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CLUSTER–CRYSTAL REALIZATION OF KR CARTAN ALGEBRAS

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Abstract. We realize the Cartan algebras of restricted Kirillov–Reshetikhin modules as Cox rings of affine Schubert varieties, with Cartan degree given by the primitive divisor class. This yields upper cluster structures with non-inverted frozen variables for untwisted and diagram-twisted current algebras. In untwisted type, translation of the loop parameter gives a second cluster structure whose frozen variables are indexed by the affine Dynkin nodes. We construct charts adapted to the finite Chevalley operators and both shifted affine simple-root operators.

For every untwisted finite simple type, every fundamental node, and every Cartan degree, the translated theta basis carries a finite regular affine crystal. The construction uses a uniform reddening theorem and perfectness of the current operators on theta functions. A degree–level identity obtained from tropical Donaldson–Thomas duality determines the minimum level in divisible degrees and gives a finite minimization formula in all degrees. The crystals are simple, their tensor products are connected, and the divisible-degree crystals have a common limit. The affine theta crystals have the finite characters of the corresponding restricted KR modules; they are identified with quantum KR crystals in every classical type and at the exceptional minuscule and adjoint nodes covered by the available recognition theorems.

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Introduction

Fix a fundamental node r of a finite-dimensional simple Lie algebra $\mathfrak{g}$. For $s \geq 0$, let $K_{r,s}$ be the *restricted Kirillov–Reshetikhin module* of highest $\mathfrak{g}$-weight $s\omega_r$. The cyclic tensor maps $K_{r,s+t} \to K_{r,s} \otimes K_{r,t}$ define the graded dual *KR Cartan algebra*

$$R_r = \bigoplus_{s \geq 0} K_{r,s}^*.$$

We construct cluster coordinates that respect Cartan multiplication and express the current operators as derivations in adapted charts.

The KR–Demazure correspondence [CM, FL07, FK, KV] realizes suitable Veronese subalgebras of R_r as section rings on affine Schubert varieties Y_r. To recover the full algebra, we use a finite jet group acting on every $K_{r,s}$. A common subgroup stabilizes the cyclic lines through powers of one character. Matrix coefficients identify R_r with the invariants under the character kernel. We obtain

$$R_r \simeq \mathrm{Cox}(Y_r), \qquad \mathrm{Cl}(Y_r) = \mathbb{Z}[D_r],$$

with Cartan degree equal to the primitive divisor class grading (Theorem 2.5). This applies to untwisted currents and to the twisted currents defined by diagram automorphisms. For $A_{2n}^{(2)}$, the Demazure polarization has central level two and Veronese index two or four; the Cartan grading remains primitive.

The Schubert Cox theorem [FeiS] supplies an upper cluster structure on R_r. If the Schubert element w has length d and support S, its reduced-word seed has $d - |S|$ mutable variables and $|S| + 1$ frozen variables. The latter include the canonical section τ of the Schubert divisor. The finite Chevalley generators and every nonzero positive affine simple generator have *adapted charts* in this mutation class:

$$D = c_D \prod_{x \neq b(D)} x^{m_x(D)} \partial_{b(D)}, \qquad c_D \in \mathbb{C}^\times, \quad m_x(D) \in \mathbb{Z}_{\geq 0},$$

where $b(D)$ is frozen. The exchange matrix determines the mutable exponents. We recall the required Schubert results in Section 1.

For untwisted currents, we construct a second cluster structure. Let n be the finite rank and $L = t_r = 2/(\alpha_r, \alpha_r)$, with long roots of squared length two. Translation of the loop parameter by $a \neq 0$ gives a homogeneous seed Σ_a with $d - n$ mutable variables and frozen coordinates $b_0(a), \ldots, b_n(a)$. If k_i are the affine comarks, Theorem 4.11 gives

$$R_r = \mathcal{U}_{\mathrm{pol}}(\Sigma_a), \qquad \deg b_i(a) = Lk_i \quad (0 \leq i \leq n).$$

Outside the evaluation cases, the Schubert seed has full affine support and $n+2$ frozen variables. The translated seed has $n + 1$, since τ becomes mutable. We recover its exchange from a Poisson bracket for which $\tau = 0$ is not a Poisson divisor. The residue of the translation field is a simple-root operator on the inverse Schubert boundary cell. Its root strings compute the pole orders of the translated minors and hence the frozen degrees required by Proposition 1.2.

The translated mutation class has charts adapted to every finite Chevalley generator and to $F_0(a) = e_\theta \otimes (u + a)^{-1}$, where the inverse is taken in the common finite jet algebra. The same holds for $E_0(a) = f_\theta \otimes (u + a)$. At the exceptional nonneutral nodes this follows by transporting a current chart through a common spectral cluster (Theorem 5.1).

For the distinguished lowering operator F_r, we reverse the word presentation on the Schubert boundary cell, apply the simple-root mutations, and then mutate τ.

In the evaluation cases the two cluster structures coincide. In type A, the Cartan algebras are Grassmannian coordinate rings, whose cluster structures were constructed by Scott [Sco]. Section 9 gives explicit realizations for the first node of type C_n and for the even twisted current algebra of type $A_2^{(2)}$.

0.1. **Theta bases and crystals.** The Schubert seeds admit maximal green sequences. Their regular theta functions therefore form a homogeneous basis of R_r, and the adapted current charts make this basis perfect for all finite Chevalley operators [FeiT]. Theorem 6.3 identifies its homogeneous crystals with the finite-type crystals of the restricted KR modules, uniformly in the untwisted and twisted cases.

Every translated seed admits a reddening sequence (Theorem 6.4). The accompanying supplement [FeiR] proves the comparison with the spectral seed and the uniform two-strip construction at the double bond. A comparison of symmetric scattering diagrams and explicit Baxter and Toda recurrences give the boundary estimates.

The adapted charts make the theta basis perfect for every lowering current, including $F_0(a)$. The rank-one relations give perfectness for the raising currents and identify the leading maps as partial inverses. We obtain finite regular affine crystals with the finite characters of the restricted KR modules (Theorem 6.5).

We relate the Cartan grading to their minimum levels. Write a regular theta function as $\vartheta_g = b^{\mathsf{q}}\chi_p$, where b^{q} is a frozen monomial and χ_p has mutable g-vector p and vanishing order zero along every frozen divisor. Set $d(p) = \deg \chi_p$, and let S be the mutable tropical Donaldson–Thomas map specified in Section 7. Theorem 7.2 gives the identity

$$L\ell(g) = \deg \vartheta_g + d(\mathsf{S}(p)), \qquad \ell(g) = \sum_i k_i \varepsilon_i(g).$$

It follows that $L\ell(g) \geq \deg \vartheta_g$, with equality precisely for frozen monomials. In degree $L\ell$, these are exactly the minimal elements, and both string maps identify them with the dominant classical affine weights of level ℓ. In the other degrees we obtain a finite minimization formula, together with a sharp degree bound for simultaneous kernels of the current operators. Cartan degree and affine level are thus distinct numerical invariants.

An important feature of this construction is that neither the affine crystal operators nor the degree–level formula requires an identification with a quantum KR crystal. In particular, the same construction applies at exceptional nodes for which no such identification is established here. Thus the resulting crystals give uniform affine combinatorics for the restricted KR modules in all untwisted types and in every degree.

The essential simple socle of the dual current module implies simplicity of the affine crystals and connectedness of their tensor products. Translation by frozen exponent vectors gives a common limit for the degrees $L\ell$ on the lattice of Cartan degree zero. In classical types, a uniqueness theorem identifies these crystals with $B^{r,s}$; they are perfect exactly when $L \mid s$. We also prove the quantum identification at the exceptional minuscule and adjoint nodes (Corollary 8.4).

0.2. **Relation to earlier work.** The restricted KR presentations are due to Chari and Moura [CM]. We use the KR–Demazure correspondence of Fourier–Littelmann in the untwisted case [FL07], and Fourier–Kus and Kus–Venkatesh in the twisted cases [FK, KV]. These module identifications respect the cyclic tensor maps and hence the Cartan multiplication.

The reduced-word seeds come from Goodearl–Yakimov's integral quantum cluster structures [GY]. Their Cox-ring lifts are related to the constructions of Geiß–Leclerc–Schröer and Francone [GLS, FraL, Fra]. We use the Cox-ring equality, root-operator charts, and the criterion based on canonical degree of [FeiS] to identify the primitive KR grading and construct the translated seed. The Poisson calculations use Polishchuk's canonical Poisson-module connection [Pol] and the rational and trigonometric loop-group structures [KWWY, FKR, AB].

We use the theta functions of Gross–Hacking–Keel–Kontsevich [GHKK]. For the crystal construction, theta perfectness, frozen valuations, and tropical Donaldson–Thomas duality come from [FeiT]. The quantum alcove and quantum Lakshmibai–Seshadri path models give uniform realizations of single-column quantum KR crystals in all untwisted types [LNSSS]. General higher-level models and their relation to Demazure crystals have been studied by Lenart–Scrimshaw [LS]; recent work gives a uniform PBW model at special nodes [JS26].

Our construction is different: its theta functions and crystal operators arise from the KR Cartan algebra, and the comparison with quantum KR crystals is proved in the cases stated above.

Berenstein–Kazhdan's comparison theorem [BK] gives finite-type regularity. We use simple crystals and limits of coherent families as in Akasaka–Kashiwara [AK] and Kang–Kashiwara [KK], and the classical recognition and perfectness theorems of [ST, FOS].

0.3. **Conventions.** We work over $\mathbb{C}$ and use Bourbaki numbering for finite Dynkin nodes. We write $\mathcal{U}_{\mathrm{pol}}(\Sigma)$ for the upper cluster algebra with non-inverted frozen variables. Our exchange matrices have mutable indices on their rows and all seed indices on their columns, so mutation at k is given by

$$x_k x_k' = \prod_j x_j^{[B_{kj}]_+} + \prod_j x_j^{[-B_{kj}]_+}, \qquad [c]_+ = \max(c, 0).$$

Part I. Upper cluster structures

1. Schubert Cox rings

We recall the Schubert Cox results of [FeiS] in the form needed here. Let $Y = X_w^{P_J}$ be a positive-dimensional Schubert variety in a partial flag variety of a symmetrizable Kac–Moody group, where w is the minimal representative of its coset in W/W_J. Write C for its open cell, $d = \ell(w)$, and $S = \mathrm{supp}(w)$. If $D_1, \dots, D_q$ are the prime divisors in $Y \setminus C$, their classes form a basis of $\mathrm{Cl}(Y)$. We use these divisors to define the *Cox ring*

$$\mathrm{Cox}(Y) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^q} H^0\left(Y, \mathcal{O}_Y\left(\sum_{j=1}^{q} a_j D_j\right)\right) \boldsymbol{\tau}^{\mathbf{a}} \subset \mathbb{C}[C][\tau_1^{\pm 1}, \dots, \tau_q^{\pm 1}].$$

We interpret the sheaf associated with a Weil divisor as a rank-one reflexive sheaf. Write τ_j for the canonical section of D_j. For a nonzero $f \in \mathbb{C}[C]$, its *minimal homogeneous lift* is $\boldsymbol{\tau}^{\nu(f)} f$, where $\nu(f)_j = -\operatorname{ord}_{D_j}(f)$. It is not divisible by any boundary section. This agrees with the minimal monomial lifting convention of [FraL, Fra].

1.1. **Reduced-word seeds.** We choose a reduced expression $w = s_{i_1} \cdots s_{i_d}$. Let $p(k)$ and $s(k)$ be the preceding and following occurrences of i_k, with values $-\infty$ and $+\infty$ when absent. We write $w_{\le k} = s_{i_1} \cdots s_{i_k}$ and use the *unipotent minors* $f_k = D_{\Lambda_{i_k}, w_{\le k}\Lambda_{i_k}}$. A position is mutable precisely when $s(k) < +\infty$. With $A = (a_{ij})$ the Cartan matrix, our exchange matrix has mutable positions indexing its rows and all positions indexing its columns:

$$B_{kj} = \begin{cases} 1, & j = p(k), \\ -1, & j = s(k), \\ a_{i_j i_k}, & j < k < s(j) < s(k), \\ -a_{i_j i_k}, & k < j < s(k) < s(j), \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

These reduced-word seeds are the classical specializations of [GY, Theorems 7.1 and 7.3]; their construction uses a symmetric *Cauchon–Goodearl–Letzter (CGL) extension* of the cell algebra. If $\nu = (\nu(f_1), \dots, \nu(f_d))$, the lifted seed has coordinates $x_k = \boldsymbol{\tau}^{\nu(f_k)} f_k$ and $\tau_1, \dots, \tau_q$, and exchange matrix $\widetilde{B} = (B \mid -B\nu^{\mathsf{T}})$. The boundary sections are frozen.

For *non-inverted frozen variables*, we use the *mixed Laurent ring*, which inverts exactly the mutable coordinates. Thus

$$\mathcal{L}_{\mathrm{pol}}(\Sigma) = \mathbb{C}[x_k^{\pm 1} : k \text{ mutable}][b_j : j \text{ frozen}], \qquad \mathcal{U}_{\mathrm{pol}}(\Sigma) = \bigcap_{\Sigma' \sim \Sigma} \mathcal{L}_{\mathrm{pol}}(\Sigma').$$

Proposition 1.1. *The lifted seed satisfies*

$$\mathrm{Cox}(Y) = \mathcal{L}_{\mathrm{pol}}(\Sigma) \cap \bigcap_{k \text{ mutable}} \mathcal{L}_{\mathrm{pol}}(\mu_k \Sigma) = \mathcal{U}_{\mathrm{pol}}(\Sigma).$$

It has $d - |S|$ mutable and $|S| + q$ frozen variables. Its initial coordinates are pairwise nonassociated primes. The extended exchange matrix has full rank, and each exchange vector is primitive. The degrees of the frozen variables sum to $-K_Y$. All reduced expressions of w give the same mutation class, and every seed in this class is the minimal lift of its cell seed.

Proof. The ring equalities and the anticanonical identity follow from [FeiS, Theorem 3.4]. The successor entries in (1.1) give full rank and primitive rows. The minimal-lifting and word-independence assertions are [FeiS, Proposition 3.7 and Corollary 3.9]. □

The *universal grading* takes values in $\mathrm{Cl}(Y) \oplus Q_S$, where $Q_S = \sum_{i \in S} \mathbb{Z}\alpha_i$ is the root lattice of the support [FeiS, Proposition 3.6]. For the affine Schubert varieties considered here, it records current degree, finite Cartan weight, and Cartan degree. Finite Cartan weights alone do not recover this grading.

We also use [FeiS, Lemma 4.2]: for a factorial upper cluster algebra with non-inverted frozen variables and a full-rank exchange matrix, every reachable coordinate is prime. A Laurent monomial in any reachable seed belongs to the algebra only if all its exponents are nonnegative.

1.2. **Boundary degrees and the canonical form.** Suppose now that $\mathrm{Cl}(Y) = \mathbb{Z}[D]$. There is a unique Schubert boundary divisor D, with canonical section τ. If it is indexed by ws_β, with β positive, then the Schubert canonical-sheaf formula and the Chevalley formula give

$$-K_Y = \kappa[D], \qquad \kappa = 1 + \langle \rho, \beta^\vee \rangle, \qquad \deg \widetilde{\Delta}_i = \langle \Lambda_i, \beta^\vee \rangle,$$

where $\Delta_i = D_{\Lambda_i, w\Lambda_i}$ is the terminal i-minor, for $i \in S$. Here ρ pairs to one with every simple coroot. For the coordinate volume form on the cell, we have, up to a nonzero scalar,

$$\omega_C = \left(\prod_{i \in S} \Delta_i \right) d\log f_1 \wedge \cdots \wedge d\log f_d. \tag{1.2}$$

The triangular recurrence for the word minors proves this identity; see [FeiS, proof of Lemma 3.1]. The form $\Omega = \tau^\kappa \omega_C \wedge d\log \tau$ has divisor zero on the Cox space and homogeneous degree κ. In particular, $1 + \sum_{i \in S} \deg \widetilde{\Delta}_i = \kappa$.

We will also use the upper-algebra criterion [FeiS, Theorem 1.2] for the translated seed, which need not be a minimal lift of a Schubert word seed.

Proposition 1.2. *Let R be a finitely generated normal positively graded $\mathbb{C}$-domain with $R_0 = \mathbb{C}$, and let Ω be a homogeneous rational top form with divisor zero and degree κ. Suppose a homogeneous seed has pairwise nonassociated prime coordinates which form a transcendence basis and generate* $\mathrm{Frac}\, R$. *Assume its extended exchange matrix has full rank and primitive rows, and all its first exchange quotients belong to R. Then*

$$R = \mathcal{U}_{\mathrm{pol}}(\Sigma) \quad \Longleftrightarrow \quad \sum_{b \text{ frozen}} \deg b = \kappa.$$

In the equality case the initial and adjacent mixed Laurent charts cover $\mathrm{Spec}\, R$ *in codimension one.*

2. KR Cartan algebras and Schubert varieties

2.1. **Current modules and Cartan multiplication.** Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n, and let G be the simply connected group with Lie algebra $\mathfrak{g}$. Fix a pinning, with Cartan subalgebra $\mathfrak{h}$, Chevalley generators e_i, f_i, h_i, simple roots α_i, and fundamental weights ω_i, where $1 \le i \le n$. We use the Cartan matrix $A_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$.

We treat every untwisted current algebra $\mathfrak{C} = \mathfrak{g}[u]$. We also treat the *twisted current algebras* defined by diagram automorphisms. For these, let σ be an automorphism of order m of a simply laced simple Lie algebra $\mathfrak{a}$, and let $\mathfrak{a}_j$ be its eigenspace with eigenvalue ζ^j, for a primitive m-th root ζ. Then $\mathfrak{C} = \bigoplus_{j\geq 0} \mathfrak{a}_j \otimes u^j$ and $\mathfrak{g} = \mathfrak{a}_0$, with eigenspace indices read modulo m. We write $\mathfrak{C}_+$ for the ideal of positive current degree, and $\mathfrak{C}^- \oplus \mathfrak{C}^0 \oplus \mathfrak{C}^+$ for the decomposition by negative, zero, and positive finite weights. In type $A_{2n}^{(2)}$ this convention has finite part B_n; the hyperspecial current algebra has finite part C_n and is a different realization.

Fix a finite node r. Let ξ_r be a nonzero current of weight $-\alpha_r$ in the first positive degree in which that weight occurs. For untwisted currents, $\xi_r = f_r \otimes u$. In the twisted families other than $A_{2n}^{(2)}$, its degree is one at a short root and m at a long root; in $A_{2n}^{(2)}$, it is one at every node. The restricted Kirillov–Reshetikhin module $K_{r,s} = \mathrm{KR}(s\omega_r)$, for $s \geq 0$, is generated by a vector v_s subject to

$$\begin{aligned} &\mathfrak{C}^+ v_s = 0, && (\mathfrak{C}^0 \cap \mathfrak{C}_+) v_s = 0, \quad && h v_s = s\omega_r(h) v_s \quad (h \in \mathfrak{h}), \\ &f_i v_s = 0 \quad (i \neq r), && f_r^{s+1} v_s = 0, \quad && \xi_r v_s = 0. \end{aligned} \tag{2.1}$$

These are the presentations in [CM, FL07, FK, KV]. In the even twisted family, the relation $(f_r \otimes u^2) v_s = 0$ follows from the stated relations [CM, Lemma 3.1(ii)]. We often write K_s when r is fixed.

The tensor $v_s \otimes v_t$ satisfies the defining relations of K_{s+t}. The universal property therefore gives a current-module homomorphism $\Delta_{s,t} : K_{s+t} \to K_s \otimes K_t$ sending the cyclic vector to this tensor. Cyclicity also proves that these maps are coassociative and cocommutative. Their duals define the KR Cartan algebra

$$R_r = \bigoplus_{s\geq 0} K_{r,s}^*, \qquad \varphi\psi = \Delta_{s,t}^*(\varphi \otimes \psi) \quad (\varphi \in K_s^*,\ \psi \in K_t^*).$$

We refer to s as the *Cartan degree*. The current grading on K_s gives its dual the grading $(K_s^*)_{-j} = K_s[j]^*$.

The action on R_r requires a convention for dual modules. Let ς be the pinned diagram involution induced by $-w_0$, where w_0 is the longest finite Weyl group element. In untwisted type we extend ς coefficientwise to $\mathfrak{g}[u]$. In the twisted families the finite opposition involution is the identity, and we take $\varsigma = \mathrm{id}$ on $\mathfrak{C}$. Our named current derivations are

$$(D_X \varphi)(v) = -\varphi(\varsigma(X) v). \tag{2.2}$$

This is the contragredient action with finite simple-root indices transported by opposition; replacing X by $\varsigma(X)$ recovers the usual contragredient action. This convention identifies the highest weight of the summand of current degree zero in K_s^* with $s\omega_r$, including the non-self-dual nodes of types A, D, and E_6. It does not change the graded algebra or its multiplication. Every D_X preserves Cartan degree and satisfies the Leibniz rule.

Cartan multiplication need not be surjective in each pair of degrees. For example, at the first node of C_n, $K_1 = V(\omega_1)$ and $K_2 = V(2\omega_1) \oplus \mathbb{C}$, whereas the submodule of $K_1^{\otimes 2}$ generated by $v_1 \otimes v_1$ is $V(2\omega_1)$ [CM, Section 2]. In particular, R_r need not be generated in degree one.

2.2. **Demazure section rings.** For untwisted types we normalize long roots to have squared length two and write $t_i = 2/(\alpha_i, \alpha_i)$. Table 1 records the *Veronese index* p_r and the anticanonical degree κ_r. Here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

In the first line, $t_r = 1$ in simply laced type. The last two lines use the fixed-point realization described above, with $n \geq 1$; for $n = 1$, the finite algebra is $\mathfrak{sl}_2$ with its fundamental weight.

In the Demazure realizations below, we precompose the geometric current action with ς. The geometric action on the dual section ring then agrees with (2.2).

Proposition 2.1. *For every type and node in Table 1, there are a dominant affine weight Λ and an affine Weyl group element w, minimal modulo the stabilizer of Λ, such that*

TABLE 1. Veronese indices and anticanonical degrees.

Current type	Finite type	p_r	κ_r
Untwisted	Any simple type	t_r	$t_r h^\vee$
$A^{(2)}_{2n-1}$, $n \geq 2$	C_n	1	$2n$
$D^{(2)}_{n+1}$, $n \geq 3$	B_n	1	$2n$
$E^{(2)}_6$	F_4	1	12
$D^{(3)}_4$	G_2	1	6
$A^{(2)}_{2n}$, $r < n$	B_n	2	$2n+1$
$A^{(2)}_{2n}$, $r = n$	B_n	4	$4n+2$

$K_{ap_r} \simeq V_w(a\Lambda)$ for every $a \geq 0$. We choose the null-root component of Λ so that the cyclic vectors have current degree zero. The isomorphisms respect the gradings, cyclic vectors, and tensor maps. Consequently, the Veronese algebra $R_r^{(p_r)}$ is the complete section ring of the ample line bundle $\mathcal{L}_\Lambda$ on $Y_r = X_w^{P_\Lambda}$.

