hexagon:2611.00792v1

ON FINITE-TIME BLOW-UP OF THE 2D INVISCID BOUSSINESQ EQUATIONS

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ABSTRACT. We construct smooth odd periodic data for the 2D inviscid Boussinesq equations whose temperature gradient becomes unbounded at the origin in finite time. On an accelerating shear with inclination δ, a wave amplifies its temperature gradient by a factor of at least $\exp(c/\sqrt{\delta})$. At the first return of its vorticity to zero, the gradient points downwards with inclination asymptotic to $\delta/2$. We use the successive zeros of the leading wave vorticity to select return times on successively shorter intervals. An upper bound on the pressure Hessian permits correction of each periodic wave with both particle endpoints prescribed. A transverse Green function bounds the contribution of late forcing to the initial velocity by the inverse of the preceding amplification, yielding smooth initial data for the limiting solution.

1. INTRODUCTION

Consider the inviscid Boussinesq equations

$$\text{(B)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \theta \mathbf{e}_2 \\ \partial_t \theta + u \cdot \nabla \theta = 0 \\ \operatorname{div} u = 0 \\ u(0) = u_0, \quad \operatorname{div} u_0 = 0, \quad \theta(0) = \theta_0 \end{cases}$$

on the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. Here u denotes the velocity, θ denotes the temperature, and p denotes the pressure. Write $\mathbf{e}_1 := (1,0)$ and $\mathbf{e}_2 := (0,1)$. The temperature is constant on each particle path. Its gradient, however, can grow as the flow brings distinct temperature layers together. The odd symmetry is defined by central inversion, so $u(t,-x) = -u(t,x)$ and $\theta(t,-x) = -\theta(t,x)$. The equations preserve this symmetry, with the mean-zero pressure even. In particular, $u(t,0) = 0$, and the particle at the origin remains there.

Theorem 1.1. *There are odd, mean-zero C^∞ initial data (u_0, θ_0) on $\mathbb{T}^2$, with $\operatorname{div} u_0 = 0$, whose unique smooth solution of (B) has maximal lifespan*

Date: November 20, 2026.

2020 *Mathematics Subject Classification.* 35Q35, 76B03, 35B44, 76B70.

Key words and phrases. Boussinesq equations, finite-time blow-up, Kelvin waves, inclined shear.

$[0, T_*)$, $0 < T_* < \infty$, *and*

$$\limsup_{t \uparrow T_*} |\nabla \theta(t, 0)| = \infty.$$

The origin is a stationary particle throughout this interval.

Write $D_t = \partial_t + u \cdot \nabla$ and $\omega = \partial_1 u_2 - \partial_2 u_1$. Taking the curl of the momentum equation and differentiating the transport equation gives

$$D_t \omega = \partial_1 \theta, \qquad D_t \nabla \theta = -(\nabla u)^T \nabla \theta.$$

Section 2 shows how a shear turns the temperature gradient and reverses the production of wave vorticity at the origin.

1.1. **Related work.** Pumir and Siggia [24] and E and Shu [15] studied the sharpening of Boussinesq fronts numerically. Córdoba and Fefferman [9] excluded collapse of a regular front of positive length when the maximum speed is time-integrable. Under symmetry and sign assumptions on the data, Kiselev, Park and Yao [21, Theorems 1.3 and 1.4] bounded $\sup_{0 \le s \le t} \|\nabla \theta(s)\|_\infty$ from below by $ct^{1/2}$ on the torus and by ct^2 in a strip, throughout smooth existence. Theorem 1.1 follows the gradient at one particle.

Chae and Nam [5] established local classical solvability for the planar Cauchy problem. Chen and Hou [6, 7] proved blow-up for smooth finite-energy data in the half-plane. Elgindi and Jeong [17] constructed finite-energy sector examples with Lipschitz data. In the plane, Elgindi and Pasqualotto [18, 19] obtained blow-up for C^{1,α_*} data, for some $\alpha_* > 0$, smooth away from the origin. Multiscale constructions for forced fluids include [11, 12]. For Boussinesq, Córdoba, Laín-Sanclemente and Martínez-Zoroa [10] constructed a forced cascade, and Alpöge and Buckmaster [1] proved blow-up with smooth forcing in both equations. Their Lemma 3.1 gives the affine Boussinesq wave equations used below, and their controlled rotation returns the wave vorticity to zero. Here the accelerating inclined shear produces the return without a control term.

The wave calculation uses the shear-wave cancellation of Craik and Criminale [13] and the short-wave equations of Lifschitz and Hameiri [22]. The endpoint formulation draws on the geometry of incompressible flow [3, 16] and Brenier's least-action principle [4]. The Euler construction in [23], also presented in [26], uses an upper pressure-Hessian bound to control a two-endpoint problem. For incompressible porous media, Dai [14] uses such an upper bound to control the preparation of a new layer. Zheng [27] combines inclined layers, one-sided profiles, terminal displacement conditions for phase means, and correction of a finite expansion. In the present endpoint problem the second-order transverse Green function estimates how late forcing contributes to the initial velocity.

1.2. **Acknowledgments.** This project began during Xi Chen and Haoling Xiang's visit to Zurich from 27 September to 2 October 2026. They thank

ETH Zurich for its hospitality. Hedong Hou was supported by Simons Initiative of Geometry of Flows.

1.3. **AI statement.** The authors proposed the underlying mechanism and used ChatGPT (6.1 Sol) to assist with the detailed computations.

2. The mechanism and the proof of Theorem 1.1

At the stationary particle $x = 0$ we follow the temperature gradient $\nabla\theta$ and the velocity gradient ∇u. Consider a layer released from rest with $\nabla\theta = -G\nu$, where $G > 0$ is the magnitude of the gradient and ν is the upward unit normal to the temperature level lines. Its horizontal component $\delta := \nu \cdot \mathbf{e}_1$ is small and positive. Thus $\nu = (\delta, \sqrt{1-\delta^2})$ and δ is the sine of its angle from the vertical. The temperature decreases upwards, so cold fluid lies above warm fluid. Pressure balances the component of buoyancy normal to the layers. The tangential component makes neighboring layers slide past one another, creating a shear.

A weak wave enters this shear with a horizontal wave vector. Where the wave carries warm fluid upwards, it reinforces the temperature perturbation that drives the upward motion. At the same time the shear turns the wave vector downwards. When its horizontal component changes sign, the horizontal temperature gradient of the wave changes sign with it, reversing the production of wave vorticity. The vorticity then decreases to zero. At this *return*, the wave's velocity gradient vanishes and its temperature gradient points downwards. The shear has subtracted about $2\nu/\delta$ from the initial unit wave vector. The spacing of the wave fronts has therefore shrunk to about $\delta/2$ of its initial value, and their new upward normal has horizontal component about $\delta/2$. To first order in time this wave acts as a top-heavy layer released from rest, with a larger temperature gradient and a smaller tilt. Its shear turns the next wave. The velocity gradient left by the older layers is small on the time scale of this new layer.

At fixed G, a smaller tilt weakens the shear and gives the wave more time in the initial regime where its growth rate is comparable to $\sqrt{G}$, the growth rate for horizontal layers. The temperature amplitude grows by a factor of at least $e^{c/\sqrt{\delta}}$, so a small incident wave can leave a gradient much larger than G. The return time is asymptotic to $2/(\delta\sqrt{G})$. In the iteration G grows fast enough that the successive return intervals have finite sum. The incident wavelength in the normalized affine model is $2\pi/N$. On the torus the background varies across a wavelength. A derivative falling on an older phase in place of the new one brings a factor of order R/N, where R is the frequency scale of the preceding waves. All waves are already present at time zero. We choose N so that the incident amplitude absorbs every fixed power of the frequency in the initial temperature. For the initial velocity the corresponding smallness follows from an estimate for the equation with prescribed particle endpoints.

