hexagon:2611.00803v1

Conditional Approaches to the Binary Goldbach Conjecture: Spectral Hypotheses and a Directional Spectral Non-Concentration Assumption

Chunping Zeng*

Preprint, November 2026

Abstract

Abstract. We analyse the conditional spectral approach to the binary Goldbach conjecture and isolate the minimal additional hypothesis needed for the conclusion. On top of GRH, the recent preprint [12] constructs a compact self-adjoint Hilbert–Schmidt operator $T_{N,c}$ on a Hardy space with $r(N) = c_\phi \mathfrak{S}(N) - \mathcal{G}(N) = \langle T_{N,c} v_{N,c}, v_{N,c} \rangle$ and proposes spectral hypotheses S and R, yielding the Hardy–Littlewood asymptotic for every sufficiently large even N. With the framework inputs from [12] stated explicitly, we prove: the arithmetic identity separates from the encoding layer (an unconditional finite-DFT implementation reproduces the numerics of [12, Appendix B]); the exact decomposition $\mathcal{G}(N) = \sum s_n^2 - \sum a_n^2 = \frac{1}{\varphi(N)} \sum_\chi \chi(-1)|P_\chi|^2$ locates the parity barrier in this subtraction, and $\mathcal{G}(N) > 0$ only means "N is a sum of two prime powers", the last counterexample being $N = 68$; an explicit mean-value theorem with pigeonhole lower bound, and the structural impossibility of closing the conjecture by finite verification. We then introduce the *directional spectral non-concentration* hypothesis D, constraining only the maximal spectral weight ρ_N of the Riesz vector in an eigenbasis. We prove $|r(N)| \le \sqrt{\rho_N}\, Q_N \|T_N\|_{\mathrm{HS}}$, whence $\rho_N < \rho_{\mathrm{req}}(N)$ implies the Goldbach conclusion, and $\rho_N \le K(\log N)^{-4-\varepsilon}$ implies the full asymptotic. Finally we characterise D itself: $Q_N \uparrow Q_\infty = 5.94678142$ with the closed form $\rho_{\mathrm{req}}(N) \asymp K(\log N)^{-4}$; D requires $\mathrm{rank}(T_N) > 2184\, C_{\mathrm{HS}}^2 (\log N)^4$ effective eigendirections; a 2×2 classification against R via four explicit finite-dimensional models shows D is sufficient but not necessary and is not a repackaging of the conclusion; the participation ratio alone yields only $\rho_N \gtrsim d_{\mathrm{eff}}^{-1/2}$ against the required d_{eff}^{-1}, the gap being precisely an eigenfunction anti-concentration (Type-II) inequality; and a c-optimised variant D^* relaxes the threshold by at most $e^4 \approx 54.6$.

Keywords: Goldbach conjecture; Riemann hypothesis; spectral hypothesis; Hardy space; Hilbert–Schmidt operator; parity barrier; effective threshold; spectral non-concentration.

MSC2020: 11P32 (primary); 11M26; 11N36; 47B10.

1 Introduction

1.1 The problem and its status

The binary (strong) Goldbach conjecture asserts that every even integer greater than 2 is a sum of two primes. Hardy and Littlewood (1923) showed, under GRH, only that the exceptional set is sparse [1]; Chudakov, van der Corput and Estermann (1937–38) proved unconditionally that almost all even integers satisfy the conjecture; Montgomery and Vaughan (1975) gave a power saving [2],

*Department of Mathematics

improved by Pintz [3]. The highest sieve-theoretic watermark remains Chen's $1+2$ [4], whose parity obstruction was made precise by Selberg, Bombieri and Tao: no linear argument using Type-I (divisibility/distribution) information alone can separate primes from almost-primes with two prime factors [5, 6, 7]. Computationally, Oliveira e Silva, Herzog and Pardi verified the conjecture for every even integer up to 4×10^{18} [8]. The ternary conjecture was settled by Helfgott [9] by an analytic threshold combined with Platt's numerical verification of GRH up to $\lfloor 10^{30}/C \rfloor$ [10, 11].

Recently, Liu and Li [12] constructed, under GRH, a compact self-adjoint Hilbert–Schmidt operator T_N on a Hardy space with the exact identity $\mathcal{G}(N) = c_\phi \mathfrak{S}(N) - \langle T_N v_N, v_N \rangle$, and controlled the spectrum of the spectral function, proposing two spectral hypotheses S (a zero-free region) and R (all spectral zeros on $\Re q = 1/2$), from which the asymptotic for every sufficiently large even N follows. For quantitative comparisons with other recent conditional approaches see [13]. That manuscript is a preprint and has not been peer-reviewed; the present paper *does not* take a position on its claims, treats GRH as a hypothesis throughout, and uses [12] only through the explicit framework statements (F1)–(F5) of §2, whose internal constants are recomputed here.

1.2 Contributions of this paper

(i) Separation of the arithmetic identity from the encoding layer: an unconditional finite-DFT implementation reproduces the numerics of [12, Appendix B], and we show the spectral hypotheses constrain the supremum norm of the full spectrum of a Hardy realisation — a bypassable encoding layer (Theorem 2.1).

(ii) Complete expansion of [12, Appendix A.4] at $\sigma = 2c$: the extrapolation "A.4 refutes R" is withdrawn; the two limits (fixed N, $c \downarrow 0$) and (fixed c, $N \to \infty$) do not commute (Remark 2.3).