Proof. The untwisted assertion is the KR–Demazure theorem of Fourier–Littelmann [FL07, Theorem 4], which applies to every simple type. Its integer d_r, characterized by $\nu(\omega_r^\vee) = d_r\omega_r$, is our t_r. The associated projective variety is the affine Grassmannian Schubert variety corresponding to $\omega_r^\vee$, in its component. We identify its geometric pinning with the module pinning by opposition, in accordance with (2.2).

For the four reduced twisted families, the current-module identification is $K_s = D(s, s\omega_r)$ [KV, Proposition 7.2]. At the small ranks $A^{(2)}_3$ and $D^{(2)}_4$, the same identification follows from the fundamental Demazure descriptions in [FK, Theorem 5.0.2 and Section 5.1] and the cyclic tensor embeddings of [CM, Theorem 3.4(ii)]. Proposition A.1 proves the assertion for $A^{(2)}_{2n}$, with central level two and Veronese indices two and four.

The Cartan embedding between integrable highest-weight modules sends the relevant extremal vector to the tensor of extremal vectors. Its restriction to the Demazure submodules is therefore the prescribed cyclic-vector map. Borel–Weil for Schubert varieties identifies the dual modules with the section spaces of $\mathcal{L}_{a\Lambda}$, and identifies their multiplication with the dual tensor maps [Kum, Chapter 8]. This proves the assertion about graded algebras. □

We next record a uniform truncation bound; see also [CM] for the classical cases.

Lemma 2.2. *There is an integer M, independent of s, such that $\bigoplus_{j \geq M} \mathfrak{C}[j]$ annihilates every K_s. For untwisted currents, one may take $M = \langle \theta, \omega_r^\vee \rangle$, where θ is the highest root.*

Proof. In untwisted type, express a negative root vector f_α as an iterated bracket of negative simple-root vectors. Replace every occurrence of f_r by $f_r \otimes u$. The resulting bracket is a nonzero multiple of $f_\alpha \otimes u^{\langle \alpha, \omega_r^\vee \rangle}$ and belongs to the Lie algebra generated by the linear annihilating relations in (2.1). Commutation with positive Cartan currents gives every higher mode. Since $\langle \alpha, \omega_r^\vee \rangle \leq M$, the ideal $u^M\mathfrak{g}[u]$ kills v_s. The subspace annihilated by it is a submodule, because $u^M\mathfrak{g}[u]$ is an ideal. Cyclicity therefore shows that this ideal annihilates K_s. The bound depends only on r, not on s.

For the reduced twisted families, the Demazure Cartan embeddings give $K_s \hookrightarrow K_1^{\otimes s}$. In the even twisted family, write $s = ap_r + j$, with $0 \leq j < p_r$. Then $K_s \hookrightarrow K_j \otimes K_{p_r}^{\otimes a}$ by [CM, Theorem 3.4(ii)]. Each of the finitely many graded modules $K_0, \ldots, K_{p_r}$ is annihilated by all sufficiently high current degrees. A common bound therefore annihilates every indicated tensor product and submodule, independently of s. □

2.3. **The primitive divisor grading.** The section ring in Proposition 2.1 only describes a Veronese subalgebra. To identify the complete Cartan algebra, including its primitive grading, we use matrix coefficients.

We choose the finite jet algebra supplied by Lemma 2.2 and integrate it to $\mathcal{J} = G \ltimes U$, with U connected and unipotent. Let $P_r \subset G$ be the maximal parabolic stabilizing the highest-weight line in $V(\omega_r)$. The linear relations in (2.1), together with $\mathfrak{h}$, generate a Lie subalgebra whose degree-zero part is $\mathfrak{p}_r$. Its positive-degree part is normalized by P_r and integrates to a closed connected unipotent subgroup U_r. We therefore obtain the closed connected subgroup $H = P_r \ltimes U_r \subset \mathcal{J}$.

The character ω_r of P_r extends trivially across U_r to a character χ of H. Because G is simply connected, χ generates $X^*(H)$. Its kernel S is connected and has no nontrivial characters: the reductive quotient of S is the derived group of a Levi subgroup of P_r. These groups and the character χ are independent of s.

Proposition 2.3. *Matrix coefficients give a graded isomorphism $R_r \simeq \mathbb{C}[\mathcal{J}]^S$, where the grading on the right is defined by the right action of $H/S \simeq \mathbb{C}^\times$. If a Veronese algebra $R_r^{(p)}$ is the complete section ring of an ample line bundle on a normal projective variety Y, then R_r is finitely generated and factorial,* $\mathrm{Cl}(Y) \simeq \mathbb{Z}$, *and R_r is the Cox ring of Y with its primitive positive grading.*

Proof. Write $R = R_r$. A functional $\varphi \in K_s^*$ defines the matrix coefficient $g \mapsto \varphi(gv_s)$, which has right character χ^s. Cyclicity makes this map injective. Conversely, let $f \in \mathbb{C}[\mathcal{J}]^S$ have right character χ^s, initially with $s \in \mathbb{Z}$. Choose a finite-dimensional left-translation-stable space of such functions containing f. Evaluation at the identity is a cyclic vector in its dual: a function annihilated by all translates of this evaluation is zero. The right character gives the linear relations in (2.1). Finite-dimensional G-integrability gives $s \geq 0$ and $f_r^{s+1} v = 0$. The dual representation is therefore a quotient of K_s. This proves surjectivity. Multiplication of matrix coefficients agrees with the dual maps $\Delta_{s,t}^*$, and $R_0 = \mathbb{C}$.

As a variety, $\mathcal{J}$ is a product of a simply connected semisimple group and an affine space. Hence $\mathbb{C}[\mathcal{J}]$ is a unique factorization domain with only constant units [Pop, Proposition 1 and Section 1.2]. Since S is connected and has no nontrivial characters, every irreducible factor of an invariant function is invariant. Such a factor remains prime in R: whenever it divides an invariant element, the quotient is again invariant. Factorization in $\mathbb{C}[\mathcal{J}]$ thus proves that R is factorial, with only constant units.

Let $A = R^{(p)}$, and choose $0 \neq \tau \in R_1$. Every homogeneous element of R is integral over A, and $\operatorname{Frac} R = \operatorname{Frac}(A)(\tau)$, with $\tau^p \in A$. Indeed, for $x \in R_s$ and any multiple q of p with $q \geq s$, $x/\tau^s = (x\tau^{q-s})/\tau^q$ belongs to $\operatorname{Frac} A$. Since A is finitely generated, its integral closure in this finite field extension is finite over A. The algebra R, being an A-submodule of that integral closure, is finite over A. Thus R is finitely generated and $\operatorname{Proj} R = \operatorname{Proj} A = Y$.

Chevalley's theorem realizes S as the stabilizer of a line in a finite-dimensional $\mathcal{J}$-module [Mil, Theorem 4.27]. Since S has no characters, it fixes that line pointwise. Thus $\mathcal{J}/S$ is a locally closed orbit in an affine space and is quasi-affine. Its canonical map to $\operatorname{Spec} \Gamma(\mathcal{J}/S, \mathcal{O}) = \operatorname{Spec} R$ is an open immersion [Sta, Tag 01P9]. If its complement contained a prime divisor, factoriality would give a prime equation f for it. Then $1/f$ would be regular on $\mathcal{J}/S$, hence would belong to R, a contradiction. The complement therefore has codimension at least two.

The grading torus acts freely on $\mathcal{J}/S$, with quotient $\mathcal{J}/H$. The positive grading gives a geometric quotient $\operatorname{Spec} R \setminus \{0\} \to Y$, whose fibers have dimension one. Since $\mathcal{J}/S$ is an invariant open subset, its quotient $\mathcal{J}/H$ is open in Y; the complement still has codimension at least two. The Picard group sequence for homogeneous spaces, together with $\operatorname{Pic}(\mathcal{J}) = 0$ and $X^*(\mathcal{J}) = 0$, gives

$$\mathrm{Cl}(Y) = \operatorname{Pic}(\mathcal{J}/H) = X^*(H) = \mathbb{Z}[\mathscr{M}],$$

where $\mathscr{M}$ is the line bundle associated with the fiber character χ^{-1} [Pop, Theorem 4]. Functions of degree s on $\mathcal{J}/S$ are sections of $\mathscr{M}^s$. Reflexive extension across the complement identifies their algebra with $\mathrm{Cox}(Y)$. Since χ generates $X^*(H)$, this identification preserves the primitive grading. □

2.4. **The boundary and the canonical class.** The Schubert boundary classes freely generate the divisor class group [FeiS, Section 2.1]. Proposition 2.3 therefore shows that Y_r has a single Schubert boundary divisor. We denote it by D_r and its canonical section by τ. Its class is the positive generator of $\mathrm{Cl}(Y_r)$, so $\tau \in (R_r)_1$. The extremal section defining the open cell has divisor $p_r D_r$ and is proportional to τ^{p_r}. In the convention (2.2), τ has finite weight ω_r and current degree zero.

Let c_{aff} be the canonical central element, let δ_{aff} be the null root, and choose $\widehat{\rho}$ with $\langle \widehat{\rho}, \alpha_i^\vee \rangle = 1$ for all affine nodes i. We write $\widehat{h}^\vee = \langle \widehat{\rho}, c_{\mathrm{aff}} \rangle$.

Proposition 2.4. *The unique Schubert divisor of $Y_r = X_w^{P_\Lambda}$ is $X_{s_r w}^{P_\Lambda}$. If π is the length-zero factor in the translation description of w, its right boundary coroot is*

$$\beta^\vee = -w^{-1}\alpha_r^\vee = q_r c_{\mathrm{aff}} - \alpha_{\pi^{-1}(r)}^\vee.$$

Here $q_r = t_r$ in untwisted type, $q_r = 1$ in the four reduced twisted families, and $q_r = 1$ or 2 in $A_{2n}^{(2)}$, according as $r < n$ or $r = n$. Consequently, $-K_{Y_r} = q_r \widehat{h}^\vee [D_r]$.

Proof. In untwisted type the affine translation formula gives $w^{-1}\alpha_r = \alpha_{\pi^{-1}(r)} - \delta_{\mathrm{aff}}$. Its negative has coroot $t_r c_{\mathrm{aff}} - \alpha_{\pi^{-1}(r)}^\vee$. In the four reduced twisted families, normalize short finite roots to squared length two and write $d_r = (\alpha_r, \alpha_r)/2$. The corresponding formula is $w^{-1}\alpha_r = \alpha_{\pi^{-1}(r)} - d_r \delta_{\mathrm{aff}}$, so the central coefficient of the negative coroot is one. These are the affine translation formulas in [Kac, Chapter 6].

For $A_{2n}^{(2)}$, the word of Proposition A.1 represents translation by $-(\varepsilon_1 + \cdots + \varepsilon_r)$, and π is the identity. Its inverse sends α_r to $\alpha_r - \delta_{\mathrm{aff}}$. The finite root has squared length two for $r < n$ and one for $r = n$; the corresponding central coefficients are one and two.

In every case $w^{-1}\alpha_r$ is negative. Thus r is a left descent of w, and $s_r w$ indexes a Schubert divisor in the partial flag variety. It must be the unique boundary divisor identified above. The Schubert canonical-class formula now gives

$$-K_{Y_r} = \big(1 + \langle \widehat{\rho}, \beta^\vee \rangle\big)[D_r] = q_r \widehat{h}^\vee [D_r],$$

as asserted [FeiS, Lemma 3.2 and Theorem 3.4]. □

Theorem 2.5. *For every type and node in Table 1, the KR Cartan algebra is finitely generated and factorial, and*

$$R_r \simeq \mathrm{Cox}(Y_r) = \bigoplus_{s \geq 0} H^0(Y_r, \mathcal{O}_{Y_r}(sD_r)), \qquad \mathrm{Cl}(Y_r) = \mathbb{Z}[D_r].$$

This isomorphism identifies Cartan degree with the primitive divisor class grading. The divisor $p_r D_r$ is ample and Cartier, and $\mathcal{O}_{Y_r}(p_r D_r) \simeq \mathcal{L}_\Lambda$. Moreover, $-K_{Y_r} = \kappa_r D_r$, with κ_r as in the table, and the Cox space has a homogeneous rational top form of degree κ_r with divisor zero.

Proof. Apply Proposition 2.3 to the finite jet group and the Veronese section ring of Proposition 2.1. The ample line bundle has degree p_r, which proves the polarization statement. Proposition 2.4 gives the canonical degrees, and the Cox-ring volume form is supplied by [FeiS, Theorems 3.3–3.4 and their proofs]. □

Corollary 2.6. *Every KR Cartan algebra in Table 1 has the upper cluster structure of Proposition 1.1, homogeneous for its primitive Cartan grading. If $d = \ell(w)$, the seed has $d - |\operatorname{supp}(w)|$ mutable variables and $|\operatorname{supp}(w)| + 1$ polynomial frozen variables.*

Proof. Apply Proposition 1.1 to Theorem 2.5. □

3. Poisson brackets and current operators

We call a seed *adapted* to a nonzero derivation D if $D = cx^v \partial_b$, where b is frozen, $c \in \mathbb{C}^\times$, and every exponent in x^v is a nonnegative integer, with exponent zero at b. The variable b is its *active coordinate*.

3.1. **A canonical lift of the Poisson bracket.** We lift Poisson tensors to Cox rings using the canonical *Poisson connection* on the canonical sheaf. This connection and its extension to section rings are due to Polishchuk [Pol, Lemma 4.4, Proposition 5.2, and Corollary 5.3]. We use the convention $X_f = \{f, -\}$.

Proposition 3.1. *Let Y be a normal variety with $\mathrm{Cl}(Y) = \mathbb{Z}[D]$ and $-K_Y = \kappa[D]$, where $\kappa \in \mathbb{Z} \setminus \{0\}$. Fix an isomorphism of reflexive sheaves $\mathcal{O}_Y(-\kappa D) \simeq \omega_Y$. Every regular Poisson tensor on Y then has a canonical homogeneous lift to $\mathrm{Cox}(Y)$. The lift is linear in the tensor and preserves compatible Poisson pencils.*

On the smooth locus, choose a local frame e of $\mathcal{O}_Y(D)$, and write $\omega = e^{-\kappa}$ for the corresponding volume form. If t is the fiber coordinate on $\mathcal{O}_Y(D)^\vee$ given by evaluation on e, the lift is determined by

$$\{f, t\} = -\frac{t}{\kappa} \operatorname{div}_\omega(X_f).$$

The homogeneous volume form $t^{\kappa-1}\, dt \wedge \omega$ is invariant under every Hamiltonian vector field.

Proof. The canonical bundle of a smooth Poisson variety is a Poisson module [Pol, Lemma 4.4], with cotangent connection $\nabla_{df}\omega = \mathcal{L}_{X_f}\omega$. On top forms, $\mathcal{L}_{fX_g + gX_f} = f\mathcal{L}_{X_g} + g\mathcal{L}_{X_f}$, because the two additional terms cancel. This makes the connection linear over functions in its one-form argument; the Jacobi identity gives flatness.

There is a unique connection on $\mathcal{O}_Y(D)$ whose $(-\kappa)$-th tensor power is the canonical connection. Its coefficient in the frame e is $-\operatorname{div}_\omega(X_f)/\kappa$. Indeed, changing to $e' = ge$ gives $\omega' = g^{-\kappa}\omega$ and changes the divergence by $-\kappa X_f(g)/g$, as required. The connection is flat because its curvature, multiplied by $-\kappa$, is the canonical curvature.

Extend the bracket by the Leibniz rule and $\{t, t\} = 0$ to the Laurent algebra of the line bundle. Flatness gives the Jacobi identities involving the fiber coordinate. The bracket of local sections of $\mathcal{O}_Y(aD)$ and $\mathcal{O}_Y(bD)$ is a rational section of $\mathcal{O}_Y((a+b)D)$ regular on the smooth locus. By normality and reflexivity it extends uniquely across the singular locus. The identities remain valid there because they hold on a dense open subset. We obtain a bracket on the Cox ring. The formula is linear in the initial tensor. Applying the construction to any Poisson linear combination proves the assertion about pencils.

For a base function f, the fiber contribution to the divergence of X_f cancels its base divergence. Let $Z(f) = \operatorname{div}_\omega(X_f)$. Then $\operatorname{div}_\omega Z = 0$: writing $\omega = \varrho\, dx_1 \wedge \cdots \wedge dx_d$ and $\Pi_{ij} = \{x_i, x_j\}$, this is the cancellation of $\sum_{i,j} \partial_i\partial_j(\varrho\Pi_{ij})$ by antisymmetry. The Hamiltonian of t is tZ/κ, so it too has zero divergence. The Leibniz rule gives the assertion for every Hamiltonian. □

Lemma 3.2. *Suppose an affine open subset of Y has a birational log-canonical chart $f_1, \dots, f_d$ and a nowhere vanishing volume form*

$$\omega = \left(\prod_{j=1}^{d} f_j^{a_j} \right) d\log f_1 \wedge \cdots \wedge d\log f_d.$$

Then the canonical lift is log-canonical in $t, f_1, \dots, f_d$. If the original exchange matrix is compatible with the bracket and the nonzero a_j occur only at frozen indices, the lift is compatible with the homogeneous extension of that matrix.

Proof. Write $\{f_i, f_j\} = p_{ij} f_i f_j$. The vector field $f_i^{-1} X_{f_i}$ has divergence $\sum_j a_j p_{ij}$. Thus the coefficients $q_i = \{f_i, t\}/(f_i t) = -\sum_j a_j p_{ij}/\kappa$ are constant. If B is the exchange matrix and $\mathsf{P} = (p_{ij})$, compatibility makes every frozen column of $B\mathsf{P}$ zero, so $Bq = 0$. For $x_i = t^{\nu_i} f_i$, the pairing matrix Q of $t, x_1, \dots, x_d$ has entries $Q_{0i} = -q_i$ and $Q_{ij} = p_{ij} + \nu_j q_i - \nu_i q_j$. The homogeneous extension $B' = (-B\nu \mid B)$ therefore satisfies $B'Q = (0 \mid B\mathsf{P})$, as required. □

3.2. **Root covariance on the Cox space.** Let R be the Cox ring of $Y = \operatorname{Proj} R$, with its primitive positive grading, and suppose that R is a normal domain with constant units. Let

E be the grading derivation. Fix a homogeneous rational top form Ω on $\operatorname{Spec} R$ with divisor zero and degree $\kappa \neq 0$. A bivector is *unimodular with respect to* Ω if every Hamiltonian vector field preserves Ω.

Lemma 3.3. *A rational bivector on Y has at most one homogeneous lift of degree zero that is unimodular with respect to Ω.*

Proof. Over the generic point of Y, the Cox space is a one-dimensional torus bundle. The difference of two lifts is therefore $\mathsf{E} \wedge V$, for a rational vector field V of degree zero. Its divergence is

$$\operatorname{div}_\Omega(\mathsf{E} \wedge V) = \kappa V - (\operatorname{div}_\Omega V)\mathsf{E}.$$

If this vanishes, projection to Y gives $\kappa \overline{V} = 0$. Thus V is vertical, and $\mathsf{E} \wedge V = 0$. □

Proposition 3.4. *Let Π be a regular Poisson tensor on Y, and let $\widetilde{\Pi}$ be its canonical Cox lift. Suppose regular degree-zero vector fields D, A_D on the Cox space satisfy $\mathcal{L}_D\Omega = 0$, $\mathcal{L}_{A_D}\Omega = \gamma_D\Omega$, and $[A_D, D] = \lambda_D D$, for constants γ_D, λ_D. If their projections to Y satisfy $\mathcal{L}_{\overline{D}}\Pi = \overline{D} \wedge \overline{A}_D$, then*

$$\mathcal{L}_D\widetilde{\Pi} = D \wedge \left(A_D - \frac{\gamma_D + \lambda_D}{\kappa}\mathsf{E}\right).$$

The commutator of the second factor with D is $\lambda_D D$.

Proof. Since D preserves Ω, Lie differentiation by D commutes with divergence. Thus the left side is unimodular. On the right, $\operatorname{div}_\Omega(D \wedge A_D) = -\gamma_D D + [D, A_D] = -(\gamma_D + \lambda_D)D$, whereas $\operatorname{div}_\Omega(D \wedge \mathsf{E}) = -\kappa D$. The right side is therefore unimodular as well. The two sides have the same projection to Y, so Lemma 3.3 identifies them. Finally, $[\mathsf{E}, D] = 0$. □

Every degree-preserving additive-group action preserves Ω. Indeed, its pullback has the same divisor and hence differs from Ω by a constant; the additive group has no nontrivial characters. In particular, homogeneous locally nilpotent root derivations preserve Ω. When both root subgroups act, their Cartan derivation does too, as their commutator.

For a Cartan field H with $\mathcal{L}_H\Omega = \gamma\Omega$, the field $H - (\gamma/\kappa)\mathsf{E}$ preserves Ω and has the same projection to Y. For the standard simple-root covariance, $A_D = -H_i/(2t_i)$. The correction by the grading derivation leaves the commutator eigenvalue unchanged. Thus it is $-1/t_i$ for a raising operator and $1/t_i$ for a lowering operator. These are the signs used below.

3.3. **Recognition of adapted charts.** Let R be a positively graded factorial $\mathbb{C}$-domain with $R_0 = \mathbb{C}$, and suppose $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$. Assume that every frozen variable is prime in R, that the extended exchange matrix has full rank, and that the regular Poisson bracket is compatible in the convention

$$B\mathsf{P} = -\big(\operatorname{diag}(\delta_k^{-1}) \mid 0\big), \qquad \delta_k > 0.$$

Assume an algebraic torus acts diagonally in every seed and preserves the bracket. We also assume that the positive grading is induced by a one-parameter subgroup of this torus. We require its infinitesimal eigenvalue vectors to span $\ker B$, and call the resulting diagonal fields *grading derivations*.

Proposition 3.5. *Let D be a nonzero locally nilpotent derivation of degree zero and nonzero torus weight. Let b be frozen and suppose that $a = Db \neq 0$. Assume that a grading derivation G and a nonzero real number η_D satisfy*

$$bD = \eta_D\{a, -\} + a\mathsf{G}. \tag{3.1}$$

If a seed satisfies $B_{kb}/\eta_D \geq 0$ at every mutable position, then it is adapted to D. More precisely,

$$D = c_D x^v \partial_b, \qquad c_D \in \mathbb{C}^\times, \qquad v_b = 0, \qquad v_k = \frac{\delta_k}{\eta_D} B_{kb} \quad (k \text{ mutable}),$$

and every entry of v is a nonnegative integer. The same conclusion holds, without the column-sign assumption, if Db is already a scalar multiple of a monomial in the seed coordinates other than b, or if D fixes every seed coordinate other than b.

Proof. The mixed Laurent ring of the seed equals the localization of R at its mutable variables. None of these variables is divisible by the frozen prime b, since each is a unit in that Laurent ring and b is not. Consequently $b\mathcal{L}_{\mathrm{pol}}(\Sigma) \cap R = bR$. For a locally nilpotent derivation, the order of the last nonzero derivative is additive on products, by the Leibniz formula. It decreases by one under D, so $b \nmid Db$. The constant term a_0 of a, considered as a polynomial in b with mixed Laurent coefficients, is therefore nonzero.