2.1. **A wave on an inclined shear.** The unit tangent to the layers is $\nu^\perp := (-\nu_2, \nu_1)$. The affine shear has velocity gradient $-tG\delta\,\nu^\perp \otimes \nu$. Superimpose a wave with an odd periodic profile of slope one at zero. Let $k(t)$ be its wave vector normalized by $k(0) = \mathbf{e}_1$. At the origin write $A(t)$ for its vorticity and $B(t)k(t)$ for its temperature gradient. Set $\beta := G(\nu \cdot \mathbf{e}_2)$ and introduce the dimensionless time $\gamma(t) := \delta\sqrt{\beta}\,t$, where t is the time since release. The shear gives $k(t) = \mathbf{e}_1 - \gamma(t)^2\nu/(2\delta)$, and the amplitudes satisfy

$$\frac{\mathrm{d}A}{\mathrm{d}t} = k_1 B, \qquad \frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\beta A}{|k|^2}, \qquad A(0) = 0, \quad B(0) = \epsilon > 0.$$

Proposition 2.1 (One-wave gain and return)**.** *For sufficiently small $\delta > 0$, the first positive zero t_δ of A is simple. At this return B attains its unique maximum on $0 \le \gamma \le 3$, and, for absolute constants $c, C > 0$,*

$$\gamma(t_\delta) = 2 + o(1), \qquad e^{c/\sqrt{\delta}} \le \frac{B(t_\delta)}{\epsilon} \le e^{C/\sqrt{\delta}}, \qquad |k(t_\delta)| = \frac{2 + o(1)}{\delta}. \tag{2.1}$$

The new upward normal $\nu_+ := -k(t_\delta)/|k(t_\delta)|$ has inclination $\delta_+ := \nu_+ \cdot \mathbf{e}_1 = (\frac{1}{2} + o(1))\delta$ as $\delta \downarrow 0$. For any $G_+ > 0$, some $0 < \epsilon \le C\delta G_+ e^{-c/\sqrt{\delta}}$ gives the wave increments at $(t_\delta, 0)$

$$\nabla\widetilde{\theta} = -G_+\nu_+, \qquad \nabla\widetilde{u} = 0, \qquad \partial_t\nabla\widetilde{u} = -G_+\delta_+\,\nu_+^\perp \otimes \nu_+.$$

The wave's contribution to the pressure Hessian at the origin is negative semidefinite up to and including the return.

In these variables the torque $k_1 B$ reverses at $\gamma = \sqrt{2}$, and A returns to zero at $\gamma = 2 + o(1)$. In the iteration $G > 1$ and we choose $G_+ := G^{11/10}$, so the old gradient is smaller by a factor $G^{-1/10}$.

2.2. **The periodic induction step.** Set $t_0 := 0$. For a large integer m, take $\delta_0 := (1 + m^2)^{-1/2}$ and $\nu_0 := (\delta_0, \sqrt{1 - \delta_0^2})$. Define the gradient magnitudes by

$$G_0 := \exp\big((\log(1/\delta_0))^2\big), \qquad G_{j+1} := G_j^{11/10}. \tag{2.2}$$

For $j \ge 1$, let t_j denote the return of the leading amplitude of wave j on its parent. Write $-G_j\nu_j$ for its leading temperature-gradient increment there and set $\delta_j := \nu_j \cdot \mathbf{e}_1$. The corresponding gradient of the exact periodic solution satisfies (2.4). The stage runs until the jth level stopping time

$$T_j := t_j + \frac{3}{\delta_j\sqrt{G_j(\nu_j \cdot \mathbf{e}_2)}}.$$

The next wave returns at $t_{j+1} < T_j$. Let $X_j(t, a)$ denote the position at time t of the particle starting at a. At the origin the incoming wave obeys the same amplitude equations with $k' = -(\nabla u_j)^T k$, $\beta := -\nabla\theta_j \cdot k^\perp$, $A(0) = 0$ and $B(0) > 0$. Its phase gives $|k(t_j) - \mathbf{e}_1| \le \delta_j^{10}$, with $k(0) = \mathbf{e}_1$ for the first wave.

Proposition 2.2 (Periodic induction step)**.** *For sufficiently small δ_0, the construction starts on $[0,T_0]$ from a smooth odd periodic shear at rest at $t=0$ with central temperature gradient $-G_0\nu_0$. Given a stage j constructed on $[0,T_j]$, one can add a localized wave and obtain a smooth odd periodic solution (u_{j+1},θ_{j+1}) of (B) on that interval. Its return satisfies $t_j < t_{j+1} < T_{j+1} \le T_j$, so the step can be repeated on $[0,T_{j+1}]$.*

(a) (Return and gain) *On stage j, t_{j+1} is the first zero of A after $t_j + G_j^{-1/2}$. It is simple, and $\nu_{j+1} := -k(t_{j+1})/|k(t_{j+1})|$ points upwards. The return time and inclination satisfy*

$$\delta_j \sqrt{G_j(\nu_j\cdot \mathbf{e}_2)}\,(t_{j+1}-t_j) = 2 + o(1), \qquad \frac{\delta_{j+1}}{\delta_j} = \tfrac{1}{2} + o(1). \tag{2.3}$$

All bounds are uniform in j, and the errors tend to zero with δ_0. The new wave satisfies

$$(Bk)(t_{j+1}) = -G_{j+1}\nu_{j+1}, \qquad \sup_{0\le t\le t_j+G_j^{-1/2}} |(Bk)(t)| \le C G_{j+1} e^{-c/\sqrt{\delta_j}}.$$

For the exact periodic solution at this return,

$$\nabla\theta_{j+1}(t_{j+1},0) = -G_{j+1}\nu_{j+1} + O(G_j), \qquad |\nabla u_{j+1}(t_{j+1},0)| \le C\sqrt{G_j}. \tag{2.4}$$

(b) (Effect on earlier times) *For either field $f = u,\theta$, the gradients at equal particle labels satisfy*

$$\sup_{\substack{0\le t\le t_j\\ a\in\mathbb{T}^2}} |\nabla f_{j+1}(t,X_{j+1}(t,a)) - \nabla f_j(t,X_j(t,a))| \le 2^{-j-1}. \tag{2.5}$$

(c) (Initial data) *For every integer $r \ge 0$, uniformly in j, the same stages satisfy*

$$\|u_{j+1}(0) - u_j(0)\|_{C^r} + \|\theta_{j+1}(0) - \theta_j(0)\|_{C^r} \le C_r\, 2^{-j}.$$

Proposition 2.1 gives the pressure sign at the origin. On the torus an upper bound on the pressure Hessian permits a correction with prescribed endpoints and initial temperature. The resulting initial velocity satisfies part (c).

Proof of Theorem 1.1 admitting Propositions 2.1 and 2.2. At the origin the periodic base shear has exactly the affine shear gradient of Proposition 2.1. Apply that proposition with $G = G_0$ and $G_+ = G_1$ to choose the incident amplitude of the first wave. At its return the wave adds gradient $-G_1\nu_1$ and zero velocity gradient. Proposition 2.2 realizes this wave on the torus and repeats the construction on each resulting solution, with the errors in (2.4).

Choose δ_0 small enough that $2/5 \le \delta_{j+1}/\delta_j \le 3/5$ for every j. Since ν_j is upward and $\delta_j \le \delta_0$, we have $\nu_j\cdot\mathbf{e}_2 \ge 1/2$. By (2.2) and (2.3),

$$0 < t_{j+1} - t_j \le C\delta_j^{-1} G_j^{-1/2} \le C\delta_0^{-1}(5/2)^j G_0^{-(11/10)^j/2}.$$

The series converges, so $t_j \uparrow T_*$ for some $0 < T_* < \infty$.