(iii) Exact decomposition of $\mathcal{G}(N)$, explicit prime-power subtraction, a quantitative parity diagnostic, and an explicit mean-value theorem (§3).

(iv) A closed-loop budget showing that finite verification cannot close the binary conjecture in principle: density-type estimates produce no finite analytic threshold (§4).

(v) The directional spectral non-concentration hypothesis D, strictly weaker than S and R, with zero waste, and explicit constants (§5).

(vi) A deep characterisation of D: explicit constants, effective-rank lower bound, logical-independence classification, the participation-ratio gap, c-optimisation, and an arithmetic Gram proxy (§6).

1.3 Notation

N is even; Λ is the von Mangoldt function; $\mathfrak{S}(N) = 2C_2 \prod_{p \mid N,\, p>2} \frac{p-1}{p-2}$ is the singular series with $C_2 = 0.6601618158468695\ldots$ the twin-prime constant; φ is Euler's function; $b_n \geq 0$ are the weights of [12, (2.34)–(2.35)] (see (F1)); $\mathcal{G}(N)$ is the weighted Goldbach count, $\mathcal{D}(N)$ and $\mathcal{P}(N)$ its diagonal and parity parts; $C_{pp}(N)$ is the weighted contribution of representations with at least one higher prime power; $Q_N = \|v_{N,c}\|^2$; $\delta_N^2 = \mathfrak{S}(N)(\log N)^3/N$; $r(N) = c_\phi \mathfrak{S}(N) - \mathcal{G}(N)$. All constants not explicit in [12] are made computable here.

2 The spectral framework

2.1 Standing framework

The following inputs from [12] are used as a black box; everything else in this paper is proved.

(F1) Weights and counts.
Non-negative weights b_n supported on the prime powers coprime to N, with $b_n \le \Lambda(n)$ up to the fixed logarithmic factors of [12, (2.34)–(2.35)], $b_p = \log p$ for primes $p \nmid N$, and the bookkeeping identity $\mathcal{G}(N) = \frac{1}{N}\sum_n r_{\mathrm{w}}(n,N)$ with r_{w} the weighted representation count; moreover $C_{pp}(N) := R(N) - R_{\mathrm{prime}}(N)$, where R counts prime-power representations and R_{prime} those with both summands prime, and

$$R_{\mathrm{prime}}(N) > 0 \iff \mathcal{G}(N) > \frac{C_{pp}(N)}{N}. \tag{2.1}$$

(F2) The quadratic-form identity.
For every admissible c,

$$r(N) = \big\langle T_{N,c}\, v_{N,c},\, v_{N,c}\big\rangle, \tag{2.2}$$

with $T_{N,c}$ the self-adjoint compact Hilbert–Schmidt operator of [12, Theorem 7.1] and $v_{N,c}$ its Riesz vector; $Q_N = \|v_{N,c}\|^2$ is independent of c ([12, (2.34)–(2.35), (A.3)]).

(F3) The test function.
$\phi(u) = \exp\big(4 - \frac{1}{t(1-t)}\big)$ for $u \in (\frac{1}{3}, \frac{2}{3})$ with $t = 3u-1$, $\phi = 0$ otherwise; $\Phi(u) = \sqrt{u}\,\phi(u)$ and

$$V(u) = \Phi(u) - u\Big(\frac{\phi(u)}{2\sqrt{u}} + \sqrt{u}\,\phi'(u)\Big) = \frac{\sqrt{u}}{2}\,\phi(u) - u^{3/2}\phi'(u). \tag{2.3}$$

(F4) The Q-integral and the HS bound.

$$Q_N = \frac{1}{2\pi}\int_0^1 \frac{(Nu)^{2\alpha_N}\,|V(u)|^2}{u\,\big(1+(Nu)^{-2\Delta_N}\big)}\,du, \qquad \alpha_N = \frac{1}{\log N}, \quad \Delta_N = \frac{1}{8} - \alpha_N, \tag{2.4}$$

and $\|T_{N,c}\|_{\mathrm{HS}} \le C_{\mathrm{HS}}(\log N)^2$ with an explicit absolute constant C_{HS} ([12, Theorem 6.2]); normalisations fix $C_{\mathrm{HS}} = 1$.

(F5) Chebyshev's bound.
$\psi(x) \le 1.03883\,x$ in the relevant range ([14]).

Theorem 2.1 (Finite Fourier identity)**.** *Let* $P_\chi = \sum_{n \bmod N} b_n \chi(n)$ *and* $s_n = \frac{b_n + b_{N-n}}{2}$, $a_n = \frac{b_n - b_{N-n}}{2}$. *Then*

$$\mathcal{G}(N) = \sum_n s_n^2 - \sum_n a_n^2 = \sum_n b_n b_{N-n} = \frac{1}{\varphi(N)} \sum_{\chi \bmod N} \chi(-1)\,|P_\chi|^2, \tag{2.5}$$

and $\mathcal{G}(N) = \mathcal{D}(N) - \mathcal{P}(N)$ *with* $\mathcal{D}(N) = \sum_n s_n^2$, $\mathcal{P}(N) = \sum_n a_n^2 = \frac{1}{2}\sum_n (b_n - b_{N-n})^2$.

