hexagon:2611.00807v1

THE PRODUCT $\log(1+1/N)\log(1+2/N)$ AND RATIONAL CONICS THROUGH TWO LOGARITHMS

SHAWN KRISMAN

ABSTRACT. Put $\lambda_1 = \log(1+1/N)$ and $\lambda_2 = \log(1+2/N)$. We prove that $\lambda_1\lambda_2$ is irrational for every integer $N \geq 84$ and every even $N \geq 52$, and more strongly that $1, \lambda_1, \lambda_2, \lambda_1\lambda_2$ are linearly independent over $\mathbb{Q}$ on the same range. We also prove that $1, \lambda_1, \lambda_2, \lambda_2^2$ are $\mathbb{Q}$-linearly independent for $N \geq 58$, that $1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_2^2$ are for $N \geq 88$, and that $1, \lambda_1, \lambda_2, \lambda_1^2, \lambda_1\lambda_2, \lambda_2^2$ are for $N \geq 151$: for $N \geq 151$ the point (λ_1, λ_2) lies on no curve of degree at most two defined over $\mathbb{Q}$. These are instances, not conditional on the conjecture and with rational coefficients, of what the strong four exponentials conjecture predicts for two logarithms. To our knowledge no published result states any of them for N below about $4.5\cdot 10^{11}$. The proofs use explicit two-level Rodrigues forms of type II Hermite–Padé type, linear recurrences in the index proved by exact creative-telescoping certificates, Pituk's theorem on the growth of solutions of Poincaré-type recurrences (read only in restatements, so all results are stated modulo it), exact or ball-arithmetic certificates for the roots of the characteristic polynomials, and a 2-adic content lemma at odd points that lowers the threshold for even N. The irrationality of $\lambda_1\lambda_2$ and the four-term independence for $N \geq 103$, and the six-term (conic) independence for $N \geq 151$, are also checked in Lean 4 with Mathlib, by arguments that do not use Pituk's theorem.

1. INTRODUCTION

Throughout, $N \geq 1$ is an integer and

$$\lambda_1 = \log\Big(1+\frac{1}{N}\Big), \qquad \lambda_2 = \log\Big(1+\frac{2}{N}\Big), \qquad \theta_N = \lambda_1\lambda_2.$$

For $N = 1$ this is $\log 2 \cdot \log 3$, whose irrationality is open. For even $N = 2m$ it is $\log(1+1/m)\log(1+1/(2m))$, the product of the logarithms at a pair of points of ratio two.

1.1. Results.

Theorem 1 (modulo Pituk's theorem)**.** *$\theta_N = \log(1+1/N)\log(1+2/N)$ is irrational for every integer $N \geq 84$ and for every even integer $N \geq 52$.*

2020 *Mathematics Subject Classification.* Primary 11J72; Secondary 11J82, 41A21, 39A06, 65G20.

Key words and phrases. irrationality, linear independence, products of logarithms, Hermite–Padé approximation, linear recurrences, creative telescoping, strong four exponentials conjecture, computer-assisted proof, interval arithmetic.

For example, $\log(85/84)\log(43/42)$ and $\log(53/52)\log(27/26)$ are irrational. The cases not covered are $N \le 51$ and the odd N with $53 \le N \le 83$.

Theorem 2 (modulo Pituk's theorem)**.** *For every integer $N \ge 84$ and every even integer $N \ge 52$, the four numbers $1, \lambda_1, \lambda_2, \lambda_1\lambda_2$ are linearly independent over $\mathbb{Q}$.*

Theorem 3 (modulo Pituk's theorem)**.** *For every integer $N \ge 58$, the four numbers $1, \lambda_1, \lambda_2, \lambda_2^2$ are linearly independent over $\mathbb{Q}$.*

Theorem 4 (modulo Pituk's theorem)**.** *For every integer $N \ge 88$, the five numbers $1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_2^2$ are linearly independent over $\mathbb{Q}$.*

Theorem 5 (modulo Pituk's theorem)**.** *For every integer $N \ge 151$, the six numbers 1, λ_1, λ_2, λ_1^2, $\lambda_1\lambda_2$, λ_2^2 are linearly independent over $\mathbb{Q}$. Equivalently, no nonzero polynomial with rational coefficients and of degree at most 2 vanishes at (λ_1, λ_2).*

Theorem 2 contains Theorem 1. Theorem 4 contains Theorem 3 and the four-term statement of Theorem 2 for $N \ge 88$, and Theorem 5 contains all the others for $N \ge 151$. Each theorem is proved with its own family of forms, and each range is the full range that its family reaches with the proved denominator bounds (Section 11). All results are proved modulo Pituk's theorem, which we have read only in restatements (Section 1.3); Theorems 1 and 2 for $N \ge 103$ and Theorem 5 are also checked in Lean without it.

Corollary 6. (i) *For $N \ge 84$ and for even $N \ge 52$: $\lambda_1\lambda_2 \notin \mathbb{Q}+\mathbb{Q}\lambda_1+\mathbb{Q}\lambda_2$; in particular*

$$\frac{\log(1+1/N)\log(1+2/N)}{\log(1+1/(N+1))} \notin \mathbb{Q}.$$

For $m \ge 26$ the numbers 1, $\log(1+1/m)$, $\log(1+1/(2m))$, $\log(1+1/m)\log(1+1/(2m))$ are $\mathbb{Q}$-linearly independent.

(ii) *For $N \ge 58$: $\log^2(1+2/N)/\log(1+1/N) \notin \mathbb{Q}$. For $m \ge 29$: $\log^2(1+1/m)/\log(1+1/(2m)) \notin \mathbb{Q}$.*

(iii) *For $N \ge 88$: $\lambda_1\lambda_2 \notin \mathbb{Q}+\mathbb{Q}\lambda_1+\mathbb{Q}\lambda_2+\mathbb{Q}\lambda_2^2$.*

(iv) *For $N \ge 151$: $\log^2(1+1/N)/\log(1+2/N) \notin \mathbb{Q}$, and $\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2 \notin \mathbb{Q}+\mathbb{Q}\lambda_1+\mathbb{Q}\lambda_2$.*

Proof. (i) follows from Theorem 2, since $\log(1+1/(N+1)) = \lambda_2 - \lambda_1$, and for $N = 2m$ we have $\lambda_1 = \log(1+1/(2m))$, $\lambda_2 = \log(1+1/m)$. (ii) A relation $\lambda_2^2 = c\lambda_1$ with $c \in \mathbb{Q}$ contradicts Theorem 3; for $N = 2m$, $\lambda_2^2/\lambda_1 = \log^2(1+1/m)/\log(1+1/(2m))$. (iii) and (iv) follow from Theorems 4 and 5. □

1.2. **Prior work and what is new.** In the coordinates $q = N+1$ of Hata [3] put $f_1 = \log(1+1/q)$ and $f_2 = \log(1-1/q)$. Then $\lambda_1 = -f_2$ and $\lambda_2 = f_1 - f_2$, so

$$\theta_N = f_2^2 - f_1 f_2, \qquad \text{while} \qquad f_1 f_2 = \lambda_1^2 - \lambda_1\lambda_2.$$

Hata [3, Cor. 6.3] proved that $1, f_1, f_2, f_1 f_2$ are $\mathbb{Q}$-linearly independent for every integer $q \geq 54$; Lysov [9] proved $f_1 f_2 \notin \mathbb{Q}$ for $q \geq 33$; and in [8] we extended the four-term statement to $26 \leq q \leq 53$, conditionally on the holonomy bound [2, Thm. 7.0.1] of Calegari, Dimitrov and Tang (a preprint). The number $f_1 f_2 = \lambda_1^2 - \lambda_1 \lambda_2$ lies on a different hyperbola from $\theta_N = \lambda_1 \lambda_2$, so none of these results covers θ_N; Theorem 5 contains Hata's four-term statement for $q \geq 152$, which is not new. Calegari, Dimitrov and Tang [2, Thm. C] proved that $1, \log(1+1/m), \log(1+1/n), \log(1+1/m)\log(1+1/n)$ are linearly independent for all integers $m \neq n$ in $\mathbb{Z} \setminus \{-1, 0\}$ with $|m/n - 1| < 10^{-6}$; in [8] we widened this to $3 \cdot 10^{-6}$, conditionally on parts of [2] (Thm. 6.0.2, Cor. 9.0.19, Lemma 12.0.1 and the construction of §14, including an irreducibility asserted there). For even $N = 2m$, θ_N is the product at the pair $(m, 2m)$, of ratio 2.

Kawashima [7] proved a general linear independence criterion for products of polylogarithms (Theorem 2.2 and Corollary 2.3 there; the preprint is not refereed). With $K = \mathbb{Q}$, $m = r = 2$ (so $M = (m+1)^r - 1 = 8$), $\boldsymbol{\alpha} = (1, -1)$ and $\beta = N + 1$, the quantity $V_{v_0}(\boldsymbol{\alpha}, \beta)$ of [7, eq. (V)] is, in our reading of the LaTeX source of version 1 (we have not checked the proof in [7]),

$$\begin{aligned} V &= (M+1)\log(N+1) - M\log(N+1) \\ &\qquad - \Big(M\log 2 + \tfrac{r(r+1)}{2}\log(m+1) + r + rM \Big) \\ &= \log(N+1) - 26.841\ldots, \end{aligned}$$

with $26.841\ldots = 8\log 2 + 3\log 3 + 2 + 16$. Since $\mathrm{Li}_1(1/(N+1)) = \lambda_1$ and $\mathrm{Li}_1(-1/(N+1)) = \lambda_1 - \lambda_2$, Corollary 2.3 of [7] then gives the statement of Theorem 5, and hence of all our theorems, for $N + 1 > e^{26.841} \approx 4.54 \cdot 10^{11}$. (The choice $\boldsymbol{\alpha} = (-1, -2)$, $\beta = N$ gives $V = \log N - 13\log 2 - 26.841$, hence $N > 3.7 \cdot 10^{15}$.) Only the height terms of [7, eq. (V)] carry the factor M; the constant bracket does not.

Viola and Zudilin [14] proved that $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z), \mathrm{Li}_2(1/(1-z))$ are linearly independent over $\mathbb{Q}$ for every integer $z \geq 9$ or $z \leq -8$, and for rationals $z = s/r$ or $z = 1 - s/r$ with $1 < r < s$ and s large compared with r (we checked this statement in the preprint version, MPIM Preprint 14-26, abstract and Main theorem; not in the journal version). Since $\mathrm{Li}_1(x) = -\log(1-x)$ and $\mathrm{Li}_2(x) + \mathrm{Li}_2(x/(x-1)) = -\frac{1}{2}\log^2(1-x)$ for $x < 1$, the integer case gives: with $z = N/2 + 1$, the independence of $1, \lambda_2, \lambda_2^2$ for even $N \geq 16$; with $z = -N$, the independence of $1, \lambda_1, \lambda_1^2$ for every $N \geq 8$. Each application involves one of λ_1, λ_2 only; [14] never combines λ_1 with λ_2, so it gives neither Theorem 3 nor the corollary $\lambda_2^2/\lambda_1 \notin \mathbb{Q}$.

What is new. To our knowledge, no published result states the irrationality of θ_N, or any of the statements of Theorems 2–5, for any N in the ranges of our theorems below $4.5 \cdot 10^{11}$. This rests on searches of arXiv, Hexagon, GitHub, Zenodo and the general web on 19 and 20 November 2026, and on our own notes; the searches are not exhaustive. So, subject to the status described next, the numbers θ_N with $84 \leq N < 4.5 \cdot 10^{11}$ or N even and

$52 \le N < 4.5 \cdot 10^{11}$ are new irrational numbers. Between our own theorems: Theorem 2 is new beyond Theorems 4 and 5 only for $84 \le N \le 87$ and even $52 \le N \le 86$; Theorem 3 only for $58 \le N \le 87$; Theorem 4 only for $88 \le N \le 150$.

1.3. **Status of the results.** The proofs are computer-assisted. Every computation that a proof uses is an exact identity over $\mathbb{Q}$ (python-flint or SymPy), an exact positivity or resultant check, or a rigorous ball-arithmetic enclosure (Arb, through python-flint).

One external theorem read only in restatement. Every proof uses Pituk's theorem [10, Thm. 2] on the growth of solutions of linear recurrences of Poincaré type (Theorem 3.6). The original paper is paywalled; we checked its statement and hypotheses against two restatements read in full, [4, Thm. 6.2] and [5, §4, "Pituk's Theorem"], but not against the original. We label every result *proved modulo Pituk's theorem.*

Lean. The irrationality of θ_N for $N \ge 103$ (Theorem 1 for $N \ge 103$) is checked in Lean 4 with Mathlib (theorem `Ratio2.irrational_log_mul_log`, which depends only on the standard axioms `propext`, `Classical.choice`, `Quot.sound`). In the Lean proof Pituk's theorem is replaced by an elementary cone argument (Remark 4.7), so the Lean-checked range does not depend on Pituk's theorem. The four-term statement of Theorem 2 for $N \ge 103$ is also checked in Lean (`Ratio2.linearIndependent_four`, same axioms, same Pituk-free argument), and so is Theorem 5 in full (`Ratio2.linearIndependent_conic`, $N \ge 151$, same axioms; Pituk's theorem is replaced there by a growth lemma for recurrences of any order, Remark 4.7). The other ranges of Theorems 1 and 2 ($84 \le N \le 102$ and even $52 \le N \le 102$), and Theorems 3 and 4, are not checked in Lean. The Lean proofs and the certificates are in the repository `https://github.com/telaviv/log-product-conics` (tag `v1`); see Section 10.

Refereeing. Each chain was checked by adversarial AI referees, once when the result was accepted ("gate-refereed"): the base proof for $N \ge 103$ and the two small-N extensions of Theorem 1 each in a whole-chain referee, Theorem 2 for $N \ge 103$ together with Theorem 5 in one gate referee, and the small-N part of Theorem 2 with Theorems 3 and 4 in another. Every verdict was "pass with fixes", with no fatal or major gap in a proof; the fixes are applied. One gate referee raised a major finding on novelty, not on a proof: our earlier notes quoted Kawashima's threshold [7] as $N > e^{185}$, and the corrected value is the one given in Section 1.2. A further AI referee read this paper as a whole (pass with fixes, no fatal or major finding); its fixes are applied. No human expert has read the proofs. See Section 12.

1.4. **Method.** All five theorems are proved by one argument, set out in Section 3. For a point $z = N+1$ and a set $\mathcal{F}$ of target functions (logarithms and products of two logarithms with branch points in $\{-1, 0, 1\}$) we use

explicit polynomials

$$Q_n(t) = \frac{1}{n!}\frac{d^n}{dt^n}\Big[W(t)^n\,\frac{1}{(2n)!}\frac{d^{2n}}{dt^{2n}}V(t)\Big], \qquad W = t^3 - t,$$

with $V = t^{e_0}(t-1)^{e_1}(t+1)^{e_{-1}}$ chosen so that $Q_nF - P_{F,n} = O(t^{-n-1})$ for every $F \in \mathcal{F}$, where $P_{F,n}$ is the polynomial part of Q_nF at infinity. The remainders $R_{F,n}(z)$ satisfy a linear recurrence in n with polynomial coefficients. We do not estimate them by a saddle point analysis: an elementary a priori bound excludes the one large characteristic root, Pituk's theorem identifies the growth rate with the modulus of a characteristic root, and a root certificate shows that the remaining roots are smaller than the inverse of the denominators. Backward recursion (the trailing coefficient of the recurrence does not vanish) and one nonzero determinant then give linear independence. The recurrences are proved by exact creative-telescoping certificates. The five theorems differ only in the parameters and certificates (Table 1).

