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# The Riemann hypothesis for Dirichlet $L$-functions

[after Levent Alpöge]

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## 1. Introduction

L. Alpöge announced on 20 November 2026 that he believes he has found a proof of the Riemann hypothesis, working with Claude, and presented it only as a formalization in Lean 4 [2, pp. 1–2], [1]. This guide follows the chain of declarations from the five theorems of the archive's file `Main.lean` back to its inputs and says what each step means in classical terms, with files and lines. In ordinary notation the five theorems say the following.

**Theorem 1.1** (Alpöge)**.** *Let $\chi$ be a Dirichlet character modulo $N \geq 1$ and $L(s,\chi)$ its $L$-function, continued to $\mathbb{C}$.*

(a) *If $\zeta(s) = 0$, $s \neq 1$ and $s \notin \{-2,-4,-6,\dots\}$, then $\operatorname{Re} s = \frac{1}{2}$.*

(b) *If $\operatorname{Re} s > \frac{1}{2}$, then $L(s,\chi) \neq 0$, unless $\chi$ is the trivial character modulo $N$ and $s = 1$.*

(c) *If $0 < \operatorname{Re} s < 1$ and $L(s,\chi) = 0$, then $\operatorname{Re} s = \frac{1}{2}$.*

In `Main.lean`, part (a) is `riemannHypothesis` (line 4), and `riemannZeta_eq_zero_re` (line 6) states it again in the words of `RiemannZetaZerosOnLine` (`Statements.lean:3`). Part (b) is `LFunction_ne_zero_of_half_lt_re` (line 8). Part (c) is `LFunction_eq_zero_re` (line 10) and, for primitive $\chi$, `LFunction_eq_zero_re_of_isPrimitive` (line 12). The five statements are Mathlib's `RiemannHypothesis` and four definitions in `Statements.lean`, a file that imports only Mathlib. As in Mathlib, $L(s,\chi)$ for imprimitive $\chi$ is the continuation of $\sum \chi(n)n^{-s}$ with $\chi(n) = 0$ for $(n,N) > 1$, so its zeros include those of the Euler factors $1 - \chi^*(p)p^{-s}$ of the primitive character $\chi^*$ inducing $\chi$, at the primes dividing the modulus but not the conductor; these zeros lie on $\operatorname{Re} s = 0$. Mathlib's $\zeta$ and the $L$-functions of trivial

characters are meromorphically continued and given a value at their pole $s = 1$, which is why $s = 1$ is set aside in (a) and (b).

Part (a) is Riemann's hypothesis for $\zeta$ [71]. Its analogue for curves over finite fields is Hasse's theorem for elliptic curves [37] and Weil's for all curves [88]. Stepanov's method [81], extended to all curves by W. Schmidt [72] and by Bombieri [14, §II, Thm 1, and §III], proves it with an auxiliary function that vanishes to high order at the rational points outside a possible pole. Deligne proved the analogue for all smooth projective varieties [24, Thm (1.6)].

Alpöge describes his proof as a descendant of Stepanov's argument in which multiplicity, hard to find over the integers, comes from a chain of theta lifts, each lift mimicking multiplicity [2, p. 1]. The comparison behind it is Chebyshev's, with a period in place of a ratio of factorials. A nonzero integral period has divisibility at most its size, the logarithm of its norm, by the product formula. In the simplest model the period is $(D^a\vartheta)(0)/\Omega^a$, for a theta function $\vartheta$ and the invariant derivation $D$ on a CM curve. Its divisibility at the primes inert in $\mathbb{Q}(\sqrt{-d})$ is, up to $O(a)$, the sum over prime powers $n \le a$ prime to $d$ of $\Lambda(n)(1-\chi_{-d}(n))\,a/n$, its size is about $\log(a!\,\Omega^{-a})$, and the two have the same main term $a\log a$. So what remains of "divisibility at most size" is an upper bound for a centered ramp sum of $\Lambda(n)$ by the same sum twisted by $\chi_{-d}$. In a chain of $m$ lifts the gains in divisibility add up, and at the scale $x$ of the ramp sums the error of the comparison is smallest at $m \approx \sqrt{x}$, one of the two places from which the exponent $\frac{1}{2}$ comes (Remark 3.6). The subsection "The idea" below works the model out.

The proof runs as follows, and Figure 1 draws the chain.

(1) Part (b) for primitive characters implies the rest of Theorem 1.1, by the Euler product and the functional equation (Lemmas 2.1 and 2.2).

(2) For a primitive $\psi$ and the primitive characters $\xi$ that induce the powers of $\psi$, Alpöge uses the centered ramp sums $G(\xi,x) = \sum_{n\le x}\Lambda(n)\xi(n)(x/n-1) + x\frac{L'}{L}(1,\xi)$ for $\xi \ne 1$ and $G(x) = \sum_{n\le x}\Lambda(n)(x/n-1) - x\log x + (1+\gamma)x$, and a positive combination $G_\psi(x)$ of their real parts. A window form of Landau's theorem excludes zeros of the $L(s,\xi)$ with real part above $\theta$, for $0<\theta<1$, when $G_\psi(x) \le x^\theta$ for all $x$ in windows $[Y, Y^\lambda]$, $\lambda > 1$, with $Y$ arbitrarily large (Theorem 3.2).

(3) For every small $\varepsilon > 0$, Alpöge bounds $G_\psi(x)$ by $x^{1/2+O(\varepsilon)}$ for all $x$ in windows $[Y, Y^{3/2}]$ with $Y$ arbitrarily large. The bound comes from an inequality $(\star)$ comparing divisibility with size on certain abelian CM fields, the host fields (Section 3.4 and Proposition 3.4), and Remark 3.6 says where the exponent $\frac{1}{2}$ enters.

(4) The inequality $(\star)$ is proved along a chain of $m \approx \sqrt{x}$ lifts, from divisibility at the inert primes, size, a slope inequality, near-orthogonality and a covolume bound (Section 4).

(5) The host fields come from a bound for Tate–Shafarevich groups of Tate twists, `VWin`, which also enters $(\star)$ through the loss of integrality at the twisted places. Section 5 states the bound and sketches how the archive obtains it.

A declaration is *on the route*, or *on the chain*, if the proof of one of the five theorems of `Main.lean` refers to it, directly or through the declarations it refers to.

Three points should be kept in mind from the start. First, for two of the twenty steps listed in Alpöge's note [2, pp. 3–4], steps 1 and 4, we found no counterpart on the chain: no Stepanov determinant, and no determinant giving the positivity of step 4 (Section 7).

Second, the periods on the route are defined from the kernels. The kernels follow Alpöge's plan and are built from derivatives at 0 of theta functions (Section 4.2). Step 12 of the note appears as an identity between the periods and inner products with Poincaré series

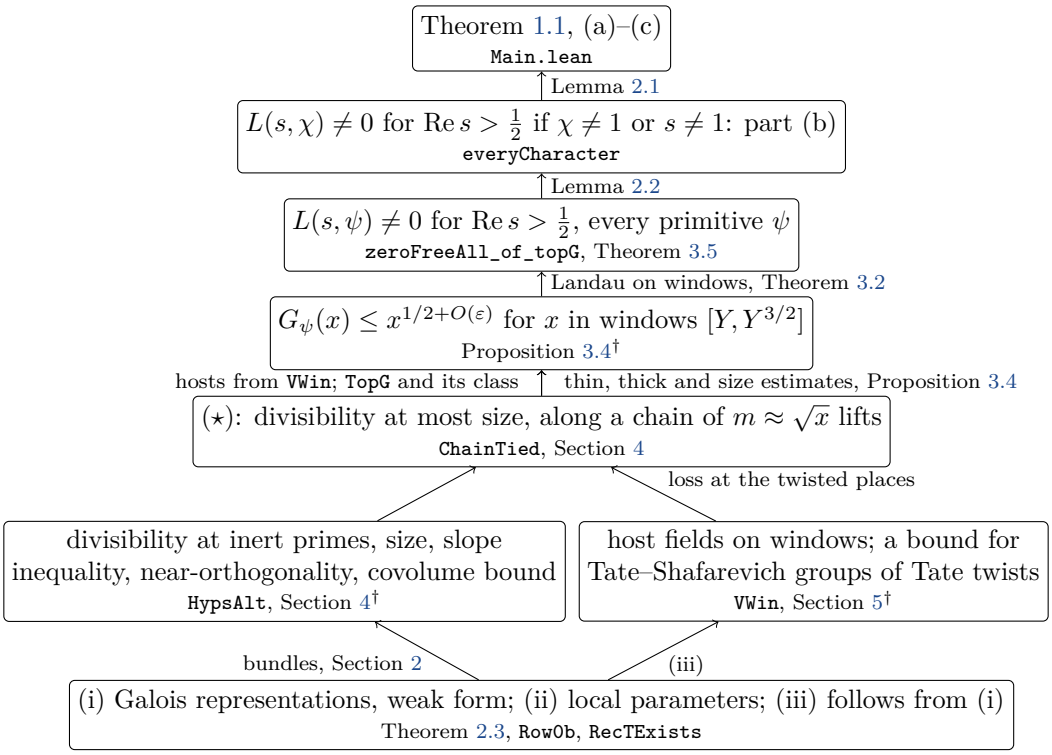



FIGURE 1. The chain from the inputs up to `Main.lean`, as in Sections 2, 3, 4 and 5; an arrow points from a statement to what it is used to prove. A dagger marks a box that holds statements for which we know of no reference; Table 1 lists these statements and Table 2 the classical inputs.

(`MId`, `BesselId`), proved by passing linear relations through Bessel periods of the row forms (`JointCondition`, Section 4.3).

Third, two inputs are weaker in the archive than the published theorems, namely the Galois representations `RowOb` and the local parameters of `RecTExists`. The laws `IsRecT` required of the second are satisfied by a map that, at each finite place $w$ of a number field $E$, sends every supercuspidal representation of $\mathrm{GL}_n(E_w)$ outside an explicit class to a real unramified twist of a single parameter $J_{w,n}$. This map is not the local Langlands correspondence (Sections 2 and 5.3).

The inputs of the proof were built by many people.

- Analytic number theory. Landau's theorem on Dirichlet series with coefficients of one sign [58, §2], and Kronecker's limit formula summed over ideal classes, as by Chowla and Selberg [74].
- Automorphic forms and the local theory. The theta correspondence and the Siegel–Weil formula [89, 45, 91], with Langlands's continuation of Eisenstein series [59]; a unitary mass formula in the tradition of Siegel's for quadratic forms [78]; local class field theory, Zelevinsky's classification for $p$-adic $\mathrm{GL}_n$ [94, Thm 6.1] and the depth-zero part of the local Langlands correspondence for $\mathrm{GL}_n$ over $\mathbb{Q}_p$ [36, 41]; Galois representations attached to polarized regular algebraic cuspidal automorphic representations of $\mathrm{GL}_n$ over CM fields [9, Thm 2.1.1], in the conjugate self-dual case [36], [77, Thm 1.2], [20, Thm 3.2.3], [17, Thm 1.1].

- Arakelov geometry. Bost's slope inequality and arithmetic intersection theory [15, 32, 66, 92], with Hironaka's resolution of singularities in characteristic zero [42].
- Iwasawa theory and Galois cohomology. Kato's Euler system [51, §§8–9] and its Selmer consequence [51, Thm 14.2(1)], the theorem of Ferrero and Washington [28] and Poitou–Tate duality [67, (8.6.10)].

Table 1 lists the statements of the chain for which we know of no reference, and Section 6 compares the classical inputs with their archive forms.

For the idea, read this introduction, Sections 3.1, 3.3, 3.5 and 3.6, Remark 3.6 and Table 1. For the inputs, read Sections 4, 5 and 6. A Lean user who follows the chain from `Main.lean` can start with Section 2, Figure 1 and the call tree of Section 2.4, and finds the file and line of each step in the section that treats it. Section 7 sets Alpöge's twenty steps against the chain, and Section 8 records other results in the archive.

**The archive and how it is read here.** The archive we read is the one posted at https://alpo.ge on 20 November 2026, in fourteen parts. Joined, the parts form `lean.tar.gz`, with SHA-256 `9631b4f020199e580ba86178cb56c6e9b09f690c2369a80dc4334aa93f49c7fc`. It uses Lean 4.33.1 and Mathlib at commit `0df444a3`, tagged v4.33.1 [1, 85]. We give statements and the order of the argument, with sketches, and complete proofs only for the elementary links. For the weak Galois representations through theta lifts, the covolume bound and the near-orthogonality bound, Theorem S (Section 4.5), we give the statements and at most a sketch. We also include some parts of the archive off the chain (Sections 5.4, 7 and 8). We read the statements of the declarations cited and selected proof terms, not the proofs below them, and we have not compiled the archive. Where we say how a proof goes, we name the lemmas its proof term applies, and the steps between them are our reconstruction. Corrections and comments are welcome.

Constants and quantifiers are simplified throughout. "For large $x$" means beyond a threshold depending on the family, on $\varepsilon$ and on the character. Families indexed by the scale and by their members are described one member at a time. A hypothesis bundle is described by what its fields assert, and a field with no short classical statement is named. But the files and lines cited lead to the archive's exact statements.

A. Sutherland has compiled the archive and reports that the five theorems of `Main.lean` depend only on the axioms `propext`, `Classical.choice` and `Quot.sound`, and that every module passes Lean's kernel and lean4lean. The independent kernels nanoda and con-leche accept the exported proof, and con-leche's comparison mode matches the five theorems with the challenge statements, which import only Mathlib [82]. Other independent checkers accept the same export [60]. The source contains no `sorry`, no `native_decide` and no declared axiom [82, 49]. Some declarations are generated by copying others with `addDecl` (`.../CutCptCopy.lean:78`, run at `.../CutMainCpt.lean:945`), and two tactics, `kexactB` and `kcopy`, assign proof terms without the elaborator's check. But in Sutherland's replay Lean's kernel checks these declarations like the rest.

**Notation.** $\Lambda$ is the von Mangoldt function and $\gamma$ Euler's constant. In ramp sums $\psi$ and $\xi$ are primitive Dirichlet characters; for other characters the group or the modulus is given. For primitive $\psi$ we write $\langle\psi\rangle$ for the set of primitive characters that induce the powers of $\psi$, and $o = |\langle\psi\rangle|$ for the order of $\psi$. For a prime $d \equiv 3 \pmod 4$, $\chi_{-d}$ is the character of $\mathbb{Q}(\sqrt{-d})$. The letter $x$ is a scale tending to infinity and $\varepsilon > 0$ is small; $x^{O(\varepsilon)}$ stands for $x^{C\varepsilon}$ with $C$ fixed before $\varepsilon$ is chosen. In the final bound $C = 9$ (Section 3.9). Table 4, at the end, lists the names that Sections 3, 4 and 5 use far from their definitions.

**The idea.** We model the note's steps 2 and 9 [2, pp. 3–4] by the following heuristic computation. Sections 3 and 4 give the archive's exact form, with weights, truncations and errors.

*Divisibility.* Let $K = \mathbb{Q}(\sqrt{-d})$ with $d \equiv 3 \pmod 4$ prime, $\chi = \chi_{-d}$, and $p$ a prime inert in $K$, so that $\chi(p^i) = (-1)^i$. Legendre's formula $v_p(a!) = \sum_{i \geq 1} \lfloor a/p^i \rfloor = (a - s_p(a))/(p-1)$, with $s_p(a)$ the sum of the digits of $a$ in base $p$, gives

$$v_p(a!) - \frac{a}{p^2-1} = \sum_{i \text{ odd}} \frac{a}{p^i} - \frac{s_p(a)}{p-1}.$$

Step 2 of the note states that $(D^a\vartheta)(0)/\Omega^a$, for a theta function $\vartheta$ and the invariant derivation $D$ on a CM curve, is divisible by $p$ to the order $a/p + a/p^3 + \cdots - O(\log a)$, which is this quantity up to $O(\log a)$. For the Weierstrass function of one CM curve a bound with this exponent is a theorem of Katz [52, §IV]; Section 4.4 places it. Weight $p$ by $2\log p$, the logarithm of the norm of $p\mathcal{O}_K$, and sum over the inert primes. As $1 - \chi(p^i)$ is 2 for odd $i$ and 0 for even $i$, and 0 at the split primes,

$$\sum_{p \text{ inert}} 2\log p \sum_{i \text{ odd}} \frac{a}{p^i} = \sum_{(n,d)=1} \Lambda(n)(1-\chi(n))\frac{a}{n}.$$

With a common cutoff $p^i = n \leq B$ the two sides agree, and both diverge as $B$ increases. The archive adds the ramified prime $d$, where $1 - \chi = 1$, and inserts a ramp, counting $(a/n - w)_+$ for a weight $w$ in place of $a/n$ (Section 3.5). With $w = 1$ the divisibility is $\mathcal{D}(a) = \sum_{n \leq a} \Lambda(n)(1-\chi(n))(a/n - 1)$.

*Size.* By Chebyshev's identity and Stirling's formula, $\log a! = \sum_n \Lambda(n)\lfloor a/n \rfloor = a\log a - a + O(\log a)$, so the divisibility of $a!$ at all primes together is its size. We model the size of the period, the logarithm of its norm from $K$, that is of its squared absolute value, by $\log(a!\,\Omega^{-a})$, for a period $\Omega$ of Chowla–Selberg shape. In the archive $\log\Omega = \gamma + \log(2\pi/\mathrm{rd}_E) - \frac{L'}{L}(1,\chi)$, with $\mathrm{rd}_E$ the root discriminant of the host field $E$ below. The terms $a\log(\mathrm{rd}_E/2\pi)$ that this puts in the sizes of the $m$ lifts behind a word $\omega$ (the index of a chain, Section 4.1) cancel together against $-a\log T_\omega$, for a prime $T_\omega$ attached to $\omega$ with $\log T_\omega \geq m\log(\mathrm{rd}_E/2\pi)$ up to terms in the weights and an error $O(\log(a\,\mathrm{rd}_E))$. What remains is the size per lift $M(a) = a\big(\log a - 1 - \gamma + \frac{L'}{L}(1,\chi)\big)$ (Section 4.1).

*Comparison.* A nonzero integral period has divisibility at most its size, by the product formula, so $\mathcal{D}(a) \leq M(a)$ up to errors. This is Chebyshev's argument, with a period in place of a ratio of factorials. Here, as the identity below shows, the two sides have the same main term. Since $\sum_{n \leq a} \Lambda(n)(a/n - 1) = a\log a - (1+\gamma)a + G(a)$, the terms $a\log a$ cancel, and by the definitions of the centered ramp sums

$$\mathcal{D}(a) - M(a) = G(a) - G(\chi, a).$$

So "divisibility at most size" is a one-sided bound for a difference of centered ramp sums. For a host field, where 2 and the auxiliary prime $q$ of Section 3.4 split in $K$, with the cutoff $N$ of ASK at least $a$ and every weight set to 1, $\mathcal{D}(a)$ and $M(a)$ are the archive's ASK and MAIN (Section 3.5). The model is a toy one, since in the archive's inequality $(\star)$ the weights exceed 1 at the thick places (Section 3.5). In the model, $\mathcal{D}(a) \leq M(a)$ bounds $G(a)$ above by $G(\chi, a)$, so an upper bound for the twisted sum bounds $G(a)$ from above. The archive keeps its twisted sums small by its choice of $d$ and $q$. A mean square over the moduli gives them square-root size on average, and by Markov's inequality most pairs are good (Section 3.8).

*From $\chi_{-d}$ to $\psi$.* In the archive the characters are those of $\mathrm{Gal}(E/\mathbb{Q})$ for a CM field $E$, the host field, the compositum of $\mathbb{Q}(\zeta_{2^{e+2}})^+$, the field of $\psi$, $\mathbb{Q}(\sqrt{-q})$ and $K$, with $2^e$ tied to

$x$ (Section 3.4). The weights vary with the place, and $(a/n - w)_+ = w\,(y/n - 1)_+$ is a ramp at $y = a/w$. At the thick places the weights are near $b = \exp(x^{\varepsilon})$, which with $a \approx xb$ puts the ramp at $y \approx x$. Orthogonality on $\mathrm{Gal}(E/\mathbb{Q})$ makes $\mathrm{ASK} - \mathrm{MAIN}$ a weighted sum of $G(\theta, y)$ over the characters $\theta$ (Proposition 3.3). After the thin places are estimated and the thick weights frozen, averaging over translates by the subgroup $A_0$ in the 2-power direction leaves $4o$ of them: those of $\langle\psi\rangle$, which the shares of the shifts combine into $G_\psi$ up to an error, and $3o$ twists of them by $\chi_{-q}$, $\chi_{-d}$ and $\chi_{qd} = \chi_{-q}\chi_{-d}$, which are bounded separately (Section 3.7).

*Why $m \approx \sqrt{x}$.* The gains of $m$ lifts add up, and the archive bounds the gain of the $m$ lifts together from below (Section 4.1). The comparison is then a sum of $m$ comparisons like the one above, with different weights. In $(\star)$, with $TT$ the set of shifts $t$ (Section 3.5), the error has a part $|TT|\,a$ that does not grow with $m$ and parts that do. Dividing by $m\,b\,|TT|$ leaves $x^{O(\varepsilon)}(x/m + m + \sqrt{x})$ (Section 3.7). This error is smallest at $m \approx \sqrt{x}$ (Remark 3.6), the note's chain of theta lifts of length $\sqrt{X}$ [2, p. 1]. The archive takes $x^{1/2-\varepsilon/16} \le m \le 2x^{1/2-\varepsilon/16}$ (`RankVal`, Section 4.2) and obtains $G_\psi(x) \le x^{1/2+O(\varepsilon)}$.

*In the archive* (Section 4) the divisible object is not one number but a map $\mathrm{Bk}_\omega \colon N_\omega \to RR^\vee$ of Hermitian lattices whose entries are periods, one for each word $\omega$, produced by a chain of $m$ steps that Alpöge describes as theta lifts between definite unitary groups [2, p. 4]. At inert $p \ge 5$ the archive has integrality of corrected kernels (`IntInert`) and, for every power series of the form this gives, a bound on its derivatives at 0 with an exponent of terms $a/p^e$, $e$ odd, with caps and some terms omitted, less a loss (`kernel_bounds_generic`). Together they make the actual periods divisible by the lattice ideals that carry the gains (Section 4.4). A lower bound for the gains of the lifts together is proved separately, without the theta kernels. Divisibility raises the slopes of the $N_\omega$. A slope inequality for all the maps at once (`SlopeIneq`) bounds the rank-weighted mean of their slopes, shifted by heights $h(\omega)$, by the slope of $RR^\vee$, which the size bound (`DIV`, (4.1)) compares with the size. A covolume bound and an approximate orthogonality of the columns control the losses. For each lift $j$, `LinkN` bounds the weighted loss of integrality at the twisted places by a dimension $d_{Z,0}(j)$ times the sum of an explicit term and a weighted sum of lengths of Tate–Shafarevich groups of Tate twists over the CM fields, when these are finite and the sum converges. The bound for that sum, `VWin`, is proved from the weak Galois inputs, Kato's Euler system and, in one of several cases, congruences between automorphic forms on $U(3)$ (Section 5).

## 2. From the half-plane to the critical line

This section derives Theorem 1.1 from its part (b) (Lemma 2.1) and part (b) from the case of primitive characters (Lemma 2.2), states the three inputs from which the archive proves (b) (Theorem 2.3), and traces the declarations that connect these statements. Lean declarations are in typewriter type. Files and lines, under the archive's folder `src/`, follow a declaration's name or, where a paragraph names several, stand in small type after the paragraph. Paths use these abbreviations:

| | | | |
|---|---|---|---|
| `DA` | `Development/AutomorphicArithmetic` | `AVT` | `Development/AbelianVarietyTheory` |
| `CB` | `DA/CharacterBoxes` | `WB` | `Development/WeilBound` |
| `PC` | `DA/PrimitiveCosets` | `FSR` | `Formalism/SharpRestriction` |
| `LT` | `DA/DedekindLFunctions/LandauTheorems` | `FLG` | `Formalism/LocalToGlobal` |
| `TPS` | `DA/ThetaCorrespondence/ThetaPairLifts/ThirdPlaceSeries` | `TGC` | `FSR/RamificationFrobenius/TotientGaloisClasses` |
| `AV` | `DA/AutomorphicVarieties` | `GTI` | `FSR/GlobalTheoremInputs.lean` |
| `SWB` | `Formalism/SizeWindowBrackets` | `LRC` | `Formalism/LocalRingCardinality` |

Lean namespaces are dropped where no confusion arises. A path beginning `.../` omits its leading folders, and a line number alone is in the file last named.

**2.1. One term behind five theorems.** The five theorems of `Main.lean` all come from a single proof of part (b) of Theorem 1.1, the statement for the half-plane $\operatorname{Re} s > \frac{1}{2}$. Parts (a) and (c), on the critical line, follow from (b) by the functional equation and the Euler product (Lemma 2.1).

The term `rhOfAll` of the first two theorems has Mathlib's type `RiemannHypothesis`, and the terms of the next two have the archive's types `UpperTheory.EveryCharacter` and `Claim.Strip`. Each type unfolds by definition to the statement of its theorem. The terms that prove the five theorems are in one file, imported through two others. There `allChi` is `UpperTheory.FinalTheorems.everyCharacter`, whose type `UpperTheory.EveryCharacter` is part (b) of Theorem 1.1. The other four theorems are derived from it: `rhOfAll` applies `Formalism.VW.rh_of_every`, `critAll` applies `Formalism.VW.critLine`, and `claimAll` applies `Claim.strip` and `Claim.zeta`. Besides (b), these derivations use short lemmas of the archive proved from Mathlib alone, and from Mathlib the functional equations of $\zeta$ and of primitive Dirichlet $L$-functions, the Euler factors relating $L(s,\chi)$ to $L(s,\chi^*)$ and, in `Claim.zeta`, the nonvanishing of $\zeta$ on $\operatorname{Re} s \geq 1$. They are the classical argument, which we give.

`rhOfAll` Supplements/Kernels/MainTheorems.lean:9, Main.lean:4, Main.lean:6
`UpperTheory.EveryCharacter` CB/EveryCharacter.lean:13
`Claim.Strip` CB/HalfPlane.lean:6
the statements `Statements.lean`
one file `Supplements/Kernels/MainTheorems.lean`
imported through `MainTheoremsAll`, `MainTheoremsCritical`
`allChi` Supplements/Kernels/MainTheorems.lean:7, Main.lean:8
`critAll` Supplements/Kernels/MainTheorems.lean:11–13
`claimAll` Supplements/Kernels/MainTheorems.lean:15–16, Main.lean:10
`Claim.strip` CB/ClaimStrip.lean:59
`Claim.zeta` CB/ClaimStrip.lean:75

**Lemma 2.1.** *Part (b) of Theorem 1.1 implies parts (a) and (c).*

*Proof.* Let $\chi$ be primitive modulo $N \geq 1$ (for $N = 1$, $L(s,\chi) = \zeta(s)$), let $a \in \{0,1\}$ with $\chi(-1) = (-1)^a$, and let $\Lambda(s,\chi) = (N/\pi)^{(s+a)/2}\Gamma(\frac{s+a}{2})L(s,\chi)$. Then $\Lambda(1-s,\bar\chi) = \epsilon(\chi)\Lambda(s,\chi)$ with $|\epsilon(\chi)| = 1$, and for $N = 1$ the function $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfies $\Lambda(1-s) = \Lambda(s)$ [23, Ch. 8, 9].

*(c) for primitive $\chi$.* Let $0 < \operatorname{Re} s < 1$ and $L(s,\chi) = 0$. If $\operatorname{Re} s > \frac{1}{2}$, this contradicts (b), since $s \neq 1$. If $\operatorname{Re} s < \frac{1}{2}$, the factors $\Gamma(\frac{s+a}{2})$ and $\Gamma(\frac{1-s+a}{2})$ are finite and nonzero, so $\Lambda(s,\chi) = 0$, hence $\Lambda(1-s,\bar\chi) = 0$ and $L(1-s,\bar\chi) = 0$ with $\operatorname{Re}(1-s) > \frac{1}{2}$ and $1-s \neq 1$; this contradicts (b). So $\operatorname{Re} s = \frac{1}{2}$.