Fix j. Summing (2.5) at a fixed label a bounds the difference of the gradients at stages $i \geq j$ and j by $\sum_{h=j}^{i-1} 2^{-h-1} \leq 2^{-j}$ throughout $[0, t_j]$. Since every $X_i(t, \cdot)$ is onto, the velocity and temperature gradients are uniformly bounded there. Part (c) gives convergence of the initial data in every C^r. For any smooth stage write $E_s = \|u\|_{H^s}^2 + \|\theta\|_{H^s}^2$, $s \geq 3$ an integer. Differentiating (B) in space and integrating by parts on the torus gives

$$E_s' \leq C_s(1 + \|\nabla u\|_\infty + \|\nabla\theta\|_\infty)E_s.$$

Incompressibility cancels the pressure and the principal transport term. Buoyancy contributes at most E_s. For $f = u, \theta$ and $1 \leq k \leq l \leq s$, each commutator product satisfies

$$\big\|(\partial^k u)\,\nabla\partial^{l-k} f\big\|_2 \leq C_s\big(\|\nabla u\|_\infty \|f\|_{H^s} + \|\nabla f\|_\infty \|u\|_{H^s}\big).$$

For $l > 1$ interpolate the two factors with Hölder exponents $2(l-1)/(k-1)$ and $2(l-1)/(l-k)$, taking ∞ at a zero denominator. For $l = 1$ use Hölder directly. Gronwall gives the uniform bound on $[0, t_j]$ [8, Lemma 3.2] (see also [20]). The projected equations bound one time derivative in H^{s-1}. Compact Sobolev embedding and Arzelà–Ascoli give a diagonal subsequence converging in $C([0, t_j]; H^{s-1})$ for every j, s. Its limit solves (B) smoothly on $[0, T_*)$ with the limiting data. Testing the difference equations in L^2 gives the same differential inequality for the squared difference norm with the gradients of the second solution. Gronwall then gives uniqueness. Oddness gives mean-zero data.

All flows fix the origin, and the subsequence converges in C^1 at each t_j. The sum of the errors at label zero and the lower bound from (2.4) therefore give

$$|\nabla\theta(t_j, 0)| \geq G_j - CG_{j-1} - 2^{-j} \longrightarrow \infty.$$

Since the return times approach T_*, the maximal smooth lifespan ends at T_*. □

3. A wave on an inclined shear

We prove Proposition 2.1 and quantify the change in the return time and gain when earlier waves perturb the shear.

For any vector $k = (k_1, k_2)$ write $k^\perp := (-k_2, k_1)$ and, for $k \neq 0$, $\widehat{k} := k/|k|$. We use $(\nabla u)_{ij} = \partial_j u_i$. The flow map $X(t, a)$ starts at a and $F = \nabla_a X$ has determinant one. Along a trajectory a prime denotes time differentiation. For symmetric matrices, $P \preceq Q$ means that $Q - P$ is positive semidefinite, and $\lambda_{\max}^+(P) = \max(0, \lambda_{\max}(P))$. The constants in $\lesssim$ and $\sim$ are uniform in the stage and the frequencies. They may depend on fixed regularity indices. A polynomial bound in nonnegative $x_1, \ldots, x_m$ has the form $C(1 + \sum_i x_i)^C$. Torus averages are normalized and Δ^{-1} annihilates constants.

3.1. **The exact wave.** Let $\nu = (\delta, \sqrt{1-\delta^2})$. For any smooth odd profile with $f'(0) = -G$, the fields

$$\theta = f(\nu \cdot x), \qquad u = t\delta\theta\nu^\perp, \qquad p = (\nu \cdot \mathbf{e}_2) \int_0^{\nu\cdot x} f(r)\,\mathrm{d}r \tag{3.1}$$

solve (B) on $\mathbb{R}^2$. Indeed u is tangent to the temperature layers, and the projection of $\theta\mathbf{e}_2$ onto ν is ∇p. At the origin

$$\nabla\theta = -G\nu, \qquad \nabla u = -tG\delta\,\nu^\perp \otimes \nu, \qquad \nabla^2 p = -G(\nu \cdot \mathbf{e}_2)\,\nu \otimes \nu. \tag{3.2}$$

We call $\delta = (1+m^2)^{-1/2}$, $m \in \mathbb{N}$, admissible. In this case $\nu/\delta = (1, m)$, so any $2\pi\delta$-periodic odd profile gives a solution on $\mathbb{T}^2$.

More generally, let $u_b = M(t)x$, $\theta_b = g(t)\cdot x$, $p_b = x\cdot\Pi_b(t)x/2$. These fields solve (B) on $\mathbb{R}^2$ when $\operatorname{Tr} M = 0$, $g' = -M^T g$, and $\Pi_b = \mathbf{e}_2 \otimes g - M' - M^2$ is symmetric. Let $k' = -M^T k$, $k(0) \neq 0$. Then $(k^\perp)' = Mk^\perp$ and $\beta = -g \cdot k^\perp$ is constant. The amplitudes of vorticity and temperature solve

$$A' = k_1 B, \qquad B' = \frac{\beta A}{|k|^2}. \tag{3.3}$$

Lemma 3.1 (Exact wave)**.** *Let ψ be smooth, odd and 2π-periodic with $\psi'(0) = 1$, and let Ψ be its mean-zero primitive. For $N > 0$ and $z = Nk \cdot x$, the affine solution admits the exact increment*

$$\begin{aligned} \widetilde{u} &= \frac{Ak^\perp}{N|k|^2}\psi(z), \quad \widetilde{\theta} = \frac{B}{N}\psi(z), \\ \widetilde{p} &= \frac{q}{N^2|k|^2}\Psi(z), \quad q = Bk_2 - \frac{2A}{|k|^2}\,k \cdot Mk^\perp. \end{aligned} \tag{3.4}$$

Proof. The background transports z, and $\widetilde{u} \cdot \nabla z = 0$. With $b = Ak^\perp/|k|^2$, substitution leaves $B' = -b \cdot g$ and $b' + Mb + qk/|k|^2 = B\mathbf{e}_2$. The first identity is (3.3). Taking the scalar product of the second with $k^\perp$ gives $A' = Bk_1$, since b is parallel to $k^\perp$ and $(k^\perp)' = Mk^\perp$. The scalar product with k gives $q = Bk_2 - 2k \cdot Mb$, since $(k \cdot b)' = 0$. □

At the origin the increments are

$$\nabla\widetilde{\theta} = Bk, \qquad \nabla\widetilde{u} = A\,\widehat{k}^\perp \otimes \widehat{k}, \qquad \nabla^2\widetilde{p} = q\,\widehat{k} \otimes \widehat{k}. \tag{3.5}$$

For a general smooth flow use $M = \nabla u$, $g = \nabla\theta$ along each trajectory and $k = F^{-T}n$, where n is the initial covector. The leading increments for the material phase $Nn \cdot a$ are

$$\begin{aligned} &F \mapsto F\big(I + \psi' c\, n^\perp \otimes n\big), \quad c(t) = \int_0^t \frac{A}{|k|^2}\,\mathrm{d}s, \\ &\nabla\theta \mapsto \nabla\theta + \psi' Bk, \qquad \nabla u \mapsto \nabla u + \psi' A\,\widehat{k}^\perp \otimes \widehat{k}, \\ &\nabla^2 p \mapsto \nabla^2 p + \psi' q\,\widehat{k} \otimes \widehat{k}. \end{aligned} \tag{3.6}$$

Here $\psi' = \psi'(Nn \cdot a)$ and $\beta c = B(t) - B(0)$. These are the terms of order one in the gradient and deformation of a periodic wave (Section 6). Indeed

$Fn^\perp = k^\perp$, so in material coordinates the velocity gradient increment is a multiple of the fixed square-zero matrix $n^\perp \otimes n$. This gives the first line by integration. The other three increments are the derivatives in (3.4).