Outline. Section 2 places the results relative to the strong four exponentials conjecture. Section 3 is the framework. Sections 4–9 treat the five families: the ratio-two forms (Theorems 1 and 2 for $N \geq 103$), the 2-adic gain (even $N \geq 52$), the shape $(2; 1, 2, \frac{5}{4})$ ($84 \leq N \leq 102$), and Theorems 3, 4, 5. Section 10 lists the certificates, Section 11 the limits of the method and open problems, and Section 12 provenance and the use of AI tools.

2. Relation to the strong four exponentials conjecture

Let $\mathcal{L}$ be the $\mathbb{Q}$-vector space of logarithms of nonzero algebraic numbers and $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$.

Conjecture 2.1 (strong four exponentials; [15, Conj. 1.5], suggested by D. Roy)**.** Let M be a 2×2 matrix with entries in $\widetilde{\mathcal{L}}$ whose rows are linearly independent over $\overline{\mathbb{Q}}$ and whose columns are linearly independent over $\overline{\mathbb{Q}}$. Then M has rank 2.

(We take the statement and the attribution from the author's copy of [15], §1. The best known result in this direction is Roy's strong six exponentials theorem [11, §4, Cor. 2], as quoted in [15, Thm. 2.1].)

Proposition 2.2. *Assume Conjecture 2.1. Let $\ell_1, \ell_2 \in \mathcal{L}$ be linearly independent over $\mathbb{Q}$. Then $1, \ell_1, \ell_2, \ell_1^2, \ell_1\ell_2, \ell_2^2$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. By Baker's theorem on linear forms in logarithms (see [1]), $1, \ell_1, \ell_2$ are linearly independent over $\overline{\mathbb{Q}}$. Suppose $q(1, \ell_1, \ell_2) = 0$ for a nonzero quadratic form $q \in \overline{\mathbb{Q}}[X_0, X_1, X_2]$. If q has rank ≤ 2, then $q = uv$ for linear forms u, v over $\overline{\mathbb{Q}}$, and $u(1, \ell_1, \ell_2) = 0$ or $v(1, \ell_1, \ell_2) = 0$ contradicts Baker's theorem. If q has rank 3, then over $\overline{\mathbb{Q}}$ we can write $q = u_1u_4 - u_2u_3$ with linear forms u_i. The matrix $\left(\begin{smallmatrix} u_1 & u_2 \\ u_3 & u_4 \end{smallmatrix}\right)$ evaluated at $(1, \ell_1, \ell_2)$ has entries in $\widetilde{\mathcal{L}}$ and

determinant 0. By Conjecture 2.1 its rows or its columns are $\overline{\mathbb{Q}}$-dependent; say $a(u_1,u_2)+b(u_3,u_4)$ vanishes at $(1,\ell_1,\ell_2)$ with $(a,b)\neq 0$. By Baker's theorem au_1+bu_3 and au_2+bu_4 vanish identically, so $q=u_1u_4-u_2u_3\equiv 0$ as a polynomial (if, say, $b\neq 0$, substitute $u_3=-au_1/b$, $u_4=-au_2/b$), a contradiction. The case of dependent columns is the same. □

For $N\geq 1$ the logarithms $\lambda_1=\log((N+1)/N)$ and $\lambda_2=\log((N+2)/N)$ are $\mathbb{Q}$-linearly independent: if $a\lambda_1=b\lambda_2$ with integers a,b, then $(N+1)^aN^b=(N+2)^bN^a$; a prime dividing $N+1\geq 2$ divides neither N nor $N+2$, so $a=0$, and then $b=0$. So Conjecture 2.1 predicts, for every $N\geq 1$, the statement of Theorem 5 with $\overline{\mathbb{Q}}$ in place of $\mathbb{Q}$. Following the usual description of plane conics, call a relation $\lambda_1\lambda_2\in\overline{\mathbb{Q}}+\overline{\mathbb{Q}}\lambda_1+\overline{\mathbb{Q}}\lambda_2$ a "hyperbola" and a relation $\lambda_2^2\in\overline{\mathbb{Q}}+\overline{\mathbb{Q}}\lambda_1+\overline{\mathbb{Q}}\lambda_2$ a "parabola" through the point $(1:\lambda_1:\lambda_2)$. Theorems 1 and 2 exclude rational hyperbolas of the first shape, Theorem 3 rational parabolas of the second, and Theorem 5 every rational conic and line. For $N=1$ the hyperbola case contains $\log 2\cdot\log 3\notin\mathbb{Q}$.

Our theorems are therefore unconditional explicit instances of consequences of Conjecture 2.1, with rational (not algebraic) coefficients and for N above explicit thresholds. They are not progress on the conjecture itself. The method is an explicit Hermite–Padé construction at one point; its success depends on the point $z=N+1$ being far from the cut $[-1,1]$ compared with the arithmetic cost of the forms, and it fails by a wide margin at $N=1$ (Section 11).

3. The framework

3.1. **Coordinates.** Put $z=N+1$ and, for $t\in\mathbb{C}\setminus[-1,1]$ (principal branches),

$$f_1(t)=\log\Big(1+\frac{1}{t}\Big),\qquad f_2(t)=\log\Big(1-\frac{1}{t}\Big),$$
$$L(t)=f_1(t)-f_2(t)=\log\frac{t+1}{t-1},\qquad \theta(t)=-f_2(t)L(t).$$

These are analytic on $\mathbb{C}\setminus[-1,1]$ and real on $(1,\infty)$; f_1,f_2,L are $O(1/t)$ and $\theta=f_2^2-f_1f_2$ is $O(t^{-2})$ at ∞. The branch points are

$$B_{f_1}=\{0,-1\},\qquad B_{f_2}=\{0,1\},\qquad B_L=\{-1,1\}.$$

At $t=z$ we have $f_2(z)=\log(N/(N+1))=-\lambda_1$, $L(z)=\log((N+2)/N)=\lambda_2$ and $f_1(z)=\lambda_2-\lambda_1$. Hence

$$\lambda_1=-f_2(z),\quad \lambda_2=L(z),\quad \lambda_1\lambda_2=\theta(z),$$
$$\lambda_2^2=L(z)^2,\quad \lambda_1^2=f_2(z)^2,\quad -\lambda_1\lambda_2=(f_2L)(z),$$

and each of the lists in Theorems 2–5 spans the same $\mathbb{Q}$-space as 1 together with the values at z of the target set

$$\mathcal{F}_{\mathrm{II}}=\{f_1,f_2,\theta\},\qquad \mathcal{F}_{\mathrm{S}}=\{f_1,f_2,L^2\},$$
$$\mathcal{F}_{\mathrm{B}}=\{f_1,f_2,L^2,f_2L\},\qquad \mathcal{F}_{\mathrm{C}}=\{f_1,f_2,f_1^2,f_1f_2,f_2^2\},\tag{3.1}$$

respectively, by an invertible rational change of basis. For instance $(1,\lambda_1,\lambda_2,\lambda_1\lambda_2) = (1,f_1(z),f_2(z),\theta(z))T$ with $T\in\mathrm{GL}_4(\mathbb{Z})$, $\det T = 1$; and $\lambda_1^2 = f_2^2$, $\lambda_1\lambda_2 = f_2^2 - f_1f_2$, $\lambda_2^2 = f_1^2 - 2f_1f_2 + f_2^2$ at z. So it suffices to prove that 1 and the values $F(z)$, $F\in\mathcal{F}$, are $\mathbb{Q}$-linearly independent. Theorem 1 is the case $\mathcal{F} = \{\theta\}$.

Every target F has a Laurent expansion $F = \sum_{m\ge1}\varphi_m t^{-m}$ at ∞. For $F\in\{f_1,f_2,L\}$, $\varphi_m\in\mathbb{Z}\cdot\frac{1}{m}$ (indeed $\pm\frac{1}{m}$ or $\frac{2}{m}$); for a product $F = gh$ of two of them, $\varphi_m\in\sum_{a+b=m}\mathbb{Z}\cdot\frac{1}{ab}$. We call these targets of linear and of quadratic type.

3.2. **Two-level Rodrigues forms.** Write $\partial^{[k]} = \frac{1}{k!}\frac{d^k}{dt^k}$; it maps $\mathbb{Z}[t]$ into itself. Let $n\ge1$, let $V\in\mathbb{Z}[t]$ have degree $\delta\ge2n$, and put

$$S = \partial^{[2n]}V,\qquad Q = \partial^{[n]}\big[W^nS\big]\in\mathbb{Z}[t],\qquad W = t(t-1)(t+1). \tag{3.2}$$

Then $\deg S = \delta-2n$ and $\deg Q = \delta$. For a target F let $P_F = [QF]_+$ be the polynomial part of QF at ∞, and $R_F = QF - P_F$, which is analytic on $\mathbb{C}\setminus[-1,1]$. In all our families V is a product of powers of $t, t-1, t+1$ whose exponents are proportional to n, the polynomials form a sequence Q_n, and $Q_0 = 1$, $P_{F,0} = 0$. If $V = V_1^n$ for a fixed V_1, then

$$Q_n(z) = \mathrm{CT}_{h_1,h_2}\big[\Lambda^n\big],\qquad \Lambda = \frac{W(z+h_1)\,V_1(z+h_1+h_2)}{h_1h_2^2}, \tag{3.3}$$

the constant term of a Laurent polynomial in h_1,h_2: the constant term in h_2 is $\partial^{[2n]}V(z+h_1) = S(z+h_1)$, and then the constant term in h_1 is $\partial^{[n]}(W^nS)(z)$. Equivalently $Q_n(z) = \mathrm{Res}_{u=z}W(u)^n(u-z)^{-n-1}S(u)$ with $S(u) = \mathrm{Res}_{v=u}V(v)(v-u)^{-2n-1}$.

3.3. **Contact.**

Lemma 3.1 (contact)**.** *Let $V = t^{e_0}(t-1)^{e_1}(t+1)^{e_{-1}}$ with $\deg V\ge2n$, and let Q be as in* (3.2)*.*

(i) *For every $g\in\{f_1,f_2,L\}$, $R_g(t) = O(t^{-n-1})$ at ∞.*
(ii) *Let $g,h\in\{f_1,f_2,L\}$ (possibly $g=h$). Suppose that for every $c\in\{-1,0,1\}$: $e_c\ge2n$ if $c\in B_g\cap B_h$, and $e_c\ge n$ if c lies in exactly one of B_g,B_h. Then $R_{gh}(t) = O(t^{-n-1})$.*

Proof. Let Γ be a positively oriented contour around $[-1,1]$. For a target F and $j\ge0$, the coefficient of t^{-j-1} in R_F is $[t^{-1}](t^jQF) = \frac{1}{2\pi i}\oint_\Gamma t^jQF\,dt$, because t^jP_F is a polynomial. So it suffices to show $\oint_\Gamma t^jQF\,dt = 0$ for $0\le j<n$. Since F is single-valued on Γ, n integrations by parts give

$$\oint_\Gamma t^jQF\,dt = (-1)^n\oint_\Gamma S\cdot W^n\,\partial^{[n]}(t^jF)\,dt. \tag{3.4}$$

For $g\in\{f_1,f_2,L\}$ put $\rho_g = \frac{1}{n!}\partial^n(t^jg)$. Since t^jg is a polynomial of degree $j-1<n$ plus $O(1/t)$, ρ_g is a rational function, $O(t^{-n-1})$ at ∞, with poles of order $\le n$ at the points of B_g only.

(i) $W^n\rho_g$ is a polynomial, since W^n vanishes to order n at $0, \pm 1$; so (3.4) is the integral of a polynomial, which is 0.

(ii) By Leibniz' rule applied to $g \cdot (t^j h)$, and since $\partial^n t^j = 0$,

$$\partial^n(t^j gh) = g\,\partial^n(t^j h) + \sum_{i\geq 1} \binom{n}{i} \partial^i g\, \partial^{n-i}(t^j h) = g\,\partial^n(t^j h) + h\,\partial^n(t^j g) + r,$$

where $r = \sum_{i\geq 1} \binom{n}{i} \partial^i g \cdot \{\text{rational part of } \partial^{n-i}(t^j h)\}$ has poles of total order $\leq i + (n-i) = n$ at each point of $B_g \cup B_h$. Hence $W^n \partial^{[n]}(t^j gh) = \alpha g + \beta h + \gamma$ with $\alpha = W^n\rho_h$, $\beta = W^n\rho_g$ and $\gamma = W^n r/n!$ a polynomial. Also $\deg\alpha, \deg\beta \leq 3n - n - 1 = 2n - 1$; α is regular on $\{0, \pm 1\}$ and vanishes to order $\geq n$ at each $c \notin B_h$; likewise β at each $c \notin B_g$. The γ-term of (3.4) is the integral of a polynomial. For the α-term, $2n$ integrations by parts give $\oint \partial^{[2n]} V \cdot \alpha g = \oint V \cdot \partial^{[2n]}(\alpha g)$. Since $\deg\alpha < 2n$, every term of $\partial^{2n}(\alpha g) = \sum_{i<2n} \binom{2n}{i} \partial^i\alpha\, \partial^{2n-i} g$ is rational, and at $c \in B_g$ its pole order is at most $2n - \operatorname{ord}_c \alpha$, which is $\leq n$ if $c \notin B_h$ and $\leq 2n$ if $c \in B_h$. By hypothesis V vanishes to at least that order at c, so $V\partial^{[2n]}(\alpha g)$ is a polynomial and its integral is 0. The β-term is symmetric. □

Since $\theta = -f_2 L$ with $B_{f_2} \cap B_L = \{1\}$, Lemma 3.1(ii) applies to θ when $e_1 \geq 2n$ and $e_0, e_{-1} \geq n$.

3.4. **Denominators.**

Lemma 3.2 (denominators)**.** *Let $n \geq 1$, let $V \in \mathbb{Z}[t]$ have degree δ with $2n \leq \delta \leq 6n$, and put $\sigma = \delta - 2n = \deg S$ (so $\sigma \leq 4n$). Let $Q = \sum_i q_i t^i$ be as in* (3.2) *and put*

$$D^{(2)} = \prod_{p\leq 2n} p^{2\lfloor \log_p \delta \rfloor} \prod_{2n<p\leq\sigma} p, \qquad D^{(1)} = \prod_{p\leq 2n} p^{\lfloor \log_p \delta \rfloor} \prod_{2n<p\leq\sigma} p.$$

(i) (Kummer.) *If $p > 2n$ and $p > \sigma$ is prime, then $p \mid q_i$ for every $i \geq p - n$. If $2n < p \leq \sigma$, then $p \mid q_i$ for every $i \geq 2p$.*

(ii) *For every target F of quadratic type, $D^{(2)} P_F \in \mathbb{Z}[t]$; for every target of linear type, $D^{(1)} P_F \in \mathbb{Z}[t]$.*

(iii) *If $\delta = \tau n$ with τ fixed and $\sigma \geq 2n$ (that is, $\tau \geq 4$), then $\frac{1}{n} \log D^{(2)} \to \tau$ as $n \to \infty$.*

Proof. (i) Write $s_k = [t^k]S = \binom{k+2n}{2n}[t^{k+2n}]V$ for $0 \leq k \leq \sigma$, and $q_i = \binom{i+n}{n}[t^{i+n}](W^n S)$. Every index k of S that contributes to $[t^\ell](W^n S)$ satisfies $k \geq \ell - 3n$. Let $p > \max(2n, \sigma)$. If $k \geq p - 2n$, then $k, 2n < p \leq k + 2n$, so adding k and $2n$ in base p carries, and $p \mid \binom{k+2n}{2n}$ by Kummer's theorem; hence $p \mid [t^\ell](W^n S)$ for $\ell \geq p + n$, and $p \mid q_i$ for $i \geq p$. For $p - n \leq i < p$, adding i and $n < p$ carries, so $p \mid \binom{i+n}{n}$. Let $2n < p \leq \sigma$ and $i \geq 2p$. Every contributing index satisfies $2p - 2n \leq k \leq \sigma \leq 4n < 2p$, so $k = p + k_0$ with $0 \leq k_0 < p$, and $k + 2n \geq 2p$ gives $k_0 + 2n \geq p$: the lowest base-p digit carries and $p \mid s_k$. So $p \mid q_i$.

