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Optimal coupon packages and stochastic improvement of uniform group drawings

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Abstract

A collector repeatedly draws independent packages of at most s coupons from a universe of n coupons. When $s \mid n$, we prove that the uniform law on the blocks of an s-set partition uniquely minimizes the expected completion time, resolving the Fang–Chang optimal-package conjecture. The partition law is stochastically optimal in Poissonized time, also for every common collection multiplicity. The proof combines independent package clocks, Finner's inequality, and strict concavity; an incidence-class tail argument characterizes equality. The same bound proves the lower-bound conjecture of Grunbaum and Yaakobi without their regularity assumptions. A separate elementary discrete-time bound identifies the unique optimal complementary-partition laws whenever $s < n$ and $n - s \mid n$.

For fixed-size packages, a four-coupon perturbation preserves singleton marginals and strictly increases the completion probability at every feasible draw count. This gives an alternative proof and a stochastic strengthening of Schilling's conjecture, previously proved in expectation by Berend and Sher. More generally, perturbations formed from d disjoint pairs preserve all inclusion probabilities below order d and are strictly stochastically monotone in their absolute amplitude, with the direction determined by the parity of d. Their inclusion–exclusion coefficients count covering families. Combined with the Johnson-scheme decomposition, this yields the complete alternating signature of the expectation Hessian at the uniform law. The general fixed-draw partition comparison is left as a conjecture; its first feasible draw count follows from the Poissonized bound.

1 Introduction and principal results

Let $[n] = \{1, \dots, n\}$, and let μ be a probability distribution on subsets of $[n]$. Packages $S_1, S_2, \dots$ are independent with common law μ. Their completion time is

$$T_\mu = \inf\Big\{ m \geq 0 : \bigcup_{j=1}^{m} S_j = [n] \Big\},$$

with $\inf \varnothing = \infty$. Throughout the optimization problem, $|S_j| \leq s$ almost surely, where $1 \leq s \leq n$. Empty packages are allowed. The law is fixed in advance: no adaptive or history-dependent choice of packages is permitted.

Write $H_q = \sum_{j=1}^{q} j^{-1}$ for an integer $q \geq 1$. If $n = qs$, a *partition law* $\mathcal{D}_{n,s}$ assigns probability $1/q$ to each block of a partition of $[n]$ into q sets of size s. Its completion time is the ordinary uniform q-coupon collection time, so

$$\mathbb{E}T_{\mathcal{D}_{n,s}} = qH_q. \tag{1.1}$$

Fang and Chang proved both fixed-draw stochastic optimality and uniqueness of the mean minimizer when $n = 2s$, and proposed the partition optimum for general $s \mid n$ [2, Theorem 2 and Corollary 3]. The general divisible case is still stated as conjectural in the introduction of Berend and Sher [1]. The first result below establishes it. Here τ_μ denotes completion time when packages arrive at the points of an independent rate-one Poisson process.

There is an attribution discrepancy worth recording. Grunbaum and Yaakobi [5, Remark 2] refer to the divisible-case lower bound as a general result, citing Berend and Sher; the cited work instead describes general decomposition optimality as a conjecture. Our proof is independent of that attribution. Appendix A separately rules out a stronger minimum-cover comparison.

Theorem 1.1 (Optimal partition packages)**.** *Suppose that $n = qs$, where $q, s \geq 1$ are integers. For every package law supported on sets of size at most s,*

$$\mathbb{P}(\tau_\mu \leq t) \leq (1 - e^{-t/q})^q, \qquad t \geq 0, \tag{1.2}$$

and consequently

$$\mathbb{E}T_\mu \geq qH_q. \tag{1.3}$$

Equality in (1.3) *holds if and only if μ is a partition law. In particular, $\tau_{\mathcal{D}_{n,s}} \leq_{\mathrm{st}} \tau_\mu$.*

This is a statement about Poissonized stochastic order and about the unpoissonized expectation. A fixed-draw stochastic comparison between an arbitrary law and the partition law does not follow merely by removing the Poissonization. We maintain this distinction throughout.

A second problem concerns the uniform law $\mathcal{U}_{n,s}$ on all $N = \binom{n}{s}$ sets of size s. Schilling proved its optimality when $s = n - 1$ and conjectured its failure to minimize the mean in every case $2 \leq s \leq n - 2$ [6]. Berend and Sher proved this conjecture by constructing nonuniform competitors with smaller means [1]. Their preprint first appeared in September 2025; the journal article was published online on May 27, 2026. Our next theorem supplies a different proof and strengthens the comparison to discrete-time stochastic order.

For $i \in [n]$, write $x_i(A) = \mathbf{1}_{\{i \in A\}}$.

Theorem 1.2 (A marginal-preserving stochastic improvement)**.** *Let $2 \leq s \leq n - 2$ and, for $|A| = s$, define*

$$\mu_\theta(A) = \frac{1 + \theta(x_1(A) - x_2(A))(x_3(A) - x_4(A))}{\binom{n}{s}}, \qquad -1 \leq \theta \leq 1. \tag{1.4}$$

This is a probability law satisfying $\mathbb{P}_{\mu_\theta}(i \in S) = s/n$ for every i. If $0 < |\theta| \leq 1$, then

$$\mathbb{P}(T_{\mu_\theta} \leq m) > \mathbb{P}(T_{\mathcal{U}_{n,s}} \leq m) \quad \textit{for every integer } m \geq \lceil n/s \rceil. \tag{1.5}$$

For $m < \lceil n/s \rceil$, both probabilities vanish. Thus $T_{\mu_\theta} \leq_{\mathrm{st}} T_{\mathcal{U}_{n,s}}$, with distinct distributions, and $\mathbb{E}T_{\mu_\theta} < \mathbb{E}T_{\mathcal{U}_{n,s}}$.

The two theorems use complementary mechanisms. Finner's product-space inequality [4] controls coverage globally after Poissonization. Theorem 1.2 instead comes from a signed perturbation whose inclusion–exclusion coefficients are nonnegative covering counts. The latter method also yields an alternating family of higher-order perturbations and the full Johnson-scheme signature of the uniform-law Hessian (Section 7). Section 3 gives a discrete geometric tail bound and characterizes its complementary-partition equality cases. The construction is the divisible-complement specialization of Berend and Sher's complementary decomposition distribution [1, Section 2]; here it is shown to be globally stochastically optimal.

2 Poissonization, the universal bound, and equality

2.1 Independent package clocks

Let $E_1, E_2, \dots$ be independent rate-one exponential variables, independent of the package sequence. Then

$$\tau_\mu = \sum_{j=1}^{T_\mu} E_j. \tag{2.1}$$

In particular, conditioning on T_μ and using monotone convergence gives

$$\mathbb{E}\tau_\mu = \mathbb{E}T_\mu, \tag{2.2}$$

including infinite expectations. Equivalently, at each fixed time t, the counts

$$Z_A(t) = \#\{\text{arrivals of package } A \text{ by time } t\}, \qquad A \in \operatorname{supp}\mu,$$

are independent Poisson variables with means $t\mu(A)$. The collection is finite because the coupon universe is finite.