3.2. **The return.** For the shear (3.2), take $k(0) = \mathbf{e}_1$, $A(0) = 0$ and $B(0) = \epsilon > 0$. Now $\beta = G(\nu \cdot \mathbf{e}_2)$ and $k \cdot \nu^\perp = -\nu \cdot \mathbf{e}_2$ is constant. With $\gamma := \delta\sqrt{\beta}\, t$ one obtains

$$k = \mathbf{e}_1 - \frac{\gamma^2}{2\delta}\nu, \qquad P_\delta(\gamma) = \delta^2|k|^2 = \frac{\gamma^4}{4} - \delta^2\gamma^2 + \delta^2. \tag{3.7}$$

In rescaled time, with $B_\gamma = \mathrm{d}B/\mathrm{d}\gamma$, (3.3) becomes

$$(P_\delta B_\gamma)_\gamma = (1 - \gamma^2/2)B, \qquad P_\delta B_\gamma = \delta\sqrt{\beta}\, A, \qquad B_\gamma(0) = 0. \tag{3.8}$$

Proof of Proposition 2.1. Normalize $B(0) = 1$. The flux $P_\delta B_\gamma$ is positive up to $\gamma = \sqrt{2}$ by (3.8), so B increases there. On $0 \le \gamma \le \sqrt{\delta}$ one has $P_\delta \le 5\delta^2/4$ and $1 - \gamma^2/2 \ge 1/2$. Integrating twice and iterating the positive integral inequality gives $B(\gamma) \ge \cosh(\sqrt{2/5}\,\gamma/\delta)$. At $\gamma = \sqrt{\delta}$ this is at least $e^{c/\sqrt{\delta}}$, and B keeps increasing until the return. For the upper bound use $\mathrm{d}\tau = \mathrm{d}\gamma/\sqrt{P_\delta}$. The system for $(B, \sqrt{P_\delta}B_\gamma)$ has bounded coefficients on $[0, 3]$, since $P_\delta \sim \delta^2 + \gamma^4$ and $|P_\delta'|/\sqrt{P_\delta} \le C$. This interval has τ-length at most $C\int_0^3(\delta + \gamma^2)^{-1}\mathrm{d}\gamma \le C/\sqrt{\delta}$. Gronwall's inequality gives the upper bound.

To locate the zero, normalize instead by $B(1)$. The ratio lies in $(0, 1]$ on $(0, 1]$, and its derivative is bounded at some point of $[1/3, 1/2]$ by the mean value theorem. On compact subintervals of $(0, 3]$ the normalized solution and its flux therefore converge along subsequences to g and $\gamma^4 g'/4$. Here $g' = \mathrm{d}g/\mathrm{d}\gamma$ and

$$(\gamma^4 g'/4)' = (1 - \gamma^2/2)g, \qquad g(1) = 1.$$

The two solutions $\gamma^{-1}e^{\pm 2/\gamma}$ form a basis. Boundedness at zero selects $g(\gamma) = \gamma^{-1}e^{2-2/\gamma}$, so the whole family converges on compact subintervals of $(0, 3]$. Its limiting flux is $\gamma^2(2 - \gamma)g/4$, as plotted in Figure 1. On $[\sqrt{2}, 3]$ the original amplitude is positive and its flux is strictly decreasing. Thus the flux vanishes exactly once there, at $\gamma = 2 + o(1)$. Since $A' = k_1 B < 0$ at the zero, it is simple. Formula (3.7) gives $k_1 \to -1$ and $\delta k_2 \to -2$ at return.

Rescale the unit solution so that $B(t_\delta)|k(t_\delta)| = G_+$. The three gradient identities follow from (3.5) and $A'(t_\delta) = B(t_\delta)k_1(t_\delta) = -G_+\delta_+$. Finally $k \cdot Mk^\perp = tG\delta(k \cdot \nu^\perp)^2 \ge 0$ and $k_2 \le 0$. Since $A \ge 0$ and $B > 0$ before return, (3.4) gives the pressure sign. □

3.3. **Comparison along a trajectory.** Let $M = \nabla u$, $g = \nabla\theta$ and $\Pi = \nabla^2 p$ along a trajectory of a smooth solution. Differentiating the momentum equation gives

$$M' + M^2 + \Pi = \mathbf{e}_2 \otimes g. \tag{3.9}$$

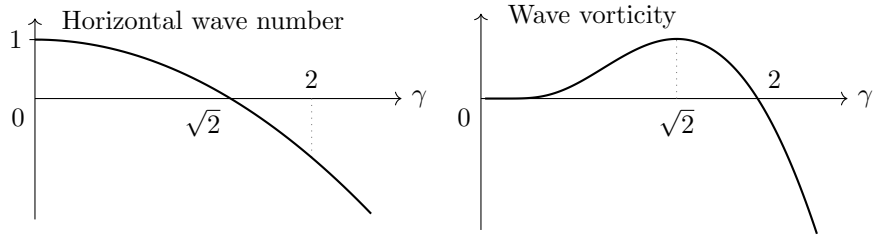


FIGURE 1. The horizontal wave number vanishes at $\gamma = \sqrt{2}$ (left). The vorticity is then maximal and returns to zero at $\gamma = 2 + o(1)$. The right panel shows its limit in the normalization $\delta\sqrt{\beta}\,A/B(1)$, with $B(1)$ evaluated at $\gamma = 1$.

Lemma 3.2 (Transverse equation)**.** *For a wave* (3.3) *with* $k' = -M^T k$*, its gradient amplitude* $d = |k|B$ *solves*

$$d'' = \big(|\widehat{k}'|^2 - \widehat{k}^{\perp} \cdot \Pi \widehat{k}^{\perp}\big) d. \tag{3.10}$$

Replacing (M, g) *by* $(M + A\widehat{k}^{\perp} \otimes \widehat{k}, g + Bk)$ *in* (3.9) *replaces* Π *by* $\Pi + q\widehat{k} \otimes \widehat{k}$*.*

Proof. Writing $w = k^{\perp}$, one has $w' = Mw$ and $w'' = -\Pi w - \beta \mathbf{e}_2$. The identity $|w|'' = \widehat{w} \cdot w'' + |w|\,|\widehat{w}'|^2$ gives

$$|k|'' = |k|\big(|\widehat{k}'|^2 - \widehat{k}^{\perp} \cdot \Pi \widehat{k}^{\perp}\big) - \beta k_1 / |k|.$$

Using $(|k|^2 B')' = \beta k_1 B$ in $(|k|B)''$ cancels the last term. For the second assertion put $b = Ak^{\perp}/|k|^2$. Since $(b \otimes k)^2 = 0$ and $k' = -M^T k$, the pressure increment equals $(B\mathbf{e}_2 - b' - Mb) \otimes k = q\,\widehat{k} \otimes \widehat{k}$ by the two projections in Lemma 3.1. □

Lemma 3.3 (Comparison)**.** *Let* $A(0) = 0$*,* $B(0) > 0$*,* $\lambda(t) = \lambda^{+}_{\max}(\Pi(t))$*. If*

$$T|M(0)| + \int_0^T (T - s)\lambda(s)\,\mathrm{d}s \le \tfrac{1}{2}, \tag{3.11}$$

then for $0 \le s \le t \le T$*,*

$$0 < d(s) \le 2d(t), \qquad d'(t)/d(t) \ge -2\big(|M(0)| + \textstyle\int_0^t \lambda(r)\,\mathrm{d}r\big).$$

Let (B_1, A_1) *be the solution with initial data* $(1, 0)$*. Every solution of* (3.3) *satisfies*

$$\frac{|(B, A)(t)|}{B_1(t)} \le C_* \frac{|(B, A)(s)|}{B_1(s)}.$$

The constant is polynomial in $1 + T$*,* $1 + |\beta|$ *and* $\sup_{[0,T]}(|k| + |k|^{-1})$*.*

Proof. Before a first zero let τ maximize d on $[0, t]$. If $\tau < t$, then $d'(\tau) = 0$ at an interior maximum and $d'(0) \ge -|M(0)|d(0)$ at the left endpoint. Integrating $d'' \ge -\lambda d(\tau)$ twice gives $d(t) \ge d(\tau)/2$ by (3.11). This excludes a zero. Integrating once from 0 gives the slope bound.