*(c) in general.* If $\chi$ modulo $N$ is induced by the primitive $\chi^*$, then $L(s,\chi) = L(s,\chi^*)\prod_{p\mid N}(1-\chi^*(p)p^{-s})$, and the factors do not vanish for $\operatorname{Re} s > 0$. A zero of $L(s,\chi)$ with $0 < \operatorname{Re} s < 1$ is therefore a zero of $L(s,\chi^*)$, and the primitive case applies.

*(a).* Let $\zeta(s) = 0$ with $s \neq 1$ and $s \notin \{-2,-4,\dots\}$. By (b) with $N = 1$, $\operatorname{Re} s \leq \frac{1}{2}$, and by (c) $\operatorname{Re} s = \frac{1}{2}$ if $\operatorname{Re} s > 0$. Suppose $\operatorname{Re} s \leq 0$. As $\zeta(0) = -\frac{1}{2}$, $s \neq 0$, so $\Gamma(s/2)$ is finite and nonzero and $\Lambda(s) = 0$. Then $\Lambda(1-s) = 0$; since $\operatorname{Re}(1-s) \geq 1$, $\Gamma(\frac{1-s}{2})$ is finite, so $\zeta(1-s) = 0$ with $\operatorname{Re}(1-s) > \frac{1}{2}$ and $1-s \neq 1$, against (b). □

In the archive, `critLine` runs the primitive case of this argument, and `rh_of_every` gives (a) through `Formalism.VW.rh_of_right`, which treats all of $\operatorname{Re} s < \frac{1}{2}$ at once by Mathlib's `riemannZeta_one_sub`.

`critLine` `Formalism/GammaFactorNonvanishing/CritLine.lean:11`
`rh_of_every` `Formalism/GammaFactorNonvanishing/CritLine.lean:29`
`Formalism.VW.rh_of_right` `Formalism/GammaFactorNonvanishing/RhOfRight.lean:5`
`riemannZeta_one_sub`, applied at `Formalism/GammaFactorNonvanishing/RhOfRight.lean:28`

**2.2. Primitive characters.** Alpöge reduces (b) to primitive characters as `UpperTheory.Top.imprimitive`. Its hypothesis `ZeroFreeAll` says that $L(s,\psi) \neq 0$ for every primitive character $\psi$, the trivial one included, and every $s$ with $\operatorname{Re} s > \frac{1}{2}$; at $s = 1$ and $\psi = 1$ this concerns Mathlib's value of $\zeta(1)$, which is not zero. The proof is the following lemma, through Mathlib's `DirichletCharacter.LFunction_changeLevel`.

`UpperTheory.Top.imprimitive` `CB/Imprimitive.lean:5`
`ZeroFreeAll` `TGC/ExplicitFormulaErrorOfHostConditions.lean:18`

**Lemma 2.2.** *If $L(s,\psi) \neq 0$ for every primitive $\psi$ and every $s$ with $\operatorname{Re} s > \frac{1}{2}$, $(\psi, s) \neq (1,1)$, then part (b) of Theorem 1.1 holds.*

*Proof.* Let $\operatorname{Re} s > \frac{1}{2}$, with $\chi \neq 1$ or $s \neq 1$, and let $\chi$ modulo $N$ be induced by $\chi^*$. If $\chi^*$ is trivial, then $\chi$ is the trivial character modulo $N$, so $s \neq 1$; in either case $(\chi^*, s) \neq (1,1)$, and $L(s,\chi^*) \neq 0$. Then $L(s,\chi) = L(s,\chi^*)\prod_{p\mid N}(1-\chi^*(p)p^{-s})$, and $|\chi^*(p)p^{-s}| \leq p^{-\operatorname{Re} s} < 1$. (This application of Mathlib's `LFunction_changeLevel` requires $\chi^* \neq 1$ or $s \neq 1$.) □

**2.3. Three inputs.** Part (b) follows from three inputs: Galois representations attached to polarized regular algebraic cuspidal automorphic representations over CM fields, local parameters for $\mathrm{GL}_n$ at the finite places of number fields, and a bound `VWin` for Tate–Shafarevich groups of Tate twists. All three are proved in the archive, the first two in forms weaker than the published theorems.

The archive proves (b) as `everyCharacter`, which is `everyCharacter_of_row` applied to `UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter`, a proof of (i) below. The theorem `everyCharacter_of_row` has the single hypothesis (i). It applies `every_character_of_literature_of_vWin`, which, with its bundle unfolded, reads as follows.

`everyCharacter` `CB/MainConclusion.lean:10`
`everyCharacter_of_row` `CB/MainConclusion.lean:7`
`UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter` `DA/Foundations/GaloisRepresentations/IntegralStructures/RowSrc.lean:6`
`every_character_of_literature_of_vWin` `TPS/EveryCharacterOfLiteratureOfVWin.lean:810`

**Theorem 2.3** (Alpöge)**.** *Suppose that*

*(i)* `RowOb IsRecT` *holds, that is, for every CM field $E$, every prime $\ell$, every isomorphism $\iota\colon \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$ and every regular algebraic cuspidal automorphic representation $\pi$ of $\mathrm{GL}_a(\mathbb{A}_E)$ polarized by a Hecke character of $E^+$, there is a continuous semisimple $r\colon \mathrm{Gal}(\bar{E}/E) \to \mathrm{GL}_a(\bar{\mathbb{Q}}_\ell)$ with the properties of Section 5.3 for every local parameter map* rc *satisfying* `IsRecT`*;*

*(ii)* `RecTExists` *holds, that is, at every finite place $w$ of every number field $E$ there is a map from isomorphism classes of smooth irreducible complex representations of $\mathrm{GL}_n(E_w)$, all $n$, to Weil–Deligne representations, satisfying* `IsRecT` *(Section 5.3);*

*(iii)* `VWin` *holds (Section 5.1).*

*Then part (b) of Theorem 1.1 holds.*

The bundle `OpenLiteratureStatements` has three fields, two copies of (i) and one of (ii). `everyCharacter_of_row` fills it with (i), `recTExists` and (i), and supplies (iii) by `vwin_of_row`, which proves (iii) from (i). The archive proves (ii) as `recTExists`. So `everyCharacter` has no hypotheses. Two further theorems also conclude `EveryCharacter`, namely `plugAll` and `everyE`$_2$, which applies `plugAll`. Neither is on the route, as the proof map of [53] also finds. In input (i) the local–global comparisons `PropsB` hold only under stated conditions, and input (ii) asserts injectivity and the laws of `IsRecT` for smooth twists other than real powers of the norm, central characters beyond their absolute value, duals and depth zero only on representations built from characters and, over $\mathbb{Q}_p$, from twists of depth-zero supercuspidals, whose inertia and zero monodromy it fixes (Section 5.3).

`OpenLiteratureStatements TPS/EveryCharacterOfLiteratureOfVWin.lean:215`
`recTExists DA/Foundations/LocalLanglands/LocalCentralUnits/RecTExists.lean:20`
`vwin_of_row AV/ModularityArgument/ThresholdArguments/DenseThresholdCases/VwinOfRow.lean:4`
`plugAll CB/EveryCharacter.lean:16`
`everyE`$_2$ `DA/ThetaCorrespondence/FockModels/EveryCharacterE.lean:67`

**2.4. The reduction.** The proof of Theorem 2.3 derives from the three inputs the two hypotheses of Theorem 3.5, namely host fields on windows for each primitive $\psi$ and the interface `TopG`, which turns such hosts into a family that satisfies the conditions of Section 4.1, and with them the inequality $(\star)$, on its windows. For every primitive $\psi$, Theorem 3.5 then excludes zeros of $L(s,\psi)$ with $\operatorname{Re} s > \frac{1}{2}$, and Lemma 2.2 gives (b).

The proof term of part (b) unfolds as follows. Under each theorem stand the terms from which its proof term is built, and in the last two lines three of the arguments of `forall_goal_of_shape_of_hyps`; a line number alone is in the file of the term it is indented under. The fifteen arguments of `topCovolSlim2_use_alt_alt`, the covolume inputs of Section 6, are not listed.

`everyCharacter CB/MainConclusion.lean:10` part (b)
  `galoisLocalGlobalStrong_recParameter .../IntegralStructures/RowSrc.lean:6` input (i)
  `everyCharacter_of_row :7`
    `recTExists .../LocalCentralUnits/RecTExists.lean:20` input (ii)
    `vwin_of_row .../DenseThresholdCases/VwinOfRow.lean:4` input (iii), from (i)
    `every_character_of_literature_of_vWin`
      `TPS/EveryCharacterOfLiteratureOfVWin.lean:810` Theorem 2.3
      `published_results_eigen :666` the first bundle
      `published_results_first :727` the second bundle
      `theta_integrals_agree_and_local_matching_of_literature :565` the two theta statements
      `topCovolSlim2_use_alt_alt TPS/TopCovolSlim2Use.lean:16` the covolume bound
      `local_langlands_gl_recT_exists_of_literature :313 RecTExists`, from (ii)
      `every_character_of_theta_integrals_local_matching`
        `TPS/SiegelEisensteinSecondTermHypotheses.lean:69`
        `green_of_regular .../SumRangeEqRoots.lean:225` Green's parametrization
        `boundAtTwoThree_linked_of_recT TPS/At23OfAll.lean:24` the bound at $\ell = 2, 3$ (Section 4.5)
        `nonempty_hyps_definite_side_of_closed_bound :44` an element of `HypsAlt`
        `every_character_of_green_v_win TPS/TopPlug.lean:16`
          `imprimitive CB/Imprimitive.lean:5` Lemma 2.2
          `zero_free_all_of_green_v_win :12`
            `zeroFreeAll_of_topG CB/ZeroFreeAllOfTopG.lean:26` Theorem 3.5 with $\lambda = \frac{3}{2}$
            `supplyChi CB/SupplyChi.lean:8` host fields on windows, from (iii)
            `forall_goal_of_shape_of_hyps TPS/ShapeOf.lean:31` the interface `TopG`
              `shape_closed_link_n_of_green TPS/TopShape.lean:33`
              `trivial`, `trivial TPS/TopPlug.lean:15` hypotheses of `TopShapeAlt` set to `True`

The two bundles are `UnitaryGroupThirdPlaceHypotheses` and `ThetaLocalInputs`, most of whose fields are named after published theorems; the first is filled from archive theorems together with (i) and (ii), the second from archive theorems that use neither. One field of the first bundle, `siegelEisensteinSimplePole`, is named after the simple pole of the Siegel Eisenstein series but has type `UnusedSlot72`, which is `True`, so it asserts nothing. From another field the archive proves Green's parametrization of the cuspidal irreducible characters of $\mathrm{GL}_d(\mathbb{F}_p)$, $p$ prime, by the monic irreducible polynomials of degree $d$ over $\mathbb{F}_p$ with nonzero constant term [33] (`GlobalTheory.GLq.Green`). The two theta statements, `ThetaIntegralsAgree` and `LocalMatching`, compare theta integrals from two spaces into one unitary group and are proved from `EisBL` and `SiegelWeilStd` (Section 4.3). The covolume bound `CovolumeBoundOfX` (Section 4.6) is proved from the covolume inputs of Section 6. The element of `HypsAlt`, the definite side (Section 4.5), is built from the bundles, the theta statements, the covolume bound, (ii) and the bound at $\ell = 2, 3$, which `boundAtTwoThree_linked_of_recT` proves from the field `nLink` of the first bundle. Green's parametrization and the element of `HypsAlt` are both used in the proof of the interface `TopG`: the first is the argument of `shape_closed_link_n_of_green`, the second the last argument of `forall_goal_of_shape_of_hyps`.

`UnitaryGroupThirdPlaceHypotheses` `TPS/BernsteinUnitaryProductIrreducible.lean:22`
`siegelEisensteinSimplePole` `TPS/BernsteinUnitaryProductIrreducible.lean:74`
`UnusedSlot72` `.../DiscreteSpectrumResidualClassification.lean:7`
`GlobalTheory.GLq.Green` `AVT/Definitions/NonGaloisDescent/LabelDatum.lean:108`
`TopG` `CB/ZeroFreeAllOfTopG.lean:12`
`TopShapeAlt` `TPS/TopShapeAlt.lean:24`

## 3. Divisibility at most size excludes zeros right of $\frac{1}{2}$

This section follows the archive from the inequality $(\star)$ to `ZeroFreeAll`. Sections 3.1 to 3.3 reduce the absence of zeros right of $\frac{1}{2}$ to a bound $G_\psi(x) \le x^{1/2+O(\varepsilon)}$ on windows, Sections 3.4 to 3.7 derive that bound from $(\star)$ on host fields, and Sections 3.8 and 3.9 supply the host fields and put the steps together. Every step of the deduction is proved in the analytic part of the archive, with three exceptions: $(\star)$ comes from the chain of kernels of Section 4, the host fields of Section 3.8 come from the bound `VWin` of Section 5, and the families on windows from the interface `TopG` and its class (Section 3.9).

**3.1. Ramp sums.** By the explicit formula the ramp sums below are sums over the nontrivial zeros of $L$-functions, up to $O(\log y)$ (Remark 3.1). An upper bound $x^\theta$ on windows for the positive combination $G_\psi$ of their real parts (Section 3.2) then excludes zeros of these $L$-functions with real part above $\theta$ (Theorem 3.2). Alpöge works with the ramp sums

$$G(y) = \sum_{n \le y} \Lambda(n)\Big(\frac{y}{n} - 1\Big) - y \log y + (1+\gamma)y, \tag{3.1}$$

$$G(\xi, y) = \sum_{n \le y} \Lambda(n)\xi(n)\Big(\frac{y}{n} - 1\Big) + y\,\frac{L'}{L}(1, \xi) \qquad (\xi \ne 1 \text{ primitive}), \tag{3.2}$$

with $G(1, y) = G(y)$. In the archive they are `Gz` and `An.gChi`.

`Gz` `Formalism/StatementOC.lean:13`
`An.gChi` `TGC/Analytic.lean:23`

*Remark* 3.1 (classical). Since $\frac{1}{2\pi i}\int_{(2)} u^s \frac{ds}{s(s-1)} = \max(u-1, 0)$ for $u > 0$, the sum $\sum_{n \le y} \Lambda(n)\xi(n)(y/n - 1)$ is $\frac{1}{2\pi i}\int_{(2)} -\frac{L'}{L}(s, \xi)\,\frac{y^s\,ds}{s(s-1)}$. Moving the line to the left, the

TABLE 1. Statements of the chain for which we know of no reference, each with what it is needed for.

| Statement | Declaration | Needed for |
|---|---|---|
| A one-sided inequality "divisibility at most size" on a 2-power cyclotomic tower, turned into a bound for ramp sums | `ch_balance`, `core_gm` (Propositions 3.3 and 3.4) | the window bound for $G_\psi$ |
| Theta integrals from two spaces on a unitary group that is not quasi-split are proportional, with the change of the constant under hyperbolic planes | `ThetaIntegralsAgree` (Section 4.3) | the reduction (Section 2) |
| Seesaw identities along the whole chain | `unfoldings_of_chain`, `iter_colPer`, `MainX` (Section 4.3) | the Bessel period of the iterated lift as an iterated integral of column periods, and the transport of linear relations between periods from the top of the chain to the bottom |
| Integrality of the corrected theta kernels at inert $p \geq 5$, over CM fields of growing degree (one curve, one variable: Katz and others, Section 4.4) | `IntInert` (Section 4.4) | divisibility of the actual periods by the lattice ideals, `LinkDIVt` |
| Near-orthogonality between different chains of lifts, off the degenerate orbits (Theorem S) | `TheoremSSelected` (Section 4.5) | `LinkS` |
| A point-sum bound for the cost of all $m$ lifts of a word | `StepThmBG` (Section 4.5) | the diagonal term of each word |
| A bound for the loss of integrality at the twisted places by lengths of Tate–Shafarevich groups | `LinkN` (Section 4.5) | `DIVinN`, and so `LinkDIV` |
| A lower bound for mean heights in signature $(p, q)$, $q \geq 2$, $p \geq 2q + 1$ | `VolumeTheoremX` (Theorem 4.2) | the covolume bound, Theorem 4.1 |
| A bound for plus parts polynomial in the modulus and the twist | `CorGK` (Section 5) | Count I of `VWin` |
| A bound for $\#H^2(\mathcal{O}_L[1/\ell], \mathbb{Z}_\ell(e+1))$ for totally complex $L$ containing $\mu_\ell$ ($\mu_4$ if $\ell = 2$), so that $\ell \leq [L : \mathbb{Q}] + 1$ | Lemma K, `lemmaK` (Section 5.2) | `ThmBab`, and so the windows of `VWin`; Counts I, II, III and V of the plus part |

residue at $s = 1$ is $y \log y - (1 + \gamma)y$ for $\xi = 1$ and $-y\frac{L'}{L}(1, \xi)$ for $\xi \neq 1$, and the residue at a nontrivial zero $\rho$ of multiplicity $m_\rho$ is $m_\rho y^\rho/(\rho(1 - \rho))$. So, as $y \to \infty$, $G(\xi, y) = \sum_\rho m_\rho\, y^\rho/(\rho(1 - \rho)) + O_\xi(\log y)$, the sum converging absolutely [23, Ch. 17, 19]. The corrections in (3.1) and (3.2) remove the contribution of $s = 1$.

**3.2. The level of a character.** Landau's argument needs one function in which the zeros of all the $L(s, \xi)$, $\xi \in \langle\psi\rangle$, appear with positive coefficients, and $G_\psi$ is that function. For a primitive $\psi$, Alpöge's `Level.cyc` keeps the characters $\langle\psi\rangle$, with $o$ their number, and its `Level.Gm` takes $G$ with coefficient 1 and the nontrivial ones with the weight `Level.wt`, which is $1/(3o + 1)$ here because `Level.cyc` has no exceptional character (`psK` is `none`):

$$G_\psi(y) = G(y) + \frac{1}{3o + 1} \sum_{\xi \in \langle\psi\rangle,\ \xi \neq 1} \operatorname{Re} G(\xi, y). \tag{3.3}$$

By Remark 3.1, $G_\psi(y)$ is a sum of the terms $\mathrm{Re}(y^\rho/(\rho(1-\rho)))$ over the nontrivial zeros of the $L(s,\xi)$, $\xi \in \langle\psi\rangle$, with positive coefficients, up to $O_\psi(\log y)$.

`Level.cyc` TGC/ExplicitFormulaErrorOfHostConditions.lean:121
`Level.Gm` TGC/ExplicitFormulaErrorOfHostConditions.lean:38
`Level.wt` TGC/ExplicitFormulaErrorOfHostConditions.lean:36

**3.3. Landau's theorem on windows.** Landau's theorem turns an upper bound for $G_\psi$ into a zero-free half-plane. The archive needs it with the bound assumed only on windows $[Y, Y^\lambda]$, since the bound `VWin` behind the host fields holds on windows that depend on the pair of primes $(q, d)$ from which the fields are built (Sections 3.4 and 3.8). The archive proves the following as `Comparison₂.formB`, through `Deduction.zeroFree_of_rpow_window`. The statement is `FormB` with `HasWindows`.

`Comparison₂.formB` PC/Imports.lean:8
`Deduction.zeroFree_of_rpow_window` LT/ZeroFreeOfRpowWindow.lean:11
`FormB` TGC/Window.lean:30,33
`HasWindows` TGC/Window.lean:7

**Theorem 3.2** (Landau's theorem, in the archive's window form)**.** *Let $\psi$ be primitive, $0 < \theta < 1$ and $\lambda > 1$. Suppose that for every $y$ there is $Y \geq y$ with $G_\psi(x) \leq x^\theta$ for all real $x \in [Y, Y^\lambda]$. Then for every $\xi \in \langle\psi\rangle$ the function $L(s,\xi)$ has no zeros with $\mathrm{Re}\, s > \theta$.*

*Sketch of proof.* Put $g(u) = G_\psi(e^u)$ and $\Phi(s) = \int_0^\infty g(u) e^{-su}\, du$ for $\mathrm{Re}\, s > 2$. What the proof uses about $\Phi$ is the archive's `PoleData`: $g$ is measurable with $|g(u)| \leq Ce^{2u}$ for $u \geq 0$; $\Phi$ continues holomorphically to $\mathrm{Re}\, s > 0$ off a set $Z$ of points with positive real part and no accumulation point in that half-plane; each point of $Z$ is a simple pole with nonzero residue, real and positive when the pole is real. For $g$ coming from $G_\psi$, $Z$ is the set of zeros of the $L(s,\xi)$ in $\mathrm{Re}\, s > 0$ and their conjugates, so not the trivial zeros, and the residue at $\rho$ is a positive combination of multiplicities times $1/(\rho(1-\rho))$, as in Remark 3.1. The proof uses no sign at a nonreal pole, but a positive residue at a real pole right of $\theta$. The window hypothesis enters as `WindowHypU`, which asks, for every $\eta > 0$ and arbitrarily large $U$, for a majorant $R \geq 0$ with $g \leq R$ on $[U, (1+\delta)U]$ and $\int_U^{(1+\delta)U} R(u) e^{-\theta u}\, du \leq e^{\eta U}$. Here $\delta = \lambda - 1$ and $R(u) = e^{\theta u}$.

`PoleData` DA/DedekindLFunctions/LandauZeroFreeCriterion.lean:15
`WindowHypU` DA/DedekindLFunctions/LandauZeroFreeCriterion.lean:25

The proof compares Taylor coefficients of $\Phi$. The $k$-th coefficient at a $\sigma$ chosen large in terms of $\delta$ is $c_k(\sigma) = \int g(u)\, u^k e^{-\sigma u}\, du/k!$ up to the sign $(-1)^k$, and for $k = \lceil (1+\delta/2)\sigma U \rceil$ the kernel $u^k e^{-\sigma u}/k!$ is concentrated on the window, the tails being bounded through $|g| \leq Ce^{2u}$. On the window $|g| \leq 2R - g$, so there the integral of $|g|$ against the kernel, which bounds the coefficient at every $\sigma + i\gamma$, is at most twice that of $R$ minus that of $g$, the window part of $c_k(\sigma)$. The result is `landau_window_key`, which gives one $\sigma > 2$ and one $C$ that serve every $\eta > 0$. For each $\eta$ there are $U$ arbitrarily large and $k \geq U$ such that, for every real $\gamma$,

$$\left\| \frac{(-1)^k \Phi^{(k)}(\sigma + i\gamma)}{k!} \right\| + \mathrm{Re}\, \frac{(-1)^k \Phi^{(k)}(\sigma)}{k!} \leq \frac{Ce^{\eta U}}{(\sigma - \theta)^k}.$$

Now suppose a pole has real part above $\theta$. By `landau_dichotomy` either a real pole $\beta > \theta$ is the only pole within some distance $r \leq \sigma - \theta$ of $\sigma$, or for some $D \leq \sigma - \theta$ the open disc of radius $D$ about $\sigma$ has no pole and a pole $\rho_1$ has $\sigma - \mathrm{Re}\, \rho_1 < D$. In the second case `landau_nearest` moves the center to $\sigma + i\gamma$, $\gamma$ near $\mathrm{Im}\, \rho_1$, where one pole is the only one within some $r \leq D$. In both cases the unique pole, at distance $d < r$ from the

center, gives the $k$-th coefficient a term of size $d^{-k-1}$, positive at a real pole, and Cauchy's estimate at a radius between $d$ and $r$ makes the rest smaller, and in the second case also the coefficient at $\sigma$. For $\eta$ small and $U$ large this contradicts the bound. The conclusion, `landau_window`, is that every $\rho \in Z$ has $\operatorname{Re}\rho \le \theta$. The range $\operatorname{Re} s \ge 1$ is treated separately, by `charFamily_LFunction_ne_zero_of_one_le_re`. □

`landau_window_key` LT/landauWindow.lean:40
`landau_dichotomy` LT/LandauDichotomy.lean:24
`landau_nearest` LT/LandauNearest.lean:36
`landau_window` LT/landauWindow.lean:83
`charFamily_LFunction_ne_zero_of_one_le_re` LT/CharFamilyLFunctionNeZeroOfOneLeRe.lean:9

Landau's theorem on Dirichlet series with coefficients of one sign is classical [58, §2]; the same argument applies to Laplace transforms of functions of one sign. For a real function whose Mellin transform has abscissa of holomorphy $\Theta$, is regular at $s = \Theta$ and is meromorphic a little to the left of the line $\operatorname{Re} s = \Theta$, Kaczorowski and Pintz find a sign change in every interval $[Y^{1-\varepsilon}, Y]$ with $Y$ large, with oscillation of size $x^{\Theta-\varepsilon}$ [50, Thm 1, Cor. 1].

**3.4. Host fields.** To bound $G_\psi$ from above, the comparison of divisibility with size needs fields in which the twisted ramp sums are small, and the archive keeps them small by its choice of two primes $q$ and $d$ (Section 3.8). These are the host fields, and ASK and MAIN are written in coordinates on their Galois groups. Alpöge's `WinPair` (`CB/ExistsPageN.lean:27`) asks, for given $\varepsilon$ and $Y$, that $q$ and $d$ be primes with $q \equiv 3 \pmod 8$, $d \equiv 7 \pmod{16}$,

$$\tfrac{1}{2}Y^{\varepsilon/5} < \log q \le Y^{\varepsilon/5}, \qquad \tfrac{1}{2}Y^{\varepsilon/5}\log^2 Y < \log d \le Y^{\varepsilon/5}\log^2 Y,$$

$q \mid d+1$, and $p \mid d+1$ for every odd prime $p \le Y^{\varepsilon/5}$. Then $-d \equiv 1$ modulo 8, modulo $q$ and modulo each odd $p \le Y^{\varepsilon/5}$, so 2, $q$ and these primes split in $K = \mathbb{Q}(\sqrt{-d})$; in particular every odd prime inert in $K$ exceeds $Y^{\varepsilon/5}$.

Given a primitive $\psi$ and an integer $e \ge 0$, a host field is a number field $E$ as in `IsChiField`. It lies in $\mathbb{Q}(\zeta_{N'})$, with $2^{e+2}$, the conductor of $\psi$, $q$ and $d$ dividing $N'$, and is the fixed field of the automorphisms $\zeta \mapsto \zeta^a$ with $a \equiv \pm 1 \pmod{2^{e+2}}$, $\psi(a) = 1$ and $(a/q) = (a/d) = 1$. So $E$ is the compositum of $\mathbb{Q}(\zeta_{2^{e+2}})^+$, the field cut out by $\psi$, $\mathbb{Q}(\sqrt{-q})$ and $\mathbb{Q}(\sqrt{-d})$. It is an abelian CM field, as the archive proves (`isCMField_of_isChiField`). For $e$ beyond a bound depending on $\psi$, the hosts that `fieldChi` supplies have the group $A$ below of order $c\,2^e$, with $c \ge 1$ depending on $\psi$. On the chain `WinDatum` ties this order to the scale, asking $x^{2+8\varepsilon} \le c\,2^e \le 2x^{2+8\varepsilon}$, which `winPairs` obtains from `PageN`, the same tie for $2^e$ alone, at the scale $x/c^{1/(2+8\varepsilon)}$.

`IsChiField` CB/ExistsPageN.lean:19
`isCMField_of_isChiField` AV/CMFieldExponentialBounds/IsCMFieldOfIsChiField.lean:7
`fieldChi` CB/FieldChi.lean:20–26
`WinDatum` CB/TwistForm.lean:19
`winPairs` CB/WinPairs.lean:5
`PageN` CB/ExistsPageN.lean:39

In the archive the Galois group is a `Frame` and a `HostRaw`: $\mathrm{Gal}(E/\mathbb{Q}) \cong \{\pm 1\}^2 \times A$ with $A$ finite abelian, a surjection $h$ from $A$ onto $(\mathbb{Z}/2^{e+2})^\times/\{\pm 1\}$, and a cyclic subgroup $A_0 \subset A$, of index $o$, on which $h$ is injective. The CM type $\Phi$ induced from $K$ is $\mathrm{Gal}(E/K)$ (`Frame.Psi`). In the additive coordinates $(\mathbb{Z}/2)^2 \times A$ in which ASK and MAIN are written (`toE`), the first coordinate is the sign on $\sqrt{-d}$, so $\Phi$ is the set where it vanishes (`InPhi`). Complex

conjugation is $c = (1, 1, 0)$, the inertia group at $d$ is $I_d = \{0, \iota_d\}$ with $\iota_d = (1, 0, 0)$, and $\chi_C$ is the character equal to 1 on $\Phi$ and $-1$ off it.

