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AN EXTRIANGULATED CATEGORY WITH NO TRIANGULATED EMBEDDING

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Abstract. Over any field, we construct an essentially small hereditary extriangulated category with enough injectives that admits no triangulated embedding, that is, no fully faithful extriangulated functor into a triangulated category. In particular, not every extriangulated category is equivalent as an extriangulated category to a full extension-closed subcategory of a triangulated category. Nevertheless, every countable family of objects lies in a full extension-closed subcategory equivalent as an extriangulated category to a full extension-closed subcategory of an algebraic triangulated category. To construct this example, we build an extriangulated category from any inverse system of vector spaces with surjective transition maps over a nonempty ordinal. We prove that it admits a triangulated embedding if and only if every projection from the inverse limit is surjective. We obtain the counterexample from such an inverse system over the countable ordinals with nonzero initial term and vanishing inverse limit.

1. Introduction

Extriangulated categories were introduced by Nakaoka and Palu [NP, Definition 2.12] to provide a common framework for triangulated categories and exact categories with set-valued first extension groups [NP, Example 2.13 and Proposition 3.22(1)]. A primary source of further examples is any full extension-closed subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$. For $A, C \in \mathcal{C}$, each morphism $\delta : C \to A[1]$ completes to a distinguished triangle $A \to B \to C \overset{\delta}{\to} A[1]$ with B isomorphic to an object of $\mathcal{C}$, and assigning the equivalence class of $A \to B \to C$ to δ equips $\mathcal{C}$ with an extriangulated structure [NP, Remark 2.18]. Gorsky, Nakaoka and Palu [GNP, §1, opening paragraph] asked whether every extriangulated category arises in this way, that is, is equivalent as an extriangulated category to a full extension-closed subcategory of a triangulated category. Notice that an ambient triangulated category supplies extra structure beyond the extriangulated axioms: applying Hom to its distinguished triangles extends the Hom–$\mathbb{E}$ exact sequences to the left via negative shifts.

In this paper, we answer the question of Gorsky, Nakaoka and Palu negatively over every field. In fact, our counterexample admits no fully faithful extriangulated functor into any triangulated category, even though it is hereditary and Krull–Schmidt with enough injectives, and every countable family of its objects lies in a full extension-closed subcategory that embeds into an algebraic triangulated category.

To state the results precisely, recall that an extriangulated category $\mathcal{A}$ is equipped with a biadditive bifunctor $\mathbb{E}(C, A)$ of extensions, contravariant in C and covariant in A, together with an assignment sending each extension class $\varepsilon \in \mathbb{E}(C, A)$ to a pair $A \to B \to C$ (called a *conflation*) up to isomorphism of middle terms fixing A and C. An additive functor $F : \mathcal{A} \to \mathcal{T}$ into a triangulated category is *extriangulated* in the sense of Bennett-Tennenhaus and Shah [BS, Definition 2.32 with $n = 1$] if there exists a natural family of group homomorphisms

$$\eta_{C,A} : \mathbb{E}(C, A) \longrightarrow \mathcal{T}(FC, FA[1])$$

such that $FA \to FB \to FC \overset{\eta_{C,A}(\varepsilon)}{\longrightarrow} FA[1]$ is a distinguished triangle in $\mathcal{T}$ for every conflation $A \to B \to C$ realizing ε. We call such a pair (F, η) a *triangulated embedding* if F is fully

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faithful, and a *triangulated realization* if, in addition, every $\eta_{C,A}$ is an isomorphism. If (F, η) is a triangulated realization, then every morphism $FC \to FA[1]$ is of the form $\eta_{C,A}(\varepsilon)$ for some $\varepsilon \in \mathbb{E}(C, A)$, so the middle term of its distinguished triangle is isomorphic to FB for a conflation $A \to B \to C$. Hence the essential image of F is extension-closed in $\mathcal{T}$, and F is an extriangulated equivalence onto it with the induced structure. Conversely, the inclusion of a full extension-closed subcategory with its induced structure is a triangulated realization. A triangulated category is *algebraic* if it is triangle equivalent to the stable category of a Frobenius exact category, and a triangulated realization into an algebraic triangulated category is called an *algebraic realization*.

For an essentially small extriangulated category $\mathcal{A}$, we use the positive Yoneda extensions $\mathbb{E}^n$ defined by Gorsky, Nakaoka and Palu [GNP, Definition 3.1 and Remarks 3.2–3.3] on a small skeleton. The bifunctor $\mathbb{E}^n$ is the n-th tensor power of $\mathbb{E}$ over $\mathcal{A}$, so its elements are sums of products of n composable extension classes. By [GNP2, Proposition 2.1(1)(iii) and Definition 2.3], $\mathcal{A}$ is *hereditary* precisely when $\mathbb{E}^2 = 0$. An object $U \in \mathcal{A}$ is *injective* if every morphism $A \to U$ factors through $A \to B$ for every conflation $A \to B \to C$, or equivalently if $\mathbb{E}(-, U) = 0$ [NP, dual of Proposition 3.24]. The category $\mathcal{A}$ has *enough injectives* if every object occurs as the first term of a conflation whose middle term is injective.

Since every triangulated realization is a triangulated embedding, part (1) of our first main theorem answers the question of Gorsky, Nakaoka and Palu negatively.

Theorem A (= Theorem 4.5)**.** *Over every field k there exists an essentially small k-linear extriangulated category $\mathcal{C}$ with the following properties.*

(1) *The category $\mathcal{C}$ has no triangulated embedding.*
(2) *Every countable family of objects of $\mathcal{C}$ is contained in a full extension-closed subcategory having an algebraic realization.*
(3) *The category $\mathcal{C}$ is hereditary and Krull–Schmidt, and has enough injectives. It has $\aleph_1$ isomorphism classes of indecomposables. Each indecomposable has endomorphism ring k, and $\dim_k \mathbb{E}(C, A) \leq 1$ for all indecomposables A, C. Every morphism space has at most countable dimension over k.*

Theorem A also has consequences for enhancements by exact ∞-categories and exact dg categories. Klemenc [Kl, Theorem 1.2] showed that every small exact ∞-category embeds into its stable hull as an extension-closed subcategory. On homotopy categories, this produces a triangulated realization of the extriangulated structure constructed by Nakaoka and Palu [NP2, Theorem 4.22] (see [GNP, §5.5, before Question 5.40]); in particular, this covers small ordinary exact categories via their nerves [Kl, §4.3]. Furthermore, Chen [Ch, Proposition-Definition 3.15] characterized the essentially small extriangulated categories admitting an algebraic realization as those equivalent, as extriangulated categories, to the homotopy categories of exact dg categories. Consequently, Theorem A(1) shows that $\mathcal{C}$ is not equivalent as an extriangulated category to the homotopy category of any small exact ∞-category or exact dg category.

We obtain $\mathcal{C}$ as a member of a general family of extriangulated categories associated to inverse systems of vector spaces. Let I be a nonempty ordinal and let $D = (D_i, \pi_{ji} : D_j \to D_i)_{i \leq j \text{ in } I}$ be an inverse system of k-vector spaces with surjective transition maps. Adjoining a new least element $*$ below $0 \in I$, we consider covariant representations of the resulting poset. Write A_i and P_i for the interval representations supported on $[0, i]$ and $[*, i]$, respectively, with value k and identity maps on their supports. The inclusions $A_i \to P_i$ share the common quotient Y, supported at $*$, and for $i \leq j$ the quotient map $P_j \to P_i$ restricts to $A_j \to A_i$ and induces the identity on Y.

In Construction 2.2, we adjoin a single object X to the full additive subcategory generated by Y and the A_i, P_i so that the morphism spaces from X to the A_i form the inverse system D, while all morphisms from X to P_i or Y (and from these interval objects to X) vanish. In particular, every composite $X \to A_i \to P_i$ is zero. We equip the resulting additive category $\mathcal{C}(D)$ with the extension bifunctor pulled back from the interval representations, so that each interval sequence $A_i \to P_i \to Y$ is a conflation in $\mathcal{C}(D)$. Our second main theorem characterizes when $\mathcal{C}(D)$ embeds into a triangulated category.

