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Factorial Moments in Two Variables: Exact Leaf Transport and Trace Orthogonality

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Abstract

Let $\mathcal{L}(x^a y^b) = a!b!$. We prove factorial-moment nonvanishing for nonzero polynomials in two variables whose highest homogeneous part is not a scalar multiple of $(x+y)^D$. The proof uses an exact radial Mittag–Leffler transform, polynomial monodromy, and Lindemann–Weierstrass. The transform produces vanishing leaf germs containing a pole density and a Volterra correction. The corrections continue with their full integration histories retained and have no negative Puiseux coefficients at a nonzero terminal critical value. The resulting weighted coefficient annihilates the top rational monodromy component, while the unweighted inverse-derivative vector belongs to that component by a zero-trace identity on every proper composition block. Conjugate pairing excludes all nonzero critical values, and endpoint leaf germs dispose of the remaining pure powers. Combining the nonflat theorem with the explicitly stated single-integral nonvanishing theorem of EIT I gives the two-variable Factorial Conjecture. As a formal consequence, the zero-integral polynomials for the product-exponential measure $e^{-x-y}\,dx\,dy$ form a Mathieu–Zhao subspace; more strongly, a polynomial whose factorial moments vanish for all sufficiently large powers is zero. Thus the two-dimensional product-exponential case of van den Essen's Integral Conjecture follows. The nonflat argument is independent of EIT I.

1 The statement and the inputs

Define the complex-linear functional $\mathcal{L} : \mathbb{C}[x,y] \to \mathbb{C}$ by $\mathcal{L}(x^a y^b) = a!b!$ for $a, b \geq 0$. The Factorial Conjecture was introduced by van den Essen, Wright and Zhao [4]. Liu and Sun proved the homogeneous two-variable case [5]. The present argument treats arbitrary lower homogeneous terms.

For $0 \neq f \in \mathbb{C}[x,y]$ of total degree $D \geq 1$, write

$$P(t,r) = f(rt, r(1-t)) = \sum_{j=0}^{D} A_j(t) r^{D-j}, \qquad \deg A_j \leq D-j, \qquad A = A_0. \tag{1.1}$$

Restriction of degree-D homogeneous forms to $x+y=1$ is injective: off $x+y=0$, homogeneity recovers the form from this restriction. Thus $A(t) = f_D(t, 1-t)$ is nonzero. Its degree d can be smaller than D. We call the cases $d > 0$ and $d = 0$ *nonflat* and *flat*, respectively. In particular,

$$A \text{ is constant} \quad \Longleftrightarrow \quad f_D(x,y) = c(x+y)^D \quad \text{for some } c \neq 0.$$

Theorem 1.1 (Nonflat nonvanishing)**.** *If $0 \neq f \in \mathbb{C}[x,y]$ has nonconstant A, then $\mathcal{L}(f^n) \neq 0$ for some $n \geq 1$, and in fact for infinitely many n.*

Theorem 1.2 (EIT I: the external nonvanishing input)**.** *Let $q \in \overline{\mathbb{Q}}[t]$ be nonconstant, let $a, b \in \overline{\mathbb{Q}}$ be distinct, and let $\xi \in \overline{\mathbb{Q}}^{\times}$. Then*

$$\int_a^b e^{\xi q(t)}\,dt \neq 0.$$

This is the nonvanishing part of the single-integral corollary of EIT I [7, Corollary 1.4, source label `cor:single`]; that result also asserts transcendence. The integral can be taken along any finite path, since its integrand is entire. No extension to arbitrary amplitudes is being used.

Corollary 1.3 (FC(2), using Theorem 1.2)**.** *For every $f \in \mathbb{C}[x,y]$,*

$$\big(\mathcal{L}(f^n) = 0 \text{ for every } n \geq 1\big) \quad \Longrightarrow \quad f = 0.$$

Equivalently, every nonzero f has infinitely many nonzero factorial moments.

The following consequence belongs to the original Image/Mathieu framework in which the Factorial Conjecture was introduced. Recall that a linear subspace M of a commutative algebra R is a Mathieu subspace (now often called a Mathieu–Zhao subspace) if, whenever $f^m \in M$ for every $m \geq 1$, one has $gf^m \in M$ for all sufficiently large m for every $g \in R$. Van den Essen's Integral Conjecture predicts this property for the kernel of integration against a positive measure with finite polynomial moments [1]. Thus the implication below is formal rather than a new connection; the substantive new input is Corollary 1.3.

Corollary 1.4 (Two-dimensional product-exponential Integral Conjecture)**.** *Let*

$$B = (0,\infty)^2, \qquad d\sigma = e^{-x-y}\,dx\,dy,$$

and let

$$M = \left\{ h \in \mathbb{C}[x,y] : \int_B h \; d\sigma = 0 \right\}.$$

Then $M = \ker \mathcal{L}$ is a Mathieu–Zhao subspace of $\mathbb{C}[x,y]$. In fact,

$$\{f \in \mathbb{C}[x,y] : \mathcal{L}(f^m) = 0 \text{ for all sufficiently large } m\} = \{0\}.$$

Consequently, the Integral Conjecture holds for the product-exponential measure $e^{-x-y}\,dx\,dy$ in dimension two.

Proof. For monomials,

$$\int_B x^a y^b e^{-x-y}\,dx\,dy = a!b!,$$

so $M = \ker \mathcal{L}$. Suppose first that $\mathcal{L}(f^m) = 0$ for all $m \geq N$. Then for every $k \geq 1$,

$$\mathcal{L}((f^N)^k) = \mathcal{L}(f^{Nk}) = 0.$$

Corollary 1.3 applied to f^N gives $f^N = 0$, hence $f = 0$. This proves the displayed eventual-power assertion. In particular, if $f^m \in M$ for every $m \geq 1$, then $f = 0$, and therefore $gf^m = 0 \in M$ for every $g \in \mathbb{C}[x,y]$ and every $m \geq 1$. Thus M is a Mathieu–Zhao subspace. Since B is open, σ is positive, and all of its polynomial moments are finite, this is precisely the stated instance of the Integral Conjecture. □

Direct dependencies and scope. The nonflat proof uses Lindemann–Weierstrass and the top-component irreducibility theorem for rational permutation modules of groups containing a full cycle [6, Theorem 3.1 and Section 4.1]. These are published inputs. We state precisely the part of that representation theorem needed below, and prove the endpoint averaging, planar separation, and resulting path-module containment in our notation. We do not reprove the underlying Schur-ring classification. For the unrestricted flat conclusion, the additional external input is Theorem 1.2. Its source is a separate paper published on Hexagon; its proof is not reproduced here. The E-function lifting and Fourier–Laplace theory used there are transitive dependencies of that input. Apart from standard complex analysis, algebraic inverse branches, and the Nullstellensatz and Lüroth theorems, the remaining main-text arguments are included. None of the appendices is a logical input to either main conclusion.

Manuscript status. This is an unrefereed research manuscript, not a Lean formalization. The EIT dependence of Corollary 1.3 is explicit; the nonflat argument is independent of EIT. The exact checks accompanying the source verify particular identities, not the universal theorem.

2 Normalization and the exact kernel

Lemma 2.1 (Algebraic specialization)**.** *A complex counterexample with specified total degree D and specified positive degree d of A would give a counterexample over $\overline{\mathbb{Q}}$ with the same degrees. The same assertion holds with the normalized flat top $A = 1$.*

Proof. Use the finitely many coefficients of a polynomial of degree at most D as variables. All equations $\mathcal{L}(f^n) = 0$ have rational coefficients. Impose the linear conditions specifying $\deg A = d$, and adjoin the inverse of its degree-d coefficient. In the flat case impose $A = 1$, after scaling the proposed counterexample by its nonzero constant leading value. Evaluation at the assumed complex point shows that the ideal over $\overline{\mathbb{Q}}$ is proper. Noetherianity and the Nullstellensatz give an algebraic point annihilating it. Every moment polynomial belongs to the ideal, so this point satisfies the entire family of moment equations. The inverse equation ensures that the degree-d coefficient of A is nonzero; hence $\deg A = d$ and $f_D \neq 0$, preserving total degree D. In the flat case $A = 1$ likewise ensures $f_D \neq 0$. Neither the monodromy group nor the decomposition lattice has to survive specialization, since the contradiction is uniform in both. □