The bound for d gives $B_1(s) \lesssim B_1(t)$ with the stated polynomial dependence. Integration of $A_1' = k_1 B_1$ then gives $|A_1(t)| \lesssim B_1(t)$. A second

solution is $(B_1 I, A_1 I + 1/B_1)$, where $I' = \beta/(|k|^2 B_1^2)$ and $I(0) = 0$. The two solutions have determinant one. In the transition matrix from s to t, the terms containing I involve only $I(t) - I(s)$, bounded by $C_*/B_1(s)^2$ after integration. The preceding bounds on B_1, A_1 give the assertion. □

3.4. **Stability of the return.** Throughout this subsection assume $0 < \delta \le 1/2$. At a return time a set $s := \sqrt{G}\,(t-a)$ and $\gamma := \delta\sqrt{\nu\cdot\mathbf{e}_2}\,s$, and let T be determined by $\gamma(T) = 3$. We first control the shear generated by the returned wave, then the next wave propagating on it.

Lemma 3.4 (Shear after return)**.** *Suppose a wave on a stationary trajectory returns at a, with $A(a) = 0$, $B(a) > 0$ and $B(a)k(a) = -G\nu$. Suppose its parent gradients M_o, g_o satisfy $|M_o| \le \eta\sqrt{G}$ and $|g_o| \le \eta G$ on $[a,T]$, with $\eta \le \delta^3/C$ for an absolute constant C. Then the updated gradients $M = M_o + A\widehat{k}^{\perp}\otimes\widehat{k}$, $g = g_o + Bk$ satisfy*

$$|M/\sqrt{G} + s\delta\,\nu^{\perp}\otimes\nu| \le C\eta/\delta^2, \quad |g/G + \nu| \le C\eta/\delta,$$
$$|A| \le C\sqrt{G}, \qquad |Bk| \le 4G.$$

Proof. The interval has length $O((\delta\sqrt{G})^{-1})$. Integrating $k' = -M_o^T k$ gives $k/|k(a)| = -\nu + O(\eta/\delta)$, so $|k_1| \le 2\delta|k(a)|$. As long as $|B| \le 2B(a)$, the equation $A' = k_1 B$ and $B(a)|k(a)| = G$ give $|A| \le C\sqrt{G}$. Then $B' = -(g_o\cdot k^{\perp})A/|k|^2$ gives $|B/B(a) - 1| \le C\eta/\delta$, closing the bound by continuity. In particular $A'/G = -\delta + O(\eta/\delta)$. Integrating once more yields $A/\sqrt{G} = -s\delta + O(\eta/\delta^2)$. Substitution in M and g proves the assertions. □

Proposition 3.5 (Return under perturbations)**.** *For sufficiently small δ, suppose the gradients M, g of a smooth solution on a stationary trajectory satisfy*

(a) $|M/\sqrt{G} + s\delta\,\nu^{\perp}\otimes\nu| + |g/G + \nu| \le \delta^{10}$ *on* $[a,T]$,
(b) (3.11) *on* $[0,T]$ *and* $|M(0)| + \int_0^T \lambda_{\max}^{+}(\Pi)\,\mathrm{d}t \le \delta\sqrt{G}$.

Let $A(0) = 0$, $B(0) > 0$, $|k(a) - \mathbf{e}_1| \le \delta^{10}$. Then A has a first zero a_+ after $a + G^{-1/2}$, with $a_+ < T$. The zero is simple and the following bounds are uniform.

(i) $\gamma(a_+) = 2 + o(1)$ *and* $\nu_+ = -\widehat{k}(a_+)$ *is upward with* $\nu_+\cdot\mathbf{e}_1 = (\frac{1}{2} + o(1))\delta$.
(ii) $d(a_+) \ge e^{c/\sqrt{\delta}}\sup_{[0,a+G^{-1/2}]} d$ *and* $\sup_{[0,T]} d \le C d(a_+)$.
(iii) $q \le 0$ *on* $[a + G^{-1/2}, T]$.

Proof. Let k^0 be the exact covector in (3.7). In time s the unperturbed covector propagator from r to s is $I + \delta(s^2 - r^2)\nu\otimes\nu^{\perp}/2$, of norm $O(\delta^{-1})$. Variation of constants thus gives $\sup|k - k^0| \le C\delta^9 + C\delta^8\sup|k| \le C\delta^7$ after absorption, since $\sup|k^0| \le C/\delta$. At a, $\beta = -g\cdot k^{\perp} = G(\nu\cdot\mathbf{e}_2)(1 + O(\delta^{10}))$. Thus $(|k|^2 B')' = \beta k_1 B$ takes the form

$$(\delta^2|k|^2 B_\gamma)_\gamma = (1 - \gamma^2/2 + O(\delta^7))B, \qquad \delta^2|k|^2 = P_\delta(\gamma)(1 + O(\delta^7)). \tag{3.12}$$

By Lemma 3.3, $B(a) > 0$ and $\sigma = G^{-1/2}B'(a)/B(a) \geq -3\delta$, since $|\partial_t \log|k|| \leq |M|$ and $|M(a)| \leq \delta^{10}\sqrt{G}$. For $0 \leq s \leq 1$ the equation $(|k|^2 B_s)_s = (\beta/G)k_1 B$ converges, in first-order form, to $B_{ss} = B$. Its normalized solutions with data $(1,0)$ and $(0,1)$ converge in C^1 to $\cosh s$ and $\sinh s$. Their combination with coefficients $1, \sigma$ therefore has positive value and derivative at $s = 1$, uniformly for $\sigma \geq -3\delta$. For $1 \leq s \leq c/\sqrt{\delta}$, one has $|k|^2 \leq 2$ and $(\beta/G)k_1 \geq 1/4$. Two integrations give $B(s) \geq B|_{s=1} \cosh((s-1)/\sqrt{8})$.

Normalize B by $B|_{\gamma=1}$. On each compact subinterval of $(0,3]$ the coefficients in (3.12) converge to those of (3.8) at $\delta = 0$. Starting at $s = 1$ the normalized amplitude is positive and increasing through $\gamma = 1$. Its flux is bounded at some point of $[1/3, 1/2]$ by the mean value theorem and the bound on $\delta^2|k|^2$. The compactness and uniqueness argument of Proposition 2.1 gives the same limit $(g, \gamma^2(2-\gamma)g/4)$, $g = e^{2-2/\gamma}/\gamma$. The flux is positive from $s = 1$ to $\gamma = 3/2$ by monotonicity away from $\sqrt{2}$ and convergence near $\sqrt{2}$. Thereafter it is strictly decreasing since $B > 0$ and $k_1 < 0$. Its unique zero has $\gamma = 2 + o(1)$ and $A' = k_1 B < 0$. The covector estimate gives the outgoing inclination. The lower gain follows from monotonicity up to return and $\sup_{t \leq a + G^{-1/2}} d(t) \leq 2d(a + G^{-1/2})$. On $[a_+, T]$ we have $B(t) \sim B(a_+)$ and $|k| \sim \delta^{-1}$. Before return, $d(t) \leq 2d(a_+)$ by Lemma 3.3.

Finally, $k \cdot Mk^\perp/\sqrt{G} = s\delta(k \cdot \nu^\perp)^2 + O(\delta^8)$ with $k \cdot \nu^\perp = -\nu \cdot \mathbf{e}_2 + O(\delta^7)$. For $s \geq 1$ this is positive, and $k_2 < 0$ by (3.7) and the covector estimate. Thus $q < 0$ until return. Thereafter $Bk_2 \leq -cB(a_+)/\delta$, whereas the other term of q has magnitude at most $CB(a_+)\delta$, since $|A| \leq CB(a_+)/(\delta\sqrt{G})$, $|k| \sim \delta^{-1}$ and $|k \cdot Mk^\perp| \leq C\sqrt{G}$. □

4. Pressure and the endpoint problem

4.1. **Coercivity and the initial velocity.** Prescribing the two endpoints of a particle path leads to an energy estimate for its displacement. The positive part of the pressure Hessian controls this energy through a one-dimensional inequality in time. In the following statements Π may be any smooth symmetric matrix field evaluated along a smooth area-preserving path $X(t,a)$ with $X(0,a) = a$. Integrals in a are over $\mathbb{T}^2$.