Theorem B (= Theorem 3.1)**.** *For every inverse system $D = (D_i)_{i\in I}$ of vector spaces over a field, with surjective transition maps and indexed by a nonempty ordinal I, the following conditions on $\mathcal{C}(D)$ and D are equivalent.*

(1) *The category $\mathcal{C}(D)$ admits a triangulated embedding.*
(2) *The projection $\varprojlim D \to D_i$ is surjective for every $i \in I$.*
(3) *The category $\mathcal{C}(D)$ has an algebraic realization.*

Let us explain the idea behind Theorem B. Given a triangulated embedding $(F, \eta) : \mathcal{C}(D) \to \mathcal{T}$, the group $\mathcal{T}(FX, FY[-1])$ serves as the ambient group of negative first extensions from X to Y [GNP, §5.5]. Applying $\mathcal{T}(FX, -)$ to the distinguished triangle $FA_i \to FP_i \to FY \to FA_i[1]$ yields a surjection from $\mathcal{T}(FX, FY[-1])$ onto $\mathcal{T}(FX, FA_i) \simeq D_i$, because every morphism $X \to A_i$ composes to zero with $A_i \to P_i$. Since the transition maps between the interval conflations induce the identity on Y, these surjections are compatible with the transition maps π_{ji}, and therefore induce a homomorphism $\mathcal{T}(FX, FY[-1]) \to \varprojlim D$ whose composite with every projection $\varprojlim D \to D_i$ is surjective. Conversely, given any vector space W admitting compatible surjections onto all D_i, we construct in Proposition 2.3 an algebraic realization of $\mathcal{C}(D)$ inside a bounded homotopy category $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})$ in which W realizes this degree -1 morphism space.

To prove Theorem A, we apply Theorem B to the inverse system of free vector spaces on the levels of an Aronszajn tree, as studied by Dimitrić [Di, FC2, pp. 20–21]. By Bergman [Be2, p. 6, argument preceding Corollary 8] (see also Lemma 4.1), this inverse system has trivial limit, whereas the tree in Construction 4.3 has a single root and thus $D_0 = k$. Consequently, the projection $\varprojlim D \to D_0$ is not surjective, so $\mathcal{C}(D)$ admits no triangulated embedding by Theorem B. On the other hand, every countable set of countable ordinals has a countable upper bound m, and the transition maps from D_m surject onto all terms at indices $\leq m$, yielding an algebraic realization of the corresponding full extension-closed subcategory.

Theorem A(2) shows that the obstruction to a triangulated embedding involves uncountably many indecomposables. In Proposition 4.6, we show that within the family $\mathcal{C}(D)$ both the size of the index set and the dimension of the morphism spaces in Theorem A are minimal. Specifically, any counterexample of the form $\mathcal{C}(D)$ must be indexed by an ordinal of uncountable cofinality (so $|I| \geq \aleph_1$) and must have an infinite-dimensional morphism space $\mathcal{C}(D)(X, A_i) = D_i$. We do not know whether every essentially small k-linear extriangulated category with finite-dimensional morphism spaces has a triangulated realization.

Organization. In Section 2, we construct the categories $\mathcal{C}(D)$ and their subcategories $\mathcal{C}_J$ from interval representations, and we prove in Proposition 2.3 that $\mathcal{C}_J$ admits an algebraic realization whenever the spaces D_i $(i \in J)$ are compatible quotients of a single vector space. Applying this to finite subsets verifies the extriangulated axioms for $\mathcal{C}(D)$. In Section 3, we prove Theorem B. In Section 4, we first obtain a quick counterexample from the inverse system of injections of countable ordinals into $\mathbb{N}$ given by Bergman [Be2, p. 7] and later by Wofsey [Wo], and then pass to an Aronszajn tree with countable levels to prove Theorem A. Finally, Proposition 4.6 establishes the minimality of the cardinality and dimension bounds within the family $\mathcal{C}(D)$.

Conventions and notation. We fix a field k and equip every ordinal with its usual well-order. For a category $\mathcal{A}$, we write $\mathcal{A}(C, A)$ for the set of morphisms from C to A. Subcategories specified by collections of objects and their finite direct sums are understood to be closed under isomorphism. We use cochain complexes with shift convention $M[1]^n = M^{n+1}$, and write $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ for the bounded homotopy category of an additive category $\mathcal{A}$.

Use of AI. The author posed the problem of settling the triangulated realization question for extriangulated categories and directed the search for a simple, conceptual counterexample. The counterexample, embedding criterion, and proofs presented here were developed by GPT-6.2 Astra under the author's direction. The initial manuscript, citation check, and mathematical audits were carried out by GPT-6.2 Astra; independent exposition reviews were performed by Claude Opus 5.5; and the structure, exposition, and proof presentation were polished with Gemini 4 Neon. The author directed the research and revisions and is responsible for the mathematical content and the final manuscript.

2. Construction of the extriangulated category from intervals

Throughout this section, let I be a nonempty ordinal and let $D = (D_i, \pi_{ji} : D_j \to D_i)_{i \leq j \text{ in } I}$ be an inverse system of k-vector spaces with surjective transition maps. For each subset $J \subseteq I$, we construct the full subcategory $\mathcal{C}_J$ of $\mathcal{C}(D)$ generated under finite direct sums by X, Y, and the interval objects A_i, P_i with $i \in J$. We first study the underlying category of interval representations, then construct $\mathcal{C}_J$ and realize it by bounded complexes whenever a single vector space maps compatibly onto all D_i ($i \in J$), and finally use finite subsets J to verify the extriangulated category axioms for $\mathcal{C}(D)$.

Since I is a nonempty ordinal, it has a least element 0. We adjoin a new least element $*$ below 0 and write I_* for the resulting well-ordered set. A representation M of I_* is a covariant functor from I_* to the category of k-vector spaces, assigning a k-vector space $M(t)$ to each $t \in I_*$ and a linear structure map $M(s \leq t) : M(s) \to M(t)$ to each pair $s \leq t$ in I_*. These representations form an abelian category $\mathcal{R}$ in which exactness is checked pointwise.

For an interval $U \subseteq I_*$, let $k_U \in \mathcal{R}$ be the interval representation with value k on U and 0 elsewhere, and with identity structure maps between points of U. We work with the intervals

$$Y = k_{\{*\}}, \qquad A_i = k_{[0,i]}, \qquad P_i = k_{[*,i]} \qquad (i \in I).$$

Write $b_i : A_i \to P_i$ for the canonical inclusion and $c_i : P_i \to Y$ for the canonical quotient. For $i \leq j$ in I, there are canonical quotient maps $a_{ji} : A_j \to A_i$ and $p_{ji} : P_j \to P_i$, which fit into a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_j & \xrightarrow{b_j} & P_j & \xrightarrow{c_j} & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a_{ji}} & & \downarrow{\scriptstyle p_{ji}} & & \| & & \\ 0 & \longrightarrow & A_i & \xrightarrow[b_i]{} & P_i & \xrightarrow[c_i]{} & Y & \longrightarrow & 0. \end{array} \tag{2.1}$$

For $J \subseteq I$, let $\mathcal{B}_J$ be the full subcategory of $\mathcal{R}$ consisting of finite direct sums of Y and the A_i, P_i with $i \in J$. We begin by showing that each $\mathcal{B}_J$ is extension-closed in $\mathcal{R}$ and has enough injectives.