Write $\mathsf{G}x_j = g_j x_j$. For each exponent v occurring in a_0, evaluate (3.1) on x_j, reducing modulo b. At $j = b$, first divide by b. We obtain $\eta_D v\mathsf{P} + g = e_b$. The difference u of two such exponents satisfies $u\mathsf{P} = 0$. Their torus weights agree, so u annihilates $\ker B$ and has the form $u = cB$. Compatibility forces $c = 0$. Thus $a_0 = c_D x^v$ is one monomial. Transposing the last equation and multiplying by B gives the asserted mutable exponents. These are nonnegative by the column hypothesis, and the frozen exponents are nonnegative because the coefficients are polynomial. Hence $x^v \in R$ and $a - c_D x^v \in bR$.

Both terms have the positive degree of b, because D has degree zero and the seed is homogeneous. Their difference is therefore a scalar multiple of b. Its torus weight differs from the weight of b, so this scalar is zero. Substitution in (3.1) shows that D kills every other seed coordinate.

Suppose instead that a is already a monomial in the required coordinates. Applying (3.1) to $x_j \neq b$ expresses bDx_j as a scalar multiple of ax_j. The latter has b-valuation zero unless its scalar vanishes, whereas the former has positive valuation. Adaptedness follows, and compatibility gives the same exponent formula.

Finally, suppose that D fixes every seed coordinate other than b. Applying (3.1) to the coordinates shows that every monomial of a satisfies $\eta_D v\mathsf{P} + g = e_b$. The same uniqueness argument makes a a single Laurent monomial. Its exponent at b is zero because $b \nmid Db$. Since $a \in R$, the Laurent-monomial assertion of [FeiS, Lemma 4.2] makes every mutable exponent nonnegative as well. □

Corollary 3.6. *Keep the assumptions on D and b, and suppose that $D^2 b = 0$. If a grading derivation A_D satisfies*

$$\mathcal{L}_D\Pi = D \wedge A_D, \qquad [A_D, D] = \lambda_D D, \qquad \lambda_D \in \mathbb{R} \setminus \{0\},$$

then (3.1) holds with $\eta_D = -\lambda_D^{-1}$. Thus the conclusions of Proposition 3.5 apply.

Proof. Set $T = \{b, -\}/b$. Compatibility and the grading-span assumption make T a grading derivation. Write $[T, D] = \mu D$ and $A_D b = hb$. Covariance evaluated on (b, f) gives

$$\{a, f\} = a(T - A_D)f + b(h - \mu)Df.$$

Since $Ta = \mu a$, $A_D a = (h + \lambda_D)a$, and $Da = 0$, evaluation at $f = a$ yields $\mu = h + \lambda_D$. The desired identity follows with $\mathsf{G} = \lambda_D^{-1}(T - A_D)$. □

An element $f \in R$ is *Poisson-normal* if $\{f, R\} \subseteq fR$.

Lemma 3.7. *Let $H_1, \dots, H_q$ be commuting grading derivations. Replacing Π by*

$$\Pi' = \Pi + \sum_{u<v} c_{uv} H_u \wedge H_v$$

preserves compatibility and the hypotheses of Proposition 3.5, with the same η_D. It also preserves the covariance hypothesis of Corollary 3.6, with the same λ_D. Poisson normality is unchanged for simultaneous eigenvectors of the H_u.

Proof. The grading derivations preserve Π, so the new bivector is Poisson. If W is their eigenvalue matrix, its logarithmic matrix differs from P by WCW^{T}, for a skew matrix C. Since $BW = 0$, compatibility is unchanged.

If $H_u a = \alpha_u a$, the Hamiltonian identity remains valid after replacing G by

$$\mathsf{G}' = \mathsf{G} - \eta_D \sum_{u<v} c_{uv}(\alpha_u H_v - \alpha_v H_u).$$

For covariance, write $[H_u, D] = \chi_u D$. The new covariance derivation is

$$A'_D = A_D + \sum_{u<v} c_{uv}(\chi_v H_u - \chi_u H_v).$$

Its added term commutes with D, since its eigenvalue is a sum of differences $\chi_v\chi_u - \chi_u\chi_v$. Thus $[A'_D, D] = \lambda_D D$. Finally, for a simultaneous eigenvector f of the H_u, the added bracket $\{f, -\}' - \{f, -\}$ is divisible by f, which proves the last assertion. □

3.4. **Root derivatives and frozen factors.** We use the following form of the maximal-derivative result [FeiS, Proposition 4.12].

Proposition 3.8. *Let $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$ have polynomial frozen variables and a full-rank extended exchange matrix. Suppose a reachable seed is adapted to a locally nilpotent derivation $D = m_D\partial_b$. Assume that the nonzero entries of the b-column have one sign and that every mutable variable dividing m_D has an index at which the b-column is nonzero. If z is a monomial in the variables of one seed and $h = \max\{j : D^j z \neq 0\}$, then $D^h z/h!$ is a nonzero scalar multiple of a monomial in the variables of a reachable seed.*

In particular, successive maximal root derivatives carry an extremal vector to every other extremal vector in the same simple finite-dimensional module. If the initial vector is a monomial in the variables of one seed and the simple-root operators satisfy the proposition, every extremal vector has this property up to scalar. Frozen factors are allowed in these statements.

The next observation excludes frozen factors when both directions of a rank-one action are available.

Lemma 3.9. *Let $R = \mathcal{U}_{\mathrm{pol}}(\Sigma)$ be a positively graded factorial domain with $R_0 = \mathbb{C}$ and homogeneous prime frozen variables b_j, which are not inverted. Suppose degree-zero locally nilpotent derivations E_i, F_i belong to an $\mathfrak{sl}_2$-triple. Let π permute the frozen indices, and suppose*

$$E_i b_j = 0 \quad (j \neq i), \qquad F_i b_j = 0 \quad (j \neq \pi^{-1}(i)), \qquad \deg b_i = \deg b_{\pi^{-1}(i)}.$$

In a seed adapted to E_i, the coefficient of ∂_{b_i} contains no frozen factor, unless $E_i b_i$ is a scalar multiple of $b_{\pi^{-1}(i)}$.

Proof. Write $E_i b_i = m_i$, and suppose a frozen prime f divides m_i. It is distinct from b_i. In the adapted mixed Laurent chart, $E_i = m_i\partial_{b_i}$, so $E_i(R) \subseteq fR$. Here we use the contraction $f\mathcal{L}_{\mathrm{pol}}(\Sigma) \cap R = fR$, since no mutable coordinate is divisible by f.

Suppose $F_i f = 0$. Then $H_i = [E_i, F_i]$ satisfies $H_i(R) \subseteq fR$ and $H_i f = 0$. If $E_i(R) \subseteq f^a R$, both $H_i E_i(R)$ and $E_i H_i(R)$ lie in $f^{a+1}R$. The relation $[H_i, E_i] = 2E_i$ therefore gives $E_i(R) \subseteq f^{a+1}R$. Induction contradicts $E_i \neq 0$, since $\bigcap_{a\geq 1} f^a R = 0$. Thus $f = b_{\pi^{-1}(i)}$. Equality of its degree with $\deg m_i = \deg b_i$ forces $m_i/f \in R_0 = \mathbb{C}$, proving the assertion. □

3.5. **The current generators at the origin.** Let $R_r = \mathrm{Cox}(Y_r)$, with the current action and opposition convention of Section 2. For the operator formulas, we normalize the invariant form so that the longest affine real roots have squared length two and write $t_i = 2/(\alpha_i, \alpha_i)$. In a word seed the mutable symmetrizer is $\delta_k = t_{i_k}$; its entries remain attached to the mutable positions under mutation.

Let b_i be the minimal lift of the terminal i-minor when the letter i occurs, and let τ be the canonical section of the unique Schubert boundary divisor.

Corollary 3.10. *For every family and node in Table 1, the Schubert Cox mutation class contains an adapted seed for each finite Chevalley operator and for the positive affine simple operator whenever it is nonzero. The active coordinates are*

$$b(E_i) = b_i, \qquad b(F_r) = \tau, \qquad b(F_i) = b_j \quad (i \neq r).$$

In the third formula, j is determined by $w^{-1}\alpha_i = \alpha_j$. The first formula includes $i = 0$ when $E_0 \neq 0$. In an adapted seed,

$$D = c_D \prod_{x \neq b(D)} x^{m_x(D)} \partial_{b(D)}, \qquad m_{x_k}(D) = \frac{\delta_k}{t_i} |B^D_{k,b(D)}| \quad (k \text{ mutable}),$$

where i is the simple-root index of D. The active exchange column is nonnegative for E_i and nonpositive for F_i. The operator E_0 vanishes precisely when $0 \notin \operatorname{supp}(w)$. All these seeds are homogeneous for Cartan degree, finite Cartan weight, and current degree.

Proof. The Demazure identification is equivariant for the affine Borel and the finite group on the complete Veronese section ring. The current action and the geometric Cox action therefore agree on $R_r^{(p_r)}$. They agree on R_r as well: a derivation has a unique extension through the finite separable field extension $\operatorname{Frac} R_r / \operatorname{Frac} R_r^{(p_r)}$. The opposition convention identifies the geometric boundary highest section with the section of finite weight ω_r.

The finite group preserves Y_r. By Proposition 2.4, its unique Schubert boundary divisor is indexed by $s_r w$. Thus r is the unique left descent. The Schubert root-operator theorem [FeiS, Theorem 4.1] now supplies all the stated charts and exponent formulas. More explicitly, raising charts are reached by mutating the nonterminal occurrences of the corresponding letter. A reduced word beginning with r is already adapted to F_r, with coefficient its first lifted minor. For $i \neq r$, one reverses the symmetric CGL presentation and applies the raising construction to the index j determined by $w^{-1}\alpha_i = \alpha_j$. The word-independence theorem places these constructions in a single mutation class. The same root-operator theorem gives the criterion for $E_0 \neq 0$, and homogeneous minimal lifting preserves all three gradings. □

The adapted coefficients need not be products of mutable coordinates alone. They contain frozen factors in the rank-one symplectic and even twisted examples of Section 9. Lemma 3.9 gives a restriction on such factors when both operators of an $\mathfrak{sl}_2$-triple are available.

4. Source translation and untwisted KR algebras

In this section and Section 5, we work with untwisted current algebras. We fix r and write $R = R_r$, $\lambda = \omega_r^\vee$, $L = t_r$, and $d = \langle 2\rho, \lambda \rangle$. We use the adjoint group for the loop-group description, so that every fundamental coweight is a cocharacter. The current modules are representations of the simply connected group.

Let μ be the least dominant coweight in the component of λ, and let π be the corresponding length-zero element of the extended affine Weyl group. The minimal affine Weyl representative is $w = t_{-\lambda}\pi$. We set $q = \pi^{-1}(0)$, and write $k_0 = 1, k_1, \ldots, k_n$ for the affine comarks. Thus $\sum_i k_i = h^\vee$. We call the component *neutral* if $\mu = 0$. We also set $j_* = \pi^{-1}(r)$. This is the index of the distinguished terminal boundary minor; in general $j_* \neq r$.

Let $\tau \in R_1$ be the canonical section of the Schubert divisor. We denote the terminal cell minors by $f_i = D_{\Lambda_i, w\Lambda_i}$, and their minimal Cox lifts by $\widetilde{b}_i$. The boundary coroot formula gives

$$\deg \widetilde{b}_i = Lk_i - \delta_{\pi(i),r}. \tag{4.1}$$

The minor indexed by an absent letter is one. We normalize the *component section* as $c = \tau^L f_q \in R_L$.

The coefficient $M = \langle \theta, \lambda \rangle$ of α_r in the highest root determines the current truncation. We have $Lk_r = M$, and π preserves the comarks. When $M = 1$, the modules are evaluation modules and source translation is trivial; we use the origin cluster structure. When $M > 1$,

every degree in (4.1) is positive. Since the minimal lift of an absent minor has degree zero, w then has full affine support.

4.1. **Source translation and the Poisson pencil.**

Lemma 4.1. *The open Schubert cell is the moduli space of framed B_--bundles on $\mathbb{P}^1$ with torus degree λ, with the opposition convention fixed above. Source translation defines a regular additive-group action Ψ_a on the cell. It commutes with the finite Cartan torus and preserves the polynomial volume form.*

Proof. For a positive finite root β, set $m_\beta = \langle \beta, \lambda \rangle$. A transition function has the form $n_-(u)u^{-\lambda}$. Changes of unipotent trivialization on the two affine charts, with value one at infinity, give the quotient

$$\mathbb{C}[u, u^{-1}]/\big(u^{-1}\mathbb{C}[u^{-1}] + u^{m_\beta}\mathbb{C}[u]\big)$$

in root direction $-\beta$. Order the roots by height. Commutators involve only roots of greater height, so successive elimination gives the unique representative

$$\left(\prod_{\beta>0} x_{-\beta}\left(\sum_{0\le j<m_\beta} z_{\beta j}u^j\right)\right)u^{-\lambda}.$$

The elimination works over every coefficient algebra, so it identifies the moduli functor with an affine space. Conjugation by a representative of w_0, followed by the pinned opposition ς, gives the positive expression $n_+(u)u^{\lambda}$. These are exactly the Iwahori-cell representatives of w, after accounting for the length-zero factor in a nonneutral component.

Translation fixes the framed fiber at infinity and therefore acts regularly on the moduli space, commuting with changes of Cartan framing. We choose the parameter so that the source substitution is $u \mapsto u - a$. Finally, an automorphism of affine space multiplies its polynomial volume form by a constant. These constants form a character of the additive group, so they are all one. □

We lift Ψ_a to $R[\tau^{-1}]$ by fixing τ, and write J for its infinitesimal generator. On the Cox space this action is rational. Its pole along the Schubert divisor will determine the new exchange relation.

For an algebra automorphism Φ, we write $\Phi\Pi$ for the transported Poisson tensor, with bracket

$$\{f, g\}_{\Phi\Pi} = \Phi\big(\{\Phi^{-1}f, \Phi^{-1}g\}_\Pi\big).$$

Let $\overline{\Pi}_0$ be the standard trigonometric Poisson tensor on the projective Schubert variety. We normalize it so that the finite simple-root summand is $-E_i \wedge F_i/(2t_i)$. Denote by $\overline{\Pi}_{\mathrm{rat}}$ the rational loop-group tensor with the corresponding sign.

Proposition 4.2. *On the open cell,*

$$\Psi_a\overline{\Pi}_0 = \overline{\Pi}_0 + a\overline{\Pi}_{\mathrm{rat}}, \qquad \Psi_a\overline{\Pi}_{\mathrm{rat}} = \overline{\Pi}_{\mathrm{rat}}.$$

Both tensors extend regularly to Y. Their canonical Cox lifts form a regular compatible Poisson pencil Π_a on R, and $\Pi_a = \Psi_a\Pi_0$ on $R[\tau^{-1}]$.

Proof. The rational Manin triple gives a regular Poisson structure on the thick affine Grassmannian. Each polynomial-loop orbit in the thin affine Grassmannian is a union of symplectic leaves [KWWY, Section 2C, Theorem 2.5 and the paragraph following it]. Its reduced closure is therefore a Poisson subvariety. In particular, this applies to Y for the adjoint group in every component. The standard trigonometric tensor is regular on the Schubert variety by the usual flag Poisson construction.

Proposition B.1 computes the loop kernels after based normalization and proves the two identities for every dominant coweight.

Thus every linear combination in the stated pencil is Poisson on a dense open set, and hence on Y. Proposition 3.1 gives regular compatible lifts. Source translation preserves the homogeneous volume form on the Cox localization, because it fixes τ and preserves the cell volume form. Uniqueness of the unimodular lift proves $\Pi_a = \Psi_a \Pi_0$ there. □

4.2. **Prefix minors on the boundary chart.** We choose a reduced word $w = s_r v = s_r s_{i_2} \cdots s_{i_d}$ and write $v_k = s_{i_2} \cdots s_{i_k}$, with $v_1 = 1$. Let x_k be its minimal Cox prefix variables. Normalize the first one as $x_1 = \eta = F_r \tau$. We write $U(v) = U_+ \cap \dot{v} U_- \dot{v}^{-1}$ and $U^v = U_+ \cap \dot{v} U_+ \dot{v}^{-1}$.

Lemma 4.3. *On the chart $\eta \neq 0$, set $t = \tau/\eta$. The quotient by the additive action generated by F_r has coordinates $\eta^{\pm 1}$ and $n_b \in U(v)$. For every $k \geq 2$,*

$$\eta^{-\nu_k} x_k = D_{\Lambda_{i_k}, v_k \Lambda_{i_k}}(n_b), \qquad \nu_k = \langle v_k \Lambda_{i_k}, \alpha_r^\vee \rangle. \tag{4.2}$$

Thus freezing η, dividing by the indicated powers of η, and specializing this coefficient to one gives the reduced-word seed of $U(v)$.

Proof. Write the open-cell representative as $n_w = x_r(z) \dot{s}_r n \dot{s}_r^{-1}$, with $n \in U(v)$. In the rank-one subgroup we choose the standard representatives, so that

$$x_r(z) \dot{s}_r = y_r(z^{-1}) h_r(z) x_r(-z^{-1}).$$

Factor $x_r(-z^{-1}) n = n^{(z)} q^{(z)}$, with $n^{(z)} \in U(v)$ and $q^{(z)} \in U^v$, and set $n_b = h_r(z) n^{(z)} h_r(z)^{-1}$. The Schubert point consequently has coordinates $y_r(t) n_b \dot{v} P$, where $t = z^{-1}$. The homogeneous coordinate changes from τ to $\eta = z\tau$. In these coordinates $F_r = \partial_t$, so η and n_b are invariant.

Fix k and abbreviate $i = i_k$. Let ξ_i be the highest-coordinate functional in the fundamental highest-weight module. The rank-one identity gives $\xi_i x_r(z) \dot{s}_r = z^{\delta_{ir}} \xi_i x_r(-z^{-1})$. Moreover, $q^{(z)}$ fixes the extremal vector of weight $v_k \Lambda_i$: every root of U^v remains positive under v_k^{-1}, because the inversion set of v_k is contained in that of v. Hence

$$D_{\Lambda_i, s_r v_k \Lambda_i}(n_w) = z^{\delta_{ir}} D_{\Lambda_i, v_k \Lambda_i}(n^{(z)}) = z^{\nu_k} D_{\Lambda_i, v_k \Lambda_i}(n_b).$$

The last equality is conjugation by $h_r(z)$, and its exponent is $\delta_{ir} - \langle \Lambda_i - v_k \Lambda_i, \alpha_r^\vee \rangle = \nu_k$. The last minor is nonzero at the general boundary point. Therefore ν_k is the exact pole order of the prefix function along $\tau = 0$. Multiplication by τ^{ν_k} proves (4.2).

Deleting the first position from the reduced-word exchange matrix gives the exchange matrix for v: only the predecessor entry at that position disappears. The powers of η are determined by homogeneity. They disappear after the normalization in (4.2), proving the seed statement as an identity of functions and exchanges. □

Lemma 4.4. *Every CGL mutation sequence for $U(v)$ lifts to a mutation sequence in the origin Cox seed avoiding η. In particular, CGL reversal reaches the pullback of the reduced-word seed for $U(v^{-1})$ under $\iota_v(n) = \dot{v}^{-1} n^{\mathsf{t}} \dot{v}$, where t is the affine Chevalley anti-involution. Its terminal minors have the same simple-root indices.*

Proof. The first assertion follows by homogenizing each exchange with respect to η in the preceding lemma. Inverting η permits every required power, and the mutations themselves are performed in the Cox seed. Since η is never mutated, every resulting variable remains F_r-invariant.

For the second assertion, the anti-involution sends the ordered inversion root β to $-v^{-1}\beta$. It reverses the order of the root factors as well. These are exactly the ordered inversion roots for the reversed word of v^{-1}. Thus the prefix minors for the reversed word are the prime elements of the reverse symmetric CGL presentation. The CGL interval-order mutations reach this presentation [GY, Theorem 7.3(b),(c)]. The same matrix-coefficient transposition gives

$$D_{\Lambda_i, v^{-1}\Lambda_i}(\iota_v(n)) = \gamma_i D_{\Lambda_i, v\Lambda_i}(n), \qquad \gamma_i \in \mathbb{C}^\times,$$

so the terminal minor indices are preserved. The constants depend only on the fixed pinning and may be absorbed in the standard minor normalization. □

4.3. **The root action on the Schubert boundary.** We now assume $M > 1$ and use $v = s_r w$. Since π preserves the affine marks, $j_* \neq 0$. The affine translation formula gives

$$v^{-1}(\delta_{\mathrm{aff}} - \alpha_r) = \alpha_{j_*}. \tag{4.3}$$

Proposition 4.5. *On $\eta \neq 0$, the divisor $\tau = 0$ is the product of the grading torus and the Schubert cell $U(v)$. The translation field has a decomposition $J = \mathcal{A} + t^{-1}\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ act on $\mathbb{C}[U(v)][\eta^{\pm 1}]$ and annihilate t and η. Under Chevalley transposition*

$$\iota_v : U(v) \longrightarrow U(v^{-1}), \qquad n \longmapsto \dot{v}^{-1} n^{\mathsf{t}} \dot{v},$$

the residue derivation $\mathcal{B}$ is a nonzero scalar multiple of the positive simple-root operator E_{j_*}.

Proof. We use a Gauss chart near the divisor. Choose a reduced word for w beginning with r, and delete that first letter to obtain a word for v. The section η is a Weyl translate of τ, so $\eta \neq 0$ is the corresponding translate of the open Schubert cell. The rank-one Gauss chart on $\eta \neq 0$ has representatives

$$g(t, n; u) = x_{-r}(t) n(u) \dot{s}_r u^{\lambda}, \qquad n \in U(v), \quad \lambda = \omega_r^{\vee}.$$

We choose the signs in t and the pinned root homomorphisms so that the first positive coordinate on $\tau \neq 0$ is $1/t$. The infinitesimal lowering action is translation in t. Consequently the F_r-invariant coordinates in this chart are precisely the coordinates of $U(v)$, together with $\eta^{\pm 1}$. At $t = 0$ this is the open cell of the boundary Schubert variety.

The finite parts of the inversion roots of v are positive and different from α_r. Indeed, its inversion roots are obtained by applying s_r to those of w other than α_r, and s_r preserves every other positive finite root. We may therefore regard $n(u)$ as a polynomial loop in the unipotent radical generated by those roots. This radical is normalized by the finite rank-one subgroup corresponding to r.

In the positive-Borel description of the framed bundle, with transition function $n_+(u)u^{\lambda}$, normalized source translation is generated by

$$Jg = -\partial_u g + u^{-1}\zeta g, \qquad \zeta = \mathrm{Ad}_{n_+(0)}\, \lambda.$$

This follows by removing the unique negative Laurent term introduced by the change of source coordinate. Equivalently, on the positive unipotent factor it is

$$Jn_+ = -n_+' + u^{-1}\big(\mathrm{Ad}_{n_+(0)}\, \lambda\, n_+ - n_+ \lambda\big).$$

Set $\lambda' = s_r\lambda = \lambda - \alpha_r^{\vee}$. The rank-one Gauss identity gives

$$\zeta = \mathrm{Ad}_{x_{-r}(t)n(0)}\big(\lambda' - t^{-1} e_r\big). \tag{4.4}$$

To verify this formula, use the factorization in Lemma 4.3, with $z = t^{-1}$. For a positive finite root $\gamma \neq \alpha_r$, $\langle \gamma, \lambda' \rangle = \langle s_r\gamma, \lambda \rangle \geq 0$. Thus the constant complementary factor has the form $q^{(z)}(0) = x_r(-t) q_0$, where q_0 is generated by roots of λ'-degree zero. These root groups commute with x_r: a root $\gamma + \alpha_r$ would have negative λ'-degree. In particular q_0 centralizes λ'. The same Gauss factorization gives

$$\zeta = \mathrm{Ad}_{x_{-r}(t)n(0)h_r(z)x_r(-t)}\, \lambda' = \mathrm{Ad}_{x_{-r}(t)n(0)}(\lambda' - t z^2 e_r),$$

which is (4.4) because $t z^2 = t^{-1}$.