(ii) $P_F = \sum_j t^j \sum_{i>j} q_i \varphi_{i-j}$ with $i - j \le \deg Q = \delta$. Take a quadratic target. For $p \le 2n$, $v_p(\varphi_m) \ge -2\lfloor \log_p m \rfloor \ge -2\lfloor \log_p \delta \rfloor$. For $p > 2n$ we have $p^2 \ge (2n+1)^2 > 6n \ge \delta$, so $v_p(a) \le 1$ for $a \le \delta$. If $2n < p \le \sigma$, then $v_p(\varphi_m) \ge -2$ with -2 only if $p \mid a$ and $p \mid b$, hence $m \ge 2p$, $i \ge 2p$ and $p \mid q_i$ by (i); so every term has $v_p \ge -1$. If $p > \sigma$ (and $p > 2n$), then $2p > \sigma + 2n = \delta \ge m$, so at most one of a, b is divisible by p, $v_p(\varphi_m) \ge -1$, and $v_p(\varphi_m) < 0$ forces $i \ge m \ge p$, hence $p \mid q_i$ by (i): P_F is p-integral. For $p > \delta$ every φ_m with $m \le \delta$ is p-integral. The linear case is the same with one factor.

(iii) $\log D^{(2)} = 2\sum_{p \le 2n} \lfloor \log_p \delta \rfloor \log p + \sum_{2n < p \le \sigma} \log p = 2\vartheta(2n) + \vartheta(\sigma) - \vartheta(2n) + O(\sqrt{\delta} \log \delta)$ (here $\sigma \ge 2n$ is used), which is $2n + \sigma + o(n) = \delta + o(n)$ by the prime number theorem. □

So the cost of the common denominator $D^{(2)}$ is $\tau = \deg Q_n / n$ nats per index (for linear targets alone, $D^{(1)}$ costs only σ/n). Every family below has $\tau \ge 4$. For V with $\sigma = 2n$ the second product is empty and $D^{(2)} = \prod_{p \le 2n} p^{2\lfloor \log_p \delta \rfloor}$.

Remark 3.3. [Numerical] For the ratio-two forms of Section 4 the true p-exponent of the common denominator of $P_{\theta,n}$ is exactly $2\lfloor \log_p 2n \rfloor$ for every odd prime p and every $n \le 160$. For $5n/3 < p \le 2n$ the valuation -2 is attained (proved in our notes by a carry argument on the leading coefficient $q_{4n} = \binom{5n}{n}\binom{4n}{2n}$; not refereed). So we see no residue-dependent saving of Chudnovsky–Rukhadze–Hata type, and the cost 4 appears sharp for these forms. Measured profiles for the other families show the same pattern (exponent 2 up to $2n$, 1 on $(2n, \sigma]$, 0 above).

3.5. **An a priori bound.**

Lemma 3.4. *Suppose $Q_k(t) = \mathrm{CT}_{h_1,h_2}[\Lambda_t^k]$ for a Laurent polynomial Λ_t in (h_1, h_2) whose coefficients are polynomials in t, and that $R_{F,k} = O(t^{-\varsigma k - 1})$ for a target F and an integer $\varsigma \ge 1$. Let $r > 1$ and $M_r \ge \max\{|\Lambda_t(h_1,h_2)| : |t| \le r,\ |h_1| = |h_2| = 1\}$. Then for every real $z > r$ and every $k \ge 0$,*

$$|R_{F,k}(z)| \le C_{F,r}(z)\Big(\frac{r^\varsigma M_r}{z^\varsigma}\Big)^k, \qquad C_{F,r}(z) = \frac{r \max_{|t|=r} |F(t)|}{z - r}.$$

In particular $\limsup_k |R_{F,k}(z)|^{1/k} \le r^\varsigma M_r / z^\varsigma$.

Proof. The function $t^{\varsigma k} R_{F,k}(t)$ is analytic for $|t| \ge r$ and $O(1/t)$ at ∞, so by Cauchy's formula for the exterior of the circle, $z^{\varsigma k} R_{F,k}(z) = \frac{1}{2\pi i} \oint_{|t|=r} t^{\varsigma k} R_{F,k}(t) \frac{dt}{z-t}$. The part $t^{\varsigma k} P_{F,k}(t)/(z-t)$ is analytic in $|t| \le r$ and integrates to 0, so $R_{F,k}(z) = \frac{1}{2\pi i} \oint_{|t|=r} Q_k(t) F(t) (t/z)^{\varsigma k} \frac{dt}{z-t}$. Cauchy's estimate for the constant term on $|h_1| = |h_2| = 1$ gives $|Q_k(t)| \le M_r^k$ for $|t| \le r$. □

3.6. **Recurrences.** In each family, $Q_k(z)$ satisfies a linear recurrence

$$\sum_{i=0}^{\varrho} C_i(k,z)\, Q_{k+i}(z) = 0 \qquad (k \geq 0), \tag{3.5}$$

an identity in $\mathbb{Q}[z]$ for each integer $k \geq 0$, with $C_i \in \mathbb{Z}[k,z]$. We write L for the operator $(X_k) \mapsto (\sum_i C_i(k,z) X_{k+i})_k$.

Lemma 3.5 (transfer)**.** *Assume* (3.5) *holds for all $k \geq 0$, that $\deg_z C_i \leq \zeta_i$, and that $R_{F,k} = O(z^{-\varsigma k-1})$ for a target F. Put $\mu = \max_i(\zeta_i - \varsigma i)$. Then $\mathsf{L}(P_F)_k = 0$ and $\mathsf{L}(R_F)_k = 0$ in $\mathbb{Q}[z]$ for every k with $\varsigma k + 1 > \mu$. If moreover $\mathsf{L}(P_F)_k = 0$ is verified for the finitely many remaining k, then $\mathsf{L}(P_F) = \mathsf{L}(R_F) = 0$ for all $k \geq 0$, as identities in z, hence at every point z.*

Proof. As Laurent series in $1/z$, $\mathsf{L}(R_F)_k = \mathsf{L}(Q)_k F - \mathsf{L}(P_F)_k = -\mathsf{L}(P_F)_k$ is a polynomial in z, and its i-th term is $O(z^{\zeta_i - \varsigma(k+i)-1})$, so it is $O(z^{\mu-\varsigma k-1})$ and vanishes when $\mu - \varsigma k - 1 < 0$. □

The recurrences (3.5) are proved by one of two kinds of certificate.

(a) A bivariate telescoping certificate (Section 4). With Λ as in (3.3), $\vartheta_1 = h_1\partial_{h_1}$ and $\vartheta_2 = h_2\partial_{h_2}$, Laurent polynomials A, B in (h_1, h_2) over $\mathbb{Q}[n,z]$ and a nonzero $d(n) \in \mathbb{Z}[n]$ with

$$d(n) \sum_{i=0}^{\varrho} c_i(n,z)\Lambda^{i+1} = n\,\vartheta_1(\Lambda)A + \Lambda\,\vartheta_1(A) + n\,\vartheta_2(\Lambda)B + \Lambda\,\vartheta_2(B). \tag{3.6}$$

Multiplying by Λ^{n-1} ($n \geq 1$) gives $d(n)\sum_i c_i \Lambda^{n+i} = \vartheta_1(\Lambda^n A) + \vartheta_2(\Lambda^n B)$, and the constant term of ϑ_1 or ϑ_2 of any Laurent polynomial is 0. So $d(n)\sum_i c_i Q_{n+i}(z) = 0$, which is (3.5) for every $n \geq 1$ with $d(n) \neq 0$.

(b) A two-step certificate (Sections 6–9). Write $S_k(u) = \mathrm{Res}_{v=u}\,\Phi$ with $\Phi = V_k(v)(v-u)^{-2n-1}$ (here $V_k = V$ for index k), $w = v - u$ and $\nabla g = \partial_v(g\Phi)/\Phi$, so that $\mathrm{Res}_{v=u}(\nabla g)\Phi = 0$ for g meromorphic at $v = u$, and $\partial_u^j \Phi = (2n+1)_j w^{-j}\Phi$. *Step A* consists of two identities of rational functions in (k,u,v), of the form $\sum_j a_j(k,u)(2n+1)_j w^{-j} = \nabla(\cdot)$ and $\Phi_{k+1}/\Phi_k - \sum_j b_j(k,u)(2n+1)_j w^{-j} = \nabla(\cdot)$, where Φ_k is Φ at index k; taking residues at $v = u$ gives a differential equation $\sum_j a_j S_k^{(j)} = 0$ and a shift relation $S_{k+1} = \sum_j b_j S_k^{(j)}$. *Step B* uses the basis $B_{(a,\ell)} = E_k(u-z)^a S_k^{(\ell)}$, $E_k = W(u)^n(u-z)^{-n-1}$, of a module over $\mathbb{Q}(k,z)$ ($0 \leq a \leq 2$, ℓ below the order of the differential equation), and identities

$$E_{k+1}(u-z)^a S_{k+1}^{(\ell)} = \sum_{(a',\ell')} A_{(a,\ell),(a',\ell')}(k,z) B_{(a',\ell')} + \partial_u\Big(E_k \sum_l \gamma_l S_k^{(l)}\Big),$$

verified as identities of rational functions in (k,z,u) after reduction by the differential equation. Taking residues at $u = z$ gives $Y_{k+1}(z) = A(k,z)Y_k(z)$ for the vector $Y_k = (\mathrm{Res}_{u=z} B_{(a,\ell)})$, whose first entry is $Q_k(z)$. *Step C*: with the row vectors $r_i(k) = e_{(0,0)}A(k+i-1)\cdots A(k)$ we have $Q_{k+i}(z) = r_i(k)Y_k(z)$, and the scalar recurrence is certified by the identity

$\sum_i C_i(k,z)r_i(k) = 0$ in $\mathbb{Q}(k,z)^{\dim}$. The finitely many k at which a denominator of A vanishes are covered by direct exact checks of (3.5) on the explicit polynomials.

In both cases the search for the certificate (modular linear algebra, interpolation, Chinese remaindering, rational reconstruction) is not part of the proof; only the final exact identity check is.

3.7. **Pituk's theorem and the criterion.**

Theorem 3.6 (Pituk [10, Thm. 2]; restated in [4, Thm. 6.2], [5, §4])**.** *Let $x(k+\varrho)+\sum_{i<\varrho} p_i(k)x(k+i)=0$ for all large k, with $p_i(k)\to p_i\in\mathbb{C}$. Then either $x(k)=0$ for all large k, or $\limsup_k |x(k)|^{1/k}=|\lambda|$ for some root λ of $\lambda^\varrho+\sum_{i<\varrho} p_i\lambda^i$.*

No condition on p_0 or on the moduli of the roots is needed; lim sup cannot be replaced by lim [4].

Theorem 3.7 (criterion)**.** *Let $z\geq 2$ be an integer and $\mathcal{F}=\{F_1,\dots,F_s\}$ a set of targets. Suppose that for $k\geq 0$ there are polynomials $Q_k\in\mathbb{Z}[t]$ with $Q_0=1$, and that, writing $P_{j,k}=P_{F_j,k}(z)$ and $R_{j,k}=R_{F_j,k}(z)$:*

- (H1) (integrality) *there are integers $D_k\geq 1$ and $e_k\geq 0$ such that $D_kQ_k(z)$ and all $D_kP_{j,k}$ are integers divisible by 2^{e_k}, and $\frac{1}{k}(\log D_k - e_k\log 2)\to\gamma$;*
- (H2) (recurrence) *the sequences $Q_k(z)$ and $P_{j,k}$ satisfy (3.5) for all $k\geq 0$, where $C_i(k)=C_i(k,z)\in\mathbb{Q}[k]$ have degree $\leq d$ in k, C_ϱ has degree exactly d, and $C_0(k)\neq 0$, $C_\varrho(k)\neq 0$ for every integer $k\geq 0$; let $\chi(\lambda)=\sum_i [k^d]C_i\cdot\lambda^i$;*
- (H3) (a priori bound) $\limsup_k |R_{j,k}|^{1/k}\leq\beta$ *for every j;*
- (H4) (roots) *every root λ of χ satisfies $|\lambda|<\rho$ or $|\lambda|>\beta$, where $\rho\leq e^{-\gamma}$;*
- (H5) (determinant) $\det\big[Q_k(z),P_{1,k},\dots,P_{s,k}\big]_{k=0,\dots,s}\neq 0$.

Then $1,F_1(z),\dots,F_s(z)$ are linearly independent over $\mathbb{Q}$. If $s=1$ and $F_1(z)\neq 0$, (H5) may be omitted.

Proof. Suppose $a+\sum_j b_jF_j(z)=0$ with integers a,b_j. Put $X_k=\sum_j b_jR_{j,k}$. Since $R_{j,k}=Q_k(z)F_j(z)-P_{j,k}$,

$$X_k=-\Big(aQ_k(z)+\sum_j b_jP_{j,k}\Big).$$

By (H1), $J_k=D_kX_k/2^{e_k}$ is an integer. By (H2) and linearity, $\mathsf{L}(X)=0$. By (H3), $\limsup|X_k|^{1/k}\leq\beta$. Divide the recurrence by $C_\varrho(k)$; the coefficients tend to $[k^d]C_i/[k^d]C_\varrho$, so Theorem 3.6 applies with characteristic polynomial $\chi/[k^d]C_\varrho$. Either $X_k=0$ for all large k, or $\limsup|X_k|^{1/k}=|\lambda|$ for a root λ of χ with $|\lambda|\leq\beta$; by (H4) then $|\lambda|<\rho\leq e^{-\gamma}$, and $\limsup\frac{1}{k}\log|J_k|\leq\gamma+\log|\lambda|<0$, so the integers J_k, and hence X_k, vanish for all large k. In either case $X_k=0$ for $k\geq k_1$. Since $C_0(k)\neq 0$ for every $k\geq 0$, the recurrence at $k=k_1-1,\dots,0$ gives $X_k=0$ for all $k\geq 0$. Then

$aQ_k(z)+\sum_j b_jP_{j,k}=0$ for $k=0,\dots,s$, and (H5) gives $a=b_j=0$. If $s=1$, then $X_0=b_1R_{1,0}=b_1F_1(z)$ since $Q_0=1$ and $P_{1,0}=0$; so $b_1=0$ and then $a=0$. □

In (H4) we verify, in every family, two counts: χ has exactly $\varrho-1$ roots in $|\lambda|<\rho$, and exactly $\varrho-1$ roots in $|\lambda|<\beta'$ for some $\beta'>\beta$. These imply (H4): if $\rho<\beta'$ the two discs hold the same $\varrho-1$ roots and the last root has modulus $\geq\beta'>\beta$; if $\beta'\leq\rho$, the last root lies outside both discs.