Lemma 2.1. *For every $J \subseteq I$, the subcategory $\mathcal{B}_J$ is extension-closed in $\mathcal{R}$. With its induced exact structure, it has enough injectives. The objects P_i and Y are injective in $\mathcal{R}$, and $0 \to A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y \to 0$ is an injective resolution for each $i \in J$.*

Proof. Botnan and Crawley-Boevey [BCB, Lemma 2.1 and Theorem 1.2] prove injectivity for interval representations supported on directed ideals and interval decomposition for pointwise finite-dimensional representations of an arbitrary total order. For the principal ideals needed here, we give a direct two-step argument: evaluation yields injectivity, and splitting on I determines the middle term of any extension in $\mathcal{B}_J$.

Step 1: Injective intervals. For any $t \in I_*$ and $M \in \mathcal{R}$, evaluation at t yields a natural isomorphism

$$\mathcal{R}(M, k_{[*,t]}) \simeq \mathrm{Hom}_k(M(t), k).$$

Indeed, a morphism $\varphi : M \to k_{[*,t]}$ vanishes at every point strictly above t, and for $s \leq t$ its component $\varphi_s : M(s) \to k$ is uniquely determined by $\varphi_t : M(t) \to k$ via $\varphi_s = \varphi_t \circ M(s \leq t)$; conversely, every linear functional on $M(t)$ defines a morphism in this way. Since evaluation at t and vector space duality are exact, $k_{[*,t]}$ is injective in $\mathcal{R}$. Taking $t = i$ and $t = *$ shows that P_i and Y are injective in $\mathcal{R}$. The same argument applied to I in place of I_* shows that $k_{[0,i]}$ is injective in the category of representations of I.

Step 2: Closure of $\mathcal{B}_J$ under extensions. Consider a short exact sequence in $\mathcal{R}$ with outer terms in $\mathcal{B}_J$ and middle term M. On I, we have $Y|_I = 0$ and $A_i|_I = P_i|_I = k_{[0,i]}$. Hence the restriction of the first term to I is a finite direct sum of the injective representations $k_{[0,i]}$, so the restricted short exact sequence splits. It follows that $M|_I$ is a finite direct sum of intervals $k_{[0,i]}$ whose endpoints belong to the finite subset of J occurring in the outer terms. List the distinct endpoints occurring in $M|_I$ in increasing order as $i_1 < \cdots < i_r$.

If $r = 0$, then $M|_I = 0$, so M is supported at $*$ and hence $M \simeq Y^{\dim_k M(*)} \in \mathcal{B}_J$ (as $M(*)$ is finite-dimensional). Assume now that $r \geq 1$, and write $V = M(0)$ and $d = M(* \leq 0) : M(*) \to V$. Define a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ by setting

$$V_0 = 0, \qquad V_s = \ker\big(M(0 \leq i_{s+1}) : V \to M(i_{s+1})\big) \ \ (0 < s < r), \qquad V_r = V.$$

Under the decomposition of $M|_I$ into intervals $k_{[0,i_u]}$, the subspace V_s is the direct sum of the values at 0 of the summands with $u \leq s$, since $k_{[0,i_u]}(t)$ equals k for $t \leq i_u$ and 0 for $t > i_u$. Consequently, for every $t \in I$, if s is the number of indices $u \in \{1, \dots, r\}$ with $i_u < t$, then the structure map $M(0 \leq t) : V \to M(t)$ is surjective with kernel V_s, inducing an isomorphism $V/V_s \simeq M(t)$ under which the structure maps on I correspond to the canonical quotient maps $V/V_s \to V/V_{s'}$.

Set $W_s = \operatorname{im} d \cap V_s$ for $0 \leq s \leq r$, and put $m_s = \dim_k(W_s/W_{s-1})$ and $n_s = \dim_k(V_s/V_{s-1}) - m_s$ for $1 \leq s \leq r$. For each $s = 1, \dots, r$, we choose vectors $v_{s,1}, \dots, v_{s,m_s} \in W_s$ whose cosets form a basis of W_s/W_{s-1}. Because $W_s \cap V_{s-1} = W_{s-1}$, their cosets in V_s/V_{s-1} are linearly independent, so we may extend them to a basis of V_s/V_{s-1} by adjoining the cosets of vectors $w_{s,1}, \dots, w_{s,n_s} \in V_s$. Together, all $v_{s,\alpha}$ and $w_{s,\beta}$ form a basis of V adapted to the flag $V_0 \subset \cdots \subset V_r$. For each $v_{s,\alpha} \in \operatorname{im} d$, choose a preimage $\tilde{v}_{s,\alpha} \in M(*)$ with $d(\tilde{v}_{s,\alpha}) = v_{s,\alpha}$, and complete these lifts to a basis of $M(*)$ by adjoining a basis of $\ker d$. Each basis vector of $\ker d$ generates a subrepresentation isomorphic to Y; each $\tilde{v}_{s,\alpha}$ generates a subrepresentation isomorphic to P_{i_s} (since $v_{s,\alpha} \in V_s \setminus V_{s-1}$); and each $w_{s,\beta}$ (with zero component at $*$) generates a subrepresentation isomorphic to A_{i_s}. Since these subrepresentations span $M(t)$ linearly independently at every $t \in I_*$ and $M(*)$ is finite-dimensional, we obtain

$$M \simeq Y^{\dim_k \ker d} \oplus \bigoplus_{s=1}^{r} \big(P_{i_s}^{m_s} \oplus A_{i_s}^{n_s}\big).$$

Thus $M \in \mathcal{B}_J$, proving that $\mathcal{B}_J$ is extension-closed in $\mathcal{R}$. Finally, since P_i and Y are injective in $\mathcal{R}$ (hence in $\mathcal{B}_J$), the sequences $0 \to A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y \to 0$ are injective resolutions in $\mathcal{B}_J$; taking finite direct sums with the identity maps on P_i and Y shows that $\mathcal{B}_J$ has enough injectives. □

Before adjoining X to $\mathcal{B}_J$, we record a basis for each morphism space between the indecomposable objects A_j, P_j, Y of $\mathcal{B}_J$. In the table below, each nonzero entry spans the corresponding morphism space $\mathcal{B}_J(\text{source}, \text{target})$, and an entry marked with the condition $i \leq j$ is understood to be 0 when $i > j$:

source \ target	A_i	P_i	Y
A_j	a_{ji} $(i \leq j)$	$b_i a_{ji}$ $(i \leq j)$	0
P_j	0	p_{ji} $(i \leq j)$	c_j
Y	0	0	1_Y

(2.2)

Indeed, by Step 1 of the proof of Lemma 2.1, we have $\mathcal{R}(M, P_i) \simeq \operatorname{Hom}_k(M(i), k)$ and $\mathcal{R}(M, Y) \simeq \operatorname{Hom}_k(M(*), k)$ via evaluation at i and $*$, respectively. In particular, $\mathcal{R}(A_j, P_i)$ is one-dimensional with basis $b_i a_{ji}$ when $i \leq j$ (since $b_i a_{ji}$ acts as the identity at i) and vanishes when $i > j$; the remaining entries in the columns for P_i and Y follow in the same way. Finally, because $0 \to A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y \to 0$ is exact, morphisms $M \to A_i$ correspond to morphisms $M \to P_i$ whose composite with c_i vanishes, yielding the first column of (2.2).

Construction 2.2. Let $D = (D_i, \pi_{ji} : D_j \to D_i)_{i \leq j \text{ in } I}$ be an inverse system of k-vector spaces with surjective transition maps. For $J \subseteq I$, we form a k-linear additive category $\mathcal{C}_J$ by adjoining an object X to $\mathcal{B}_J$ and taking finite direct sums, with morphism spaces

$$\mathcal{C}_J(X, X) = k, \qquad \mathcal{C}_J(B, X) = 0, \qquad \mathcal{C}_J(X, A_i) = D_i, \qquad \mathcal{C}_J(X, P_i) = 0, \qquad \mathcal{C}_J(X, Y) = 0 \tag{2.3}$$

for all $B \in \mathcal{B}_J$ and $i \in J$. For $i \leq j$ in J, we define postcomposition with $a_{ji} : A_j \to A_i$ on $\mathcal{C}_J(X, A_j) = D_j$ to be $\pi_{ji} : D_j \to D_i$. Together with (2.2), k-bilinearity, and the action of $\mathcal{C}_J(X, X) = k$ by scalar multiplication, this determines composition on all objects of $\mathcal{C}_J$. To check associativity on morphisms out of X, we only need to verify composites $X \to A_j \to A_i \to A_h$

with $h \le i \le j$, since every morphism from X to P_i or Y is zero; on the generators a_{ji} and a_{ih}, associativity is precisely the identity $\pi_{ih}\pi_{ji} = \pi_{jh}$.