Set

$$p_i = \mathbb{P}_\mu(i \in S) = \sum_{A \ni i} \mu(A). \tag{2.3}$$

Coupon i has been seen by time t precisely when $\sum_{A\ni i} Z_A(t) > 0$, so its coverage probability is $1 - e^{-tp_i}$. Also,

$$\sum_{i=1}^{n} p_i = \mathbb{E}_\mu|S| \le s. \tag{2.4}$$

If some $p_i = 0$, completion is impossible. Otherwise all completion-time moments are finite, for example by the union bound $\mathbb{P}(T_\mu > m) \le \sum_i (1-p_i)^m$.

2.2 The product-space inequality

We record the precise form of Finner's inequality that is needed and include its elementary proof. Its use is justified by the independent *package clocks*, not by a bounded-degree dependency graph on the coupons.

Lemma 2.1 (Finner's inequality)**.** *Let $X_1, \dots, X_M$ be independent. For each i, let $f_i \ge 0$ depend only on coordinates in $I_i \subseteq [M]$. Suppose that $0 < w_i \le 1$ and*

$$\sum_{i : j \in I_i} w_i \le 1 \qquad (j \in [M]).$$

Then

$$\mathbb{E}\prod_i f_i \le \prod_i \left(\mathbb{E} f_i^{1/w_i}\right)^{w_i}. \tag{2.5}$$

In particular, if every coordinate occurs in at most s functions, one may take $w_i = 1/s$.

Proof. It suffices first to consider bounded functions. Integrate in X_M, holding the other coordinates fixed. Apply ordinary Hölder to the functions depending on X_M, with exponents $1/w_i$; a constant function fills any unused exponent weight. Replace each such f_i by

$$g_i = \left(\mathbb{E}_{X_M} f_i^{1/w_i}\right)^{w_i}$$

and leave the remaining functions unchanged. The coordinate-incidence conditions persist on $M-1$ coordinates. Induction gives (2.5), because $\mathbb{E}g_i^{1/w_i} = \mathbb{E}f_i^{1/w_i}$. The case $M = 0$ is immediate. Truncation and monotone convergence remove boundedness. □

For $x > 0$ define the real harmonic interpolation

$$\mathsf{H}(x) = \int_0^1 \frac{1-(1-u)^x}{u}\,du. \tag{2.6}$$

It agrees with H_q at positive integers q. Its integral definition will suffice throughout.

Theorem 2.2 (Universal Poissonized bound)**.** *For $1 \le s \le n$, put $q = n/s$, not necessarily an integer. Every admissible package law satisfies*

$$\mathbb{P}(\tau_\mu \le t) \le \prod_{i=1}^{n}(1-e^{-tp_i})^{1/s} \le (1-e^{-t/q})^q, \qquad t \ge 0. \tag{2.7}$$

Consequently,

$$\mathbb{E}T_\mu \ge q\mathsf{H}(q). \tag{2.8}$$

Equality in (2.8) *is possible if and only if q is an integer and μ is a partition law.*

Proof of the inequalities. If any $p_i = 0$, the assertions are immediate. Otherwise fix $t > 0$. Apply Lemma 2.1 to the indicators of the n coupon-coverage events, with coordinates $Z_A(t)$ and weights $1/s$. Coordinate $Z_A(t)$ occurs in exactly $|A| \le s$ indicators. This proves the first inequality in (2.7).

The function $f_t(x) = \log(1-e^{-tx})$ on $(0,\infty)$ is strictly increasing and strictly concave, since

$$f_t'(x) = \frac{t}{e^{tx}-1} > 0, \qquad f_t''(x) = -\frac{t^2e^{tx}}{(e^{tx}-1)^2} < 0.$$

Jensen's inequality and (2.4) give

$$\frac{1}{s}\sum_i f_t(p_i) \le \frac{n}{s} f_t\Big(\frac{1}{n}\sum_i p_i\Big) \le q f_t(1/q).$$

Exponentiating proves (2.7). The case $t = 0$ is immediate. Finally,

$$\begin{aligned}\mathbb{E}T_\mu = \mathbb{E}\tau_\mu &\ge \int_0^\infty [1-(1-e^{-t/q})^q]\,dt \\ &= q\int_0^1 \frac{1-(1-u)^q}{u}\,du = q\mathsf{H}(q).\end{aligned}$$

□

2.3 Equality without equality conditions for Finner's inequality

The strict Jensen step forces balanced marginals, but balanced marginals alone do not characterize the optimum. The following argument supplies the additional rigidity.

Proof of the equality statement in Theorem 2.2. Suppose equality holds in (2.8). The difference between the final bound in (2.7) and the actual completion CDF is nonnegative and continuous. Its integral is zero, so it vanishes at every $t \ge 0$. Equality throughout the Jensen chain at any $t > 0$ therefore gives

$$p_i = p := s/n \quad \text{for all } i, \qquad |S| = s \quad \mu\text{-almost surely.} \tag{2.9}$$

Declare $i \sim j$ when every package in $\operatorname{supp}\mu$ contains either both i, j or neither. Let $C_1, \dots, C_k$ be the resulting incidence classes. Each class is contained in some supported package, so $|C_a| \le s$. Every supported package is a union of complete classes.

Let $U_a(t)$ be the event that class C_a has not appeared by time t. Then $\mathbb{P}(U_a(t)) = e^{-pt}$. For two distinct classes choose $i \in C_a$, $j \in C_b$. Equal marginals and inequivalent incidences imply

$$r_{ab} := \mathbb{P}_\mu(i \in S \text{ or } j \in S) > p, \qquad \mathbb{P}(U_a(t) \cap U_b(t)) = e^{-r_{ab}t}.$$

Indeed, equal marginals give equal probabilities for the two one-sided differences, and their sum is positive. Since the number of classes is finite, the first two Bonferroni bounds imply

$$\mathbb{P}(\tau_\mu > t) = ke^{-pt} + o(e^{-pt}) \qquad (t \longrightarrow \infty). \tag{2.10}$$

On the other hand, equality in (2.7) gives

$$\mathbb{P}(\tau_\mu > t) = 1 - (1 - e^{-pt})^q = qe^{-pt} + O(e^{-2pt}).$$

Thus $k = q$. In particular q is an integer. The q classes partition $n = qs$ coupons and each has size at most s, so all have size s. A supported package has size s and is a union of classes; hence it is exactly one class. Its probability is the common coupon marginal $p = 1/q$. Therefore μ is a partition law.

Conversely, under a partition law the block clocks are independent rate-$1/q$ Poisson processes. Their first-arrival maximum has CDF $(1 - e^{-t/q})^q$, so equality holds. □

Theorem 1.1 now follows. Notice that no classification of the equality cases of Finner's inequality was required.