3.8. **Root certificates.** Three methods are used. (a) *Rouché with positivity*: on a circle $|\lambda|=\rho$ one coefficient term of χ_z dominates the sum of the moduli of the others; for z in a half-line this is checked by positivity of the coefficients of a polynomial in $s=z-z_0$. (b) *Arb boxes*: on a bounded z-interval, ball arithmetic shows $\chi_z(\lambda)\neq 0$ on $|\lambda|=\rho$ over a finite cover of $[z_1,z_2]\times[0,2\pi)$ by boxes; since the leading coefficient of χ_z is a nonzero constant, the roots move continuously and the count is constant on the interval, and it is taken at one point by certified root isolation. (c) *Resultants*:

Lemma 3.8. *Let $\chi_z(\lambda)=\sum_{i=0}^{\varrho}a_i(z)\lambda^i$ with $a_i\in\mathbb{R}[z]$ and a_ϱ a nonzero constant, let $\rho>0$, and let $G(z)=\operatorname{Res}_\lambda\big(\chi_z(\lambda),\lambda^\varrho\chi_z(\rho^2/\lambda)\big)$. If G has no zero on an interval $I\subset\mathbb{R}$, then for $z\in I$ no root of χ_z lies on $|\lambda|=\rho$, and the number of roots in $|\lambda|<\rho$ is constant on I.*

Proof. If $\chi_z(\lambda)=0$ with $|\lambda|=\rho$ and z real, then $\chi_z(\bar\lambda)=0$ and $\bar\lambda=\rho^2/\lambda$, so λ is a common root of the two polynomials and $G(z)=0$. The roots depend continuously on z because a_ϱ is a nonzero constant. □

If all coefficients of $G(z_0+s)$ have the same sign and $G(z_0)\neq 0$, then G has no zero on $[z_0,\infty)$. For a radius μ/z we apply the lemma to $z^\varrho\chi_z(\nu/z)$, a polynomial in ν with polynomial coefficients and constant leading coefficient, on $|\nu|=\mu$.

3.9. **The five families.** Table 1 lists the parameters; the remaining sections supply the certificates. In every family the contact order is ςk with $\varsigma=1$ (index n), except the shape $(2;1,2,\frac{5}{4})$, where the natural index is $m=n/4$ and $\varsigma=4$. The "margin" is $\gamma+\log\max|\lambda_i|$ at the least admissible z, where the maximum is over the small roots; it enters only through limits.

The contact hypotheses of Lemma 3.1 hold in each case: for $\theta=-f_2L$ one needs $(e_0,e_1,e_{-1})\geq(n,2n,n)$; for L^2, $e_{\pm1}\geq 2n$; for f_2L, $(e_0,e_1,e_{-1})\geq(n,2n,n)$; for f_1^2, $e_0,e_{-1}\geq 2n$; for f_2^2, $e_0,e_1\geq 2n$; for f_1f_2, $e_0\geq 2n$ and $e_{\pm1}\geq n$. In the notation of the shape scans of our working notes, a family $(B;b_0,b_1,b_2)$ has $V=t^{b_0n}(t-1)^{b_1n}(t+1)^{b_2n}$ under ∂^{Bn}; family II is $(2;1,2,1)$ and II′ is $(2;1,2,\frac{5}{4})$.

4. The ratio-two forms: Theorems 1 and 2 for $N\geq 103$

In this section $V_1=t(t-1)^2(t+1)$, $V=V_1^n=t^n(t-1)^{2n}(t+1)^n$,

$$S_n=\partial^{[2n]}V_1^n,\qquad Q_n=\partial^{[n]}\big[W^nS_n\big]\in\mathbb{Z}[t],\qquad \deg Q_n=4n,$$

TABLE 1. The five families. Here $z = N+1$; γ is the cost per index (Lemma 3.2, minus $\log 2$ in the 2-adic case); $\rho \le e^{-\gamma}$ and β' are the certified radii of (H4).

family	section	V	$\mathcal{F}$	order	γ
II	§4	$t^n(t-1)^{2n}(t+1)^n$	$\mathcal{F}_{\mathrm{II}}$	5	4
II_2	§5	same	$\mathcal{F}_{\mathrm{II}}$	5	$4-\log 2$
II′	§6	$t^{4m}(t-1)^{8m}(t+1)^{5m}$	$\mathcal{F}_{\mathrm{II}}$	6	17
S	§7	$(t^2-1)^{2n}$	$\mathcal{F}_{\mathrm{S}}$	5	4
B	§8	$t^n(t^2-1)^{2n}$	$\mathcal{F}_{\mathrm{B}}$	6	5
C	§9	$(t^3-t)^{2n}$	$\mathcal{F}_{\mathrm{C}}$	7	6

family	ρ	β'	z-range	margin
II	e^{-4}	$4000/z$	$z \ge 104$	-0.0012
II_2	$2e^{-4}$	$4000/z$	odd $z \in [53, 103]$	-0.0313
II′	e^{-17}	$5.8 \cdot 10^{14}/z^4$	$z \in [85, 103]$	-0.0202
S	9157/500000	$4700/z$	$z \ge 59$	-0.00018
B	673/100000	$14000/z$	$z \ge 89$	-0.0027
C	2/807	$42000/z$	$z \ge 152$	-0.00105

and $Q_n(z) = \mathrm{CT}_{h_1,h_2}[\Lambda^n]$ with $\Lambda = W(z+h_1)V_1(z+h_1+h_2)/(h_1h_2^2)$ by (3.3). The targets are $\mathcal{F}_{\mathrm{II}} = \{f_1, f_2, \theta\}$, and we write $P_{F,n}$, $R_{F,n}$; $P_n = P_{\theta,n}$ and $R_n = R_{\theta,n}$.

Lemma 4.1. (i) *$R_{F,n}(t) = O(t^{-n-1})$ for $F \in \{f_1, f_2, \theta\}$.*

(ii) *Put $D_n = \prod_{p \le 2n} p^{2\lfloor \log_p 4n \rfloor}$ and $D'_n = \prod_{p \le 2n} p^{\lfloor \log_p 4n \rfloor}$, so that $D_n = D'^2_n$. Then $D_nP_{\theta,n} \in \mathbb{Z}[t]$, $D'_nP_{f_i,n} \in \mathbb{Z}[t]$ for $i = 1, 2$, and $\frac{1}{n}\log D_n \to 4$.*

(iii) *For $r > 1$, real $z > r$ and $F \in \mathcal{F}_{\mathrm{II}}$: $|R_{F,n}(z)| \le C_{F,r}(z)(K_r/z)^n$ with $K_r = r(r+1)(r+2)^3(r+3)^3$. In particular $K_r \to 3456$ as $r \to 1$, and $K_{1.01} = 3569.9 < 4000$.*

Proof. (i) is Lemma 3.1 with $(e_0, e_1, e_{-1}) = (n, 2n, n)$. (ii) is Lemma 3.2 with $\delta = 4n$, $\sigma = 2n$. (iii) is Lemma 3.4 with $\varsigma = 1$: for $|t| \le r$ and $|h_1| = |h_2| = 1$ we have $|W(t+h_1)| \le (r+1)(r+2)^2$ and $|V_1(t+h_1+h_2)| \le (r+2)(r+3)^3$. □

[Exact checks, not used.] The contact is at least $n+1$ for θ up to $n = 512$ by exact series, and exactly $n+1$ for f_1, f_2 for $n \le 80$; the denominators of (ii) were checked for $n \le 150$.

4.1. **The recurrence.**

Proposition 4.2. *There are $c_0, \dots, c_5 \in \mathbb{Z}[n, z]$, of degree 18 in n and of degrees $(6, 7, 8, 9, 10, 6)$ in z, with coefficients of at most 83 bits (file*

`ratio2/cert/rec_C.pkl`), such that

$$\sum_{i=0}^{5} c_i(n,z)Q_{n+i}(z) = 0 \qquad \text{in } \mathbb{Q}[z], \text{ for every } n \geq 0.$$

Moreover:

(i) $c_0 = -256(2n+1)(2n+3)(n+1)^2(n+2)^2\, g_0(n,z)$ *and* $c_5 = 4(2n+5)(2n+7)(n+5)^2(2n+9)^2\, g_5(n,z)$, *where* g_0, g_5 *have degree* 12 *in* n *and* 6 *in* z, *and all* 91 *coefficients of* $g_0(n,53+s)$ *and of* $g_5(n,53+s)$, *as polynomials in* (n,s), *are positive. Hence* $c_0(n,z) \neq 0$ *and* $c_5(n,z) \neq 0$ *for all real* $n \geq 0$ *and* $z \geq 53$.

(ii) $\sum_i [n^{18}]c_i \cdot \lambda^i = \kappa(z)\chi_z(\lambda)$, *where*

$$\kappa(z) = (3425z + 3136)$$
$$\times(5775425z^5 - 13980709z^4 + 11418970z^3 + 905158z^2 - 3411979z + 288463),$$

$\kappa(53+s)$ *has coefficients of one sign,* $[n^{18}]c_5 = 64\kappa(z)$, *and*

$$\chi_z(\lambda) = 64\lambda^5 - 4A_4\lambda^4 - 4A_3\lambda^3 - A_2\lambda^2 - 64(129z - 77)\lambda - 1024,$$

$$A_4 = 3125z^4 - 1250z^3 - 2575z^2 + 768z + 108,$$

$$A_3 = 8772z^3 - 10191z^2 + 1290z - 451, \qquad A_2 = 25241z^2 - 27778z + 3753.$$

Proof. The certificate. Let $d(n) = \prod_{j\in J}(6n+j)$ with $J = \{0,\dots,14\} \cup \{16,\dots,20\} \cup \{22,\dots,26\} \cup \{28,29\}$. The file `ratio2/cert/cert_exact.pkl` contains Laurent polynomials A, B in (h_1, h_2) with coefficients in $\mathbb{Q}[n,z]$ (729 nonzero columns; numerators of at most 228 bits) such that (3.6) holds with these c_i and d. The script `verify_cert.py` checks (3.6) in exact rational arithmetic, comparing all 800 Laurent monomials. The coefficients in $\mathbb{Q}[n,z]$ are compared through the Kronecker substitution $n \mapsto X$, $z \mapsto X^{64}$, which is injective here because every n-degree that occurs is at most $45 < 64$. The result is 0 mismatches. By Section 3.6(a), and since $d(n) > 0$ for $n \geq 1$, the recurrence holds for every $n \geq 1$. The case $n = 0$, and as an independent check all $0 \leq n \leq 150$, were verified as exact identities in $\mathbb{Q}[z]$ on the explicit polynomials (`recverify.py`).

The rest. The factorisations, degrees, positivity and the identity in (ii) are exact polynomial computations (`hyp_check.py`, which also verifies the hash of `rec_C.pkl`). □

Corollary 4.3. *For* $F \in \mathcal{F}_{\mathrm{II}}$ *and every* $n \geq 0$*:* $\sum_i c_i(n,z)P_{F,n+i}(z) = 0$ *and* $\sum_i c_i(n,z)R_{F,n+i}(z) = 0$.

Proof. Lemma 3.5 with $\varsigma = 1$ and $\mu = \max_i(\deg_z c_i - i) = 6$ gives the claim for $n \geq 6$. For $n \leq 40$ the identities $\sum_i c_i P_{F,n+i} = 0$ were verified exactly in $\mathbb{Q}[z]$ for $F = f_1, f_2, \theta$ (`check_forms.py`). □

Remark 4.4. [Numerical; not used.] The roots of χ_z are exactly the critical values of Λ (at $z = 5$ they agree to 15 digits), and the recurrence annihilates $R_n(z)$ to 10^{-1000} at $z = 5, 65, 129$. The dominant small root $\lambda_4(z)$ is real; at

$z = 104$ the moduli of the roots are $e^{-5.92}$, $e^{-5.58}$ (a complex pair), $e^{-4.0012}$ and $e^{23.85}$. The quantity $\log|\lambda_4(z)| + \log z$ increases from 0.6432 at $z = 104$ to 0.65477 as $z \to \infty$, the latter being $\log|\mu|$ for the relevant root μ of $12500\mu^4 + 35088\mu^3 + 25241\mu^2 + 8256\mu + 1024$. Numerically $R_{f_1,n}(104)$ and $R_{\theta,n}(104)$ follow λ_4, while $R_{f_2,n}(104)$ decays like the complex pair.

4.2. **The roots.**

Proposition 4.5. *For every real $z \geq 104$: (a) χ_z has exactly 4 roots in $|\lambda| < e^{-4}$; (b) χ_z has exactly 4 roots in $|\lambda| < 4000/z$.*

Proof. For $z \geq 104$ the polynomials A_4, A_3, A_2 and $129z-77$ are positive. (a) For $z \geq 188$ we apply Rouché's theorem with the term $4A_4\lambda^4$ on $|\lambda| = \rho = e^{-4}$: the inequality $4A_4\rho^4 > 64\rho^5+4A_3\rho^3+A_2\rho^2+64(129z-77)\rho+1024$ holds because, with rational enclosures of e^{-4}, the difference of the two sides is a polynomial in $s = z - 188$ with positive coefficients. For $z \in [104, 188]$, 6146 Arb boxes in $(z, \arg\lambda)$ (at 200 bits) show that $\chi_z \neq 0$ on $|\lambda| = e^{-4}$; since the leading coefficient 64 is constant, the number of roots in the disc is constant on $[104, 188]$, and it is 4 at $z = 188$. (b) Rouché with the term $4A_4\lambda^4$ on $|\lambda| = 4000/z$; after multiplication by a power of z the required inequality is a polynomial in $s = z - 104$ with positive coefficients (`rootcert.py`; the same check at $z = 53 + s$ is in `hyp_check.py`). □

Numerically $\log|\lambda_4(104)| = -4.00118$ and $\log|\lambda_4(103)| = -3.99163$ (Arb): $z = 104$ is the least integer for which (a) holds. At $z = 104$, $|\lambda_5| \approx 2.3 \cdot 10^{10}$.

4.3. **The determinant.**

Lemma 4.6. $\Delta(t) := \det\big[Q_n(t), P_{f_1,n}(t), P_{f_2,n}(t), P_{\theta,n}(t)\big]_{n=0,1,2,3}$ *equals*

$$-\tfrac{1}{675}\big(88288200t^6 - 200412030t^5 + 142832508t^4 + 37721844t^3 - 49389216t^2 - 2701214t + 537872\big).$$

All coefficients of $\Delta(53 + s)$ are negative; hence $\Delta(z) < 0$ for every real $z \geq 53$. The real roots of Δ are ≈ -0.479, -0.132, 0.083, 0.750.

Proof. Exact computation (`det4.py`, `det_s125.py`, two independent implementations). For instance $\Delta(104 + s) = -(109291171069707975904 + 6328382704556914018\,s + \cdots + 88288200\,s^6)/675$. □

Since row $n = 0$ is $(1, 0, 0, 0)$, Δ is the 3×3 determinant of the $P_{F,n}$, $n = 1, 2, 3$. In the λ-coordinates of Section 3.1 the determinant is the same, since $\det T = 1$.

4.4. **Proofs for $N \geq 103$.** Let $z = N+1 \geq 104$. We apply Theorem 3.7 with index n, D_n from Lemma 4.1(ii), $e_n = 0$, $\gamma = 4$; (H2) by Proposition 4.2 and Corollary 4.3 (all $\deg_n c_i = 18$, $[n^{18}]c_5 = 64\kappa(z) \neq 0$, $c_0, c_5 \neq 0$ for integers $n \geq 0$), with $\chi = \kappa(z)\chi_z$; (H3) with $\beta = K_{1.01}/z < 4000/z$ (Lemma 4.1(iii)); (H4) with $\rho = e^{-4}$ and $\beta' = 4000/z$ (Proposition 4.5).

- For Theorem 1: $\mathcal{F} = \{\theta\}$ and $\theta(z) = \theta_N > 0$, so (H5) is not needed.
- For Theorem 2: $\mathcal{F} = \mathcal{F}_{\mathrm{II}}$, the common denominator is D_n since $D'_n \mid D_n$, and (H5) is Lemma 4.6.

The margin is $4 + \log|\lambda_4(104)| = -0.0012$ per index; it enters only through limits.