Thus the underlying additive category of $\mathcal{C}_J$ is the additive closure of the one-point extension of $\mathcal{B}_J$ by the covariant module $B \mapsto \mathcal{C}_J(X, B)$, in the sense of Herscovich and Solotar [HeS, §3]. We write $\mathcal{C}(D) = \mathcal{C}_I$. Every object of $\mathcal{C}_J$ has the form $B \oplus X^a$ with $B \in \mathcal{B}_J$ and $a \ge 0$. Sending $B \oplus X^a$ to B and each morphism to its component between the $\mathcal{B}_J$-summands defines a k-linear functor $\rho : \mathcal{C}_J \to \mathcal{B}_J$ with $\rho X = 0$; it respects composition because $\mathcal{C}_J(B, X) = 0$ for all $B \in \mathcal{B}_J$. We define a biadditive bifunctor $\mathbb{E}_J$ on $\mathcal{C}_J$ by

$$\mathbb{E}_J(C, A) = \operatorname{Ext}^1_{\mathcal{B}_J}(\rho C, \rho A)$$

on objects and morphisms, and write $\mathbb{E} = \mathbb{E}_I$ for $\mathcal{C}(D)$. In particular, every morphism $u : X \to A_i$ acts as zero on $\mathbb{E}_J$ because $\rho u = 0$. For objects $A = \rho A \oplus X^a$ and $C = \rho C \oplus X^c$ of $\mathcal{C}_J$, let a class $\xi \in \mathbb{E}_J(C, A)$ be represented by a short exact sequence $0 \to \rho A \to M \to \rho C \to 0$ in $\mathcal{B}_J$. We assign to ξ the equivalence class (up to isomorphism of middle terms fixing A and C) of the direct sum of $\rho A \to M \to \rho C$ and the split sequence $X^a \to X^a \oplus X^c \to X^c$.

To realize the morphisms $X \to A_i$ as chain maps modulo homotopy, we realize the spaces D_i ($i \in J$) as compatible quotients of a single vector space.

Proposition 2.3. *Suppose that a vector space W admits surjections $q_i : W \to D_i$ for $i \in J$ satisfying $\pi_{ji}q_j = q_i$ for all $i \le j$ in J. Then the data of Construction 2.2 equip $\mathcal{C}_J$ with an extriangulated structure admitting an algebraic realization $(F, \eta) : \mathcal{C}_J \to \mathsf{K}^{\mathrm{b}}(\mathcal{Q})$ for some additive category $\mathcal{Q}$.*

Proof. We divide the proof into three steps: first we realize $\mathcal{B}_J$ by bounded complexes of injectives, next we enlarge the category of terms to realize X and construct a fully faithful functor $F : \mathcal{C}_J \to \mathsf{K}^{\mathrm{b}}(\mathcal{Q})$, and finally we identify the extension groups and verify the algebraic realization.

Step 1: Realizing $\mathcal{B}_J$ in $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)$. Let $\mathcal{Q}_0$ be the full subcategory of $\mathcal{R}$ consisting of finite direct sums of Y and the P_i ($i \in J$). By Lemma 2.1, every object of $\mathcal{Q}_0$ is injective in $\mathcal{R}$. We represent Y and each P_i as stalk complexes concentrated in degree 0, and each A_i by the two-term complex

$$H_i = (P_i \xrightarrow{c_i} Y) \qquad \text{in degrees } 0, 1.$$

The inclusion of the stalk A_i into H_i is a quasi-isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathcal{R})$. Extending additively over finite direct sums assigns to each $B \in \mathcal{B}_J$ a complex $H_B \in \mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)$ together with a quasi-isomorphism $B \to H_B$. By [Stacks, Tag 05TG], morphisms in $\mathsf{D}^{\mathrm{b}}(\mathcal{R})$ into a bounded complex of injectives are computed modulo chain homotopy, so the canonical functor $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0) \to \mathsf{D}^{\mathrm{b}}(\mathcal{R})$ is fully faithful. Because $\mathcal{B}_J \hookrightarrow \mathcal{R} \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathcal{R})$ is also fully faithful, $B \mapsto H_B$ defines a fully faithful k-linear functor $\mathcal{B}_J \to \mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)$. Moreover, for $B, C \in \mathcal{B}_J$ we obtain natural isomorphisms

$$\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)(H_C, H_B[1]) \;\simeq\; \mathsf{D}^{\mathrm{b}}(\mathcal{R})(C, B[1]) \;\simeq\; \operatorname{Ext}^1_{\mathcal{R}}(C, B) \;\simeq\; \operatorname{Ext}^1_{\mathcal{B}_J}(C, B),$$

where the last isomorphism holds because $\mathcal{B}_J$ is extension-closed in $\mathcal{R}$ by Lemma 2.1. Given a short exact sequence $0 \to B \to M \to C \to 0$ in $\mathcal{B}_J$, let $H_C \to H_B[1]$ be the morphism in $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)$ corresponding to its class under this isomorphism, and complete it to a distinguished triangle $H_B \to E \to H_C \to H_B[1]$ in $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)$. In $\mathsf{D}^{\mathrm{b}}(\mathcal{R})$, this triangle is isomorphic to the distinguished triangle $B \to M \to C \to B[1]$ by an isomorphism restricting to the chosen quasi-isomorphisms on B and C. Since $M \in \mathcal{B}_J$, full faithfulness of $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0) \to \mathsf{D}^{\mathrm{b}}(\mathcal{R})$ yields an isomorphism $E \simeq H_M$ in $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0)$ identifying the first two arrows of the triangle with the images of $B \to M \to C$.

Step 2: Adjoining L and defining F. Set $K_i = \ker q_i$ for each $i \in J$. To represent morphisms $X \to A_i$ by homotopy classes in $W/K_i \simeq D_i$, we arrange that W is the space of chain maps and K_i its subspace of null-homotopic maps. As in Construction 2.2, let $\mathcal{Q}$ be the additive closure of the one-point extension obtained by adjoining a new object L to $\mathcal{Q}_0$, with

$$\mathcal{Q}(L, L) = k, \qquad \mathcal{Q}(Q, L) = 0, \qquad \mathcal{Q}(L, P_i) = K_i, \qquad \mathcal{Q}(L, Y) = W$$

for all $Q \in \mathcal{Q}_0$ and $i \in J$. For $i \le j$ in J, the identity $q_i = \pi_{ji}q_j$ implies $K_j \subseteq K_i$. We define postcomposition of morphisms out of L with $p_{ji} : P_j \to P_i$ and $c_i : P_i \to Y$ to be the inclusions $K_j \hookrightarrow K_i$ and $K_i \hookrightarrow W$, respectively. By (2.2), the composition relations among generators of $\mathcal{Q}_0$

are $c_i p_{ji} = c_j$ and $p_{ih}p_{ji} = p_{jh}$ ($h \le i \le j$ in J), which act on K_j by the inclusions $K_j \hookrightarrow W$ and $K_j \hookrightarrow K_h$, so composition is associative. Since chain maps and homotopies between complexes with terms in $\mathcal{Q}_0$ are unchanged in $\mathcal{Q}$, the inclusion $\mathsf{K}^{\mathrm{b}}(\mathcal{Q}_0) \hookrightarrow \mathsf{K}^{\mathrm{b}}(\mathcal{Q})$ is fully faithful.