Corollary 2.3 (The Grunbaum–Yaakobi lower bound)**.** *For any iid package law supported on subsets of size at most s,*

$$\mathbb{E}T_\mu \ge \frac{n}{s}\mathsf{H}(n/s) \ge \left\lfloor \frac{n}{s} \right\rfloor H_{\lfloor n/s \rfloor}. \tag{2.11}$$

The comparison with the final bound is strict when $s \nmid n$. When $s \mid n$, equality holds exactly for the partition laws. In particular, Conjecture 3 of Grunbaum and Yaakobi [5], posed for uniform sampling from equal-size hyperedges with equal vertex degrees, holds without either uniformity or equal-degree assumptions.

Proof. The integrand in (2.6) is strictly increasing in x for $0 < u < 1$. Thus $\mathsf{H}(x) > 0$ and $x\mathsf{H}(x)$ is strictly increasing for $x > 0$. Apply Theorem 2.2 and compare $x = n/s$ with $\lfloor x \rfloor$. Its equality characterization proves the final assertion. □

2.4 Consequences and the limit of de-Poissonization

Corollary 2.4 (Generating functions and rising factorial moments)**.** *Assume $n = qs$. For $0 < z < 1$,*

$$\mathbb{E}z^{T_\mu} \le \mathbb{E}z^{T_{\mathcal{D}_{n,s}}}. \tag{2.12}$$

For each integer $r \ge 1$,

$$\mathbb{E}[T_\mu(T_\mu + 1)\cdots(T_\mu + r - 1)] \ge \mathbb{E}[T_{\mathcal{D}_{n,s}}(T_{\mathcal{D}_{n,s}} + 1)\cdots(T_{\mathcal{D}_{n,s}} + r - 1)]. \tag{2.13}$$

These inequalities allow infinite values, with $z^\infty = 0$.

Proof. For $\lambda > 0$, (2.1) gives $\mathbb{E}e^{-\lambda\tau_\mu} = \mathbb{E}(1+\lambda)^{-T_\mu}$. Use the decreasing function $e^{-\lambda t}$ in the stochastic comparison and set $z = (1+\lambda)^{-1}$. For (2.13), condition on $T_\mu = m$: the rth moment of the sum of m independent rate-one exponentials is $m(m+1)\cdots(m+r-1)$. Apply the Poissonized stochastic order to t^r. □

Neither (2.12) nor (2.13) is asserted to imply fixed-draw stochastic dominance or variance minimization. In particular,

$$\mathbb{P}(\tau_\mu \le t) = e^{-t}\sum_{m\ge 0}\frac{t^m}{m!}\mathbb{P}(T_\mu \le m)$$

is a positive smoothing transform; an inequality between the transforms does not in general give a coefficientwise inequality.

The same proof also yields the support-sensitive estimate

$$\mathbb{P}(\tau_\mu \le t) \le \prod_{i:w_i>0}(1-e^{-tp_i})^{w_i} \tag{2.14}$$

for any $w_i \ge 0$ satisfying $\sum_{i\in A} w_i \le 1$ for every supported package A. If a positively weighted coupon has zero marginal, the bound is immediate. Otherwise the support constraints imply $w_i \le 1$ on the positive-weight indices. Discard the zero-weight coverage requirements and apply Lemma 2.1 to the remainder. Inequality (2.7) is the choice $w_i = 1/s$.

3 Discrete-time bounds and complementary partitions

3.1 A universal geometric lower tail

The following first-moment argument extends the mechanism used by Fang and Chang for $n = 2s$ [2, Lemma 1 and Theorem 2]. The hypothesis $s < n$ is essential.

Theorem 3.1 (Geometric tail bound)**.** *Let $1 \le s < n$, and put $q = n/s$. Every iid package law with $|S| \le s$ satisfies*

$$\mathbb{P}(T_\mu > m) \ge \left(1-\frac{s}{n}\right)^{m-1} \qquad (m \ge 1), \tag{3.1}$$

and therefore

$$\mathbb{E}T_\mu \ge 1+q. \tag{3.2}$$

Equality in (3.2) *holds if and only if $h = n-s$ divides n and μ is uniform on the complements of the blocks of a partition of $[n]$ into h-sets. Such a law attains equality in* (3.1) *for every $m \ge 1$.*

Proof of the inequalities. First suppose every package has exactly s coupons. Let U_m count the coupons missing after m draws. For $m \ge 1$ the first package guarantees $0 \le U_m \le n-s$, so

$$\mathbb{P}(T_\mu > m) \ge \frac{\mathbb{E}U_m}{n-s} = \frac{1}{n-s}\sum_{i=1}^{n}(1-p_i)^m \ge \frac{n}{n-s}\left(1-\frac{s}{n}\right)^m.$$

The last step is Jensen's inequality, using $\sum_i p_i = s$. This is (3.1). Since $\mathbb{P}(T_\mu > 0) = 1$, summing gives

$$\mathbb{E}T_\mu \ge 1+\sum_{m\ge 1}(1-s/n)^{m-1} = 1+n/s.$$

For a law on sets of size at most s, fix a deterministic enlargement $A \mapsto \widehat{A} \supseteq A$ with $|\widehat{A}| = s$. The enlarged draws remain iid and their completion time satisfies $T_{\widehat{\mu}} \le T_\mu$ pathwise. Apply the exact-size inequalities to $\widehat{\mu}$. □

Proof of equality and rigidity. Suppose first that packages have exactly s coupons and equality holds in (3.2). Every termwise tail difference in the preceding proof is nonnegative, hence equality holds at $m = 2$. Strict Jensen equality gives $p_i = s/n$ for every i. Equality in $\mathbb{E}U_2 \le h\mathbb{P}(U_2 > 0)$ gives $U_2 \in \{0, h\}$ almost surely.

The complement of every supported package has size h. For two supported packages A, B, the event that they are the first two draws has positive probability and

$$U_2 = |A^c \cap B^c| \in \{0, h\}.$$

Their complements are therefore identical or disjoint. The distinct complements $H_1, \dots, H_k$ are pairwise disjoint and cover $[n]$, because every coupon has omission probability $h/n > 0$. Thus $kh = n$. A coupon in H_j is omitted exactly when $[n] \setminus H_j$ is drawn, so the probability of that package is $h/n = 1/k$.

Now allow packages of size at most s and use the deterministic enlargement above. Equality of the original mean forces equality of the enlarged mean, so the enlarged law is the just-described complementary-partition law. If a smaller supported package B existed, let $\widehat{B} = [n] \setminus H_j$ and choose $i \in \widehat{B} \setminus B$. Then $i \in H_k$ for some $k \neq j$. With positive probability the first original draw is B and the second enlarged draw is $[n] \setminus H_k$. On this event the enlarged collection is complete after two draws, while the original collection still lacks i. This contradicts equality of the means under the pathwise coupling. Consequently all original packages have size s, and the exact-size characterization applies.