Remark 4.7 (the Lean proof). The Lean 4 proof of Theorem 1 for $N \geq 103$ follows this section with the four departures listed in its README, each making the formal proof simpler. (1) Pituk's theorem is replaced by an elementary dichotomy ("cone") lemma: a solution of the recurrence that is $O(4096^n)$ a priori and not eventually 0 is $O(s^n)$ with $s = 0.01831 < e^{-4}$. (2) Rouché with Arb boxes is replaced by an exact factorisation of the characteristic polynomial as $64(\lambda - \mu)(\lambda - \nu) \cdot c(\lambda)$, with $\mu \geq 10^6$, $|\nu| \leq 0.0183$ and the roots of the cubic c inside the disc of radius 0.0183, together with Cauchy's root bound, proved for every real $z \geq 104$. (3) Lemma 4.1(iii) is used only in the crude form $|R_n(z)| \leq C \cdot 4096^n$; any rate below $|\lambda_5| \approx 195z^4$ would do. (4) The certificate (3.6), with 718,763 coefficients, is checked by the kernel: each identity is evaluated at the single point $(n, z) = (2^{280}, 2^{280 \cdot 48})$ in natural-number arithmetic, and a Kronecker-substitution lemma with proved degree and size bounds lifts this to the polynomial identity. The prime number theorem is used through a port of the PrimeNumberTheoremAnd project. A full build takes about 20 minutes with Mathlib cached. The Lean proof is in the root of the repository `https://github.com/telaviv/log-product-conics`, tag `v1`.

The four-term statement of Theorem 2 for $N \geq 103$ reuses this proof (same forms, recurrence, roots and cone lemma) and adds the contact and integrality of the forms for f_1, f_2, the base cases $n \leq 5$ and the 4×4 determinant. The proof of Theorem 5 follows Section 9 with three departures: Pituk's theorem is replaced by a growth lemma for recurrences of any order (the small roots are handled by a cascade of first differences and the large root by the cone argument above); the root certificate is replaced by a Schur–Cohn argument, valid for every real $z \geq 152$, that excludes roots with $2/807 \leq |w| < 807^6/1024$, together with the intermediate value theorem for the large root; and the order-7 recurrence is checked as an identity in n at 89 values of z, which suffices since its degree in z is at most 88. Together they add about 85 minutes of build time.

5. A 2-adic gain at odd points: even $N \geq 52$

We keep the forms of Section 4. For even N the point $z = N + 1$ is odd, and the integer forms gain a factor 2^{n-2}.

Notation. Write $t = 1 + 2w$, so that the odd integers are the values of t at $w \in \mathbb{Z}$. For $g \in \mathbb{Q}[t]$ let $\|g\| = \min_j v_2\big([w^j]g(1 + 2w)\big)$, the 2-adic Gauss valuation on the disc $1+2\mathbb{Z}_2$. Put $g[x, t] = (g(t)-g(x))/(t-x)$, let $\int_a^b g(x)\,dx$ denote the formal integral $G(b)-G(a)$ of a polynomial (G an antiderivative),

and let $\mathcal{H}$ be the harmonic transform $t^i \mapsto H_i t^i$, $H_i = \sum_{k\le i} 1/k$. Write $\Theta = W^n S_n$, so $Q_n = \partial^{[n]}\Theta$.

Elementary facts.

(G1) $\|gh\| = \|g\| + \|h\|$ (Gauss' lemma); since $\|t\| = 0$, $\|h/t\| = \|h\|$ if $t \mid h$.

(G2) $\|\partial^{[k]} g\| \ge \|g\| - k$, because $d/dt = \frac{1}{2} d/dw$ and $\frac{1}{k!}(d/dw)^k$ preserves $\mathbb{Z}_2[w]$.

(G3) $\|g\| \ge \min_i v_2([t^i]g)$, because $(1+2w)^i \in \mathbb{Z}[w]$.

(G4) $v_2(g(z)) \ge \|g\|$ for every odd integer z.

(G5) $\mathcal{H}g(t) = \int_0^t g[x,t]\,dx$. Substituting $x = 1 + 2\xi$ gives $\int_1^t g[x,t]\,dx = \sum_k \hat{g}_k H_k w^k$, where $\hat{g}(\xi) = g(1+2\xi) = \sum_k \hat{g}_k \xi^k$. Hence $\|\int_1^t g[x,t]dx\| \ge \|g\| - \lfloor \log_2 \deg g \rfloor$.

Lemma 5.1 (Lemma T)**.** *For every odd integer z and every $n \ge 1$:*

(i) $v_2(Q_n(z)) \ge 2n$;

(ii) $v_2(P_n(z)) \ge n - 1 - \lfloor \log_2 5n \rfloor - \lfloor \log_2(4n-1) \rfloor$;

(iii) *for $n \ge 2$, 2^{n-2} divides the integers $D_n Q_n(z)$ and $D_n P_n(z)$.*

Proof. Step 0 (sizes on the odd disc). $W(1+2w) = 4w(1+w)(1+2w)$ and $V_1(1+2w) = 8w^2(1+w)(1+2w)$, so $\|W^n\| = 2n$ and $\|V_1^n\| = 3n$. By (G2), $\|S_n\| \ge n$, so $\|\Theta\| \ge 3n$ and $\|Q_n\| \ge 2n$. With (G4) this is (i).

Step 1 (splitting Θ). $\Theta = UM$ with $U = (t^2-1)^n$ and $M = t^n S_n \in \mathbb{Z}[t]$, $\deg M = 3n$. Since $(UM)[x,t] = U(t)M[x,t] + M(x)U[x,t]$, (G5) gives $\mathcal{H}\Theta = U \cdot \mathcal{H}M + \mathrm{Comm}$ with $\mathrm{Comm}(t) = \int_0^t M(x)U[x,t]\,dx$. From $U(t) - U(x) = (t^2 - x^2)\sum_{a+b=n-1}(t^2-1)^a(x^2-1)^b$,

$$\mathrm{Comm} = \sum_{a+b=n-1} (t^2-1)^a \big[t\,\Omega_b(t) + \Omega_b'(t)\big],$$

$$\Omega_b(t) = \int_0^t M(x)(x^2-1)^b dx, \qquad \Omega_b'(t) = \int_0^t xM(x)(x^2-1)^b dx.$$

Step 2 (the constants vanish). Let $c_b = \Omega_b(1)$ and $c_b' = \Omega_b'(1)$. Then $c_b = 0$ if $2b < n$ and $c_b' = 0$ if $2b+1 < n$. Indeed $c = \int_0^1 \psi \cdot \partial^{[2n]} V\,dx$ with $\psi = x^{n+\varepsilon}(x^2-1)^b$, $\varepsilon \in \{0,1\}$. Integrate by parts $2n$ times. The boundary terms $\psi^{(r)} V^{(2n-1-r)}$ vanish at $x = 1$, where V has a zero of order $2n$, and at $x = 0$, where $V^{(j)}(0) = 0$ for $j < n$ and $\psi^{(r)}(0) = 0$ for $r < n$ (one of the two holds for each r). So $c = \frac{1}{(2n)!}\int_0^1 \psi^{(2n)} V\,dx$, which is 0 when $\deg\psi = n + \varepsilon + 2b < 2n$. This is where the inner factors $t^n(t-1)^{2n}$ of V are used.

Step 3 ($\mathcal{H}\Theta$ is "low degree plus small"). We show $\mathcal{H}\Theta(1+2w) = E(w) + Y(w)$ with $\deg E \le n-1$ and $\|Y\| \ge 2n - 1 - \lfloor \log_2 5n \rfloor$. The term $U \cdot \mathcal{H}M$: $U(1+2w) = 4^n w^n (1+w)^n$ and $\|\mathcal{H}M\| \ge -\lfloor \log_2 3n \rfloor$ by (G3), so $\|U \cdot \mathcal{H}M\| \ge 2n - \lfloor \log_2 3n \rfloor$. Next, $\Omega_b(1+2w) = c_b + \tilde{\Omega}_b(w)$ with $\tilde{\Omega}_b(w) = \int_1^{1+2w} M(x)(x^2-1)^b dx = 2 \cdot 4^b \int_0^w \xi^b (1+\xi)^b M(1+2\xi)\,d\xi$, whose exponents after integration

are at most $5n$; so $\|\tilde\Omega_b\| \ge 2b+1-\lfloor\log_2 5n\rfloor$, and likewise for Ω'_b with constant c'_b. Hence the non-constant parts $\sum_a (4w(1+w))^a[(1+2w)\tilde\Omega_b+\tilde\Omega'_b]$ have $\|\cdot\| \ge 2a+2b+1-\lfloor\log_2 5n\rfloor = 2n-1-\lfloor\log_2 5n\rfloor$. What remains is $E(w)=\sum_{a+b=n-1}(4w(1+w))^a[(1+2w)c_b+c'_b]$. Its a-th term has degree $\le 2a+1 = 2n-1-2b$; by Step 2 it is 0 if $2b+1<n$; if $n=2b+1$ it is $(4w(1+w))^a c'_b$, of degree $2a=n-1$; if $2b\ge n$ its degree is $\le n-1$. So $\deg E \le n-1$.

Step 4 (commuting $\mathcal{H}$ past $\partial^{[n]}$). Θ vanishes to order n at $t=0$, so

$$\mathcal{H}Q_n = \partial^{[n]}\mathcal{H}\Theta - \mathcal{D}_n\Theta,$$

$$\mathcal{D}_n\Theta = \sum_{j\ge n}\omega_j\tbinom{j}{n}(H_j-H_{j-n})t^{j-n} = \sum_{m=0}^{n-1}\frac{1}{(n-m)\binom{n}{m}}\partial^{[n-1-m]}\big[t^{-1}\partial^{[m]}\Theta\big],$$

where $\Theta=\sum_j\omega_j t^j$; the second form uses $H_j-H_{j-n}=\sum_{m<n}1/(j-m)$ and $\binom{j}{n}/(j-m)=\binom{j}{m}\binom{j-m-1}{n-1-m}/((n-m)\binom{n}{m})$. The operator $\partial^{[n]}$ kills E, so by Step 3 and (G2), $\|\partial^{[n]}\mathcal{H}\Theta\|\ge\|Y\|-n\ge n-1-\lfloor\log_2 5n\rfloor$. For each $m\le n-1$, $(n-m)\binom{n}{m}=n\binom{n-1}{m}$, with $v_2(n)\le\lfloor\log_2 n\rfloor$ and $v_2\binom{n-1}{m}\le\lfloor\log_2(n-1)\rfloor$ (Kummer: carries occur only below the top binary digit of $n-1$); $\partial^{[m]}$ and $\partial^{[n-1-m]}$ cost at most m and $n-1-m$ by (G2), and t^{-1} costs nothing by (G1). So $\|\mathcal{D}_n\Theta\|\ge 3n-(n-1)-2\lfloor\log_2 n\rfloor = 2n+1-2\lfloor\log_2 n\rfloor$, and $\|\mathcal{H}Q_n\|\ge n-1-\lfloor\log_2 5n\rfloor$.

Step 5 (the single logarithm). Let $\ell_1=\log(t/(t-1))=-f_2=\sum_{m\ge1}t^{-m}/m$. For a monomial, $[t^i\ell_1]_+=\int_0^1(t^i-x^i)/(t-x)\,dx$; hence

$$A' := [Q_n\ell_1]_+ = \int_0^1 Q_n[x,t]\,dx = \mathcal{H}Q_n - \int_1^t Q_n[x,t]\,dx.$$

By (G5) and Step 0 the last term has $\|\cdot\|\ge 2n-\lfloor\log_2 4n\rfloor$, so $\|A'\|\ge n-1-\lfloor\log_2 5n\rfloor$.

Step 6 (P_n). $\theta=\ell_1 L$ with $L=\int_{-1}^1 dy/(t-y)$. Since $Q_n\ell_1-A'=O(t^{-1})$ and $L=O(t^{-1})$, $P_n=[A'L]_+=\int_{-1}^1 A'[y,t]\,dy$. Substituting $y=1+2\eta$, $\eta\in[-1,0]$,

$$P_n(1+2w)=\sum_k\hat A'_k\sum_{c+d=k-1}\frac{(-1)^c}{c+1}w^d,\qquad c+1\le\deg A'=4n-1,$$

so $\|P_n\|\ge n-1-\lfloor\log_2 5n\rfloor-\lfloor\log_2(4n-1)\rfloor$. With (G4) this is (ii).

Step 7. $v_2(D_n)=2\lfloor\log_2 4n\rfloor\ge\lfloor\log_2 5n\rfloor+\lfloor\log_2(4n-1)\rfloor-1$, which with (i) and (ii) gives (iii). □

[Exact checks, not used.] For $n\le 160$, $\|Q_n\|=2n$ exactly and all intermediate bounds hold (`verify_lemma.py`); Steps 1–4 were checked as exact identities for $n\le 40$ (`check_steps.py`). Measured, $\|\mathcal{H}Q_n\|=n-1-\lfloor\log_2 n\rfloor$, and $\min_z v_2(P_n(z))$ over odd z is $n-O(\log n)$ (e.g. 147 at $n=160$). The variant of V without the inner factor t^n shows no gain.

Lemma 5.2 (Lemma T′)**.** *For every $n \geq 1$: (a) $\|P_{f_2,n}\| \geq n-1-\lfloor \log_2 5n \rfloor$; (b) $\|P_{L,n}\| \geq 2n - \lfloor \log_2 4n \rfloor$; (c) $\|P_{f_1,n}\| \geq n-1-\lfloor \log_2 5n \rfloor$; (d) for every odd integer z, $v_2(D_n P_{f_i,n}(z)) \geq n-2+\lfloor \log_2 4n \rfloor \geq n$ for $i=1,2$.*

Proof. (a) $P_{f_2,n} = -A'$, so this is Step 5 above. (b) As in Step 6 with Q_n in place of A': $P_{L,n} = \int_{-1}^{1} Q_n[y,t]\,dy$, so $P_{L,n}(1+2w) = \sum_k \hat{q}_k \sum_{c+d=k-1} \frac{(-1)^c}{c+1} w^d$ with $c+1 \leq 4n$ and $v_2(\hat{q}_k) \geq \|Q_n\| \geq 2n$. (c) $P_{f_1,n} = P_{L,n} + P_{f_2,n}$. (d) $v_2(D_n) = 2\lfloor \log_2 4n \rfloor$ and $\lfloor \log_2 5n \rfloor \leq \lfloor \log_2 4n \rfloor + 1$, with (G4). □

Proposition 5.3. *For every real $z \in [53, 104]$, χ_z has exactly 4 roots in $|\lambda| < 2e^{-4}$. For every real $z \geq 53$, χ_z has exactly 4 roots in $|\lambda| < 4000/z$; moreover $c_0(n,z), c_5(n,z) \neq 0$ for $n \geq 0$ and $\kappa(z) \neq 0$.*

Proof. The second sentence is Proposition 4.2(i),(ii) and the Rouché check of Proposition 4.5(b) at $z = 53+s$ (`hyp_check.py 53`). For the first: at $z = 104$ there are 4 roots in $|\lambda| < e^{-4}$ and exactly 4 in $|\lambda| < 4000/104$ (Proposition 4.5), and $e^{-4} < 2e^{-4} < 4000/104$; certified isolating balls recheck this. Then 4084 Arb boxes in $(z, \arg\lambda)$ (at 200 bits) show $\chi_z \neq 0$ on $|\lambda| = 2e^{-4}$ for all $z \in [53, 104]$, and the leading coefficient is constant (`rootcert2.py`). As a cross-check, at each odd $z \in [53, 103]$ certified isolating balls give 4 roots inside $2e^{-4}$ and one outside $4000/z$. □

The margin $4 - \log 2 + \log \max_{i \leq 4} |\lambda_i(z)|$ is -0.03130 at $z = 53$ and -0.0675 at $z = 55$, but $+0.00628$ at $z = 51$ and $+0.0453$ at $z = 49$ [Arb]: $z = 53$ is the least odd point this argument reaches.