`Frame` `TGC/ZsmulGeneratorEqZeroOfProjection.lean:15`
`HostRaw` `TGC/ConjEMemHostFIff.lean:12`
$\mathrm{Gal}(E/\mathbb{Q}) \cong \{\pm 1\}^2 \times A$ `TGC/ConjEMemHostFIff.lean:27`
`Frame.Psi` `TGC/ZsmulGeneratorEqZeroOfProjection.lean:112`
`toE` `PC/RecH.lean:10`
`InPhi` `Formalism/ThickAverageLoss/PartnersToFactorsAt.lean:20`
`toE_bij` `PC/ToEBij.lean:37`

**3.5. The inequality ($\star$) between divisibility and size.** The inequality ($\star$) is the archive's form of the comparison $\mathcal{D}(a) \le M(a)$ of Section 1, with weights and shifts, summed over a family, and with an error term. Its two sides are ASK, for divisibility, and MAIN, for size. Fix a host field. For $a > 0$, $t \in A$, a weight $w > 0$ on $\Phi$ and a cutoff $N \ge 1$, the archive defines

$$\mathrm{ASK}(a, t, w) = \sum_p 2 \log p \sum_{\substack{1 \le i \le N \\ \mathrm{Frob}_p^i \notin \Phi}} \max\Big(\frac{a}{p^i} - w(c\,\mathrm{Frob}_p^i\,\tau_t), 0\Big) + \log d \cdot \mathrm{ASK}_d,$$

$$\mathrm{MAIN}(a, w) = a\Big(\log a - 1 - \gamma + \frac{L'}{L}(1, \chi_{-d}) - \frac{1}{|\Phi|}\sum_{g \in \Phi} \log w(g)\Big),$$

where $\mathrm{Frob}_p$ is a Frobenius element, at a ramified prime a chosen lift, $p$ runs over the primes up to $N$ other than 2, $q$ and $d$ with $\mathrm{Frob}_p \notin \Phi$ (`IsInert`), $\tau_t$ is the element that `topPl` attaches to $t$, with $g \mapsto cg\tau_t$ the archive's `partner`, and $\mathrm{ASK}_d$ the corresponding term at the ramified prime $d$. The archive writes $q_1$ for $q$ (`arith`). The primes $p$ in the sum are those inert in $K$, and since $\Phi$ has index 2, $\mathrm{Frob}_p^i \notin \Phi$ means that $i$ is odd. The value $\frac{L'}{L}(1, \chi_{-d})$ is $\ell(\chi_C)$ for a function $\ell$ on characters that, at every nontrivial $\theta$, is $\frac{L'}{L}(1, \cdot)$ of the matching primitive character (`HostAddCharCompatible`).

ASK and MAIN `SWB/FirstPrimeNeSecondPrime.lean:19–89`
`IsInert` `SWB/FirstPrimeNeSecondPrime.lean:72`
`partner` `SWB/FirstPrimeNeSecondPrime.lean:19`
`arith` `PC/RecH.lean:23`
`HostAddCharCompatible` `PC/CoreAtH.lean:18–20`

At the scale $x$ put $b = \exp(x^\varepsilon)$ and $X' = \exp(x^{\varepsilon/3})$. The family consists of a set $TT \subset A$ with $|TT| \ge x^{-\varepsilon/8}|A|/2$ and sizes $a_t$, $|a_t/b - x| \le 1$; a chain length $m$ and an integer $k$ with

$$x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \qquad xm \le k \le |A|^2 x^{1/2-\varepsilon};$$

a set $T_2 \subset A_0$ with $|T_2| \ge x^{-\varepsilon}|A|$; and, for each $u \in A_0$ and $j < m$, a weight $w_{u,j}$ on $\Phi$ with $b \le w_{u,j} \le b + \eta + 1$, where $\eta \ge 0$ and $6|\Phi|(x+1)(\eta+1) \le b/x$ (`CostWindows`), at the thick places $c\,\iota_d(s+u)$, $s \in T_2$ (`IsThickAt`), and $w_{u,j} \le 2b'$ with $b'X' \le xb$ at the other places of $\Phi$, the thin places. In the archive these conditions are `RestSize` and, from `ErrWindows`, the sizes of $m$ and $k$, which `ch_core` assembles into `SizeWindows`, and `SizesA`, which includes $N \ge a_t/\min w$; the inclusion $T_2 \subset A_0$ is a separate hypothesis of `ch_core`. The inequality is then, for every $u \in A_0$,

$$\sum_{t \in TT} \sum_{j < m} \big[\mathrm{ASK}(a_t, t, w_{u,j}) - \mathrm{Re}\,\mathrm{MAIN}(a_t, w_{u,j})\big] \le x^{O(\varepsilon)}\Big[|TT|\,a\Big(1 + \frac{m^2}{k}\Big) + m\Big(2|A|\,m\,b\,x^\varepsilon + \frac{|T_2|\,k\,b\,x^{2\varepsilon}}{|A|^2}\Big)\Big], \quad (\star)$$

where $a \geq 0$ is a real parameter with $a \leq 2xb$ (`RestSize`); in `CoreAtH` (Section 3.7) it also satisfies $xb \leq a$ and $0 \leq a_t \leq 2a$. In the archive the left side is `gapA`, the bracket is `errorBracketSizeWindows`, and $(\star)$ is the last hypothesis of `ch_core`. The analytic part takes from Section 4 the host field and the compatibility of $\ell$, the sizes above, the thin estimates, the cost bounds and $(\star)$, but nothing about the lattices, since `tiedCore_of_chainTied` drops the clauses that tie the data to them.

$b$ and $X'$ `LRC/DataParams₂.lean:29–33`
`IsThickAt` `Formalism/ThickAverageLoss/PartnersToFactorsAt.lean:97`
`CostWindows` `SWB/SChain.lean:50`
`RestSize` `SWB/SChain.lean:53`
`ErrWindows` `SWB/ErrorFormsWeakWindowsFalse.lean:43`
the sizes of $m$ and $k$ `SWB/ErrorFormsWeakWindowsFalse.lean:32`
`ch_core` `LT/ChCore.lean:343`
assembled into `SizeWindows` `LT/ChCore.lean:347–370`
`SizesA` `DA/DedekindLFunctions/SplitInertia.lean:42`
$N \geq a_t/\min w$ `DA/DedekindLFunctions/SplitInertia.lean:48`
`gapA` `DA/DedekindLFunctions/SplitInertia.lean:50`
`errorBracketSizeWindows` `SWB/ErrorFormsWeakWindowsFalse.lean:133`
`tiedCore_of_chainTied` `PC/LevelGmLeOnFamilyScales.lean:13`
the hypothesis $T_2 \subset A_0$ of `ch_core` `LT/ChCore.lean:349`
the bounds on $a$ in `CoreAtH` `SWB/ErrorFormsWeakWindowsFalse.lean:30, 37`

**3.6. The balance identity.** Orthogonality on $\mathrm{Gal}(E/\mathbb{Q})$ makes $\mathrm{ASK} - \mathrm{MAIN}$ a weighted sum of the ramp sums $G(\theta, y)$ over the characters $\theta$ of the Galois group, so that $(\star)$ bounds such a sum from above. The archive proves the following as `Deduction.ch_balance`, under `HostFactsS`, conditions on abstract inertia groups and Frobenius lifts that the host fields satisfy: $d$ is prime; a finite set $S$ of natural numbers, which are primes on the host fields, contains 2 and $q_1$ but not $d$; every prime outside $S \cup \{d\}$ has trivial inertia; the members of $S$ have inertia and Frobenius in $\Phi$; and the inertia group at $d$ is $I_d = \{0, \iota_d\}$. For a character $\theta$ of $\mathrm{Gal}(E/\mathbb{Q})$ write $G(\theta, y)$ for $G(y)$ if $\theta = 1$ and otherwise for $\sum_{n \leq y} \Lambda(n)\theta(n)(y/n - 1) + y\,\ell(\theta)$, with $\theta(p^i)$ the mean of $\theta$ over the coset $\mathrm{Frob}_p^i I_p$ of the inertia group, extended multiplicatively (`primVal`, `GPsi`). On the host fields these are the values of the matching primitive character.

`Deduction.ch_balance` `LT/ChBalance.lean:200`
`HostFactsS` `DA/DedekindLFunctions/SplitInertia.lean:14`
`primVal`, `GPsi` `SWB/FirstPrimeNeSecondPrime.lean:65–68`

**Proposition 3.3** (Alpöge)**.** *Under* `HostFactsS`*, for every function $\ell$ with $\ell(\chi_C) = \frac{L'}{L}(1, \chi_{-d})$, $a > 0$, $w > 0$ on $\Phi$ and $N \geq a/\min w$,*

$$\mathrm{ASK}(a, t, w) - \mathrm{MAIN}(a, w) = \frac{1}{|\Phi|} \sum_{g \notin \Phi} \sum_{\theta} \overline{\theta(g)}\, w(g')\, G(\theta, a/w(g')), \qquad g' = c\, g\, \tau_t,$$

*the inner sum over all characters $\theta$ of* $\mathrm{Gal}(E/\mathbb{Q})$.

*Sketch of proof.* For fixed $g$, the kernel $|\Phi|^{-1} \sum_\theta \overline{\theta(g)}\theta(p^i)$, an average over the inertia coset $\mathrm{Frob}_p^i I_p$, is $2/|I_p|$ on that coset and 0 off it. Since $w(g')(y/p^i - 1) = a/p^i - w(g')$ for $y = a/w(g')$, the prime-power parts of the right side are the sums of $\log p\,(a/p^i - w(g'))_+$ over the $g \notin \Phi$ in these cosets, with weight $2/|I_p|$. At $p \notin S \cup \{d\}$ the coset is $\{\mathrm{Frob}_p^i\}$, with weight 2; at $p \in S$ it lies in $\Phi$ and gives nothing; at $d$ it has one element outside $\Phi$, with weight 1. Adding these gives ASK. Of the main terms, those of $\theta = 1$ give $-a \log a + a\,|\Phi|^{-1} \sum_\Phi \log w + (1+\gamma)a$ (as $g \mapsto g'$ maps the complement of $\Phi$ onto $\Phi$), those of $\theta = \chi_C$ give $-a\,\ell(\chi_C)$, and those

of the other $\theta$ cancel, since $\sum_{g\notin\Phi}\overline{\theta(g)} = 0$ unless $\theta$ is trivial on $\Phi$. Together they are $-$ MAIN. $\square$

The identity is elementary. It is also where the Euler product enters. For the Davenport–Heilbronn function $f$, which has zeros with real part above $\frac{1}{2}$, the ramp sum of $-f'/f$, once centered by adding $y\,f'(1)/f(1)$, has the properties of `PoleData` used in Theorem 3.2, but $f$ has no Euler product (the coefficient of $-f'/f$ at 6 is not zero), and the orthogonality step above fails for it.

**3.7. From ($\star$) to the window inequality.** The window inequality $G_\psi(x) \le x^{1/2+O(\varepsilon)}$, from which Theorem 3.2 excludes zeros right of $\frac{1}{2}$, follows from ($\star$) on the host fields. By the balance identity, averaging the left side of ($\star$) over $A_0$ and over the shifts leaves, up to errors, $G_\psi(x)$ and $3o$ twisted sums that are bounded separately. The statement in the archive is `core_gm`, proved through `ch_core`.

`core_gm` PC/CoreGm.lean:12
`ch_core` LT/ChCore.lean:343

**Proposition 3.4** (Alpöge)**.** *For every primitive $\psi$, all real $e_D$, $e_L$ and $C$, all values of three further real parameters described below, and every $\varepsilon > 0$, there is $x_1$ such that for every real $x \ge x_1$, every frame, every host $H$ satisfying* `HostAx` *and every subset $TT$ of $A$, if* `EAtH` *holds with the constant $C$ and* `CoreAtH` *with the five other parameters, for these data at scale $x$, then $G_\psi(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$.*

`CoreAtH` asserts that there exist the data of Section 3.5 on $H$, with $b$, $X'$ and $k$ any real numbers that satisfy the conditions there, and a record of real numbers of the chain at the scale $x$, with the constants of its conditions, identified with ASK and MAIN (`IdentU`, `SizeData`$_2$, `ErrWindows`, `BNBoundsU`). The constants of the chain's conditions are thus part of what is asserted to exist, and may change with $x$ and with the host; the parameters of the proposition only bound them. The exponents satisfy $e_{\mathrm{Div}} \le e_D$ and $e_{\mathrm{Loss}} \le e_L$; of the three further parameters, one bounds the exponent $c$ in the chain's height bound $x^c b$ (`HHBound`), one is an exponent of $x$ that bounds $k$, and one bounds seven nonnegative constants of the conditions, $c_{\mathrm{Div}}$ among them. So $x_1$ depends only on $\psi$, $\varepsilon$ and the six parameters. ($\star$) is not among the conditions: for $x \ge x_1$ and $u \in A_0$, `ch_top` derives ($\star$) at $u$ from them by real algebra, and `core_gm` enlarges the exponent to $(e_D + e_L + 3)\varepsilon$. Among the conditions `EAtH`, three relate $A_0$ to $\psi$ (`o_eq`, `kept`, `ps`), and two matter here. The first, on the shares, says that for every $g \in A$ the number of $t \in TT$ with $t - g \in A_0$ is $\lambda(g)|TT|$ up to $x^{-1+\varepsilon}|TT|$, where $\lambda(g) = 4/(3o+1)$ on $A_0$ and $3/(3o+1)$ off it (`Frame.lam`, `Frame.Shares`). The second gives the bounds $|G(\theta, x)| \le x^{1/2+\varepsilon}$ and $|\frac{L'}{L}(1,\theta)| \le C \log x$ for the characters trivial on $A_0$ that are not kept.

`HostAx` TGC/ConjEMemHostFIff.lean:174
`CoreAtH` PC/CoreAtH.lean:29
`IdentU` DA/DedekindLFunctions/SplitInertia.lean:67
`BNBoundsU` DA/DedekindLFunctions/SplitInertia.lean:74
the parameters PC/CoreAtH.lean:35, SWB/ErrorFormsWeakWindowsFalse.lean:23–45
`HHBound` Formalism/RankGapBounds/MeanBoundSecondForm.lean:158
($\star$) derived PC/CoreGm.lean:57–60
`ch_top` LT/ChTop.lean:37
enlarges the exponent PC/CoreGm.lean:38–60
`EAtH` TGC/ExplicitFormulaErrorOfHostConditions.lean:144, its fields `o_eq`, `kept`, `ps` 145–147
`Frame.lam` TGC/ExplicitFormulaErrorOfHostConditions.lean:135
`Frame.Shares` TGC/ExplicitFormulaErrorOfHostConditions.lean:139

*Sketch of proof.* By Proposition 3.3, $(\star)$ bounds from above a weighted sum of ramp sums $G(\theta, y)$, at $y = a_t/w_{u,j}(g')$. At the thin places $y = a_t/w \geq X'/3$, and `ThinScales`, one of the conditions `CoreAtH`, gives $|G(\theta, y)| \leq y\,\theta_0$ for a $\theta_0$ with $2|\Phi|(x+1)\theta_0 \leq 1/x$ (`CostWindows`). On the actual hosts, which lie in $\mathbb{Q}(\zeta_{Q_S qd})$ with $Q_S$ prime to $qd$, the bound holds with $\theta_0 = \exp(-cx^{\varepsilon/8})$ by `ThinH`, which the archive proves as `thinH` for $0 < \varepsilon < \frac{1}{52}$, a bounded odd part of $Q_S$, $\log(Q_S qd) \leq 2x^{\varepsilon/5}\log^2 x$ and `RealBox`, which says that no real character of $E$ whose conductor is divisible by $q$ or $d$ has an $L$-function with a real zero above $1 - \frac{1}{400}$. The bound of `thinH` is the prime number theorem for characters, through $G(\chi, y) = -y\int_y^\infty \Psi(u,\chi)\,u^{-2}\,du$, $\Psi(u,\chi) = \sum_{n\leq u}\Lambda(n)\chi(n)$, less $u$ when $\chi = 1$, and de la Vallée Poussin's zero-free region, since $\log y/\log(Q_S qd) \gg x^{2\varepsilon/15}/\log^2 x$. `RealBox` excludes an exceptional zero when $q$ or $d$ divides the conductor, and the other real characters have bounded conductor. The total over the thin places is absorbed in the error.

At the thick places $w \in [b, b+\eta+1]$, so replacing $w$ by $b$ costs an error that is absorbed (`LipW`; `CostWindows`, also in `CoreAtH`). Summing over $u \in A_0$ then leaves only the $4o$ characters trivial on $\{0\} \times A_0$ (`sum_Th`). A thick place $g' = c\,\iota_d(s+u)$ comes from $g = \iota_d(s+u-t)$, so the thick part becomes $|A_0|\,b\,(\delta_2/2)\sum_\theta \theta(\iota_d t)\,G(\theta, a_t/b)$ with $\delta_2 = |T_2|/|A|$, and $\theta(\iota_d t) = \theta(t)$ for $\theta \in \langle\psi\rangle$, which are unramified at $d$. Placing the thick places in the coset $c\,\iota_d A$ removes the factor $\theta(c) = \psi(-1)^j$ that complex conjugation brings when $\psi$ is odd; with the thick places in $A$, the trivial coefficient would stay 1 and that of a nontrivial $\psi^j$ would be within $o\,x^{-1+\varepsilon}$ of $\psi(-1)^j/(3o+1)$. The archive applies $(\star)$ for each $u$, sums over $A_0$ and divides by $|A_0|$, and combines this with a separate lower bound in which $a_t/b$ is moved to $x$ (`move`; `one_char`), at a cost absorbed in the term $4x^{1/2+\varepsilon}$ below.

Summing over $t \in TT$ and dividing by $|TT|$, the trivial character keeps the weight 1, and the shares give the other kept characters weights within $o\,x^{-1+\varepsilon}$ of $1/(3o+1)$, the Fourier transform of $\lambda$ on $A/A_0$ at a nontrivial character being $\frac{4}{3o+1} - \frac{3}{3o+1}$. The discrepancies go to the error, and the sum with the ideal weights is $G_\psi$ (`gmA_eq`). This step, like the averaging, is elementary. The $3o$ characters trivial on $A_0$ that are not kept are bounded in absolute value, $\mathrm{SigA} \leq 4o\,x^{1/2+\varepsilon}$ (`sigA_le`). The result is

$$|TT|\,m\,b\,\frac{\delta_2}{2}\,(G_\psi(x) - \mathrm{SigA} - 4x^{1/2+\varepsilon}) \leq \text{right side of } (\star) + \frac{2|TT|mb}{x},$$

and dividing out, the sizes of Section 3.5 turn the right side into $x^{O(\varepsilon)}(x/m + m + \sqrt{x})$, which is $x^{1/2+O(\varepsilon)}$ on the range of $m$. In the exponent the losses add up to $(e_D + e_L + 7)\varepsilon$. □

`ThinScales` SWB/GrowthBoundGivesRiemannHypothesis.lean:55
`CostWindows` SWB/SChain.lean:50
the actual hosts TGC/ConjEMemHostFIff.lean:21,33,54
`ThinH` CB/PairGood.lean:36–39
`thinH` DA/MeanSquareBounds/GaussSumFunctionEq.lean:55
`LipW` SWB/GrowthBoundGivesRiemannHypothesis.lean:58
`sum_Th` LT/ChCore.lean:56
divides by $|A_0|$ LT/ChCore.lean:283
`move` LT/ChCore.lean:154
`one_char` LT/ChCore.lean:202
`gmA_eq` PC/GmAEq.lean:61
`sigA_le` PC/SigALe.lean:38

**3.8. Pairs and host fields on windows.** The window inequality needs a host field at every scale $x$ of a window, built from a pair $(q, d)$ for which the twisted ramp sums are small. The archive obtains the host fields from the bound `VWin` of Section 5, and the pairs

from a mean square over the moduli and Markov's inequality. Alpöge's `supplyChi` turns `VWinAt` $\psi$ 401 (Section 5.1) into host fields on windows, `HostExW` $\frac{3}{2}$ (`GChi` $\psi$)`.req`. This says that for small $\varepsilon$ there are windows $[Z, Z^{3/2}]$ arbitrarily far out on which every real $x$ has ranks and a host field at scale $x$ meeting `RankVal` and `HostGood` with common constants, the conditions `EAtH` of Section 3.7 and the bound $\sqrt{b}$ for the weighted lengths of Section 5 over $1 \le e < t$, $t$ the host's rank parameter (`GChi`). The exponent 401 of `VWin` at an auxiliary scale gives 400 at the rescaled point (`WinDatum`), and $t \le x^{400}$ by the equation for $t$ in `RankVal` (Section 4.2), so this covers $1 \le e < t$ (`supplyOfAdmOf`). The bound `VWin` holds on windows $[Y, Y^2]$ that depend on the pair $(q, d)$.

`supplyChi` `CB/SupplyChi.lean:8`
`GChi` `CB/Supply.lean:42`
`WinDatum` `CB/TwistForm.lean:25–26`
`supplyOfAdmOf` `DA/ShortNorms/supplyOfAdmOf.lean:13`, the step at `129–134`

The pairs are those of `PairGood`. For a modulus $M$ prime to $qd$, it asks `WinPair`; for every character $\xi$ modulo $M$ and $i, j \in \{0, 1\}$ not both 0, that the primitive character $\chi$ inducing $\xi \cdot (\cdot/q)^i (\cdot/d)^j$ satisfy $|G(\chi, y)| \le y^{1/2+\varepsilon/2}$ and $|\frac{L'}{L}(1, \chi)| \le 4 \log y$ for $Y \le y \le X$; and, for every real primitive $\rho$ of conductor dividing $8M$ and such $(i, j)$, that the corresponding twist have no zero with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s|$ at most a height depending on $M, \varepsilon, Y, j$ (`ZeroBox`). Their existence, with $X = Y^A$, for every $Y$ sufficiently large in terms of $M \ge 1$, $0 < \varepsilon < \frac{1}{52}$ and $A \ge 1$ (`PairEx`), is `pairEx`. Since $G(\chi, y) = -\sum_n \Lambda(n)\chi(n) \min(1, y/n)$, the proof cuts the sum at a fixed power $Y^{c_0}$ of $Y$. For $n \le Y^{c_0}$, a mean square of the sums of $\Lambda(n) \min(1, y/n)\kappa(n)\,(n/q)$, $|\kappa(n)| \le 1$, over prime moduli $q$ far larger than $Y^{c_0}$ (`meanSqK`), and Markov's inequality (`bad_share`) leave few bad moduli, first among the $q$, then among the $d$. The tail $n > Y^{c_0}$ is small when $L(s, \chi)$ has no zero with $\operatorname{Re} s > 1 - \frac{1}{400}$ up to a height, and `count_bad` shows that few moduli have one, through a zero-density estimate proved in the archive: at most $CQ^{1/4}$ primitive characters of conductor at most $Q \ge 3$ have a zero with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \le Q$ (`zero_density_box`), an estimate of the classical kind [65]. The primes are counted in their progressions, and the sums of Jacobi symbols over them bounded (`twistK`), by a prime number theorem for progressions with a term for an exceptional zero (`PNTAPRelW`), Page's theorem (`page_across`) and the bound $\beta \le 1 - c(\varepsilon) q^{-\varepsilon}$ for real zeros of real characters modulo $q$ (`ExcBound`), which the archive derives from Siegel's lower bound for $L(1, \chi)$ (`siegelLower`).

`PairGood` `CB/PairGood.lean:18`
`PairEx` `CB/PairGood.lean:28`
`pairEx` `DA/MeanSquareBounds/pairEx.lean:112`
`meanSqK` `DA/MeanSquareBounds/meanSqK.lean:51`
`count_bad` `DA/MeanSquareBounds/CountBad.lean:9`
`zero_density_box` `Formalism/GL3MeanSquares/GenusMeanAsymptotics/JacobiCharacterSums/CountZAll.lean:13`
`twistK` `DA/MeanSquareBounds/twistK.lean:148`
`PNTAPRelW` `Formalism/GL3MeanSquares/GenusMeanAsymptotics/JacobiCharacterSums/MeanSquareHypWIff.lean:38`
`page_across` `Formalism/GL3MeanSquares/GenusMeanAsymptotics/JacobiCharacterSums/MSqPage.lean:37`
`ExcBound` `Formalism/LandauZeroFreeBounds/LogDerivRegionChiOf.lean:50`
`siegelLower` `Formalism/LandauZeroFreeBounds/FinalSiegel.lean:17`

The set $T_2$ of thick places is the image under $s \mapsto s\,a_0$, for a generator $a_0$ of $A_0$ of order $n_0$, of a set of natural numbers without short relations. `hostThickExists` gives one with `NoRel`, that is, with no relation $j(s_2 - s_1) = i(s_3 - s_1)$ among distinct elements $s_1, s_2, s_3$

of the set, for integers $i \neq j$ with $0 < |i|, |j| \leq L$, where $L \geq 2$ is a given natural number, and with $4Ls < n_0$ for every element $s$ of the set, so that short relations modulo $n_0$ hold in the integers. Its proof uses Mathlib's `Behrend.sphere` and `Behrend.map`, Behrend's construction [10]. The set comes from `thickFrame2` through `hof_thick` and is installed in the host by `hof_exists_host` and mapped into $A_0$. The analytic part uses only $T_2 \subset A_0$, the density of $T_2$ and $T_2$ as the set of thick places; `NoRel` is used by the other parts.

`hostThickExists` `Formalism/FrobeniusOrbits/FrobeniusOrbitsProofs/ThickExists.lean:181`
`NoRel` `Formalism/FrobeniusOrbits/HostThickExists.lean:13`
`thickFrame2` `DA/ThickFrames/ThickFrame2Large.lean:29`
`hof_thick` `DA/ShortNorms/hostOfFieldOf.lean:71`
`hof_exists_host` `DA/ShortNorms/hostOfFieldOf.lean:97`
the set `DA/ShortNorms/hostOfFieldOf.lean:116`
mapped into $A_0$ `FSR/GlobalSchurBounds/QuadraticLogDerivCongr.lean:428`

**3.9. Assembly.** For each primitive $\psi$, the families on windows come from the interface `TopG`, in the class of families of the requirement `GChi` $\psi$, and the host fields from Section 3.8. Proposition 3.4 then gives the window inequality on these families, and Theorem 3.2 the zero-free half-plane $\operatorname{Re} s > \frac{1}{2}$. In the archive the assembly is `zeroFreeAll_of_topG` (`CB/ZeroFreeAllOfTopG.lean:26`).

**Theorem 3.5** (Alpöge)**.** *Let $\lambda > 1$. Assume* `TopG`*, and that for every primitive $\psi$ host fields meeting* `GChi` *$\psi$ exist, with common constants, on windows of exponent $\lambda$* (`HostExW` $\lambda$ (`GChi` $\psi$)`.req`)*. Then for every primitive $\psi$ the function $L(s, \psi)$ has no zeros with* $\operatorname{Re} s > \frac{1}{2}$.

The interface `TopG` says that for every $\lambda > 1$ and every requirement `FReq` on the scale, the frame, the data `HostAx`, $TT$ and the rank parameter, if hosts meeting it exist on windows and every family in the requirement's class satisfies `HypV` (the bound $\sqrt{b}$ of `GChi` at the family's hosts), then `FamilyClassConnected` holds. This says that there are $e_D, e_L$ and $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$ some family of the class satisfies `ChainTied` (Section 4.1) for all large $x$ in its windows, with $e_{\mathrm{Div}} \leq e_D$ in (4.1) and $e_{\mathrm{Loss}} \leq e_L$ for the losses. The other constants of `ChainTied` may depend on $\varepsilon$ but move only $x_1$ in Proposition 3.4. The archive proves `TopG` with $e_D = 2$ and $e_L = 0$, so the bound is $G_\psi(x) \leq x^{1/2+9\varepsilon}$.