Set

$$M_n = \frac{\mathcal{L}(f^n)}{\Gamma(Dn+2)}, \qquad \widehat{G}(w) = \frac{1}{w}\sum_{n\geq 0} M_n w^{-n}, \tag{2.1}$$

$$E_{D,2}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(Dn+2)}, \qquad \mathcal{C}(w,t) = \frac{1}{w}\int_0^\infty e^{-r} r E_{D,2}\Big(\frac{P(t,r)}{w}\Big)\, dr. \tag{2.2}$$

Termwise integration of a polynomial gives

$$\mathcal{L}(f^n) = \int_0^\infty\int_0^\infty e^{-x-y} f(x,y)^n\, dx\, dy = \int_0^1\int_0^\infty e^{-r} r P(t,r)^n\, dr\, dt.$$

The last equality uses the Jacobian r of $x = rt$, $y = r(1-t)$. For sufficiently large $|w|$ the resulting generating integral is normally convergent, and

$$\widehat{G}(w) = \int_0^1 \mathcal{C}(w,t)\, dt. \tag{2.3}$$

Indeed, for $x = |z|^{1/D}$ the defining series gives the global bound

$$|E_{D,2}(z)| \leq \sum_{n\geq 0} \frac{x^{Dn}}{(Dn+1)!} \leq \sum_{m\geq 0} \frac{x^m}{m!} = e^x. \tag{2.4}$$

Since $|P(t,r)|^{1/D} \leq C(1+r)$ on a compact t-set, this estimate also justifies summation under the integral for sufficiently large $|w|$. All positive moments vanish exactly when $\widehat{G}(w) = 1/w$ near infinity.

Put $V = \{t : A(t) \neq 0\}$. We need a coefficient on a finite cover, not a single-valued logarithmic coefficient on the entire base.

Lemma 2.2 (Exact local kernel and a continued coefficient)**.** *Near $w = A(t) \neq 0$, a continued germ of the kernel has the form*

$$\mathcal{C}(w,t) = \frac{E(t)}{w - A(t)} + B_0(w,t)\log(w - A(t)) + H(w,t), \qquad E(t) = \exp\Big(\frac{A_1(t)}{DA(t)}\Big), \tag{2.5}$$

where B_0, H are locally jointly holomorphic. The residue is unchanged by continuation. For fixed $t \in V$, the only possible finite exceptional values for continuation in w are $0, A(t)$, locally uniformly in t.

The distinguished diagonal germ of B_0 extends to a holomorphic function $\mathscr{B}$ on

$$\mathscr{X} = \{(w,t,\rho) : w \neq 0,\ t \in V,\ \rho^D = A(t)/w\}. \tag{2.6}$$

It is given by

$$\mathscr{B}(w,t,\rho) = \frac{\rho^{D-1}}{DA(t)} \sum_{m\geq 1} \frac{(-1)^m F_m(t)}{(m-1)!}(1-\rho)^{m-1}, \tag{2.7}$$

where F_m are defined in (2.9)*. The series is entire in $1-\rho$, locally uniformly on V. The initial diagonal germ is the restriction with ρ near 1.*

Proof. Let $\omega = e^{2\pi i/D}$. Expansion of the exponentials gives the exact root filter

$$E_{D,2}(X^D) = \frac{1}{DX}\sum_{k=0}^{D-1} \omega^{-k} e^{\omega^k X} - C_{D,1}(X), \quad C_{D,1} = \begin{cases} X^{-1}, & D = 1, \\ 0, & D \geq 2. \end{cases} \tag{2.8}$$

Fix a compact set $K \subset V$. Split the radial integral at a sufficiently large R, chosen uniformly for $t \in K$, and apply the filter only to the exterior part. Leave the compact part in its entire Mittag–Leffler form. Initially impose $|w| > |A(t)|$, so $|\rho| = |A(t)/w|^{1/D} < 1$ and all D exponential summands converge. Each summand is then continued separately from this common domain.

On V, the exterior root with leading term r and its inverse are

$$\Phi_t(r) = (P(t,r)/A(t))^{1/D} = r + p_1(t) + O(r^{-1}), \quad \chi(t,Y) = Y - p_1(t) + O(Y^{-1}), \quad p_1 = \frac{A_1}{DA}.$$

The analytic inverse theorem at infinity gives convergent Laurent series on common exterior disks for $t \in K$; one first works on a neighborhood of K and then uses compactness. The chosen leading terms specify the roots and inverses uniquely, so their coefficients are single-valued and holomorphic on V. Define

$$F(t,Y) = e^{Y-\chi(t,Y)}\frac{\chi(t,Y)\chi_Y(t,Y)}{Y} = \sum_{m\geq 0} F_m(t)Y^{-m}. \tag{2.9}$$

Here $F_0 = E$. By a uniform Cauchy estimate at infinity, there are $C_K, R_0 > 0$ such that $|F_m(t)| \leq C_K R_0^m$ for every $t \in K$ and $m \geq 0$. Enlarge R so that the image paths lie in this common exterior domain. With $\rho = (A(t)/w)^{1/D}$ near 1, the summand singular at this diagonal is

$$\frac{\rho^{D-1}}{DA(t)}\int_{\Phi_t([R,\infty))} e^{-(1-\rho)Y} F(t,Y)\,dY. \tag{2.10}$$

In $\Re(1-\rho) > 0$ the exterior image ray can be moved to a common positive ray. The far connector tends to zero exponentially, and the finite connector is entire in $1-\rho$, locally uniformly in t.

Choose a starting point $R_1 > R_0$ on that ray and put

$$J_m(\eta) = \int_{R_1}^{\infty} e^{-\eta Y} Y^{-m}\,dY.$$

Direct integration and integration by parts give

$$J_0 = e^{-\eta R_1}/\eta,$$
$$J_1 = -\log\eta - \gamma - \log R_1 - \sum_{j\ge 1} \frac{(-\eta R_1)^j}{j\, j!},$$
$$J_{m+1} = \frac{e^{-\eta R_1} R_1^{-m} - \eta J_m}{m} \qquad (m \ge 1).$$

The logarithmic coefficient of J_m is $(-1)^m \eta^{m-1}/(m-1)!$. For $m \ge 1$ put

$$H_m(\eta) = J_m(\eta) - \frac{(-1)^m \eta^{m-1}}{(m-1)!} \log\eta.$$

The displayed formulas show that H_m is entire and obeys the same recurrence as J_m. On $|\eta| \le L$, set $B_m = R_1^{m-1} \sup |H_m|$. Then

$$B_{m+1} \le \frac{e^{LR_1}}{m} + \frac{LR_1}{m} B_m.$$

For $m \ge 2LR_1$ the last coefficient is at most $1/2$; enlarging a constant to cover the finitely many earlier terms proves $\sup_{|\eta|\le L} |H_m| \le C_{L,R_1} R_1^{1-m}$. Together with $R_1 > R_0$ and the bound on F_m, this proves normal convergence of the entire remainders. The logarithmic coefficients converge normally by their factorial denominators. Thus the tail is

$$\frac{E(t)}{\eta} + \left(\sum_{m\ge 1} \frac{(-1)^m F_m(t)}{(m-1)!} \eta^{m-1} \right) \log\eta + \text{a function holomorphic near } \eta = 0.$$

This supplies exact continuation on the universal cover of $\eta \ne 0$. Since

$$1 - \rho = \frac{w - A(t)}{DA(t)} + O((w - A(t))^2),$$

(2.10) has residue $E(t)$ and logarithmic coefficient (2.7). Changing $\log(1-\rho)$ to $\log(w - A(t))$ changes only a holomorphic remainder.

The entireness used below also has a contour formulation. For any $R_2 > R_0$, orient $|Y| = R_2$ counterclockwise. Termwise integration gives

$$\Lambda(t,\eta) := \sum_{m\ge 1} \frac{(-1)^m F_m(t)}{(m-1)!} \eta^{m-1} = -\frac{1}{2\pi i} \int_{|Y|=R_2} e^{-\eta Y} F(t,Y)\, dY.$$

Thus $|\Lambda(t,\eta)| \le C_{K,R_2} e^{R_2|\eta|}$, uniformly on K, and $\mathscr{B} = \rho^{D-1}\Lambda(t, 1-\rho)/(DA(t))$ is holomorphic on the stated cover. This contour formula is for Λ; the prefactor must be retained when recovering $\mathscr{B}$.