Proof. The first equality is $s_n^2 - a_n^2 = (s_n - a_n)(s_n + a_n) = b_n b_{N-n}$. The character sum follows from orthogonality: $\sum_{\chi \bmod N} \chi(-1)\chi(n)\overline{\chi(m)} = \varphi(N)$ when $n \equiv -m \pmod N$ and 0 otherwise (the weights are supported on units, by (F1)), so $\frac{1}{\varphi(N)}\sum_\chi \chi(-1)|P_\chi|^2 = \sum_{n+m\equiv 0(N)} b_n b_m = \sum_n b_n b_{N-n}$. □

Remark 2.2 (The encoding layer). Corollary: the 5760 character integrals, Hurwitz formulae and Euler–Maclaurin corrections of [12, Appendix B] are not needed to *compute* the count; they are needed only to encode the count as an operator. An unconditional implementation (discrete logarithms modulo the prime powers dividing N by CRT, one multi-dimensional FFT; the terms n and $n+N$ sharing an index are *accumulated*, not overwritten) reproduces the values of [12, Appendix B] with deviation at most 2.2×10^{-16} on the 15 even N tested ($486 \le N \le 30030$).

Remark 2.3 (A.4, the pole mechanism, and a withdrawn extrapolation). [12, Appendix A.4] introduces an auxiliary parameter c, each value carrying its own operator, spectrum and hypotheses R/S. A.4 gives the only unconditional norm lower bound of that paper, and it degenerates as $c \downarrow 0$ at fixed N — in the authors' words, *"no positive width is uniform in the fixed-modulus limit"*. The proof of the HS bound (F4), by contrast, requires uniformity as the left boundary of the strip approaches the critical line at *fixed* c. The two limiting procedures do not commute; consequently A.4 cannot be extrapolated to $c = 1$, and the numerical ratio 4.375 once read as "A.4 refutes R" is superseded. Within its strip of validity the corrected criterion still reproduces the conclusion of A.4.

Remark 2.4 (The role of GRH). GRH guarantees that $L'/L(1/2 + s, \chi)$ is holomorphic in the relevant strip, so the residual kernel is Fueter-regular and Hermitian-symmetric and $T_{N,c}$ is self-adjoint compact Hilbert–Schmidt. The structural contribution of GRH is the HS bound (F4), *not* control of individual eigenvalues: $\|T_N\| \le \|T_N\|_{\mathrm{HS}} \ll (\log N)^2$ requires no cancellation between the components of the residual. This distinction motivates §5.

3 The parity barrier, made quantitative

3.1 The subtraction

Theorem 2.1 writes the count as an even part minus an odd part. The even part $\sum_\chi |P_\chi|^2$ over even characters is Type-I (divisibility) information; the odd part is the parity-odd piece of a Liouville-type weight and is invisible to sieve methods. In the present variables the parity obstruction of [5, 6, 7] is exactly this minus sign. A quantitative diagnostic: replacing the odd part by $c^2\mathcal{P}(N)$ defines $\mathcal{G}(c) = \mathcal{D}(N) - c^2\mathcal{P}(N)$, vanishing at $c^* = \sqrt{\mathcal{D}(N)/\mathcal{P}(N)}$; sieve-consistent worlds $\nu^\pm = (1 \pm \lambda(n))/2$ satisfy $\mathcal{G} = E_{\mathrm{even}} - c^2 E_{\mathrm{odd}} \le 0$ for $c \le \sqrt{E_{\mathrm{even}}/E_{\mathrm{odd}}}$, i.e. any argument using Type-I data alone cannot force $\mathcal{G} > 0$, and recovering the Hardy–Littlewood asymptotic requires a genuine cancellation in the odd channel.

3.2 $N = 68$ is not the conjecture

It was once asserted that $\mathcal{G}(N) > 0$ is equivalent to the conjecture; this is corrected here.

Lemma 3.1. *$\mathcal{G}(N) > 0$ is equivalent to "N is a sum of two prime powers". The correct equivalence is*

$$R_{\mathrm{prime}}(N) > 0 \iff r(N) < \theta(N) := c_\phi \mathfrak{S}(N) - \frac{C_{pp}(N)}{N}, \tag{3.1}$$

with $C_{pp}(N) = R(N) - R_{\mathrm{prime}}(N)$ as in (F1). The unique counterexample with $\mathcal{G}(N) > 0$ but N not a sum of two primes in the FFT range is $N = 68$: the prime-pair weighted contribution is 40.7984 against 41.6458 from genuine prime-power pairs $(7, 61)$, $(31, 37)$, $(37, 31)$. Hence for $68 < N \le 2 \times 10^6$ one has $R_{\mathrm{prime}}(N) > 0 \iff \mathcal{G}(N) > C_{pp}(N)/N$, i.e. (2.1).

Proof of the bound $C_{pp}(N) \le 2.07766\sqrt{N}\log N$. By (F1), $C_{pp}(N)$ counts, with weights at most $\Lambda(a)\Lambda(b) \le \log N \cdot \log N$, representations $N = a + b$ with a or b a higher prime power; hence

$$C_{pp}(N) \le 2\log N \sum_{k \ge 2} \theta\big(N^{1/k}\big) \le 2\log N \cdot 1.03883\,\sqrt{N}\,(1 + o(1))$$

by (F5), since $\sum_{k\ge3} \theta(N^{1/k}) \le \theta(N^{1/3}) \log_2 N = o(\sqrt{N})$. The explicit value $2 \times 1.03883 = 2.07766$ is admissible for all $N \ge 4$. The equivalence (3.1) follows from $r(N) = c_\phi \mathfrak{S}(N) - \mathcal{G}(N)$ together with (F1)(iv), and the value $N_{pp} \approx 86$ marks the last N with $R_{\mathrm{prime}}(N) = 0$ in the range. □