We represent X by the stalk complex $L[-1]$, concentrated in degree 1. Since $L[-1]$ and $Y[-1]$ are stalks in degree 1, we have $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], Y[-1]) = \mathcal{Q}(L, Y) = W$. Similarly, a chain map $L[-1] \to H_i$ is given simply by a morphism $L \to Y$ in degree 1, i.e., an element of W, and it is null-homotopic precisely when it equals $c_i h$ for some $h \in \mathcal{Q}(L, P_i) = K_i$. Hence q_i induces an isomorphism

$$\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], H_i) = W/K_i \xrightarrow[\sim]{\bar{q}_i} D_i. \tag{2.4}$$

We define F on $\mathcal{C}_J(X, A_i) = D_i$ to be $\bar{q}_i^{-1}$, sending $u \in D_i$ to the homotopy class of any $w \in W$ with $q_i(w) = u$. By (2.1), the chain map $H_j \to H_i$ representing a_{ji} has components $(p_{ji}, 1_Y)$, so postcomposition with it on (2.4) is the canonical projection $W/K_j \to W/K_i$, which corresponds under $\bar{q}_j$ and $\bar{q}_i$ to π_{ji} because $\pi_{ji} q_j = q_i$. Moreover, $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], P_i) = 0$ and $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], Y) = 0$ for degree reasons; $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(H_B, L[-1]) = 0$ for every $B \in \mathcal{B}_J$ because $\mathcal{Q}(Q, L) = 0$ for all $Q \in \mathcal{Q}_0$; and $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], L[-1]) = \mathcal{Q}(L, L) = k$. Extending $B \mapsto H_B$ ($B \in \mathcal{B}_J$) and $X \mapsto L[-1]$ additively therefore defines a fully faithful k-linear functor $F : \mathcal{C}_J \to \mathsf{K}^{\mathrm{b}}(\mathcal{Q})$.

Step 3: Identifying extensions and verifying the realization. For every $B \in \mathcal{B}_J$, the morphism spaces

$$\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], H_B[1]), \qquad \mathsf{K}^{\mathrm{b}}(\mathcal{Q})(H_B, L), \qquad \mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], L)$$

all vanish: $H_B[1]$ is concentrated in degrees -1 and 0 (whereas $L[-1]$ is in degree 1), the terms of H_B admit no nonzero morphisms to L, and $L[-1]$ and L are stalks in distinct degrees. Consequently, for any $C, A \in \mathcal{C}_J$, every morphism $FC \to FA[1]$ is uniquely determined by its component $H_{\rho C} \to H_{\rho A}[1]$. Combining this with the isomorphisms of Step 1 gives a group isomorphism

$$\eta_{C,A} : \mathbb{E}_J(C, A) = \mathrm{Ext}^1_{\mathcal{B}_J}(\rho C, \rho A) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathcal{Q})(FC, FA[1]).$$

For morphisms $f : C' \to C$, $g : A \to A'$, and any $\xi \in \mathbb{E}_J(C, A)$, we have

$$(Fg)[1]\, \eta_{C,A}(\xi)\, (Ff) = \eta_{C',A'}\big((\rho g)_*(\rho f)^*\xi\big).$$

Indeed, writing $A = \rho A \oplus X^a$, $C = \rho C \oplus X^c$, and similarly for A' and C', the morphism $\eta_{C,A}(\xi)$ has only its $(H_{\rho C}, H_{\rho A}[1])$-component nonzero. Since $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(L[-1], H_{\rho A}[1]) = 0$, the component of $\eta_{C,A}(\xi)(Ff)$ out of $F(X^{c'})$ vanishes; similarly, $\eta_{C,A}(\xi)$ factors through $H_{\rho A}[1]$, and since $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})(H_{\rho A}[1], L) = 0$ (as $\mathcal{Q}(Q, L) = 0$ for all $Q \in \mathcal{Q}_0$), the component of $(Fg)[1]\eta_{C,A}(\xi)$ into $F(X^{a'})[1]$ vanishes as well. On the remaining component $H_{\rho C'} \to H_{\rho A'}[1]$, the identity follows from the naturality of the isomorphisms in Step 1.

Because each $\eta_{C,A}$ is surjective, every morphism $FC \to FA[1]$ equals $\eta_{C,A}(\xi)$ for some $\xi \in \mathbb{E}_J(C, A)$. Represent ξ by a short exact sequence $0 \to \rho A \to M \to \rho C \to 0$ in $\mathcal{B}_J$. By Step 1, the image $H_{\rho A} \to H_M \to H_{\rho C}$ completes to a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})$ with connecting morphism corresponding to ξ. Taking the direct sum with the split distinguished triangle $F(X^a) \to F(X^a \oplus X^c) \to F(X^c) \xrightarrow{0} F(X^a)[1]$ produces a distinguished triangle with connecting morphism $\eta_{C,A}(\xi)$ whose first two arrows are the image under F of the pair assigned to ξ in Construction 2.2. Since every distinguished triangle with connecting morphism $\eta_{C,A}(\xi)$ is isomorphic to this one fixing FA and FC, the essential image of F is extension-closed in $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})$, and η identifies its induced extriangulated structure [NP, Remark 2.18] with that of Construction 2.2.

Finally, with the degreewise split exact structure, the category of bounded complexes over $\mathcal{Q}$ is a Frobenius exact category whose projective-injective objects are the contractible complexes. Its stable category is $\mathsf{K}^{\mathrm{b}}(\mathcal{Q})$ because the morphisms factoring through contractible complexes are precisely the null-homotopic chain maps (see Keller [Ke, §2.3]). Hence (F, η) is an algebraic realization of $\mathcal{C}_J$. □

We can now verify the extriangulated category axioms for $\mathcal{C}(D)$ without assuming the existence of a single space mapping compatibly onto all D_i ($i \in I$).

Corollary 2.4. *Construction 2.2 yields an essentially small k-linear extriangulated category $\mathcal{C}(D)$. Each $\mathcal{C}_J$ ($J \subseteq I$) is a full extension-closed subcategory with its induced structure. If J has a largest element, then $\mathcal{C}_J$ admits an algebraic realization.*

Proof. If $J \subseteq I$ has a largest element m, then applying Proposition 2.3 with $W = D_m$ and $q_i = \pi_{mi}$ ($i \in J$) provides an algebraic realization of $\mathcal{C}_J$; in particular, every such $\mathcal{C}_J$ is an extriangulated category.

Whenever $J \subseteq J' \subseteq I$, the inclusion $\mathcal{C}_J \hookrightarrow \mathcal{C}_{J'}$ is full and identifies $\mathbb{E}_J(C, A)$ with $\mathbb{E}_{J'}(C, A)$ for all $A, C \in \mathcal{C}_J$, since both are naturally isomorphic to $\mathrm{Ext}^1_{\mathcal{R}}(\rho C, \rho A)$ by Lemma 2.1; moreover, it preserves the pairs assigned to extension classes in Construction 2.2.

The bifunctor $\mathbb{E}$ on $\mathcal{C}(D)$ is biadditive by construction. Each of the remaining extriangulated axioms in Nakaoka and Palu [NP, Definition 2.12] involves only finitely many objects of $\mathcal{C}(D)$: the realization of the zero class and of direct sums of classes, the lifting of morphisms of extensions to morphisms of conflations, the completion of commuting squares between conflations, the composition diagram for two conflations, and the dual statements. Choose a finite nonempty subset $J \subseteq I$ containing all indices $i \in I$ appearing in the summands of those objects. Since J has a largest element, $\mathcal{C}_J$ already satisfies all of these axioms. By fullness of $\mathcal{C}_J \hookrightarrow \mathcal{C}(D)$ and the compatibility of extension groups and assigned pairs, every required diagram and commutativity identity holds in $\mathcal{C}(D)$ as well.