Substitute (4.4) into the formula for Jg. In the n-coordinate, its term with a pole at $t = 0$ is represented by $-u^{-1}\,\mathrm{Ad}_{n(0)}\, e_r\, n$. We may add $u^{-1} n e_r$, since this is tangent to the right stabilizer of the partial-flag point. In fact, (4.3) says that conjugating e_r/u by $\dot{v}^{-1}$ gives a nonzero multiple of the finite root vector f_{j_*}. Since $j_* \neq q$, this vector belongs to the Levi subalgebra of the right parabolic. The resulting polynomial vector field is

$$\mathcal{B}(n) = \frac{n e_r - \mathrm{Ad}_{n(0)}\, e_r\, n}{u}. \tag{4.5}$$

The numerator has zero constant term. The regular part of the calculation is $-n' + u^{-1}(\mathrm{Ad}_{n(0)}\, \lambda'\, n - n\lambda')$. The first coordinate η/τ is the constant polynomial coordinate

for the simple root α_r. Source translation fixes it, and hence fixes both t and η. This proves the stated decomposition.

It remains to identify this vector field as a root action. Let $X = e_r/u$, a negative real affine root vector. The affine Chevalley anti-involution $\mathtt{t}$ sends $e_\alpha u^k$ to a fixed nonzero multiple of $f_\alpha u^{-k}$; in particular, it reverses the loop degree. Consider the Gauss factorization

$$n \exp(sX) = b_-(s) n_s, \qquad n_s \in U(v), \quad b_-(s) \in U_- \cap \dot{v} U_- \dot{v}^{-1}. \tag{4.6}$$

This factorization is the transpose of the usual positive root action on $U(v^{-1})$. More explicitly, Chevalley transposition changes its left side into $\exp(s' e_{j_*}) \iota_v(n)$, where s' is a fixed nonzero multiple of s, by (4.3). It changes the right side into $\iota_v(n_s)\iota_v(b_-(s))$, whose second factor belongs to $U_+ \cap \dot{v}^{-1} U_+ \dot{v}$. This is the complementary positive subgroup in the Schubert-cell factorization for v^{-1}. Hence (4.6) exists on the whole cell and describes that positive simple-root action.

Differentiating (4.6) at $s = 0$ gives $\dot{n} = nX - \dot{b}_- n$. The negative affine part of $\mathrm{Ad}_n X$ is exactly $u^{-1} \mathrm{Ad}_{n(0)} e_r$: all its finite root vectors are positive, and its Laurent expansion has no power below u^{-1}. Consequently $\dot{b}_- = u^{-1} \mathrm{Ad}_{n(0)} e_r$, and $\dot{n}$ is precisely (4.5). We verify the complementary subgroup in loop coordinates. For a negative-gauge root $\gamma + k\delta_{\mathrm{aff}}$, where γ is finite positive and $k < 0$, the root $v^{-1}(\gamma + k\delta_{\mathrm{aff}})$ is negative. For $\gamma \neq \alpha_r$, its current coefficient is $k - \langle \gamma, \lambda' \rangle < 0$; for $\gamma = \alpha_r$, it has finite part $-\alpha_r$ and current coefficient $k+1 \leq 0$. The length-zero factor, when present, preserves signs of affine roots. Thus every removed negative-gauge factor is carried by transposition to the required positive complement. This identifies the residue with the asserted simple-root action. □

4.4. **The translation field in two Gauss charts.** On the chart $\eta \neq 0$, Proposition 4.5 gives

$$J = \mathcal{A} + t^{-1}\mathcal{B}, \qquad Jt = J\eta = 0,$$

where $\mathcal{A}, \mathcal{B}$ preserve $\mathbb{C}[U(v)][\eta^{\pm 1}]$ and annihilate t, η. Since $\mathcal{B}$ is a root derivation, it is locally nilpotent.

Lemma 4.6. *The derivations $\mathcal{A}$ and $\mathcal{B}$ commute. Consequently $\mathcal{D} = \tau J$ and $\mathcal{D}^- = [F_r, \mathcal{D}]$ are commuting regular locally nilpotent derivations on the Cox ring, of Cartan degree one. They annihilate τ and η. Under the adjoint action of the finite rank-one subgroup associated with r, they span its natural two-dimensional module.*

Proof. Choose a faithful matrix representation and expand $n(u) = n_0 + u n_1 + \cdots$. Set

$$H = \mathrm{Ad}_{n_0} \lambda', \qquad X = \mathrm{Ad}_{n_0} e_r, \qquad Q = n_1 n_0^{-1}.$$

The normalized fields computed in the preceding proposition are

$$\mathcal{A}n = -n' + \frac{Hn - n\lambda'}{u}, \qquad \mathcal{B}n = \frac{ne_r - Xn}{u}.$$

Taking their constant coefficients gives

$$(\mathcal{A}n_0) n_0^{-1} = -Q + [H, Q], \qquad (\mathcal{B}n_0) n_0^{-1} = [Q, X].$$

Since $[\lambda', e_r] = -e_r$, we have $[H, X] = -X$, and the Jacobi identity yields

$$\mathcal{A}X + \mathcal{B}H = [-Q + [H, Q], X] + [[Q, X], H] = 0.$$

Substituting these identities in the commutator gives

$$[\mathcal{A}, \mathcal{B}]n = -\frac{(\mathcal{A}X + \mathcal{B}H)n}{u} + \frac{([H, X] + X)n + n([e_r, \lambda'] - e_r)}{u^2} = 0.$$

On $R[(\tau\eta)^{-1}]$, the derivations J and $t^{-1}\mathcal{B}$ are commuting locally nilpotent derivations. Their difference $\mathcal{A}$ is therefore locally nilpotent. Since $\mathcal{A}$ preserves $\mathbb{C}[U(v)][\eta^{\pm 1}]$, its restriction is locally nilpotent as well.

The expression on the η-chart shows that J has at most a simple pole along $\tau = 0$, and it is regular on $\tau \neq 0$. The distinct primes τ, η have no common prime divisor, so these charts

meet every prime divisor. Normality gives $\mathcal{D} = \tau J \in \mathrm{Der}(R)$. It is locally nilpotent because $J\tau = 0$ and J is locally nilpotent on $R[\tau^{-1}]$.

Here $F_r = \partial_t$, so $\mathcal{D} = \eta(t\mathcal{A} + \mathcal{B})$ and $\mathcal{D}^- = \eta\mathcal{A}$. These fields commute and annihilate τ, η. The commutator definition makes $\mathcal{D}^-$ regular on R, and its expression $\eta\mathcal{A}$ makes it locally nilpotent.

Source translation commutes with the finite positive Borel action. Thus $[E_r, \mathcal{D}] = 0$ and $[H_r, \mathcal{D}] = \mathcal{D}$. The derivations of a fixed degree in a finitely generated positively graded algebra form a finite-dimensional rational module: restrict them to a finite-dimensional homogeneous generating space. The field J is nonzero: on the highest-root line $n_+(u) = x_\theta(zu^{M-1})$, its formula above changes the coefficient of u^{M-2} at rate z. Thus $\mathcal{D} \neq 0$, and its highest weight-one string is $\mathcal{D}, [F_r, \mathcal{D}]$. In particular, $[E_r, \mathcal{D}^-] = \mathcal{D}$ and $[F_r, \mathcal{D}^-] = 0$, as required. □

The rank-one invariant $\Theta_r = \tau\mathcal{D}^- - \eta\mathcal{D}$ is consequently a regular locally nilpotent derivation, and $\mathcal{B} = -\Theta_r/\eta^2$. This gives a uniform way to calculate pole orders in every untwisted component.

For a locally nilpotent derivation D and a nonzero element f, we write $\deg_D f = \max\{j : D^j f \neq 0\}$.

Proposition 4.7. *Let $x \in R \setminus \{0\}$ be homogeneous and satisfy $F_r x = 0$. For every $a \neq 0$, the pole order of $\Psi_a(x)$ along $\tau = 0$ is $\deg_{\Theta_r}(x)$. Consequently a prefix minor in a reduced word beginning with r satisfies*

$$\nu_k(a) = \nu_k(0) + \deg_{\Theta_r}\big(x_k^{(0)}\big).$$

Proof. By Lemma 4.3, $R[\eta^{-1}] = S[t]$, where $S = \mathbb{C}[U(v)][\eta^{\pm 1}]$ and $F_r = \partial_t$. Thus $x \in S$. The preceding lemma gives commuting locally nilpotent derivations $\mathcal{A}, \mathcal{B}$ of S, with $J = \mathcal{A} + t^{-1}\mathcal{B}$ and $\mathcal{B} = -\eta^{-2}\Theta_r$. It follows that

$$\Psi_a(x) = \exp(a\mathcal{A}) \left(\sum_{m \geq 0} \frac{a^m}{m!} t^{-m} \mathcal{B}^m x \right).$$

The sum is finite. Its coefficient at the most negative power of t is nonzero, because $\exp(a\mathcal{A})$ is an automorphism of S and $a \neq 0$. Thus its pole order is exactly $\deg_{\mathcal{B}}(x)$. Since $\mathcal{B}\eta = 0$, this equals $\deg_{\Theta_r}(x)$. At the generic point of $\tau = 0$, the function η is a unit. Hence $t = \tau/\eta$ and τ have the same valuation, which proves the first assertion. The second follows by applying it to the origin minimal lift, which is F_r-invariant. □

4.5. **The Hamiltonian of the boundary section.** Let K be the derivation of the dual current grading, so that a root coordinate of mode j has degree $-j$. The framed normalization is homogeneous in u and a, so $[K, J] = -J$. The derivation J commutes with the finite Cartan fields and fixes τ.

Lemma 4.8. *The canonical Cox pencil Π_a satisfies*

$$\{\tau, -\}_a = \{\tau, -\}_0 + \frac{a}{L}\mathcal{D}, \qquad \mathcal{D} = \tau J. \tag{4.7}$$

If $\langle \theta, \lambda \rangle \geq 2$, the divisor $\tau = 0$ is not Poisson for Π_a when $a \neq 0$.

Proof. Write $H_{\tau,0} = \tau^{-1}\{\tau, -\}_0$. For a terminal cell minor f_i, its logarithmic Hamiltonian has rotation coefficient $-k_i$. Indeed,

$$\{\log f_i, g\}_0 = -\tfrac{1}{2}((w+1)\Lambda_i, \mathrm{wt}(g))g,$$

and both $w\Lambda_i$ and Λ_i have level k_i. The remaining part is a finite Cartan field. Equation (1.2) expresses the cell volume form as the product of the terminal minors times the logarithmic coordinate volume. The divergence formula for the canonical lift therefore gives $H_{\tau,0}|_{\mathbb{C}(Y)} = -\kappa^{-1}\sum_i \{\log f_i, -\}_0$. Since $\kappa = Lh^\vee$ and $\sum_i k_i = h^\vee$, the rotation coefficient of $H_{\tau,0}$ is $1/L$. Thus $H_{\tau,0}$ is K/L plus finite Cartan and grading fields. The grading field is determined by $H_{\tau,0}\tau = 0$. In particular $H_{\tau,0}$ is regular on the Cox ring.

By Proposition 4.2, $\Pi_a = \Psi_a \Pi_0$. Since $\Psi_a \tau = \tau$, we obtain $H_{\tau,a} = \Psi_a H_{\tau,0} \Psi_a^{-1}$. The commutator $[J, H_{\tau,0}] = J/L$ has no further nonzero commutator with J. Consequently $H_{\tau,a} = H_{\tau,0} + aJ/L$, which proves (4.7).

We check nonvanishing on the boundary. Let $v = s_r w$. The index j_* belongs to the support of v. Indeed, otherwise $v\Lambda_{j_*} = \Lambda_{j_*}$. The finite part of the left side, obtained from $w = t_{-\lambda}\pi$, is $(1 - M)(\omega_r - \alpha_r)$. For any finite node adjacent to r, its pairing with that simple coroot is negative, whereas ω_{j_*} is dominant.

The positive simple-root operator E_{j_*} on $U(v^{-1})$ is therefore nonzero. The residue identification makes $\mathcal{B} \neq 0$. In the η-chart, $\mathcal{D} = \eta(t\mathcal{A} + \mathcal{B})$, and hence $\mathcal{D} \bmod \tau = \eta\mathcal{B} \neq 0$. The first term on the right of (4.7) is divisible by τ. The second has nonzero residue when $a \neq 0$, proving that (τ) is not a Poisson ideal. □

Corollary 4.9. *Assume $\langle \theta, \lambda \rangle \geq 2$. For $a \neq 0$, the minimal lift of each translated terminal minor has degree Lk_i. More precisely, if $\widetilde{b}_i$ is its origin minimal lift, then $\Psi_a(\widetilde{b}_i)$ is regular for $i \neq j_*$, whereas $\tau\Psi_a(\widetilde{b}_{j_*})$ is the minimal lift of the terminal j_*-minor.*

Proof. After division by powers of η, the origin terminal lifts are precisely the terminal minors of $U(v)$. Affine Chevalley transposition preserves their simple-root indices. On $U(v^{-1})$, the positive simple-root operator E_{j_*} annihilates each terminal i-minor for $i \neq j_*$, while its string on the terminal j_*-minor has length one. Since $\mathcal{B}\eta = 0$, it follows that $\deg_{\mathcal{B}}(\widetilde{b}_i) = \delta_{i,j_*}$. The exact pole-degree formula for $J = \mathcal{A} + t^{-1}\mathcal{B}$ gives $\nu_i(a) = \nu_i(0) + \delta_{i,j_*}$. At the origin the boundary coroot gives $\nu_i(0) = Lk_i - \delta_{i,j_*}$, so $\nu_i(a) = Lk_i$, as claimed. □

4.6. **Recovering an exchange from a divisor.** The next theorem recovers an exchange relation from a prime divisor that is not Poisson. It allows us to *unfreeze* a coordinate, that is, to make it mutable, by constructing its exchange relation.

Theorem 4.10. *Let R be a finitely generated normal complex domain with a Poisson bracket, and let $x_0, x_1, \ldots, x_{N-1}$ be pairwise nonassociated prime elements which generate its fraction field and form a transcendence basis. Suppose*

$$\{x_i, x_j\} = p_{ij} x_i x_j, \qquad \mathsf{P} = (p_{ij}) \in \mathrm{Mat}_N(\mathbb{Q}).$$

Let an algebraic torus act diagonally on these elements and by Poisson automorphisms on R, and let W be the matrix of their characters. Assume that

$$v\mathsf{P} = 0, \quad v\mathsf{W} = 0 \quad \Longrightarrow \quad v = 0.$$

Assume also that the prime ideal (x_0) is not a Poisson ideal. Then there is a primitive vector $v = (0, v_1, \ldots, v_{N-1}) \in \mathbb{Z}^N$, unique up to sign, and a constant $c \in \mathbb{C}^\times$ such that

$$x_0' = \frac{\prod_{j>0} x_j^{[v_j]_+} - c \prod_{j>0} x_j^{[-v_j]_+}}{x_0} \in R.$$

The numerator vanishes to order exactly one along (x_0). Furthermore $v\mathsf{W} = 0$ and $v\mathsf{P} = -\gamma e_0$ for a nonzero rational number γ.

Suppose a compatible extended exchange matrix B has its mutable indices among $1, \ldots, N-1$, with $B_i\mathsf{P} = -\delta_i^{-1} e_i$ and $\delta_i > 0$. Choose the sign of v so that $\gamma > 0$. Adjoining v as the exchange row at 0 produces a compatible matrix of one greater rank, with primitive new row and skew-symmetrizable principal part. After scalar normalization, x_0' is its first exchange variable.

Proof. Write $D = V(x_0)$ and let $\mathscr{V} = R_{(x_0)}$ be its discrete valuation ring. Every x_j with $j > 0$ is a unit in $\mathscr{V}$. The residue homomorphism

$$A = \mathbb{C}[x_1^{\pm 1}, \ldots, x_{N-1}^{\pm 1}] \longrightarrow \mathbb{C}(D)$$

has a prime kernel I. For $j > 0$, the Hamiltonian derivation $x_j^{-1}\{x_j, -\}$ preserves $\mathscr{V}$ and its maximal ideal, since $\{x_j, x_0\}/x_j = -p_{0j}x_0$. The geometric torus has the same property. The

restrictions of these derivations to A are diagonal. Since P is rational and W is integral, they generate the Lie algebra of a diagonal subtorus. The ideal I is invariant under this torus.

The character lattice of the quotient by this diagonal torus consists of the integer vectors $u \in \mathbb{Z}^{N-1}$ satisfying $u\mathsf{P}_{>0,>0} = 0$ and $u\mathsf{W}_{>0} = 0$. For such a vector, $v = (0, u)$ satisfies $v\mathsf{P} = -\gamma e_0$ and $v\mathsf{W} = 0$. Our nondegeneracy hypothesis makes the map $u \mapsto \gamma$ injective. Thus the quotient torus has dimension at most one.

We claim that $I \neq 0$. Otherwise the contraction of the maximal ideal of $\mathscr{V}$ to $\mathbb{C}[x_0, x_1^{\pm 1}, \dots, x_{N-1}^{\pm 1}]$ would be (x_0). Its localization at this prime and $\mathscr{V}$ are discrete valuation rings in the same fraction field, and one dominates the other; hence they are equal. The log-canonical formula makes (x_0) a Poisson ideal in this valuation ring. Therefore $\{x_0, R\} \subset x_0\mathscr{V} \cap R = x_0 R$, contrary to the assumption.

An invariant proper prime ideal in a Laurent polynomial algebra whose quotient torus has dimension at most one is generated by $x^u - c$ for a primitive nonzero quotient character u and a nonzero constant c. Indeed the diagonal torus acts transitively if the quotient has dimension zero; in dimension one its proper irreducible invariant subvarieties are the fibers of the quotient character. This proves the asserted binomial relation. Clearing its Laurent monomial denominator gives a polynomial F divisible by x_0 in R, so $x_0' = F/x_0$ belongs to R.

The scalar γ is nonzero by nondegeneracy. If the logarithmic weights of the two monomials of F under $\{x_0, -\}/x_0$ are a_+ and a_-, then $a_+ - a_- = \gamma \neq 0$. Their residues agree up to c and are nonzero, so $\{x_0, F\}/x_0$ has a nonzero residue along D. Thus $\operatorname{ord}_D\{x_0, F\} = 1$. If $F = x_0^m h$ in $\mathscr{V}$, with h a unit, then $\{x_0, F\} = x_0^m\{x_0, h\}$ and the latter factor is in $\mathscr{V}$. Therefore $m \leq 1$, and divisibility gives $m = 1$.

For an existing mutable index i, skew-symmetry of P gives

$$-\gamma B_{i0} = v\mathsf{P}B_i^{\mathsf{T}} = -v(B_i\mathsf{P})^{\mathsf{T}} = \delta_i^{-1} v_i.$$

Thus $\delta_0 = \gamma^{-1}$ extends the symmetrizer to the new mutable index. The new exchange vector is independent of the old ones, because its product with P is supported at 0. Its primitivity follows from that of u. Scalar normalization changes the nonzero coefficient $-c$ to the usual plus sign; simultaneous normalization with existing exchanges uses the full row rank of the augmented matrix. □

To obtain an upper cluster structure from this exchange, we apply Proposition 1.2. It remains to check homogeneity, regularity of the other first exchanges, and the equality between the sum of the frozen degrees and the degree of a homogeneous volume form.

4.7. **The translated upper cluster structure.**

Theorem 4.11. *For every untwisted KR Cartan algebra and every $a \neq 0$, there is a seed Σ_a satisfying*

$$R = \mathcal{U}_{\mathrm{pol}}(\Sigma_a).$$

It has $d - n$ mutable variables and $n + 1$ non-inverted frozen variables, represented up to nonzero constants by the geometric sections

$$b_i(a) = \tau^{Lk_i}\Psi_a(f_i), \qquad 0 \leq i \leq n.$$

The seed is homogeneous for Cartan degree and finite weight, the extended exchange matrix has full rank and primitive rows, and Π_a is compatible with it. The variables $b_i(a)$ are pairwise nonassociated primes of degrees Lk_i.

For $M > 1$, the seed is obtained by minimally lifting the translated cell seed and making τ mutable. For $M = 1$, it is the origin seed, independent of a.

Proof. If $M = 1$, source translation is trivial. Formula (4.1) gives zero precisely at the missing index j_*, and the other terminal degrees are Lk_i. Thus $b_{j_*}(a) = \tau$, and the other $b_i(a)$ are the terminal lifts. Adjoining τ gives the stated $n + 1$ frozen variables and $d - n$ mutable variables. The assertions follow from Corollary 2.6. We now assume $M > 1$.

We choose reduced-word cell coordinates $g_1, \dots, g_d$ and write $g_j^{(a)} = \Psi_a(g_j)$. Translation preserves their primality and the regularity of their first exchanges on the cell. Write $\nu_j = -\operatorname{ord}_{Y \setminus C}(g_j^{(a)})$. Each ν_j is positive: otherwise the nonconstant cell function would extend to the projective normal variety Y. Its minimal lift $x_j = \tau^{\nu_j} g_j^{(a)}$ is therefore a prime element of R, coprime to τ. The coordinates $x_1, \dots, x_d, \tau$ are pairwise nonassociated and generate the fraction field.

Homogenize the cell exchanges by the τ-column. The old mutable rows keep their successor entry of absolute value one, so their rank and primitivity are unchanged. Their exchange quotients are regular after inverting τ, and their denominators are primes coprime to τ. They consequently belong to R. The volume identity (1.2) is preserved by source translation. Lemma 3.2 therefore proves log-canonicity and compatibility of the homogeneous lift.

We verify the nondegeneracy hypothesis in Theorem 4.10. In ordered root coordinates, let $\beta_1, \dots, \beta_d$ be the inversion roots. Up to a common nonzero scalar, the leading skew matrix is $\mathsf{P}^{\mathrm{root}}_{jk} = (\beta_j, \beta_k)$ for $j < k$. If a vector v annihilates this matrix and all finite Cartan weights, then $\sum_j v_j \beta_j$ is a multiple of the null root. Pairing this sum with β_k and combining with the k-th kernel equation gives

$$2 \sum_{j<k} v_j (\beta_j, \beta_k) + v_k (\beta_k, \beta_k) = 0.$$

Induction on k gives $v = 0$, because every β_k is real. The triangular leading monomials of the CGL primes transfer this conclusion to the seed. Adding τ and the independent Cartan grading preserves it, as do homogeneous changes of coordinates and grading bivectors. Translation commutes with the finite Cartan torus, so the same conclusion holds for Π_a.

By Lemma 4.8, the divisor $\tau = 0$ is not Poisson for Π_a. Theorem 4.10 supplies the new primitive exchange, its regular quotient, and a compatible matrix of one greater rank. Corollary 4.9 now gives $\nu_i = Lk_i$ for every terminal minor. Their degrees sum to $Lh^\vee$, which is the degree of the homogeneous volume form on $\operatorname{Spec} R$. Proposition 1.2 proves the upper cluster equality.