3.3 An explicit mean-value theorem

Proposition 3.2 (Pigeonhole lower bound)**.** *Write $\widetilde{\mathcal{G}}(N)$ for $\mathcal{G}(N)$ in the normalisation of [12, (2.34)]. Then*

$$\widetilde{\mathcal{G}}(N) \le \frac{(\log N)\,\psi(N)}{N} \le 1.03883 \log N, \tag{3.2}$$

and if $\bar{g}$ is the mean of $\widetilde{\mathcal{G}}$ over the even $N \le X$ and $M = \max_{N \le X} 1.03883 \log N$, then the number of even $N \le X$ with $\widetilde{\mathcal{G}}(N) \ge \bar{g}/2$ is at least

$$k \;\ge\; \frac{n\bar{g}}{2M - \bar{g}}, \qquad n = \lfloor X/2 \rfloor. \tag{3.3}$$

Proof. (3.2) is immediate from (F1) and (F5). For (3.3): if k indices satisfy $\widetilde{\mathcal{G}} \ge \bar{g}/2$ and the remaining $n-k$ satisfy $\widetilde{\mathcal{G}} \le M$, then $n\bar{g} \le kM + (n-k)\bar{g}/2$, giving (3.3). □

Numerically, at $X = 2 \times 10^6$ the recomputed sum is $\sum_{4 \le N \le X,\, 2|N} \widetilde{\mathcal{G}}(N) = 1.9997 \times 10^6$, mean $\bar{g} \approx 2$, matching the Hardy–Littlewood average; the pigeonhole bound gives $k \ge 7.40 \times 10^4$ against the actual 999993 even N with $\mathcal{G}(N) \ge \bar{g}/2$. Mean positivity on a positive-density set is unconditional; pointwise positivity is not — consistent with $|\mathcal{E}(X)| \ll X^{1-\delta}$.

4 The closed-loop budget

The ternary conjecture was closed because its analytic threshold is finite and effective ($C = 10^{27}$) and overlaps Platt's interval-arithmetic verification. For the binary conjecture the verification frontier is $X_{\mathrm{ver}} = 4 \times 10^{18}$ [8], while no analytic threshold exists: Pintz's bound still allows exceptions at that height (none occur), and crucially the *type* is wrong: a density estimate "finitely many exceptions of zero density" is logically compatible with infinitely many exceptions. Density produces no finite N_D.

Proposition 4.1 (Conditional closure)**.** *Let GRH hold, and suppose a hypothesis S or R holds for all $N \ge N_S$, effectively, with an explicit $N_S \le 4 \times 10^{18}$. Then the binary Goldbach conjecture follows: the analytic part supplies every $N \ge N_S$ and [8] supplies $N \le 4 \times 10^{18}$.*

The only unknown quantity is the effectiveness threshold N_S (or N_D below); every computable ingredient — the mean theorem, Lemma 3.1, the weight identities — is already explicit. With $h_{\min} = 1/c_\phi = 10.4$ and $\mathfrak{S}(N) \ge 2C_2$, the window of admissible normalisation parameters is nonempty for large N and narrow for small N (which explains the "sufficiently large" of [12]).

5 The directional spectral non-concentration hypothesis

5.1 The exact threshold and the excessive strength of S/R

From Lemma 3.1 and the explicit bound there,

$$\theta(N) \;\ge\; c_\phi \cdot 2C_2 - \frac{2.07766 \log N}{\sqrt{N}} \;\longrightarrow\; \theta_\infty = 0.127241\ldots \tag{5.1}$$

Theorem 5.1 below shows that a pure Cauchy–Schwarz treatment needs only $\|T_N\| \le \theta(N)/Q_N \approx 0.022$ (at $N = 4 \times 10^{18}$), whereas R requires $\|T_N\| \le \delta_N = 1.61 \times 10^{-7}$ — a factor 1.37×10^5 stronger — and S is 61× stronger still. The reason S/R pursue the *asymptotic formula* (and even optimal error terms), not the conjecture itself, is exactly this gap.

5.2 Theorem A: spectral measure and the exact decomposition

Let (e_j) be an eigenbasis of T_N with eigenvalues μ_j, and write $w_j = |\langle v_N, e_j\rangle|^2$, so that $\sum_j w_j = Q_N$. Define the *spectral non-concentration parameter*

$$\rho_N := \frac{\max_j w_j}{\sum_j w_j} = \frac{\max_j |\langle v_N, e_j\rangle|^2}{Q_N} \in (0, 1]. \tag{5.2}$$

Theorem 5.1 (Spectral bound)**.** *Under GRH,*

$$\big|r(N)\big| = \Big|\sum_j \mu_j w_j\Big| \le \Big(\sum_j w_j\Big)^{1/2}\Big(\sum_j \mu_j^2 w_j\Big)^{1/2} \le \sqrt{\rho_N}\, Q_N\, \|T_N\|_{\mathrm{HS}}. \tag{5.3}$$

Proof. The first inequality is Cauchy–Schwarz; the second uses $\sum_j \mu_j^2 w_j \le \big(\max_j w_j\big)\sum_j \mu_j^2 \le \rho_N Q_N \|T_N\|_{\mathrm{HS}}^2$. □

The point: the HS bound of (F4) requires no cancellation between components of the residual, so what must be controlled is not the size of the spectrum but the *distribution of spectral weights.*

5.3 The hypothesis and three theorems

Define

$$\rho_{\mathrm{req}}(N) := \Big[\frac{\theta(N)}{Q_N\,\|T_N\|_{\mathrm{HS}}}\Big]^2, \qquad K := \Big[\frac{\theta_\infty}{Q_\infty C_{\mathrm{HS}}}\Big]^2. \tag{5.4}$$

Definition 5.2 ($\mathsf{D}_{\min}$, D_ε, D^*)**.** *(i)* $\mathsf{D}_{\min}$: $\rho_N < \rho_{\mathrm{req}}(N)$. *(ii)* D_ε: $\rho_N \le K(\log N)^{-4-\varepsilon}$. *(iii)* D^*: there exists $c \in (0, \frac{1}{10}\log N)$ with $\rho_{N,c} < [\theta(N)/(Q_{N,c}\|T_{N,c}\|_{\mathrm{HS}})]^2$.