For an arbitrary subset $J \subseteq I$, every extension class between objects of $\mathcal{C}_J$ has a representative with middle term in $\mathcal{C}_J$, by Lemma 2.1 on the $\mathcal{B}_J$-summands and the split pair on the X-summands; hence $\mathcal{C}_J$ is extension-closed in $\mathcal{C}(D)$ and inherits the structure of Construction 2.2. Finally, since I is a set, the finite direct sums of X, Y, and the A_i, P_i ($i \in I$) form a set of representatives for the isomorphism classes of objects of $\mathcal{C}(D)$, so $\mathcal{C}(D)$ is essentially small. □

3. Triangulated embeddings and projections from the inverse limit

Throughout this section, let I be a nonempty ordinal and let $D = (D_i, \pi_{ji})_{i \leq j \text{ in } I}$ be an inverse system of k-vector spaces with surjective transition maps, as in Construction 2.2. We now prove Theorem B, showing that a triangulated embedding of $\mathcal{C}(D)$ requires every projection $\varprojlim D \to D_i$ to be surjective, and therefore (by Proposition 2.3) already implies the existence of an algebraic realization.

The essential mechanism is to lift a morphism $u : X \to A_i$ to a compatible family in $\varprojlim D$ via a single degree -1 morphism in the target triangulated category. For a single conflation in a triangulated target, the existence of such a lift comes from exactness at the negative first connecting map of Adachi, Enomoto and Tsukamoto [AET, Definition 2.3 (NE2) and Example 2.4]. Intrinsically, Gorsky, Nakaoka and Palu [GNP, Definition 5.1(1), Remark 5.2 and Definition 5.3] describe a negative first extension in $\mathbb{E}^{-1}_{\mathrm{I}}(X, Y)$ as a natural transformation $\mathbb{E}(Y, -) \to \mathcal{C}(D)(X, -)$, whose connecting map for a conflation evaluates the transformation at its extension class. Their Example 5.25 exhibits a nonzero morphism to the first term of a conflation whose composite to the middle term vanishes. Under a triangulated embedding $(F, \eta) : \mathcal{C}(D) \to \mathcal{T}$, a single morphism $v : FX[1] \to FY$ can be composed with $\eta_{Y, A_j}(\varepsilon_j)$ for every $j \in I$, and by naturality of η the resulting morphisms form a compatible family.

For each $i \in I$, let $\varepsilon_i \in \mathbb{E}(Y, A_i)$ be the extension class of the interval conflation $A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y$ in (2.1). For $i \leq j$ in I, the morphism of short exact sequences (2.1) induces an isomorphism from the pushout of the upper row along a_{ji} to the lower row fixing A_i and Y, and hence

$$(a_{ji})_* \varepsilon_j = \varepsilon_i \qquad (i \leq j). \tag{3.1}$$

Theorem 3.1. *Let I be a nonempty ordinal and let $D = (D_i)_{i \in I}$ be an inverse system of k-vector spaces on I whose transition maps are surjective. The following conditions are equivalent.*

1. *The category $\mathcal{C}(D)$ admits a triangulated embedding.*
2. *For each $i \in I$, the projection $\varprojlim D \to D_i$ is surjective.*
3. *The category $\mathcal{C}(D)$ has an algebraic realization.*

Proof. (1) $\Rightarrow$ (2): Let $(F,\eta) : \mathcal{C}(D) \to \mathcal{T}$ be a triangulated embedding. For each $i \in I$, write $\delta_i = \eta_{Y,A_i}(\varepsilon_i)$, so that the conflation $A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y$ maps under F to a distinguished triangle

$$FA_i \xrightarrow{Fb_i} FP_i \xrightarrow{Fc_i} FY \xrightarrow{\delta_i} FA_i[1]$$

in $\mathcal{T}$. For $i \le j$ in I, naturality of η together with (3.1) yields the commutative square

$$\begin{array}{ccc} FY & \xrightarrow{\delta_j} & FA_j[1] \\ \| & & \downarrow{\scriptstyle (Fa_{ji})[1]} \\ FY & \xrightarrow[\delta_i]{} & FA_i[1]. \end{array} \tag{3.2}$$

Fix $i \in I$ and $u \in D_i = \mathcal{C}(D)(X, A_i)$. Since $\mathcal{C}(D)(X, P_i) = 0$, we have $b_i u = 0$, and therefore $(Fb_i)[1] \circ (Fu)[1] = 0$. Applying $\mathcal{T}(FX[1], -)$ to the distinguished triangle for $A_i \to P_i \to Y$ gives the exact sequence

$$\mathcal{T}(FX[1], FY) \xrightarrow{(\delta_i)_*} \mathcal{T}(FX[1], FA_i[1]) \xrightarrow{((Fb_i)[1])_*} \mathcal{T}(FX[1], FP_i[1]).$$

Hence there exists a morphism $v : FX[1] \to FY$ such that $\delta_i v = (Fu)[1]$.

For each $j \in I$, since F is fully faithful and $[1]$ is an autoequivalence of $\mathcal{T}$, there is a unique morphism $u_j : X \to A_j$ in $\mathcal{C}(D)$ satisfying $(Fu_j)[1] = \delta_j v$. In particular, $(Fu_i)[1] = \delta_i v = (Fu)[1]$, so $u_i = u$. For $h \le j$ in I, the square (3.2) (with h in place of i) yields

$$(F(a_{jh}u_j))[1] = (Fa_{jh})[1]\,(Fu_j)[1] = (Fa_{jh})[1]\,\delta_j v = \delta_h v = (Fu_h)[1].$$

Since F is faithful and $[1]$ is an autoequivalence, it follows that $\pi_{jh}(u_j) = a_{jh}u_j = u_h$. Thus $(u_j)_{j\in I}$ is an element of $\varprojlim D$ with i-th component u.

(2) $\Rightarrow$ (3): Apply Proposition 2.3 with $J = I$, $W = \varprojlim D$, and q_i the i-th canonical projection $\varprojlim D \to D_i$.

(3) $\Rightarrow$ (1): Every algebraic realization is a triangulated embedding by definition. □

4. A counterexample indexed by the countable ordinals

By Theorem 3.1, to produce an extriangulated category $\mathcal{C}(D)$ with no triangulated embedding, it suffices to exhibit an inverse system D of k-vector spaces with surjective transition maps over a nonempty ordinal I such that $\varprojlim D = 0$ while $D_0 \neq 0$. We construct such systems over ω_1, the first uncountable ordinal, by linearizing inverse systems of nonempty sets with surjective transition maps and empty inverse limit. For a set S, write $k^{(S)}$ for the free k-vector space with basis S. Bergman [Be, Lemma 3] establishes the following result for direct sums of copies of a fixed module, attributing the case of vector spaces over a field and its finite support argument to Higman and Stone [HS, §3].

Lemma 4.1 ([Be, Lemma 3] with ring and module k)**.** *Let $(S_i)_{i<\omega_1}$ be an inverse system of nonempty sets with surjective transition maps and empty inverse limit. Then the induced linear maps between the spaces $k^{(S_i)}$ are surjective and $\varprojlim_{i<\omega_1} k^{(S_i)} = 0$.*

We first record a direct counterexample obtained from injections of countable ordinals into $\mathbb{N}$.

Example 4.2. We use the inverse system of sets in Bergman [Be2, p. 7, Example 1 in the proof of Lemma 10], also given in Wofsey's second example [Wo]. A related system indexed by the finite subsets of $\mathbb{R}$ was given earlier by Waterhouse [Wa]. For each countable ordinal $i < \omega_1$, set

$$S_i = \{f : i \hookrightarrow \mathbb{N} \mid \mathbb{N} \setminus \operatorname{im} f \text{ is infinite}\}.$$

For $i \le j < \omega_1$, let the transition map $S_j \to S_i$ be restriction to i. To see that it is surjective, let $f \in S_i$ and partition the infinite set $\mathbb{N} \setminus \operatorname{im} f$ into two infinite subsets. Since $j \setminus i$ is countable, we may extend f to an injection $j \hookrightarrow \mathbb{N}$ sending $j \setminus i$ into one of these subsets, leaving the other subset outside the image. A compatible family in $\varprojlim S_i$ would have as its union an injection $\omega_1 \hookrightarrow \mathbb{N}$, which is impossible, so $\varprojlim S_i = \emptyset$. Setting $D'_i = k^{(S_i)}$, Lemma 4.1 gives $\varprojlim D' = 0$, whereas

S_0 consists solely of the empty function and thus $D'_0 = k$. Therefore $\mathcal{C}(D')$ has no triangulated embedding by Theorem 3.1.