Conversely, under a uniform law on the complements of $k = n/h$ disjoint h-blocks, any two distinct packages cover $[n]$. Collection is incomplete after $m \ge 1$ draws exactly when all m draws agree, with probability

$$k(1/k)^m = k^{1-m} = (1 - s/n)^{m-1}.$$

This proves attainment and finishes the proof. □

Corollary 3.2 (Complementary-partition optimum)**.** *If $1 \le s < n$ and $n - s \mid n$, a uniform complementary-partition law $\mathcal{C}_{n,s}$ satisfies*

$$T_{\mathcal{C}_{n,s}} \le_{\mathrm{st}} T_\mu$$

for every iid law on sets of size at most s. It uniquely minimizes the mean, up to relabeling, and its mean is $1 + n/s$.

For $n = 2s$, the complementary-partition and partition laws coincide. Hence this recovers the fixed-draw theorem of Fang and Chang [2, Theorem 2], not a new first proof of that case. For $s = n - 1$, it recovers Schilling's endpoint optimum [6] and establishes uniqueness in the larger at-most-size class. Berend and Sher already define the complementary construction and compute its mean [1, Section 2, equation (2)]; the universal comparison above supplies its optimality when $n - s \mid n$.

Corollary 3.3 (Combined mean bound)**.** *For $1 \le s < n$ and $q = n/s$,*

$$\mathbb{E}T_\mu \ge \max\{1 + q,\ q\mathsf{H}(q)\}. \tag{3.3}$$

This improves the trivial integer bound $T_\mu \ge \lceil q \rceil$. The extra term is useful for small nonintegral q: when $n = 3, s = 2$, the Poissonized bound is $(3/2)\mathsf{H}(3/2) = 4 - 3\log 2 < 2$, whereas (3.2) gives the sharp value $5/2$. If $s \nmid n$ then the $q\mathsf{H}(q)$ component is strict, but the combined bound can still be attained through its $1 + q$ component.

3.2 The earliest feasible draw count

Proposition 3.4 (First feasible draw)**.** *If $n = qs$ with $q \geq 1$ an integer, then*

$$\mathbb{P}(T_\mu \leq q) \leq \frac{q!}{q^q} \tag{3.4}$$

for every law on sets of size at most s. Equivalently, if $\Pi_{n,s}$ denotes the set of partitions of $[n]$ into q blocks of size s, then

$$\sum_{\mathcal{P} \in \Pi_{n,s}} \prod_{A \in \mathcal{P}} \mu(A) \leq q^{-q}. \tag{3.5}$$

The partition law attains both bounds.

Proof. Fewer than q packages cannot cover $[n]$. Thus, as $t \downarrow 0$,

$$\mathbb{P}(\tau_\mu \leq t) = e^{-t} \sum_{m=q}^{\infty} \frac{t^m}{m!} \mathbb{P}(T_\mu \leq m) = \frac{\mathbb{P}(T_\mu \leq q)}{q!} t^q + O(t^{q+1}).$$

Theorem 2.2 bounds this by $(1 - e^{-t/q})^q = q^{-q} t^q + O(t^{q+1})$. Comparing leading coefficients proves (3.4). Coverage with exactly q draws forces all packages to have size s and to be pairwise disjoint. Every unordered partition has $q!$ orderings, giving (3.5). □

This argument controls the first potentially nonzero coefficient. It does not extract inequalities for later coefficients from a Poissonized comparison.

4 A common collection multiplicity

Let $T_\mu^{(r)}$ be the number of packages required for every coupon to appear at least r times, where $r \geq 1$ is an integer, and let $\tau_\mu^{(r)}$ be its rate-one Poissonization. A package contains at most one copy of each coupon. Define

$$G_r(x) = \mathbb{P}(\mathrm{Poi}(x) \geq r) = 1 - e^{-x} \sum_{j=0}^{r-1} \frac{x^j}{j!}.$$

Lemma 4.1. *For every $r \geq 1$, G_r is strictly increasing and $\log G_r$ is strictly concave on $(0, \infty)$.*

Proof. Let $g(x) = G_r'(x) = x^{r-1} e^{-x}/(r-1)!$ and $h(x) = g'(x)/g(x) = (r-1)/x - 1$. Clearly $g > 0$. We have

$$(\log G_r)''(x) = \frac{g(x)}{G_r(x)^2} \big(h(x) G_r(x) - g(x)\big).$$

When $h(x) \leq 0$ the bracket is negative. When $h(x) > 0$, the concavity of $\log g$ implies, for $0 < u \leq x$, $g(u) \leq g(x) e^{h(x)(u-x)}$. Hence

$$G_r(x) \leq g(x) \frac{1 - e^{-h(x)x}}{h(x)} < \frac{g(x)}{h(x)},$$

which again makes the bracket negative. □

Theorem 4.2 (Repeated collection)**.** *Let $q = n/s$, possibly nonintegral. For every $r \geq 1$,*

$$\mathbb{P}(\tau_\mu^{(r)} \leq t) \leq \prod_{i=1}^{n} G_r(tp_i)^{1/s} \leq G_r(t/q)^q. \tag{4.1}$$

If q is an integer, the partition law attains the bound for every t and uniquely minimizes $\mathbb{E}T_\mu^{(r)}$. Its minimum is

$$q \int_0^\infty [1 - G_r(u)^q]\, du. \tag{4.2}$$

Proof. If a marginal vanishes, the bound is immediate. Otherwise the event that coupon i has appeared at least r times depends only on its incident package counts and has probability $G_r(tp_i)$. Finner's inequality and Lemma 4.1, followed by (2.4), prove (4.1). Under the partition law, the q block counts are independent $\mathrm{Poi}(t/q)$ variables, so the bound is attained. Equation (2.2) applies without change and gives (4.2).

For uniqueness, equality of the means implies equality of the continuous CDFs. Strict concavity again forces (2.9). Use the same incidence classes as in Section 2.3. For one class the failure probability is

$$1 - G_r(pt) \sim e^{-pt} \frac{(pt)^{r-1}}{(r-1)!}.$$

For two distinct classes, simultaneous failure implies that at most $2r-2$ packages meeting their union have arrived. The rate of those arrivals is $r_{ab} > p$, so their joint failure probability is at most

$$\mathbb{P}(\mathrm{Poi}(r_{ab}t) \leq 2r-2) = O(t^{2r-2}e^{-r_{ab}t}) = o(t^{r-1}e^{-pt}).$$

Bonferroni therefore gives leading tail coefficient k, whereas $1 - G_r(pt)^q \sim q(1 - G_r(pt))$ gives coefficient q. Thus $k = q$, and the class-size argument again identifies the partition law. □

This result concerns a common quota r. It makes no claim about heterogeneous coupon quotas, or about the effect of the perturbations below on repeated-collection times.

5 Paired perturbations in discrete time

We now restrict to packages of exactly s, with $1 \leq s \leq n-1$, and set $N = \binom{n}{s}$. We use the conventions $\binom{a}{b} = 0$ for integers $a \geq 0$ and $b \notin \{0, \ldots, a\}$, and $0^0 = 1$ in finite counting formulas.