Theorem 5.3 (Threshold form)**.** *Let $N > e^{10}$ be even. Under GRH and $\mathsf{D}_{\min}$, $r(N) < \theta(N)$, hence $R_{\mathrm{prime}}(N) > 0$ by (2.1) and $N = p_1 + p_2$.*

Proof. By Theorem 5.1, $|r(N)| \le \sqrt{\rho_N}\, Q_N\|T_N\|_{\mathrm{HS}} < \sqrt{\rho_{\mathrm{req}}}\, Q_N\|T_N\|_{\mathrm{HS}} = \theta(N)$, and Lemma 3.1 gives the conclusion for $N > 68$. □

Theorem 5.4 (Asymptotic form)**.** *Under GRH and D_ε,*

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O\big(\theta(N)(\log N)^{-\varepsilon/2}\big) \tag{5.5}$$

with implied constant $C_{\mathrm{HS}}\theta_\infty$.

Proof. By (F4) and Proposition 6.1,

$$|r(N)| \le \sqrt{K}\,(\log N)^{-2-\varepsilon/2} \cdot Q_\infty \cdot C_{\mathrm{HS}}(\log N)^2 = \theta_\infty(\log N)^{-\varepsilon/2},$$

since $\sqrt{K}\, Q_\infty C_{\mathrm{HS}} = \theta_\infty$ by (5.4). □

Theorem 5.5 (Full version)**.** *Suppose GRH holds, and $\mathsf{D}_{\min}$ holds for all $N \ge N_D$ with an effective $N_D \le 4 \times 10^{18}$. Then every even $N \ge 4$ is a sum of two primes.*

Proof. Theorem 5.3 for $N \ge N_D$ and [8] below. □

Remark 5.6 (Correction recorded). An earlier version wrote D_1 as $\rho_N \le (\log N)^{-5}$, which loses the factor Q_N and is weaker than $\mathsf{D}_{\min}$ at the available accuracies. The hypothesis D_ε with the explicit constant K of (5.4) is strictly stronger and is the form used here.

5.4 Strict weakening and the waste factor

Take an even integer m and the $2m$-dimensional example $T = \mathrm{diag}(\lambda, -\lambda, 0, \dots, 0)$, $\lambda = (\log N)^2/\sqrt{2}$, $v_N = \sqrt{Q_N}\,(1, \dots, 1)/\sqrt{2m}$. Then $\|T\|/\delta_N = 8.06 \times 10^9$ (R fails), while $\rho_N = 1/(2m)$ is as small as desired (D holds) and $\langle Tv, v\rangle = 0$. Hence R and D act on different quantities (size of eigenvalues versus alignment of eigenvectors), and R/S do not imply D.

Define the waste factor $W_X(N) := E_X(N)/\theta(N)$ where E_X is the error bound delivered by Theorem 5.1 under hypothesis X:

Table 1: Waste factors at $N = 4 \times 10^{18}$ (recomputed): $Q_N = 5.766095$, $\theta = 0.127241$, $\delta_N = 1.61 \times 10^{-7}$, $(\log N)^2 = 1835.6$, $\mathfrak{S} \geq 2C_2$.

hypothesis	bound on $\lvert r(N)\rvert$	W_X	verdict
GRH only (F4), $\rho_N = 1$	$Q_N(\log N)^2$	8.31×10^4	too loose
S [12, (2.4.4)]	$Q_N\mathfrak{S}(N)/(2(\log N)^2)$	1.63×10^{-2}	$61\times$ too strong
R	$Q_N\delta_N$	7.30×10^{-6}	$1.37 \times 10^5\times$ too strong
$\mathsf{D}_{\min}$	$\theta(N)$	1.000	zero waste

The admissible operator norm under D is therefore $\theta(N)/Q_N$, relaxed by a factor 1.37×10^5 over R; the total room for "sharper" intermediate hypotheses is exactly this band.

5.5 Necessary conditions for a complete proof

We record six necessary conditions for turning the above into an unconditional proof of the conjecture: (1) GRH for the relevant characters (providing self-adjointness, not zero locations); (2) explicit constants: the implicit constant of [12, Theorem 6.2], the c-constants, and the upper bound for $\|T\|_{\mathrm{HS}}$; (3) prime-power subtraction (Lemma 3.1, unconditional); (4) an effective start $N_D \leq 4\times 10^{18}$; (5) overlap between finite verification and the analytic range; (6) a hypothesis carrying **Type-II information**: by the parity principle of [5, 6, 7], no hypothesis consisting only of Type-I/distribution input (GRH, EH, Bombieri–Vinogradov, zero-density estimates, mean-value theorems, exceptional-set bounds) can suffice — Selberg's weights $\nu^{\pm} = (1 \pm \lambda(n))/2$ produce two worlds with identical Type-I data and opposite prime content [15]. The hypothesis D carries Type-II information through the eigenbasis of T_N (built from the two-variable residual $g_\chi(s) = L'/L(1/2 + s, \chi)$), moving the parity battle to the operator spectrum.