Lemma 4.1 (Endpoint coercivity)**.** *If*

$$\kappa = 1 - \sup_a \int_0^T \frac{t(T-t)}{T} \lambda^+_{\max}(\Pi(t,a))\,\mathrm{d}t > 0,$$

then for every $Y \in H^1_0((0,T); L^2)$,

$$\int_0^T \int (|Y_t|^2 - Y \cdot \Pi Y)\,\mathrm{d}a\,\mathrm{d}t \geq \kappa \int_0^T \int |Y_t|^2\,\mathrm{d}a\,\mathrm{d}t.$$

Proof. At each label, $Y(t) = \int_0^T (\mathbb{1}_{[0,t]}(s) - t/T)Y_s(s)\,\mathrm{d}s$. The squared L^2 norm of the kernel is $t(T-t)/T$. Apply Cauchy–Schwarz and integrate the resulting bound against $\lambda^+_{\max}(\Pi)$. □

For stage j, let $T_j \le T_0$ be the endpoint in (7.2). The bound

$$b_j = T_0^2 \sup \lambda_{\max}^{+}(\nabla^2 p_j) + T_0 \sup |\nabla u_j(0)| \le \tfrac{1}{8} \tag{4.1}$$

implies both (3.11) and $\kappa \ge 1 - b_j/6$ on each trajectory. The suprema range over all positions and, for the pressure, all times in $[0, T_j]$.

Let $F = \nabla_a X$ and set $Y = Fw$. By change of variables, $Y \circ X^{-1}$ is divergence free in space precisely when w is divergence free in the labels. The linear endpoint problem is

$$Y_{tt} + \Pi Y + F^{-T}\nabla_a \pi = f, \qquad \nabla_a \cdot w = 0, \qquad w(0) = w(T) = 0. \tag{4.2}$$

Testing against Fz for divergence-free $z \in H_0^1((0,T); L^2)$ gives

$$\mathcal{B}(w,z) = \int_0^T \int \big[(Fw)_t \cdot (Fz)_t - \Pi Fw \cdot Fz\big]\, \mathrm{d}a\, \mathrm{d}t = -\int_0^T \int f \cdot Fz\, \mathrm{d}a\, \mathrm{d}t. \tag{4.3}$$

Lemma 4.2 (Linear endpoint problem)**.** *For $\kappa > 0$ and $f \in L^2$, (4.3) has a unique solenoidal solution $w \in H_0^1((0,T); L^2)$, and*

$$\|w\|_{H_t^1 L^2} \le \frac{T}{\kappa} \|F^{-1}\|_\infty \big(1 + T + T\|F_t\|_\infty \|F^{-1}\|_\infty\big) \|f\|_{L^2}.$$

If $\mathbb{P}$ is the Leray projection in the labels, the flux $\Phi = \mathbb{P}(F^T Y_t)$ satisfies

$$\Phi_t = \mathbb{P}(F^T f + F_t^T Y_t - F^T \Pi Y). \tag{4.4}$$

Both Y_t and w_t are continuous in time with values in L^2, and $Y_t(0) = w_t(0) = \Phi(0)$.

Proof. Since $w = F^{-1}Y$ and $w_t = F^{-1}(Y_t - F_t w)$, the inequality $\|Y\|_{L^2} \le T\|Y_t\|_{L^2}$ controls $\|w\|_{H_t^1 L^2}$ by $\|Y_t\|_{L^2}$ with the displayed deformation factor. Lemma 4.1 gives coercivity in w, so Lax–Milgram gives a unique solution. Testing with w gives $\kappa \|Y_t\|_{L^2}^2 \le \|f\|_{L^2}\|Y\|_{L^2}$ and hence the bound. Expanding $(Fz)_t$ gives (4.4) distributionally. Its right side belongs to L^2, hence Φ is continuous. On solenoidal fields the operator $\mathbb{P}F^T F$ is bounded below by $\|F^{-1}\|_\infty^{-2}$ as a quadratic form. Thus $w_t = (\mathbb{P}F^T F)^{-1}(\Phi - \mathbb{P}F^T F_t w)$ is continuous, as is $Y_t = Fw_t + F_t w$. At time zero $F = I$ and $w = 0$. □

To recover the Eulerian perturbation of an exact background (U, Θ), write $D_t = \partial_t + U \cdot \nabla$. Its linearized equations are

$$D_t v + v \cdot \nabla U + \nabla p' = h\mathbf{e}_2 + f_u, \qquad D_t h + v \cdot \nabla \Theta = f_\theta, \qquad \nabla \cdot v = 0. \tag{4.5}$$

The displacement satisfies $Y_t = v \circ X + (\nabla U \circ X)Y$. Since $(h \circ X + (\nabla\Theta \circ X) \cdot Y)_t = f_\theta \circ X$, differentiation of the displacement equation and (3.9) give (4.2) with

$$\begin{aligned} H = h(0) + \int_0^t f_\theta \circ X\, \mathrm{d}s, \qquad f = f_u \circ X + H\mathbf{e}_2, \\ v \circ X = Fw_t, \qquad h \circ X = H - \nabla_a \Theta(0) \cdot w. \end{aligned} \tag{4.6}$$

These identities also recover a solution of (4.5) from the endpoint solution. The condition $Y(T) = 0$ determines $v(0) = \Phi(0)$.

4.2. **Periodic profiles and the initial shear.** An amplifying wave lowers the pressure Hessian at its centre. We choose its periodic profile so that at other phases the opposite increment is small.

Lemma 4.3 (One-sided profiles)**.** *For each integer $m \geq 2$ there is an odd trigonometric polynomial ψ_m of degree $m-1$ with $\psi_m'(0) = 1$ and $-1/(m-1) \leq \psi_m' \leq 1$. Its derivative has nonnegative Fourier coefficients summing to one. For every symmetric matrix Π, unit vector e and real q,*

$$\lambda_{\max}(\Pi + \psi_m' q\, e \otimes e) \leq \max\{\lambda_{\max}(\Pi), \lambda_{\max}(\Pi + q\, e \otimes e)\} + |q|/(m-1).$$

Proof. Use the Fejér kernel $K_m(z) := m^{-1}|\sum_{l=0}^{m-1} e^{ilz}|^2$. Define ψ_m by $\psi_m(0) = 0$ and $\psi_m' = (K_m - 1)/(m-1)$. The kernel is even, has mean one, attains its maximum m at zero and has nonnegative Fourier coefficients. The matrix inequality follows by convexity of $\lambda_{\max}$ when $0 \leq \psi_m' \leq 1$ and by the operator norm bound $|q|/(m-1)$ otherwise. □

In particular $|\partial_z^r \psi_m'| \leq (m-1)^r$ and its norm (5.1) is at most 2 at radius $1/(2m)$.