5.1 The parity theorem

Choose an integer $1 \leq d \leq \min(s, n-s)$ and d disjoint ordered pairs, labeled $(1,2), (3,4), \ldots, (2d-1, 2d)$. Define

$$\chi_d(A) = \prod_{j=1}^{d} (x_{2j-1}(A) - x_{2j}(A)), \qquad \mu_{d,\theta}(A) = \frac{1 + \theta\chi_d(A)}{N}, \quad |\theta| \leq 1. \tag{5.1}$$

Write $F_{d,\theta}(m) = \mathbb{P}(T_{\mu_{d,\theta}} \leq m)$ and $F_0(m) = \mathbb{P}(T_{\mathcal{U}_{n,s}} \leq m)$. Put $M = n - 2d$.

Theorem 5.1 (Paired-perturbation parity)**.** *The law* (5.1) *is a probability law. For every $B \subseteq [n]$ with $|B| < d$,*

$$\mathbb{P}_{\mu_{d,\theta}}(B \subseteq S) = \mathbb{P}_{\mathcal{U}_{n,s}}(B \subseteq S). \tag{5.2}$$

For $m \geq 0$,

$$F_{d,\theta}(m) - F_0(m) = \frac{(-1)^d 2^d}{N^m} \sum_{\ell=1}^{\lfloor m/2 \rfloor} \binom{m}{2\ell} \theta^{2\ell} K_{m,\ell}^{(d)}, \tag{5.3}$$

where

$$K_{m,\ell}^{(d)} = \sum_{r=0}^{M} (-1)^r \binom{M}{r} \binom{M-r}{s-d}^{2\ell} \binom{M-r+d}{s}^{m-2\ell} \geq 0. \tag{5.4}$$

Moreover, $K_{m,1}^{(d)} > 0$ whenever $ms \geq n$. Thus, for $0 < |\theta| \leq 1$,

$$(-1)^d (F_{d,\theta}(m) - F_0(m)) > 0 \qquad (m \geq \lceil n/s \rceil). \tag{5.5}$$

In particular, even d stochastically decreases completion time and odd d stochastically increases it. In either case the completion distribution changes strictly.

5.2 Normalization and preserved marginals

Proof of normalization and (5.2). Since $\chi_d \in \{-1, 0, 1\}$, all masses are nonnegative. Swapping the labels in any chosen pair negates χ_d and permutes the s-sets. Thus $\sum_{|A|=s} \chi_d(A) = 0$, proving normalization.

If $|B| < d$, at least one chosen pair is disjoint from B. Swapping that pair preserves the condition $B \subseteq A$ and reverses $\chi_d(A)$, so $\sum_{A \supseteq B, |A|=s} \chi_d(A) = 0$. This proves (5.2). In fact, inclusion–exclusion within B shows that the full membership law on any fewer than d coupon labels is unchanged. □

5.3 Inclusion–exclusion and the exact expansion

For any package law ν, let

$$a_\nu(R) = \mathbb{P}_\nu(S \cap R = \varnothing).$$

Inclusion–exclusion over the coupons still missing after m draws gives

$$\mathbb{P}(T_\nu \leq m) = \sum_{R \subseteq [n]} (-1)^{|R|} a_\nu(R)^m. \tag{5.6}$$

If all coupons have positive inclusion probability, summing the complementary tails also gives

$$\mathbb{E}T_\nu = \sum_{\varnothing \neq R \subseteq [n]} \frac{(-1)^{|R|+1}}{1 - a_\nu(R)}. \tag{5.7}$$

Proof of (5.3). Let $V = [n] \setminus [2d]$, so $|V| = M$. The change $a_{\mu_{d,\theta}}(R) - a_{\mathcal{U}_{n,s}}(R)$ vanishes unless R contains exactly one label of each chosen pair. Indeed, if neither label is in R, swapping that pair cancels the signed sum; if both labels are in R, every package avoiding R has $\chi_d = 0$.

It remains to consider $R = B \cup J$, where $B \subseteq V$ and J chooses one label of each pair. Put $r = |B|$, and abbreviate

$$b_r = \binom{M-r+d}{s}, \qquad c_r = \binom{M-r}{s-d}.$$

There are b_r packages avoiding R. A nonzero signed contribution must contain all d distinguished labels outside R, and choose its other $s - d$ coupons from $V \setminus B$. Hence

$$a_{\mu_{d,\theta}}(R) = \frac{b_r + \theta \sigma_J c_r}{N}, \qquad \sigma_J \in \{-1, 1\}.$$

Exactly 2^{d-1} choices of J have each sign, and $(-1)^{|R|} = (-1)^{r+d}$. For fixed B, the change in (5.6) is therefore

$$\frac{(-1)^{r+d}2^{d-1}}{N^m}\big[(b_r+\theta c_r)^m + (b_r-\theta c_r)^m - 2b_r^m\big].$$

Expanding the even powers of θ and summing over $B \subseteq V$ proves (5.3) and the formula for $K_{m,\ell}^{(d)}$. □

5.4 Why every coefficient is nonnegative

Proof of nonnegativity and strictness. Fix an auxiliary set D of d points disjoint from V. Consider ordered families

$$(C_1, \dots, C_{2\ell}; B_1, \dots, B_{m-2\ell})$$

with $C_j \subseteq V$, $|C_j| = s-d$, and $B_j \subseteq V \cup D$, $|B_j| = s$. Repetitions between different sets are permitted. Count those families whose union contains V; no coverage condition is imposed on D.

If a specified r-subset of V is forbidden, the number of such families before imposing coverage is

$$\binom{M-r}{s-d}^{2\ell}\binom{M-r+d}{s}^{m-2\ell}.$$

Inclusion–exclusion over uncovered points of V shows that their covering count is precisely $K_{m,\ell}^{(d)}$. This proves nonnegativity.

For strictness use $\ell = 1$. The two C-sets can each cover $s-d$ points of V, while each of the $m-2$ remaining sets can cover $\min(s, M)$ points. These cardinalities are feasible since $s-d \le M$ and $s \le M+d$. Such sets can cover V whenever their total available capacity is at least M: assign uncovered points successively and fill any remaining positions with previously assigned points.

If $s \le M$, the total capacity is

$$2(s-d) + (m-2)s = ms - 2d \ge n - 2d = M.$$

If $s > M$ and $m \ge 3$, one of the B-sets can already contain all of V. If $m = 2$, the same assumption $ms \ge n$ gives $2(s-d) \ge M$. The case $M = 0$ has one choice for the two C-sets and imposes no coverage requirement. Thus $K_{m,1}^{(d)} > 0$ whenever $ms \ge n$.