The other root summands have parameters $1 - \omega^k \rho$; their zeros all project to $w = A(t)$. At $\rho = 1$ they are locally holomorphic for $k \ne 0$. The compact part and the $D = 1$ correction are holomorphic at every nonzero diagonal. This proves the claimed exceptional set and uniformity. A change of logarithm branch adds a function holomorphic on the cover, and cannot alter the pole residue. □

Remark 2.3 (Canonical local coefficient versus continued germ). The local logarithmic coefficient of the complete kernel is canonical at its nonzero diagonal. The continuation of its distinguished germ, however, is $\mathscr{B}(w,t,\rho)$ on $\mathscr{X}$ and need not descend to the base (w,t)-space. If another root summand becomes singular, the identity $\omega^{-k}\rho^{D-1} = (\omega^k\rho)^{D-1}$ gives the same canonical local coefficient after reselecting that summand. It does not identify different continued root sheets. Every integral below retains its root parameter. Appendix A gives an exact sign-changing example.

Lemma 2.4 (Normality near zeros of the top)**.** *Near a zero of A, the original kernel integral is normally convergent for w on any fixed compact set away from zero. On every bounded t-disk it is holomorphic in t for all sufficiently large $|w|$.*

Proof. For $D > 1$, locally uniformly,

$$|P(t,r)/w|^{1/D} \le |A(t)/w|^{1/D} r + C(1+r)^{1-1/D}.$$

For $D = 1$ the second term is bounded. Near a zero of A, or for $|w|$ sufficiently large on a bounded t-disk, make the first coefficient less than $1/2$. By (2.4), on compact w-sets away from zero the kernel integrand is bounded by

$$C_1 r \exp(-r/2 + C_2(1+r)^{1-1/D}),$$

with the last contribution bounded when $D = 1$. This is integrable and gives joint holomorphy locally in (w,t), including at zeros of A. To compare with a continued germ, first fix the entire finite continuation path in w away from zero. Its modulus has a positive minimum, so near a zero of A one can shrink the t-neighborhood until $|A(t)/w|^{1/D} < 1/2$ along the whole path. The single-valued integral agrees with the germ near infinity, hence with its continuation along this path by uniqueness. No uniform claim over an unbounded t-set is needed. □

3 Leaf identities and continuation with retained paths

Choose a regular value b different from $0, A(0), A(1)$. Let T be an embedded finite star centered at b, with one arm to each distinct member of

$$\{0, A(0), A(1)\} \cup \mathrm{CritVal}(A).$$

The arms have disjoint interiors and are straight near their leaves. The inverse graph $\Gamma = A^{-1}(T)$ is a tree. Here is a useful direct justification. If T has s arms, then Γ has ds edges, d center vertices, and $ds - (d-1)$ leaf vertices: the total finite ramification of a degree-d polynomial is $d-1$. It is connected because the local monodromies about the leaf values generate a transitive covering group; their cycles join precisely the incident lifted stars. Thus its Euler characteristic is 1, and it is a tree. This is the usual polynomial inverse-tree construction [6, Section 2.1].

Let μ be its unique simple path from the actual point 0 to the actual point 1. Both μ and $[0,1]$ lie in a bounded disk. Lemma 2.4 therefore permits replacing the interval by μ in (2.3) for large w. Append a simple arc from b to infinity, disjoint from T except at b and straight sufficiently near infinity. Label the inverse branches over b so that the geometric monodromy obtained by following this arc, making one counterclockwise turn on a large circle, and returning is $g_\infty = (1\ 2\ \cdots\ d)$. We retain this geometric labeling throughout; an arbitrary generator of the same cyclic subgroup need not preserve the circular order used below. Orient each target arm from its leaf v toward b. The path gives a vector $n_v \in \{0,1,-1\}^d$, recording the signed use of the lift in each sheet. With these orientations,

$$\widehat{G}(w) = \sum_v \int_v^b \sum_{i=1}^d n_{v,i} \mathcal{C}(w, t_i(u)) t_i'(u)\, du. \tag{3.1}$$

The complement $\widehat{\mathbb{C}} \setminus T$ in the Riemann sphere is simply connected. For $t \in \mu$, the possible kernel singularities $0, A(t)$ lie in T, and the kernel is holomorphic at infinity. Lemmas 2.2 and 2.4 therefore determine its single-valued continuation off T, locally uniformly along the compact path μ. Consequently (3.1) holds there by continuation from infinity. It retains every lift over each leaf.

Fix $v \neq 0$ and write $c_i = \lim_{u \to v} t_i(u)$ on its arm. For w near v, define the germ

$$\mathscr{L}_v(w) = \sum_i n_{v,i} \left\{ E(t_i(w)) t_i'(w) + \int_{c_i}^{t_i(w)} \mathscr{B}(w, t, \rho)\, dt \right\}. \tag{3.2}$$

Initially the integrals are local near c_i. Their lifts to $\mathscr{X}_w$ are selected by $\rho = 1$ at the upper endpoint. Since $A(c_i) = v \neq 0$, regular and ramified endpoints are both allowed.

Lemma 3.1 (Complete local singular extraction)**.** *If $\widehat{G}$ is rational, then $\mathscr{L}_v \equiv 0$ for every nonzero leaf v.*

Proof. Only meromorphy of the chosen continued germ of $\widehat{G}$ at v is used in this local argument. Put $z = w - v$ and $x = u - v$. The germ arriving from the complement of T uses the diagonal root $\rho \to 1$; its local logarithm may differ from another choice only by a holomorphic term. Near the leaf, set

$$\begin{aligned} a(x) &= \sum_i n_{v,i} E(t_i(v + x)) t_i'(v + x), \\ b_0(z, x) &= \sum_i n_{v,i} \mathscr{B}(v + z, t_i(v + x), \rho) t_i'(v + x), \qquad \rho^D = (v + x)/(v + z), \quad \rho \sim 1. \end{aligned}$$

These have convergent Puiseux expansions in x with a common finite denominator; b_0 is also holomorphic in z. Every x-exponent is greater than -1, since a local inverse of degree k has derivative $O(x^{1/k-1})$.

All other edges in (3.1) are holomorphic in w near v on the chosen continuation. This follows from Lemma 2.2 where $A(t) \neq 0$, and from Lemma 2.4 near zero top values. The integrated local holomorphic remainder is holomorphic as well. After straightening the short arm, its relevant part is

$$\int_0^\epsilon \left(\frac{a(x)}{z - x} + b_0(z, x) \log(z - x) \right) dx.$$

A common nonzero linear rescaling of x, z does not affect the following vanishing argument. Take z in a disk of radius less than ϵ, slit along its positive radius. Then $z - x \neq 0$ for $0 \leq x \leq \epsilon$, and $\log(z - x)$ has a continuous choice on this product, holomorphic in z; at $x = \epsilon$ it is holomorphic on the full smaller disk. Put $q(z, x) = \int_0^x b_0(z, s)\, ds$. This tends to zero at the lower endpoint. Integration by parts gives a holomorphic boundary term plus

$$\int_0^\epsilon \frac{a(x) + q(z, x)}{z - x}\, dx. \tag{3.3}$$

In particular the sign of the correction is positive.

There are finitely many fractional classes. Choose their representatives $-1 < \delta \leq 0$ and write

$$a(x) + q(z, x) = \sum_\delta x^\delta h_\delta(z, x),$$

where each h_δ is holomorphic in the ordinary variables z, x. Divide $h_\delta(z, x) = h_\delta(z, z) + (x - z) k_\delta(z, x)$. The k_δ-term in (3.3) is holomorphic in z, by integrability of x^δ. The only nonholomorphic term from this class is therefore

$$h_\delta(z, z) I_\delta(z), \qquad I_\delta(z) = \int_0^\epsilon \frac{x^\delta}{z - x}\, dx.$$

For $-1 < \delta < 0$, the singular part of I_δ is a nonzero multiple of z^δ on a fixed slit; scaling and the beta integral give a factor proportional to $\pi / \sin(\pi\delta)$. For $\delta = 0$ it is $\log z$, up to a holomorphic function. Let T_0 denote continuation once around $z = 0$, and choose N divisible

by all Puiseux denominators. Applying $T_0^N - 1$ to a meromorphic germ kills the fractional terms and isolates $2\pi i N h_0(z,z)$, so $h_0(z,z) = 0$. The remaining terms $z^\delta h_\delta(z,z)$ have distinct monodromy eigenvalues $e^{2\pi i\delta} \neq 1$. The corresponding polynomial projectors in T_0 isolate each term from the others and from the meromorphic part. Since the constants multiplying z^δ are nonzero, every $h_\delta(z,z)$ vanishes. Hence

$$a(z) + \int_0^z b_0(z,x)\,dx = 0.$$

The change of variables $t = t_i(v+x)$ identifies this with (3.2). This proves vanishing of the whole corrected germ, not only of its leading coefficient. □

Remark 3.2 (The triangular coefficient equations). If $a(x) = \sum a_\alpha x^\alpha$ and $b_0(z,x) = \sum b_{p,\beta} z^p x^\beta$, the preceding identity is equivalent to

$$a_\alpha + \sum_{p+\beta+1=\alpha} \frac{b_{p,\beta}}{\beta+1} = 0 \qquad (\alpha > -1). \tag{3.4}$$

It is not an identity for a alone. For example, $-\sqrt{x}/(z-x) + (2\sqrt{x})^{-1}\log(z-x)$ is $\partial_x(\sqrt{x}\log(z-x))$, and its integral is holomorphic near $z = 0$, although both densities are nonzero.