Lemma 4.4 (Gevrey cutoffs)**.** *Fix a Gevrey order $\sigma > 1$. There are $a_\sigma > 0$ and an even function $0 \leq \varphi \leq 1$, equal to one on $[-1/2, 1/2]$ and supported in $[-7/8, 7/8]$, with $|\varphi^{(r)}| \leq a_\sigma^{-r}(r!)^\sigma$. A product $\prod_i \varphi(y_i/\ell)$ obeys the same bound for every multi-index of order r, with $a_\sigma \ell$ in place of a_σ.*

Proof. Convolve $\mathbb{1}_{[-3/4,3/4]}$ successively with the uniform probability measures on $[-a_\sigma j^{-\sigma}, a_\sigma j^{-\sigma}]$, choosing $\sum_j a_\sigma j^{-\sigma} < 1/8$ [25, Lemma 3.3]. The rth derivative of the limit is bounded by differentiating the first r measures once each, at total variation cost $a_\sigma^{-r}(r!)^\sigma$. For products use $\prod_i \alpha_i! \leq |\alpha|!$. □

Lemma 4.5 (Base shear)**.** *Let δ_0 be admissible and sufficiently small, $\nu_0 = (\delta_0, \sqrt{1-\delta_0^2})$, and $T_0 = 3/(\delta_0\sqrt{G_0\, \nu_0 \cdot \mathbf{e}_2})$. There is an odd shear (3.1) released from rest, with $\nabla\theta(0,0) = -G_0\nu_0$, such that on $[0, T_0]$*

$$|\nabla\theta| + |\nabla^2 p| \leq 4G_0, \qquad |\nabla u| \leq C\sqrt{G_0},$$
$$\lambda_{\max}^+(\nabla^2 p) \leq CG_0\delta_0^4, \quad |F| + |F^{-1}| \leq C/\delta_0.$$

In particular $b_0 \leq C\delta_0^2$. For G_0 as in (2.2), the norms (5.1) of these fields $F^{\pm 1}, \nabla u, \nabla\theta, \nabla^2 p$ are at most G_0^2 at radius $\rho_0 = a_\sigma \delta_0^5/2 \geq G_0^{-1}$.

Proof. Let χ be the 2π-periodization of $\varphi(\cdot/\delta_0^4)$, with mean $\bar\chi \leq \delta_0^4$. Choose $f(0) = 0$ and

$$f'(s) = -G_0\, \frac{\chi(s/\delta_0) - \bar\chi}{1 - \bar\chi}.$$

The derivative is even and has zero mean on a period, so f is odd and $2\pi\delta_0$-periodic. Its derivative has magnitude at most $2G_0$ and positive part at most $2G_0\delta_0^4$. In (3.1), $\nabla^2 p$ is this derivative times $(\nu_0 \cdot \mathbf{e}_2)\nu_0 \otimes \nu_0$. The remaining bounds follow from

$$F^{\pm 1} = I \pm \tfrac{1}{2} t^2 \delta_0\, \nu_0^\perp \otimes \nabla\theta(0), \qquad \nabla u = t\delta_0 \nu_0^\perp \otimes \nabla\theta.$$

One label derivative of $\chi(\nu_0 \cdot a/\delta_0)$ costs at most $(a_\sigma \delta_0^5)^{-1}$ in addition to the Gevrey factorial. At half this radius the norm is bounded by the geometric sum 2. □

5. The nonlinear endpoint problem

We correct the particle equation while preserving the endpoint maps. The estimates distinguish derivatives of amplitudes from derivatives of phases.

5.1. **Derivatives on translated phases.** For $M_i \in \mathbb{Z}^2$ write $z_i = M_i \cdot a + \zeta_i$. Each $\zeta \in \mathbb{T}^j$ defines a translated torus (a *fiber*), with $\zeta = 0$ giving the physical phases. For a function of $y = (a, z_1, \dots, z_j)$, denote its restriction to this torus by f_ζ. Then $(Df)_\zeta = \nabla_a f_\zeta$, where $D = \nabla_a + \sum_i M_i \partial_{z_i}$. For the Gevrey order σ fixed above, set

$$\|f\|_r = \sum_{m \geq 0} \frac{r^m}{(m!)^\sigma} \max_{|\alpha|=m} \|\partial_y^\alpha f\|_\infty. \tag{5.1}$$

The supremum includes time. Write $\|f\|_{r;E}$ if each derivative is measured in the fiber norm E before taking the supremum over ζ. Brackets $[f]_{r;E}$ omit the zeroth order.

Lemma 5.1 (Gevrey estimates)**.** *For $0 < s < r \leq 1$, the following estimates hold.*

(a) *If $\|hg\|_E \leq \|h\|_\infty \|g\|_E$ for scalar multipliers h, then $\|fg\|_{r;E} \leq \|f\|_r \|g\|_{r;E}$.*
(b) *$[f]_s \leq (s/r)\|f\|_r$.*
(c) *$\|\partial_i f\|_{s;E} \leq C_\sigma r^{-1}(1 - s/r)^{-\sigma}\|f\|_{r;E}$.*
(d) *If $g \geq g_* > 0$ and $[g]_r \leq g_*/2$, then $\|g^{-1}\|_r \leq 2/g_*$.*

Proof. In Leibniz's rule the terms of orders $i, m-i$ carry total coefficient $\binom{m}{i}$, and $\binom{m}{i}(i!)^\sigma((m-i)!)^\sigma \leq (m!)^\sigma$. The same estimate applied to $gg^{-1} = 1$, first up to a finite order, gives $g_*[g^{-1}]_r \leq [g]_r \|g^{-1}\|_r$. The two radius estimates follow from $(s/r)^m \leq s/r$ for $m \geq 1$ and $\sup_m (m+1)^\sigma (s/r)^m \lesssim_\sigma (1 - s/r)^{-\sigma}$. □

Lemma 5.2 (Fiber estimates)**.** *Let $R = 1 + \sum_i |M_i|$ and $d = j + 2$. Write $\Delta_D = D \cdot D$ and take Δ_D^{-1} to vanish on constants on each physical fiber.*

(a) *The fiberwise Leray projection $\mathbb{P}_D$, the fiber mean and $D\Delta_D^{-1}$ commute with all independent derivatives. Their operator norms on a fiber are those on $\mathbb{T}^2$.*
(b) *$\|Df\|_{s;E} \leq C_\sigma R r^{-1}(1 - s/r)^{-\sigma}\|f\|_{r;E}$.*
(c) *For $h = r - s$, $\|f\|_s \leq (C_\sigma d/h)^{C_\sigma d}\|f\|_{r;L^2(\mathbb{T}^d)}$, where the last norm takes all d variables in L^2.*

Proof. On the Fourier mode (m, ℓ) the physical frequency is $m + \sum_i \ell_i M_i$. This proves the multiplier statement, including its action at zero frequency. The derivative estimate follows from Lemma 5.1. For the last inequality

use Fourier Cauchy–Schwarz with $d+2$ derivatives and apply the derivative estimate on $d+2$ equal subdivisions of the radius loss h. □

5.2. **The inverse of the Jacobi operator.** For divergence-free label fields, the coefficients in (4.3) are the blocks of the matrix in

$$\mathcal{B}(w,z)=\int_0^T\int \begin{pmatrix} w_t \\ w \end{pmatrix}\cdot\begin{pmatrix} F^TF & F^TF_t \\ F_t^TF & F_t^TF_t-F^T\Pi F \end{pmatrix}\begin{pmatrix} z_t \\ z \end{pmatrix}\mathrm{d}a\,\mathrm{d}t. \tag{5.2}$$

Lemma 5.3 (Gevrey regularity of the endpoint inverse)**.** *Suppose on every fiber that $\mathcal{B}(w,w)\ge L^{-1}\|w\|^2_{H^1_tL^2}$ and $F^TF\ge L^{-1}I$. Suppose each block in (5.2) has positive-order norm at radius r at most c/L for a sufficiently small absolute c. Then*

$$\|w\|_{r;H^1_tL^2}+\|w_t\|_{r;C_tL^2}\le C\|F^Tf\|_{r;L^2_tL^2},$$

where C is polynomial in L, $T+T^{-1}$ and one plus the norms of the blocks at radius r. For continuous forces, $\|w_{tt}\|_{r;C_tL^2}\le C'\|F^Tf\|_{r;C_tL^2}$, with the same polynomial dependence for C' and additional polynomial dependence on the time derivatives of the two upper blocks.