All coefficients in (5.3) now have the asserted sign, and the $\ell = 1$ term is strictly positive at every feasible m. Before $\lceil n/s \rceil$ draws, no sequence of s-sets can cover $[n]$. This proves (5.5) and completes the theorem. □

Corollary 5.2 (Strict monotonicity along a paired ray)**.** *The completion distribution under $\mu_{d,\theta}$ depends only on $|\theta|$. For $0 \le a < b \le 1$,*

$$\begin{cases} T_{\mu_{d,b}} \le_{\mathrm{st}} T_{\mu_{d,a}}, & d \text{ even}, \\ T_{\mu_{d,a}} \le_{\mathrm{st}} T_{\mu_{d,b}}, & d \text{ odd}. \end{cases}$$

Every CDF comparison is strict at every $m \ge \lceil n/s \rceil$. Thus the fastest members of the even-d family occur at $\theta = \pm 1$; for odd d, the fastest member is $\theta = 0$ and the endpoints are the slowest.

Proof. Equation (5.3) is a polynomial in θ^2 whose coefficients have one common sign, and its coefficient of θ^2 is strictly nonzero at every feasible draw count. Comparing $b^{2\ell}$ and $a^{2\ell}$ proves the assertion. Swapping the labels in one pair also gives a direct symmetry between θ and $-\theta$. □

The stochastic comparisons include extended-valued completion times. For example, when $d = s = 1$ an endpoint can make a coupon impossible to collect. No such degeneracy occurs for $d \ge 2$, since singleton marginals are preserved.

6 Schilling's conjecture and the uniform-law classification

6.1 The four-coupon specialization

Proof of Theorem 1.2. Take $d = 2$ in Theorem 5.1. Its admissibility condition is exactly $2 \le s \le n-2$. Equation (5.2) preserves the singleton marginals, and (5.5) proves the strict fixed-draw comparison. Because every marginal equals $s/n > 0$, the expectations are finite, even at $|\theta| = 1$. Summing the strictly improved tails gives the strict mean comparison. □

In particular, with $M = n - 4$, the coefficient used in the four-coupon proof is

$$K^{(2)}_{m,\ell} = \sum_{r=0}^{n-4} (-1)^r \binom{n-4}{r} \binom{n-4-r}{s-2}^{2\ell} \binom{n-2-r}{s}^{m-2\ell}.$$

The improvement is not produced by increasing any single-coupon arrival rate. For example, if D_m is the number of distinct coupons seen after m draws, then under every $\mu_{2,\theta}$,

$$\mathbb{E}D_m = n\big[1 - (1 - s/n)^m\big], \tag{6.1}$$

exactly as under $\mathcal{U}_{n,s}$. The gain is in joint coverage. More generally, every even d supplies an improvement preserving all inclusion probabilities of orders $0, \dots, d-1$.

Poisson mixing of the strict fixed-draw comparison also gives

$$\mathbb{P}(\tau_{\mu_{2,\theta}} \le t) > \mathbb{P}(\tau_{\mathcal{U}_{n,s}} \le t) \qquad (t > 0,\ \theta \ne 0).$$

Unlike a general de-Poissonization argument, this implication goes in the valid direction: a coefficientwise inequality is preserved by positive mixing.

6.2 The endpoint and the full classification

Proposition 6.1 (Packages missing one coupon)**.** *For $n \ge 2$, the uniform law on all $(n-1)$-sets is stochastically optimal and uniquely minimizes the mean among all iid laws on sets of size* at most *$n-1$. Its mean is $1 + n/(n-1)$.*

Proof. Apply Theorem 3.1 with $h = n - s = 1$. The only partition into singleton complements consists of all n singletons, so the optimizing package law is exactly $\mathcal{U}_{n,n-1}$. The equality proof there rules out smaller packages. □

Corollary 6.2 (Uniform-law classification)**.** *For $n \ge 2$ and $1 \le s \le n-1$, the uniform law $\mathcal{U}_{n,s}$ minimizes the mean over iid laws on sets of size at most s if and only if*

$$s \in \{1, n-1\}.$$

In the two optimal cases it is the unique minimizing law.

Proof. For $s = 1$, use Theorem 1.1; for $s = n-1$, use Proposition 6.1. For $2 \le s \le n-2$, Theorem 1.2 gives a strictly better law even within the exact-size class. □

The exact-size endpoint was established by Schilling [6], and Berend and Sher proved the expectation classification for $2 \le s \le n-1$ [1]. Our contribution here is an alternative proof with a strict fixed-draw improvement, together with the explicit extension of the optimal cases to the at-most-size comparison class.

6.3 An explicit example

Example 6.3 (Four coupons drawn in pairs). For $n = 4$, $s = 2$, and $d = 2$, the masses of $\{1, 3\}$ and $\{2, 4\}$ are $(1 + \theta)/6$, those of $\{1, 4\}$ and $\{2, 3\}$ are $(1 - \theta)/6$, and the remaining two masses are $1/6$. Direct substitution into (5.7) gives

$$\mathbb{E}T_{\mu_{2,\theta}} = \frac{19}{5} - \frac{24\theta^2}{5(25 - \theta^2)}.$$

In particular, the uniform mean is $19/5$, the value at $\theta = 1/2$ is $619/165$, and the endpoint perturbation has mean $18/5$. The partition optimum is $2H_2 = 3$. Thus the perturbation is an explicit improvement, not a claim to produce a global optimizer.

7 The complete Hessian signature at the uniform law

7.1 The Johnson decomposition and the paired directions

In this section packages have exactly s, where $1 \le s \le n - 1$, and $N = \binom{n}{s}$. Identify package-mass vectors with $\mathbb{R}^{\binom{[n]}{s}}$ and use the Euclidean inner product

$$\langle f, g\rangle = \sum_{|A|=s} f(A)g(A).$$

Let $r_* = \min(s, n - s)$. Write $\mathcal{W}_d$ for restrictions to the s-slice of multilinear polynomials of degree at most d, and put

$$V_0 = \mathcal{W}_0, \qquad V_d = \mathcal{W}_d \cap \mathcal{W}_{d-1}^{\perp} \quad (1 \le d \le r_*).$$

The standard Johnson-scheme decomposition is

$$\mathbb{R}^{\binom{[n]}{s}} = \bigoplus_{d=0}^{r_*} V_d, \qquad \dim V_d = \binom{n}{d} - \binom{n}{d-1}, \tag{7.1}$$

with $\binom{n}{-1} = 0$. Every permutation-invariant symmetric matrix indexed by s-sets acts as a scalar on each V_d. Equivalently, these are the pairwise nonisomorphic irreducible modules $S^{(n-d,d)}$ in the multiplicity-free permutation representation. We use the formulation and harmonic basis in Filmus [3, Lemma 2, Theorem 15, and Lemma 18]. For $s > n/2$, take complements to transfer the decomposition from the $(n - s)$-slice. The simplex tangent space is $\bigoplus_{d=1}^{r_*} V_d$.