Write G_A for the inverse-branch monodromy group. Fix a terminal path from b toward a nonzero critical value λ, with a final straight segment, avoiding zero and all critical values before its endpoint. For any negative Puiseux exponent α, define the linear functional

$$\Phi_{\lambda,\alpha}(e_i) = [(w-\lambda)^\alpha]\frac{\exp(A_1(t_i(w))/(Dw))}{A'(t_i(w))} \tag{3.5}$$

using continuation along that path and a common Puiseux parameter.

Theorem 3.3 (Transport of the full leaf identity). *If $\widehat{G}$ is rational, $\Phi_{\lambda,\alpha}$ annihilates $\mathbb{Q}[G_A]n_v$ for every nonzero leaf v and every $\alpha < 0$.*

Proof. Let $U = \mathbb{C} \setminus (\{0\} \cup \mathrm{CritVal}(A))$. Continue the zero germ $\mathscr{L}_v$ along its arm to b, and then along any path in U. The initial local integration paths lie in V, and every moving inverse branch stays there because $A(t_i(w)) = w \neq 0$. For an individual integral in (3.2), retain its path from the fixed c_i and append the path of its moving upper endpoint. Lift the accumulated path to $\mathscr{X}_w$ with $\rho = 1$ at that upper endpoint. The fixed lower root is allowed to change after a loop; it remains part of the data.

Here is a precise local description. On a small w-disk W away from zero choose a holomorphic root $s(w)^D = w$, and put

$$\Sigma = \{(t,\sigma) : t \in V,\ \sigma^D = A(t)\}, \qquad \rho = \sigma/s(w).$$

Then $\mathscr{X}|_W$ is identified with $W \times \Sigma$. The curve Σ is locally parametrized by t, since $\sigma \neq 0$, even at a critical point of A. The differential $\mathscr{B}(w,t,\sigma/s(w))\,dt$ is holomorphic on this product. A fixed compact lifted path integrates to a holomorphic function of w; a short appended piece has a jointly holomorphic local primitive. This constructs the continuation along each compact portion of the chosen path. At the moving endpoint $\sigma = s(w)$, so $\rho = 1$ is preserved. Different roots at a fixed lower endpoint are never reset.

Now approach $\lambda \neq 0$, and suppose an endpoint tends to c with $A(c) = \lambda$. Split the accumulated history into a fixed compact lifted path ending near c, and a short terminal piece. The first part is holomorphic on a disk about λ by the product description. A point with $A(t) = w$ on that history causes no singularity of $\mathscr{B}$, because its coefficient series is entire in $1-\rho$. No extension across $A(t) = 0$ is asserted or used. Near (λ, c) the root with $\rho \sim 1$ is

holomorphic. A primitive $J(w,t)$ of the corresponding differential is jointly holomorphic, and the terminal piece is

$$J(w,t_i(w)) - J(w,t_*).$$

If A has local degree k at c, then $t_i(w) = c + O((w-\lambda)^{1/k})$. Thus this primitive has only nonnegative Puiseux exponents. The same holds for every retained history. Every negative coefficient of the continued identity comes solely from $E(t_i(w))t_i'(w)$.

For each $g \in G_A$, choose a loop at b realizing g, and then follow the fixed terminal path. The continued $\mathscr{L}_v$ is still zero, so its negative coefficient is $\Phi_{\lambda,\alpha}(gn_v) = 0$. Linear extension proves the theorem. Removing the extra regular point zero from the base, when zero is not critical, does not remove any permutation from its monodromy image. The complete histories can depend on more than g; no action of G_A on those histories is asserted. Only the negative Puiseux coefficients factor through the permutation action: the history terms contribute none, while the pole contribution depends only on g. □

4 The path module and the simultaneous trace identity

We use the bilinear pairing $\langle x, y\rangle = \sum_i x_i y_i$. Vector spaces are over $\mathbb{Q}$ until explicitly complexified. For the actual signed vectors in (3.1), put

$$M = \sum_v \mathbb{Q}[G_A]n_v, \qquad M_{\neq 0} = \sum_{v\neq 0} \mathbb{Q}[G_A]n_v.$$

Lemma 4.1 (The zero leaf is algebraically redundant)**.** *One has $\sum_v n_v = 0$ coordinatewise, and $M_{\neq 0} = M$.*

Proof. Each visited center vertex of μ has one entering and one leaving edge, contributing opposite signs to its sheet coordinate. Neither actual endpoint lies over b. Thus $n_0 = -\sum_{v\neq 0} n_v$; apply all monodromy elements. This makes no analytic assertion at $w = 0$. □

Use the geometric labeling of Section 3, with $g_\infty = (1\ 2\ \cdots\ d)$, and identify sheet i with $i-1$ in $\mathbb{Z}/d\mathbb{Z}$. For $k \mid d$, let $V_k \subset \mathbb{Q}^d$ consist of k-periodic vectors, and set

$$D(G_A) = \{k \mid d : V_k \text{ is } G_A\text{-invariant}\}, \qquad U_d = \bigcap_{\substack{k\in D(G_A)\\ k<d}} V_k^{\perp}, \qquad U_d^{\mathbb{C}} = U_d \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.1}$$

The indicators of residue classes modulo k span V_k. Because the cyclic subgroup is regular, every block system is of this form: the block containing the label 0 in $\mathbb{Z}/d\mathbb{Z}$ is a subgroup, and its translates are all blocks. Invariance of its indicator span is equivalent to invariance of the partition. In polynomial monodromy these blocks correspond to decompositions $A = P \circ R$. The intermediate-field correspondence and Lüroth's theorem give a rational generator; the unique point over infinity permits choosing that generator R with its only pole at infinity. Then R, and also P, are polynomials. For V_k one has $\deg P = k$ and $\deg R = d/k$.

Theorem 4.2 (Rational cyclic-module input, Pakovich–Muzychuk)**.** *For a permutation group $G \leq S_d$ containing a full d-cycle, the space U_d defined as in* (4.1) *is an irreducible rational G-module.*

This is the top-component case of [6, Theorem 3.1 and Section 4.1]. All numbering in references to [6] refers to arXiv:0710.4085v2. Theorem 3.1 there uses the maximal proper elements below d in $D(G)$. Intersecting over these or over all proper elements gives the same space: every proper k divides some maximal proper f, and $V_k \subseteq V_f$. Section 4.1 explicitly identifies the latter intersection with the irreducible top rational component. Its proof uses rational Schur rings over a cyclic group. That representation-theoretic theorem, not the whole polynomial moment problem, is retained as an external input. The next lemmas establish its application to the actual path, following [6, Section 2.2, Propositions 2.4, 2.6 and 4.1].