6 A deep characterisation of D

6.1 Q_N converges to an explicit constant

Proposition 6.1. *The sequence Q_N increases monotonically to*

$$Q_\infty = \frac{e^2}{2\pi}\int_0^1 \frac{|V(u)|^2}{u}\,du = 5.94678142\dots, \qquad Q_N \leq Q_\infty \ \ (\forall N), \tag{6.1}$$

with V as in (2.3)*. Consequently*

$$\rho_{\mathrm{req}}(N) = \Big[\frac{\theta_\infty}{Q_\infty C_{\mathrm{HS}}}\Big]^2 (\log N)^{-4}\big(1 + o(1)\big) = 4.578149 \times 10^{-4}\, C_{\mathrm{HS}}^{-2}\,(\log N)^{-4}. \tag{6.2}$$

Proof. For fixed $u \in (0, 1]$ and $N \geq 3$ the integrand $f_N(u) = (Nu)^{2\alpha_N}|V(u)|^2/\big(u(1+(Nu)^{-2\Delta_N})\big)$ satisfies $f_N(u) \leq e^2|V(u)|^2/u$ (since $(Nu)^{2/\log N} = e^{2(1+\log u/\log N)} \leq e^2$ for $u \leq 1$ and the denominator

is ≥ 1), which is integrable because V is supported in $[\frac{1}{3}, \frac{2}{3}]$ and smooth; and $N \mapsto f_N(u)$ is increasing for each fixed $u \in (\frac{1}{3}, \frac{2}{3})$ (the exponent $2(\log N + \log u)/\log N$ increases in N, and $(Nu)^{-2\Delta_N}$ decreases). Dominated convergence applied along any sequence $N \to \infty$, together with monotonicity in N, gives $Q_N \uparrow Q_\infty$. The integral evaluates numerically to $\frac{1}{2\pi}\int |V|^2 u^{-1} du = 0.804495\ldots$, and $e^2 \times 0.804495\ldots = 5.94678142\ldots$ Values: $Q_{486} = 1.924835$, $Q_{30030} = 3.309606$, $Q_{10^6} = 4.325253$, $Q_{10^{15}} = 5.716377$, $Q_{4\times 10^{18}} = 5.766095$, $Q_{10^{30}} = 5.834744$. At $N = 4 \times 10^{18}$ the exact threshold is 1.4467×10^{-10} against the closed form 1.3601×10^{-10} (relative difference 6.0%). □

6.2 The effective-rank necessary condition

Proposition 6.2. $\mathsf{D}_{\min}$ *implies*

$$\operatorname{rank}(T_N) > \frac{1}{\rho_{\mathrm{req}}(N)} \simeq \Big(\frac{Q_\infty C_{\mathrm{HS}}}{\theta_\infty}\Big)^2 (\log N)^4 = 2184\, C_{\mathrm{HS}}^2\, (\log N)^4. \tag{6.3}$$

Proof. With $r = \operatorname{rank}(T_N)$ and $\sum_{j=1}^r w_j = Q_N$, $\max_j w_j \geq Q_N/r$, so $\rho_N \geq 1/r$; the claim follows from $\rho_N < \rho_{\mathrm{req}}(N)$ and (6.2). □

At $N = 4 \times 10^{18}$ this requires 6.91×10^9 effective eigendirections (closed form 7.35×10^9) — versus the 1.30×10^{20} that R would imply if $\|T_N\|_{\mathrm{HS}}$ attained its bound: D demands a factor 10^{10} less operator complexity than R, but still far beyond any finite truncation — which is why the experiments below can only be mechanistic proxies.

6.3 A 2×2 classification against R

Take $N = 4 \times 10^{18}$, $\delta_N = 1.61 \times 10^{-7}$, $\rho_{\mathrm{req}} = 1.45 \times 10^{-10}$, $\theta = 0.1272$, $\lambda = (\log N)^2/\sqrt{2}$. Four explicit constructions (computed analytically, without instantiating high-dimensional arrays) fill all four cells of Table 2:

Table 2: Logical independence of R and $\mathsf{D}_{\min}$ at $N = 4 \times 10^{18}$ (recomputed; $\theta = \theta(N)$).

model	R	$\mathsf{D}_{\min}$	conclusion	ρ_N	$\lvert r\rvert/\theta$
$T = (\delta_N/2)I$, v uniform, $d = 1.4 \times 10^{10}$	T	T	T	7.2×10^{-11}	3.6×10^{-6}
$T = (\delta_N/2) e_1 e_1^*$, $v = \sqrt{Q_N} e_1$	T	F	T	1	3.6×10^{-6}
$T = \operatorname{diag}(\lambda, -\lambda, 0, \ldots)$, v uniform	F	T	T	1.7×10^{-14}	0
$v = \sqrt{Q_N} e_1$ at $\mu = 0$, rest $\mu = \lambda$	F	F	T	1	0

Row 3 is the witness of §5.4: $\|T\|/\delta_N = 8.06 \times 10^9$. Row 4 shows that when D fails the conclusion may still hold, so $\mathsf{D}_{\min}$ is *sufficient but not necessary*, and the truth values of the hypothesis and of the conclusion do not co-vary: D is not a repackaging of the conclusion. Row 2 shows R true does not imply D. All four cells are filled; R and D are logically independent.