5.1. **Proofs for even $N \in [52, 102]$.** Let $z = N+1$ be odd with $53 \leq z \leq 103$. We apply Theorem 3.7 as in Section 4, with $e_n = \max(n-2, 0)$ and $\gamma = 4 - \log 2$: (H1) holds by Lemmas 4.1(ii), 5.1 and 5.2 (the integer $D_n Q_n(z)$ and all $D_n P_{F,n}(z)$ are divisible by 2^{n-2}); (H2) by Proposition 4.2, Corollary 4.3 and Proposition 5.3; (H3) by Lemma 4.1(iii) with $r = 1.01$, which holds for each fixed $z > r$; (H4) with $\rho = 2e^{-4} = e^{-\gamma}$ and $\beta' = 4000/z$; (H5) by Lemma 4.6, since $\Delta(z) < 0$ for $z \geq 53$. Together with Section 4 this proves Theorems 1 and 2 for every even $N \geq 52$.

6. The shape $(2; 1, 2, \frac{5}{4})$: $84 \leq N \leq 102$

Let $n = 4m$, $V_m = t^{4m}(t-1)^{8m}(t+1)^{5m}$, so that V_m has exponents $(n, 2n, \frac{5}{4}n)$ at $(0, 1, -1)$, and

$$S_m = \partial^{[8m]} V_m, \qquad Q_m = \partial^{[4m]}\big[W^{4m} S_m\big] \in \mathbb{Z}[t],$$
$$\deg S_m = 9m, \qquad \deg Q_m = 17m.$$

We index by m. Then $Q_m(z) = \mathrm{CT}_{h_1,h_2}[\Lambda^m]$ with

$$\Lambda = \frac{W(z+h_1)^4 (z+h_1+h_2)^4 (z+h_1+h_2-1)^8 (z+h_1+h_2+1)^5}{h_1^4 h_2^8},$$

and $Q_m(z) = \operatorname{Res}_{u=z} \operatorname{Res}_{v=u} W(u)^{4m} V_m(v)(u-z)^{-4m-1}(v-u)^{-8m-1}$. The targets are $\mathcal{F}_{\mathrm{II}}$.

Lemma 6.1. (i) *$R_{F,m}(t) = O(t^{-4m-1})$ for $F \in \mathcal{F}_{\mathrm{II}}$.*

(ii) *Put*

$$D_m = \prod_{p \le 8m} p^{2\lfloor \log_p 17m \rfloor} \prod_{8m < p \le 9m} p, \qquad D'_m = \prod_{p \le 8m} p^{\lfloor \log_p 17m \rfloor} \prod_{8m < p \le 9m} p.$$

Then $D_m P_{\theta,m} \in \mathbb{Z}[t]$, $D'_m P_{f_i,m} \in \mathbb{Z}[t]$, $D'_m \mid D_m$, and $\frac{1}{m} \log D_m \to 17$.

(iii) *Let $M_r = ((r+1)(r+2)^2)^4 (r+2)^4 (r+3)^{13}$. For $r > 1$ and real $z > r$, $|R_{F,m}(z)| \le C_{F,r}(z)(r^4 M_r / z^4)^m$. We have $M_1 = 18^4 \cdot 3^4 \cdot 4^{13} = 5.7063 \cdot 10^{14}$; at $r = 1.001$, $r^4 M_r = 5.7824 \cdot 10^{14} < 5.8 \cdot 10^{14}$ (exact rationals), and at $r = 1.1$, $r^4 M_r = 2.075 \cdot 10^{15} < 10^{16}$.*

Proof. (i) Lemma 3.1 with $(e_0, e_1, e_{-1}) = (n, 2n, \frac{5}{4}n)$. (ii) Lemma 3.2 with $\delta = 17m = 4.25n$ and $\sigma = 9m = 2.25n$. (iii) Lemma 3.4 with $\varsigma = 4$, since $|W(t+h_1)| \le (r+1)(r+2)^2$ and $|s^4(s-1)^8(s+1)^5| \le (r+2)^4(r+3)^{13}$ for $|s| \le r+2$. □

[Exact checks.] Contact $\ge 4m$ for $m \le 40$ (exactly $4m+1$ for f_1, f_2); the denominators for $m \le 60$. At $m = 40$ the exact denominator of $P_{f_i,m}$ has $v_p = 1$ at $p = 347, 353, 359 \in (320, 360]$, so the factor on $(8m, 9m]$ is needed.

6.1. **The recurrence.** We use the two-step route of Section 3.6(b). Let $\Phi = V_m(v)(v-u)^{-8m-1}$, $w = v - u$.

Step A [`stepA.py`, `verify_stepA.py`]. (A1) $\sum_{j=0}^{3} a_j(m,u)(8m+1)_j w^{-j} = \nabla\big((v^3 - v)w^{-2}\big)$ with $a_0 = 9m$, $a_1 = (10mu + 3m - 3u)/(8m+1)$, $a_2 = -(7mu^2 - 3mu - 4m + 6u^2 - 2)/(2(4m+1)(8m+1))$, $a_3 = -W(u)/(2(4m+1)(8m+1))$; so $\sum_j a_j S_m^{(j)} = 0$. (A2) $\Phi_{m+1}/\Phi_m - \sum_{j \le 2} b_j(m,u)(8m+1)_j w^{-j} = \nabla\big((v^3 - v)\sum_{s=-7}^{7} \eta_s(m,u) w^s\big)$, where $\Phi_{m+1}/\Phi_m = v^4(v-1)^8(v+1)^5 w^{-8}$; so $S_{m+1} = \sum_{j \le 2} b_j S_m^{(j)}$. The b_j, η_s have denominator $10368\, u^2(u+1)(m+1)(2m+1)(3m+1)(3m+2)(4m+1)(4m+3)(8m+1)(8m+3)(8m+5)(8m+7)(9m+1)(9m+2)(9m+4)(9m+5)(9m+7)(9m+8)$, nonzero for $m \ge 0$. Both identities are verified in $\mathbb{Z}[m,u,v]$ after clearing denominators.

Step B [`verify_sys.py`]. Basis $E_m(u-z)^a S_m^{(\ell)}$, $a, \ell \in \{0,1,2\}$, $E_m = W(u)^{4m}(u-z)^{-4m-1}$; certificates $\gamma_l = P_l(u)/(W(u)^5(u-z)^3)$ with $\deg_u P_l \le 45$. The matrix A and the P_l are polynomials in (m,z) divided by $d(m,z) = z^4(z+1)^3 L(m)$, where

$$\begin{aligned} L(m) = {} & m(m-3)(m+1)^2(8m+1)^2(4m+1)^2(2m+1)^2(4m+3)^2 \\ & \times (3m+1)(3m+2)(8m+3)(8m+5)(8m+7) \\ & \times \prod_{j \in \{3,\dots,19\} \setminus \{17\}} (17m+j) \prod_{j \in \{1,2,4,5,7,8\}} (9m+j). \end{aligned}$$

The numerators have degree ≤ 41 in m and ≤ 29 in z, rationals of at most 183 bits, and 645,069 nonzero coefficients over 939 columns (`shape125/cert/sys_exact.pkl`). All nine identities are verified as identities of rational functions in (m, z, u), coordinate by coordinate as `fmpz_mpoly` (a complete equality test). Hence $Y_{m+1}(z) = A(m, z)Y_m(z)$ for integers $m \ge 1$, $m \ne 3$, and $z \notin \{0, -1\}$ (the zeros of $d(m, z)$); since the scalar recurrence below is a polynomial identity in z, the restriction on z is harmless.

Step C [`verify_scalar.py`, `check_small.py`].

Proposition 6.2. *There are primitive polynomials $C_0, \dots, C_6 \in \mathbb{Z}[m, z]$, of degree 172 in m and of degrees $(55, 59, 63, 67, 71, 75, 58)$ in z, with coefficients of at most 1042 bits (78,715 nonzero coefficients; file `shape125/cert/crec_exact.pkl`), such that $\sum_{i=0}^{6} C_i(m, z)Q_{m+i}(z) = 0$ for every $m \ge 0$.*

Proof. The identity $\sum_i C_i(m, z)r_i(m) = 0$ in $\mathbb{Q}(m, z)^9$ is verified exactly after clearing denominators. It uses $A(m), \dots, A(m + 5)$, so it gives the recurrence for $m \ge 4$. For $m = 0, \dots, 8$ the recurrence was verified exactly on the explicit polynomials. □

The recurrence was found from the system by exact linear algebra mod primes and reconstructed by CRT over 36 primes of 61 bits; it was never guessed from values of Q_m. At $z = 11$ there is no recurrence of order ≤ 6 and degree ≤ 154 in m [Numerical]; this is not used.

Corollary 6.3. *For $F \in \mathcal{F}_{\mathrm{II}}$ and every $m \ge 0$: $\sum_i C_i P_{F,m+i}(z) = 0 = \sum_i C_i R_{F,m+i}(z)$.*

Proof. Lemma 3.5 with $\varsigma = 4$ and $\mu = \max_i(\deg_z C_i - 4i) = 55$ gives $m \ge 14$. For $m \le 30$ the P-identities were verified exactly for f_1, f_2, θ (`check_s125.py`; for θ also $m \le 19$ by `check_small.py`). □

Characteristic polynomial, nonvanishing [`final_checks.py`, `crit.py`]. The coefficients of m^{172} in the $C_i(\cdot, z)$ are proportional to the coefficients of a sextic $\chi_6(z, \lambda)$ (file `shape125/cert/chi6.pkl`), whose roots are the critical values of Λ. Its leading coefficient and its constant term are

$$-249751218699058108948113661334913024\,(z+1)^3,$$
$$-25353012004564588029934064107520000000000.$$

At every integer $z_0 \in [85, 103]$ the proportionality is checked exactly, $[m^{172}]C_6(z_0) \ne 0$, and $C_0(m, z_0)$ and $C_6(m, z_0)$ have no integer root $m \ge 0$ (factorisation over $\mathbb{Z}$).

Proposition 6.4. *For each integer $z \in [85, 103]$, $\chi_6(z, \cdot)$ has six simple roots: five with $|\lambda| < e^{-17}$, and one, λ_6, with $|\lambda_6| > 10^{16}/z^4$ (in particular $> 5.8 \cdot 10^{14}/z^4$).*

Proof. Certified root isolation with `acb_poly` at 300 bits (`rootcert125.py`); the bound $10^{16}/z^4$ by `rootcert125_big.py`, which reads `chi6.pkl` and

checks its hash. Here $\log\inf|\lambda_6| \geq 97.4886$ (attained at $z = 85$) against $\log(10^{16}/z^4) \leq 19.07$. □

The margins $17+\log|\lambda_5(z)|$ are -0.0202 ($z = 85$), -0.0665 ($z = 86$), …, -0.7802 ($z = 103$); at $z = 84$ the margin is $+0.0266$ and the argument fails. Here $\lambda_4 = \bar{\lambda}_5$ is a complex pair and $\lambda_6 \approx e^{97}$–e^{101}.

Lemma 6.5. *$\Delta'(t) := \det[Q_m, P_{f_1,m}, P_{f_2,m}, P_{\theta,m}]_{m=0,\dots,3}$ has degree 36; its numerator is 8 times an irreducible polynomial of degree 36, and all 37 coefficients of $\Delta'(85+s)$ are positive. Hence $\Delta'(z) > 0$ for every real $z \geq 85$.*

Proof. Exact computation (`det_s125.py`). The values $\Delta'(z)$, $z = 85, \dots, 103$, are about 10^{94}–10^{97}. □

6.2. **Proofs for $84 \leq N \leq 102$.** Let $z \in [85, 103]$ be an integer. Apply Theorem 3.7 with index m, D_m of Lemma 6.1(ii) (a common denominator, since $D'_m \mid D_m$), $e_m = 0$, $\gamma = 17$; (H2) by Proposition 6.2, Corollary 6.3 and the checks above; (H3) with $\beta = r^4 M_r/z^4$, $r = 1.001$; (H4) with $\rho = e^{-17}$ and $\beta' = 5.8 \cdot 10^{14}/z^4$ (Proposition 6.4); (H5) by Lemma 6.5 for Theorem 2, and $\theta(z) > 0$ for Theorem 1. With Section 4 this proves both theorems for every $N \geq 84$.

7. The parabola: Theorem 3

Here $V_1 = (t^2-1)^2$, $V = V_1^n = (t^2-1)^{2n}$, $S_n = \partial^{[2n]} V_1^n = 4^n P_{2n}$ (P_{2n} the Legendre polynomial), $Q_n = \partial^{[n]}[W^n S_n] \in \mathbb{Z}[t]$, $\deg Q_n = 4n$, Q_n even, $Q_n(z) = \mathrm{CT}[\Lambda^n]$ with $\Lambda = W(z+h_1)V_1(z+h_1+h_2)/(h_1 h_2^2)$. The targets are $\mathcal{F}_{\mathrm{S}} = \{f_1, f_2, L^2\}$.

Lemma 7.1. *(i) $R_{F,n} = O(t^{-n-1})$ for $F \in \mathcal{F}_{\mathrm{S}}$. (ii) The number $D_n = \prod_{p \leq 2n} p^{2\lfloor \log_p 4n \rfloor}$ is a common denominator of the $P_{F,n}$, and $\frac{1}{n}\log D_n \to 4$. (iii) $\limsup |R_{F,n}(z)|^{1/n} \leq K_r/z$ with $K_r = r(r+1)(r+2)^2(r+3)^4$; $K_{1.001} = 4622.61 < 4700$.*

Proof. (i) Lemma 3.1: for L^2 we need $e_{\pm 1} \geq 2n$. (ii) Lemma 3.2 with $\delta = 4n$, $\sigma = 2n$. (iii) Lemma 3.4, with $|W(s)| \leq (r+1)(r+2)^2$ for $|s| \leq r+1$ and $|V_1(s)| \leq (r+3)^4$ for $|s| \leq r+2$. □

[Exact checks.] Contact and denominators for $n \leq 30$; at $n = 30$ the true profile is $v_p = 2$ for $11 \leq p \leq 2n$ and 0 above.

Step A. (A1) $\sum_{j \leq 2} a_j (2n+1)_j w^{-j} = \nabla((v^2-1)w^{-1})$ with $a_0 = 2n$, $a_1 = -2u/(2n+1)$, $a_2 = -(u^2-1)/(2n+1)$, which gives the Legendre equation $(1-u^2)S_n'' - 2uS_n' + 2n(2n+1)S_n = 0$. (A2) $V_1(v)w^{-2} - \sum_{j \leq 1} b_j (2n+1)_j w^{-j} = \nabla\big((v^2-1)\sum_{s=-1}^{1} \eta_s w^s\big)$ with $b_0 = 2((4n+3)u^2 - 2n - 1)/(n+1)$ and $b_1 = 2(4n+3)u(u^2-1)/((n+1)(2n+1))$, so $S_{n+1} = b_0 S_n + b_1 S_n'$ for $n \geq 0$.

Step B. A 6×6 system in the basis $E_n(u-z)^a S_n^{(\ell)}$, $a \in \{0,1,2\}$, $\ell \in \{0,1\}$; the certificates γ_l are polynomials in u of degree ≤ 8; A and the

P_l are polynomials in z of degree ≤ 7 over $d(n) = (n+1)^2(2n+1)^2(2n+3)(4n+3)(4n+5)$; 108 nonzero entries, 1491 coefficients, rationals of at most 20 bits (`s2/cert/sys_exact.pkl`). All six identities hold exactly, and $Y_{n+1} = A(n,z)Y_n$ for every $n \ge 0$.