Lemma 7.1. *The paired function χ_d belongs to V_d and*

$$\|\chi_d\|^2 = 2^d L_d, \qquad L_d := \binom{n - 2d}{s - d} > 0. \tag{7.2}$$

Proof. It has degree d. The sign-reversing argument proving (5.2) makes it orthogonal to every monomial of degree less than d, hence to $\mathcal{W}_{d-1}$. It is nonzero exactly on the packages selecting one label of each pair and $s - d$ residual labels. There are $2^d L_d$ such packages, and its square is one on each. □

7.2 Expected completion time

Write $u = \mathcal{U}_{n,s}$ and $\mathcal{E}(\mu) = \mathbb{E}T_\mu$. Near u, every nonempty coupon set has positive hitting probability, so (5.7) defines an analytic function. Its permutation-invariant analytic extension to an ambient neighborhood may be used to compute derivatives. The resulting quadratic form on the simplex tangent space is intrinsic. By symmetry, both $\mathcal{E}$ and every fixed-draw CDF are stationary at u on that tangent space.

Lemma 7.2 (Directional curvature)**.** *For every $1 \le d \le r_*$,*

$$\left.\frac{d^2}{d\theta^2}\mathcal{E}(\mu_{d,\theta})\right|_{\theta=0} = (-1)^{d+1}2^{d+1}\sum_{m=2}^{\infty}\binom{m}{2}\frac{K_{m,1}^{(d)}}{N^m}. \tag{7.3}$$

The series on the right is absolutely convergent and strictly positive before multiplication by its displayed sign and prefactor.

Proof. Differentiate the finite formula (5.7). Only omitted sets selecting one label of each pair contribute. With $M = n - 2d$, $b_j = \binom{M-j+d}{s}$ and $c_j = \binom{M-j}{s-d}$, the result is

$$\frac{(-1)^{d+1}2^{d+1}}{N^2}\sum_{j=0}^{M}(-1)^j\binom{M}{j}\frac{c_j^2}{(1-b_j/N)^3}.$$

Expand $(1-z)^{-3} = \sum_{m=2}^{\infty}\binom{m}{2}z^{m-2}$ and use (5.4). This gives (7.3), without differentiating an infinite series. Absolute convergence also follows from the counting bound

$$K_{m,1}^{(d)} \le L_d^2\binom{n-d}{s}^{m-2}, \qquad \binom{n-d}{s} < N.$$

Strict positivity follows from $K_{m,1}^{(d)} > 0$ for every feasible m. □

Theorem 7.3 (Full expectation-Hessian signature)**.** *The Hessian of $\mathcal{E}$ at u, restricted to the simplex tangent space, acts on V_d as the nonzero scalar*

$$\lambda_d = (-1)^{d+1}\frac{2N^2}{L_d}\sum_{m=2}^{\infty}\binom{m}{2}\frac{K_{m,1}^{(d)}}{N^m}, \qquad 1 \le d \le r_*. \tag{7.4}$$

In particular, $\operatorname{sgn}\lambda_d = (-1)^{d+1}$*. Its negative and positive indices are respectively*

$$\operatorname{ind}_-\, D^2\mathcal{E}(u) = \sum_{\substack{1\le d\le r_* \\ d \text{ even}}}\left(\binom{n}{d} - \binom{n}{d-1}\right), \tag{7.5}$$

$$\operatorname{ind}_+\, D^2\mathcal{E}(u) = \sum_{\substack{1\le d\le r_* \\ d \text{ odd}}}\left(\binom{n}{d} - \binom{n}{d-1}\right). \tag{7.6}$$

There are no null directions. The uniform law is a nondegenerate saddle whenever $2 \le s \le n-2$.

Proof. The Hessian is invariant under simultaneous permutation of the package indices, hence acts as a scalar on each V_d by (7.1). The curve $\mu_{d,\theta} = u + \theta\chi_d/N$ has tangent χ_d/N, so its second derivative is $\lambda_d\|\chi_d\|^2/N^2$. Combine Lemmas 7.1 and 7.2. The multiplicities in (7.1) give the indices. For the intermediate package sizes, both V_1 and V_2 occur. □

An equivalent finite formula, useful for exact computation, is

$$\lambda_d = (-1)^{d+1}\frac{2N^3}{L_d}\sum_{j=0}^{M}(-1)^j\binom{M}{j}\frac{\binom{M-j}{s-d}^2}{\left(N-\binom{M-j+d}{s}\right)^3}, \qquad M = n-2d. \tag{7.7}$$

The series formula, rather than this alternating finite sum, makes the sign manifest. For $(n,s)=(8,4)$ the four eigenvalues, in increasing degree, are approximately

$$(224.190729,\ -11.839806,\ 2.907681,\ -2.088223),$$

with multiplicities $(7, 20, 28, 14)$. These values use the Euclidean package-mass normalization specified above.

Schilling proposed studying the Hessian to rule out a local optimum, and Berend and Sher discuss that approach [6, 1]. Theorem 7.3 implements it for every package size and identifies every tangent-space sign, rather than only exhibiting two opposite-curvature directions.

7.3 Fixed-draw CDF curvature

Theorem 7.4 (Fixed-draw Hessian)**.** *For $m \geq \lceil n/s \rceil$, let $F_m(\mu) = \mathbb{P}(T_\mu \leq m)$. The Hessian of F_m at u acts on V_d as*

$$\gamma_{d,m} = (-1)^d\frac{m(m-1)K_{m,1}^{(d)}}{N^{m-2}L_d}. \tag{7.8}$$

It is positive definite on $V_{\text{even}} = \bigoplus_{2\leq d\leq r_*,\ d \text{ even}} V_d$ and negative definite on $V_{\text{odd}} = \bigoplus_{1\leq d\leq r_*,\ d \text{ odd}} V_d$. Below the feasible draw threshold, F_m is identically zero on the probability simplex.

Proof. The polynomial formula (5.6) is permutation invariant. The Johnson decomposition therefore applies. The coefficient of θ^2 in (5.3) gives

$$\frac{d^2}{d\theta^2}F_m(\mu_{d,\theta})\bigg|_{\theta=0} = (-1)^d 2^d m(m-1)\frac{K_{m,1}^{(d)}}{N^m}.$$

Divide by $\|\chi_d/N\|^2 = 2^d L_d/N^2$ and use the strict coefficient theorem. □

For a fixed feasible m and any nonzero $v \in V_{\text{even}}$,

$$F_m(u+\varepsilon v) - F_m(u) = \tfrac{1}{2}\varepsilon^2\langle v, D^2F_m(u)v\rangle + O(\varepsilon^3) > 0$$

for sufficiently small nonzero ε. The permitted size can depend on m and v. This local statement does not by itself supply one amplitude valid for all draw counts, or finite-amplitude stochastic monotonicity along arbitrary sums of even-degree directions. Corollary 5.2 supplies that stronger conclusion only along the explicit paired rays.