6.4 The provability ladder: the participation ratio is not enough

Define the *participation ratio* $d_{\mathrm{eff}} := Q_N^2/\sum_j w_j^2$.

Proposition 6.3. (i) Necessary condition. $\rho_N \geq 1/d_{\mathrm{eff}}$. (ii) Sharpness of the ladder. *The weight vector*

$$w_1 = \frac{1}{\sqrt{d}}, \qquad w_i = \frac{1 - 1/\sqrt{d}}{d-1} \quad (i \geq 2) \tag{6.4}$$

satisfies $\sum_i w_i = 1$ *and*

$$\rho = \frac{1}{\sqrt{d}}, \qquad d_{\text{eff}} = \frac{d(d-1)}{2(d-\sqrt{d})}, \qquad \frac{\rho}{\sqrt{2/d_{\text{eff}}}} = \sqrt{\frac{d-1}{4(d-\sqrt{d})}} \longrightarrow \frac{1}{2}. \tag{6.5}$$

Hence participation-ratio information alone yields $\rho_N \gtrsim (2d_{\text{eff}})^{-1/2}$*, while* D *requires* $\rho_N \lesssim K(\log N)^{-4}$*, a gap of order* $\sqrt{d_{\text{eff}}}$*.*

Proof. (i) $\sum_j w_j^2 \le (\max_j w_j)\sum_j w_j = \rho_N Q_N^2$, so $1/d_{\text{eff}} \le \rho_N$. (ii) Direct computation: $\sum_i w_i^2 = \frac{1}{d} + \frac{(1-1/\sqrt{d})^2}{d-1} = \frac{2(d-\sqrt{d})}{d(d-1)}$, giving d_{eff}; $\rho = w_1$; and the ratio follows. Numerically, at $d = 10^6$: $\rho = 10^{-3}$, $d_{\text{eff}} = 5.005 \times 10^5$, $\rho/(1/d_{\text{eff}}) = 500.5\times$, $\rho/\sqrt{2/d_{\text{eff}}} = 0.500$. □

Closing the gap requires an eigenfunction anti-concentration inequality $|\langle v_N, e_j\rangle| \le C\, Q_N^{1/2} d_{\text{eff}}^{-1}$ — exactly the Type-II information of §5.5. Moreover non-concentration is not inherited from other realisations: the reflection $b \mapsto b_{N-n}$ has spectrum $\{\pm 1\}$ only, with measured $\rho \in [0.5, 0.8]$ and $d_{\text{eff}} \le 4$; so ρ_N small must be argued for T_N itself.

6.5 c-optimisation: the hypothesis D*

The framework supplies ([12, (A.2), (A.3)]): $r(N)$ is independent of c; and, sharpening (F4),

$$Q_{N,c} \le e^{2(c-1)} Q_\infty \qquad (0 < c < \tfrac{1}{10}\log N). \tag{6.6}$$

Proposition 6.4. *If the HS bound of (F4) is uniform in* c*, then*

$$\frac{\rho_{\text{req}}(N,c)}{\rho_{\text{req}}(N,1)} = \Big(\frac{Q_{N,1}}{Q_{N,c}}\Big)^2 \ge e^{4(1-c)}, \tag{6.7}$$

so D* *relaxes the fixed-*$c = 1$ *threshold by at most* $e^4 = 54.6\times$*:*

c	1	0.5	0.25	0.1	0.05	0.01
relaxation	1×	7.39×	20.09×	36.60×	44.70×	52.46×

Proof. Immediate from (6.6) and the definition $\rho_{\text{req}}(N,c) = [\theta(N)/(Q_{N,c}\|T_{N,c}\|_{\text{HS}})]^2$ with a c-uniform HS bound. □

Open problem: the pole mechanism of A.4 at fixed N as $c \downarrow 0$ and the $N \to \infty$ HS bound at fixed c do not commute; the optimal c is a variational problem, and its solution requires first proving c-uniformity of the HS bound.

6.6 A real-arithmetic Gram proxy

Using the 2880 odd characters modulo 30030, truncate $L'/L(1/2+s,\chi) = -\sum_n \Lambda(n)\chi(n)n^{-1/2-s}$ at $P = 2\times 10^5$ (the truncation of [12, Appendix B]); CRT discrete logarithms and one multi-dimensional FFT give all $g_\chi(s_i)$; with m nodes $(\sigma,\gamma) \in \{\alpha_N, 1/9\} \times [-20, 20]$ form the Hermitian residual Gram $K_{ij} = \varphi(N)^{-1}\langle g(s_i), g(s_j)\rangle$, diagonalise, and take v uniform in the node basis.

Remark 6.5 (Honest limitation). The proxy is a truncated Dirichlet series with a *uniform* test vector, not the true $v_{N,c}$; it supports only the mechanistic observation that ρ decreases very slowly with dimension, and is neither evidence for D nor a measurement of the true T_N. The earlier version's assignment-overwrite bug folded the Dirichlet series, produced spuriously low-rank Grams ($\rho \sim m^{-1.06}$) and a misleadingly optimistic extrapolation; after the fix the decay is far slower, *strengthening* rather than weakening the conclusion that increasing the node count cannot manufacture delocalisation: the required anti-concentration inequality does not come for free.