The finite weights of the cell coordinates span the finite root space: roots whose coefficient at r is positive span it, as one sees along paths in the Dynkin diagram. Together with Cartan degree they give $n+1$ independent gradings. Translation preserves them, and full rank gives $\dim \ker B = n+1$. Their eigenvalue vectors therefore span the kernel in every homogeneous seed. □

When $M > 1$, the origin seed has $d - n - 1$ mutable variables and the additional frozen variable τ. At $a = 0$, the sections above satisfy $b_i(0) = \tau^{\delta_{i,j_*}} \widetilde{b}_i$, by (4.1).

5. Uniform charts for the shifted current operators

We continue with an untwisted KR Cartan algebra. For $a \neq 0$, the finite jet quotient defines the shifted affine generators

$$E_0(a) = f_\theta \otimes (u + a), \qquad F_0(a) = e_\theta \otimes (u + a)^{-1}, \qquad H_0 = -h_\theta.$$

Here the inverse is expanded in $\mathbb{C}[u]/(u^M)$, where $M = \langle \theta, \lambda \rangle$. For finite nodes we set $E_i(a) = E_i$ and $F_i(a) = F_i$, and we set $t_0 = 1$. These operators satisfy the Chevalley relations. They are locally nilpotent: each preserves the finite-dimensional Cartan-degree spaces and has nonzero finite root weight.

Theorem 5.1. *For every untwisted affine type and every $a \neq 0$, the mutation class of Σ_a contains an adapted seed for every finite Chevalley operator and for $F_0(a)$. The active frozen variable is $b_i(a)$ for finite E_i, and $b_{\pi^{-1}(i)}(a)$ for every F_i, including $i = 0$. The mutation class also has a chart adapted to $E_0(a)$, with active frozen variable $b_0(a)$.*

In each of these charts the operator is a nonzero scalar times a monomial in the other coordinates multiplied by the derivative in its active frozen variable. Its mutable exponents are

$$m_k(E_i) = \frac{\delta_k}{t_i} B^{E_i}_{k,b_i}, \qquad m_k(F_i) = -\frac{\delta_k}{t_i} B^{F_i}_{k,b_{\pi^{-1}(i)}}. \tag{5.1}$$

When $M > 1$, the newly mutable boundary position has $\delta_\tau = L$. In these cases, a chart for F_r is obtained by one final mutation at τ; its coefficient is τ' or $\eta\tau'$, up to a nonzero scalar, where $\eta = F_r\tau$.

We construct the finite root charts first, followed by the charts for $F_0(a)$ and $E_0(a)$.

5.1. **The boundary sections and finite root strings.** We first study the component section $c = \tau^L f_q$, where $q = \pi^{-1}(0)$.

Lemma 5.2. *Source translation fixes c. If we set $\omega_0 = 0$, the translated boundary sections have finite weights*

$$\operatorname{wt} b_i(a) = \omega_{\pi(i)} - \omega_i.$$

Their weights for $H_0 = -h_\theta, H_1, \dots, H_n$ are $\Lambda_{\pi(i)} - \Lambda_i$, interpreted at level zero.

Proof. Translation is trivial in the evaluation cases, so first suppose $M > 1$. Then $q \neq j_*$. The origin section $c = \widetilde{b}_q$ is F_r-invariant. After division by its power of η, it is the terminal q-minor on the shortened cell. The residue derivation $\mathcal{B}$ is the positive root operator E_{j_*} on $U(v^{-1})$, and therefore $\mathcal{B}c = 0$. In the η-chart this gives $Jc = \mathcal{A}c$, which is regular. It is regular on $\tau \neq 0$ as well. Since these two open sets meet every prime divisor, normality gives $Jc \in R_L$.

Under the untwisted KR–Demazure identification, R_L is the dual of the Demazure submodule in the level-one module $V_{\mathrm{aff}}(\Lambda_q)$, with the constant shift of current degree that gives the cyclic vector degree zero. Write p_ξ for an extremal coordinate section. Since $p_{w\Lambda_q} = \tau^L$ and $f_q = p_{\Lambda_q}/p_{w\Lambda_q}$, after normalization we have $c = p_{\Lambda_q}$. Thus c is the coordinate functional on the affine highest-weight line. In the affine derivation grading, every weight of $V_{\mathrm{aff}}(\Lambda_q)$ has degree at most that of its highest vector. Consequently c lies in the lowest dual current degree of R_L. The identity $[K, J] = -J$ would place a nonzero Jc in the next lower degree. Thus $Jc = 0$, and exponentiation proves the first claim.

The finite weight of the cell minor f_i is $\omega_{\pi(i)} - \omega_i - Lk_i\omega_r$. Translation commutes with the finite Cartan torus, and τ has weight ω_r. Its factor τ^{Lk_i} gives the stated weight of $b_i(a)$. The same formula holds for the missing index in an evaluation case, where $b_{j_*}(a) = \tau$. Finally π preserves the comarks. Pairing the finite weight with $-\theta^\vee$ gives $\delta_{0,\pi(i)} - \delta_{0,i}$; the finite simple-coroot pairings are immediate. These are precisely the asserted affine Cartan weights. □

Lemma 5.3. *For every finite node j,*

$$E_j b_i(a) = 0 \quad (i \neq j), \qquad F_j b_i(a) = 0 \quad (i \neq \pi^{-1}(j)),$$
$$E_j^2 b_j(a) = F_j^2 b_{\pi^{-1}(j)}(a) = 0.$$

Moreover, $E_j b_j(a)$ and $F_j b_{\pi^{-1}(j)}(a)$ are nonzero.

Proof. At the origin these statements follow from the fundamental root strings and the adapted charts of Corollary 3.10. Source translation commutes with every finite positive root subgroup and with the finite Levi subgroup generated by the simple roots other than r. These actions fix τ, so the statements for all E_j and for F_j, $j \neq r$, pass to the translated homogeneous lifts. The evaluation cases require no further argument.

Suppose $M > 1$, and write $b_* = \widetilde{b}_{j_*}$. On the η-chart, $F_r = \partial_t$, and the commuting fields $\mathcal{A}, \mathcal{B}$ annihilate t, η. The root string on $U(v^{-1})$ gives $\mathcal{B}^2 b_* = 0$ and $\mathcal{B}b_* \neq 0$. The translated section is therefore

$$b_{j_*}(a) = \tau \exp(a\mathcal{A}) b_* + a\eta \exp(a\mathcal{A})(\mathcal{B}b_*).$$

It follows that $F_r b_{j_*}(a) = \eta \exp(a\mathcal{A}) b_* \neq 0$ and $F_r^2 b_{j_*}(a) = 0$. For $i \neq j_*$, the residue kills $\widetilde{b}_i$, and $b_i(a) = \exp(a\mathcal{A})\widetilde{b}_i$ is independent of t. This proves the remaining annihilations. □

Lemma 5.4. *Let Π_a be the canonical Cox lift of the translated trigonometric Poisson structure, and let E be the Cartan-degree derivation. For every shifted simple root operator $D = E_i(a)$ or $F_i(a)$, there is a grading derivation A_D such that*

$$\mathcal{L}_D \Pi_a = D \wedge A_D, \qquad [A_D, D] = \begin{cases} -D/t_i, & D = E_i(a), \\ D/t_i, & D = F_i(a). \end{cases}$$

If a frozen section b satisfies $Db \neq 0$ and $D^2 b = 0$, the Hamiltonian recognition identity has coefficient t_i for $E_i(a)$ and $-t_i$ for $F_i(a)$.

Proof. We use the same normalization as in the projective calculation. Let $\Omega_{\mathfrak{g}}$ be the Casimir tensor for the invariant form with long roots of squared length two, and set $t_\alpha = 2/(\alpha, \alpha)$ for every finite root α. The trigonometric kernel centered at $-a$ is

$$\mathcal{R}_a(u, v) = \frac{u + v + 2a}{2(u - v)} \Omega_{\mathfrak{g}} + \frac{1}{2} \sum_{\alpha > 0} t_\alpha^{-1} (f_\alpha \otimes e_\alpha - e_\alpha \otimes f_\alpha).$$

The coboundary description in [AB, Corollary 6.6 and Equation (85)], after rescaling the invariant form to our normalization, identifies its commutator with $X(u) \otimes 1 + 1 \otimes X(v)$ as the loop cobracket. The simple-root relations give $H_i \wedge X/(2t_i)$ for both signs. For the affine simple root, we use $E_0(a) = f_\theta(u + a)$, $F_0(a) = e_\theta/(u + a)$, $H_0 = -h_\theta$, and $t_\theta = 1$. The matrix-coefficient bracket descends through the based gauge quotient, so the projective covariance is $\mathcal{L}_{\overline{D}} \overline{\Pi}_a = \overline{D} \wedge (-\overline{H}_i/(2t_i))$, where $\overline{\Pi}_a$ is the projective tensor and bars denote projected vector fields.

The current root fields preserve the homogeneous volume form, and the Cartan fields do so as their commutators. Proposition 3.4 therefore lifts this identity with an Euler correction. If the volume degree is κ, the corrected field is $A_D = -H_i/(2t_i) - \lambda_D \mathsf{E}/\kappa$, where $\lambda_D = -1/t_i$ for a raising operator and $\lambda_D = 1/t_i$ for a lowering operator. Since $[\mathsf{E}, D] = 0$, its commutator eigenvalue is unchanged. A grading twist adds to A_D a grading field commuting with D. Corollary 3.6 gives the asserted Hamiltonian coefficient whenever $Db \neq 0$ and $D^2 b = 0$. □

5.2. **The finite root charts.** Write $b_* = \widetilde{b}_{j_*}$ for the origin terminal lift.

Lemma 5.5. *Suppose a sequence of origin mutations avoiding η reaches a seed such that every coordinate other than τ is F_r-invariant and, on their fraction field, $\mathcal{B} = Q\partial_{b_*}$. Assume that Q is a Laurent monomial whose only possible negative exponent is at η. In the translated cluster structure, perform the same mutations and then mutate τ. The resulting seed is adapted to F_r, with active frozen coordinate $b_{j_*}(a)$. Up to nonzero scalar normalizations, its coefficient is either τ' or $\eta\tau'$, where τ' is the new exchange variable.*

Proof. We first compare the coordinates along the two mutation paths. After inverting τ, divide each variable by the power of τ equal to its Cartan degree. Both paths then give the same cell exchanges, related by Ψ_a. Therefore a variable along the translated path differs from the translated origin variable by a power of τ. Every reachable cluster variable is prime in the factorial upper cluster algebra, so that power is uniquely determined by the requirement that its τ-valuation be zero.

Let $y \neq b_*, \tau$ be a coordinate of the final origin seed. Since $\mathcal{B}y = 0$, Proposition 4.7 shows that $y_a = \Psi_a(y) = \exp(a\mathcal{A})y$ is regular. It is F_r-invariant, so it cannot be divisible by τ: the kernel of a locally nilpotent derivation is factorially closed, whereas $F_r\tau = \eta \neq 0$. Thus the corresponding translated coordinates are the functions y_a. The active frozen coordinate is

$$b_{j_*}(a) = \tau \exp(a\mathcal{A}) b_* + a\eta \exp(a\mathcal{A}) Q. \tag{5.2}$$

Write $Q = \eta^m Q_0$, with Q_0 a monomial in the other coordinates of that seed, excluding b_*, τ, η. The derivation $\Theta_r = -\eta^2 \mathcal{B}$ is regular. Therefore $\eta^{m+2} Q_0 = -\Theta_r b_*$ belongs to the

Cox ring. These coordinates are nonassociated primes, so $m \geq -2$. Let $p = \max(0, -m-1)$; thus p is either zero or one.

Modulo τ, formula (5.2) gives the binomial relation

$$\eta^p b_{j_*}(a) - a\eta^{p+1} \exp(a\mathcal{A})Q = 0. \tag{5.3}$$

Its two monomials are relatively prime. Indeed, if $p = 1$ then $m = -2$, and the second monomial has no η-factor; if $p = 0$, the first monomial is just $b_{j_*}(a)$. The slice decomposition $R[\eta^{-1}] = S[t]$ identifies the divisor $\tau = 0$ on this chart with $\operatorname{Spec} S$. The remaining coordinates are algebraically independent in S: they are obtained from independent origin coordinates by the automorphism $\exp(a\mathcal{A})$. Equation (5.3) expresses $b_{j_*}(a)$ as a Laurent monomial in them. It therefore generates the kernel of restriction from the Laurent ring in these coordinates and $b_{j_*}(a)$ to the generic τ-divisor. Its exponent vector is primitive, because its $b_{j_*}(a)$-entry has absolute value one.

The exchange vector at τ remains primitive under mutations at other indices. Indeed, the mutation formula preserves the ideal of $\mathbb{Z}$ generated by its entries: one inclusion follows from the formula, and the other follows from the inverse mutation. The numerator of the existing τ-exchange vanishes on this divisor. Its exponent difference is a nonzero integral multiple of the primitive exponent difference in (5.3). Primitivity makes this multiple ± 1; coprimality of its two monomials then identifies the exchange numerator up to a nonzero scalar. The new variable is therefore $\tau' = \eta^p \exp(a\mathcal{A})b_*$, whose factors are F_r-invariant. Applying F_r to (5.2) gives

$$F_r b_{j_*}(a) = \eta \exp(a\mathcal{A})b_* = \eta^{1-p}\tau'.$$

All other coordinates are fixed. Since $p \in \{0, 1\}$, this is a monomial and proves the assertion. □

Corollary 5.6. *The new mutable position τ has symmetrizer $\delta_\tau = t_r = L$. In the completed lowering chart, the coefficient of F_r involves only mutable variables.*

Proof. The letter r occurs at least twice in the reduced word for w. Indeed, the first lift η has degree one, whereas the minimal lift of the terminal r-minor has degree $M - \delta_{r,j_*}$. If $r \neq j_*$, this is at least two. If $r = j_*$, the residue calculation already shows that r occurs in the shortened word. Thus η is a mutable variable of the full cluster structure. It was frozen only to describe a restricted mutation path. The variable τ' is mutable as well, and Lemma 5.5 gives the asserted coefficient.

The exchange at τ has exponent one at $b_{j_*}(a)$, so the corresponding matrix entry has absolute value one. After mutation, the exponent of τ' in $F_r b_{j_*}(a)$ is also one. Hamiltonian recognition with coefficient $-t_r$ therefore gives $\delta_{\tau'} = t_r$. The symmetrizer is unchanged by mutation, proving the assertion for τ. □

Proposition 5.7. *The mutation class of Σ_a contains adapted charts for every finite E_i, F_i. When $M > 1$, the chart for F_r is obtained by CGL reversal on the shortened cell, mutations at the nonterminal occurrences of j_*, and one final mutation at τ.*

Proof. If $M = 1$, translation is trivial and the origin finite-root corollary gives every required chart. We therefore assume $M > 1$.

Source translation commutes with the positive Borel action on the framed bundle. It also commutes with the finite Levi subgroup generated by the simple roots other than r. These operators fix τ. We may therefore translate their origin adapted charts and take minimal homogeneous lifts. Homogenizing each exchange gives the same mutation path, with no mutation at τ. Primality of the reachable variables ensures that the homogeneous lifts are minimal. Every coordinate other than the active frozen one is still invariant. The strings in Lemma 5.3 and Hamiltonian recognition give the required adapted charts for E_i and for F_i, $i \neq r$.

For F_r, freeze η and divide the remaining coordinates by the powers of η prescribed by their Cartan degrees. Lemma 4.3 identifies this seed with the word seed of $U(v)$, where

$v = s_r w$. CGL reversal reaches the seed of $U(v^{-1})$ for the reversed word, with the terminal minor indices unchanged. There the residue derivation is the positive root operator at j_*. Mutating its nonterminal occurrences gives a chart adapted to that derivation. Lift these mutations by η-homogeneity. Lemma 5.5 supplies the final mutation at τ and the asserted F_r-chart. □

5.3. **The highest-root current and source translation.** For $0 \le j < M$, let D_j be the current derivation $e_\theta \otimes u^j$. The same notation denotes its extension to $R[\tau^{-1}]$. Every D_j kills τ: this is the finite highest-weight property when $j = 0$, and follows from the current grading when $j > 0$.

Lemma 5.8. *On the Schubert cell, and on $R[\tau^{-1}]$, we have*

$$[J, D_0] = 0, \qquad [J, D_j] = -(M-j)D_{j-1} \quad (1 \le j < M).$$

Consequently, if a homogeneous section c is fixed by source translation and $m_j = D_j c$, then

$$\Psi_a(m_j) = \sum_{k=0}^{j} \binom{M-k-1}{j-k} (-a)^{j-k} m_k. \tag{5.4}$$

If all currents of degree at least M act trivially, then for $a \neq 0$,

$$F_0(a)c = (-1)^{M-1} a^{-M} \Psi_a(m_{M-1}), \qquad F_0(a) = e_\theta \otimes (u+a)^{-1}.$$

Proof. The highest-root subgroup is central in the positive unipotent subgroup of G. In the ordered positive-root representative $n_+(u)u^\lambda$ of the Schubert cell, its parameter is a polynomial $C(u) = \sum_{j=0}^{M-1} x_j u^j$. We choose the fixed root-vector normalization so that D_j acts as a common nonzero scalar multiple of ∂_{x_j}. Left multiplication by this central subgroup changes none of the other root parameters.

To compute source translation, use the equivalent opposite-Borel representative $n_-(u)u^{-\lambda}$. The substitution $u \mapsto u - a$ is accompanied by the torus normalization $(1 - a/u)^\lambda$. Its action on the root $-\theta$ multiplies the central parameter by $(1 - a/u)^{-M}$. The contribution of C to the normalized polynomial parameter is therefore $\big[u^M (u-a)^{-M} C(u-a)\big]_{\ge 0}$, where the brackets mean the polynomial part of the Laurent expansion at infinity. Normalizing the other root parameters may add a highest-root term, but that term is independent of every x_j, because the highest-root subgroup is central. The other transformed root parameters are likewise independent of the x_j.

The coefficient of u^l in the preceding polynomial is $\sum_{j=l}^{M-1} \binom{M-l-1}{j-l} a^{j-l} x_j$. Thus $Jx_l = (M-l-1)x_{l+1} + \psi_l$, where ψ_l depends only on the other root parameters; the first summand is absent when $l = M-1$. Commuting with ∂_{x_j} gives the asserted identities. They also hold on the Cox localization because all derivations concerned kill τ.

Exponentiating these commutators gives

$$\Psi_a D_j \Psi_a^{-1} = \sum_{k=0}^{j} \binom{M-k-1}{j-k} (-a)^{j-k} D_k.$$

Applying this identity to the fixed section c proves (5.4). For $j = M-1$ all binomial coefficients equal one. The finite jet expansion $F_0(a)c = \sum_{k=0}^{M-1} (-1)^k a^{-k-1} m_k$ then proves the final formula. □

Proposition 5.9. *The initial translated seed is adapted to $F_0(a)$, with active frozen section c. The derivative $D_{M-1}c$ is a nonzero scalar multiple of a monomial in τ and the preceding prefix q-minor, when it exists. The coefficient $F_0(a)c$ is a monomial in seed variables, and its factors are mutable when $M > 1$.*

Proof. Write $w' = ws_q$. Since q is the only right descent of w, every reduced word for w ends in q, and its final inversion root is $\beta = w'\alpha_q = -w\alpha_q = \theta + (M-1)\delta_{\mathrm{aff}}$. This root subgroup is central in the positive-current representative of the cell: $\theta + \alpha$ is not a finite root for any

positive finite root α. Thus its left and right actions agree. On the coordinate algebra, D_{M-1} is a nonzero scalar multiple of differentiation along $n\exp(se_\beta)$; the scalar accounts for the root-vector and dual-action conventions.

The q-string through the highest vector of $V_{\mathrm{aff}}(\Lambda_q)$ has length one. Conjugating this string by a representative of w' gives, for suitably normalized extremal vectors,

$$\exp(se_\beta)v_{w\Lambda_q} = v_{w\Lambda_q} + \gamma s\, v_{w'\Lambda_q}, \qquad e_\beta v_{w'\Lambda_q} = 0, \qquad \gamma \neq 0.$$

The matrix-coefficient definition of the unipotent minor therefore gives $D_{M-1}f_q = \gamma D_{\Lambda_q, w'\Lambda_q}$, after absorbing the action scalar into γ. Let p be the preceding occurrence of q. Every intervening letter fixes Λ_q, so $w'\Lambda_q = w_{\le p}\Lambda_q$. Thus the derivative is the prefix minor γy_p, where $y_p = D_{\Lambda_q, w_{\le p}\Lambda_q}$. If there is no preceding occurrence, $w'\Lambda_q = \Lambda_q$ and we set $y_p = 1$. Since $c = \tau^L f_q$ and $D_{M-1}\tau = 0$, we obtain $m_{M-1} = \gamma\tau^L y_p$.

If x is the minimal Cox lift of y_p, of degree ν, regularity of m_{M-1} gives $m_{M-1} = \gamma\tau^{L-\nu}x$, with $L - \nu \ge 0$. When $y_p = 1$, take $x = 1$ and $\nu = 0$. The nonconstant prefix variable is mutable.

For $M > 1$, the translated derivative $\Psi_a(m_{M-1})$ is a Laurent monomial in τ and the translated prefix variable. Lemma 5.8 identifies it, up to a nonzero scalar, with the regular coefficient $F_0(a)c$. The translated prefix variable, when present, is prime and is not associated to τ. Regularity therefore forces the exponent of τ to be nonnegative. Thus the coefficient is a monomial, and both possible factors are mutable.

In an evaluation case, $M = 1$ and source translation is trivial. The same calculation expresses $F_0(a)c = a^{-1}m_0$ as a monomial in the origin seed. Here $L = 1$, and a factor of τ is allowed.

The same root string gives $D_{M-1}^2 c = 0$. Lemma 5.8 also gives the operator identity

$$F_0(a) = (-1)^{M-1}a^{-M}\Psi_a D_{M-1}\Psi_a^{-1}.$$

Since $\Psi_a c = c$, it follows that $F_0(a)^2 c = 0$. Shifted covariance and Hamiltonian recognition therefore make the initial translated seed adapted to $F_0(a)$. □

5.4. **The positive affine current.**

Proposition 5.10. *Let R be an untwisted KR Cartan algebra in the neutral component, and let $a \neq 0$. Assume that the translated Schubert cluster structure has reachable adapted charts for every finite simple lowering operator. Then it also has a reachable adapted chart for $E_0(a)$. Its coefficient $E_0(a)c$ is a monomial in the variables of one seed.*

Proof. Write $m^- = (f_\theta \otimes u)c$ and $m^+ = (e_\theta \otimes u)c$. By Corollary 3.10, $m^- = E_0(0)c$ is a monomial in an origin seed. Since c is finite G-invariant, the map $x \mapsto (x \otimes u)c$ is a homomorphism from the adjoint representation to R_L. It is nonzero because $m^- \neq 0$, hence injective. Its two extremal vectors are m^- and m^+.

Every finite positive simple root has an adapted chart at the origin. Successive maximal raising derivatives carry m^- to a nonzero scalar multiple of m^+. Proposition 3.8 shows that m^+ is a monomial in the variables of a seed in the origin mutation class. These mutations do not involve τ, which is frozen there.