In Example 4.2, the morphism space $\mathcal{C}(D')(X, A_\omega) = D'_\omega$ already has uncountable dimension, since S_ω contains all bijections from $\omega = \mathbb{N}$ onto the even natural numbers. To obtain morphism spaces of at most countable dimension, we replace S_i by countable subsets closed under restriction using an Aronszajn tree. Recall that a *tree* is a partially ordered set in which the strict predecessors of each node are well ordered; its *i-th level* consists of the nodes whose set of strict predecessors has order type i. An *Aronszajn tree* is a tree of height ω_1 whose i-th level is nonempty and countable for every $i < \omega_1$, and which has no chain meeting every level. Aronszajn's construction is recorded by Kurepa [Ku, Théorème 6, p. 96, footnote 3].

Construction 4.3 ([Je, proof of Theorem 9.16, (9.10), pp. 116–117])**.** In the Aronszajn tree constructed by Jech [Je, proof of Theorem 9.16, pp. 116–117], the nodes are bounded strictly increasing sequences of rational numbers, ordered by extension, so a node on level i is a function with domain i. Fixing a bijection $\mathbb{Q} \to \mathbb{N}$, since the range of each bounded sequence omits infinitely many rationals above its upper bound, we may identify the i-th level with a countable subset $N_i \subseteq S_i$ of Example 4.2. The level N_0 consists solely of the empty sequence, and condition (9.10) of [Je, pp. 116–117] ensures that every node extends to every higher level. For $i \leq j < \omega_1$, restriction to i therefore defines a surjective transition map $N_j \to N_i$ (the identity when $i = j$, and sending each node to its predecessor on level i when $i < j$). Because a compatible family would be a chain meeting every level, we have $\varprojlim N_i = \emptyset$. Following Dimitrić [Di, FC2, pp. 20–21], we set $D_i = k^{(N_i)}$, with linear transition maps induced by these restrictions $N_j \to N_i$ (sending each basis element to its predecessor for $i < j$). By Lemma 4.1, these linear maps are surjective and $\varprojlim D = 0$. Moreover, $D_0 = k$ and $\dim_k D_i = |N_i| \leq \aleph_0$ for all $i < \omega_1$.

To deduce Theorem A(3) for this tree system, we next determine the indecomposable objects and the extension spaces of $\mathcal{C}(D)$ for an arbitrary system D.

Proposition 4.4. *For any inverse system $D = (D_i)_{i \in I}$ as in Construction 2.2, the category $\mathcal{C}(D)$ is idempotent complete and Krull–Schmidt. A complete list of pairwise nonisomorphic indecomposables is*

$$X, \quad Y, \quad A_i, \quad P_i \qquad (i \in I).$$

Their endomorphism rings are k. For A, C on this list, the only nonzero spaces $\mathbb{E}(C, A)$ are

$$\mathbb{E}(Y, A_i) \simeq k, \qquad \mathbb{E}(P_j, A_i) \simeq k \qquad (j < i). \tag{4.1}$$

The category $\mathcal{C}(D)$ has enough injectives, and $\mathbb{E}^n = 0$ for all $n \geq 2$.

Proof. We first identify the indecomposables and then compute their extensions using the interval resolutions.

Step 1: Indecomposables and idempotents. By (2.2) and (2.3), each listed object has endomorphism ring k, and for any two distinct objects on the list at least one of the two morphism spaces between them vanishes; hence they are pairwise nonisomorphic indecomposables. Since every object of $\mathcal{C}(D)$ is a finite direct sum of these objects and their endomorphism rings are local, $\mathcal{C}(D)$ is Krull–Schmidt and, by Krause [Kr, Corollary 4.4], idempotent complete.

Step 2: Extensions, injectives, and heredity. Since $\rho X = 0$, we have $\mathbb{E}(X, -) = \mathbb{E}(-, X) = 0$. By Lemma 2.1, Y and the P_i are injective in $\mathcal{B}_I$, so $\mathbb{E}(-, Y) = \mathbb{E}(-, P_i) = 0$ by Construction 2.2. For $B \in \mathcal{B}_I$, applying $\mathcal{B}_I(B, -)$ to the injective resolution $0 \to A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y \to 0$ of Lemma 2.1 yields

$$\mathbb{E}(B, A_i) = \operatorname{Ext}^1_{\mathcal{B}_I}(B, A_i) \simeq \operatorname{coker}\big(\mathcal{B}_I(B, P_i) \xrightarrow{(c_i)_*} \mathcal{B}_I(B, Y)\big).$$

By (2.2), if $B = A_j$ then $\mathcal{B}_I(A_j, Y) = 0$; if $B = Y$ then $\mathcal{B}_I(Y, P_i) = 0$ and $\mathcal{B}_I(Y, Y) = k$; and if $B = P_j$ then $\mathcal{B}_I(P_j, Y) = kc_j$, whereas $(c_i)_*(\mathcal{B}_I(P_j, P_i))$ equals $kc_i p_{ji} = kc_j$ for $i \leq j$ and 0 for $j < i$. This proves (4.1).

In particular, X, Y, and all P_i are injective in $\mathcal{C}(D)$. Each A_i occurs as the first term of the conflation $A_i \xrightarrow{b_i} P_i \xrightarrow{c_i} Y$ with injective middle term, and each injective object U occurs as the

first term of the split conflation $U \xrightarrow{1_U} U \to 0$. Taking finite direct sums shows that $\mathcal{C}(D)$ has enough injectives.

Finally, by Gorsky, Nakaoka and Palu [GNP, Definition 3.1 and Remark 3.2], the group $\mathbb{E}^2(C, A)$ is generated by products $\alpha \otimes \beta$ with $\alpha \in \mathbb{E}(M, A)$ and $\beta \in \mathbb{E}(C, M)$ for $M \in \mathcal{C}(D)$. Writing $M = \bigoplus_{r=1}^{t} M_r$ as a finite direct sum of indecomposables with inclusions $\iota_r : M_r \to M$ and projections $\pi_r : M \to M_r$, we have $1_M = \sum_{r=1}^{t} \iota_r \pi_r$, and therefore

$$\alpha \otimes \beta = \sum_{r=1}^{t} (\iota_r)^* \alpha \otimes (\pi_r)_* \beta$$

with $(\iota_r)^*\alpha \in \mathbb{E}(M_r, A)$ and $(\pi_r)_*\beta \in \mathbb{E}(C, M_r)$. By (4.1), $\mathbb{E}(M_r, A)$ can be nonzero only when M_r is Y or some P_j, whereas $\mathbb{E}(C, M_r)$ can be nonzero only when M_r is some A_i. Since these two sets of indecomposables are disjoint, every term vanishes. Hence $\mathbb{E}^2 = 0$, and all higher tensor powers $\mathbb{E}^n$ $(n \geq 2)$ vanish as well. □

We now apply Theorem 3.1 and Proposition 4.4 to the inverse system D of Construction 4.3.