Lemma 4.3 (A rational invariant subspace dichotomy)**.** *If $N \subset \mathbb{Q}^d$ is G_A-invariant, then either $U_d \subset N$ or $N \perp U_d$.*

Proof. The orthogonal projectors P_N and P_U have rational entries and commute with G_A. They therefore commute with the full cycle. Its centralizer consists of circulant matrices and is commutative, so $P_N P_U = P_U P_N$. The restriction of P_N to U_d is a G_A-equivariant idempotent. By Theorem 4.2 it is either zero or the identity. These are the two assertions. No absolute irreducibility is used. □

For an actual endpoint $p \in \{0, 1\}$, let I_p be the set of sheet labels whose inverse branches on its arm tend to p, and let $e_p = |I_p|$ be its local degree. Write

$$m_p = \frac{1}{e_p} \sum_{i \in I_p} e_i \in \mathbb{Q}^d.$$

Lemma 4.4 (Endpoint averaging)**.** *If $A(0) = A(1)$, then $m_0 - m_1 \in M$. If $A(0) \neq A(1)$, then $m_0, m_1 \in M$.*

Proof. Average the vector n_v over the finite cyclic group generated by the local monodromy at its leaf. On the labels tending to one point above v, the averaging replaces every coordinate by the mean over that local cycle. Equivalently, the average of each basis vector is the uniform vector on its local-monodromy orbit, even when the monodromy has several cycles of different lengths. At an interior visit of μ, one edge enters and another leaves the point; their opposite signs therefore cancel after averaging. At an actual endpoint the sum is $+1$ for 0, and -1 for 1, under our orientations. Thus if the endpoint values agree, the average is $m_0 - m_1$. Otherwise the two separate averages are m_0 and $-m_1$. All averages belong to M. □

Lemma 4.5 (Planar separation of endpoint labels)**.** *Let $\zeta = e^{2\pi i/d}$, and identify a label i with ζ^{i-1}. If $A(0) = A(1)$, the two endpoint label sets lie in disjoint open arcs of the circle. If their values differ, the sets have this property or share one point and, after that point is removed, lie on opposite open arcs cut out by it and one further circle point. In particular, their convex hulls are disjoint in the first case and meet at most in that single circle point in the second.*

Proof. Use the appended arc already chosen in Section 3. Its lifts are disjoint rays from the d center vertices to infinity. Because the arc is straight near infinity, its lifted ends have asymptotic directions in the geometric cyclic order and each crosses a sufficiently large circle just once. Truncate there. The resulting boundary order is the order of g_∞, up to reversal, which does not affect separation. For an endpoint p, take the union K_p of its incident arm lifts and the attached rays. It is an embedded star joining p to exactly the boundary labels I_p; intermediate center vertices have degree two.

If the endpoints have the same value, K_0 and K_1 are disjoint. Between two consecutive rays of K_0, the two rays and the intervening boundary arc enclose a sector by the Jordan curve theorem. The connected star K_1, disjoint from K_0, lies in a single such sector. Its boundary labels consequently occur in one gap between consecutive labels of I_0. Choose one chord endpoint between the first label of I_1 in this gap and its preceding I_0 label, and the other between the last I_1 label and its following I_0 label. This chord strictly separates the two finite sets. If one set is a singleton, a sufficiently small arc about that label gives the same conclusion.

For different endpoint values, two stars can share a center only if one lifted star contains both endpoints. There cannot be two such centers: the two paths from 0 through these centers to 1 would form a cycle in $A^{-1}(T)$. If there is no common center, the preceding argument applies. Otherwise $K_0 \cap K_1$ is the one common ray from that center to the boundary. After this common ray, including its center, is removed, the rest of K_1 is connected, disjoint from K_0, and has that center in its closure. It lies in one sector of K_0 adjacent to the common

ray. If both label sets have other labels, traverse this sector from the common label through the labels of I_1. Join the common label to a point in the gap between the last such I_1 label and the next I_0 label. The remaining labels are strictly on opposite sides of this chord; each convex hull touches it only at the common label. If either set has no other labels, its hull is that single point and the separation assertion is immediate. Thus the two hulls meet at most at the common label in all cases. □

Proposition 4.6 (The actual path contains the top component)**.** *For $d \geq 2$, $U_d \subset M$.*

Proof. Put $z = (1, \zeta, \ldots, \zeta^{d-1})$. The sum of its entries on every residue class modulo a proper divisor k is zero. Thus $z \in U_d^{\mathbb{C}}$. The number $\langle m_p, z \rangle$ is the arithmetic mean of the roots of unity labeled by I_p. If the endpoint values agree, Lemma 4.5 gives disjoint convex hulls, so these two means differ. Lemma 4.4 therefore supplies a vector in M pairing nontrivially with z. If the values differ, both means cannot be zero: their convex hulls are disjoint or meet only on the unit circle. At least one of the two vectors in Lemma 4.4 again pairs nontrivially with z. Orthogonality of the rational spaces would imply orthogonality of their complexifications, so the complex witness z suffices. Hence M is not orthogonal to U_d, and Lemma 4.3 gives $U_d \subset M$. □

This is the needed path-module statement of [6, Proposition 4.1], with its endpoint and sign conventions established here. It allows ramified endpoints and equal or zero endpoint values. A regular extra leaf at zero contributes no active lift unless it is an endpoint value.

Corollary 4.7. *If $\widehat{G}$ is rational, every functional* (3.5) *with $\lambda \neq 0$ and $\alpha < 0$ annihilates $U_d^{\mathbb{C}}$.*

Proof. Theorem 3.3 gives annihilation of $M_{\neq 0}$. Lemma 4.1 and Proposition 4.6 give $U_d \subset M = M_{\neq 0}$. Extend scalars. □

Lemma 4.8 (Simultaneous trace orthogonality)**.** *For every regular w, in the continued sheet ordering,*

$$u(w) = \left(\frac{1}{A'(t_1(w))}, \ldots, \frac{1}{A'(t_d(w))} \right) \in U_d^{\mathbb{C}}. \tag{4.2}$$

Every Puiseux coefficient vector of u also belongs to $U_d^{\mathbb{C}}$.

Proof. Take any proper $k \in D(G_A)$ and the corresponding decomposition $A = P \circ R$, with $r = \deg R = d/k \geq 2$. A block over w is a fiber $R^{-1}(\xi)$ with $P(\xi) = w$. Regularity gives $P'(\xi) \neq 0$ and r distinct points in that fiber. The block trace is

$$\sum_{i \text{ in block}} \frac{1}{A'(t_i(w))} = \frac{1}{P'(\xi)} \sum_{R(c)=\xi} \frac{1}{R'(c)} = 0. \tag{4.3}$$

Indeed, in the partial fractions

$$\frac{1}{R(t) - \xi} = \sum_{R(c)=\xi} \frac{1}{R'(c)(t - c)},$$

the coefficient of t^{-1} is zero since the left side is $O(t^{-r})$. Block indicators span V_k, so $u \perp V_k$. Intersect over all proper k. The relations are constant rational equations, and consequently hold coefficientwise in every continued Puiseux expansion. This uses the rational block-indicator equations, not any reality assumption on the coefficients of A. □

The intersection in this argument treats all proper decompositions at once. It imposes no constancy condition on a weighted vector along any one chosen block system.

5 The arithmetic obstruction and the nonflat conclusion

The next proposition applies to the top two layers of any algebraically specialized polynomial; it does not impose genericity on their coefficients.

Proposition 5.1 (Nonzero critical values are detected)**.** *Suppose $A, A_1 \in \overline{\mathbb{Q}}[t]$, $\deg A = d \geq 2$, and all negative coefficient functionals in* (3.5) *at nonzero critical values annihilate U_d. Then A has no nonzero critical value.*

Proof. Suppose $\lambda \neq 0$ is a critical value. Let $K \geq 2$ be the largest local degree of A among the points above λ, and put $\alpha = 1/K - 1 < 0$. At a point c of local degree K, write

$$A(c+s) = \lambda + a_c s^K + O(s^{K+1}), \qquad a_c \neq 0.$$

The K inverse branches there have convergent expansions

$$t_i(w) = c + b_c \zeta_i (w-\lambda)^{1/K} + O((w-\lambda)^{2/K}), \qquad b_c^K = a_c^{-1}, \quad \zeta_i^K = 1.$$

Let β be the coefficient vector at exponent α in $u(w)$ from (4.2). On these branches, $\beta_i = b_c \zeta_i / K$. At a point of smaller local degree $k < K$, the first exponent $1/k - 1$ is strictly larger than α, so $\beta_i = 0$. In particular, $\beta \neq 0$, its entries are algebraic, and Lemma 4.8 gives $\beta \in U_d^{\mathbb{C}}$.

Since $A(t_i(w)) = w$ and $\lambda \neq 0$, the exponential weight is holomorphic in the local inverse coordinate. Its constant term is $\exp(A_1(c_i)/(D\lambda))$, and all other terms have positive Puiseux exponent. Thus the coefficient at α in the weighted inverse-derivative vector is exactly

$$\phi_i = \begin{cases} \beta_i \exp(A_1(c_i)/(D\lambda)), & \beta_i \neq 0, \\ 0, & \beta_i = 0. \end{cases} \tag{5.1}$$

The maximal choice of K is what excludes contributions from smaller ramification indices or nonconstant terms of the weight.