Table 3: The real Gram proxy after correction of the CRT index folding (the terms n and $n+N$ sharing a CRT index must be accumulated, not overwritten): the Gram has full numerical rank, and ρ decays only as $m^{-0.52}$ (recomputed fit $\rho \approx 1.031\, m^{-0.519}$). Reaching $\rho_{\mathrm{req}}(30030) = 1.065 \times 10^{-10}$ would require $m^* \approx 1.7 \times 10^{19}$, against the hard rank bound $1/\rho_{\mathrm{req}} = 9.39 \times 10^9$ of Proposition 6.2.

m	8	16	32	64	128
ρ_N	0.336	0.158	0.295	0.176	0.0527
d_{eff}	3.79	9.67	7.76	11.19	39.18

7 Discussion and open problems

All theorems here are conditional, on GRH and the stated spectral or non-concentration hypotheses; they do not constitute a proof of the binary Goldbach conjecture, nor do they refute any claim of [12]: S and R remain valid and stronger hypotheses, merely $61\times$ and $1.37 \times 10^5\times$ stronger than the conclusion requires. With $C_{\mathrm{HS}} = 10$ the threshold shrinks by $100\times$ and all qualitative conclusions stand. Q_N below 10^5 is covered by finite verification; the window for S is void for $N \lesssim 10^{10}$ since $\eta_N \le 1/4$ there.

Open problems: (O1) prove the c-uniformity of the HS bound and solve for the optimal c; (O2) supply the eigenfunction anti-concentration inequality $|\langle v_N, e_j\rangle| \le C Q_N^{1/2} d_{\mathrm{eff}}^{-1}$, or prove its equivalence to an approachable bilinear estimate; (O3) make the implicit constants of [12, Theorem 6.2] explicit so that $\mathsf{D}_{\min}$ becomes effective; (O4) provide interval-arithmetic certificates for T_N in a verifiable range (analogous to Platt's finite-height GRH verification); (O5) find an eigenbasis-free formulation of D to reduce circularity risk.

8 Conclusion

The conditional proof of the binary Goldbach conjecture has been advanced to a precise interface. Everything unconditional — the identities, the encoding-layer bypass, the A.4 expansion, the parity decomposition with quantitative barrier, the explicit mean theorem, the prime-power subtraction, the closed-loop budget — is now explicit. On top, D suffices for the conclusion, is strictly weaker than S and R with zero waste and fully explicit constants, and the full proof is reduced to six necessary conditions, the hardest being that the additional hypothesis must carry Type-II information. The threshold is a clean $(\log N)^{-4}$ law; the effective-rank lower bound, the logical-independence classification, the participation-ratio gap and the c-optimisation are in place. What remains is not another computation but a not-yet-formulated eigenfunction anti-concentration inequality and an effective start N_D. Until then the binary Goldbach conjecture remains open — but with a sharper coordinate system for it.

A Verification of constants

Every numerical constant in this paper was recomputed from its definition before inclusion: the Q-integral (2.4) on a 4×10^5-point grid (values in Proposition 6.1; $Q_N \le Q_\infty$ at every tested N, consistent with (6.6)); the threshold ρ_{req} and rank bounds of Propositions 6.2 and 6.1; the 2×2 models of §6.3 (analytically, without array instantiation); the ladder of §6.4; the relaxation factors of Proposition 6.4; and the Gram proxy of §6.6 (with the CRT accumulation fix). Power-law fits over $m \le 128$ are descriptive, not theorems.

References

[1] G. H. Hardy, J. E. Littlewood, *Some problems of 'Partitio numerorum' III*, Acta Math. **44** (1923), 1–70.

[2] H. L. Montgomery, R. C. Vaughan, *The exceptional set of Goldbach's problem*, Acta Arith. **27** (1975), 353–370.

[3] J. Pintz, *A new explicit formula in the additive theory of primes with applications II*, arXiv:1804.09084, 2018.

[4] J. R. Chen, *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, Scientia Sinica **16** (1973), 157–176.

[5] A. Selberg, *On elementary methods in prime number theory and their limitations*, 11th Skand. Mat.-Kongr., Trondheim, 1949, 13–22.

[6] E. Bombieri, *The asymptotic sieve*, Rend. Accad. Naz. Lincei **60** (1976), 243–252.

[7] T. Tao, *The parity problem in sieve theory*, blog essay, 2014; Polymath8b, Res. Math. Sci. **1** (2014).

[8] T. Oliveira e Silva, S. Herzog, S. Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to* 4×10^{18}, Math. Comp. **83** (2014), 2033–2060.

[9] H. A. Helfgott, *The ternary Goldbach conjecture is true*, arXiv:1312.7748, 2013/14.

[10] H. A. Helfgott, D. J. Platt, *Numerical verification of the ternary Goldbach conjecture up to* 8.875×10^{30}, Exp. Math. **22** (2013), 406–409.

[11] D. J. Platt, *Numerical computations concerning the GRH*, Math. Comp. **85** (2016), 3009–3041.

[12] J. M. Li, J. Y. Liu, *A Conditional Proof of the Goldbach Conjecture*, Shandong University, 2026, 63 pp. (preprint, not peer-reviewed; the framework items (F1)–(F5) are quoted from this manuscript).

[13] *Theorem (1+1.9) on the Goldbach conjecture*, arXiv:2606.05224v2, 2026 (preprint; quantitative reference only).

[14] P. Dusart, *Estimates of some functions over primes without R.H.*, arXiv:1002.0442, 2010.

[15] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS Colloq. Publ. **53**, 2004.