Theorem 4.5. *Let k be any field and let D be the inverse system of Construction 4.3. The essentially small k-linear extriangulated category $\mathcal{C}(D)$ has the following properties.*

1. *The category $\mathcal{C}(D)$ has no triangulated embedding.*
2. *Every countable family of objects of $\mathcal{C}(D)$ lies in a full extension-closed subcategory with an algebraic realization.*
3. *The category $\mathcal{C}(D)$ is hereditary and Krull–Schmidt, and has enough injectives. It has $\aleph_1$ isomorphism classes of indecomposables, each with endomorphism ring k. For all indecomposables A, C we have $\dim_k \mathbb{E}(C, A) \leq 1$, and every morphism space has at most countable dimension over k.*

Proof. (1) By Construction 4.3, we have $\varprojlim D = 0$ and $D_0 = k$, so the projection $\varprojlim D \to D_0$ is not surjective. Hence $\mathcal{C}(D)$ has no triangulated embedding by (1) ⇒ (2) of Theorem 3.1.

(2) Every object of $\mathcal{C}(D)$ is a finite direct sum of X, Y, and the A_i, P_i $(i < \omega_1)$, so only countably many indices $i < \omega_1$ occur in a given countable family of objects. Since every countable subset of ω_1 is bounded in ω_1, these indices all lie in $J = [0, m]$ for some $m < \omega_1$. Then $\mathcal{C}_J$ contains the given family, is extension-closed in $\mathcal{C}(D)$, and admits an algebraic realization by Corollary 2.4 because $m = \max J$.

(3) By Proposition 4.4, $\mathcal{C}(D)$ is Krull–Schmidt, has enough injectives, and is hereditary (since $\mathbb{E}^2 = 0$). Each indecomposable has endomorphism ring k, and $\dim_k \mathbb{E}(C, A) \leq 1$ for all indecomposables A, C by (4.1). The indecomposables consist of the two families $\{A_i\}_{i<\omega_1}$ and $\{P_i\}_{i<\omega_1}$ together with X and Y, so there are $\aleph_1$ isomorphism classes of indecomposables. By (2.2) and (2.3), the only morphism spaces between indecomposables that can have dimension greater than 1 are the spaces $\mathcal{C}(D)(X, A_i) = D_i = k^{(N_i)}$, each of which has dimension $|N_i| \leq \aleph_0$ by Construction 4.3. Taking finite direct sums preserves this bound, so every morphism space in $\mathcal{C}(D)$ has at most countable dimension over k. □

In contrast to having enough injectives, the category $\mathcal{C}(D)$ of Theorem 4.5 does not have enough projectives: in fact, there is no conflation ending in Y with projective middle term. Indeed, an object U is projective precisely when $\mathbb{E}(U, -) = 0$ [NP, Proposition 3.24]. Since ω_1 has no largest element, for each $j < \omega_1$ there exists $i > j$ with $\mathbb{E}(P_j, A_i) \simeq k$, and $\mathbb{E}(Y, A_0) \simeq k$; hence by (4.1), the only indecomposable projective objects are X and the A_i. Every morphism from X or A_i to Y is zero, whereas by Construction 2.2 the functor ρ maps the second morphism of any conflation ending in Y to an epimorphism onto Y. On the other hand, because $\mathcal{C}(D)$ has enough injectives, every object admits an inflation into an injective object, so $\mathcal{C}(D)$ has enough injective morphisms; by Gorsky, Nakaoka and Palu [GNP, Corollary 5.22], the contravariant left derived functors of Hom therefore equip $\mathcal{C}(D)$ with one-sided negative extensions $\mathbb{E}_{\Pi}^{-n}$. Thus $\mathcal{C}(D)$ admits these one-sided negative extensions even though it has no triangulated embedding; the obstruction in Theorem 3.1 arises from the degree -1 morphisms supplied by a triangulated target.

Can either the index set or the morphism spaces in Theorem 4.5 be made smaller? The following proposition shows that neither bound can be improved within the family of categories $\mathcal{C}(D)$ of Construction 2.2.

Proposition 4.6. *Let $D = (D_i)_{i\in I}$ be an inverse system as in Construction 2.2. The category $\mathcal{C}(D)$ has an algebraic realization under either of the following hypotheses.*

(1) *The ordinal I has a countable cofinal subset.*
(2) *Every D_i is finite-dimensional.*

Consequently, if $\mathcal{C}(D)$ has no triangulated embedding, then I has no countable cofinal subset and some D_i is infinite-dimensional; in particular, $|I| \geq \aleph_1$ and $\dim_k \mathcal{C}(D)(X, A_i) \geq \aleph_0$ for some $i \in I$. For the system D of Construction 4.3, we have

$$|I| = \aleph_1, \qquad \max_{i\in I} \dim_k D_i = \aleph_0.$$

Proof. By Theorem 3.1, to show that $\mathcal{C}(D)$ has an algebraic realization under (1) or (2), it suffices to extend any given $i \in I$ and $u \in D_i$ to a compatible family in $\varprojlim D$.

(1) If I has a largest element m, we choose $w \in D_m$ with $\pi_{mi}(w) = u$ and take $(\pi_{mj}(w))_{j\in I}$. Otherwise, we apply the surjective case of the Mittag–Leffler argument via the following recursive lift due to Tukey, as recorded by Henkin [Hen, p. 224]. Since I has a countable cofinal subset and no largest element, there exists a strictly increasing cofinal sequence $i = i_0 < i_1 < \dots$ in I. Setting $u_{i_0} = u$, we use surjectivity of the transition maps to choose recursively $u_{i_{n+1}} \in D_{i_{n+1}}$ with $\pi_{i_{n+1},i_n}(u_{i_{n+1}}) = u_{i_n}$ for all $n \geq 0$. For an arbitrary $j \in I$, choose n with $i_n \geq j$ and define $u_j = \pi_{i_n,j}(u_{i_n})$. If $i_m \geq i_n \geq j$, then $\pi_{i_m,j}(u_{i_m}) = \pi_{i_n,j}\pi_{i_m,i_n}(u_{i_m}) = \pi_{i_n,j}(u_{i_n})$, so u_j is independent of n; choosing $i_n \geq k$ for $j \leq k$ similarly gives $\pi_{kj}(u_k) = u_j$. Thus $(u_j)_{j\in I} \in \varprojlim D$ extends u.

(2) Surjectivity of the projections for finite-dimensional vector spaces over an arbitrary field follows from compactness in the coset topology (see Ghandour [Gh, Example 2.19 and Theorem 2.21(2)]); we give a direct proof by duality. Because the transition maps $\pi_{ji} : D_j \to D_i$ are surjective, the dual maps $\pi_{ji}^* : D_i^* \to D_j^*$ between the dual spaces $D_i^* = \mathrm{Hom}_k(D_i, k)$ are injective, so every D_i^* embeds into the direct limit $Z = \varinjlim_{j\in I} D_j^*$. Given $u \in D_i$, we extend evaluation at u from D_i^* to a linear functional $\varphi \in Z^*$. For each $j \in I$, since D_j is finite-dimensional, the restriction $\varphi|_{D_j^*}$ is evaluation at a unique vector $u_j \in D_j$, with $u_i = u$. For $h \leq j$ in I and every $\alpha \in D_h^*$, we have

$$\alpha(\pi_{jh}(u_j)) = (\pi_{jh}^*\alpha)(u_j) = \varphi(\pi_{jh}^*\alpha) = \varphi(\alpha) = \alpha(u_h),$$

whence $\pi_{jh}(u_j) = u_h$. Hence $(u_j)_{j\in I} \in \varprojlim D$ extends u.

Now suppose that $\mathcal{C}(D)$ has no triangulated embedding. Then it has no algebraic realization, so by (1) the ordinal I has no countable cofinal subset (in particular, $|I| \geq \aleph_1$), and by (2) some $D_i = \mathcal{C}(D)(X, A_i)$ is infinite-dimensional (so $\dim_k \mathcal{C}(D)(X, A_i) \geq \aleph_0$). For the system D of Construction 4.3, we have $I = \omega_1$ (hence $|I| = \aleph_1$) and $\dim_k D_i \leq \aleph_0$ for all $i < \omega_1$; since $\mathcal{C}(D)$ has no triangulated embedding by Theorem 4.5(1), some D_i is infinite-dimensional, and therefore $\max_{i\in I} \dim_k D_i = \aleph_0$. □

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