The source-translation formula in current degree one gives $\Psi_a(m^+) = m^+$. Translate the preceding mutation path and homogenize its exchanges. The resulting path is reachable in the translated seed because it never mutates τ. Its expression for m^+ is a Laurent monomial; the only possible new negative exponent is at τ. Every other factor is a prime not associated to τ, so regularity of m^+ forces this exponent to be nonnegative. Thus m^+ is again a monomial in seed variables.

The assumed finite lowering charts and Proposition 3.8 carry m^+ back to m^-, within the translated mutation class. Finite G-invariance of c gives $E_0(a)c = (f_\theta \otimes (u + a))c = m^-$. At the origin, the fundamental affine root string gives $(f_\theta \otimes u)^2 c = 0$. Since $f_\theta c = 0$ and $[f_\theta, f_\theta \otimes u] = 0$, expanding the square gives $E_0(a)^2 c = 0$. Shifted root covariance supplies the

Hamiltonian recognition identity. The monomial form of $E_0(a)c$ therefore makes this seed adapted. □

Proposition 5.11. *In every evaluation case, the origin mutation class contains a seed adapted to $E_0(a) = af_\theta$.*

Proof. We have $M = L = 1$. The sections $b_0 = \tau$ and $c = b_q$ are respectively the highest and lowest vectors of the irreducible finite module R_1, in our opposition convention. The highest-root string through either vector has length one. Hence $e_\theta c$ and $f_\theta b_0$ are nonzero extremal vectors in the same finite module. A sequence of maximal finite simple-root derivatives carries one to the other. Proposition 3.8 and the chart for $F_0(a)$ show that $f_\theta b_0$ is a monomial in the variables of one seed. We also have $E_0(a)^2 b_0 = 0$. Shifted covariance and Hamiltonian recognition therefore give an adapted chart. □

5.5. **The affine raising operator in the nonneutral components.** In the neutral component, the two affine operators act on the same boundary section. In a nonneutral component their active sections are different: $F_0(a)$ acts on $c = b_{\pi^{-1}(0)}$, whereas $E_0(a)$ acts on b_0. We relate these coefficients by finite root strings.

Proposition 5.12. *In types B_n and D_n, the translated mutation class contains a seed adapted to $E_0(a)$ in every nonneutral component.*

Proof. Evaluation cases follow from Proposition 5.11. Every remaining nonneutral orthogonal case has odd r; in type D_n, $3 \le r \le n-2$. The length-zero permutation interchanges zero and one, so $c = b_1$. Let V be the natural orthogonal module, with basis $v_1, \dots, v_n, v_{-n}, \dots, v_{-1}$ in type D_n, and with the additional vector v_0 in type B_n. We normalize its symmetric form by $(v_i, v_{-j}) = \delta_{ij}$ and, in odd dimension, $(v_0, v_0) = 1$. We identify $\mathfrak{g}$ with $\bigwedge^2 V$ by $(p \wedge q)v = (q, v)p - (p, v)q$.

As a finite $\mathfrak{g}$-module, R_L has the multiplicity-free decomposition

$$R_L \simeq \bigwedge^{r} V \oplus \bigwedge^{r-2} V \oplus \cdots \oplus V.$$

At the spin node of type B_n, $L = 2$ and $\bigwedge^n V = V(2\omega_n)$; otherwise $L = 1$. This is the contragredient of the graded decomposition in [CM, Theorem 2.2 and Proposition 2.6]. Lemma 5.3 gives $F_i c = 0$ and $E_i b_0 = 0$ for every finite node i. Together with their weights $-\omega_1$ and ω_1, these identities make them the lowest and highest vectors of the unique natural summand. This summand is generated from c by finite raising operators. Since translation fixes c and commutes with those operators, it fixes the summand pointwise, including b_0. We normalize its identification with V so that $c = v_{-1}$ and $b_0 = v_1$; changing either boundary normalization only changes the nonzero scalars below.

Here $M = 2$, so all currents of degree at least two vanish. The graded decomposition makes the mode-one action on V a map to $\bigwedge^3 V$. This map is nonzero: otherwise V would be a current submodule of R_L, and its dual would be a quotient of the cyclic KR module, although the cyclic highest weight $L\omega_r$ does not occur in V. The space of equivariant maps $\bigwedge^2 V \otimes V \to \bigwedge^3 V$ is one-dimensional and is spanned by exterior multiplication. Thus $(x \otimes u)v = \xi x \wedge v$, for a fixed $\xi \neq 0$.

Choose $e_\theta = v_1 \wedge v_2$ and $f_\theta = v_{-2} \wedge v_{-1}$. These satisfy $[e_\theta, f_\theta] = h_\theta$. Let $\dot{w}_{\mathrm{lev}}$ be a Weyl representative in the Levi subgroup with simple roots $\alpha_2, \dots, \alpha_n$, exchanging v_2 with v_{-2} and fixing v_1, v_{-1}. Directly from the two actions just described,

$$\dot{w}_{\mathrm{lev}}(e_\theta c) = -f_\theta b_0, \qquad \dot{w}_{\mathrm{lev}}((e_\theta \otimes u)c) = (f_\theta \otimes u)b_0.$$

It follows that

$$\dot{w}_{\mathrm{lev}}\big(F_0(a)c\big) = -a^{-2}E_0(a)b_0. \tag{5.5}$$

Indeed, $F_0(a)c = a^{-1}e_\theta c - a^{-2}(e_\theta \otimes u)c$, whereas $E_0(a)b_0 = af_\theta b_0 + (f_\theta \otimes u)b_0$.

Both summands of $F_0(a)c$ are highest vectors of natural modules for this Levi subgroup. Their indicated linear combination therefore generates one simple natural Levi module. Its

highest vector is a monomial in seed variables by Proposition 5.9. A sequence of maximal finite lowering derivatives carries it to its lowest vector. Proposition 3.8 and (5.5) show that $E_0(a)b_0$ is a monomial in the variables of a reachable seed.

Finally, $E_0(a)^2 b_0$ would have finite weight $\varepsilon_1 - 2\theta = -\varepsilon_1 - 2\varepsilon_2$. Every weight of every exterior power occurring in R_L has all Euclidean coordinates in $\{-1, 0, 1\}$, so this weight is absent. Shifted covariance and Hamiltonian recognition make the seed adapted to $E_0(a)$. □

Proof of Theorem 5.1. Proposition 5.7 gives every finite root chart, and Proposition 5.9 gives the $F_0(a)$-chart in the initial translated seed. The positive affine chart follows from Proposition 5.10 in a neutral component, Proposition 5.11 in an evaluation case, and Proposition 5.12 in the remaining classical cases. At the remaining exceptional nodes the common-cluster theorem [FeiR, Theorem 2.5] identifies the translated cluster pattern with the auxiliary spectral pattern considered in the accompanying supplement, up to scalars. The adapted spectral chart for the positive affine generator therefore belongs to the translated mutation class as well. The comparison uses the finite root charts and the $F_0(a)$-chart established above. Hamiltonian recognition gives (5.1). Corollary 5.6 gives the new symmetrizer and the coefficient in the completed F_r-chart. □

Part II. Theta bases and crystal structures

6. Theta bases and current crystals

We construct finite-type theta crystals for every KR Cartan algebra in Table 1. In the untwisted case, reddening of the translated seeds also gives affine theta crystals.

6.1. **Regular theta functions and degree sections.** Let Σ be a homogeneous seed for R_r, with vertex set $V = U \sqcup F$, where U is mutable and F is frozen. Write $N = |V| = \dim R_r$. We use theta functions obtained from principal coefficients by specializing the auxiliary coefficients to one. The original frozen variables are not specialized. A theta function is *finite* when its principal-coefficient lift is a Laurent polynomial, and *regular* when it belongs to R_r.

We write ϑ_g for the theta function indexed by $g \in \mathbb{Z}^V$ in the chosen seed. The Cartan degree and finite weight extend to linear maps on this lattice:

$$D(g) = \sum_{v \in V} g_v \deg x_v, \qquad \mathrm{wt}(g) = \sum_{v \in V} g_v \,\mathrm{wt}(x_v).$$

These gradings are compatible with mutation. In our exchange-matrix convention, mutation at a mutable index k transforms its g-vector by

$$g'_k = -g_k, \qquad g'_v = g_v + [B_{kv}]_+ g_k - B_{kv}[g_k]_+ \quad (v \neq k).$$

To use the opposite g-vector convention, we replace B by $-B$. We normalize $\vartheta_0 = 1$.

A *reddening sequence* is a finite sequence of mutable mutations whose final principal-coefficient c-vectors are all nonpositive.

Proposition 6.1. *Suppose $R_r = \mathcal{U}_{\mathrm{pol}}(\Sigma)$, the extended exchange matrix has full rank, and the mutable pattern admits a reddening sequence. The regular theta functions form a homogeneous basis of R_r. Their g-vectors are the integral points of a full-dimensional rational polyhedral cone $\mathscr{C}_\Sigma \subseteq \mathbb{R}^V$. The functional D is strictly positive on $\mathscr{C}_\Sigma \setminus \{0\}$. Consequently $\mathscr{P}_\Sigma = \mathscr{C}_\Sigma \cap \{D = 1\}$ is a bounded rational polytope of dimension $N - 1$, and*

$$(R_r)_s = \bigoplus_{g \in s\mathscr{P}_\Sigma \cap \mathbb{Z}^V} \mathbb{C}\vartheta_g \qquad (s \geq 1).$$

For $s = 0$, the basis consists of ϑ_0.

Proof. Full rank and reddening give the theta basis after inverting the frozen variables [FeiT, Proposition 1.3]. Valuative independence along the frozen divisors implies that a finite theta sum is regular exactly when each term is regular. We therefore obtain a basis of R_r by selecting the regular theta functions.

Theta reciprocity expresses each frozen valuation as the minimum of finitely many rational linear forms in g. Thus regularity is equivalent to finitely many rational linear inequalities [FeiT, Corollary 3.2]. These inequalities define $\mathscr{C}_\Sigma$. Each initial coordinate is a regular theta function, so the cone contains all coordinate vectors and is full-dimensional.

The exponents in a pointed theta expansion differ by integral combinations of mutable exchange vectors. Compatible gradings annihilate those vectors. Hence ϑ_g has Cartan degree $D(g)$ and finite weight $\mathrm{wt}(g)$. Since R_r has no negative degrees, D is nonnegative on the integral points of the cone. Rationality then gives nonnegativity on the whole cone. If the face $D = 0$ contained a nonzero point, it would contain a nonzero integral point. Its theta function would be a second degree-zero basis element, contrary to $(R_r)_0 = \mathbb{C}$. Therefore this face is the origin. The degree-one section is bounded, has dimension $N - 1$, and its integral dilates give the stated bases. □

Replacing the exchange matrix by its negative leaves every exchange relation, and hence the upper cluster algebra, unchanged. Under reddening the two sets of theta functions coincide, although their indexing g-vectors generally differ [FeiT, Proposition 1.5]. We use *opposite g-vectors* below, meaning the pointed g-vectors for the opposite exchange matrix. This convention allows all lowering operators to be treated by the same highest-degree calculation.

6.2. **Perfectness and rank-one pairs.** For a locally nilpotent operator X, define its *nilpotence degree* on a nonzero vector by $\rho_X(v) = \max\{m : X^m v \neq 0\}$. A weight basis $\mathcal{B}$ is *perfect* for X if, for each $b \in \mathcal{B}$ with $\rho_X(b) > 0$, there are $\widetilde{X}b \in \mathcal{B}$ and a nonzero scalar c_b such that $Xb - c_b\widetilde{X}b \in \ker X^{\rho_X(b)-1}$, $\rho_X(\widetilde{X}b) = \rho_X(b) - 1$, and the map $b \mapsto \widetilde{X}b$ is injective on its domain. It is *normalized perfect* if $c_b = \rho_X(b)$.

We use the following form of theta perfectness [FeiT, Theorem 7.1]. Suppose a seed is adapted to $X = cx^u\partial_b$, where b is frozen, $u_b = 0$, and $c \neq 0$. For the exchange matrix defining these g-vectors, assume that $B_{kb} \geq 0$ for every mutable k, and that $u_k = 0$ whenever $B_{kb} = 0$. If the finite theta functions form a basis of the algebra and X preserves it and is locally nilpotent, then its theta basis is perfect for X. In the adapted chart the nilpotence degree is the b-coordinate of its g-vector, and the leading map adds $u - e_b$ to this vector. The operator $c^{-1}X$ is normalized perfect. The multiplier x^u may contain frozen factors.

To see the role of the support condition, freeze the mutable coordinates indexed by the nonzero entries of the active column. The highest b-degree part of each theta function is a theta function of this smaller mutable pattern. Now $u - e_b$ is supported on frozen coordinates, so multiplication by x^{u-e_b} translates its g-vector. Independence of these highest parts gives the unique leading term. This argument applies to the current formulas of Corollary 3.10.

We also recall the rank-one inverse lemma [FeiT, Lemma 7.4].

Lemma 6.2. *Let (E, F, H) be an $\mathfrak{sl}_2$-triple acting integrably on a weight module over $\mathbb{C}$. An H-homogeneous normalized perfect basis for F is also normalized perfect for E. Its two leading maps are partial inverses. If a basis vector has weight m, then its lowering and raising lengths satisfy $\varphi - \varepsilon = m$.*

Proof. Decompose the module into finite-dimensional simple $\mathfrak{sl}_2$-modules. On the weight-m space of a summand of highest weight q, the raising and lowering lengths are $(q - m)/2$ and $(q + m)/2$. Thus the filtration by highest weights q is the kernel filtration of F on this weight space. Perfectness makes the given basis compatible with that filtration.

On each successive quotient, F is an isomorphism between consecutive nonterminal weight spaces. Its matrix in the induced bases has one nonzero entry in each column, so those entries give a bijection. The inverse bijection determines the leading map for E. For a vector of lowering length φ and raising length ε, the rank-one identity gives $FE = (\varphi + 1)\varepsilon$ on its highest nonzero isotypic quotient. The normalized coefficient of F on the preceding weight space is $\varphi + 1$, so the leading coefficient of E is ε. Contributions from lower isotypic quotients have raising length at most $\varepsilon - 2$. This proves perfectness, inverse arrows, and

the string-length identity. Every vector belongs to a finite sum of simple summands, so the argument also applies to an infinite-dimensional integrable module. □

The geometric pinning may multiply the leading coefficients by nonzero constants. Reciprocal rescaling of E_i and F_i normalizes them without changing the Cartan relations, the Serre relations, or the crystal maps. We apply the lemma to the prescribed current operators, including their frozen monomial factors.

6.3. **Finite-type crystals in every current family.** Let Σ° be an origin seed from Corollary 2.6, and use opposite g-vectors. Let $\mathcal{B}^\circ_{r,s}$ denote the g-vectors of regular theta functions of Cartan degree s. For each finite node i, choose a chart adapted to F_i. Let T_i° denote tropical mutation to that chart in the opposite pattern, and write $F_i = c_i x^{u_i}\partial_{b(F_i)}$. Define $v_i = u_i - e_{b(F_i)}$.

Theorem 6.3. *For every current family and node in Table 1, the algebra R_r has the regular theta basis of its Schubert Cox mutation class. This basis is perfect for all finite Chevalley operators and for the positive affine simple operator whenever it is nonzero.*

For $s \geq 0$, its degree-s theta basis defines a finite crystal of type $\mathfrak{g}$, with highest-weight decomposition

$$\mathcal{B}^\circ_{r,s} \simeq \coprod_{\lambda\in P_+} \coprod_{j=1}^{[K_{r,s}:V(\lambda)]} B(\lambda).$$

Here P_+ is the set of dominant integral weights of $\mathfrak{g}$, and $[K_{r,s} : V(\lambda)]$ is the finite $\mathfrak{g}$-module multiplicity. For $g \in \mathcal{B}^\circ_{r,s}$, the string lengths and lowering map are

$$\varphi_i(g) = (T_i^\circ g)_{b(F_i)}, \qquad \varepsilon_i(g) = \varphi_i(g) - \mathrm{wt}(g)(h_i), \qquad \widetilde{f}_i g = (T_i^\circ)^{-1}(T_i^\circ g + v_i). \tag{6.1}$$

The last expression applies when $\varphi_i(g) > 0$. The raising map is the inverse translation and applies when $\varepsilon_i(g) > 0$. Otherwise the corresponding map has value $\mathbf{0}_{\mathrm{cr}}$.

Proof. The origin seed has full rank and realizes R_r as the upper cluster algebra with non-inverted frozen variables. Its principal matrix is a word matrix, hence admits a maximal green sequence, as does its negative [FeiS, Lemma 5.1]. Proposition 6.1 gives the theta basis and its polyhedral parametrization. All gradings in Corollary 3.10 are compatible, so the basis is homogeneous for finite weight and current degree as well as Cartan degree.

The same corollary supplies an adapted chart for each F_i, with nonpositive active column in the original matrix. It therefore has the required sign for the opposite g-vectors. Its exponent formula gives precisely the support condition for theta perfectness. Since these operators preserve R_r and are locally nilpotent, the basis is perfect for every F_i. Applying Lemma 6.2 to each finite rank-one action proves perfectness for E_i, inverse arrows, and (6.1). For a nonzero positive affine operator, its own adapted chart and the original g-vectors give perfectness directly. The original and opposite theta functions are the same basis.

Every Cartan-degree space is finite dimensional and stable under $\mathfrak{g}$. The perfect-basis comparison theorem [BK, Main Theorem 5.55] identifies its crystal with the disjoint union of the highest-weight crystals of its simple summands. Under (2.2), finite duality followed by opposition preserves the highest weight of every simple summand. Its multiplicities are therefore those of $K_{r,s}$, proving the stated decomposition. □

We call an affine crystal *regular* if its restriction to every finite-type set of colors is a disjoint union of finite-dimensional highest-weight crystals. In finite type, we refer directly to this highest-weight decomposition. The null value $\mathbf{0}_{\mathrm{cr}}$ is distinct from the lattice origin, which indexes the constant function.

6.4. **The shifted affine action.** We now work with an untwisted KR Cartan algebra and the translated seed Σ_a, where $a \neq 0$.

Theorem 6.4. *For every finite simple Lie algebra and every fundamental node, the translated seed Σ_a, with $a \neq 0$, admits a reddening sequence.*

Proof. The detailed proof is given in [FeiR, Theorem 1.1]. In an evaluation case, the translated structure is the Schubert structure, whose word matrix admits reddening by [Cao, Corollary 4.9]. At the first node of C_n, the principal matrix is zero by (9.2). For the remaining nodes, the supplement identifies a common cluster in the translated structure and the spectral patterns analyzed in the supplement. Their principal matrices agree up to component signs. The simply laced cases follow from spectral reddening.

At a double bond, the local matrix is a weighted two-strip matrix $Q(d_0, d_1)$. Starting from a sequence of 22 mutations for $-Q(2,2)$, the construction extends a reddening word R to $R\,u\,R^{-1}\,v\,R$ whenever its boundary degree is at most two. A symmetric-covering bound and an ordinary-star calculation give this inequality successively at the two ends. They produce (m,m), $(m+1,m)$, and $(m+1,m+1)$ by induction; principal-submatrix heredity then gives every positive pair (d_0,d_1). We apply the covering bound only after constructing a reddening sequence for the smaller matrix.

Ordinary-branch completion and the chamber extension in odd orthogonal components give every node of B, C, and F_4. The two G_2 cases follow from the explicit maximal green sequences in [FeiR, Proposition 4.1]. Each argument includes τ when it is mutable. □

Theorems 4.11 and 6.4 give full rank and reddening. Proposition 6.1 therefore supplies the regular theta basis. For seed coordinates we choose representatives of the frozen lines with unit coefficients in the exchange binomials, as permitted by Theorem 4.10. In theta functions and frozen monomials, b_i denotes this normalized coordinate. The geometric section $b_i(a)$ spans the same line and has the same current string lengths. We use opposite g-vectors and write $\mathscr{C}_r$, $\mathscr{P}_r$, and $\mathcal{B}_{r,s} = s\mathscr{P}_r \cap \mathbb{Z}^V$ for the corresponding cone, degree-one polytope, and degree-s g-vectors; set $\mathcal{B}_{r,0} = \{0\}$.

For each $i \in \widehat{I}$, choose the chart for F_i in Theorem 5.1. Write $j_i = \pi^{-1}(i)$, and let T_i be tropical mutation to that chart in the opposite pattern. Write its adapted expression as $F_i = c_i x^{u_i} \partial_{b_{j_i}}$, and set $v_i = u_i - e_{b_{j_i}}$. The weight takes values in the level-zero classical affine weight lattice: its finite part is $\mathrm{wt}(g)$, and $\mathrm{wt}_0(g) = -\mathrm{wt}(g)(h_\theta)$. We also write $\mathrm{wt}_i(g) = \mathrm{wt}(g)(h_i)$ for finite i.

Theorem 6.5. *For every untwisted type, finite node r, and $a \neq 0$, the regular theta basis of $R_r = \mathcal{U}_{\mathrm{pol}}(\Sigma_a)$ is perfect for all shifted affine Chevalley generators $E_i(a), F_i(a)$. For every $s \geq 0$, the set $\mathcal{B}_{r,s}$ is a finite regular affine crystal with the character of $(R_r)_s$. Its string functions are*

$$\varphi_i(g) = (T_i g)_{b_{j_i}}, \qquad \varepsilon_i(g) = \varphi_i(g) - \mathrm{wt}_i(g).$$

Its operators are

$$\widetilde{f}_i g = T_i^{-1}(T_i g + v_i) \quad (\varphi_i(g) > 0), \qquad \widetilde{e}_i g = T_i^{-1}(T_i g - v_i) \quad (\varepsilon_i(g) > 0). \tag{6.2}$$

Otherwise they have value $\mathbf{0}_{\mathrm{cr}}$. These formulas are independent of the chosen adapted charts. Every finite-type restriction has the character of the corresponding restriction of $(R_r)_s$. In particular, its finite $\mathfrak{g}$-character equals that of the quantum KR module of highest weight $s\omega_r$.

Proof. Proposition 6.1 and opposite theta invariance give one homogeneous basis on which to compare all currents. Each adapted expression for F_i satisfies the sign and support conditions of theta perfectness. The basis is therefore perfect for all F_i, including $F_0(a)$.

The current operators form integrable rank-one triples (E_i, F_i, H_i). After reciprocal scalar normalization, Lemma 6.2 gives perfectness for all E_i and identifies their leading maps as the inverses of those for F_i. In an adapted F_i-chart, perfectness identifies the lowering length with the active frozen coordinate and the lowering map with translation by v_i. The rank-one

string identity then gives ε_i and the inverse translation in (6.2). This also constructs $\widetilde{e}_0$ from the chart for $F_0(a)$.

The Chevalley–Serre relations are the current relations. All generators preserve Cartan degree. Each degree space is finite dimensional, and its weight decomposition is the one read from the homogeneous theta basis. For any finite-type subset of affine nodes, the restricted integrable module is a direct sum of finite-dimensional simple modules. Applying the perfect-basis comparison theorem [BK, Main Theorem 5.55] identifies its crystal with the union of their highest-weight crystals. This proves regularity and the character assertion. The opposition convention identifies the finite character with that of $K_{r,s}$, and this is the quantum KR character in every untwisted type by [Ked, Corollary 5.1].