By hypothesis ϕ is bilinearly orthogonal to $U_d^{\mathbb{C}}$. Since U_d is defined by rational linear equations, its complexification is preserved by coordinatewise conjugation; A itself need not have real coefficients. Consequently $\overline{\beta} \in U_d^{\mathbb{C}}$, and

$$0 = \langle \phi, \overline{\beta} \rangle = \sum_{\beta_i \neq 0} |\beta_i|^2 \exp\!\left(\frac{A_1(c_i)}{D\lambda}\right). \tag{5.2}$$

All exponents and coefficients in this expression are algebraic. After equal exponents are grouped, each coefficient is a strictly positive real algebraic number. At least one group occurs. Lindemann–Weierstrass [2] makes the exponentials of distinct algebraic numbers linearly independent over $\overline{\mathbb{Q}}$, contradicting (5.2). □

Remark 5.2. Grouping by critical points instead gives

$$\sum_{\substack{A(c)=\lambda \\ e_c = K}} \frac{(a_c \overline{a_c})^{-1/K}}{K} \exp\!\left(\frac{A_1(c)}{D\lambda}\right) = 0,$$

where the positive real root is intended. This records explicitly that local root choices do not affect the positive coefficients. The positivity is created by the conjugate pairing, not by the original oriented interval or by the complex exponentials.

Lemma 5.3 (Only zero critical values)**.** *If all critical values of a polynomial A of degree $d \geq 2$ are zero, then $A(t) = a(t-c)^d$ for some $a \neq 0$.*

Proof. Every root of A' is a root of A. If the distinct roots of A have multiplicities $m_1, \dots, m_q$, their multiplicities in A' are exactly $m_j - 1$. They therefore account for $d - q$ roots of A'. By hypothesis A' has no zeros outside the zero set of A, so these account for all $d-1$ zeros counted with multiplicity. Hence $q = 1$. □

Lemma 5.4 (The endpoints exclude pure powers)**.** *Let $f \in \overline{\mathbb{Q}}[x, y]$ have $A(t) = a(t - c)^d$, $a \neq 0$, $d \geq 1$. Then $\widehat{G}$ is not rational.*

Proof. Every nonzero value of A is regular. At an actual endpoint p with $A(p) = v \neq 0$, the corresponding summand of the corrected leaf germ extends holomorphically to $w = v$. Its Volterra integral vanishes there, because the integrand is holomorphic and the integration path shrinks to p. Its value is therefore, with the path orientation, $E(p)/A'(p)$. Unramified preimages of a leaf have graph degree one, so only actual endpoints can be active above this regular value. By Lemma 3.1, rationality would force the sum of these signed endpoint values to be zero.

If $A(0) \neq A(1)$, one endpoint has a nonzero value different from the other endpoint value. Its nonzero leaf-germ value cannot cancel. If the endpoint values agree, their common value v cannot be zero, since A has just one distinct root. Rationality would give

$$\frac{\exp(A_1(0)/(Dv))}{A'(0)} = \frac{\exp(A_1(1)/(Dv))}{A'(1)}.$$

The exponential of the algebraic difference of the two exponents is therefore algebraic. Hermite–Lindemann forces that difference to be zero; the equality then forces $A'(0) = A'(1)$. But

$$A'(0) = \frac{dv}{-c}, \qquad A'(1) = \frac{dv}{1 - c},$$

which are unequal. When $d = 1$, equal endpoint values are already impossible. These cases exhaust the possibilities. □

Proof of Theorem 1.1. Suppose all positive factorial moments of a nonflat f vanish. Lemma 2.1 gives such a polynomial with algebraic coefficients. Its transform is $\widehat{G} = 1/w$. If $d \geq 2$, Corollary 4.7 and Proposition 5.1 exclude every nonzero critical value. Lemma 5.3 then makes A a pure power. If $d = 1$, it is already a translated pure power. Lemma 5.4 contradicts rationality in both cases.

Finally, if only finitely many moments of a nonflat f were nonzero, choose $N \geq 1$ such that $\mathcal{L}(f^n) = 0$ for every $n \geq N$. The nonzero polynomial f^N has nonconstant projective top A^N and every positive moment zero, contrary to the first assertion. □

6 The flat case and the EIT input

The radial estimate below is independent of the monodromy argument. It shows exactly where the EIT theorem enters.

Lemma 6.1 (Uniform radial limit)**.** *On every compact t-set on which A does not vanish,*

$$\frac{1}{\Gamma(Dn + 2)} \int_0^\infty e^{-r} r P(t, r)^n \, dr = A(t)^n \left[\exp\left(\frac{A_1(t)}{DA(t)} \right) + o(1) \right], \tag{6.1}$$

with a uniform error inside the brackets.

Proof. For $\mathbf{k} = (k_1, \dots, k_D) \in \mathbb{N}^D$, put $K = \sum_j k_j$ and $J = \sum_j j k_j$. The multinomial expansion and termwise radial integration give terms

$$A^{n-K} \prod_{j=1}^{D} \frac{A_j^{k_j}}{k_j!}\, C_{n,\mathbf{k}}, \qquad C_{n,\mathbf{k}} = \frac{n!}{(n-K)!} \frac{\Gamma(Dn - J + 2)}{\Gamma(Dn + 2)} \quad (K \le n).$$

Set $C_{n,\mathbf{k}} = 0$ for $K > n$. The inequalities $K \le J \le DK$ show $0 \le C_{n,\mathbf{k}} \le 1$. Indeed, the smallest K factors in the denominator product dominate the numerator factors:

$$Dn - J + 2 + j \;\ge\; D(n-K) + 2 + j \;\ge\; n - K + 1 + j \qquad (0 \le j < K).$$

These denominator factors are present because $J \ge K$; all remaining ones are at least one. The empty-product case $K = 0$ is immediate. For fixed $\mathbf{k}$,

$$C_{n,\mathbf{k}} \sim D^{-J} n^{K-J}.$$

After division by A^n, only the terms $k_2 = \cdots = k_D = 0$ survive, and their sum is $\exp(A_1/(DA))$. On the compact set, all absolute terms are bounded by

$$\prod_{j=1}^{D} \frac{(b_j/a_*)^{k_j}}{k_j!}, \qquad b_j = \max |A_j|, \quad a_* = \min |A| > 0.$$

This is summable over $\mathbb{N}^D$ and proves uniform dominated convergence, including $D = 1$. □

Proof of Corollary 1.3. The nonflat case is Theorem 1.1, and a nonzero constant polynomial is immediate. Suppose $D \ge 1$ and A is constant. It is nonzero, so scaling f permits $A = 1$. If all positive moments vanish, specialize algebraically with this normalization retained. Equation (6.1) gives

$$\lim_{n\to\infty} M_n = \int_0^1 \exp\!\left(\frac{A_1(t)}{D}\right) dt.$$

When A_1 is constant this limit is a nonzero exponential. Otherwise Theorem 1.2 applies with $q = A_1$, $a = 0$, $b = 1$ and $\xi = 1/D$. Its hypotheses hold over $\overline{\mathbb{Q}}$, and the amplitude is exactly dt. The limit is again nonzero, contradicting $M_n = 0$ for every $n \ge 1$. The infinitely-many assertion follows by applying the result to f^N if a tail of its factorial moments were zero. □

An elementary flat subcase. The EIT input is unnecessary when A is constant and $\deg A_1 \le 1$. After normalizing $A = 1$, retain the condition $\deg A_1 \le 1$ as rational linear equations in the specialization argument. Write $A_1(t)/D = \alpha t + \beta$ over $\overline{\mathbb{Q}}$. Then

$$\int_0^1 e^{\alpha t + \beta}\, dt = \begin{cases} e^{\beta}, & \alpha = 0, \\ e^{\beta}(e^{\alpha} - 1)/\alpha, & \alpha \ne 0. \end{cases}$$

This is nonzero by Hermite–Lindemann [2]. Thus the same radial limit proves nonvanishing in this subcase using published inputs only; in particular it handles the flat case in total degree $D \le 2$. No rate of convergence beyond the uniform $o(1)$ in Lemma 6.1 is needed.