`TopG` `CB/ZeroFreeAllOfTopG.lean:12`
`FReq` `CB/Supply.lean:10`
`HypV` `CB/DivinNOfHypV.lean:10–11`
`FamilyClassConnected` `CB/DivinNOfHypV.lean:13`
$e_{\mathrm{Div}} \leq e_D$ and $e_{\mathrm{Loss}} \leq e_L$ `FSR/GlobalSchurBounds/TieGlobal.lean:107`
$e_D = 2$ and $e_L = 0$ `TPS/TopShape.lean:52`

*Sketch of proof.* Fix $\psi$ and $s$ with $\operatorname{Re} s > \frac{1}{2}$, and take $\theta = \frac{1}{2} + \frac{1}{2}\min(\operatorname{Re} s - \frac{1}{2}, \frac{1}{2})$ and $\varepsilon$ with $(e_D+e_L+7)\varepsilon \leq \frac{1}{2}\min(\operatorname{Re} s-\frac{1}{2}, \frac{1}{2})$ (`level_of_tiedWin`). By `TopG`, applied to the requirement `GChi` $\psi$, the supplied hosts and `hypV_of_cls`, there is for $0 < \varepsilon < \varepsilon_1$ a family of the class with `ChainTied` on its windows, and the condition `GChi` $\psi$ that its class imposes at every large scale gives it `EAtH` (`eFamC_of_cls`). Proposition 3.4 then gives $G_\psi(x) \leq x^{1/2+(e_D+e_L+7)\varepsilon} \leq x^\theta$ there (`level_gm_le_on_family_scales`, of type `ChebyshevInequalityOnWindows`), and Theorem 3.2 shows that $L(s, \xi)$ has no zeros with $\operatorname{Re} s > \theta$. In the archive this is `zeroFreeOn_of_tiedWin_cyc` applied to `Level.cyc` $\psi$. □

`zeroFreeOn_of_tiedWin_cyc` `TGC/TopWin.lean:90`
`level_of_tiedWin` `TGC/TopWin.lean:68`
`hypV_of_cls` `CB/ZeroFreeAllOfTopG.lean:16`

`eFamC_of_cls` `CB/ZeroFreeAllOfTopG.lean:19`
`level_gm_le_on_family_scales` `PC/LevelGmLeOnFamilyScales.lean:28`
`ChebyshevInequalityOnWindows` `TGC/TopWin.lean:26`

*Remark* 3.6 (Where the exponent $\frac{1}{2}$ enters). The exponent $\frac{1}{2}$ comes from two places. The first is the length of the chain. With $m$ lifts the right side of $(\star)$, divided by $|TT|\,m\,b$, is $x^{O(\varepsilon)}(x/m+m+\sqrt{x})$: $x/m$ from the term $|TT|\,a(1+m^2/k)\le 2|TT|\,a$, which does not grow with $m$; $m$ from the error $B_0\le 2|A|\,m\,b\,x^{\varepsilon}$ that each of the $m$ lifts pays in (4.1); $\sqrt{x}$ from the term in $k$, as $k\le |A|^2x^{1/2-\varepsilon}$. The first two balance at $m\approx\sqrt{x}$, which makes it $x^{1/2+O(\varepsilon)}$ (Proposition 3.4). The second is the twisted ramp sums. `EAtH` asks $|G(\theta,x)|\le x^{1/2+\varepsilon}$ for the $3o$ characters that are not kept, the bound that the Riemann hypothesis for their $L$-functions would give at this scale. Under that hypothesis Remark 3.1, made uniform in the conductor $Q$, gives $|G(\theta,x)|\ll\sqrt{x}\log Q+\log x$, and here $Q$ divides $Q_Sqd$, so that $\log Q\le 2x^{\varepsilon/5}\log^2 x$. The archive obtains the bound of `EAtH` not for every character but for the pairs $(q,d)$ of `PairGood`, from a mean square over the moduli and Markov's inequality (Section 3.8). Theorem 3.2 then turns the bound on windows into zero-free half-planes with no loss in the exponent. The window bound is sharp up to the factor $x^{O(\varepsilon)}$. Indeed, since $1\in\langle\psi\rangle$ and $\zeta$ has zeros on the line $\operatorname{Re}s=\frac{1}{2}$, Theorem 3.2 shows that for every $0<\theta<\frac{1}{2}$ and $\lambda>1$ the bound $G_\psi(x)\le x^{\theta}$ fails at some $x$ in every window $[Y,Y^{\lambda}]$ with $Y$ large.

*Remark* 3.7. We know of no reference for the deduction as a whole, in which a one-sided inequality "divisibility at most size" over a family on a 2-power cyclotomic tower of CM fields is turned, by an identity and an averaging, into a one-sided bound for the positive combination $G_\psi$ of ramp sums.

## 4. Where divisibility at most size comes from

This section follows the archive from its chain of kernels, which Alpöge describes as theta lifts and which are built from theta functions (Section 4.2), to $(\star)$. Section 4.1 gives the interface with Section 3 and the conditions behind it, Section 4.2 the datum and its kernels, Sections 4.3 and 4.4 the lifts and their integrality at the inert primes, Section 4.5 what supplies the conditions, and Section 4.6 the covolume bound.

**4.1. The tied chain.** In the archive the divisible object is, for each word $\omega$, a map $\mathrm{Bk}_\omega\colon N_\omega\to RR^\vee$ of Hermitian lattices whose entries are periods. This subsection describes these lattices and three of the conditions on them, and states the hypotheses from which Alpöge's `ch_top` derives $(\star)$. The interface with Section 3 is `ChainTied`. At the scale $x$ it gives a host field with the data of Section 3.5 and, for each $u\in A$, a member of the family. A member carries a totally complex Galois number field $LL$, a finite set of *words* $\omega$ (the columns), Hermitian $\mathcal{O}_{LL}$-lattices $N_\omega$ and $RR$, linear maps $\mathrm{Bk}_\omega$ from the ambient space of $N_\omega$ to the dual of that of $RR$, integral by `LinkDIV` below, and real numbers $h(\omega)$ (`DivData`). The clauses `TieLocal` identify the numbers that `ChainTied` speaks about with the ranks, slopes and maximal slopes of these lattices and with the $h(\omega)$, and `SizeData`$_2$ lists eleven conditions on them. The archive computes slopes with squared norms, so that $\mu=2\widehat{\mu}$ for Bost's normalized slope $\widehat{\mu}$.

`ChainTied` `FSR/GlobalSchurBounds/TieGlobal.lean:97`
`DivData` `LRC/DataArith.lean:139`
`TieLocal` `FSR/GlobalSchurBounds/TieGlobal.lean:63`
`SizeData`$_2$ `SWB/SChain.lean:17`

We describe three of the conditions.

(a) The size bound (`DIV`) says that for every word $\omega$,

$$\mu(RR^{\vee}) + h(\omega) - F_o \le \mathrm{MAIN}_\omega + c_{\mathrm{Div}} |TT|\, a \log(a\, \mathrm{rd}_E) + c_{\mathrm{Div}}\, m\, (B_0 + N_c)\, x^{e_{\mathrm{Div}}\varepsilon}, \quad (4.1)$$

where $\mathrm{MAIN}_\omega = \sum_{t \in TT} [m\,(a_t \log a_t - a_t - a_t \log \Omega) - a_t \log T_\omega]$, $T_\omega$ is a prime attached to $\omega$, $B_0$ and $N_c$ are the two error sizes of `BNBounds`, and $F_o$ is the size of a correction that the archive fixes place by place from the weights and the Fock degrees of the lifts (`Fval`).

`DIV` `Formalism/RankGapBounds/MeanBoundSecondForm.lean:118`
`BNBounds` `SWB/SChain.lean:33`
$F_o$ `LRC/DataParams`$_2$`.lean:218`
`Fval` `LRC/DataForced.lean:126–132`

(b) The slope inequality (`SlopeIneq`) says that if $s$ is the mean of $\mu(N_\omega) - h(\omega)$ weighted by rank, $u_F = \max_\omega(\mu_{\max}(N_\omega) - h(\omega)) - s$, and $H$ is the member's height bound, with $0 < H$ and $s - \mu_{\min}(RR^{\vee}) \le H$ (`HHBound`), then

$$s - \log \#\{\omega\} - (1-\delta)^{-1}\big(4u_F \log(1 + H/u_F) + (\eta + 4\sqrt{e})H\big) \le \mu(RR^{\vee}),$$

with parameters $0 \le \delta < 1$, $\eta \ge 0$, $0 \le e \le \frac{1}{4}$ (`FParams`, with $\delta < 1$ and $e \le \frac{1}{4}$ from `HidaSetting`), the term $u_F \log(1 + H/u_F)$ read as 0 at $u_F = 0$. Here $\delta$ is the defect of `LinkIRR` below.

`SlopeIneq` `Formalism/RankGapBounds/MeanBoundSecondForm.lean:151`
$\delta$ `FSR/GlobalSchurBounds/TieGlobal.lean:82`
`HidaSetting` `Formalism/RankGapBounds/MeanBoundSecondForm.lean:111`

(c) Two-sided bounds for the slopes of the lattice of stage-zero forms (`Slopes`), and a comparison of the slopes of the $N_\omega$ with them, shifted by the gains (`QuotientSlopeBoundsWith`).

`Slopes` `Formalism/RankGapBounds/MeanBoundSecondForm.lean:139`
`QuotientSlopeBoundsWith` `SWB/QuotientSlopeBoundsWith.lean:19`

For an injective map $L \to M$ of Hermitian lattices of norm at most $C > 0$, with $L \neq 0$, the classical inequality is, in Bost's slopes, $\widehat{\mu}_{\max}(L) \le \widehat{\mu}_{\max}(M) + \log C$ [15, Prop. 4.3(1)], which the archive proves as `slope_inequality_general`. Alpöge's `ch_top` derives ($\star$) by real algebra from `SizeData`$_2$, the error windows, the bounds `BNBoundsU`, the largeness condition `LargeX` and the identification `IdentU` (`Ident`, with a map of index sets) of the archive's ASK and MAIN with those of Section 3.5. This is the computation of "The idea" (Section 1), done with weights and errors. For a fixed member one takes the rank-weighted mean of the shifted slopes of the $N_\omega$, uses the comparison of slopes and the counting estimate for $\log T_\omega$, and bounds the gains of the lifts together from below. These estimates bound the left side of ($\star$) above (`ch_top`), and Proposition 3.3 expresses each of its terms ASK $-$ MAIN by centered ramp sums, the ramified contribution included.

`slope_inequality_general` `Formalism/HarderNarasimhan/HarderNarasimhanProofs/SlopeIneq.lean:507`
statement `Formalism/HarderNarasimhan/Statement.lean:91`
`ch_top` `LT/ChTop.lean:37`
`Ident` `SWB/SChain.lean:26`
map of index sets `DA/DedekindLFunctions/SplitInertia.lean:67`

In $\mathrm{MAIN}_\omega$, by `logOm` and `Host.v`,

$$\log \Omega = \gamma + \log \frac{2\pi}{\mathrm{rd}_E} - \frac{L'}{L}(1, \chi_{-d}), \qquad \mathrm{rd}_E = |\mathrm{disc}\, E|^{1/[E:\mathbb{Q}]}.$$

The shape of $\log\Omega$ is that of the Chowla–Selberg constant of $K$. The archive proves Kronecker's first limit formula [26, Thm 1] averaged over the class group of $K$ (Table 2), as `classMean_log_norm_disc_chowlaSelberg`, in the form

$$\frac{1}{h_K}\sum_{c}\log\bigl(N(\mathfrak{a}_c)^6|\Delta(L_c)|\bigr) = 12\log 2\pi - 6\log d + 6\gamma - 6\frac{L'}{L}(1,\chi_{-d})$$

for $K=\mathbb{Q}(\sqrt{-d})$ of discriminant $-d$, where $d>4$ is the host prime of a datum (so that $d\equiv 7 \bmod 16$), $L_c=\iota(\mathfrak{a}_c)$ is the embedded ideal of a representative of the class $c$, and $\Delta(L_c)$ is its Weierstrass discriminant. Chowla and Selberg obtain this class sum from the factorization of $\zeta_K$ and then express $L'(1,\chi_{-d})$ through values of $\Gamma$ [74, §12, pp. 109–110]. The functional equation of $L(s,\chi_{-d})$ gives $\gamma+\log(2\pi/d)-\frac{L'}{L}(1,\chi_{-d})=\frac{L'}{L}(0,\chi_{-d})$, so that $\log\Omega=\frac{L'}{L}(0,\chi_{-d})-\log(\mathrm{rd}_E/d)$.

`logOm` `LRC/DataParams₂.lean:224`
`Host.v` `LRC/DataHost.lean:140–143`
`classMean_log_norm_disc_chowlaSelberg` `AVT/Proofs/DivisorProofs/ClassMeanLogNormDiscChowlaSelbergStatement.lean:9`

`ChainTied` is produced by Alpöge's `LinkGLOBAL`, which says that for a family satisfying the bundle of conditions `ChosenE`, among them the bounds $|G(\chi,x)|\le x^{1/2+\varepsilon}$ and $|\frac{L'}{L}(1,\chi)|\le C\log x$ for $\chi=\chi_{-q}$, $\chi_{-d}$ and $\chi_{qd}$, the five conditions `LinkG`, `LinkS`, `LinkIRR`, `LinkDIV`, `LinkMu` (`GlobalTheoremInputs`) give constants and $x_0$ such that `ChainTied` holds for every $x\ge x_0$ in the family's windows. The five conditions are bounds from above and below on the gains at the inert primes (`LinkG`), near-orthogonality of the column spaces (`LinkS`), a bound on the defect of the slope inequality (`LinkIRR`), the size bound with the inclusion $\mathrm{Bk}_\omega(N_\omega)\subset RR^\vee$ (`LinkDIV`) and a comparison of the slope of each $N_\omega$ with that of a reference lattice and the gains (`LinkMu`). For the families used, `link_gl` proves `LinkGLOBAL` under two size conditions, from the condition `Duty` on every family of the class, the real-place conditions, a volume ratio and a genus mass (Section 4.5).

`LinkGLOBAL` `GTI:162`
`ChosenE` `FSR/GlobalSchurBounds/Chosen.lean:83`
the bounds `FSR/GlobalSchurBounds/Chosen.lean:42`
`GlobalTheoremInputs` `GTI:158`
`link_gl` `AVT/Proofs/GlobalInterpolation/LinkGl.lean:14`

`LinkG` bounds the gains at the places $w$ over primes $p\le X''=x\,n\exp(x^{\varepsilon/3})$ inert in $K$, with $f=2n$ the size of the CM type (Section 4.2). It gives $0\le \mathrm{gain}_j(w)\le 4|TT|a/p$ and

$$\sum_j\sum_{t\in TT_w}\bigl(\textstyle\sum_{i\ \mathrm{odd}}\max(0,\,a_t/p^i-w_{j,t,p,i})-\mathrm{lo}\bigr)-m\,\mathrm{fl}\le\textstyle\sum_j\mathrm{gain}_j(w)-\tau(w),$$

the inner sum over odd $i<\lfloor\log_p a_t\rfloor+2$, where the $w_{j,t,p,i}\ge 1$ are real numbers given with the size data (its field `wPart`), $TT_w\subset TT$ is a set of shifts, all of $TT$ when $p\ge X'$ (`TTw`), lo and fl are the explicit losses `loG` and `flG`, and $\tau(w)$ is a loss at $w$ that the total gain `Ga` also subtracts, so that the lifts together gain at least $\sum_j\sum_t(a_t/p+a_t/p^3+\cdots)$ at $p$, less the weights and explicit losses (`vonMangoldtClassSum`). In `LinkG` each lift alone is bounded only as above.

`LinkG` `GTI:122`, `GTI:138`
`loG`, `flG` `GTI:109`, `100`
`vonMangoldtClassSum` `LRC/DataParams₂.lean:240`
`wPart` `LRC/DataParams₂.lean:178–180`

`LinkMu` says $\left|\mu(N_\omega) - \mu(N_\omega^H) - \sum_w \mathrm{sz}_w \mathrm{sg}_w\right| \le CX''$ (`NprimeMuR`). Here $N_\omega^H$ is a reference lattice, $w$ runs over the finite set of places of $LL$ given with the size data, $\mathrm{sz}_w = 2e_w \log N\mathfrak{P}_w/[LL : \mathbb{Q}]$ with $e_w$ the ramification index of $\mathfrak{P}_w$ over $\mathbb{Q}$, and $\mathrm{sg}_w = \sum_{j=1}^m (\mathrm{gain}'_j(w) + \mathrm{F}_j(w) - \mathrm{E}_{\omega,j}(w))$. In this sum $\mathrm{gain}'_j(w)$ is $\mathrm{gain}_j(w)$, replaced by 0 when the prime below $w$ lies in a set of dropped primes given with the size data, and $\mathrm{F}_j(w)$ and $\mathrm{E}_{\omega,j}(w)$ are real numbers given with the size data (its fields `F` and `E`), which the conditions `FDef` and `EDef` equate with the explicit `Fval` and `Eval`; $\sum_j \sum_w \mathrm{sz}_w \mathrm{F}_j(w)$ is the $F_o$ of (4.1). By definition the lattices carry the gains. The lattice $N_\omega$ is spanned by the $ay$ with $a \in I_\omega = \prod_w \mathfrak{P}_w^{-\lfloor e_w \mathrm{sg}_w \rfloor}$ and $y$ in the base lattice $N_\omega^H$ (`NprimeDef`), and `LinkMu` follows from this and `ZeroAbove` when the $N_\omega$ are finitely generated, span their ambient spaces and have positive rank, and $\log T_\omega$ and $\log D_\Gamma$ (Section 4.2) are at most a constant times $X''$ (`linkMu_of_linkDIV`).

`LinkMu` GTI:74
`NprimeMuR` GTI:69
`NprimeDef` LRC/DataDIVDefs.lean:17–24
`linkMu_of_linkDIV` FSR/AdmissibleParameterJoins/SumLogPrimeFactorsLe.lean:775
`sz`, `gainD`, `sg` LRC/DataParams$_2$.lean:205, 212, 215
$e_w$ LRC/DataArith.lean:39–42
`FDef` LRC/DataForced.lean:134
`EDef`, `Eval` LRC/DataErr.lean:171, 159

`LinkS`, the near-orthogonality, says that if $C_0^{-1} \le d_\omega \Gamma^2/D^* \le C_0$ for every $\omega$, where $d_\omega$ is the dimension of the stage-zero column space of $\omega$ and $\Gamma = \exp(Cm^5 f x^\varepsilon)$, then the kernel between the column spaces of two words $\omega \ne \omega'$ has $\|K_{\omega\omega'}\|_{HS}^2 \le (1+\eta_0) d_\omega d_{\omega'}/D^*$, the diagonal defect is at most $e_0 d_\omega$ and the total dimension at most $(1+\eta_1)D^*$, with $\eta_0, \eta_1, \sqrt{e_0} \le \exp(-x^{\varepsilon/8})$. `LinkIRR` bounds the defect $\delta$ of `SlopeIneq`, given by $D_{\mathrm{rows}} = (1-\delta)D_{\mathrm{all}}$ with $D_{\mathrm{rows}}$ the dimension of the space of rows and $D_{\mathrm{all}}$ the total dimension just bounded (the datum's fields `Drows`, `Dall`), by $\exp(-x^{\varepsilon/(7J)})$, with $J \ge 1$ a constant of the family. Here $D^*$ is the product of a volume, an index of level subgroups at the two auxiliary primes and the formal degrees of the real discrete series of the weights (`DstarD`; `dsigTie`).

`LinkS` GTI:145–152
`LinkIRR` GTI:55
`DstarD` FSR/GlobalSchurBounds/GeneralLinearActions/GapAtLeast.lean:97
`dsigTie` LRC/DataTies.lean:17
`Drows`, `Dall` LRC/DataRows.lean:66–71

`LinkDIV` carries the size bound (4.1), with the inclusion $\mathrm{Bk}_\omega(N_\omega) \subset RR^\vee$, the height bound $h(\omega)$ for $\mathrm{Bk}_\omega \ne 0$, a bound $x^{C_H} b$ for $s + \mu_{\max}(RR)$, with $s$ the mean of the slope inequality above and $C_H$ a real parameter of the condition, a condition `ZeroAbove` on the gains above $X''$, ties between lifts and words, and the identity `BesselId` (Section 4.3). It is used in the form `LinkDIVt`, which derives it from `LinkEta` and `DIVinN`. `DIVinN` bounds the loss at the twisted places, those at which the tuple $\mu_w$ of Hecke characters is not the reference tuple $\mu_0$ (`etaSum`), by $D_0 C x^\varepsilon (B_0 + N_c)$, with $D_0$ the common value of the dimensions $d_{Z,0}(j)$ of `LinkN` (`SmallLevelRankEq`).

`LinkDIV` GTI:77, GTI:91
`LinkDIVt` GTI:94
`LinkEta` GTI:49
`DIVinN` GTI:62

**4.2. The datum.** Each member of the family of Section 4.1 is a datum: ranks, Hermitian spaces over the host field $E$ and a chain of kernels on finite index sets. Theta functions

enter through the kernels. In the archive a datum is a `D7`. Its ranks, given by `Ranks`, are integers $k, m$ even and $t$ odd with $2 \le m$, $m+3 \le k$, $r+5 \le t$ where $r = k+m-1$, $3m < t$, $m \le f \le k$ for the size $f = 2n$ of the CM type, and $m^2 \le N_m$, where $N_j = k + j(t+1)$. At the scale $x$, `RankVal` fixes $t$ by $r + t = 8fr + 4 + \lceil x^{377} \rceil\, n_W$, with $n_W$ an explicit integer (`nWv`), which gives $t \le x^{400}$ for $x \ge 2^{100}$ when $\varepsilon < 1/52$ (`eventually_rank_le_rpow_of_lt_one_div_fiftytwo`), and fixes

$$x^{2+8\varepsilon} \le n \le 2x^{2+8\varepsilon}, \qquad x^{1/2-\varepsilon/16} \le m \le 2x^{1/2-\varepsilon/16}, \qquad x^{9/2+10\varepsilon} \le k \le 2x^{9/2+10\varepsilon}.$$

The spaces are given by `Spaces`: Hermitian spaces $X_j$ over $E$, definite of rank $N_j$ for $j \le m$; a space $X'$ embedded isometrically in $X_m$ as the orthogonal complement of an anisotropic vector $z$; for $0 \le j \le m$, spaces $Y_j$ of rank $r + jt$; and a prime $T$. The chain is a `ColumnChain`: finite index sets at each stage $i \le m$ with specified finite stabilizers whose orders divide a common $D_\Gamma$, value spaces, and kernels. The $i$-th map sends $\varphi$ to $L' \mapsto \sum_L |\mathrm{Stab}\, L|^{-1} K_i(L, L')\varphi(L)$, and the column map is the composite of the $m$ maps (`kerOpM`, `chainOpM`).

`D7` `LRC/RhoDetPrimeEqMuTildeMulChiPow.lean:76`
`Ranks` `LRC/Params.lean:6`
`RankVal` `AVT/Definitions/GlobalTwisting/PairRowIffQdRows.lean:28–32`
`nWv` `AVT/Definitions/GlobalTwisting/PairRowIffQdRows.lean:19`
`eventually_rank_le_rpow_of_lt_one_div_fiftytwo` `CB/EventuallyRankLeRpowOfLtOneDivFiftytwo.lean:5`
`Spaces` `LRC/DataSpaces.lean:77`
`ColumnChain` `LRC/DataKernels.lean:14`
`kerOpM`, `chainOpM` `Formalism/ThetaLiftIndependence/Statement.lean:47, 52`

The kernels are built from a field of the datum, an arbitrary linear map `unf` (`Kmap`). On the route to the five theorems `unf` applies the derivatives at 0 of a function $\Theta$ to polynomial functions of the frame, and $\Theta$ is a finite sum of translates of a theta function for the CM lattice. This is the shape of $(D^a\vartheta)(0)$ in the note's step 2. The theta function is nonzero under the lattice hypotheses, while single derivatives may vanish. Comparison data match the index sets with lattice classes and the stabilizer orders with those of level-preserving stabilizers, and at stage zero the stabilizers themselves. The archive's `PeriodMap` is abstract. We read the final pairing as one with forms on $U(X')$, which is the shape of Alpöge's chain of theta lifts.

`unf` `LRC/DataKernels.lean:39`
`Kmap` `LRC/DataKernels.lean:63–65`
the derivatives at 0 `.../AutomorphicMembership/LatAtLattice.lean:68–73`, `.../PredMemGamRanks.lean:438–459`
$\Theta$ `.../XiAtPenultimateEqXiCpl1.lean:157–162, 265–284`
nonzero `.../KernelLocalFactors/IsAdmissibleFlagPairOfUnitriangularLeftAutomorph.lean:75–80, 218–222`
comparison data `FLG/GroupLattices/ComparisonMaps/ClsStab.lean:43–51`, `FLG/InertKernelTheory/SecondKernelEntrywiseLastLevel.lean:27–34`
`PeriodMap` `LRC/DataKernels.lean:163`

The existence proof of `HypsAlt` passes through placeholder rows, with the pairings `Bu` zero and a spectral space of dimension zero (`StandIn`), which are replaced before Theorem S is applied. Theorem S enters as `LinkS` (Section 4.5), and wherever `LinkS` is applied its diagonal bound excludes a zero spectral map. At every sufficiently large scale, each member of the selected family has a word $\omega$ with nonzero period matrix $\mathrm{Bk}_\omega$, since the frame estimates

force the columns to span at least 4/11 of the target dimension while the extra rows occupy less.

`StandIn` `.../AutomorphicMembership/SameForm.lean:40–61`, `.../SCuIsom.lean:16`
replaced `.../ExistsSRestRe.lean:29–38`,
`.../PeriodFactorisationScalarReferenceTupleEqOne.lean:353–379`
diagonal bound `GTI:142–156`, `FSR/GlobalSchurBounds/TopDensityNGeSixteen.lean:1322–1346`
nonzero period matrix `FSR/GlobalSchurBounds/TopDensityNGeSixteen.lean:519–531`, `596–600`, `629–646`, `1245–1258`, `SWB/ChainSlopeWindows/JointDModelAppliesUnitLattice.lean:243–252`, `GTI:55–57`

**4.3. One lift, and the seesaw.** This subsection treats the comparison of theta integrals from two spaces, with the change of the constant under hyperbolic planes, and the form that step 12 of the note takes on the route. The first is needed for the reduction of Section 2. Alpöge's `ThetaIntegralsAgree` compares two theta integrals. Let $E/F$ be a CM extension, Galois over $\mathbb{Q}$ and unramified over $F$ at the finite places; let $V$ and $V'$ be Hermitian of ranks $a > a'$ and $W$ skew-Hermitian of rank $a + a'$, with Weil representations attached to Hecke characters $\alpha, \beta$ of large weight difference and with $V$, $V'$ nearly definite, among the further conditions of `IsTriple`. The theta integrals $I(\Phi)(h) = \int_{[U(V)]} \theta(g, h; \Phi)\, dg$ and $I'(\Phi')(h)$ converge on the Schwartz space and on its primed subspace, $I'$ is not identically zero, and there is a constant $c \neq 0$ with $I(\Phi) = c\, I'(\Phi')$ on $U(W)(\mathbb{A})$ for matched $\Phi, \Phi'$ (`Decay`, `OneConst`, `Coinv`, `Match`). Removing a hyperbolic plane from $V'$ and from $W$, compatible with the lattices, with the Cartan involutions and with the chosen bases at a finite set of finite places of $F$ (`Plane.IsAdapted`), which lowers $a'$ by 2 and leaves $V$ unchanged, multiplies the constant by $C\, L^{S_0}(\alpha/\beta, \frac{a-a'}{2} + 1)$ times local factors that are explicit up to three signs, the same for all fields, where $a$ and $a'$ are the ranks before the plane is removed and $L^{S_0}$ is the $L$-function of the Hecke character $\alpha/\beta$ of $E$ without its Euler factors above a finite set $S_0$ of places of $F$ given with the characters (`StepBeta`). The constant $C \neq 0$ depends only on the base, which fixes the weights of $\alpha$ and $\beta$ at the infinite places, on the spaces before and after the removal with the coordinates, vacuum vectors, Haar measures and fundamental domains chosen for them (`Geo`), on the planes and on the matching maps at the infinite places; it does not depend on $\alpha$ and $\beta$, on the Weil representations or on the matching maps at the finite places. Removing a hyperbolic plane from $V$ and from $W$, which lowers $a$ by 2 and leaves $V'$ unchanged, where $V'$ is definite and $V$ is not, divides the constant by $C\, L^{S_0}(\alpha/\beta, \frac{a-a'}{2})$ times factors of the same kind, with a constant $C \neq 0$ of the same kind (`StepAlpha`). None of these constants is explicit. `LocalMatching` says that the local maps used in the matching exist.