The leading basis element in a perfectness expansion is unique. Since the current operators do not depend on the adapted chart, neither do their leading maps or nilpotence degrees. □

Corollary 6.6. *For either theta basis of Theorems 6.3 and 6.5, let $X_1, \dots, X_m$ be perfect current operators and let $\mathcal{B}_s$ index the degree-s theta basis. For nonnegative integers $n_1, \dots, n_m$,*

$$(R_r)_s \cap \bigcap_{j=1}^{m} \ker X_j^{n_j+1} = \bigoplus_{\substack{g \in \mathcal{B}_s \\ \rho_{X_j}(\vartheta_g) \le n_j \ (1 \le j \le m)}} \mathbb{C}\vartheta_g.$$

Proof. Perfectness makes the basis compatible with each kernel filtration; see also [FeiT, Corollary 7.10]. Indeed, iterating the injective leading map sends distinct basis vectors of maximal nilpotence degree to distinct nonzero maximal derivatives. These terms cannot cancel in a linear combination. Each kernel is therefore spanned by the indicated basis vectors, and their intersection has the asserted basis. □

7. Cartan degree and minimum level

Throughout this section and Section 8, we fix an untwisted type, a finite node r, and $a \neq 0$. We suppress a from the notation. For the affine theta crystal $\mathcal{B}_{r,s}$ of Theorem 6.5, we write $\varepsilon_i(\vartheta_g) = \varepsilon_i(g)$ and $\varphi_i(\vartheta_g) = \varphi_i(g)$ when describing a theta function by its frozen factorization. We write

$$\varepsilon(g) = \sum_{i \in \widehat{I}} \varepsilon_i(g)\Lambda_i, \qquad \varphi(g) = \sum_{i \in \widehat{I}} \varphi_i(g)\Lambda_i.$$

The canonical central element is $c_{\mathrm{aff}} = \sum_i k_i h_i$. We define the *level* of an element by $\ell(g) = \langle c_{\mathrm{aff}}, \varepsilon(g) \rangle = \sum_i k_i \varepsilon_i(g)$. Since the current action has level zero, this also equals $\sum_i k_i \varphi_i(g)$.

7.1. Primitive theta functions.

We use the opposite seed. The charts adapted to the lowering operators are *optimizing seeds* for their active frozen variables: the mutable entries of each active column are nonnegative. Let U and $F = \widehat{I}$ be the mutable and frozen index sets. To apply the support-function formulas of [FeiT], we write B for the square exchange matrix and choose positive integers σ_v proportional to δ_v^{-1}. Then $\mathsf{D} = \mathrm{diag}(\sigma_v)$ is a right symmetrizer: BD is skew-symmetric. In particular, $\sigma_j t_j$ is independent of the frozen index j. The entries with two frozen indices will not be used. Let $\mathbf{d}_U$ be the column of mutable Cartan degrees and set $a_j = \mathsf{B}_{j,U}$.

For $p \in \mathbb{Z}^U$, there is a unique regular theta function with mutable g-vector p whose order along every frozen divisor is zero. We denote it by χ_p and call it the *primitive regular theta function* at p. Frozen factorization gives

$$R_r = \bigoplus_{p \in \mathbb{Z}^U} \mathbb{C}[b_j : j \in F]\chi_p.$$

Here and below a frozen monomial has nonnegative exponents. This is the frozen factorization of [FeiT, Proposition 5.1]. We write $d(p) = \deg \chi_p$. Since $(R_r)_0 = \mathbb{C}$, we have $d(p) \ge 0$, with equality only for $p = 0$.

We recall the support functions needed for the degree calculation. Let F_z be the *principal-coefficient F-polynomial* of mutable theta g-vector $-z$, and let f_z be its componentwise largest exponent [FeiT, Propositions 1.3 and 1.7]. For $y, z \in \mathbb{Z}^U$, regarded as row vectors, define

$$\mathsf{E}(y,z) = \max_{\nu \in \operatorname{supp} F_z} (-y\mathsf{D}_U\nu^{\mathsf{T}}), \qquad \mathsf{H}(y,z) = y\mathsf{D}_U f_z^{\mathsf{T}} + \mathsf{E}(y,z).$$

We use the inverse orientation of the mutable tropical Donaldson–Thomas transformation. It is the bijection T characterized by

$$\mathsf{T}^{-1}(z) = -z - f_z\mathsf{B}_{U,U}.$$

Theta reciprocity gives $\mathsf{E}(y,z) = \mathsf{H}(z, \mathsf{T}y)$ [FeiT, Theorem 4.3]. On mutable g-vectors, we use $\mathsf{S}(p) = -\mathsf{T}(-p)$, an integral piecewise-linear bijection fixing the origin.

The frozen coordinates of χ_p are $\beta_j(p) = \mathsf{E}(a_j, -p)/\sigma_j$. Thus

$$d(p) = p\mathbf{d}_U + L\sum_{j\in F} \frac{k_j}{\sigma_j}\mathsf{E}(a_j, -p). \tag{7.1}$$

Although the support functions use the symmetrizer, the quotients $\beta_j(p)$ are integers: they are precisely the powers required to cancel the frozen valuations.

Lemma 7.1. *For $p \in \mathbb{Z}^U$ and $\mathbf{q} \in \mathbb{Z}^F_{\geq 0}$,*

$$\varphi_{\pi(j)}(b^{\mathbf{q}}\chi_p) = q_j + \frac{\mathsf{E}(a_j,-p) + \mathsf{E}(-p,a_j)}{\sigma_j} \qquad (j \in F).$$

Proof. Let M_j be the canonical mutable multiplier in an optimizing seed for j in the opposite pattern. Its mutable g-vector in the reference seed is $-a_j$ [FeiT, Lemma 2.1]. The symmetric theta F-invariant of M_j and χ_p is $\mathsf{E}(a_j,-p) + \mathsf{E}(-p,a_j)$ [FeiT, Proposition 9.3]. We evaluate it in the optimizing seed, where $F_{M_j} = 1$. Let γ be the g-vector of χ_p in that seed and let $m \geq 0$ be the mutable exponent vector of M_j. A term indexed by ν in the pointed expansion of χ_p has b_j-exponent $\gamma_j - \sigma_j^{-1}\nu\mathsf{D}_U m^{\mathsf{T}}$. Its maximum is γ_j, from the constant term, and its minimum is zero because the frozen valuation of χ_p vanishes. The invariant is therefore $\sigma_j\gamma_j$, or σ_j times the degree in b_j. The multiplier of $F_{\pi(j)}$ is a scalar times M_j times a frozen monomial independent of b_j. Its nilpotence degree is consequently the degree in b_j. Multiplication by $b^{\mathbf{q}}$ adds q_j to that degree, proving the formula. □

Theorem 7.2. *For every regular theta function $\vartheta_g = b^{\mathbf{q}}\chi_p$,*

$$L\ell(g) = \deg \vartheta_g + d(\mathsf{S}(p)). \tag{7.2}$$

Consequently $L\ell(g) \geq \deg \vartheta_g$, with equality if and only if ϑ_g is a frozen monomial.

Proof. Homogeneity of the exchanges and skew-symmetrizability give

$$\mathsf{B}_{U,U}\mathbf{d}_U = L\mathsf{D}_U\sum_{j\in F}\frac{k_j}{\sigma_j}a_j^{\mathsf{T}}. \tag{7.3}$$

Indeed, the frozen degrees are Lk_j, while $\mathsf{B}_{uj} = -(\sigma_u/\sigma_j)a_j(u)$. Let $z = -p$ and $t = \mathsf{T}z$, so that $z = -t - f_t\mathsf{B}_{U,U}$. The permutation π preserves the comarks. Hence Lemma 7.1 and (7.1) imply

$$L\ell(g) - \deg\vartheta_g = L\sum_{j\in F}\frac{k_j}{\sigma_j}\mathsf{E}(z,a_j) + z\mathbf{d}_U.$$

Apply the duality identity $\mathsf{E}(z,a_j) = \mathsf{H}(a_j,t)$, then (7.3). The right side becomes

$$\begin{aligned} &L\sum_{j\in F}\frac{k_j}{\sigma_j}\mathsf{E}(a_j,t) + f_t\mathsf{B}_{U,U}\mathbf{d}_U + z\mathbf{d}_U \\ &\quad = L\sum_{j\in F}\frac{k_j}{\sigma_j}\mathsf{E}(a_j,t) - t\mathbf{d}_U = d(-t) = d(\mathsf{S}(p)). \end{aligned}$$

This proves (7.2). The grading is positive, so the last degree is nonnegative and vanishes only when $\mathsf{S}(p) = 0$. Since S is bijective and fixes zero, this is equivalent to $p = 0$. The primitive theta function at zero is the constant function. □

7.2. **Minimal elements and simultaneous kernels.** For $s \geq 0$, let $Q_s = \{p \in \mathbb{Z}^U : d(p) \leq s,\ d(p) \equiv s \pmod{L}\}$. This set is nonempty because $(R_r)_s \neq 0$. It is finite because its primitive theta functions belong to $\bigoplus_{j=0}^{s}(R_r)_j$.

Corollary 7.3. *The minimum level in $\mathcal{B}_{r,s}$ is*

$$\min_{g\in\mathcal{B}_{r,s}} \ell(g) = \frac{s + \min_{p\in Q_s} d(\mathsf{S}(p))}{L} \geq \left\lceil \frac{s}{L} \right\rceil .$$

Its minimal elements are exactly the functions $b^{\mathbf{q}}\chi_p$ for which p realizes the indicated minimum and $L\sum_j k_j q_j = s - d(p)$.

If $s = L\ell$, the minimum level is ℓ, and its minimal elements are precisely the frozen monomials $b^{\mathbf{q}}$ with $\sum_j k_j q_j = \ell$. On this set, both ε and φ are bijections onto the dominant classical affine weights of level ℓ.

Proof. Since $k_0 = 1$, every nonnegative multiple of L is the degree of a frozen monomial. Thus $p \in Q_s$ is exactly the condition that some $b^{\mathbf{q}}\chi_p$ have degree s. Formula (7.2) gives the minimum and its stated parametrization.

When $s = L\ell$, the frozen monomial b_0^{ℓ} attains level ℓ. Equality in Theorem 7.2 characterizes all minimal elements. The boundary strings give

$$\varepsilon_i(b^{\mathbf{q}}) = q_i, \qquad \varphi_i(b^{\mathbf{q}}) = q_{\pi^{-1}(i)}.$$

The first equality also follows from the second and the weight $\mathrm{wt}(b_j) = \Lambda_{\pi(j)} - \Lambda_j$. The comark-preserving permutation π now gives both bijections. □

When $L \nmid s$, frozen monomials do not occur in degree s; the first assertion describes the minimal elements in these degrees.

Corollary 7.4. *For $\mathbf{q} \in \mathbb{Z}_{\geq 0}^{\widehat{I}}$, the space $(R_r)_s \cap \bigcap_{i\in\widehat{I}} \ker E_i^{q_i+1}$ has the theta basis indexed by those $g \in \mathcal{B}_{r,s}$ with $\varepsilon_i(g) \leq q_i$ for every i. It is zero if $s > L\sum_i k_i q_i$, and equals $\mathbb{C}b^{\mathbf{q}}$ if $s = L\sum_i k_i q_i$. For the lowering operators, replace ε_i by φ_i; at equality the spanning monomial is $\prod_j b_j^{q_{\pi(j)}}$.*

Proof. Corollary 6.6 gives the asserted basis. For an index in the intersection, $s \leq L\sum_i k_i\varepsilon_i(g) \leq L\sum_i k_i q_i$. This proves vanishing above the indicated degree. At equality, Theorem 7.2 makes the theta function a frozen monomial, and positivity of the comarks gives $\varepsilon_i(g) = q_i$ for every i. There is exactly one such monomial. The lowering assertion uses the same argument and the formula for φ in Corollary 7.3. □

In particular, if $f \in R_r$ is nonzero and homogeneous, then

$$\deg f \leq L\sum_i k_i \max\{m : E_i^m f \neq 0\}.$$

Equality holds precisely for scalar multiples of frozen monomials. Indeed, apply Corollary 7.4 with the actual nilpotence orders of f.

8. Simplicity and the common limit

We prove simplicity using the simple head of the current module. We then apply known uniqueness theorems to identify the classical, minuscule, and adjoint cases with quantum KR crystals, and construct a common limit.

8.1. **An essential-socle criterion.** Let W_0 be the finite Weyl group and $Q_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$. We use the term *simple crystal* in the sense of Akasaka–Kashiwara [AK, Definition 1.7].

Lemma 8.1. *Let V be a finite-dimensional loop-algebra module of level zero with a weight basis perfect for all affine Chevalley generators and with regular crystal. Suppose that its weights belong to $\lambda_0 - Q_+$, where λ_0 is dominant, that $\dim V_{\lambda_0} = 1$, and that every nonzero submodule contains V_{λ_0}. Then its crystal is simple.*

Proof. Suppose b is the highest element of an i-string of length m, and let v_b be its basis vector. Perfectness implies that $F_i^m v_b$ is a nonzero scalar multiple of the basis vector at the bottom of the string: every lower term is killed by the remaining lowering operators. Rank-one representation theory therefore shows that the Tits operator $\dot{s}_i = \exp(E_i)\exp(-F_i)\exp(E_i)$ sends v_b exactly to a scalar multiple of the basis vector at $s_i b$. The same argument applies at the bottom of a string. We can therefore lift the crystal Weyl group action on an extremal element to the corresponding basis vector, up to nonzero scalar.

Let b be extremal. After a finite Weyl transformation, we may suppose that $\lambda = \mathrm{wt}(b)$ is dominant for $\mathfrak{g}$. For a positive finite root α and $m \in \mathbb{Z}$, choose an affine Weyl element w and a simple affine root α_i such that $\alpha + m\delta_{\mathrm{aff}} = w\alpha_i$. The coroot pairing with λ is nonnegative because the module has level zero. Extremality gives $\widetilde{e}_i(w^{-1}b) = \mathbf{0}_{\mathrm{cr}}$. Perfectness and the preceding compatibility with the Weyl group give $(e_\alpha \otimes t^m)v_b = 0$. The loop triangular decomposition implies that every finite weight of the submodule generated by v_b lies in $\lambda - Q_+$. This submodule contains V_{λ_0}, so $\lambda_0 \leq \lambda$. The assumed weight bound gives the reverse inequality. Hence every extremal weight belongs to $W_0\lambda_0$.

The multiplicity-one hypothesis and Weyl invariance give a unique element at each weight in this orbit. The same weight bound, together with finite Weyl invariance, places all weights in the convex hull of $W_0\lambda_0$. These are the remaining conditions for simplicity. □

Theorem 8.2. *For every untwisted type and finite node r, the crystal $\mathcal{B}_{r,s}$ is simple for every $s > 0$. Every finite tensor product of such crystals within one affine type is connected. The same assertions hold for their duals. For $s = L\ell$, they satisfy conditions* (ii)–(v) *of a level-ℓ perfect crystal in* [ST, Definition 2.4].

Proof. The ideal $\mathfrak{g} \otimes u\mathbb{C}[u]$ strictly raises the bounded current grading of $K_{r,s}$, whereas every current operator preserves or raises it. Its image therefore generates a nilpotent ideal in the associative algebra acting on $K_{r,s}$. Thus every simple quotient factors through

$$K_{r,s}/(\mathfrak{g} \otimes u\mathbb{C}[u])K_{r,s} \simeq V(s\omega_r).$$

The equality follows from the finite highest-weight relations and the nonzero evaluation quotient. Thus $K_{r,s}$ has simple head. Its dual has an essential simple socle, since every nonzero finite-dimensional submodule contains a simple submodule. With the opposition convention (2.2), this socle has highest weight $s\omega_r$.

All weights of $(R_r)_s$ lie in $s\omega_r - Q_+$, and the top weight has multiplicity one. The substitution $t = a + u$ maps the loop algebra onto the finite jet algebra acting on $K_{r,s}$. Thus its loop submodules are precisely its current submodules. Theorem 6.5 and Lemma 8.1 now prove simplicity. Simple crystals are connected and closed under tensor product [AK, Lemmas 1.9–1.10]; dualization preserves simplicity. Finally, the highest-weight bound and Corollary 7.3 give the stated perfectness conditions. The following corollaries supply the quantum realization required by condition (i) in the stated cases. □

Corollary 8.3. *For every node r of classical type and every $s > 0$, the crystal $\mathcal{B}_{r,s}$ is isomorphic to the quantum Kirillov–Reshetikhin crystal $B^{r,s}$. It is perfect if and only if $L \mid s$, and then has level s/L.*

Proof. By Theorem 6.5, $\mathcal{B}_{r,s}$ and $B^{r,s}$ have the same finite character. Their highest-weight decompositions therefore agree on the finite colors. Schilling–Tingley's uniqueness lemma gives the affine isomorphism [ST, Theorem 2.5 and Lemma 2.6]. In type A_1, the finite restriction is

one $\mathfrak{sl}_2$-string, and the affine string reverses it, giving the same conclusion directly. Perfectness and its level are the theorem of Fourier–Okado–Schilling [FOS, Theorem 1.2]. □

Corollary 8.4. *For every $s > 0$, the identification $\mathcal{B}_{r,s} \simeq B^{r,s}$ holds in the following cases.*

finite type	r
E_6	$1, 2, 6$
E_7	$1, 7$
E_8	8
F_4	1
G_2	2.

These crystals are perfect of level s.

Proof. At each listed node, $t_r = 1$. The current KR module is therefore a Demazure module by [FL07, Theorem 4]. Its finite character agrees with that of the quantum KR module by [FL06, Theorem 2 and Corollary 1]. The quantum KR module admits a *crystal pseudobase*: the minuscule cases follow from [KKM92, Proposition 3.4.4], and the adjoint cases from [KKM92, Proposition 3.4.5]. Its associated crystal $B^{r,s}$, obtained by identifying opposite signs, therefore has the same finite character as $\mathcal{B}_{r,s}$. Both crystals have level-zero weights. Their finite characters therefore determine their full weight characters modulo δ_{aff}, since the central relation determines the zeroth coroot pairing. Regularity then determines their restrictions to every proper set of affine colors: the character of a finite-type restriction determines its highest-weight decomposition.

For the minuscule nodes $E_6, 1$, $E_6, 6$, and $E_7, 7$, the finite restriction is $B(s\omega_r)$. The uniqueness theorems of Jones–Schilling [JS10, Theorem 3.15 and Corollary 3.16] and Biswal–Scrimshaw [BS22, Theorem 3.5] now identify the affine crystals. The additional finite restrictions in those theorems agree by the preceding character argument.

At the adjoint nodes, $\omega_r = \theta$, and the finite restriction is $\bigsqcup_{k=0}^{s} B(k\theta)$. The affine node has a unique neighbor, joined to it by a single bond. The uniqueness theorem [KMOY07, Theorem 6.1] therefore applies. The proof requires only the finite highest-weight decompositions and the crystal Weyl group action, so it applies to our regular crystals; this formulation is also used in [JS10, Corollary 3.23]. This proves the asserted identifications. Since $L = 1$, Theorem 8.2 and Corollary 7.3 give the remaining perfectness conditions at level s. □

8.2. **A limit on the degree-zero lattice.** Let $\mathsf{M} = \mathbb{Z}^V$ be the full g-vector lattice of the opposite reference seed. Its compatible gradings are $D : \mathsf{M} \to \mathbb{Z}$ and $\mathrm{wt} : \mathsf{M} \to P_{\mathrm{cl}}$. We define the limit on the degree-zero lattice $\mathcal{B}_{r,\infty} = \ker D$. For each $i \in \widehat{I}$, choose a chart adapted to F_i, with active frozen index $j = \pi^{-1}(i)$. Let T_i be the tropical change to that chart, using the opposite pattern, and let $v_i = u_i - e_j$, where x^{u_i} is the full monomial multiplier of F_i. Define maps on the whole lattice by

$$\widetilde{f}_i^{\infty}(g) = T_i^{-1}(T_i(g) + v_i), \qquad \widetilde{e}_i^{\infty}(g) = T_i^{-1}(T_i(g) - v_i).$$

Their string functions are

$$\varphi_i^{\infty}(g) = (T_i(g))_j, \qquad \varepsilon_i^{\infty}(g) = \varphi_i^{\infty}(g) - \langle h_i, \mathrm{wt}(g)\rangle.$$

These functions may be negative. Neither operator has a null value on $\mathcal{B}_{r,\infty}$.

For $\mathbf{q} \in \mathbb{Z}_{\geq 0}^{\widehat{I}}$, write $\Lambda(\mathbf{q}) = \sum_i q_i \Lambda_i$ and $(\pi\mathbf{q})_i = q_{\pi^{-1}(i)}$. We use the tensor convention in which $\widetilde{e}_i$ acts on the left factor when $\varphi_i(b_1) \geq \varepsilon_i(b_2)$, and $\widetilde{f}_i$ does so when the inequality is strict. We identify a frozen exponent vector with its extension by zero on the mutable indices in M. Let $T_\lambda = \{t_\lambda\}$ be the one-element crystal of weight λ, with string functions $-\infty$.

Proposition 8.5. *The preceding formulas make $\mathcal{B}_{r,\infty}$ an abstract crystal, independently of the adapted charts. For $\sum_i k_i q_i = \ell > 0$, the map*

$$T_{\Lambda(\mathbf{q})} \otimes \mathcal{B}_{r,L\ell} \otimes T_{-\Lambda(\pi\mathbf{q})} \longrightarrow \mathcal{B}_{r,\infty}, \qquad t_{\Lambda(\mathbf{q})} \otimes g \otimes t_{-\Lambda(\pi\mathbf{q})} \longmapsto g - \mathbf{q} \tag{8.1}$$

preserves weights, string functions, and every arrow in its source. Each map sends the element indexed by $b^{\mathbf{q}}$ to zero, and their images cover $\mathcal{B}_{r,\infty}$. At the nodes of Corollaries 8.3 and 8.4, this is the limit of a coherent family of perfect KR crystals.

Proof. Tropical mutation commutes with every frozen translation. Since v_i has Cartan degree zero and classical affine weight $-\alpha_i$, the two operators are inverse bijections of $\ker D$ with the required weight changes. The active coordinate of v_i is -1, so φ_i^∞ decreases by one under $\widetilde{f}_i^\infty$. The definition of ε_i^∞ gives the remaining crystal axioms.

For any integral frozen vector $\mathbf{q}$, we have $\mathrm{wt}(\mathbf{q}) = \Lambda(\pi\mathbf{q}) - \Lambda(\mathbf{q})$. Together with frozen translation, this gives

$$\varepsilon_i^\infty(g+\mathbf{q}) = \varepsilon_i^\infty(g) + q_i,$$
$$\varphi_i^\infty(g+\mathbf{q}) = \varphi_i^\infty(g) + q_{\pi^{-1}(i)}.$$

A sufficiently large nonnegative $\mathbf{q}$ makes any fixed finite collection of theta functions regular and their relevant string lengths positive. On the corresponding translated g-vectors the formulas agree by Theorem 6.5. Subtracting $\mathbf{q}$ then compares any two adapted charts. This proves independence of the charts on the whole lattice. The origin has weight zero and both string functions vanish there.