Research provenance and assistance. During the development of this argument, the `openai/math` release suggested examining local jet separation, differential-system multiplicity, and continuation of complete branch data [11, 10, 12]. These were prompts for proof exploration, not theorems transferred into the FC(2) argument. In particular, local étale rigidity does not imply injectivity of a sum over sheets, and the cited billiard continuation results are not applied here. The main proof uses none of the release's results. Appendix B records the separate finite-jet application of Moura's classical estimate, together with its formal proof in the release. ChatGPT assisted with exploration, drafting, symbolic checking and review; the author is responsible for the mathematical claims and their attribution.

A Exact continuation examples

The examples in this appendix illustrate the continuation conventions. None is a logical input to the main proof in Sections 2–6.

A.1 The pole alone has the wrong jump

For $f = 3x + 2y + 1$, one has $D = 1$ and $A(t) = 2 + t$. Its exponential generating series is

$$\sum_{n\geq 0} \frac{\mathcal{L}(f^n)}{n!} z^n = \frac{e^z}{(1-3z)(1-2z)}.$$

Integrating the series and substituting $z = 1/w$ gives the exact identities

$$\widehat{G}(w) = \int_0^{1/w} \frac{e^z}{(1-3z)(1-2z)}\, dz, \qquad \widehat{G}'(w) = -\frac{e^{1/w}}{(w-3)(w-2)}.$$

Counterclockwise continuation around 3 therefore gives $\Delta_3 \widehat{G} = -2\pi i e^{1/3}$. In contrast, the pole integral alone is

$$C_0(w) = \int_2^3 \frac{e^{1/u}}{w-u}\, du, \qquad \Delta_3 C_0 = -2\pi i e^{1/w}.$$

The difference is a nonconstant analytic germ. For this family, $\mathscr{B}(w,t) = e^{1/A(t)}/A(t)^2$, and the corrected branch germ at a lower endpoint with value v is

$$e^{1/w} + \int_v^w \frac{e^{1/u}}{u^2}\, du = e^{1/v}.$$

Thus the correction, while one local singular order softer, changes the entire jump. A leading-pole identity cannot be substituted for (3.2).

A.2 A coefficient that changes root sheets

Take $D = 2$ and $P = Ar^2 + c$, where $c \neq 0$ and $A = A(t) \neq 0$. The coefficient formula can be computed independently of radial reversion. On writing $n = m + k$, the duplication formula gives

$$\binom{m+k}{k} \frac{\Gamma(2m+2)}{\Gamma(2m+2k+2)} = \frac{4^{-k}\Gamma(m+3/2)}{k!\,\Gamma(m+k+3/2)}.$$

For $k \geq 1$, use the beta integral for the ratio and sum the geometric series over m. For large $|w|$ this gives

$$\mathcal{C}(w,t) = \frac{1}{w-A} + \int_0^1 \frac{u^{1/2}}{w-Au} \sum_{k\geq 1} \frac{(c/(4w))^k (1-u)^{k-1}}{k!(k-1)!}\, du. \tag{A.1}$$

All exchanges converge normally there. At the endpoint $u = 1$, the logarithmic coefficient is minus $1/A$ times the analytic numerator evaluated at $u = w/A$. Reindexing $k = j + 1$ yields

$$\mathscr{B}(w,t,\rho) = -\frac{c\rho}{4A^2} \sum_{j\geq 0} \frac{[c(\rho^2-1)/(4A)]^j}{j!(j+1)!}, \qquad \rho^2 = A/w. \tag{A.2}$$

This is an all-orders convergent identity. In particular, $\mathscr{B}(A,t,1) = -c/(4A^2)$, agreeing with (2.7).

The right side of (A.2) is odd in ρ. A fixed-t loop around zero changes the sign of the continued distinguished coefficient. When the loop returns to a neighborhood of the diagonal, the canonical coefficient of the complete kernel is unchanged because the singular root-filter summand has been reselected. This distinguishes the two objects in Remark 2.3 concretely.

A.3 A moving endpoint with a changed lower root

Let

$$P = (L(t)r + c)^D, \qquad L(t) = at + b, \qquad ac \neq 0, \quad D \geq 2.$$

This is the radial polynomial of $[ax + b(x + y) + c]^D$. Fix a lower endpoint p with $L(p) \neq 0$, and write $s = L(t_i(w))$, so $s^D = w$. On the lifted integration path,

$$\Phi = r + c/L, \qquad \chi = Y - c/L, \qquad F = e^{c/L}(1 - c/(LY)), \qquad \rho = L/s.$$

Formula (2.7) now reads

$$\mathscr{B}\, dt = -\frac{d_t(e^{c/L(t)})}{Da\, s^{D-1}}.$$

Since $A'(t_i) = Da\, s^{D-1}$, the full corrected expression is

$$E(t_i(w))t_i'(w) + \int_p^{t_i(w)} \mathscr{B}\, dt = \frac{e^{c/L(p)}}{Da\, s^{D-1}}. \tag{A.3}$$

For a positive loop around zero, s changes to ωs, where $\omega = e^{2\pi i/D}$. The lower root parameter changes to $\omega^{-1}\rho$, and (A.3) changes by the factor ω. Resetting the lower root, or deleting the integral, would both give the wrong continuation. The general transport proof retains exactly these data.

B Finite certificates for branch densities

This appendix concerns auxiliary differential systems and is not an input to the main proof. In particular, its branch-density tests do not bound the first nonzero factorial moment. The analytic multiplicity estimate is due to Moura [8, 9]. We prove the precise polynomial-presentation, formal-power-series version below; the proof is adapted from the version in [10], which also gives a Lean statement. The formal Wronskian step is explained in Bostan–Dumas [3]. These attributions concern the auxiliary estimate, not a formal verification of FC(2).

B.1 The classical multiplicity estimate

Theorem B.1 (Moura's estimate in a polynomial presentation)**.** *Let F have characteristic zero, $m, n \geq 1$, $q \in F[z]$ with $q(0) \neq 0$ and $\deg q \leq n$, and $P \in M_m(F[z])$ with $\deg P_{ij} \leq n - 1$. Suppose*

$$G \in \mathrm{GL}_m(F[[z]]), \qquad qG' = PG.$$

For a constant row u and constant column v, a nonzero series $y = uGv$ satisfies

$$\mathrm{ord}_0\, y \leq (n - 1)\binom{m}{2} + m - 1. \tag{B.1}$$

Proof. Put $h = uG$, and let V be the F-span of its component series. Its dimension k is between 1 and m. Gaussian elimination on successive coefficients gives a basis $b_1, \ldots, b_k$ of V with strictly increasing orders $a_1 < \cdots < a_k$. The filtration by order is eventually zero: in a finite-dimensional space any stabilizing nonzero tail would lie in every power of z, which is impossible. A nonzero element of V has order at most a_k.

The Wronskian $W = \det(b_\ell^{(j)})_{0 \leq j < k,\, 1 \leq \ell \leq k}$ has order

$$\mathrm{ord}_0\, W = \sum_{i=1}^{k} a_i - \binom{k}{2}. \tag{B.2}$$

Indeed, its leading coefficient after the powers of z are factored is the product of the leading coefficients of the b_i times $\det((a_i)_j)$, a nonzero Vandermonde determinant in characteristic zero. Changing the basis of V multiplies W by a nonzero constant.

Define rows $r_0 = u$, $r_{j+1} = r_j' + r_j P/q$. Then $h^{(j)} = r_j G$. Writing $r_j = Z_j/q^j$ gives

$$Z_{j+1} = qZ_j' - jq'Z_j + Z_j P, \qquad \deg Z_j \le j(n-1).$$

Let R_* have rows $r_0, \dots, r_{k-1}$ and put $S = \binom{k}{2}$. Some k components of h form a basis of V, so R_*G has a nonzero k-minor by (B.2). Thus R_* also has such a minor. Each k-minor of R_* has denominator q^S and a polynomial numerator of degree at most $(n-1)S$. Since q is a unit at zero, the minimum order δ of its nonzero k-minors obeys $0 \le \delta \le (n-1)S$.