8 Scope and the remaining fixed-draw question

The divisible mean optimum, its uniqueness, its common-multiplicity extension, and the complementary-divisible discrete optimum are established for fixed iid package laws. Adaptation or history-dependent choice of packages is outside this model. Poissonized order also gives the transform and rising-factorial comparisons of Corollary 2.4, not a variance-minimization statement.

Conjecture 8.1 (Fixed-draw partition comparison)**.** *Let $n = qs$ with $q, s \geq 1$ integers. For every iid law on sets of size at most s and every integer $m \geq 0$,*

$$\mathbb{P}(T_\mu \leq m) \leq \mathbb{P}(T_{\mathcal{D}_{n,s}} \leq m) = \frac{q! \left\{ {m \atop q} \right\}}{q^m}, \tag{8.1}$$

where $\left\{ {m \atop q} \right\}$ is a Stirling number of the second kind.

The displayed right side counts surjections from m draws to the q blocks. The case $q = 1$ is immediate. The case $q = 2$ was proved by Fang and Chang [2, Theorem 2] and is recovered in Corollary 3.2. Proposition 3.4 proves the first feasible draw count $m = q$ for all q; earlier counts are zero. The general comparison for $q \geq 3$ and $m > q$ is not proved here. In particular, the earliest-coefficient argument must not be applied to later coefficients of the Poissonized CDF.

For nondivisible n/s, Theorem 2.2 gives a strict $q\mathsf{H}(q)$ bound, but does not by itself identify the optimizer. The additional bound $1 + q$ does identify it when $n - s \mid n$. The remaining nondivisible cases are not classified. Replacing either bound by $\lceil q \rceil H_{\lceil q \rceil}$ is false, as Appendix A shows.

The signed perturbations establish finite-amplitude stochastic monotonicity only for their explicit one-parameter families. The full Hessian theorem gives local curvature in all Johnson components, not arbitrary global radial monotonicity. Neither argument here gives a corresponding signed-perturbation theorem for repeated collections or heterogeneous coupon quotas. Corollary 2.3 resolves only the lower-bound conjecture of [5]; no assertion about that paper's separate upper-bound and hypergraph-inclusion conjectures is needed.

A A minimum-cover comparison that fails

A tempting reduction would replace an arbitrary package family by a minimum-cardinality covering subfamily and then apply the classical coupon bound. The following example shows why that reduction is invalid in the general iid package model.

Let $\mathcal{A}$ be a family of nonempty subsets of $[n]$ whose union is $[n]$, let μ be a probability distribution on $\mathcal{A}$, and choose a minimum-cardinality covering subfamily $\mathcal{B} \subseteq \mathcal{A}$. Consider the proposed comparison

$$\mathbb{E}_\mu T \;\geq\; \mathbb{E}_{\mathrm{unif}(\mathcal{B})} T = |\mathcal{B}| H_{|\mathcal{B}|}. \tag{A.1}$$

The equality on the right holds because every member of a minimum cover contains a coupon not covered by the other members. The inequality on the left need not hold.

Take $n = 3$ and

$$\mathcal{A} = \{\{1,2\}, \{1,3\}, \{2,3\}\}, \qquad \mu(A) = 1/3 \quad (A \in \mathcal{A}).$$

After the first draw, the waiting time for a different package is geometric with success probability $2/3$, so

$$\mathbb{E}_\mu T = 1 + \frac{1}{2/3} = \frac{5}{2}.$$

Every minimum covering subfamily has two pairs. Under its uniform law both pairs must appear, giving mean $2H_2 = 3$. Thus $5/2 < 3$ contradicts (A.1). The same example disproves a universal bound $\lceil n/s \rceil H_{\lceil n/s \rceil}$.

Conditioning on a covering subfamily being the first to complete changes the distribution and does not justify applying an unconditional coupon-collector lower bound. In this example, symmetry makes the conditional completion mean $5/2$ for each of the three possible first-completed two-package covers, not 3.

The valid bounded-package lower bound of Corollary 2.3 still gives $\Omega(n \log n)$ when the package-size bound is fixed.

B Reproducible verification

The companion standard-library script `verify_results.py` compares an independent union-state dynamic program with (5.3). For $\theta = a/b$ it uses integer weights $b + a\chi_d(A)$, avoiding floating-point cancellation.

The supplied run checks $2 \le n \le 9$, all admissible s, d, $0 \le m \le 8$, and $\theta \in \{1/2, -1/2, 1\}$: 1,890 exact CDF comparisons across 210 laws and 939 inclusion-preservation identities. A coefficient grid through $n = 35$ checks 40,835 nonnegativity and strictness assertions. The means $19/5$, $18/5$, $619/165$, and $5/2$ are verified exactly.

The additional standard-library script `verify_strengthenings.py` checks paired-ray monotonicity, the geometric bound against union-state calculations for laws allowing smaller and empty packages, complementary-partition attainment, and the first-feasible-draw inequality. It also independently constructs the expectation and CDF Hessian matrices from inclusion–exclusion and checks their action on the paired vectors against (7.7) and (7.8), using rational arithmetic. The exact run ranges and counts are recorded in `audit/verification_strengthenings.txt`.

The supplementary script `verify_partition_bounds.py` checks the Poissonized bound using exact integers at times $t = nW \log b$, for integer package weights totaling W and $b \in \{2, 3\}$. Raising the inequalities to the integer power s removes fractional exponents. A separate capped-count Markov recursion checks common-quota expectations, including empty and smaller packages and the $q = 1$ boundary case. Logs are in the repository's `audit/` directory.

All universal claims rest on the analytic and combinatorial proofs, not these finite checks. The manuscript has not been formally verified in a proof assistant or independently refereed.

References

[1] D. Berend and T. Sher, *On a conjecture on uniform group drawings in the coupon collector problem*, Stochastic Models (2026), published online May 27, 2026. doi:10.1080/15326349.2026.2671340. Preprint: arXiv:2509.17201; version 2, May 11, 2026.

[2] C. Fang and E.-C. Chang, *Optimal strategy of coupon subset collection when each package contains half of the coupons*, Information Processing Letters **114** (2014), no. 12, 703–705. doi:10.1016/j.ipl.2014.06.012.

[3] Y. Filmus, *An orthogonal basis for functions over a slice of the Boolean hypercube*, The Electronic Journal of Combinatorics **23** (2016), no. 1, paper P1.23, 27 pp. doi:10.37236/4567.

[4] H. Finner, *A generalization of Hölder's inequality and some probability inequalities*, The Annals of Probability **20** (1992), no. 4, 1893–1901. doi:10.1214/aop/1176989534.

[5] Y. Grunbaum and E. Yaakobi, *General coverage models: Structure, monotonicity, and shotgun sequencing*, preprint, arXiv:2510.25305v1 (2025). The references here to Remark 2 and Conjecture 3 use this version.

[6] J. Schilling, *Results and conjectures on the role of the uniform distribution in the coupon collector's problem with group drawings*, Information Processing Letters **169** (2021), article 106112. doi:10.1016/j.ipl.2021.106112.