`ThetaIntegralsAgree` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:585`
`IsTriple` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:388`
`Decay`, `OneConst`, `Coinv`, `Match` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:483–507`
`StepBeta` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:559`
`StepAlpha` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:571`
`Geo` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:314`
`Plane.IsAdapted` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:532`
$L^{S_0}$ `AVT/Definitions/Foundations/NotIsSplitOfIsInertAt.lean:369`
weights in the base `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:146–148`
`LocalMatching` `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:581`

The inputs are the Siegel–Weil formula, as `SiegelWeilStd`, and a weak form of the meromorphic continuation of Siegel Eisenstein series on $U(n, n)$ (`EisBL`). `SiegelWeilStd` says that under its hypotheses, for $A$ definite of rank $m$, $Y$ of rank $n$ and a $K$-finite Schwartz function, the Siegel Eisenstein series on $U(n, n)$ at $s_0 = (m - n)/2$ is $\kappa$ times the normalized

average of the theta function over $[U(A)]$, with $\kappa = 1$ for $n < m$ and $\kappa = 2$ otherwise. The archive proves `ThetaIntegralsAgree` and `LocalMatching` by applying `theoremK_head2` to `eisBL` and `siegelWeilStd`, which it proves outright.

`SiegelWeilStd` `WB/ThetaLifts/LatticeBasics/HypothesesS.lean:11`

Here $I$ and $I'$ are the theta lifts to $U(W)$ of the trivial representations of $U(V)$ and $U(V')$, and $\operatorname{rk} V + \operatorname{rk} V' = \operatorname{rk} W$. The Siegel–Weil formula used as input is known: for an anisotropic Hermitian space of rank $m$ and the quasi-split group $U(n,n)$ it is due to Weil for $m > 2n$ [89], to Ichino for $n < m \le 2n$ [45] and to Yamana for $m \le n$ [91, Thm 2.2], and the three cases are stated together in [31, Thm 7.1]. The doubling method is that of Piatetski-Shapiro and Rallis [69]. These theorems concern Eisenstein series on $U(n,n)$. In `ThetaIntegralsAgree` the form on $W$ is nearly definite at each infinite place, so $U(W)$ is not quasi-split, and $V$ and $V'$ do not lie in one Witt tower. The comparison of $I$ and $I'$ is thus not an instance of these theorems, nor of the first-term identity [31, Thm 7.3], in which the group is $U(n,n)$ and the two spaces differ by hyperbolic planes. For the proportionality of $I$ and $I'$ on $U(W)$, and for the change of the constant under hyperbolic planes, we know of no reference.

signatures `AVT/BlockTheory/DivisorBase/WeilTensorModels.lean:364–368`

The note's step 12 says that the period of the iterated lift is a Bessel period independent of the Hecke characters [2, p. 4]. On the route to the five theorems this takes the following form. For every word $\omega$, every tuple $t$ of characters among those used ($\mu_0$ and the $\mu_w$) and rational vectors $\varphi$, $F$, the column period $\mathrm{per}_{\omega,t}(\varphi, F)$ satisfies

$$\zeta_{\omega,t}\,\mathrm{per}_{\omega,t}(\varphi,F) = a_\omega\,\langle V\Theta_t F,\ P_\omega(U_\omega\varphi)\rangle,$$

where $\Theta_t F$ is the row vector that the row map for $t$ attaches to $F$, $U_\omega$ and $V$ are linear maps, $a_\omega \neq 0$, $\zeta_{\omega,t} \neq 0$ and $\zeta_{\omega,\mu_0} = 1$ (`MIid`, `MIUnit2`, `MIstar2`; these periods are tied to the datum's by `DcPtG`). The inner product is that of the row-side Hilbert space, to which the datum's section carries the row vector and the spectral map $P_\omega$ of the word carries the column vector. In the archive's model of the rows by automorphic forms on adelic groups these inner products are, up to one nonzero constant, inner products of the row forms with Poincaré series (`PairingFormCondition`). The tuple enters only through $\Theta_t F$ and the constant $\zeta_{\omega,t}$. `BesselId`, described below, is the statement of this shape for the periods $\mathrm{Bu}_\omega$.

`MIid` `FLG/IndefiniteReal/IsometricPeriodsOfHypotheses.lean:99`
`MIUnit2` `FLG/IndefiniteReal/IsometricPeriodsOfUnitHypotheses.lean:21`
`MIstar2` `FLG/IndefiniteReal/IsometricPeriodsOfUnitHypotheses.lean:27`
`DcPtG` `AVT/Definitions/LatticeFrames/CxTiesOf.lean:23`
`PairingFormCondition` `TPS/FormsAtZero.lean:89`
Poincaré series `FLG/SiegelLatticeTheory/HaarTransversals/PoincareInHH.lean:185`

The identity is proved through Bessel periods (`miStar2_at_realisation_alt`). First, for constants $c_t \neq 0$ and a similitude $U'_\omega$, a linear relation between $c_t\,\mathrm{per}_{\omega,t}(\varphi,F)$ and a reference value, attached to a reference row form $T$ and a reference column vector $\varphi_0$, forces the same relation between two Bessel periods: that of the row form of $\Theta_t F$ against the column form of $U'_\omega\varphi$, and that of $T$ against the column form of $U'_\omega\varphi_0$ (`JointCondition`). A Bessel period here is an integral over the smaller unitary group and a unipotent subgroup against its character (`besPer`). This comes from relations transported along the whole chain (`MainX`). Second, a linear relation between two such Bessel periods forces the same relation between the inner products of the row forms with the Poincaré series (`BottomRelation`). Both are transfers of relations in one direction. The proof applies the

first to the relation whose coefficients are the two values themselves, and so obtains an equation: $c_t \operatorname{per}_{\omega,t}(\varphi, F)\, B_0 = p\, B_t$ for all $t$, $\varphi$, $F$ and all further arguments of the Bessel periods, where $B_t$ and $B_0$ are the two Bessel periods above, that of $\Theta_t F$ and that of $T$, and $p \neq 0$ is the reference value (`anchor_word_alt`).

`miStar2_at_realisation_alt` `TPS/MiStar2AtRealisation.lean:84`
`anchor_word_alt` `TPS/MiStar2AtRealisation.lean:48`, the step at `76`
`JointCondition` `TPS/FormsAtZero.lean:67`
`besPer` `AVT/Definitions/TwistedInnerProducts/WittSmallGroupEmbeddingMemPointwiseStabilizerIso.lean:179`
`MainX` `AVT/Definitions/LatticeFrames/ValLastX.lean:244`, used at `TPS/JointOfFamAllQ.lean:26`
`BottomRelation` `TPS/FormsAtZero.lean:100`

The archive's model of the rows by automorphic forms on adelic groups, which is also on the route, has an equation between Bessel periods and its own column periods: under continuity and decay hypotheses, the Bessel period of the iterated lift of a form is an iterated integral of the model's column periods (`iter_colPer`), and Bessel periods of iterated lifts of the row forms are the functionals that test the rationality of the rows (`InjK`, `FormulaAt`). The theta record of the datum has a Bessel integral of its own (`BesselTied.bper`); on the route that record comes from an existence theorem whose proof takes a witness with zero column and row forms and then replaces the row forms, at the tuples of characters that the chain uses (`Used0`), by cusp forms that keep inner products (`rows_of_ties`); the column forms stay zero, and the statement includes a condition on the row forms (`TieIo`). The tie `TieBm`, which would equate the datum's periods with a real factor of the record (`mY`) times the top period of the column forms against the row forms (`Btop`), is not on the route.

`iter_colPer` `AVT/Proofs/TwistedOperators/QEqColPer.lean:73`
`InjK`, `FormulaAt` `AVT/Statements/HeckeThetaIntegrals/FormulaAt.lean:76`, `82`
`BesselTied.bper` `FLG/BesselHermitianTheta/TiedBessel.lean:72`
the witness `.../AdelicDataRowMeasureScaleTrivialTuple.lean:553`, through `.../ExistsSplittingCharacterSelf.lean:145`
`rows_of_ties` `AVT/Proofs/IwasawaDecomposition/FunctionLevelDataOfEqualTuplesZeroIntertwiners.lean:29`
`Used0` `FLG/IndefiniteReal/RecTie.lean:189`
`TieIo` `LRC/DataTheta.lean:743`
`TieBm` `LRC/DataTheta.lean:747`
`Btop` `LRC/DataTheta.lean:665`

On the datum, for $y$ in the space of rows, $\mathrm{Bu}_\omega(\varphi, y)$ is the period of the column map against the form $\mathrm{sec}(y)$ given by the datum's section, for the reference tuple $\mu_0$ of characters (`Rows.BuDefQ`; the field `bu_def` of `Ties`). `BesselId` says that the $\mathrm{Bu}_\omega$ take $LL$-rational values on the lattices and that there are an isometry $V$ of the space of rows, the same for every word, and for each word $a_\omega \in \mathbb{C}^\times$ and an isometry $U_\omega$ of its column space with $\mathrm{Bu}_\omega(\varphi, y) = a_\omega\, \mathrm{besS}_\omega(U_\omega \varphi, V y)$, where the pairing $\mathrm{besS}_\omega$ depends on the word through its spectral map $P_\omega$ on the stage-zero column space.

`BesselId` `GTI:65`
projected Poincaré series `FLG/SiegelLatticeTheory/FlagStabilizerTensors/KerJoinS.lean:133–145`
`Rows.BuDefQ` `LRC/DataBu.lean:32`
`bu_def` `LRC/DataTies.lean:42`
`besS` `LRC/RhoDetPrimeEqMuTildeMulChiPow.lean:353`
spectral map $P_\omega$ `LRC/RhoDetPrimeEqMuTildeMulChiPow.lean:354`

The archive states independence of the tuple $\mu$ of Hecke characters as a condition, namely that for the tuples used ($\mu_0$ and the $\mu_w$), the period for $\mu$ against $F$ in a subspace $V_0(\mu)$ of the space on which the periods for $\mu$ are defined is $\zeta_{\omega,\mu}\, \mathrm{Bu}_\omega(\varphi, \iota_\mu F)$, where $\iota_\mu$ is the

datum's linear map from that space to the space of rows (`V0`, `iota`), with $\zeta_{\omega,\mu_0} = 1$ and $\zeta_{\omega,\mu_w}$ a unit at $w$ and $\bar{w}$ (`LocalQuadraticParts.adm1`). This condition is not on the route. The statement of this kind on the route is the identity above, in which $\zeta_{\omega,\mu_w}$, when it lies in $LL$, has valuation zero at $w$ (`MIUnit2`); there the periods for the other tuples also enter through their own denominators (`aPerTw`). That $\zeta_{\omega,\mu}$ is a root of unity for every $\mu$ appears under further hypotheses, as a conclusion of `MultiplicityOneWithZeta` and of `RowsMultiplicityClosed`, which is derived from it.

`LocalQuadraticParts.adm1` `FSR/Divisibility/LocalQuadraticParts.lean:42–51`
`aPerTw` `TPS/Hyps.lean:69`, which states for all large $x$ the condition
`TwistedPeriodAdmissible` `.../AdmissibleAtGeneratorPlaces.lean:14`
`MultiplicityOneWithZeta` `FSR/Divisibility/ZetaProductPairs.lean:137`
`V0`, `iota` `LRC/DataRows.lean:190`, `184`

The model's equation is proved through unfoldings of the flag integrals (`unfoldings_of_chain`), an induction on the chain from the double step (`DoubleStep`) and an integrated seesaw at the last step (`LastLaw`). `SeeLast`, the last seesaw identity, is a pointwise identity of Weil-representation values proved for every datum (`lastSeeSaw_thetaIdentity`). Seesaw identities between these periods are known one step at a time [55], with the local Hom spaces in [29, §14]; along the whole chain we know of no reference. At the real place the archive proves uniqueness of Bessel models in a restricted form (`uniqGap_holds`). Under its hypotheses, for a discrete series whose Harish-Chandra parameter has gaps at least $h \geq 120(2r + k + 1)^3$ between entries of opposite sign type, the restrictions of any two such Bessel maps to the smooth vectors are linearly dependent. This is known in greater generality [48, Thm A].

`unfoldings_of_chain` `AVT/Proofs/DecoratedLatticeProofs/UnfoldingsOfChain.lean:23`
the induction `AVT/Proofs/DecoratedLatticeProofs/IteratedThetaUnfoldingOfDoubleStepLastTwoStepsStatement.lean:13`
`DoubleStep` `AVT/Definitions/LatticeFrames/IsQuasiFam.lean:77`
`LastLaw` `AVT/Definitions/LatticeFrames/IsQuasiFam.lean:71`, proved at `AVT/Proofs/DecoratedLatticeProofs/DoubleLiftPeriodRankZeroFubiniCompactQuotientStatement.lean:13`
`lastSeeSaw_thetaIdentity` `AVT/Proofs/DecoratedLatticeProofs/LastSeeSawThetaIdentityStatement.lean:13`
`uniqGap_holds` `CompanionTheory/AutomorphicTheory/LocalBesselUniqueness/BesselModels/MultiplicityOneUnitaryBranchingOfGapHyp.lean:8`

**4.4. Divisibility at the inert primes.** The note's step 2 says that $(D^a\vartheta)(0)/\Omega^a$ is divisible by each prime $p$ inert in $K$ to the order $a/p + a/p^3 + \cdots - O(\log a)$. The archive has this in two parts: integrality of corrected kernels at inert $p \geq 5$, and a bound of the note's shape for every power series of the form that integrality gives. This subsection states both and where the chain uses them. Alpöge's `IntInert`, proved as `kernels_integral_inert`, is used through the selection hypothesis `hServ` (`SelectionServed`), by `servH_row_of_artin`, and makes the actual periods divisible by the lattice ideals (`LinkDIVt`). Let $p \geq 5$ be a prime with $(-d/p) = -1$, and take a CM elliptic curve attached to a nonzero ideal of $K$ of norm prime to $2d$, with a model over $\mathbb{C}_p$ with $p$-integral coefficients and discriminant 1 and formal parameter $-x/y$. For admitted $\mathcal{O}_E$-lattices $L, L'$, every $p$-integral combination of the power series at 0 of the kernels translated by prime-to-$p$ torsion points of the matrix coordinate space modulo its lattice, each with its exponential factor (`IsComb`) becomes, after substituting the formal logarithm of the curve along an orthonormal local basis and multiplying by an explicit factor, a power series with $p$-integral coefficients (`IntFirst`). The theorem assumes that for CM by squarefree $d \equiv 3 \pmod 4$, at every $\ell \geq 5$ with $\ell \nmid d$, the period lattice has, after

rescaling in $\mathbb{C}_\ell$, a good model with coefficients in the completed maximal unramified extension (CM2). On the chain CM2 is supplied by cmGoodModel_of_globalArtinReciprocity.

IntInert FLG/InertKernelTheory/StatementInt.lean:18
kernels_integral_inert FLG/InertKernelTheory/KernelsIntegralInert.lean:9
hServ TPS/Hyps.lean:76
servH_row_of_artin AVT/Proofs/SatakeParameters/BundleBothPartsOfStandardKernelOfGlobalArtinReci.lean:15–24
periods divisible TPS/TopShape.lean:40–47
IsComb FSR/GlobalSchurBounds/GeneralLinearActions/IsDiscriminantTwelfthRootOfIdealLattice.lean:280–295
IntFirst FLG/InertKernelTheory/StatementInt.lean:9
cmGoodModel_of_globalArtinReciprocity AVT/Proofs/DecoratedLatticeProofs/CmGoodModelOfGlobalArtinReciprocity.lean:10, applied at AVT/Proofs/SatakeParameters/BundleBothPartsOfStandardKernelOfGlobalArtinReci.lean:21–23

At an inert $p$ the curve is supersingular. CM3 gives integrality of the elliptic formal parameter after substituting a unit times the height-2 Lubin–Tate logarithm, and IntFirst that of the corrected matrix kernel in formal-log coordinates.

CM3 FLG/TorsionIntegrality/CMGoodModels/Statement.lean:49

The bound of the note's shape is a theorem about power series (kernel_bounds_generic). Under hypotheses on a formal group law and its periods, if a series is, in logarithm coordinates, an integral series times $\exp(-\sum_e \beta(x_e, y_e))$, with $e$ running over the pairs of coordinates and $\beta$ a series in two variables attached to the law (BetaHyp), then every coefficient of its frame jets has norm at most $p^{-g}$. Here $g$ is the sum, over the positions $s$ in a chosen set and the odd $e$, of $\max(a_s/p^e - c_{s+e}, 0)$, omitting the terms in which $e$ is less than the number of positions and $s+e$ is in the set, less a loss built from logarithms of the sizes, where $a_s$ is the number of columns of height one at $s$ if $s$ is in the set and 0 otherwise, and $c_s$ is the number of the other columns at $s$.

Integrality is what puts the kernel series of the datum in this form. IntInert is a hypothesis of the theorem that applies the bound (at_of_true_columns), where the model of the curve gives the unit and the series in the exponential, and IntInert gives the condition that the kernel series of the datum have that form (hostAt_inert, EvenConditionsInert).

The bound is on the route as one condition of hServ (OrbitBoundWithoutOperator, by orbit_bound_rec). At the primes $p \le X''$ of the host's class inertClass, which are inert in $K$, under the conditions OrbOK and hP $\neq 0$ on $p$, it says that for every word, every step $i < m$ and every series of that form, the derivatives at 0 applied to the frame polynomials have norm at most $p^{-\max(g_0,0)}$, where $g_0$ is a sum of terms of the same kind over the tops of an orbit, less a combination of $\log_p(2a)$ and $\log_p(2a)^2$ (g0). With conditions on the other orbits, the result is a bound for the derivatives at 0 of the kernel series of the datum at those of these places whose residue characteristic avoids the primes excluded for the word and the tuple and two of the host's level primes (its field g) (LevelFree; EntriesAtInert, by entI_of). The bound with $g_0$ is the form of the note's step 2 on the route, with caps, omitted terms, and a loss in $\log_p(2a)$ and its square where the note has $O(\log a)$.

kernel_bounds_generic Formalism/ThetaLogCoordinates/UniformThetaSubstitution/FrobeniusLiftNorms/EGaussK.lean:255
BetaHyp Formalism/ThetaLogCoordinates/UniformThetaSubstitution/IsLawOfLog.lean:89
at_of_true_columns AVT/Proofs/SatakeParameters/AtOfTrueColumns.lean:30–47
hostAt_inert AVT/Proofs/SatakeParameters/JacobiOfClass.lean:63
EvenConditionsInert FSR/Divisibility/OrbitPlaces.lean:303
OrbitBoundWithoutOperator FSR/Divisibility/OrbitPlaces.lean:333
orbit_bound_rec AVT/Proofs/SatakeParameters/OrbitBoundStandardSeriesInertPlaces.lean:22

`g0` FSR/GrowthBounds/IsResidueClassZero.lean:247
`LevelFree` FSR/Divisibility/OrbitPlaces.lean:64
`EntriesAtInert` FSR/Divisibility/OrbitPlaces.lean:362
`entI_of` FSR/Divisibility/OrbitPlaces.lean:1090
`inertClass` LRC/DataHost.lean:33–37

The size of the gains is bounded below for the lifts together, by `LinkG`, whose proof sets every kernel chain map to zero and so does not use the theta kernels (`linkG_without_kernels`). Its proof uses a theorem of the archive about the families (`FirstTheoremAtFamily`), and this is where the archive's divisibility of minors is used: if $w^2 = p$ in a commutative ring, $u$ and $u'$ are vectors whose coordinates other than the first are divisible by $w$, and $M$, $N'$ are matrices with $M_{ba}^{p^e} \equiv N'_{ba} \pmod{p}$, then $w$ divides $(u'N')_c\,(uM)_0^{p^e} - (u'N')_0\,(uM)_c^{p^e}$ for every column $c$ (`key_minor_dvd`). It gives order at least one in $w$ for explicit polynomials over a discrete valuation ring with uniformizer $w$ (`CoarseMultiplicityOrderBound`), a step in the bounds that the theorem takes as an input (`ExtendedClosedForm`). The same theorem gives the weight vectors on the hosts (`WClosed`) used for `classSel`. The proofs of `IntInert` and `kernel_bounds_generic` do not use the lemma.

`linkG_without_kernels` FLG/GroupLattices/AdmissibleFamilyNoKernel.lean:166–186, 224–239
`FirstTheoremAtFamily` FSR/GrowthBounds/PlaneCountTheoremWithConstants.lean:979, used
  at FLG/GroupLattices/AdmissibleFamilyNoKernel.lean:241–249
`key_minor_dvd` FSR/GrowthBounds/CoarseMultiplicity/KeyMinorDvd.lean:132
`CoarseMultiplicityOrderBound` FSR/GrowthBounds/CoarseMultiplicity/KeyOne.lean:114
`ExtendedClosedForm` FSR/GrowthBounds/ColumnSystems.lean:511, supplied
  at FLG/GroupLattices/AdmissibleFamilyNoKernel.lean:65
`WClosed` AVT/Definitions/GlobalTwisting/AuxiliaryPrimeMemAvoidedPrimes.lean:142, proved
  at AVT/Proofs/GlobalInterpolation/WindowShareRequirementClosedFormProof.lean:40

Potential good reduction of CM curves is known [75]. For functions of one variable on the formal group of one curve there is a classical bound with this exponent. Let $E$ be an elliptic curve with multiplication by $\mathcal{O}_K$, defined over $K$, with a Weierstrass model $y^2 = 4x^3 - g_2x - g_3$, $g_2, g_3 \in \mathcal{O}_K$; let $p$ be an odd prime inert in $K$ that does not divide the discriminant of the model, $\wp$ the Weierstrass function of the model, and $b \in \mathcal{O}_K$ prime to $p$. Then the $n$th derivative at 0 of $\wp(z) - z^{-2}$, multiplied by $(1 - b^{n+2})(1 - p^n)$, is divisible by $p^{[np/(p^2-1)]}$, and $np/(p^2-1) = n/p + n/p^3 + \cdots$; for $n \geq 2$ these derivatives are the Bernoulli–Hurwitz numbers $\mathrm{BH}(n+2)/(n+2)$. This is a theorem of Katz [52, §IV], who takes $p \geq 5$ and $E$ over the Hilbert class field of $K$; we state it as recalled in [7, Thm 5.1]. Katz deduces it from a theorem on Lubin–Tate formal groups of height two: if a power series $f$ with integral coefficients has translates by the $p^2$ points of order dividing $p$ that sum to zero, then the derivatives $D^n f(0)$ along the invariant derivation are divisible by $p^{[np/(p^2-1)]}$ [52, Main Theorem 3.11]. Chellali refined the accompanying congruences [19], and Bannai and Kobayashi reproved both by $p$-adic Fourier theory for the Lubin–Tate group of height two [7, Thm 5.8]. For the theta function of a CM curve itself, Bannai and Kobayashi observed that at a supersingular prime its power series, and that of the Kronecker theta function in two variables, is not integral and has radius of convergence less than one [6, §4]. The radius of convergence of the $p$-adic sigma function is the subject of [8]. The archive's statements are for theta kernels in several variables, over CM fields of growing degree and along the $m$ lifts, with caps and a loss in the exponent; for these we know of no reference.

**4.5. The definite side.** The conditions of Section 4.1 are supplied, for one class of families, by Alpöge's `HypsAlt`. `classSel` picks one class on which all its rows hold at once.

`HypsAlt` `TPS/Hyps.lean:22`
`classSel` `CB/ClassSel.lean:14`

The main conditions of `HypsAlt` are these. For three of them, Theorem S, the cost of a lift and the Selmer bound, we know of no reference.

- Masses and volumes. The existence proof of `HypsAlt` assumes `SiegelMassRatioCM`, which says that for a CM field $E$ unramified over its real subfield and $M \geq 3$, the mass of the standard lattice of rank $M$ divided by that of rank $M-1$ is $\big((M-1)!\,(v/\pi)^M\big)^f (2v)^{-f/2} L_F(M, \eta^M)$, with $v = \mathrm{rd}_E/2$, $f$ the number of infinite places of $F$ and $\eta$ the quadratic character of $E/F$. The archive proves it as `massRatio_all_unramified_cmFields`. Under its hypotheses, among them $T$ split in $F$, the class number of the stage-zero lattices of the level equals the standard mass times explicit indices at two auxiliary primes and at $T$ (`ClassNumberFormula`$_2$). The codimension-one volume ratio and the mass of rank-2 definite lattices are fields `hU` and `hG`. These are known [30], as are their analogues for quadratic forms [78].

  `SiegelMassRatioCM` `FSR/GlobalSchurBounds/StandardLattice.lean:281`
  right side `Formalism/HermitianLattices/StatementMass.lean:40`
  `massRatio_all_unramified_cmFields` `AVT/Proofs/GlobalInterpolation/MassRatioAllUnramifiedCmFields.lean:8`
  `ClassNumberFormula`$_2$ `FSR/GlobalSchurBounds/StandardLattice.lean:453`

- Theorem S (field `sO`; `TheoremSSelected`). Under the standing hypotheses, for words $\omega \neq \omega'$ the non-degenerate part $K_0$ of the kernel between their column spaces satisfies $\|K_0(\omega,\omega')\|^2 \leq (1+b^{-6})\, d(\omega) d(\omega')/D^*$, where $d(\omega)$ is the dimension of the stage-zero column space. Here $K_0$ is the orbital expansion of the kernel without the degenerate orbits (`Terms.K0`), and $\|K_0\| = \|K_{\omega\omega'}\|_{HS}$ for a tied datum when the orbital expansion represents the kernel and converges absolutely, the invariant `Gterm` of the terms vanishes on the identity terms and the degenerate orbits contribute zero. `theoremS_transport_link_zero` turns it into `LinkS`, using orbit bounds for the diagonal and the other parts, local identifications and the family's size conditions. Its inputs include the Siegel–Weil formula in the convergent range, finite dimensionality and Bessel uniqueness.

  `TheoremSSelected` `FLG/SiegelLatticeTheory/FlagStabilizerTensors/ThmSOwn.lean:669`
  `Terms.K0` `Formalism/HermitianCurvature/StatementPB.lean:61`
  for a tied datum `FLG/SiegelLatticeTheory/FlagStabilizerTensors/PackageTie.lean:208`
  `Gterm` `FSR/CuspSpanNormBounds/DataS.lean:61`
  `theoremS_transport_link_zero` `FLG/SiegelLatticeTheory/FlagStabilizerTensors/LinkSPos.lean:131`

  *Idea of the proof.* Up to isometries, the column map of a word $\omega$ is a Poincaré series on the unitary group $G$ of the setting over $F$: for $a$ in the column space, $P_\omega a(g) = \sum_{\gamma \in S_\omega \backslash G(F)} \phi_{\omega,a}(\gamma g)$, where $S_\omega$ is the group of rational points of the subgroup attached to the seed (`Srat`) and the seed $\phi_{\omega,a}$ is built from a unipotent character, the level character and an archimedean Bessel vector. Unfolding writes the kernel $K_{\omega\omega'} = (c_\omega c_{\omega'})^{-1/2} P_\omega^* P_{\omega'}$, with $c_\omega$ a product of three volumes divided by the dimension of the compact representation of the word, as an absolutely convergent sum of orbital integrals over the double cosets $S_\omega \backslash G(F) / S_{\omega'}$. The orbits are sorted by a discriminant formed from the leading minors of the cross-pairing matrix of the two flags, and $K_0$ is the sum over the identity cell and the orbits of

nonzero discriminant. For $\omega \neq \omega'$ the identity cell contributes zero, and every other orbit of zero discriminant has orbital integral zero, by the character of its stabilizer or by Schur's lemma at a real place where the two words have distinct highest weights; so $K_0(\omega, \omega')$ is the whole kernel.