Multiplication by G preserves the ideal of k-minors in $F[[z]]$: apply Cauchy–Binet first with G and then with G^{-1}. Hence the minimum order for the minors of R_*G is also δ. A nonzero minor of that order is a Wronskian of k components of h; those components form a basis of V. Equation (B.2) therefore gives

$$\delta = \sum_{i=1}^{k} (a_i - (i-1)), \qquad \operatorname{ord}_0 y \le a_k \le \delta + k - 1 \le (n-1)\binom{k}{2} + k - 1.$$

Each summand in the first expression is nonnegative. Finally $k \le m$ proves (B.1). □

Comparison with the analytic statement. Over $\mathbb{C}$, Moura's formulation [9, Section 2, Theorem 1] uses the total pole order σ of the matrix differential form $\Omega = (P/q)\,dz$ on the projective line. If $P \neq 0$, put $s = \deg q$ and $p = \max_{P_{ij} \neq 0} \deg P_{ij}$. Finite poles contribute at most s, and the pole order at infinity is at most $\max(0, p - s + 2)$. Therefore

$$\sigma \le s + \max(0, p - s + 2) = \max(s, p + 2) \le n + 1.$$

Moura's expression $m - 1 + (\sigma - 2)\binom{m}{2}$ gives (B.1). If $P = 0$, the solution matrix is constant and the bound is immediate. This comparison does not assume $\deg q = n$; the direct proof above applies over every characteristic-zero field. The corresponding pinned Lean declaration is `OAI.NCHitting.system_multiplicity`. Citing or reading that declaration does not claim that the present FC(2) paper, or the density application below, has been formally verified.

B.2 An explicit rational connection

Fix $A, B \in \mathbb{C}[t]$, $\deg A = d \ge 1$, and an integer $D \ge 1$. On a disk avoiding zero and the critical values of A, put

$$h_i(w) = \frac{\exp(B(t_i(w))/(Dw))}{A'(t_i(w))}, \qquad \Psi_{ji}(w) = t_i(w)^j h_i(w) \quad (0 \le j < d).$$

The determinant of Ψ is a Vandermonde determinant times nonzero column factors, so it is invertible. In $\mathbb{C}(w)[t]/(A(t) - w)$, reduce

$$\frac{jt^{j-1}}{A'} - \frac{t^j A''}{(A')^2} + \frac{t^j B'}{DwA'} - \frac{t^j B}{Dw^2} = \sum_{k=0}^{d-1} R_{jk}(w) t^k. \tag{B.3}$$

The first term is understood as zero for $j = 0$. Differentiating $t_i' = 1/A'(t_i)$ and h_i proves the exact system $\Psi' = R\Psi$.

Let $\Delta(w) = \operatorname{Res}_t(A(t) - w, A'(t))$. It is a nonzero polynomial. The resultant identity provides a representative of $1/A'$ whose only parameter denominator is Δ. Reduction modulo $A - w$ introduces only constant leading-coefficient denominators. Consequently

$$q(w) = Dw^2 \Delta(w)^2, \qquad P(w) = q(w) R(w)$$

has polynomial entries. Define

$$n = \max\{1, \deg q, 1 + \max_{P_{ij}\neq 0} \deg P_{ij}\}, \qquad J = (n-1)\binom{d}{2} + d - 1,$$

omitting the last entry of the maximum when $P = 0$.

Corollary B.2 (Finite test for an identically zero density)**.** *For a constant coefficient vector c, a point w_0 with $q(w_0) \neq 0$ (equivalently $w_0 \notin \{0\} \cup \mathrm{CritVal}(A)$ for the displayed q), and $h(w) = \sum_i c_i h_i(w)$ on a sufficiently small disk about w_0, one has*

$$h \equiv 0 \quad \Longleftrightarrow \quad h^{(j)}(w_0) = 0 \quad (0 \le j \le J).$$

Proof. Translate w_0 to zero and normalize the fundamental matrix to $G(w) = \Psi(w)\Psi(w_0)^{-1}$. Then $h = e_0^{\mathsf{T}} G(\Psi(w_0)c)$ is a constant scalar output. The denominator q is nonzero at w_0. Apply Theorem B.1 contrapositively. □

For algebraic A, B, w_0, c, the derivatives are finite algebraic linear combinations of exponentials of algebraic numbers. They can be tested exactly by grouping equal exponents and using Lindemann–Weierstrass. This certifies a particular density identity. It neither makes every sheet sum nonzero nor bounds the first nonzero moment of an arbitrary factorial polynomial.

For example, $A = t^2 + 1$, $B = t$ and $D = 2$ give

$$R = \begin{pmatrix} -\dfrac{1}{2(w-1)} & \dfrac{2-w}{4w^2(w-1)} \\ \dfrac{2-w}{4w^2} & 0 \end{pmatrix}.$$

One may take $q = 4w^2(w-1)$, $n = 3$ and $J = 3$. The first two observation rows e_0^{T} and $e_0^{\mathsf{T}} R$ are independent over $\mathbb{C}(w)$. By contrast, for $A = t(1-t)$ and constant $B = c$,

$$\frac{e^{c/(2w)}}{A'(t_+(w))} + \frac{e^{c/(2w)}}{A'(t_-(w))} = 0, \qquad t_\pm(w) = \frac{1 \pm \sqrt{1-4w}}{2}.$$

Both summands are nonzero. All jets of their sum vanish, even on the étale locus. The actual interval uses opposite orientations on these two branches; this is why the signed path data in the main proof cannot be replaced by local jet rigidity.

B.3 A different, conditional certificate for a complete system

If an exact full transform is embedded in a rational system $Y' = MY$ and $h = \ell_0 Y$, define observation rows $\ell_{j+1} = \ell_j' + \ell_j M$. Then $h^{(j)} = \ell_j Y$. If a coordinate row e_r^{T} lies in the rational span of these rows and $Y_r \not\equiv 0$, one necessarily has $h \not\equiv 0$. Indeed, $h \equiv 0$ would force $Y_r \equiv 0$ by the displayed row relation. This criterion requires actual observation of a nonzero coordinate; finiteness of the system alone is not sufficient.

For a nonconstant linear polynomial $f = ax + by + c$, let $q = (w-a)(w-b)$ and take

$$Y = (\widehat{G}, e^{c/w}, 1)^{\mathsf{T}}, \qquad M = \begin{pmatrix} 0 & -1/q & 0 \\ 0 & -c/w^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \ell_0 = (1, 0, -1/w).$$

The system follows by integrating the exponential generating series, and $h = \widehat{G} - 1/w$ is the exact positive-moment output. Direct differentiation gives

$$\det \begin{pmatrix} \ell_0 \\ \ell_1 \\ \ell_2 \end{pmatrix} = \frac{-(a+b+c)w^2 + (2ab + c(a+b))w - cab}{q^2 w^4}.$$

Its numerator is identically zero only if $a = b = c = 0$: from $c = -a - b$, the middle coefficient gives $a^2 + b^2 = 0$, and the last gives $ab(a + b) = 0$, forcing $a = b = 0$. Thus in the nonconstant case all coordinates, including 1, are observed and h cannot vanish identically. No analogous universal observation assertion in higher degree is needed or claimed here.

C A collision with incomparable decompositions

Here is a concrete instance of Proposition 5.1. Let $D \geq 6$ and suppose

$$A(t) = T_6(2t - 1) - 1, \qquad T_6(z) = 32z^6 - 48z^4 + 18z^2 - 1.$$

It admits both $T_6 = T_2 \circ T_3$ and $T_6 = T_3 \circ T_2$, where $T_2(z) = 2z^2 - 1$ and $T_3(z) = 4z^3 - 3z$. The corresponding block sizes are incomparable, and $A(0) = A(1) = 0$. The three points over the nonzero critical value -2 are

$$c_- = \frac{2 - \sqrt{3}}{4}, \qquad c_0 = \frac{1}{2}, \qquad c_+ = \frac{2 + \sqrt{3}}{4}.$$

They have local degree two, with quadratic coefficients $A''(c)/2$ equal to $288, 72, 288$, respectively. The positive critical-point weights in (5.2) are therefore $1/576, 1/144, 1/576$. Rationality of $\widehat{G}$ would force

$$e^{-A_1(c_-)/(2D)} + 4e^{-A_1(1/2)/(2D)} + e^{-A_1(c_+)/(2D)} = 0.$$

For algebraic layers this contradicts Lindemann–Weierstrass after equal exponents are grouped. There are no restrictions on the subsequent layers. Algebraic specialization gives the corresponding nonvanishing statement with arbitrary complex coefficients. The computation illustrates why neither colliding critical points, incomparable block systems, nor zero endpoint values are omitted by the proof.

Reproducibility. The accompanying repository provides an exact $\mathbb{Q}(\sqrt{3})$ calculation of the identities in this appendix, and symbolic checks of the rational connection, observation determinant, root filter, radial normalization, and branch-change examples. These finite checks are separate from the analytic and representation-theoretic proofs.

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