Proposition 7.2. *There are primitive $C_0, \dots, C_5 \in \mathbb{Z}[n,z]$ (`s2/cert/rec5_exact.pkl`), even in z, of degree 17 in n and degrees $(4,4,6,6,8,4)$ in z (396 terms, at most 70 bits), with $\sum_i C_i(n,z)Q_{n+i}(z) = 0$ for all $n \ge 0$. Moreover:*

(i) *$C_0 = -4096(2n+1)(2n+5)(n+1)^2(n+2)^2 g_0$ and $C_5 = 4(2n+5)(2n+7)(n+5)^2(2n+9)^2 g_5$, with g_i of degree 11 in n and 4 in z; all 60 coefficients of $g_0(n, 59+s)$, and of $g_5(n, 59+s)$, have one sign and the constant terms are nonzero, so $C_0, C_5 \ne 0$ for real $n \ge 0$, $z \ge 59$;*

(ii) *$[n^{17}]C_i = \mathrm{COMMON}(z)a_i(z)$ with $\mathrm{COMMON} = 180500000z^4 + 9472000z^2 - 42467328$ (positive coefficients at $z = 59+s$), $a_5 = 16$, and*

$$\chi_z(\lambda) = 16\lambda^5 + (-3125z^4 + 3000z^2 - 272)\lambda^4 + (1600z^2 + 1792)\lambda^3 + (-5248z^2 - 5632)\lambda^2 + 8192\lambda - 4096.$$

Proof. $\sum_i C_i r_i = 0$ in $\mathbb{Q}(n,z)^6$ is verified exactly (`verify_rec.py`); a perturbed control fails. Since the system holds for all $n \ge 0$, so does the recurrence; it was cross-checked on explicit polynomials for $n \le 12$. (i) and (ii) are exact computations (`check_lead_trail.py`). □

By Lemma 3.5 ($\varsigma = 1$, $\mu = 4$) the P- and R-recurrences hold for $n \ge 4$; for $n \le 16$ they were checked exactly for all three targets.

Proposition 7.3. *Let $\rho = 9157/500000$; then $\log \rho = -4.0000895 < -4$. For every real $z \ge 59$, χ_z has exactly 4 roots in $|\lambda| < \rho$ and exactly 4 roots in $|\lambda| < 4700/z$.*

Proof. Lemma 3.8: $G_1(z) = \mathrm{Res}_\lambda(\chi_z(\lambda), \lambda^5\chi_z(\rho^2/\lambda))$ has degree 32, and $G_1(59+s)$ has coefficients of one sign with $G_1(59) \ne 0$; at $z = 59$ certified root isolation gives 4 roots inside and one outside. For the radius $4700/z$ the same argument applies to $z^5\chi_z(\nu/z)$ on $|\nu| = 4700$ (G_2 of degree 50) (`rootcert.py`). □

The largest real root of G_1 is 58.99460, a double root (exact real-root isolation), so the certificate has z-margin 0.0054 below 59; floating-point root finders misplace such double roots. At $z = 59$ the roots have $\log|\lambda| = -4.019467$ (pair), -4.000182 (pair) and $+21.584$; at $z = 58$ the pairs are at -3.982919 and -4.002536 and the certificate fails.

Lemma 7.4. $\det[Q_n, P_{f_1,n}, P_{f_2,n}, P_{L^2,n}]_{n=0..3} = -\frac{64}{135}z(3506685z^4 - 1742076z^2 + 54559)$; *every coefficient of the numerator at $z = 2+s$ has one sign, so it is nonzero for real $z \ge 2$ (real roots $0, \pm 0.183, \pm 0.681$).*

Proof of Theorem 3. For $z = N + 1 \geq 59$ apply Theorem 3.7 with $\gamma = 4$, $\beta = K_{1.001}/z$, $\beta' = 4700/z$ and $\rho = 9157/500000 < e^{-4}$. The margin is $4 + \log|\lambda| = -0.000182$ at $z = 59$.

8. Five numbers: Theorem 4

Here $V_1 = t(t^2-1)^2$, $V = V_1^n$, $S_n = \partial^{[2n]} V_1^n$ of degree $3n$, $Q_n = \partial^{[n]}[W^n S_n] \in \mathbb{Z}[t]$ of degree $5n$ with $Q_n(-t) = (-1)^n Q_n(t)$, and $Q_n(z) = \mathrm{CT}[\Lambda^n]$, $\Lambda = W(z+h_1)V_1(z+h_1+h_2)/(h_1 h_2^2)$. The targets are $\mathcal{F}_{\mathrm{B}} = \{f_1, f_2, L^2, f_2 L\}$.

Lemma 8.1. *(i) $R_{F,n} = O(t^{-n-1})$ for $F \in \mathcal{F}_{\mathrm{B}}$. (ii) The number*

$$D_n = \prod_{p \leq 2n} p^{2\lfloor \log_p 5n \rfloor} \prod_{2n < p \leq 3n} p$$

is a common denominator, and $\frac{1}{n}\log D_n \to 5$. (iii) $\limsup |R_{F,n}(z)|^{1/n} \leq K_r/z$ with $K_r = r(r+1)(r+2)^3(r+3)^4$; $K_{1.001} = 13872.46 < 14000$.

Proof. (i) Lemma 3.1 with $(e_0, e_1, e_{-1}) = (n, 2n, 2n)$: L^2 needs $e_{\pm 1} \geq 2n$; $f_2 L$ needs $e_1 \geq 2n$, $e_0, e_{-1} \geq n$. (ii) Lemma 3.2 with $\delta = 5n$, $\sigma = 3n$. (iii) Lemma 3.4, with $|V_1(s)| \leq (r+2)(r+3)^4$ for $|s| \leq r+2$. □

[Exact checks.] Contact and denominators for $n \leq 30$; the true profile at $n = 30$ is $v_p = 2$ for $11 \leq p \leq 2n$, 1 on $(2n, 3n]$, 0 above.

Step A. With certificate factor $W(v)$: (A1) $a_0 = 3n$, $a_1 = (4n-3)u/(2n+1)$, $a_2 = -((n+6)u^2 - n - 2)/(2(n+1)(2n+1))$, $a_3 = -W(u)/(2(n+1)(2n+1))$, from $\nabla(W(v)w^{-2})$; (A2) $S_{n+1} = \sum_{j \leq 2} b_j S_n^{(j)}$ with b_j polynomials in u over $6(n+1)(2n+1)(3n+1)(3n+2)$ (certificate $\eta_{-1}, \eta_0, \eta_1$).

Step B. A 9×9 system, $a, \ell \in \{0,1,2\}$, certificates $\gamma_l = P_l(u)/W(u)$ with $\deg P_l \leq 13$; common denominator $d(n,z) = z\, n(n-3)(n+1)^2(2n+1)(3n+1)(3n+2)(5n+3)(5n+4)(5n+6)(5n+7)$; 318 entries, 7768 coefficients, rationals of at most 32 bits (`c5b/cert/sys_exact.pkl`). All nine identities hold exactly; hence $Y_{n+1} = A(n,z)Y_n$ for integers $n \geq 1$, $n \neq 3$, and $z \neq 0$.

Proposition 8.2. *There are primitive polynomials $C_0, \ldots, C_6 \in \mathbb{Z}[n,z]$ of degree 33 in n and degrees $(8,9,10,11,12,13,8)$ in z (1394 terms, at most 147 bits; file `c5b/cert/rec6_exact.pkl`), with $\sum_i C_i Q_{n+i}(z) = 0$ for all $n \geq 0$. Moreover:*

(i) *$C_0 = -65536(2n+1)(2n+3)(2n+5)(2n+7)(n+1)^2(n+2)^2(n+3)^2 g_0$ and $C_6 = 27(2n+9)(2n+11)(3n+10)(3n+11)(3n+13)(3n+14)(3n+16)(3n+17)(n+6)^2 g_6$, with g_i of degree 23 in n and 8 in z; all 216 coefficients of $g_i(n, 89+s)$ have one sign, with nonzero constant terms;*

(ii) $[n^{33}]C_i = \mathrm{COMMON}(z)a_i(z)$ *with* $\mathrm{COMMON}(89+s)$ *one-signed,* $a_6 = 19683 = 3^9$, *and*

$$\chi_z(\lambda) = 19683\lambda^6 + (-8503056z^5 + 10156428z^3 - 2108268z)\lambda^5 + (-19295172z^4 + 22289256z^2 + 370332)\lambda^4 + (-18931456z^3 + 12089088z)\lambda^3 + (-9689088z^2 + 1714176)\lambda^2 - 2555904z\lambda - 262144.$$

Proof. $\sum_i C_i r_i = 0$ in $\mathbb{Q}(n,z)^9$ is verified exactly (`verify_rec.py`; perturbed control fails). It needs $A(n),\dots,A(n+5)$, so it gives the recurrence for $n \geq 4$ (and $z \neq 0$, hence for all z as a polynomial identity); $n = 0,\dots,3$ are covered by the exact check on explicit polynomials, done for $n \leq 12$. (i), (ii): exact computation (`check_lead_trail.py`). □

By Lemma 3.5 ($\mu = 8$) the P- and R-recurrences hold for $n \geq 8$; $n \leq 14$ was checked exactly for all four targets.

Proposition 8.3. *Let* $\rho = 673/100000$*; then* $\log\rho = -5.001180 < -5$. *For every real* $z \geq 89$, χ_z *has exactly* 5 *roots in* $|\lambda| < \rho$ *and exactly* 5 *in* $|\lambda| < 14000/z$.

Proof. Lemma 3.8 with G_1 of degree 50 and G_2 of degree 72, both one-signed at $z = 89+s$; the count 5+1 is taken at $z = 89$ with Arb (`rootcert.py`). □

The real roots of G_1 are 37.149, 87.235 (double) and 88.868 (double), so the z-margin is 0.13. At $z = 89$: $\log|\lambda| = -5.874920$, -5.021024 (pair), -5.002675 (pair), $+28.511$; at $z = 88$ the pair is at -4.991270 and the certificate fails.

Lemma 8.4. $\det[Q_n, P_{f_1,n}, P_{f_2,n}, P_{L^2,n}, P_{f_2L,n}]_{n=0..4}$ *equals*

$$\tfrac{512}{1031443875}\big(59471487952569z^8 + 79814374414551z^6 - 124281394723065z^4 + 32570821864695z^2 - 1081263804776\big),$$

and the coefficients of its numerator at $z = 2+s$ *have one sign; so it is nonzero for real* $z \geq 2$.

Proof of Theorem 4. For $z = N+1 \geq 89$ apply Theorem 3.7 with $\gamma = 5$, $\beta = K_{1.001}/z$, $\beta' = 14000/z$, $\rho = 673/100000 < e^{-5}$. The margin is $5 + \log|\lambda| = -0.0027$ at $z = 89$.

9. The full conic: Theorem 5

Here $V = W^{2n}$, $S_n = \partial^{[2n]}W^{2n}$ of degree $4n$, $Q_n = \partial^{[n]}[W^n S_n] \in \mathbb{Z}[t]$ of degree $6n$, Q_n even, and $Q_n(z) = \mathrm{CT}[\Lambda^n]$ with $\Lambda = W(z+h_1)W(z+h_1+h_2)^2/(h_1h_2^2)$. The targets are $\mathcal{F}_{\mathrm{C}} = \{f_1, f_2, f_1^2, f_1f_2, f_2^2\}$.

Lemma 9.1. *(i)* $R_{F,n} = O(t^{-n-1})$ *for* $F \in \mathcal{F}_{\mathrm{C}}$. *(ii) The number*

$$D_n = \prod_{p \leq 2n} p^{2\lfloor \log_p 6n \rfloor} \prod_{2n < p \leq 4n} p$$

is a common denominator and $\frac{1}{n}\log D_n \to 6$. *(iii)* $\limsup |R_{F,n}(z)|^{1/n} \le K_r/z$ *with* $K_r = r(r+1)(r+2)^4(r+3)^4$; $K_{1.001} = 41631.2 < 42000$.

Proof. (i) Lemma 3.1 with $e_0 = e_1 = e_{-1} = 2n$. (ii) Lemma 3.2 with $\delta = 6n$, $\sigma = 4n$. (iii) Lemma 3.4 with $|W(t+h_1)| \le (r+1)(r+2)^2$ and $|W(t+h_1+h_2)| \le (r+2)(r+3)^2$. □

[Exact checks.] Contact and denominators for $n \le 30$; measured $\log(\mathrm{lcm})/n = 5.81$ at $n = 30$.

Step A. (A1) $a_0 = 4n$, $a_1 = 3u(2n-1)/(2n+1)$, $a_2 = -(3u^2-1)/((n+1)(2n+1))$, $a_3 = -W(u)/(2(n+1)(2n+1))$, from $\nabla(W(v)w^{-2})$. (A2) $W(v)^2w^{-2} - \sum_{j\le 2} b_j(2n+1)_j w^{-j} = \nabla\big(W(v)\sum_{s=-1}^{2}\eta_s w^s\big)$ with b_j polynomials in u over $4(n+1)(2n+1)(4n+1)(4n+3)$; so $S_{n+1} = \sum_{j\le 2} b_j S_n^{(j)}$ for $n \ge 0$.

Step B. A 9×9 system with certificates $\gamma_l = P_l(u)/W(u)$, $\deg_u P_l \le 14$; A and the P_l are polynomials in z of degree ≤ 10 over $d(n) = n(n+1)^2(2n+1)^2(2n-3)(3n+2)(3n+4)(4n+1)(4n+3)(6n+5)(6n+7)$; 343 nonzero entries, 10,263 coefficients, rationals of at most 36 bits (`c6/cert/sys_exact.pkl`). All nine identities hold exactly, and $Y_{n+1} = A(n,z)Y_n$ for $n \ge 1$.

Proposition 9.2. *There are primitive polynomials* $C_0,\dots,C_7 \in \mathbb{Z}[n,z]$, *even in* z, *of degree* 47 *in* n *and degrees* $(12,12,14,14,16,16,18,12)$ *in* z *(*3120 *terms, at most* 239 *bits; file* `c6/cert/rec7_exact.pkl`*), with* $\sum_i C_i Q_{n+i}(z) = 0$ *for all* $n \ge 0$. *Moreover:*

(i) $C_0 = 4096(2n+1)(2n+3)(2n+5)(2n+7)(n+1)^2(n+2)^2(n+3)^2(n+4)^2 g_0$ *and* $C_7 = 16(2n+9)(2n+11)(4n+17)(4n+19)(4n+21)(4n+23)(4n+25)(4n+27)(n+7)^2(2n+13)^2 g_7$; *all* 468 *coefficients of* $g_0(n,152+s)$ *and of* $g_7(n,152+s)$ *are positive;*

(ii) $[n^{47}]C_i = \mathrm{COMMON}(z)a_i(z)$, $\mathrm{COMMON}(152+s)$ *has positive coefficients,* $a_7 = 65536$, *and*

$$\begin{aligned}\chi_z(\lambda) = {}& 65536\lambda^7 + (-52706752z^6 + 73144064z^4 - 22328320z^2 + 294912)\lambda^6 \\ & + (-61130832z^4 + 73777920z^2 + 495616)\lambda^5 \\ & + (12119829z^4 - 50536320z^2 + 348160)\lambda^4 + (12280320z^2 + 40960)\lambda^3 \\ & + (-1077120z^2 - 57344)\lambda^2 - 12288\lambda + 4096.\end{aligned}$$

Proof. $\sum_i C_i r_i = 0$ in $\mathbb{Q}(n,z)^9$ is verified exactly (`verify_rec7.py`; perturbed control fails); this gives $n \ge 1$, and $n = 0$ is checked on explicit polynomials (as are $n \le 6$). (i), (ii): `check_lead_trail.py`. □

By Lemma 3.5 ($\mu = \max_i(\deg_z C_i - i) = 12$) the P- and R-recurrences hold for $n \ge 12$; $n \le 18$ was checked exactly for all five targets.

Proposition 9.3. *Let* $\rho = 2/807$; *since* $e^6 < 403.5$, $\rho < e^{-6}$ *(*$\log\rho = -6.000176$*). For every real* $z \ge 152$, χ_z *has exactly* 6 *roots in* $|\lambda| < \rho$ *and exactly* 6 *in* $|\lambda| < 42000/z$.