A Gelfand–Graev character estimate at the two level primes leaves, as the main term of $\|K_0(\omega, \omega')\|^2$, a signed sum over the pairs of terms of corank one that are generic and scalar. A recursion over the levels of the flag by Poisson summation, and an exact cancellation of the local masses against the formal degree, bound it by $(1+\delta)e_1\, d(\omega)d(\omega')/D^*$, where $\delta$ is the error of the recursion and $e_1$ is the larger of 1 and the supremum of the lattice densities divided by the Euler product. The pairs of corank at least two and the generic pairs of corank one that are not scalar each contribute at most $\exp(-kf\log k/3)\, d(\omega)d(\omega')/D^*$, and the non-generic pairs of corank one at most $2^{-(k-m-3)}\exp(C_N m^5 f\log b)\, d(\omega)d(\omega')/D^*$ for a constant $C_N$. The size conditions then bound all the factors together by $1+16b^{-7}$, which is at most $1+b^{-6}$ since $b \geq 16$.

On the diagonal the identity cell gives the identity operator, the other orbits of nonzero discriminant have squared norm at most $\Gamma\, d(\omega)^2/D^*$ using the cost of a lift (the next item), and the remaining orbits at most $d(\omega)e^{-kf\log k}$. In the window $d(\omega) \leq 3D^*/\Gamma^2$ of `LinkS` this gives $\|K_{\omega\omega} - I\|_{HS}^2 \leq e_0\, d(\omega)$ with $e_0 = 6/\Gamma + 2e^{-kf\log k}$.

Poincaré series `FLG/SiegelLatticeTheory/HaarTransversals/PoincareInHH.lean:149–186`
$c_\omega$ and the kernel `257–263`
unfolding `FLG/SiegelLatticeTheory/ClosedLocalOrbits/GaugeLatticeShifts/DoubleCosetSingleUnfolding.lean:31–191`
vanishing `FLG/SiegelLatticeTheory/CokernelGLCounts/RankPointCounts/FixedDecomposition.lean:744–775`
identity cell for $\omega \neq \omega'$ `FLG/SiegelLatticeTheory/FlagStabilizerTensors/RootLatticeThresholds/NuQPinExistsHolds.lean:57–69`
Gelfand–Graev `FLG/SiegelLatticeTheory/ClosedLocalOrbits/GGMirabolic.lean:64–120`
main term and $1+b^{-6}$ `Formalism/HermitianCurvature/HermitianCurvatureProofs/GrowthOfBasic.lean:525–550, 1070–1183`
$1+16b^{-7}$ `232–297`
the constant $1/3$ `1191–1235`
identity cell `FLG/SiegelLatticeTheory/ClosedLocalOrbits/GaugeLatticeShifts/IdBlocks.lean:735–829`
diagonal `FLG/SiegelLatticeTheory/FlagStabilizerTensors/RowsPos.lean:95–245`, `FSR/CuspSpanNormBounds/StandingHypothesesHoldFamily.lean:144–184, 628–651`

- The cost of a lift. This is the statement `StepThmBG`. It is not itself a field of `HypsAlt`: the field `sM`, of type `RowMainEqP`, is derived from it (`rowMainEqP_of_siegelWeil_of_stepChoice`). Let $r_S = m/2 - 1$. A *point* is a pair $\gamma = (c, z)$ of a composition $c$ of $r_S + 1$, with blocks of sizes $s_1, s_2, \ldots$, and an element $z_i$ of $E$ for each block; put $\delta_i = \prod_{i' \leq i}(-1)^{s_{i'}(s_{i'}-1)/2} z_{i'}^{s_{i'}}$. Each term of the orbital expansion of the kernel of a word with itself has a point. Under the standing hypotheses, for every finite set $S$ of points of generic terms with nonzero invariant (the `G` of `Terms`) and nonzero vector, and every integer $A \geq 0$,

$$\sum_{\gamma \in S} \mathrm{wt}(\gamma)\, n(\gamma)\, \big(N_G(\gamma) N_D(\gamma)\big)^{A(r_S+1)^2} J(\gamma) \leq b^{C_B m^5 f} I_{\mathrm{re}}$$

(`PointSumInput`), with $C_B \geq 0$ depending only on $A$ and on six constants of the standing hypotheses. Here $\mathrm{wt}(\gamma) = \prod_i \prod_v (1 + |\delta_i|_v^2)^2$, over the blocks and the infinite

places $v$ of $E$; $N_G$, $N_D$ and $n$ are explicit products over the infinite places of powers of the $|\delta_i|_v$, of the $|z_i|_v$ and of one quadratic expression in the last two $\delta_i$ and their conjugates, each divided by a power of $\mathrm{rd}_E$; $J(\gamma)$ is the squared Hilbert–Schmidt norm of an operator that the archive attaches to $\gamma$ (`Jw`); and $I_{\mathrm{re}}$ is the quantity `IntC` of the archive's mass data divided by two of its constants, by $2^{f(r_S+1)}$ and by an explicit power of $\mathrm{rd}_E$ (`FrameInputs`).

`StepThmBG` `FLG/SiegelLatticeTheory/FlagStabilizerTensors/RootLatticeThresholds/BDataOf.lean:641`
`rowMainEqP_of_siegelWeil_of_stepChoice` `AVT/Proofs/FlagMeasures/RowMainEqPOfSiegelWeilConstants.lean:14`
`sM` `TPS/Hyps.lean:51`
$r_S$ `LRC/Params.lean:38`
points `Formalism/HermitianCurvature/StatementPN.lean:169`, 204–207
generic terms `FLG/SiegelLatticeTheory/CokernelGLCounts/RankPointCounts/MainEqCut.lean:26–33`, `Formalism/HermitianCurvature/StatementPB.lean:51`
`PointSumInput` `Formalism/HermitianCurvature/StatementPN.lean:240`
$\delta_i$, wt `Formalism/FlagUnipotentLattices/UnipotentStabiliserCharacterisation.lean:77–93`
$N_G$, $N_D$, $n$ `FLG/SiegelLatticeTheory/CokernelGLCounts/RowsConc.lean:345–376`
$J$ `FLG/SiegelLatticeTheory/CokernelGLCounts/NGData.lean:29–37`
$I_{\mathrm{re}}$ `Formalism/HermitianCurvature/StatementPN.lean:30–35`, `FLG/SiegelLatticeTheory/CokernelGLCounts/RankPointCounts/StepHeightOfNormOne.lean:20`

- Periods. The periods of the stage-$m$ lifts lie in $LL$, with explicit denominators at the places named, for the reference tuple (`aPt0`, `aPer0`) and, away from their exceptional places, for the other tuples used (`aPtTw`, `aPerTw`). At the places $w$ over $T$ the column map sends a lattice $N_\lambda(w)$ into $\varpi_w^{-\lceil e_w m E_T \rceil}$ times the stage-$m$ lattice (`aP10`, `Conclusion5Sharp`), where $e_w$ is the ramification index of $w$ and $E_T$ grows with $m$ at the places seen by the word and is 0 at the others (`EatT`). Explicit denominators for a single theta lift are known [70].

  `Conclusion5Sharp` `FSR/GlobalSchurBounds/ProductFirstZerothParts.lean:315`
  `EatT` `LRC/DataErr.lean:103`

- Analytic inputs. `LitThin`: zero density for primitive $L(s,\chi)$ near $\mathrm{Re}\, s = 1$ (at most $C_1 Q^{1/4}$ zeros with conductor at most $Q$, $\mathrm{Re}\,\rho \geq 1 - \frac{1}{400}$, $|\mathrm{Im}\,\rho| \leq Q$, for $Q \geq 3$), the classical zero-free region with at most one exceptional zero for each primitive character of conductor at least 3, a truncated explicit formula and the local count of zeros. The zero-free region, the explicit formula and the local count are known [23, Ch. 13–20], and the zero-density estimate is of the classical kind [65].

  `LitThin` `FSR/GlobalSchurBounds/MainSup.lean:88`, proved at `AVT/Proofs/GlobalInterpolation/LitThin.lean:8`

- The Selmer bound (field `nRest`, through `LinkN`). If the groups $\mathrm{Sha}_\ell(e)$ are finite and $\sum_\ell \log \ell \sum_e \text{length } \mathrm{Sha}_\ell(e)$ converges, then for each lift $j$

$$\sum_w \frac{\log N w}{[LL:\mathbb{Q}]}\, e_w\, \eta_j(w) \leq d_{Z,0}(j)\Big(C(fmb\, x^{2\varepsilon} + N_c \log^2 x) + \sum_\ell \log \ell \sum_{1 \leq e < t} \text{length } \mathrm{Sha}_\ell(e)\Big),$$

  where $w$ runs over the twisted places, $\eta_j(w)$ is the divisibility defect of the $j$-th lift at $w$, and $\mathrm{Sha}_\ell(e)$ is the Tate–Shafarevich group of $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$ over $E$. It comes from bounds at the twisted places, which the archive states by ranges of

the residue characteristic $\ell$ (the fields `large`, `mid`, `two`, `ram` and `typ` of `Concl`). At two primes $g_0 \neq g_1$ that the host carries among its level primes (field `g`) the defect is bounded directly (`StarH`); in the other ranges the bound has the shape of one inequality, `Bd`. Let $w$ be a twisted place of residue characteristic $\ell$. The datum gives a complete discrete valuation ring $\mathcal{O}$, finite over $\mathbb{Z}_\ell$, whose fraction field $\mathcal{K}$ receives $LL$ compatibly with $w$, and a finite set of Hecke eigensystems $\pi$ with values in $\mathcal{O}$. For each good $\pi$, `Bd` says that the length of $\mathcal{O}/\eta_\pi$, with $\eta_\pi$ the congruence ideal, is at most that of a Selmer group $\mathrm{Sel}(\pi)$, both normalized by the length of $\mathcal{O}/\ell\mathcal{O}$, plus $\mathrm{err} + \sum_{1\le e<t} \text{length } \mathrm{Sha}_\ell(e) + v_\ell(h_E) + \ell^{-2}$, with $h_E$ the class number of $E$. Here $\mathrm{Sel}(\pi) \subset H^1(G_E, W_\pi \otimes \mathcal{K}/\mathcal{O})$ with $W_\pi = \rho_\pi \otimes \chi_{w,1}^{-1}$, where $\chi_{w,1}$ is the first of the datum's characters $G_E \to \mathcal{O}^\times$ at $w$ and the representations $\rho_\pi \colon G_E \to \mathrm{GL}_{k+m-1}(\mathcal{O})$ belong to auxiliary data over which the statements are quantified (`BoundAtTwoThreeFamilyFonAlt`, `ConclF`). A class lies in $\mathrm{Sel}(\pi)$ if it is unramified at the places not above $\ell$ outside a finite set of split places of $E$, with no condition on that set, and if at each place above $\ell$ it lies in the image of the cocycles of $W_\pi \otimes \mathcal{K}$ that become coboundaries over the archive's period ring, the shape of the finite part of Bloch and Kato (`selPi`, `bkCond`, `HfD`). The cases $\ell = 2$, and $\ell = 3$ when 3 is not small for the host, enter this step as a separate hypothesis, with $\mathrm{err} = C\, fmb \log x + C\, x^{C+2\varepsilon}$ and $b = \exp(x^\varepsilon)$ (`BoundAtTwoThree`), which the chain proves as `boundAtTwoThree_linked_of_recT`. The step supplies the other ranges itself, with $\mathrm{err} = 0$ at the large $\ell$ outside an exceptional set (`conclF_of_family_facts_and_bound_at_two_three`). These bounds are combined with Chebotarev, local Tate duality and the local Euler characteristic, among other inputs (`final_alt_alt`). The shape of `Bd` is the inequality between congruence module and Selmer group of Wiles and Lenstra [90, 61], which is known in another setting.

`LinkN` GTI:40–47
`Bd` AVT/Definitions/NonGaloisDescent/BoundAtTwoThreeFamily.lean:17
`BoundAtTwoThree` 33–53
`StarH` 82–85
`Concl`, `ConclF` 124–143, FSR/LevelLetterCocycles/JointConditionsZpowers/CommutingEndomorphismCondition.lean:132–135
`g` LRC/DataHost.lean:71
`conclF_of_family_facts_and_bound_at_two_three` TPS/ImportsFurtherTailStatement.lean:11
`BoundAtTwoThreeFamilyFonAlt` TPS/BoundAtTwoThreeFamilyFon.lean:25–28
the ring $\mathcal{O}$, the eigensystems,
$\chi_{w,1}$ AVT/Definitions/AutomorphicMembership/ExistsSameData.lean:236–287
`selPi` FSR/LevelLetterCocycles/QuotientHomNonvanishingBound.lean:857, $W_\pi$ 271
`bkCond` Formalism/GaloisCohomology/Statement.lean:162
`HfD` Formalism/WeaklyAdmissibleFiltrations/StatementTor.lean:126, Formalism/WeaklyAdmissibleFiltrations/Statement.lean:190
`boundAtTwoThree_linked_of_recT` TPS/At23OfAll.lean:24
`final_alt_alt` TPS/CFinal.lean:19

`HypV` says that for all large $x$ and every member the $\mathrm{Sha}_\ell(e)$ are finite and the series $\sum_\ell \log \ell \sum_{1\le e<t} \text{length } \mathrm{Sha}_\ell(e)$ has sum at most $\sqrt{b}$. With it, `LinkN` and `SmallLevelRankEq` give `DIVinN` for $\varepsilon > 0$ (`divinN_of_hypV`).

`HypV` CB/DivinNOfHypV.lean:10–11
`vvRecord` CB/DivinNOfHypV.lean:17
`divinN_of_hypV` CB/DivinNOfHypV.lean:117

**4.6. The covolume bound.** The slope of $RR^\vee$ in (4.1) is bounded through the covolumes of lattices of automorphic forms. The archive states the following as `CovolumeBoundOfX` and proves it as `topCovolSlim2_use_alt_alt` from the fifteen covolume inputs listed in Section 6, all theorems of the archive.

`CovolumeBoundOfX` `TPS/LiftLambdaBound.lean:60`
`topCovolSlim2_use_alt_alt` `TPS/TopCovolSlim2Use.lean:16`

**Theorem 4.1** (Alpöge)**.** *For all small $\varepsilon$ there are largeness thresholds such that, for every family satisfying* `FamHyp` *and every $C_H$ that together meet them, there is $C_0$ such that, for all large $x$, every member and every point at the member (defined below), every $C \geq C_0$ and every $j = 1, \dots, m$,*

$$-\mu(M_j) \leq \sum_{\rho} (p_j - q_j)\, e_j(\rho) \log \Omega + C\, f m b\, x^{\varepsilon} \log\log x.$$

Here $W_j$ is the $j$-th seesaw space, of signature $(p_j, q_j) = (k + j - 1, m - j)$; $M_j$ is the row lattice in its frame on $U(W_j)$, which the archive compares with the lattice of integral automorphic forms of the good Hecke type (`RecAt`); $\rho$ runs over an index set of $f$ elements, each sent by the archive to an element of the CM type $\Phi$ (`RPl`, `tauPl`), and $e_j(\rho)$ is the $j$-th entry of the part $e$ of the datum's Harish-Chandra parameter there (`HCParam`); a *point* at a member is the archive's record of the choices made there (`PtAt`), and the theorem is about the points that satisfy the conditions `RowsTAlt`; and $\mu = -\mathrm{lcv}/\mathrm{rank}$ is the archive's slope (`LcvMu`). The bound enters once, as the field `l3`, of type `L3`, used by `IndefSide.boundsMu`, which bounds $\mu(RR^\vee)$ by the sum over the lifts of these bounds, the theta indices, ramification terms and errors.

`RowsTAlt` `TPS/LiftLambdaBound.lean:17–19`
`PtAt` `AVT/Definitions/Divisors/TupleReadBaseTupleEqFamily.lean:145`
main term `FLG/Divisibility/ZscOfNum.lean:64–66`
`RPl` `FLG/Divisibility/Numbers.lean:13`, `160–163`
`tauPl` `LRC/DataForced.lean:15`, `FLG/Divisibility/NumbersLemmas.lean:604–613`
`HCParam` `LRC/Params.lean:50`
`RecAt` `AVT/Definitions/Divisors/CovolXOfCovol.lean:57`
`LcvMu` `Formalism/MeanHeightDescent/MuMaxMinkowski.lean:13`
`L3` `FLG/IndefiniteForms/IndefSide.lean:315`
`IndefSide.boundsMu` `FLG/IndefiniteForms/IndefUse.lean:271`
the sum `FLG/IndefiniteForms/IndefUse.lean:286–288`
`FamHyp` `AVT/Definitions/Divisors/TupleReadBaseTupleEqFamily.lean:503`

For $j < m$ the proof compares $M_j$ with sections of automorphic bundles on integral toroidal models of the unitary Shimura variety of $W_j$ at a deep level. Bost's slope inequality [15, Props 4.3, 4.4] enters as `DescentSlope`, which says that $\mathrm{rk}A\,(\mu(A) - h(\varphi)) + (\mathrm{rk}B - \mathrm{rk}A)\,\mu_{\min}(B) \leq \mathrm{rk}B\,\mu(B)$ for finitely generated Hermitian lattices $A \neq 0$ and $B$ over a totally complex field and an injective linear map $\varphi$ of their ambient spaces with $\varphi(A) \subset B$. Applied to the combined map $A \to B^n$ given by multiplication by Hecke translates of one small integral section, each followed by the trace to a character eigenspace, with its height bound, $n\,\mathrm{rk}B \leq (1 + \delta)\,\mathrm{rk}A$ with $0 < \delta \leq 1$ and a lower bound for $\mu_{\min}(B)$ from small spanning vectors, it moves a lower bound for slopes from a lower weight to a higher one.

the steps `AVT/Definitions/Divisors/CovolXOfCovol.lean:53–160`
`DescentSlope` `Formalism/MeanHeightDescent/MuMaxMinkowski.lean:19`

`FlatDegree` states that on every component and for every embedding $\tau$ in a fixed CM type, flat sections of the CM part of the weight exist and all have Arakelov degree $\frac{p-q}{2} \log \Omega + q(\log d - \log \mathrm{rd}_E)$. On the route the archive derives its form at good covers

from embedded resolution in characteristic zero and GAGA for sections, through a constant section with the Kronecker character (`flatDeg_alt_alt`; `flatDegreeSt_of_resK_gaga`, specialized). The archive has a second derivation (`flatDeg`), not on the route, from hypotheses (`ScholzeWeinsteinB`, `Hypotheses2`) that include its form of prismatic Dieudonné theory over $\mathcal{O}_C$. That form enters the route through input (6) of Section 6. The first term has the shape of the main term above and of the Chowla–Selberg constant (compare Colmez's formula for the Faltings heights of abelian varieties with complex multiplication [22, Thm 0.3, Conj. 0.4], a theorem when the CM field is abelian over $\mathbb{Q}$, as $E$ is [22, Thm 0.5], [68, Thm 3.9]). The second is at most $3q\,x^{\varepsilon}$ in absolute value and goes into the error term of Theorem 4.1. The lower bound for the rest is the volume theorem `VolumeTheoremX`, obtained, with the existence of good towers, from ten statements of arithmetic geometry by `good_tower_volume_theorem_bis`.

`FlatDegree` `AVT/Definitions/Foundations/ExistsFlatSecRankZero.lean:746–754`
`flatDeg_alt_alt` `TPS/FlatDegreeStatement.lean:12`
`flatDegreeSt_of_resK_gaga` `Development/AutomorphicSpectrum/Proofs/DivisibilityProofs/FlatDegreeStOfResKGaga.lean:18`
specialized `AVT/Proofs/DivisorProofs/FlatDegreeStatement.lean:15–17`
`flatDeg` `AVT/Proofs/DivisorProofs/FlatDegreeStatement.lean:25–29`
the second term `TPS/FlatValStatement.lean:11`
`VolumeTheoremX` `Development/AutomorphicSpectrum/Statements/TowerExistence/GoodTowerExistsX.lean:28`
`good_tower_volume_theorem_bis` `WB/CoreStatements/AutomorphicKernels/MoriwakiBigness/HilbertSamuelApplications/GoodTowerVolumeTheoremBis.lean:9`

**Theorem 4.2** (Alpöge)**.** *There is $C$ such that for every abelian CM field $E$ with a host, every global model, every root tower with a good origin satisfying* `GoodOrigin.Extra` *and every full intersection system, of signature $(p,q)$ with $q \geq 1$ and $p \geq 2q+1$ and with at most $4f(p+q)^2$ bad primes, $f = [E:\mathbb{Q}]/2$, and for every strictly dominant weight $u$ with a gap $g > 0$ satisfying* `WeightHyp`$(u,g)$ *and every cutoff $B \geq 128$ with $B+1 > \mathrm{bigB}_0(u,g)$, where $\mathrm{bigB}_0(u,g) = 4f(p+q)^2(1+\|u\|/g)$ with $\|u\|$ the norm of $u$ in the archive's weight space* (`bigB0`)*, the mean slope satisfies the bound* `MeanSlope`

$$\frac{1}{\#\mathrm{Comp}}\sum_{c}\widehat{h}_{B,c}(u) \geq -C\Big[\Big(\log(B\,\mathrm{rd}_E) + \Big|\frac{L'}{L}(1,\chi_{-d})\Big|\Big)\ell_1(u) + q\,\mathrm{spread}(u)\Big].$$

Here $\widehat{h}_{B,c}(u)$ is the archive's slope of the weight $u = (u_{\tau,i})$, $\tau$ in the CM type and $i < p+q$, on the component $c$ of the $B$-th model, for an abstract tower and intersection system, and $\ell_1(u) = \sum_{\tau,i}|u_{\tau,i}|$, $\mathrm{spread}(u) = \sum_{\tau}(\max_i u_{\tau,i} - \min_i u_{\tau,i})$. For $q = 1$, exact formulas for the metrized Hodge bundle are known: over an imaginary quadratic field of odd discriminant and for a self-dual lattice, by Bruinier and Howard in signature $(n-1,1)$ [16, Thm A] and by Howard for an anisotropic space of signature $(1,1)$ [43, Thm A]; and over a CM extension of a totally real field $F \neq \mathbb{Q}$, by Guo, in signature $(n,1)$ at one real place and definite at the others, with level maximal at the nonsplit places and under the lattice and local hypotheses stated there [34, Thm 1.1]. For $q \geq 2$ we know of no reference for a lower bound of this shape for mean heights in signature $(p,q)$. For $j = m$, the definite last case, `covering_low_defect_alt` supplies the low-defect estimate the bound needs, from the Hodge-bundle, Gram and anchor hypotheses.

`MeanSlope` `Development/AutomorphicSpectrum/Definitions/DivisibilityTowers/HeckeSectionsIsotropicCommutatorPairingPower.lean:91`
`covering_low_defect_alt` `TPS/CoveringLowDefectBound.lean:17`
`bigB0` `Development/AutomorphicSpectrum/Definitions/DivisibilityTowers/CompactGapsGE.lean:77`

## 5. A bound for Tate–Shafarevich groups of Tate twists

**5.1. The statement.** For each lift the chain bounds the loss of integrality at the twisted places in terms of a weighted sum of lengths of Tate–Shafarevich groups of Tate twists over the host fields. The comparison of divisibility with size needs a bound for that sum, and the host fields on windows come from the same bound (Section 3.8). For a number field $E$, a prime $\ell$ and an integer $e$, let $\mathrm{Sha}^1(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ be the group of classes in $H^1(\mathcal{O}_E[1/\ell], \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ that are trivial at every place above $\ell$. It is the archive's `Sha` (`Formalism/GaloisCohomology/Statement.lean:75`), taken in the continuous cohomology of the Galois group of the maximal extension unramified outside $\ell$, with the action through the $(-e)$-th power of the cyclotomic character. For $e \geq 1$, Frobenius at $v \nmid \ell$ acts by $Nv^{-e} \neq 1$ on these coefficients, so an unramified class there is locally trivial. For $e \geq 1$ and $E$ totally complex, as below, this is the usual Tate–Shafarevich group.

The archive states the following as `VWin` (with `VWinAt`, `VAt`, `VVle`).

`VWin` `CB/ExistsPageN.lean:49`
`VWinAt`, `VAt`, `VVle` `CB/ExistsPageN.lean:12–47`

**Theorem 5.1** (Alpöge)**.** *Let $\psi$ be primitive, $0 < \varepsilon < \frac{1}{52}$, and $k(x)$ integers with $x^{2+8\varepsilon} \leq 2^{k(x)} \leq 2x^{2+8\varepsilon}$ for $x \geq 1$. There are $J \geq 1$ and $X_0$ such that for every $X \geq X_0$ and every pair $(q, d)$ satisfying* `WinPair` *at $X^{1/J}$ (Section 3.4) there is $Y$ with $X^{1/J} \leq Y$ and $Y^2 \leq X$ such that, for every $x \in [Y, Y^2]$, the host field $E$ attached to $\psi$, $k(x)$, $q$ and $d$ has $\mathrm{Sha}^1(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ finite for every prime $\ell$ and every $1 \leq e < \lfloor x^{401} \rfloor$, and*

$$\sum_{\ell} \log \ell \sum_{1 \leq e < \lfloor x^{401} \rfloor} \mathrm{length}_{\mathbb{Z}_\ell} \mathrm{Sha}^1(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)) \leq \exp(x^{\varepsilon}/2).$$

Here $\exp(x^{\varepsilon}/2) = \sqrt{b}$. The bound enters through `supplyChi` (Section 3.8) and `HypV` (Section 4.5).

**5.2. The proof.** For odd $\ell$ the proof bounds the minus and plus parts of $\mathrm{Sha}^1$ under complex conjugation polynomially in $[E : \mathbb{Q}]$, the conductor and the number of twists, except for a term in the plus part from finitely many small primes. For that term and the part at $\ell = 2$ the proof has its bound for scales in windows. Alpöge proves `VWin` as `vwin_of_row`, whose only hypothesis is `RowOb`. The other inputs of the proof are theorems of the archive, among them Poitou–Tate duality, the global and local Euler characteristic formulas, local Tate duality and base change. For odd $\ell$ complex conjugation splits $\mathrm{Sha}^1$ into a minus part and a plus part, the plus part being the $(-1)^e$-eigenspace (`shaLen_eq_add`). The total is bounded by the part at $\ell = 2$ and the minus and plus parts (`vvle_of_parts`).

`vwin_of_row` `AV/ModularityArgument/ThresholdArguments/DenseThresholdCases/VwinOfRow.lean:4`
`shaLen_eq_add` `AV/LengthEqAddEigen.lean:88`
`vvle_of_parts` `AV/LengthEqAddEigen.lean:127`

*The minus part.* Let $E \subset \mathbb{Q}(\zeta_N)$ be a CM field, $\ell \nmid [E : \mathbb{Q}]$ an odd prime, $e \geq 1$, and $\eta$ a character of $\mathrm{Gal}(E/\mathbb{Q})$, a quotient of $(\mathbb{Z}/N)^\times$, with values in the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$ and with $\eta(-1) = (-1)^{e+1}$. The archive bounds the length of the $\eta$-part of the minus part by $\mathrm{ord}_\pi(N \prod_{p \leq k+1} p \cdot B_{k,\eta^*})$, $\pi$ a uniformizer of $\mathcal{O}$, the valuation of an integral representative of a generalized Bernoulli number of the primitive character $\eta^*$, with $k = e + 1$ (`PsiBound`). At the $\ell$ dividing $[E : \mathbb{Q}]$ it uses a cruder bound, by which $\log \ell$ times the length of the whole minus part is at most $C\, \ell\, [E : \mathbb{Q}]^2 (e+1) \log(e\ell N)^2$, for $\ell$ odd or $\mu_4 \subset E$ (`StickBound`). Summed, for the host fields, the minus part is polynomial in $[E : \mathbb{Q}]$, the conductor and the number of twists (`ThmBcFree`). Bounds of minus parts by

Bernoulli numbers are of the kind the Iwasawa main conjecture gives (compare [62, Ch. 1, §10, Thms 2 and 6]). The archive states them for these twists.

