT}}(D) \geq \mathcal{B}(d) \geq \frac{2}{\pi} s^2.$$

The cut has zero area. Positive continuous Finsler norms on a compact manifold are uniform limits on tangent unit vectors of smooth strongly convex norms. This follows by local convolution in the base and angular variables, adding a vanishing Euclidean norm in each chart and combining the norms by a partition of unity. The approximating norms are bounded above and below by multiples of F with constants tending to one. Thus their systoles converge to $\mathrm{sys}(F)$, and the areas converge by the corresponding inclusions of the unit codiscs. This proves (1.1) for continuous metrics.

We next prove rigidity for a smooth strongly convex metric. Let $\Sigma = S^*_F \mathbb{RP}^2$ be its unit cosphere bundle and α the restriction to Σ of the canonical one-form. The Reeb flow of α is the lift of the unit-speed geodesic flow. Its periods are geodesic lengths, and Stokes' theorem gives

$$\mathrm{Area}_{\mathrm{HT}}(F) = \frac{1}{2\pi} \int_\Sigma \alpha \wedge \mathrm{d}\alpha. \tag{4.3}$$

Let Z be the union of the lifts of the noncontractible closed geodesics of length s. This is closed by compactness and continuous dependence of the geodesic flow. Noncontractibility of the projections is locally constant under uniform convergence of closed curves.

Suppose F attains equality and $Z \neq \Sigma$. Choose a smooth nonnegative nonzero function χ on Σ with support disjoint from Z. The contact form $\alpha_\epsilon = (1 - \epsilon\chi)\alpha$ is the pullback of the canonical one-form on the radial deformation $p \mapsto (1 - \epsilon\chi(p))p$ of the cosphere. For sufficiently small positive ϵ this deformation defines a smooth strongly convex Finsler metric F_ϵ. Since the perturbation is small in C^∞, the positive curvature of the original fiberwise cospheres persists. As $\alpha_\epsilon \wedge \mathrm{d}\alpha_\epsilon = (1 - \epsilon\chi)^2 \alpha \wedge \mathrm{d}\alpha$, the area strictly decreases by (4.3).

For all sufficiently small ϵ the new systole is still s. Orbits comprising Z are unchanged. If along a sequence tending to zero there were a shorter noncontractible geodesic, its lift to the deformed flow on Σ would meet $\mathrm{supp}\, \chi$. Otherwise it would be an unperturbed orbit. Start each such trajectory at a point of that support. By uniform multiplicative comparison of F_ϵ with F,

the shorter periods tend to s. After taking a subsequence, the trajectories converge to a closed orbit of period s for α, with noncontractible projection, that meets $\operatorname{supp}\chi$. This contradicts the choice of support. The smaller area at unchanged systole would contradict (1.1). Hence $Z = \Sigma$ and the geodesic flow is Zoll. Every prime orbit projects to a noncontractible loop and has length s.

Conversely, suppose that F is Zoll, of least period T. The Reeb flow gives a free circle action on Σ with oriented surface quotient B. The total space Σ has fundamental group $\mathbb{Z}/4$. Indeed, the unit tangent bundle of S^2 is $SO(3)$, and the deck transformation of the unit bundle over $\mathbb{RP}^2$ sends the frame $(x, v, x \times v)$ to $(-x, -v, x \times v)$. This rotation of order two lifts to an element of order four in the double cover of $SO(3)$. The resulting fundamental group is the same for every Finsler unit cosphere bundle over $\mathbb{RP}^2$. It follows that $\pi_1(B)$ is finite and B is a 2-sphere. The Euler number of the Reeb circle bundle has absolute value four by its homotopy sequence. A Reeb fiber generates $\pi_1(\Sigma)$, and its image in $\pi_1(\mathbb{RP}^2)$ is nontrivial because the bundle projection $\Sigma \to \mathbb{RP}^2$ surjects on fundamental groups. Thus $s = T$. The connection form α/T, with integral one along each fiber, then gives

$$\int_\Sigma \alpha \wedge \mathrm{d}\alpha = 4T^2.$$

Together with (4.3) this gives equality in (1.1) and completes Theorem 1.

5. The four-dimensional capacity consequence

We give the constants in Corollary 2. For smooth X, origin symmetry follows from $i^2 = -1$, and the origin is in the interior by convexity. The quaternionic construction of Chaidez and Tanny [3, Section 4 and Theorem 10] gives a strict contact four-sheeted covering of the unit cosphere bundle of a smooth strongly convex, possibly asymmetric Finsler metric F_X on $\mathbb{RP}^2$ by ∂X. The form on ∂X is the standard radial Liouville form. In particular,

$$\mathrm{vol}_4(X) = 4\pi \operatorname{Area}_{\mathrm{HT}}(F_X).$$

A systolic geodesic on $(\mathbb{RP}^2, F_X)$, lifted to the unit cosphere bundle and then to ∂X, has a closed lift after at most four iterations. The capacity is the least positive Reeb action on ∂X, so Theorem 1 gives

$$c_{\mathrm{EHZ}}(X)^2 \le 16 \operatorname{sys}(F_X)^2 \le 8\pi \operatorname{Area}_{\mathrm{HT}}(F_X) = 2\,\mathrm{vol}_4(X).$$

Equality implies that F_X is Zoll and that $c_{\mathrm{EHZ}}(X) = 4 \operatorname{sys}(F_X)$. Every prime Reeb orbit downstairs then has a lift closing first after four iterations, since any shorter closing lift would contradict this capacity value. All Reeb orbits upstairs therefore have the same least period. The smooth Zoll ball characterization [1, Proposition 4.3] identifies the interior with a symplectic ball. Conversely, a symplectomorphism to a ball preserves capacity and volume and hence gives equality. Finally, every convex X with $iX = X$ is a Hausdorff limit of smooth, uniformly convex bodies with that same symmetry. One can smooth its support function by rotational convolution,

average over the cyclic group, and add a vanishing positive multiple of the ball. The volume and c_{EHZ} are continuous under this approximation, so (1.2) holds for every body in the stated class.

Declaration on generative AI

ChatGPT (OpenAI) was used to search the literature, develop and check the boundary-distance argument, and draft the manuscript.

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