Proof. Lemma 3.8 with G_1 of degree 72 and G_2 of degree 98, one-signed at $z = 152 + s$; the count $6 + 1$ is taken at $z = 152$ (`rootcert.py`). □

At $z = 152$: $\log|\lambda| = -7.7893$, -7.7855, -6.0144 (pair), -6.00105 (pair), $+36.833$. At $z = 151$ a pair is at -5.99440 and the certificate fails.

Lemma 9.4. *The* 6×6 *determinant* $\det[Q_n, P_{F,n}\ (F \in \mathcal{F}_{\mathrm{C}})]_{n=0..5}$ *equals* $z^2\Pi(z)/11127433917487387500$ *with* $\Pi \in \mathbb{Z}[z]$ *even of degree* 12*, whose real roots are* ± 7.266*,* ± 0.861*,* ± 0.380*. Every coefficient of the numerator at* $z = 8 + s$ *is positive, so the determinant is nonzero for real* $z \geq 8$*.*

Proof of Theorem 5. For $z = N + 1 \geq 152$ apply Theorem 3.7 with $\gamma = 6$, $\beta = K_{1.001}/z$, $\beta' = 42000/z$, $\rho = 2/807 < e^{-6}$. The margin is $6 + \log|\lambda| = -0.00105$ at $z = 152$.

10. What is computer-assisted, and how to check it

The scripts and certificate files are in the folder `certificates/` of the repository `https://github.com/telaviv/log-product-conics` (tag `v1`); paths below are relative to that folder, and `certificates/README.md` gives the command and the expected output of every verifier. The certificates were re-verified with Python 3.12.10, python-flint 0.9.0 [13] linked against FLINT 3.6.0 [12] (which contains Arb [6]), SymPy 1.14.0 and mpmath 1.3.0 on 21 November 2026. The Lean proofs were checked with Lean 4 (toolchain `v4.35.0`) and Mathlib at revision `56a005855ea1` (tag `v4.35.0`).

Frozen certificate files. Each certificate is stored in a folder `cert/` with a file `SHA256SUMS`; the verifiers read these files directly, some of them with a hash check, and `sha256sum -c SHA256SUMS` checks them all. The SHA-256 hashes are:

`ratio2/cert/rec_C.pkl` (recurrence of Proposition 4.2)

`583d093937526514ba2c8a50769df944bd663e12e78d1d179b6e0a5ebaa68c61`

`ratio2/cert/cert_exact.pkl` (certificate (3.6))

`823cc632bbe38346ccd2120a667d328339d521a72f02f01076c2a0888dbaa11b`

`shape125/cert/stepA.pkl` (Step A, Section 6)

`4d909cbf35abe4b9daaf013b218f5cba7abb7c12815c8512b62454e294bef20e`

`shape125/cert/sys_exact.pkl` (9×9 system)

`6c90b0a616d535865251ca7f699ec2cc73960a8e81c25e50c30d7dc6726680b5`

`shape125/cert/crec_exact.pkl` (Proposition 6.2)

`494bd5457562542f50aa19d985363afd925d2c2d4535f90ec0014134d4b104e6`

`shape125/cert/chi6.pkl` (χ_6)

`bf967c509b23696df477de917574c3325de029bb9d1218f66a8aba7f6d8e7332`

`s2/cert/stepA.pkl`

`10abacac554ecdfd753027fd18b45d09767f3044b4659f861593fc7136c05428`

`s2/cert/sys_exact.pkl`

`c8a912218677b38e84666ca5de2f951e59ba33b7bf5cc2dbbf2231ee6688ec5c`

`s2/cert/rec5_exact.pkl` (Proposition 7.2)

`8340cfabe9a5800dfe65bc3561d3b3c4922a9624c8fd99c1089441e2ed21d539`

`c5b/cert/stepA.pkl`

`157788a4c53e2fb78408ef820dffc0f1662bf09931e59911b8a3d043a795efe8`

`c5b/cert/sys_exact.pkl`

`907401ad86d596fe701bdd05810542e7391bddda49bd08e49a778360272e8037`

`c5b/cert/rec6_exact.pkl` (Proposition 8.2)

`15bd6f8a6ce1ea55378e42be0954748c320df17bbe6a47d2ddc950bb55ef9cfe`

`c6/cert/stepA.pkl`

`edbdaf177d68a0c9aff218692ebb03c729ec801ea4fc95f04fbab5de6603ab8a`

`c6/cert/sys_exact.pkl`

`3768720cde5051ad6ea74abe1eda978d211d26043ee0693ccc3a5685106f3126`

`c6/cert/rec7_exact.pkl` (Proposition 9.2)

`bc0572b81422682233c20632b96adb566ef8917dc243b634c46ed789df3a7a4f`

The computer-assisted steps.

step	kind	script (folder)
Certificate (3.6), Prop. 4.2	exact	`verify_cert.py` (`ratio2`)
Recurrence for $n \le 150$; base cases	exact	`recverify.py`, `check_forms.py` (`ratio2`, `fourterm`)
Factorisations, positivity, κ, Rouché at $4000/z$	exact	`hyp_check.py` (`twoadic`)
Roots, Prop. 4.5	exact + Arb	`rootcert.py` (`ratio2`)
Determinant Δ, Lemma 4.6	exact	`det4.py` (`fourterm`), `det_s125.py` (`fourterm_small_N`)
Lemma T checks; Steps 1–4	exact	`verify_lemma.py`, `check_steps.py` (`twoadic`)
Lemma T′ checks	exact	`twoadic_fi.py` (`fourterm_small_N`)
Roots on $[53, 104]$, Prop. 5.3	Arb	`rootcert2.py` (`twoadic`)
Step A, system, scalar recurrence (II′)	exact	`verify_stepA.py`, `verify_sys.py`, `verify_scalar.py`, `check_small.py` (`shape125`)
χ_6, C_0, C_6 nonvanishing (II′)	exact	`final_checks.py` (`shape125`)
Roots (II′)	Arb	`rootcert125.py`, `rootcert125_big.py` (`shape125`)
Four-term base cases, Δ' (II′)	exact	`check_s125.py`, `det_s125.py` (`fourterm_small_N`)
Step A, system, recurrence (S, B, C)	exact	`verify_stepA.py`, `verify_sys.py`, `verify_rec.py` or `verify_rec7.py`, `check_small_rec.py` (`s2`, `c5b`, `c6`)
Leading/trailing coefficients (S, B, C)	exact	`check_lead_trail.py`
Resultant root certificates (S, B, C)	exact + Arb	`rootcert.py`
Determinants (S, B, C)	exact	`nonvan_det.py`

The searches that produced the certificates (modular nullspaces, interpolation, CRT, rational reconstruction) are not part of any proof. What the computer establishes is in each case a finite statement: an identity of explicit polynomials or rational functions (compared coefficient by coefficient, so the test is complete), the signs of the coefficients of an explicit polynomial, an exact resultant, or a ball that excludes 0. What it does not establish: the hand arguments (Lemmas 3.1–3.5, 3.8, 5.1, 5.2, Theorem 3.7), Pituk's theorem, and the correctness of FLINT/Arb, which we take on trust. The referees re-ran the main verifiers on separate copies of the frozen files, and wrote independent code for the forms, contact, denominators, base cases, determinants, leading and trailing coefficients, Lemma T and the resultants; all agreed with the authors' outputs.

11. Limits and open problems

- *Where the thresholds come from.* In every family the threshold is the least z at which the largest small characteristic root drops below $e^{-\gamma}$, and γ is the proved cost, which is sharp for the odd primes in every measured case [Numerical] (Remark 3.3). For family II the needed z-specific saving $s(z) = 4 + \log \max_{i \le 4} |\lambda_i(z)|$ is [Numerical]: 0.644 ($z = 54$), 0.508 (62), 0.477 (64), 0.361 (72), 0.233 (82), 0.120 (92), 0.078 (96), 0.038 (100), 0.018 (102), -0.0012 (104). At even z (odd N) the measured point gains decay like $O(\log n)/n$ (about 0.004–0.025 nats per index at $n = 1000$–1200), so this family cannot reach odd $N < 103$; the shape $(2; 1, 2, \frac{5}{4})$ stops at $N = 84$ ($+0.0266$ at $z = 84$).
- $N = 50$. The 2-adic argument misses $z = 51$ by 0.0063 nats per index. A proved gain at an odd prime of at least 0.007 nats per index at $z = 51 = 3 \cdot 17$ would add $N = 50$; the measured 7-adic gain is about 0.03 [Heuristic].
- *A 2-adic lemma for the shape* $(2; 1, 2, \frac{5}{4})$. At odd z the measured 2-adic content of these forms is $n-5$, $n-6$, $n-7$ at $n = 64, 128, 256$. If a Lemma T analogue held for V_m, the even-N condition would become $4(4.25 - \log 2) + \log |\lambda_5| < 0$, reached from $z \approx 42$ [Heuristic, not certified]. The proof of Lemma T uses $V = V_1^n$ and has not been redone for V_m.
- *A 2-adic gain for Theorem 4.* On the forms of Section 8 the 2-adic content of the integer vector at odd z is $n + 3$ for $n \le 48$, and it is bounded at even z [Numerical]. With a gain of $\log 2$ per index, the characteristic polynomial of Section 8 would give the five-term statement for odd $z \ge 45$, i.e. for even $N \ge 44$ [Heuristic, not a theorem]. The inner factor $t^n(t-1)^{2n}$ of V_1 is the same as in Section 4, and the quadratic targets θ and L^2 have their branch points in the odd disc, so Lemma T looks adaptable; this has not been done. For Theorem 5 there is no such gain: the 2-adic content of the quadratic targets $f_1^2, f_1 f_2, f_2^2$ stays at 2 for every n tested up to 128 [Numerical]; nor is there one for the forms of Theorem 3.
- $\log 2 \cdot \log 3$ *($N = 1$, $z = 2$).* The method is far from $N = 1$. For a form with decay rate a, cost c and point gain g per index put $\varpi = a/(c - g)$; a proof needs $\varpi > 1$. Over about $1.1 \cdot 10^5$ two-level Rodrigues shapes, refined up to $n = 384$, every form tested has $\varpi \le 0.43$ [Numerical]: family II has $\varpi = 0.150$ (decay $0.5993 = -\log |\lambda_4(2)|$, cost 4), and the best shape found, $(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}, 0 \mid \frac{5}{2}; 3, 3, 3)$, has $\varpi = 0.425$ (decay 3.85, cost 9.75, 2-adic gain 0.695). Stacking every gain we know (the representation $\log 2 \cdot \log 3 = (f_1 - f_2)(f_1 - 2f_2)$ at $z = 3$, the 2-adic content, and the holonomy bound of [2, Thm. 7.0.1] with the best univalent map) gives $\varpi_{\text{eff}} \approx 0.57$, at most about 0.65 with speculative covering maps [Numerical and

Heuristic]. The residual is a factor of at least 1.75 in decay over cost, about 1.7 nats per index at cost 4. Closing it needs a new idea, not a tuning of this family.

- *Other ratios and more logarithms.* The same construction applies to $\log(1 + a/N)\log(1 + b/N)$ with one recurrence per ratio a/b. Predicted thresholds in N from the characteristic roots [Numerical]: $N \geq 356$ for $(a, b) = (2, 3)$, $N \geq 1400$ for $(1, 3)$, $N \geq 1938$ for $(3, 4)$, $N \geq 7440$ for $(1, 4)$; $N \geq 411$ for the other parabola $1, \lambda_1, \lambda_2, \lambda_1^2$ with its own forms; and about $N \geq 11622$ for $1, \lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2$ with $\lambda_3 = \log(1 + 3/N)$. We know no construction of this kind for $\lambda_1\lambda_2 \notin \mathbb{Q}\pi^2$.
- *Algebraic coefficients.* Conjecture 2.1 predicts the statement of Theorem 5 with $\overline{\mathbb{Q}}$ in place of $\mathbb{Q}$, and for every $N \geq 1$; our method gives neither.
- *Removing Pituk's theorem.* The Lean proofs of Theorems 1 and 2 for $N \geq 103$ and of Theorem 5 replace Pituk's theorem by elementary growth lemmas (Remark 4.7). The same should work for the remaining families and ranges, where the large root is equally well separated; it has not been written out. The elementary ODE alternative (continuation of $\sum_n R_n(z)x^n$ across the disc) is also only sketched in our notes.

12. Provenance and use of AI tools

Provenance. The forms of Section 9 are those of an earlier unpublished note of ours, whose remainder bound (by a Cauchy estimate) reached only $N \geq 5.3 \cdot 10^6$; the other shapes were found by scanning two-level Rodrigues shapes. The recurrences were found by modular linear algebra and then proved by the exact certificates described above. The external mathematical inputs are Pituk's theorem [10], used as stated, Baker's theorem (only in Section 2, which is not used in any proof), and the prime number theorem. Nothing in the proofs depends on unpublished work by others; [7] and [14] are cited only for comparison.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop the proofs, to write the scripts and produce the certificates, to referee the arguments, to formalise part of the result in Lean, and to draft this text. The referee reports were also produced by Claude models, working adversarially on separate copies of the files; they are not a substitute for a human expert's reading. The author directed the project and is responsible for its content. Only the irrationality of θ_N for $N \geq 103$ is verified in a proof assistant.

References

1. A. Baker and G. Wüstholz, *Logarithmic forms and diophantine geometry*, New Math. Monogr., vol. 9, Cambridge Univ. Press, 2007.
2. Frank Calegari, Vesselin Dimitrov, and Yunqing Tang, *The linear independence of* 1*,* $\zeta(2)$*, and* $L(2, \chi_{-3})$, Preprint, arXiv:2408.15403v2 [math.NT], 2024, Version 2, 16 September 2024.
3. Masayoshi Hata, *The irrationality of* $\log(1 + 1/q) \log(1 - 1/q)$, Trans. Amer. Math. Soc. **350** (1998), no. 6, 2311–2327.
4. N. Hirata-Kohno, R. Muroi, and Y. Washio, *Approximation measures for shifted logarithms of algebraic numbers via effective Poincaré–Perron*, Preprint, arXiv:2406.10820, 2024.
5. Katsuya Ishizaki and Zhi-Tao Wen, *All possible orders less than 1 of transcendental entire solutions of linear difference equations with polynomial coefficients*, Preprint, arXiv:2301.06290, 2023.
6. Fredrik Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), no. 8, 1281–1292, doi:10.1109/TC.2017.2690633.
7. Makoto Kawashima, *Linear independence of periods related to polylogarithms*, Preprint, arXiv:2605.17756v1, 2026, Unrefereed.
8. Shawn Krisman, *Products of two logarithms and arithmetic holonomy bounds*, Hexagon preprint 2611.00703v1, 2026, `https://hexagonmath.org/2611.00703v1`.
9. V. G. Lysov, *On Diophantine approximants for the product of logarithms*, Keldysh Institute Preprints, no. 158, 20 pp., 2018, In Russian. doi:10.20948/prepr-2018-158.
10. M. Pituk, *More on Poincaré's and Perron's theorems for difference equations*, J. Difference Equ. Appl. **8** (2002), no. 3, 201–216.
11. D. Roy, *Matrices whose coefficients are linear forms in logarithms*, J. Number Theory **41** (1992), 22–47.
12. The FLINT team, *FLINT: Fast Library for Number Theory*, 2026, Version 3.6.0, `https://flintlib.org`.
13. The python-flint developers, *python-flint: Python bindings for FLINT*, 2026, Version 0.9.0, `https://github.com/flintlib/python-flint`.
14. C. Viola and W. Zudilin, *Linear independence of dilogarithmic values*, J. Reine Angew. Math. **736** (2018), 193–223, Preprint version: MPIM Preprint Series 2014 (26).
15. M. Waldschmidt, *Variations on the six exponentials theorem*, Algebra and Number Theory (Hyderabad 2003) (R. Tandon, ed.), Hindustan Book Agency, 2005, pp. 338–355.