`PsiBound`, `StickBound` `AV/OddResidueBounds/ZeroMemUnrHom.lean:91–104`
`ThmBcFree` `AV/PrimeSetBoundDataOfPrimitiveChar.lean:64`

*The plus part.* The plus part is bounded through characters. For each character $\eta$ of $(\mathbb{Z}/N)^\times$ trivial on the subgroup $H$ that fixes $E$, with $\eta(-1) = (-1)^e$, the archive takes the length over $\mathcal{O}$, normalized by that of $\mathcal{O}/\ell\mathcal{O}$, of the group of classes in $H^1(\mathbb{Q}, (\operatorname{Frac}\mathcal{O}/\mathcal{O})(\eta^{-1}\chi_{\mathrm{cyc}}^{-e}))$ that are unramified everywhere and trivial at the primes dividing $\ell N$, and proves that the plus part is at most the sum of these over $\eta$ when $\ell \nmid [(\mathbb{Z}/N)^\times : H]$ (`redPrime`), and for every odd $\ell$ at most this sum with an extra term lossAll added to each character's length (`redAll`). The proof uses the first outside the finite set $B_\psi$ below and the second at every prime of $B_\psi$.

`redPrime` `AV/HeightAndVanishing/RedPrime.lean:11`
`redAll` `AV/HeightAndVanishing/RedAll.lean:6`

The triples $(\ell, e, \eta)$ fall into cases according to whether $\ell$ lies in a finite set $B_\psi$ of small primes attached to the host's character $\psi$ of Theorem 5.1, to whether $e = 1$ or $e \geq 2$ and whether $\ell - 1$ divides $2e$, and to tests on the order and components of $\eta$ (`Host.CaseI`–`CaseV`).

`Host.CaseI–CaseV` `.../DirichletPrimeToLPartMulLPowerPart.lean:208–219`
the counts `AV/HeightAndVanishing/Counts.lean:53–73`
Count IV `.../LayerLe.lean:16–20`

The main case is Count I, which the archive derives from Alpöge's `CorGK` by grouping characters by kernel (`U3.countI`). `CorGK` says that for every number field $K$ there are $c_5, A$ such that, for all $e \geq 0$, $N \geq 1$, every subgroup $H \subset (\mathbb{Z}/N)^\times$, two primes $p \neq p'$ not dividing $30N|\operatorname{disc} K|$ and every finite set of primes $\ell$ at which some character $\eta$ with kernel $H$ satisfies the conditions `GD`,

$$\sum_{\ell} \log \ell \sum_{\ker \eta = H} \mathrm{length}^* \leq 2c_5\, \varphi\big([(\mathbb{Z}/N)^\times : H]\big)\big(N(e+1)\big)^A (\log p + \log p').$$

The conditions `GD` include $K$ imaginary quadratic with $\ell \nmid |\operatorname{disc} K| h_K$, $\ell$ odd with $\ell \nmid \operatorname{ord}\eta$ and $\ell - 1 \nmid 2e$, $e \geq 1$ and $\eta(-1) = (-1)^e$. Here length$^*$ is the length over $\mathcal{O}$ divided by that of $\mathcal{O}/\ell\mathcal{O}$; the length is finite under `GD`. The constant $c_5$ is not explicit.

`CorGK` `AV/UnitaryThreeGroups/KatoUse.lean:44`
`U3.countI` `AV/UnitaryThreeGroups/CountI.lean:22`
`GD` `AV/HeightAndVanishing/Counts.lean:9`

On the chain `CorGK` comes from `propGK_of_numberU3U_of_oneCharU`, by which a nonzero class forces congruences between the traces of Frobenius of irreducible three-dimensional $\rho$ with $\rho \otimes \chi_{\mathrm{cyc}}^{-1}$ attached to a regular algebraic conjugate self-dual cuspidal representation of $\mathrm{GL}_3$ over $K$, or of $1 \oplus W$ restricted to $G_K$ with a newform of weight $2e+1$ attached to $W \otimes \chi_{\mathrm{cyc}}^{-e}$, and those of the reducible type $\Psi + 1 + \Psi^{-1}$, $\Psi = \eta\chi_{\mathrm{cyc}}^{e}$, and the sum of normalized lengths is at most twice the valuation of a nonzero integer attached to $p$ when $\ell \neq p$, and to $p'$ when $\ell = p$ (`PropGK`; the integer from `NumberU3U_s`). The inputs are the vanishing of rational Bloch–Kato Selmer groups of $W$ and $W \otimes \theta$ ($\theta$ the quadratic character of $K$) under attachment hypotheses (`KatoTwoW`), which the archive proves from its theorem `zetaUr` (applied by `twoS`; compare Kato's finiteness theorem [51, Thm 14.2(1)]), and rigidity and automorphy statements for $U(3)$, all proved in the archive. For odd $\ell$, Poitou–Tate duality identifies the dual of $\mathrm{Sha}^1(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ with a subgroup of $H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(e+1))$. Soulé bounds the product of the orders of these groups over $\ell \geq \max(4e+3, [E:\mathbb{Q}]+2)$ by a number doubly exponential in $e$ and in the discriminant

of $E$ [80, Thm 3, Prop. 4]. For abelian $E$, Huber and Kings prove that, up to sign and powers of 2, $\zeta_E^*(-e) = R_{e+1}(E)\prod_\ell \#H^2(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(e+1))/\#H^1(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(e+1))_{\mathrm{tors}}$ [44, Thm 1.4.1], with $\zeta_E^*(-e)$ the leading coefficient and $R_{e+1}(E)$ the Beilinson regulator; a bound for these orders from it needs a lower bound for the regulator. We know of no reference giving a bound for plus parts polynomial in the conductor and the twist. Bellaïche and Chenevier use congruences between non-tempered and tempered forms on $U(3)$ to construct Selmer classes [11, Thm 1.1]; here such congruences bound them.

`propGK_of_numberU3U_of_oneCharU` AV/ModularityArgument/ThresholdArguments/DenseThresholdCases/FPropGKU.lean:110
$1 \oplus W$ AV/ModularityArgument/AutomorphyLifting/AutomorphyDescent/AutNewformW.lean:14–20
`PropGK` AV/UnitaryThreeGroups/KatoUse.lean:36–42
`NumberU3U_s` .../DenseThresholdCases/NumberU3US.lean:30–45
`zetaUr` .../UnramifiedZeta/ZetaUr.lean:9
`twoS` .../SphericalDecomposition/SphAtTw.lean:16–19

The other cases are each bounded by $(\ell Ne)^C$ per character, for finitely many $\ell$: II and III by `countII_and_countIII_of_printsH_of_sp4In`, from a rank-four input (`SymplecticFourImageDense`) about a deformation problem of $\mathrm{GSp}_4$-valued lifts over $\mathbb{Q}$ of a residual sum of four characters (`Raw.IsPt`): finiteness of a coefficient algebra over $\mathcal{O}$, a proper span for one diagonal block, and automorphy of the continuous irreducible $\overline{\mathbb{Q}}_\ell$-points, each attached, over every imaginary quadratic field $K$ where it stays irreducible, to a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_K)$ (`IsAutQA`); V and Vs, the two branches of `CaseV`, from a unitary lift and from Lemma K; and $\eta = 1$ with $\ell \notin B_\psi$ and $\ell - 1 \mid 2e$ from `PropZ`. The archive proves the rank-four input (`sp4In0_alt`) from the same bundle of hypotheses as the counts (`vwin_of_hypotheses_H_E_alt`). Count IV, at the primes of $B_\psi$, is bounded by $B + (Nt)^C$ summed over $e$, given a bound $B$ on the chosen layer of the cyclotomic tower (`LayerLe`). Assembled, the plus part is at most $(\mathrm{cond}(E)\,t)^C + C\exp(x^{4\varepsilon/5})$ with $t = \lfloor x^{401} \rfloor$ (`HVAt`, `ThmHV`), which is polynomial except for the term from $B_\psi$.

`SymplecticFourImageDense` .../DenseThresholdCases/IsAut4.lean:14
`Raw.IsPt` AV/SymplecticGSp4/DecompSigmaOrd.lean:155
`IsAutQA` AV/ModularityArgument/AutomorphyLifting/IsAutCsdA.lean:16
`sp4In0_alt` .../DenseThresholdCases/Sp4In0.lean:18
`vwin_of_hypotheses_H_E_alt` .../DenseThresholdCases/WindowBoundOfLiteratureLevel.lean:25
`HVAt`, `ThmHV` AV/PrimeSetBoundDataOfPrimitiveChar.lean:75–86
`PropZ` AV/HeightAndVanishing/Counts.lean:49

*The prime 2 and the small primes.* `ThmBab`, proved from Lemma K, Ferrero–Washington and a bound along towers (`thmBab_of_lemmaK`), gives $Y \in [X^{1/J}, X^{1/2}]$ such that, for every $x \in [Y, Y^2]$, the part at $\ell = 2$ over $E$, and the parts at the small primes over one layer of the $\ell$-cyclotomic tower over each field isomorphic to $\mathbb{Q}(\zeta_{\ell N_\psi})$, $\ell \in B_\psi$, are at most $\exp(x^{4\varepsilon/5})$. Here $N_\psi$ is the conductor of $\psi$; `ThmBab` itself is stated for any finite set of odd primes $\ell$ and abelian fields containing $\mu_\ell$, and `ThmHV` takes the instance with $B_\psi$ and these fields (`Bchi`, `Lchi`). This is where the windows of `VWin` come from. Lemma K says that there is an absolute $C_K$ such that for every prime $\ell$, every totally complex $L$ containing $\mu_\ell$ ($\mu_4$ if $\ell = 2$) and every $e \geq 1$, $H^2(\mathcal{O}_L[1/\ell], \mathbb{Z}_\ell(e+1))$ is finite and $\log \#H^2 \leq (2|D_L|)^{(2e[L:\mathbb{Q}])^{C_K}}$. `lemmaK` derives it under its hypotheses, among them `BOdd`, a bound by $r_2(L)$ on the rank over $\mathbb{Z}$ of the span of the images under `zSeq` of the homology of $\mathrm{GL}_n(\mathcal{O}_L)$ in each odd degree $k+1 < r_2(L)$, for $n \geq 2k+6$, and it is applied with `bOdd`. Since $\mu_\ell \subset L$, Lemma K concerns

only $\ell \le [L:\mathbb{Q}]+1$, and Soulé's theorem [80, Thm 3] requires $\ell \ge [L:\mathbb{Q}]+2$. Lemma K is therefore not Soulé's bound.

`ThmBab` `AV/PrimeSetBoundDataOfPrimitiveChar.lean:49`
`Bchi` `AV/StmtFields.lean:66`
`Lchi` `AV/PrimeSetBoundDataOfPrimitiveChar.lean:71`
Lemma K `AV/GeneratorCountTuples/LemmaK.lean:13`
`thmBab_of_lemmaK` `AV/LayeredIsomorphisms/ThmBabOfLemmaK.lean:9`
`lemmaK` `AV/ModularityArgument/ClassCounting/LemmaKOfLitN.lean:28`
`BOdd` `.../ClassCounting/LemmaKOfSteps.lean:11`
applied with `bOdd` `.../DenseThresholdCases/KFinal.lean:11–12`

**5.3. The two weak inputs.** The proof of Theorem 5.1 rests on the Galois representations of input (i) of Theorem 2.3, attached to regular algebraic polarized cuspidal representations of $\mathrm{GL}_a$ over CM fields and compared with the local parameters of input (ii). Both inputs are weaker than the published theorems their names recall, and both are proved in the archive. Input (i), `RowOb`, takes a CM field $E$, a prime $\ell$, an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and a cuspidal automorphic representation $\pi$ of $\mathrm{GL}_a(\mathbb{A}_E)$ that is regular algebraic and polarized, and gives a continuous semisimple $r\colon G_E \to \mathrm{GL}_a(\overline{\mathbb{Q}}_\ell)$ and, for each embedding $\tau$, a multiset $H_\tau$ of integers with entries $\frac{a-1}{2} - z$, $z$ in the infinitesimal character, with $H_{\tau c} = \{k-n : n \in H_\tau\}$ for one integer $k$ and the complex conjugation $c$ of $E$, such that, with $\sigma = \pi_w \otimes |\det|^{(1-a)/2}$ and every map rc satisfying `IsRecT`: at $w \nmid \ell$, the Frobenius semisimplification of the Weil–Deligne representation of $r|_{G_w}$ equals $\mathrm{rc}(\sigma)$ when this has abelian image and $N = 0$, and shares with it $N = 0$ and the characteristic polynomials on inertia when $\mathrm{rc}(\sigma)$ is tame with $N = 0$, its eigenvalues of multiplicity at least two on a generator modulo wild inertia have order dividing $Nw - 1$, and $E_w \cong \mathbb{Q}_p$ for a prime $p$, with $a = 1$ or $p > 16a^2$; at $w \mid \ell$, given the archive's tied period data (`LocTied`) and a compatible local bridge, $r$ is crystalline when $\pi_w$ is spherical, and potentially semistable in the abelian and tame cases, with its Weil–Deligne representation compared with $\mathrm{rc}(\sigma)$ as at $w \nmid \ell$; in these three cases the multiset gives the Hodge–Tate jumps at the embeddings induced by the bridge. No de Rham assertion is made at the other $w \mid \ell$.

`RowOb` `DA/Foundations/GaloisRepresentations/TypeConditions/WeightsConjBis.lean:69`
$\sigma$ `DA/Foundations/GaloisRepresentations/TypeConditions/WeightsConjBis.lean:59`

For polarized $\pi$ the published theorem gives local–global compatibility up to Frobenius semisimplification at $w \nmid \ell$ [9, Thm 2.1.1(2)] and, when $\pi_w$ has an Iwahori-fixed vector, at $w \mid \ell$ [9, Thm 2.1.1(4)]. For cohomological conjugate self-dual $\pi$ of rank at least 2, Shin proves existence with a slight regularity hypothesis in even rank [77, Thm 1.2], which Chenevier and Harris remove [20, Thm 3.2.3]. For such $\pi$ Caraiani proves the same compatibility at $w \mid \ell$ [18, Thm 1.1] and at all $w \nmid \ell$ [17, Thm 1.1]. Taylor and Yoshida had proved the compatibility at $w \nmid \ell$ when $\pi$ is square integrable at a finite place [84, Thm 1.2]. `RowOb` is a weak form of these theorems. At a place $w \nmid \ell$ where $\pi_w$ is unramified it says as much as they do, since the laws of `IsRecT` make $\mathrm{rc}(\sigma)$ diagonal with the inverse Satake parameters of $\sigma$ (`SphLaw.sat`), so that $r$ is unramified at $w$ with these Frobenius eigenvalues (`AwayB`, field `tor`).

`SphLaw.sat` `AVT/Proofs/InertiaEigenvalues/IsSphericalIffUnrOfLawsT.lean:73`

The archive proves it as `galoisLocalGlobalStrong_recParameter` from two theorems, combined by `galoisRepresentations_of_thetaLift`. The first, `GeoOne`, gives for restricted data of odd rank auxiliary Galois representations $\rho$, continuous and semisimple, such that at almost all split $w \nmid \ell$ over primes $p$ with $-D$ a square in $\mathbb{Q}_p$, for an integer $D$ with $-D$ a

square in $E$, at which the data have Satake parameters, $\rho$ is unramified and $\rho(\mathrm{Frob}_w^{-1})$ is annihilated by the polynomial whose roots are the normalized Satake parameters (`VBase`, `ESAt`). In its proof they are twists of graded pieces of the Galois action on a common eigenspace of finite étale correspondences in the $\ell$-adic cohomology of a scheme (`RepEx`, `HTower`). The second, `TLiftK`, gives under its hypotheses, for an automorphic representation of the standard definite unitary group $U(n)$ and every $d$ beyond a threshold, one of $U(d)$ with the weights of a theta lift and, at each split place, local components linked by a nonzero map from the Weil representation.

`galoisLocalGlobalStrong_recParameter` `.../IntegralStructures/RowSrc.lean:6`
`galoisRepresentations_of_thetaLift` `.../GaloisRepresentationsOfThetaLift.lean:6`
`RepEx` `.../GeometricFlagBases/HodgeTrAt.lean:103`
`HTower` `AVT/Definitions/Divisors/IdTower.lean:9`
`TLiftK` `.../HalfIndexLattices/TLiftK.lean:12`
proved `.../ThetaKernels/TLiftK.lean:4`

Input (ii), `RecTExists`, asserts at every finite place a local Artin map and a map from irreducible representations of $\mathrm{GL}_n$ to Weil–Deligne representations that is compatible with local class field theory on characters, with the Langlands classification, with twists by real powers of the norm and with the absolute value of the central character. Compatibility with other smooth twists, central characters and duals, injectivity, and the passage from depth zero to tame parameters are asserted only at representations whose supercuspidal support consists of characters, or of twisted depth-zero cuspidals when the field is some $\mathbb{Q}_p$. It gives depth-zero cuspidals over $\mathbb{Q}_p$ their Green-parametrized inertia and zero monodromy, and it has no $L$- or $\varepsilon$-factors. The archive proves it as `recTExists` (Section 2) from local class field theory, Zelevinsky's classification [94, Thm 6.1] and the depth-zero parametrization over $\mathbb{Q}_p$.

`RecTExists` `DA/Foundations/LocalLanglands/IsRecTOfIsRec.lean:78`

The map it exhibits satisfies `IsRecT` and, for a finite place $w$ of a number field $E$, sends all supercuspidal representations of $\mathrm{GL}_n(E_w)$ outside `GoodCusp`, the class of the characters and, when $E_w$ is some $\mathbb{Q}_p$, of the twists by smooth characters of representations of depth zero, to real unramified twists of a single parameter $J_{w,n}$ (`cuspEx`). Each $J_{w,n}$ is irreducible of dimension $n \geq 2$ and, when $E_w$ is some $\mathbb{Q}_p$, not a twist of a tame parameter (`JunkEx`). When $\sigma = \pi_w \otimes |\det|^{(1-a)/2}$ is supercuspidal and outside `GoodCusp`, `RowOb` asks nothing of $r|_{G_w}$, whichever map satisfying `IsRecT` is taken: the laws make $\mathrm{rc}(\sigma)$ irreducible of dimension $a \geq 2$, which excludes the abelian case (`dim_one_of_dvd`), and when $E_w$ is some $\mathbb{Q}_p$ a supercuspidal representation with tame parameter has depth zero (`dz_of_cusp`) and so lies in `GoodCusp`, which excludes the tame case. The exhibited map is not the local Langlands correspondence [36, 41].

`GoodCusp` `DA/Foundations/LocalLanglands/IsRecTOfIsRec.lean:20`
`cuspEx` `DA/Foundations/LocalLanglands/LocalCentralUnits/CuspEx.lean:168`
$J_{w,n}$ `DA/Foundations/LocalLanglands/LocalCentralUnits/CuspEx.lean:176–177`
`JunkEx` `DA/Foundations/LocalLanglands/RecTTransport.lean:244`
`dim_one_of_dvd`, `dz_of_cusp` `DA/Foundations/GaloisRepresentations/IntegralStructures/InertCharpoly/CharPolOfInert.lean:39, 114`

A full form, with existence and uniqueness of a map satisfying all the laws, bijectivity, $\varepsilon$-factors of pairs and compatibility between fields, but no $L$-factors, is stated as `LocalLanglandsGL` and proved only conditionally, on `LitLoc` and `LitAut`. It is carried as a hypothesis, directly or derived conditionally from fields of a bundle `LitN` of about 65 published theorems, among them `llc`. Since `everyCharacter` has no hypotheses and the

archive has no unconditional proof of `LocalLanglandsGL`, the five theorems do not rest on it.

`LocalLanglandsGL` `AVT/PublishedTheorems/AutomorphicForms₂/LocalLanglandsGL.lean:96`
proved conditionally `AVT/Proofs/IwasawaDecomposition/LocalLanglandsOfLiterature.lean:21`
`LitN` `AVT/Definitions/AutomorphicMembership/Brauer.lean:133`
`llc` `AVT/Definitions/AutomorphicMembership/Brauer.lean:76`

**5.4. A specialization over $\mathbb{Q}$.** The estimate `CorGK` behind Count I of the plus part also gives, over $\mathbb{Q}$, for positive even $e$ and primes $\ell \geq 5$ with $\ell - 1 \nmid 2e$, the vanishing of the $\omega^{-e}$-eigenspace of the class group of $\mathbb{Q}(\zeta_\ell)$ modulo $\ell$, with $\omega$ the Teichmüller character of $\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$, once $\log \ell > C(e+1)^A$, for constants $C$ and $A$. The specialization is off the chain. Specialized to $K = \mathbb{Q}(\sqrt{-3})$, $N = 2$ and $\eta = 1$, the estimate `CorGK` becomes Alpöge's `UpperTheory.V.SmallClassNumbers.bound`, which says that there are constants $C, A$ such that for every positive even $e$ and every finite set of primes $\ell \geq 5$ with $\ell - 1 \nmid 2e$,

$$\sum_{\ell} \log \ell \cdot \mathrm{length}\, S_\ell(e) \leq C(e+1)^A$$

(for coefficients $\mathbb{Z}_\ell$; the statement allows finite extensions, with normalized lengths), where $S_\ell(e) \subset H^1(\mathbb{Q}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ is the group of classes unramified everywhere and trivial at 2 and $\ell$.

`UpperTheory.V.SmallClassNumbers.bound` `AV/SmallClassNumbers/CorGKOf.lean:27`
statement `AV/SmallClassNumbers/ClassNumberThree.lean:16`

The statement `Large`, which follows from it by `large_of`, says that $S_\ell(e) = 0$ when $\log \ell > C(e+1)^A$. This is the vanishing of the $\omega^{-e}$-eigenspace of the class group of $\mathbb{Q}(\zeta_\ell)$ modulo $\ell$. With $n = e + 1$, Soulé proved such vanishing for $\log \ell > n^{224n^4}$ and odd $n \geq 5$ [79], and Kurihara for $n = 3$ with no condition on $\ell$ [56]. The statement `Bound` also asserts that $S_\ell(e)$ is finite for positive even $e$, $\ell \geq 5$ and $\ell - 1 \nmid 2e$. Since $C$ is not explicit, the vanishing test $\log \ell > C(e+1)^A$ is not effective at a given prime.

`Large` `AV/SmallClassNumbers/LenEqZeroIff.lean:65`
`large_of` `AV/SmallClassNumbers/LenEqZeroIff.lean:70`
`Bound` `AV/SmallClassNumbers/ClassNumberThree.lean:16`
$S_\ell(e)$ finite `AV/SmallClassNumbers/ClassNumberThree.lean:19`

## 6. Classical inputs and their archive forms

Table 2 sets the classical inputs on the chain beside their archive forms, all proved in the archive, with two further statements that the archive does not prove unconditionally. Most entries open with one word for how the two compare: "Full" for the published statement, "Special" for a special case, "Weak" for a weaker statement and "Different" for a statement of another shape. Green's parametrization (Section 2), the slope inequality (Section 4.1), `LitThin` and the mass formulas (Section 4.5) are also proved in the archive. The statements of resolution, of principalization and of the Riemann hypothesis over finite fields are on Mathlib's schemes and their morphisms, but blowups, defined by their universal property, simple normal crossings and étale cohomology are the archive's own definitions.

TABLE 2. Classical inputs on the chain, and two statements that the archive does not prove unconditionally.

| Theorem | In the archive | Declaration |
|---|---|---|
| Local Langlands for $\mathrm{GL}_n$ [36, 41] | Weak (Section 5.3): a map with the rank-one and segment laws and Green-parametrized depth-zero inertia and zero monodromy over $\mathbb{Q}_p$; the laws hold for a map sending the supercuspidals of $\mathrm{GL}_n(E_w)$, $w$ finite, outside `GoodCusp` to real unramified twists of a single parameter $J_{w,n}$; no bijectivity, no $L$- or $\varepsilon$-factors | `recTExists`, `DA/Foundations/LocalLanglands/LocalCentralUnits/RecTExists.lean:20` |
| Galois representations over CM fields [9, 77, 20, 17, 18] | Weak (Section 5.3), against $\mathrm{rc}(\sigma)$, $\sigma = \pi_w \otimes \lvert\det\rvert^{(1-a)/2}$: at $w \nmid \ell$, equality when $\mathrm{rc}(\sigma)$ is abelian with $N = 0$ (so at every unramified $\pi_w$), and $N = 0$ and the characteristic polynomials on inertia when it is tame with $N = 0$, its eigenvalues of multiplicity at least two on a generator modulo wild inertia have order dividing $Nw - 1$, and $E_w \cong \mathbb{Q}_p$ for a prime $p$, with $a = 1$ or $p > 16a^2$; at $w \mid \ell$, given tied period data and a compatible bridge, crystalline when $\pi_w$ is spherical, the comparison in the abelian and tame cases, and in all three cases the prescribed Hodge–Tate jumps at every embedding induced by the bridge | `galoisLocalGlobalStrong_recParameter`, `DA/Foundations/GaloisRepresentations/IntegralStructures/RowSrc.lean:6` |
| Resolution of singularities in characteristic 0, with principalization [42] | Resolution without the assertion that the centers are smooth; principalization with smooth centers | `resolutionCharZero`, `WB/CoreStatements/ResolutionCharZero/ResolutionCharZero.lean:6`; `principalisationCharZero_closed`, same folder, `PrincipalisationCharZero Closed.lean:7` |
| Arithmetic Hilbert–Samuel [32, Thm 8], [95, Thm (1.4)] | Different: `HSUse` gives upper bounds and the arithmetic Kleiman inequality; the asymptotic for nef bundles is proved separately, as `ArithHilbertSamuel`, the field `q9` of the Hodge-bundle inputs (`TPS/EveryCharacterOfLiteratureOfVWin.lean:803`) | `hsUse_outright`, `WB/CoreStatements/AutomorphicKernels/MoriwakiBigness/MoriwakiBigTheorem/Outright.lean:13`; `ArithHilbertSamuel`, `AVT/PublishedTheorems/Divisors$_2$/SmallSecTensTriv.lean:14` |
| Arithmetic bigness [66, Thms A, B], [92, Thm 2.2] | For $X$ in a product of projective spaces with the restricted metric, under the stated semipositivity and nefness assumptions: the arithmetic volume is at least the top self-intersection | `moriwakiBig_outright`, `WB/CoreStatements/AutomorphicKernels/MoriwakiBigness/MoriwakiBigTheorem/Outright.lean:9` |
| Siegel–Weil for unitary dual pairs [89, 45, 91] | Special: $V$ definite, with self-dual lattices and a $K$-finite Schwartz function; $E/F$ unramified at all finite places and Galois over $\mathbb{Q}$, its inert places not above 2 and of norm $> 3$, the additive character unramified there | `siegelWeilStd`, `WB/CoreStatements/AutomorphicKernels/SiegelWeilFormula/SiegelWeilStandard/SiegelWeilStd.lean:8` |
| Continuation of Siegel Eisenstein series on $U(n,n)$ [59] | Weak: meromorphic continuation after pole-clearing, against the specified functionals, with the moderate-growth bound; no functional equation; over the base of the Siegel–Weil row | `eisBL`, `WB/RepresentationTheoryGL/EisensteinSeries/SiegelEisensteinHalfPlane/SchwartzBruhatMajorants/SiegelMultOf.lean:101` |
| Kronecker's limit formula, summed over the classes [26, Thm 1], [74, §12, pp. 109–110] | Special: for $\mathbb{Q}(\sqrt{-d})$ with $d$ the host prime, $d \equiv 7 \bmod 16$, in terms of ideal lattices and $\frac{L'}{L}(1, \chi_{-d})$ (Section 4.1); no evaluation by values of $\Gamma$ | `classMean_log_norm_disc_chowlaSelberg`, `AVT/Proofs/DivisorProofs/ClassMeanLogNorm DiscChowlaSelbergStatement.lean:9` |

| Theorem | In the archive | Declaration |
|---|---|---|
| Discrete series [35] and lowest $K$-types [38] | Special: $U(p,q)$ only; the lowest $K$-type with multiplicity one, and a bound on the other $K$-types that allows all positive roots, not only the noncompact ones | `discreteSeriesClassification`, `CompanionTheory/RealGroups/DiscreteSeries/DiscreteSeriesClassification/DiscreteSeriesExhaustion/DsExhaust.lean:17` |
| Ferrero–Washington [28] | Bounded $\ell$-rank up the cyclotomic tower, for abelian $K$ containing $\mu_\ell$ ($\mu_4$ if $\ell = 2$) | `FerreroWashington`, `AV/PublishedTheorems/FerreroWashington.lean:7`, proved as `ferreroWashington`, `AV/GaloisLibrary/FerreroWashingtonIndex/FerreroWashington.lean:104` |
| Kato's Euler system [51, §§8–9] and Selmer finiteness [51, Thm 14.2(1)] | Different: the cocycles, and the vanishing of rational Selmer groups of $W$ and of a twist $W \otimes \theta$ by the quadratic character of an imaginary quadratic field, under `Used`, a newform attachment and `WitAQ` | `zetaUr`, `AV/ModularityArgument/KatoQuadraticCharacters/UnramifiedZeta/ZetaUr.lean:9`; `KatoTwoW`, `.../SphericalDecomposition/KatoTwoW.lean:10`; `katoTwoW_of_printsH`, `AV/ModularityArgument/ThresholdArguments/DenseThresholdCases/KatoTwoWOfPublishedH.lean:48` |
| Poitou–Tate [67, (8.6.10)] | Special: finite modules killed by $\ell^r$, at the allowed places, $\ell$ odd or the field totally complex | `poitouTateNormalised_holds`, `WB/CoreStatements/PublishedTheorems/PoitouTateDuality/PoitouTateNineTerm/PoitouTateNormalisedHolds.lean:18` |
| Local Tate duality and local Euler characteristic [67, (7.2.6), (7.3.1)] | Special: perfect cup-product pairings with an isomorphism $H^2(\mu_{\ell^r}) \cong \mathbb{Z}/\ell^r$ whose normalization is not specified, and the Euler characteristic formula, for finite modules killed by $\ell^r$ at the finite places | `localTateDuality`, `AVT/Proofs/DivisorProofs/ExistsMulEquiv.lean:115`; `loceuler_holds`, `AVT/Proofs/DivisorProofs/LoceulerHolds.lean:8` |
| Artin reciprocity [5] | Full | `globalArtinReciprocity_holds`, `AVT/Proofs/IwasawaDecomposition/GlobalArtinReciprocityHolds.lean:5` |
| Chebotarev's density theorem [87] | Full, in Dirichlet density | `hasDirichletDensity_chebotarevSet`, `GroundTheory/GlobalFields/TateThesis/Chebotarev.lean:198` |
| The Riemann hypothesis over finite fields [88], [24, Thm (1.6)] | Stated, and no other declaration names them: Deligne's purity bound for smooth projective varieties of every dimension (Weil's for curves) | `WeilOneD`, `WeilOneP`, `DA/CommutativeAlgebra/EtaleCohomology/WeilOneD.lean:17,21` |
| Local Langlands, the full correspondence | Stated; proved conditionally on `LitLoc` and `LitAut` (`localLanglandsGL_and_prN_of_lit`), not unconditionally; the main route uses `RecTExists` | `LocalLanglandsGL`, `AVT/PublishedTheorems/AutomorphicForms₂/LocalLanglandsGL.lean:96`; `AVT/Proofs/IwasawaDecomposition/LocalLanglandsOfLiterature.lean:21` |