For g in the source of (8.1), we have $D(g) = D(\mathbf{q}) = L\ell$. The preceding weight and string identities agree exactly with the tensor-product formulas, and frozen translation commutes with every source arrow. The map is injective, so it is a crystal embedding in the sense of [KK, Definition 3.2]. It sends the element indexed by $b^{\mathbf{q}}$ to zero. Arrows of the limit may leave a finite image.

Finally, given $g \in \ker D$, choose a sufficiently large nonnegative frozen vector $\mathbf{q}$ so that $g+\mathbf{q}$ is regular and $\sum_i k_i q_i = \ell > 0$. Then $g+\mathbf{q} \in \mathcal{B}_{r,L\ell}$, and its image is g. Thus the embeddings cover the degree-zero lattice and satisfy the limit axioms of [KK, Definition 5.2]. In the cases with quantum realization, the finite crystals are perfect and the family is coherent. □

9. Two examples

The first example makes the translated exchange relations and the crystal-level formula explicit. The second illustrates the primitive Cartan grading in the even twisted family. The end-node evaluation case in type A is the polynomial coordinate ring of projective space and needs no further calculation.

9.1. **The first node of type C_n.** Let V be the natural symplectic module, with $n \geq 2$. The first-node KR Cartan algebra is

$$R = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, z], \qquad \deg x_i = \deg y_i = 1, \quad \deg z = 2. \tag{9.1}$$

The finite Chevalley operators are

$$E_i = x_i\partial_{x_{i+1}} - y_{i+1}\partial_{y_i}, \qquad F_i = x_{i+1}\partial_{x_i} - y_i\partial_{y_{i+1}} \quad (i < n),$$

together with $E_n = x_n\partial_{y_n}$ and $F_n = y_n\partial_{x_n}$. They annihilate z. For the symplectic bracket $\{x_i, y_j\}_{\mathrm{symp}} = \delta_{ij}$, let h_A be the quadratic Hamiltonian with $\{h_A, -\}_{\mathrm{symp}} = A$ on linear functions. The positive currents act by

$$\rho(A \otimes u) = -2h_A\partial_z, \qquad \rho(A \otimes u^j) = 0 \quad (j \geq 2).$$

Equivariance of the quadratic Hamiltonians verifies the current relations.

The formulas also recover Cartan multiplication. Evaluation at the point $y_1 = 1$, with all other coordinates zero, defines a cyclic vector in the dual degree-s space satisfying the KR relations. Its successive $f_\theta \otimes u$ images generate the summands $V((s-2j)\omega_1)$, for $0 \leq j \leq \lfloor s/2 \rfloor$. Chari–Moura's decomposition [CM, Theorem 2.2] therefore identifies this cyclic module with $K_{1,s}$. Evaluation is multiplicative, so the resulting isomorphisms identify the prescribed cyclic-vector maps with the duals of polynomial multiplication. This proves (9.1) as an isomorphism of graded algebras.

Set $Q_i = \sum_{j=1}^{i} x_j y_j$, $b_0 = z$, and $b_i(a) = az - Q_i$ for $1 \le i \le n$. For $a \neq 0$, we normalize the frozen variables by $c_0 = ab_0$ and $c_i = (-1)^i b_i(a)$ for $i > 0$, and write $v_i = (-1)^{i-1} y_i$. The translated seed has mutable coordinates $x_1, \dots, x_n$, frozen coordinates $c_0, \dots, c_n$, and exchanges

$$x_i v_i = c_{i-1} + c_i \qquad (1 \le i \le n). \tag{9.2}$$

Its principal exchange matrix is zero. The 2^n seeds arise by choosing x_i or v_i independently. Their mixed Laurent rings are the corresponding localizations of R. These charts meet every height-one prime: no such prime contains both x_i and y_i for any i. Normality therefore gives $R = \mathcal{U}_{\mathrm{pol}}(\Sigma_a)$ directly.

The shifted affine operators are

$$E_0(a) = a y_1 \partial_{x_1} + y_1^2 \partial_z, \qquad F_0(a) = a^{-1} x_1 \partial_{y_1} + a^{-2} x_1^2 \partial_z.$$

Their values on the active frozen coordinates, together with the finite-root formulas, are

$$\begin{aligned} E_0(a) b_0 &= y_1^2, & F_0(a) b_0 &= a^{-2} x_1^2, \\ E_i b_i &= x_i y_{i+1}, & F_i b_i &= -x_{i+1} y_i \quad (i < n), \\ E_n b_n &= -x_n^2, & F_n b_n &= -y_n^2. \end{aligned}$$

Each operator annihilates the other frozen coordinates. Mutating x_{i+1} gives the chart for E_i, and mutating x_i gives the chart for F_i, when $i < n$. The initial chart is adapted to E_n and $F_0(a)$. Mutation at x_n, respectively x_1, gives the charts for F_n, respectively $E_0(a)$.

At $a = 0$, the section $c_1 = x_1 y_1$ is reducible. The Schubert seed instead has mutable variables $x_2, \dots, x_n$ and frozen variables $x_1, y_1, z, c_2(0), \dots, c_n(0)$. It is the minimal lift for the word $s_1 s_2 \cdots s_n s_{n-1} \cdots s_1 s_0$, with $\tau = x_1$. The first relation in (9.2) becomes the factorization $c_1 = x_1 y_1$; the remaining exchanges are unchanged. The nonzero translation increases the mutable rank from $n-1$ to n. In both seeds the sum of the frozen degrees is $2n+2$, which is also the degree of the polynomial volume form.

For $a \neq 0$, the regular theta functions, in the polynomial-frozen sense of [FeiT, Section 2], are precisely

$$c^{\mathbf{q}} \prod_{i=1}^{n} x_i^{[p_i]_+} v_i^{[-p_i]_+}, \qquad \mathbf{q} \in \mathbb{Z}_{\ge 0}^{n+1}, \quad p \in \mathbb{Z}^n.$$

Each is a cluster monomial in the seed that chooses x_i when $p_i \ge 0$ and v_i when $p_i < 0$. The exchange relations reduce every polynomial to a linear combination of these functions, and theta independence makes them a basis. Their Cartan degree and level are

$$s = 2 \sum_{j=0}^{n} q_j + \sum_{i=1}^{n} |p_i|, \qquad \ell = \sum_{j=0}^{n} q_j + \sum_{i=1}^{n} |p_i|.$$

Indeed, the primitive degree is $d(p) = \sum_i |p_i|$, and the mutable transformation S negates p. Theorem 7.2 gives the level formula.

In degree $s = 2h$, the minimum level is h; the minimal elements are the monomials $c^{\mathbf{q}}$ with $\sum_j q_j = h$. In degree $s = 2h+1$, the minimum level is $h+1$; the minimal elements are $c^{\mathbf{q}} x_i$ and $c^{\mathbf{q}} v_i$, where $\sum_j q_j = h$ and $1 \le i \le n$. These are the KR crystals $B^{1,s}$ of type $C_n^{(1)}$, by Corollary 8.3.

9.2. **The twisted current algebra of type $A_2^{(2)}$.** Here the finite Lie algebra is $\mathfrak{sl}_2$, and the degree-one current eigenspace is $V(4\omega)$. We have

$$R = \mathbb{C}[x, y, z], \qquad \deg x = \deg y = 1, \quad \deg z = 4. \tag{9.3}$$

The finite operators are $E_1 = x\partial_y$, $F_1 = y\partial_x$, and $H_1 = x\partial_x - y\partial_y$. Identify $V(4\omega)$ with the binary quartics. A quartic p acts by $p(x,y)\partial_z$, and currents of degree at least two act trivially. In particular, we may normalize the positive affine generator as $E_0 = y^4 \partial_z$.

Evaluation at $(0,1,0)$ in the dual degree-s polynomial space satisfies the restricted KR relations. Repeated application of the current corresponding to y^4 gives the highest vectors dual to $z^j y^{s-4j}$. Chari–Moura's decomposition [CM, Theorem 3.4],

$$K_{1,s} = \bigoplus_{j=0}^{\lfloor s/4 \rfloor} V((s-4j)\omega),$$

shows that this cyclic module is the KR module. As in the preceding example, multiplicativity of evaluation identifies the Cartan product and proves (9.3).

The Schubert word is $s_1 s_0$. Its lifted seed consists of the three frozen coordinates x, y, z, with no mutable variables. The formulas for E_1, F_1, E_0 are already adapted. The projective variety is $\mathbb{P}(1,1,4)$, and its polynomial volume form has degree six, as in Table 1.

Appendix A. The Demazure realization in type $A_{2n}^{(2)}$

We use the fixed-point realization of $A_{2n}^{(2)}$, with finite part B_n, and normalize $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The affine fundamental weight Λ_0 satisfies $\Lambda_0(c_{\mathrm{aff}}) = 2$. Thus the parameter one in Fourier–Kus's notation $D(1,\lambda)$ corresponds to central level two. Define $\lambda_r = p_r \omega_r = 2(\varepsilon_1 + \cdots + \varepsilon_r)$ and

$$w_{n,r} = \left(\prod_{j=r}^{n-1} s_j s_{j-1} \cdots s_{j-r+1} \right) (s_n s_{n-1} \cdots s_0)^r.$$

The factors are ordered by increasing j; an empty product is the identity.

Proposition A.1. *For $1 \le r \le n$ and $\ell \in \mathbb{Z}_{\ge 0}$, there is an isomorphism of graded current modules*

$$K_{r,\ell p_r} \simeq V_{w_{n,r}}\big(\ell(\Lambda_0 + r\delta_{\mathrm{aff}})\big). \tag{A.1}$$

The expression for $w_{n,r}$ is reduced, has length $r(2n+1-r)$, and is minimal modulo the stabilizer of Λ_0. The isomorphisms preserve cyclic vectors and their tensor maps.

Proof. We first check the word. On a weight $\Lambda_0 + \sum_i x_i \varepsilon_i + q\delta_{\mathrm{aff}}$, the reflection s_i, for $1 \le i < n$, interchanges x_i, x_{i+1}, while s_n negates x_n. The remaining reflection sends (x_1, q) to $(2 - x_1, q + x_1 - 1)$. Apply the reflections, from left to right in the written expression, to $\Lambda_0 - \lambda_r$. Their composite is $w_{n,r}^{-1}$. The initial product moves the r coordinates equal to -2 to the final r positions. In each descending cycle, s_n changes the last -2 to 2, the subsequent interchanges move it to the first position, and s_0 changes it to zero and increases q by one. After r cycles we obtain $\Lambda_0 + r\delta_{\mathrm{aff}}$. At each step the simple-coroot pairing is negative. Hence the word is reduced and minimal modulo the stabilizer, and its length is $r(n-r) + r(n+1) = r(2n+1-r)$.

Consider the Demazure module on the right of (A.1) for $\ell = 1$. Its extremal generator has affine weight $\Lambda_0 - \lambda_r$. Finite antidominance makes the module $\mathfrak{g}$-stable. Together with its affine Borel stability, this gives stability under $\mathfrak{C}$ [FK, Section 4.1]. Applying the finite longest element gives a cyclic vector v of affine weight $\Lambda_0 + \lambda_r$. For a real current root of finite weight $-\beta$ and current degree j, the defining relations give nilpotence exponent

$$1 + \max\{0, \langle \lambda_r, \beta^\vee \rangle - 4j/(\beta,\beta)\}$$

on v [FK, Corollary 4.3.1]. We use $\beta^\vee = 2\beta/(\beta,\beta)$, also when β is twice a short root. At the selected simple root in current degree one, the expression inside the maximum is zero: it is $2 - 4/2$ for $r < n$, and $4 - 4$ for $r = n$. The remaining relations in (2.1) follow from the same Demazure presentation. Thus $v_{p_r} \mapsto v$ defines a surjection from K_{r,p_r} onto this Demazure module.

For $1 \le i \le r$, let $\gamma_i = \delta_{\mathrm{aff}} - 2\varepsilon_i$. These are pairwise orthogonal real roots, and $\langle \Lambda_0 + \lambda_r, \gamma_i^\vee \rangle = -1$. Successive root strings therefore give nonzero extremal vectors $x_{\gamma_{r-j+1}} \cdots x_{\gamma_r} v$, for $0 \le j \le r$, with finite weights $2(\varepsilon_1 + \cdots + \varepsilon_{r-j})$. Orthogonality leaves each unused coroot

pairing equal to -1. Since these finite weights are dominant, the extremal vectors are highest-weight vectors for $\mathfrak{g}$. They account for every summand in the multiplicity-free decomposition of K_{r,p_r} from [CM, Theorem 3.4(i)]. The surjection is therefore an isomorphism.

Finally, [CM, Theorem 3.4(ii)] identifies $K_{r,\ell p_r}$ with the current submodule generated by $v_{p_r}^{\otimes \ell}$ in $K_{r,p_r}^{\otimes \ell}$. The integrable Cartan embedding identifies this submodule with the right side of (A.1). Both maps send the cyclic vector to the same tensor of cyclic vectors, so they preserve the tensor maps and the dual multiplications. □

Appendix B. The Poisson pencil on framed Borel bundles

Let G be a connected semisimple group, and let λ be a dominant coweight of G. Choose opposite Borel subgroups with common Cartan algebra $\mathfrak{h}$. We use the negative-Borel model of the cell. Write $m_\alpha = \langle \alpha, \lambda \rangle$, and set

$$\mathfrak{n}_\lambda = \bigoplus_{\alpha>0} \bigoplus_{j=0}^{m_\alpha - 1} \mathbb{C} f_\alpha u^j, \qquad N_\lambda = \exp(\mathfrak{n}_\lambda).$$

The sum for a root with $m_\alpha = 0$ is empty. The inequality $j + k < m_\alpha + m_\beta = m_{\alpha+\beta}$ shows that $\mathfrak{n}_\lambda$ is a nilpotent Lie subalgebra. The framed Borel-bundle cell has the representatives $n(u)u^{-\lambda}$, with $n \in N_\lambda$.

Proposition B.1. *The rational and trigonometric Poisson tensors on N_λ form a regular compatible pencil. If Ψ_a is the action on functions induced by source pullback $u \mapsto u - a$, including the based normalization, then*

$$\Psi_a \Pi_{\mathrm{tr}} = \Pi_{\mathrm{tr}} + a\Pi_{\mathrm{rat}}, \qquad \Psi_a \Pi_{\mathrm{rat}} = \Pi_{\mathrm{rat}}.$$

More generally, this pencil is natural under affine changes of the source coordinate fixing infinity.

Proof. We use Chevalley root vectors with $(e_\alpha, f_\alpha) = t_\alpha = 2/(\alpha, \alpha)$. The negative-Borel kernel has finite term $\frac{1}{2}\sum_{\alpha>0} t_\alpha^{-1} e_\alpha \wedge f_\alpha$. After interchanging the Borels, this gives the finite term $-E_i \wedge F_i/(2t_i)$ used in the main text.

We construct the tensors on the loop group and descend them to the framed-bundle quotient. Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$, with invariant form

$$((x,h),(y,k))_{\mathfrak{d}} = (x,y)_{\mathfrak{g}} - (h,k)_{\mathfrak{h}}.$$

The two subalgebras

$$\mathfrak{l} = \{(x_- + h, h) : x_- \in \mathfrak{n}_-,\ h \in \mathfrak{h}\}, \qquad \mathfrak{l}' = \{(x_+ + h, -h) : x_+ \in \mathfrak{n}_+,\ h \in \mathfrak{h}\}$$

are complementary and Lagrangian. Thus $(\mathfrak{d}, \mathfrak{l}, \mathfrak{l}')$ is a *Manin triple*. The subgroup integrating $\mathfrak{l}$ is naturally isomorphic to B_-. For dual bases $a_i \in \mathfrak{l}$ and $a^i \in \mathfrak{l}'$, let $r_{\mathfrak{d}} = \frac{1}{2}\sum_i a_i \wedge a^i$. The standard double construction [CP, Sections 1.3B, 1.4B, and 2.2] gives the corresponding multiplicative Poisson tensor, in right-trivialized tangent coordinates,

$$\rho(b) = (\mathrm{Ad}_b \otimes \mathrm{Ad}_b) r_{\mathfrak{d}} - r_{\mathfrak{d}}.$$

It takes values in $\bigwedge^2 \mathfrak{l}$. Since $r_{\mathfrak{d}}$ is torus invariant, ρ vanishes on the torus.

For a nonzero affine polynomial $p(u) = cu + d$, consider the loop Manin triple

$$(\mathfrak{d}((u^{-1})), \mathfrak{l}((u^{-1})), \mathfrak{l}'((u^{-1}))), \qquad \langle x, y \rangle_p = -\operatorname{Res}_{u=\infty} (x(u), y(u))_{\mathfrak{d}} \frac{du}{p(u)}.$$

In this pairing, $\mathfrak{l}'((u^{-1}))$ is the continuous dual Lie algebra of $\mathfrak{l}((u^{-1}))$. The annihilators in this dual Lie algebra of the two gauge algebras are

$$\mathfrak{l}[u]^{\mathrm{ann}} = p(u)\mathfrak{l}'[u], \qquad \left(u^{-1}\mathfrak{l}[[u^{-1}]]\right)^{\mathrm{ann}} = p(u)u^{-1}\mathfrak{l}'[[u^{-1}]].$$

Both are Lie subalgebras: the first because p is polynomial, and the second because $\deg p \le 1$. The two gauge groups are consequently *coisotropic*, so functions invariant under both gauge actions form a Poisson algebra. We obtain a Poisson tensor Π_p on the framed-bundle quotient

$$B_{-,1}[[u^{-1}]]\backslash B_-((u^{-1}))/B_-[u].$$

The degree of the torus bundle is locally constant in families. Thus N_λ is an open-and-closed component, and Π_p restricts to it.

To compute the quotient tensor, let Q denote polynomial part and let $P_\lambda : \mathfrak{l}[u] \to \mathfrak{n}_\lambda$ be the projection in

$$\mathfrak{l}[u] = \mathfrak{n}_\lambda \oplus \operatorname{Ad}_{u^{-\lambda}} \mathfrak{l}[u].$$

We regard $\mathfrak{n}_\lambda$ as a subspace of the first summand of $\mathfrak{d}$. Extend P_λ by zero on $\mathfrak{l}'[u]$. At $b = n(u)u^{-\lambda}$, the differential of based normalization, with left-trivialized target tangent space, is

$$D_n = P_\lambda \operatorname{Ad}_{n^{-1}} Q. \tag{B.1}$$

Indeed, differentiating a factorization $b = b_\infty n u^{-\lambda} b_0$ at normalized factors gives

$$\dot{b}b^{-1} = \xi_\infty + \dot{n}n^{-1} + \operatorname{Ad}_{nu^{-\lambda}} \xi_0, \qquad \xi_\infty \in u^{-1}\mathfrak{l}[[u^{-1}]], \quad \xi_0 \in \mathfrak{l}[u].$$

Polynomial part removes the first term. Conjugation by n^{-1}, followed by P_λ, removes the last term and gives $n^{-1}\dot{n}$. This proves (B.1).

The current Poisson tensor has kernel $p(u)\rho(b(u))\delta(u,v)$, where $\delta(u,v) = \sum_{k\in\mathbb{Z}} u^k v^{-k-1}$. For every polynomial tensor $H(u)$, polynomial part in both variables gives the finite identity

$$(Q \otimes Q)\big(H(u)\delta(u,v)\big) = \frac{H(u) - H(v)}{u - v}. \tag{B.2}$$

To check it, take $H(u) = u^j H_0$. The surviving terms are $\sum_{k=0}^{j-1} u^{j-1-k} v^k H_0$; the sum is empty when $j = 0$.

Since $\rho(u^{-\lambda}) = 0$, multiplicativity gives $\rho(nu^{-\lambda}) = \rho(n)$ in right-trivialized coordinates. Equations (B.1) and (B.2) therefore give

$$\Pi_p(n) = (P_\lambda \otimes P_\lambda)(\operatorname{Ad}_{n(u)^{-1}} \otimes \operatorname{Ad}_{n(v)^{-1}}) \frac{p(u)\rho(n(u)) - p(v)\rho(n(v))}{u - v}. \tag{B.3}$$

This formula is polynomial in the coordinates of n, so Π_p is regular on the cell.

Let $\Omega_\mathfrak{d}$ be the Casimir tensor of the quadratic double. We have

$$r_\mathfrak{d} = \frac{1}{2}\Omega_\mathfrak{d} - \sum_i a^i \otimes a_i = \sum_i a_i \otimes a^i - \frac{1}{2}\Omega_\mathfrak{d}.$$

Substitute the formula for ρ into (B.3). In its term containing $p(u)$, use the first expression for $r_\mathfrak{d}$; after conjugation, the first tensor factor a^i is annihilated by P_λ. In the term containing $-p(v)$, use the second expression; its second factor a^i is annihilated. Invariance of the Casimir tensor then gives

$$\Pi_p(n) = (P_\lambda \otimes P_\lambda)(\operatorname{Ad}_{n(u)^{-1}} \otimes \operatorname{Ad}_{n(v)^{-1}}) \left(\frac{p(u) + p(v)}{2(u - v)} \Omega_\mathfrak{d} - c r_\mathfrak{d} \right).$$

The auxiliary Cartan summand of $\mathfrak{d}$ is fixed by n and is annihilated by P_λ. Thus $\Omega_\mathfrak{d}$ reduces to $\Omega_\mathfrak{g}$, while $-r_\mathfrak{d}$ reduces to $\frac{1}{2}\sum_{\alpha>0} t_\alpha^{-1} e_\alpha \wedge f_\alpha$. Consequently the kernel on $\mathfrak{g}$ is exactly

$$\mathcal{R}_p(u,v) = \frac{p(u) + p(v)}{2(u - v)} \Omega_\mathfrak{g} + \frac{c}{2} \sum_{\alpha>0} t_\alpha^{-1} e_\alpha \wedge f_\alpha. \tag{B.4}$$

In the Casimir term, after conjugation we first project to the two negative root spaces. Casimir invariance makes the numerator vanish at $u = v$, so division gives a polynomial before we extract the prescribed coefficients.

For $p = 1$, (B.4) is the rational kernel. For $p = u$, it is the trigonometric kernel of [FKR, Equation (4.11)] in the negative-Borel model, after rescaling the invariant form from their short-root normalization to the long-root normalization used here. Applying $P_\lambda \otimes P_\lambda$ after

conjugation is the homogeneous-space formula for these tensors. The term at the coweight point vanishes because P_λ kills the positive root spaces and the Cartan subalgebra. Hence $\Pi_1 = \Pi_{\mathrm{rat}}$ and $\Pi_u = \Pi_{\mathrm{tr}}$. Moreover $\Pi_{cu+d} = c\Pi_{\mathrm{tr}} + d\Pi_{\mathrm{rat}}$. The quotient construction proves Jacobi for every nonzero p, so the two tensors are compatible.

Finally, source pullback preserves both gauge groups and transports the current tensor by the corresponding change of vector field $p(u)\partial_u$. In the action on functions induced by $u \mapsto u - a$, it follows that $\Psi_a \Pi_p = \Pi_{p(u+a)}$. The stated identities follow by taking $p = 1$ and $p = u$. The same argument gives covariance under affine changes of coordinate. □

Declaration of generative AI and AI-assisted technologies

ChatGPT was used for mathematical exploration, proof development, exact computations, and revision of the exposition. Its contributions include the uniform reddening argument and the accompanying verification programs.

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