At the call site, the covolume inputs of Section 4.6 are: (1) resolution and (2) principalization in characteristic 0; (3) an integrated Poincaré–Lelong formula; (4) arithmetic Hilbert–Samuel bounds with arithmetic Kleiman and (5) arithmetic bigness, as in Table 2; (6) under its hypotheses, among them mixed characteristic, for a $p$-divisible group, $\ker(\mathrm{HT}) = (\ker \mathrm{dlog})^\perp$ under the Weil pairing (middle exactness of the rational Hodge–Tate sequence is [73, Prop. 4.3.6]), which the archive derives from its form of prismatic

Dieudonné theory over $\mathcal{O}_C$, a functor from $p$-divisible groups to finite free $A_{\mathrm{inf}}$-modules with Frobenius and Hodge filtration (`PrismaticDieudonneHodge`, by `farguesExact_of_hodge`); (7) Zarhin's trick [93, Lemma 2.5], in the form that every polarized abelian scheme over such a valuation ring is a retract of a principally polarized one; (8) asymptotic Riemann–Roch for nef weights; (9) toroidal compactification of Siegel modular varieties over $\mathbb{Z}[\zeta_n, 1/n]$ for odd $n \geq 3$ [27] and (10) of unitary ones on the generic fiber; (11) seminegativity of the Lie algebra of a semi-abelian scheme over a smooth projective geometrically connected curve in characteristic 0; (12) under positivity and klt hypotheses, a section vanishing at exactly one of two given points (compare [54, Thm 5.9]; simultaneous surjectivity onto both fibers, for smooth varieties, is [3, Thm 0.3, Cor. 0.4]); (13) birational invariance of arithmetic degrees, with the Chern–Weil degree formula; (14) 28 further inputs to the Hodge-bundle argument, among them Kawamata–Viehweg vanishing for a nef and big line bundle on a smooth projective variety over an algebraically closed field of characteristic 0 (`KawamataViehweg`, field `q6`), which the archive proves from Nadel vanishing for pairs over $\mathbb{C}$ (`NadelVanishingPair`; `kawamata_viehweg_of_literature`); and (15) deformation of abelian schemes and their homomorphisms over nilpotent divided-power thickenings, given a reference lift [64, 63].

the call site `TPS/EveryCharacterOfLiteratureOfVWin.lean:812`
`PrismaticDieudonneHodge` `AVT/PublishedTheorems/Divisors₂/InjectiveOfSmulMemRange.lean:62`
`farguesExact_of_hodge` `WB/CoreStatements/TemperedRepresentationTheory/FarguesExactness/FarguesExactOfPrismatic.lean:56`
`kawamata_viehweg_of_literature` `TPS/EveryCharacterOfLiteratureOfVWin.lean:447`

Other published theorems enter the route under their usual names. The archive proves the main theorem of complex multiplication for elliptic curves with multiplication by the maximal order of an imaginary quadratic field, in idelic form [76] (`main_theorem_complexMultiplication`), which serves the first bundle of Section 2.4. Base change is used in several forms, among them Arthur–Clozel's cyclic base change of prime degree over totally complex fields, for cuspidal representations of $\mathrm{GL}_a$ not isomorphic to their twists by the nontrivial characters of the extension [4] (`ArthurClozelStrongTC`), in the proof of input (i) (Section 5.3), and weak base change from definite unitary groups to $\mathrm{GL}_n$, for a CM field unramified over its real subfield [57] (`DefiniteN`), in the first bundle. Both theorems of Jacquet and Shalika are used over totally complex fields: strong multiplicity one for isobaric sums [46] (`JacquetShalikaRigidity`) and the bound $Nw^{-1/2} < |\alpha| < Nw^{1/2}$ on Satake parameters of unitary cuspidal representations [47] (`JacquetShalikaBound`). They serve the proof of input (i) and the first bundle. Clozel's purity lemma [21] appears as `ClozelPurity`, in the first bundle: for an algebraic cuspidal representation of $\mathrm{GL}_a$ over a totally complex field, the infinitesimal characters at conjugate embeddings are related by $z \mapsto w_0 - z$. Besides Zelevinsky's classification, the route uses the Bernstein–Zelevinsky finiteness theorems for $\mathrm{GL}_n$ of a non-archimedean local field [12, 13] (`BernsteinZelevinskyFiniteness`), in both bundles. Deligne–Ribet [25] appears as an integrality statement (`DeligneRibet`), in the first bundle: for totally real $F$ and an $\ell$-integral rational-valued function $\varepsilon$ on the ray classes modulo $\ell^n$, $n \geq 1$, for $\mathfrak{c}$ prime to $\ell$ and $k \geq 1$, $L(\varepsilon, 1-k) - N\mathfrak{c}^k L(\varepsilon_{\mathfrak{c}}, 1-k)$ is $\ell$-integral, where $\varepsilon_{\mathfrak{c}}(I) = \varepsilon(\mathfrak{c}I)$.

`main_theorem_complexMultiplication` `GroundTheory/GlobalFields/ComplexMultiplication/MainTheorem.lean:66`
`ArthurClozelStrongTC` `AVT/PublishedTheorems/GaloisGeometry₂/ArthurClozelStrongTC.lean:13`
`DefiniteN` `DA/Foundations/BaseChangeIdentities/Narrow.lean:28`
`JacquetShalikaRigidity` `AVT/PublishedTheorems/GaloisCohomology₂/StrongMultiplicityOne.lean:189`
`JacquetShalikaBound` `AVT/PublishedTheorems/GaloisCohomology₂/StrongMultiplicityOne.lean:197`

`ClozelPurity` `AVT/PublishedTheorems/GaloisGeometry₂/ClozelPurity.lean:10`
`BernsteinZelevinskyFiniteness` `AVT/PublishedTheorems/TwistedTraceFormula/HasFiniteLengthOfIsIrreducible.lean:49`
`DeligneRibet` `AVT/PublishedTheorems/Divisors₂/SiegelKlingenDeligneRibet.lean:45`

## 7. Alpöge's twenty steps against the chain

Alpöge's note lists twenty steps, dated from 6 October to 16 November 2026, and says that some of their descriptions paraphrase summaries and may be revised [2, pp. 3–4 and n. 4]. Table 3 sets each step, in our words and with the note's numbering, against the chain to the five theorems. Some entries open with "In part", "By role" or "Not found". "In part" means that the counterpart named matches part of the step, "By role" that a declaration plays the step's part without stating what the step describes, and "Not found" that we found no counterpart to the step on the chain.

TABLE 3. The note's steps [2, pp. 3–4] and the chain.

| Step | The note | On the chain | § |
|---|---|---|---|
| 1 | A Stepanov determinant | Not found. | – |
| 2 | The invariant derivation $D$ on a CM curve at an inert $p$; $p$-divisibility of $(D^a\vartheta)(0)/\Omega^a$ | `IntInert` and `kernel_bounds_generic`. The first is integrality of the corrected theta kernels at inert $p \geq 5$ after substituting the formal logarithm, with `CM2` from Artin reciprocity. The second is a bound for the derivatives at 0 of every power series of that form, with exponent a sum of terms $\max(a_s/p^e - c, 0)$, $e$ odd, some omitted, less a loss. Together they bound the derivatives of the kernel series of the datum at the inert places of residue characteristic at most $X''$, under conditions (`EntriesAtInert`) and make the actual periods divisible by the lattice ideals. The lower bound for the gains of the lifts together is `LinkG`, proved without the theta kernels. | 4.1, 4.4 |
| 3 | An identity $qN = M^k + R$ with a path quadric | In part: the unweighted count identity occurs by specialization of `count_mul_card` (`CompanionTheory/SpectralGeometry/LocalGlobalProofs/AdelicAnalysis/FiniteFieldQuadraticFormPointCount.lean:84`); the full weighted match was not found. | – |
| 4 | A positivity $W(1_S) \geq 0$ from a determinant | Not found. The sign on the chain is that of `ch_top`, by which divisibility minus size is at most an error, with ramp test functions. | 3.5, 4.1 |
| 5 | The weighted sum $\sum_{n \leq X} \Lambda(n)(X/n - 1) - X \log X + (1+\gamma)X$; "Guinand-Weil had the sign" | In part: `Gz` is exactly this function. The "variance test function trick" and the integral over twists were not found. | 3.1 |
| 6 | $W\left(1_{(0,1]} + (\sim X^\eta) \cdot 1_{(0,1/X]}\right) \geq 0$ "gives $> 1 - \eta$" (in our reading, no zeros with $\operatorname{Re} s > 1 - \eta$) | By role: Landau's theorem on windows, with exponent $\frac{1}{2} + C\varepsilon$ and $\varepsilon$ taken small for each $\operatorname{Re} s > \frac{1}{2}$. The positivity heuristic is the note's. | 3.3 |
| 7 | Determinants containing Gan–Gross–Prasad periods; interlacing weights | In part: `comult_unitary` (`AVT/Proofs/FoundationProofs/BranchesAlongOfIrreducible.lean:123`), the space of intertwiners for $U(n) \subset U(n+1)$ has dimension one for interlacing weights and zero otherwise. | – |

| Step | The note | On the chain | § |
|---|---|---|---|
| 8 | Iterated theta lifts between definite unitary groups, multiplicity $m$, with $m$ like $\sqrt{X}$ | The column chains of a datum (`D7`): $m$ kernel maps, each raising the definite rank by $t+1$; `RankVal` fixes $x^{1/2-\varepsilon/16} \le m \le 2x^{1/2-\varepsilon/16}$. | 4.2 |
| 9 | Norms of the lifts, cancellation of the terms of order $m$, divisibility; Chowla–Selberg, Weyl, Siegel's mass | Kronecker's limit formula averaged over the class group, through $\log\Omega$; ratios of Siegel masses (`SiegelMassRatioCM`). The size estimates bound the left side of $(\star)$ above, and Proposition 3.3 expresses each term ASK − MAIN by centered ramp sums, which we read as the note's cancellation. | 4.1, 4.5, 4.6 |
| 10 | A lower bound for the divisibility of the $m$-fold lift | `LinkG` bounds the gains of the lifts together from below and is proved without the theta kernels; `IntInert` and the bound for a single kernel series (Section 4.4) make the actual periods divisible by the lattice ideals (`LinkDIVt`), and `LinkMu` compares slopes; at the level primes, the denominator bound `aP10`, of exponent $\lceil e_w m E_T \rceil$, with $E_T$ growing with $m$ at the seen places and 0 at the others. | 4.1, 4.5 |
| 11 | The Iwasawa tower, by analogy with a pigeonhole on $\mathbb{F}_{q^n}/\mathbb{F}_q$ | The host fields, composita with $\mathbb{Q}(\zeta_{2^{e+2}})^+$; the average over the subgroup $A_0$. | 3.4, 3.7 |
| 12 | Seesaw: the period of the iterated lift is a Bessel period, independent of the Hecke characters; Rallis | In part: an identity $\zeta_{\omega,t}\,\mathrm{per}_{\omega,t}(\varphi,F) = a_\omega \langle V\Theta_t F, P_\omega(U_\omega\varphi)\rangle$ between the periods and inner products with Poincaré series, with $a_\omega \neq 0$ and $\zeta_{\omega,\mu_0} = 1$ (`MIstar2`), proved by passing linear relations through Bessel periods of the row forms (`JointCondition`); `BesselId`, the comparison of this shape for $\mathrm{Bu}_\omega$; in the adelic model, the Bessel period of the iterated lift as an iterated integral of column periods (`iter_colPer`); `SeeLast`; theta integrals compared through Siegel–Weil (`ThetaIntegralsAgree`). The condition `LocalQuadraticParts.adm1`, which states the independence in terms of $\mathrm{Bu}_\omega$, and the tie `TieBm` are not on the route (Section 4.3). | 4.3 |
| 13 | Bost's slope inequality | `slope_inequality_general`; in the covolume bound, the separate statement `DescentSlope`. | 4.1, 4.6 |
| 14 | The cost of each lift; stopping at $\frac{2}{3}$ a priori; the need for division | By role: the point-sum bound `StepThmBG`, with factor $b^{C_B m^5 f}$. The $\frac{2}{3}$ and the transfer to a function field were not found. | 4.5 |
| 15 | An integral structure across Hecke characters, for extra averaging | The integral structure: `RR_spec` (`FLG/IndefiniteForms/IndefUse.lean:219`) specifies every localization of the lattice $RR$, which glues the row lattices selected by the Hecke-character tuples $\mu(w)$. The weighted sum over the characters generated by $\psi$ (`Gm`, `TGC/ExplicitFormulaErrorOfHostConditions.lean:38`), with weight 1 at the trivial character and $1/(3o+1)$ at the others, $o$ the order of $\psi$, is separate. | 4.1, 3.7 |
| 16 | Relative trace formula; projected Poincaré series; approximate orthogonality of the columns | In part: Theorem S, through the double-coset expansion of the Gram operator of projected Poincaré series (`kExpTiedS_holds`, `FLG/SiegelLatticeTheory/FlagStabilizerTensors/TheoremSConclusion.lean:84`), for unitary groups and Siegel lattices; a sixfold cover of $\mathrm{GL}_5$ realizing the note's "automorphy of $\mu$" was not found. | 4.5 |
| 17 | Classes in $H^1(\mathcal{O}_E[1/\ell], \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ not killed by twisting; a weakening of Vandiver | `VWin`, proved as `vwin_of_row`, bounds weighted lengths of localization kernels on selected windows; `CorGK` contributes on the plus side. | 5 |

| Step | The note | On the chain | § |
|---|---|---|---|
| 18 | Covolumes, the index of the theta series, congruences between lifts and forms | `CovolumeBoundOfX`, an upper bound for the normalized log covolume and so a lower bound for the slope; `ON` (`FLG/IndefiniteForms/IndefSide.lean:318`) bounds the signed indices of the theta images, summed over the lifts and the conjugate places. A Hecke congruence module also occurs. | 4.6 |
| 19 | The choice of tower, using Behrend | `WinPair` constrains the primes; inside the given frame, `hostThickExists` selects the thick set $T_2$, built on Mathlib's `Behrend.sphere` and `Behrend.map`. | 3.4, 3.8 |
| 20 | GRH | `everyCharacter`, `imprimitive`, `rh_of_every`. | 2 |

## 8. Other results in the archive

Several results of the archive are of interest on their own, among them the following.

- Ferrero–Washington in Iwasawa's form, as in Table 2, and the class-number specialization of `CorGK` (Section 5.4).
- Split primes in short intervals. `normPrimesImagQuad_short_interval_lower_bound` says that there are $A, c, c' > 0$ such that, for every prime $d \equiv 3 \pmod 4$ with $d \geq 100$, every $M \geq 3$, and all $X \geq 1$ and $h$ with $\log(4dM^2) \leq c\sqrt{\log X}$ and $e^{-c\sqrt{\log X}} \leq h \leq 1$, the number of primes $p \in (X, X(1+h)]$ with $4p = a^2 + db^2$, $M \mid b$ and $2M \mid a - 2 - b$ is at least $c'hX/\big((4dM^2)^A \log X\big)$. These are the primes $p = N\alpha$ with $\alpha \in \mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-d})$, $\alpha \equiv 1 \pmod{M\mathcal{O}_K}$. Thorner and Zaman prove a lower bound of this shape for the prime ideals of degree one in a ray class of any number field [86, Thm 3.1]. The archive's bound is its analogue for short intervals, in the narrower range $\log(4dM^2) \leq c\sqrt{\log X}$. The statement gives no values for the constants. The theorem is used on the chain (`shortNorm`).

  `normPrimesImagQuad_short_interval_lower_bound` `AVT/Proofs/GlobalInterpolation/NormPrimesImagQuadShortIntervalLowerBoundStatement.lean:8`
  `shortNorm` `DA/ShortNorms/Primes.lean:5`

- Bernoulli units for 2-power twists. One file states two results. The first (proved as `lemmaX`) says that $B_{1,\varphi\xi}$ is an $\ell$-adic unit when $\ell$ is prime, $r \geq 1$ satisfies $4 \mid \ell^r - 1$, $\varphi$ is a primitive character of odd conductor $f$ and order dividing $\ell^r - 1$, and $\xi$ is a primitive character of conductor $2^n \geq f \cdot 4^{v_2(\ell^r - 1)}$ with $\varphi\xi$ odd. The second (proved as `residual_bernoulli_nonvanishing`) is a vanishing criterion. For odd $\ell$ and a character $\chi \neq 1$ modulo $N$, not necessarily primitive, of order prime to $\ell$, whose reduction $\bar{\chi}$ differs from the reduction of the cyclotomic character, if the Bernoulli number at level $N$, $\sum_{a=1}^{N} \chi^{-1}(a) B_1(a/N)$, is an $\ell$-adic unit, then every cocycle of $G_{\mathbb{Q}}$ with values in $\bar{\chi}$ that is unramified everywhere, in the form `UnrEv`, is a coboundary. `corX` combines the two, and is used on the chain, in Counts II and III of Section 5 and in Count V.

  two results `AV/LevelCohomology/UnramifiedCocycleIsCoboundaryOfBernoulliUnit.lean`
  the first `AV/LevelCohomology/UnramifiedCocycleIsCoboundaryOfBernoulliUnit.lean:24`
  `lemmaX` `AV/LevelCohomology/LemmaX.lean:78`
  the second `AV/LevelCohomology/UnramifiedCocycleIsCoboundaryOfBernoulliUnit.lean:43`
  `residual_bernoulli_nonvanishing` `AV/LevelCohomology/ResidualBernoulliNonvanishing.lean:174`
  `UnrEv` `AV/LevelCohomology/UnramifiedCocycleIsCoboundaryOfBernoulliUnit.lean:37`
  `corX` `AV/LevelCohomology/ChrInvNeg.lean:197`

Counts II and III `AV/ModularityArgument/ThresholdArguments/DenseThresholdCases/AdjointLiftingOfCohomology.lean:33–37`
Count V `.../DenseThresholdCases/LinkedUnlE.lean:61`

- Dedekind and Hecke $L$-functions over any number field $K$. The Dedekind zeta function continues meromorphically to $\mathbb{C}$, holomorphic away from $s = 1$, where it has a simple pole with the residue of the class number formula. Its completion satisfies $\Lambda_K(1-s) = \Lambda_K(s)$ for $s \neq 0, 1$, and $\zeta_K(s)/s^{r_1+r_2-1} \to -hR/w$ as $s \to 0$. For a unitary Hecke character not of the form $|\cdot|^{i\tau}$, the $L$-function continues to an entire function and does not vanish on $\operatorname{Re} s \geq 1$, and its completion satisfies $\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\chi^{-1})$. These are classical [39, 40, 83]. Chebotarev's theorem is in Table 2.

  Dedekind and Hecke `GroundTheory/GlobalFields/TateThesis`
  continues meromorphically `.../TateThesis/DedekindZetaContinuation.lean:47–93`
  $\Lambda_K(1-s) = \Lambda_K(s)$ `.../TateThesis/DedekindContinuation.lean:181`
  $\zeta_K(s)/s^{r_1+r_2-1}$ `GroundTheory/GlobalFields/AnalyticClassNumber/ClassNumberFormula.lean:40`
  entire function `.../TateThesis/HeckeTheorem.lean:47`
  does not vanish `.../TateThesis/Nonvanishing.lean:193`
  $\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\chi^{-1})$ `.../TateThesis/HeckeTheoremFE.lean:86`

- Quadratic fields. For quadratic $K$ the archive proves $\zeta_K(s) = \zeta(s)L(s,\chi_K)$ for $s \neq 1$ (`continuedDedekindZeta_eq_mul`, `GroundTheory/GlobalFields/AnalyticClassNumber/Quadratic.lean:165`). We deduce by Theorem 1.1 that the zeros of $\zeta_K$ in $0 < \operatorname{Re} s < 1$ lie on the critical line. Beyond $\mathbb{Q}$ the archive states no Riemann hypothesis for a Dedekind zeta function, and the factorization after continuation only for quadratic fields. Alpöge expects the extended hypothesis to follow with more time [2, p. 1].
- A frame for the Weil conjectures. The archive states, without proof, a conditional goal for the Frobenius weights, `EWeil.Goal`, under about 35 hypotheses `Base` on an étale cohomology theory, and a separate affine Fermat estimate, `EWeil.FermatCount` (`LateralTheory.EWeil`, `Development/EtaleCohomology/WeilConjectures/FermatCount.lean:75,13,170`). Neither is on the route. The two theorems named `weil_bound` are bounds for theta and Haar integrals, and the archive does not prove the Weil bound (Table 2).

The bundles `GeneralHyps`, `GeneralHypU` and `GeneralHypV` of published statements are declared in the archive, and the five theorems do not refer to them.

`GeneralHyps` `AVT/Definitions/GlobalTwisting/GeneralHyps.lean:10`
`GeneralHypU` `AVT/Definitions/GlobalTwisting/FinitelyManyWithInfinitesimalCharacter.lean:67`
`GeneralHypV` `AVT/Definitions/GlobalTwisting/SiegelWeilUnitaryConvergent2.lean:6`

## Index of names

TABLE 4. Names used in Sections 3, 4 and 5, what each says, and the section that defines it.

| Name | What it says | § |
|---|---|---|
| `WinPair` | Primes $q \equiv 3 \bmod 8$, $d \equiv 7 \bmod 16$, $\log q \asymp Y^{\varepsilon/5}$, $\log d \asymp Y^{\varepsilon/5}\log^2 Y$; $q$ and every odd prime $p \leq Y^{\varepsilon/5}$ divide $d+1$ | 3.4 |
| `EAtH` | Shares of $TT$ in $A_0$-cosets; $\lvert G(\theta,x)\rvert \leq x^{1/2+\varepsilon}$, $\lvert\frac{L'}{L}(1,\theta)\rvert \leq C\log x$ for the $\theta$ trivial on $A_0$ that are not kept | 3.7 |

| Name | What it says | § |
|---|---|---|
| `GChi` | Host requirement: `EAtH` for $\langle\psi\rangle$, $C = 4$; weighted Sha lengths $\le \sqrt{b}$ over $1 \le e < t$ | 3.8 |
| `HostExW` | For each $0 < \varepsilon < \frac{1}{52}$, each $x$ in windows $[Z, Z^{\lambda}]$ arbitrarily far out has ranks and a host meeting the requirement | 3.8 |
| `PairGood` | `WinPair`; $M$ prime to $qd$; for $(i,j) \neq (0,0)$: on $[Y,X]$, $\lvert G(\chi,y)\rvert \le y^{1/2+\varepsilon/2}$ for the twists of characters mod $M$ by $(\cdot/q)^i(\cdot/d)^j$; the twists of real characters of conductor dividing $8M$ zero-free in boxes right of $1 - \frac{1}{400}$ | 3.8 |
| `ChainTied` | At scale $x$, the data and size conditions of $(\star)$, tied to the lattices of family members | 4.1 |
| `LinkG` | At inert places over $p \le X''$: each gain in $[0, 4\lvert TT\rvert a/p]$, the total over lifts bounded below | 4.1 |
| `LinkS` | Near-orthogonality: if $C_0^{-1} \le d_\omega \Gamma^2/D^* \le C_0$ for every $\omega$, then $\lVert K_{\omega\omega'}\rVert_{HS}^2 \le (1+\eta_0) d_\omega d_{\omega'}/D^*$ for words $\omega \neq \omega'$, with $\eta_0 \le e^{-x^{\varepsilon/8}}$ | 4.1 |
| `LinkIRR` | The defect $\delta$ in $D_{\text{rows}} = (1-\delta) D_{\text{all}}$ satisfies $0 \le \delta \le \exp(-x^{\varepsilon/(7J)})$, with $J \ge 1$ a constant of the family | 4.1 |
| `LinkDIV` | Size bound (4.1); $\mathrm{Bk}_\omega(N_\omega) \subset RR^\vee$; height bounds; `BesselId`; ties of lifts and words | 4.1 |
| `LinkMu` | $\mu(N_\omega)$ is a reference slope plus the local shifts of the lifts, up to $CX''$ | 4.1 |
| `RankVal` | At scale $x$: $n \asymp x^{2+8\varepsilon}$, $m \asymp x^{1/2-\varepsilon/16}$, $k \asymp x^{9/2+10\varepsilon}$, and an equation fixing $t$ | 4.2 |
| `IntInert` | At inert $p \ge 5$, $p$-integral kernel combinations in formal-log coordinates, times an explicit factor, are $p$-integral | 4.4 |
| `LinkN` | For each lift $j$, if the $\mathrm{Sha}_\ell(e)$ are finite and their weighted lengths summable: weighted divisibility defects at the twisted places $\le d_{Z,0}(j)$ $\big($errors + weighted $\mathrm{Sha}_\ell(e)$ lengths$\big)$ | 4.5 |
| `HypV` | For large $x$, every member's $\mathrm{Sha}_\ell(e)$, $1 \le e < t$, are finite, with weighted lengths $\le \sqrt{b}$ | 4.5 |
| `CorGK` | Under `GD`, lengths for $\ker \eta = H$, summed over $\ell$: polynomial in $N(e+1)$, times $\log pp'$ | 5.2 |
| `IsRecT` | Parameter-map laws: class field theory on characters, Langlands classification, real twists; smooth twists, duals, injectivity only on supports made of characters and, over $\mathbb{Q}_p$, of twisted depth-zero cuspidals | 5.3 |

**Use of AI.** This guide was researched and written with Anthropic's Claude Opus 5.5 in Claude Code and OpenAI's GPT-6.2 Astra in Codex. The author takes full responsibility for its contents.

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