Proof. Independent translations carry fibers isometrically to fibers and commute with D. Take their difference quotients in the weak equation. Coercivity gives the differentiated equation at every finite order by weak compactness and preserves the solenoidal constraint. Apply coercivity to the differentiated form (5.2) and sum over the orders. By the product estimate of Lemma 5.1, the terms in which a derivative hits a coefficient contribute at most $4c\|w\|_{r;H^1_tL^2}$ after multiplication by L. Absorption gives the $H^1_tL^2$ estimate. For $Y=Fw$ the flux $\Phi=\mathbb{P}_D(F^TY_t)$ obeys

$$\Phi_t=\mathbb{P}_D\big(F^Tf+F_t^TFw_t+(F_t^TF_t-F^T\Pi F)w\big).$$

The same blocks thus bound Φ in $H^1_tL^2$, hence in C_tL^2. Solve

$$\mathbb{P}_D(F^TFw_t)=\Phi-\mathbb{P}_D(F^TF_tw)$$

on solenoidal fields, using the lower bound on F^TF and the same commutator estimate. This gives the asserted bound for w_t, including its initial trace. Differentiating this identity in time and using (4.4) gives the stated bound for w_{tt}. □

5.3. **Correction of an approximate flow.** With prescribed initial temperature θ_0 and terminal map $X^0(T)$, the material equations are

$$\begin{aligned} X_{tt}+(DX)^{-T}Dp&=\theta_0\mathbf{e}_2, \qquad \det DX=1,\\ X(0,a)&=a, \qquad X(T,a)=X^0(T,a). \end{aligned} \tag{5.3}$$

Write $|f|_{r,m}=\|f\|_{r;C_tH^m(\mathbb{T}^2)}$. Products satisfy the algebra estimate for $m\ge 2$. Passing from a fiber L^2 estimate to H^m on radius $r-h$ costs at most $C_{m,\sigma}(R/h^\sigma)^m$ by Lemma 5.2. Set

$$F=DX,\qquad g=F^{-T}Dp,\qquad \Pi=(Dg)F^{-1},\qquad \mathcal{R}(X,p)=X_{tt}+g-\theta_0\mathbf{e}_2.$$

Here p is the material pressure, so Π is its Eulerian Hessian pulled back by X. In DX the derivative of the identity lift is I.

Proposition 5.4 (Correction with prescribed endpoints)**.** *Fix $m \geq 5$ and $\sigma > 1$. Let $X^0(0) = a$, let $X^0 - a$ be periodic, and assume $\det DX^0 = 1$ on each fiber. Fix the terminal map $X^0(T)$ and the material temperature θ_0. Suppose the coefficients at (X^0, p^0) satisfy Lemma 5.3 at radius $r \leq 1$, with a sufficiently small c. Let $P \geq 2$ bound $L, T + T^{-1}, r^{-1}, R$, the norms $|X^0 - a|_{r,m}$ including its first two time derivatives, and $|p^0|_{r,m}, |\theta_0|_{r,m}$. Take pressures of fiber mean zero. If $|\mathcal{R}(X^0, p^0)|_{r,m} \leq c_m P^{-C_m}$, there is an exact solution of (5.3) with*

$$|X - X^0|_{r/2,m} + |p - p^0|_{r/2,m} \leq C_m P^{C_m} |\mathcal{R}(X^0, p^0)|_{r,m},$$

including the first two time derivatives of $X - X^0$. Here C_m, c_m may depend on σ. On the untranslated fiber, if X^0 and θ_0 are odd in the label and p^0 is even, the corrected X is odd and p is even.

Proof. A tangent displacement $Y = Fw$ has $D \cdot w = 0$. For a material pressure variation δp put $\pi = \delta p - g \cdot Y$. Differentiating g gives $\delta g = \Pi Y + F^{-T} D\pi$. The Newton equation is thus (4.2) with $f = -\mathcal{R}(X, p)$ and zero endpoint values. Lemma 5.3 bounds Y, Y_t, Y_{tt}, while $D\pi = F^T(f - Y_{tt} - \Pi Y)$ determines the pressure up to its mean. After shrinking the radius by h we obtain

$$|Y|_{s-h,m} + |\delta p|_{s-h,m} \leq C_m (P/h)^{C_m} |\mathcal{R}(X, p)|_{s,m}, \tag{5.4}$$

including Y_t and Y_{tt}. To keep the inverse uniform at the current radius s, measure the distance to the starting pair by

$$\sum_{q=0}^{2} |\partial_t^q (X - X^0)|_{s,m} + |p - p^0|_{s,m}.$$

On area-preserving maps, $F^{-T} = \operatorname{cof} F$ is linear in F. The product estimates and the fiber embedding $H^m \hookrightarrow W^{2,\infty}$ therefore control the changes of the form blocks and of $F^T F$ by $C_m P^{C_m}$ times this distance, for distance at most one. Choosing the initial positive-order block bound with a factor of two to spare, and keeping the distance below $c_m P^{-C_m}$, preserves the coercivity and absorption bounds with a fixed change of constants.

To restore the determinant, put $Z = Y + FDb$. The Piola identity and the two-dimensional determinant formula give $\det(F + DZ) = 1 + \Delta_D b + \det DZ$. Thus $Db = -D\Delta_D^{-1} \det D(Y + FDb)$, whose source has zero fiber mean by integration by parts. On radius $s - 2h$ one additional derivative bounds F, F_t, F_{tt} in the fiber H^m norm by $C_m(P/h)^{C_m}$. The multiplier $D\Delta_D^{-1}$ maps H^{m-1} to H^m. Thus the quadratic determinant map for Db has Lipschitz constant at most $C_m(P/h)^{C_m}(|Y|_{s-h,m} + |Db|_{s-2h,m})$ on its ball about zero. It gives $Z - Y$ quadratic in Y for a sufficiently small Newton step. Differentiating this fixed-point equation once or twice in time gives

the same invertible linear part and quadratic sources. At both endpoints $Z = 0$. The new map is locally invertible and has degree one.

On area-preserving maps $F^{-T} = \operatorname{cof} F$ is linear in F. After the update $(Z, \delta p)$ the surviving residual is therefore

$$(Z - Y)_{tt} + \operatorname{cof}(D(Z - Y))Dp + \operatorname{cof}(DZ)D\delta p.$$

It is bounded on radius $s - 3h$ by $C_m(P/h)^{C_m}$ times the squared old residual. Taking $s_k = r(1/2 + 2^{-k-1})$ and $3h_k = s_k - s_{k+1}$, the errors $e_k = |\mathcal{R}(X_k, p_k)|_{s_k,m}$ satisfy $e_{k+1} \le C_m P^{C_m} 2^{C_m k} e_k^2$. After increasing C_m, the rescaled errors $C_m P^{C_m} 2^{C_m(k+1)} e_k$ are bounded at each step by the square of the preceding one. For sufficiently small $P^{C_m} e_0$ the corrections therefore sum to the claimed bound by (5.4). Every step preserves odd symmetry. □

6. Realization of a periodic wave

We construct an exact periodic wave with the four prescribed increments in (3.6). Fix its parent flow X on $[0, T]$. Its fields below are pulled back by X. The new angle z has carrier $Nn \in \mathbb{Z}^2$. Write $D_+ = D + Nn\partial_z$, $k = F^{-T}n$, and $\mu = R/N$, where R is the sum of the old frequencies plus one.

6.1. **Recovering spatial gradients.**

Lemma 6.1 (Material div–curl estimate)**.** *Suppose X is an area-preserving diffeomorphism on each fiber, $F = DX$, and $\|F\|_r + \|F^{-1}\|_r \le M$ for $0 < r \le 1$. For a periodic vector field v, let $H = (\nabla v) \circ X$ and let d, c denote the pullbacks of its divergence and curl. If $p \ge C_\sigma(1 + \log(R(1 + M)/r))$ and $s = r/(C_\sigma p M^2)$, then $\|H\|_s \le C_\sigma p(\|d\|_r + \|c\|_r)$.*

Proof. The identities $\Delta v_1 = \partial_1 \operatorname{div} v - \partial_2 \operatorname{curl} v$ and $\Delta v_2 = \partial_2 \operatorname{div} v + \partial_1 \operatorname{curl} v$ express each gradient entry through two operators $\partial_i \partial_j \Delta^{-1}$ on the physical torus. Each has L^p norm at most $p - 1$ for $p \ge 2$ by [2, Corollary 4.1], applied to the compact group $\mathbb{T}^2$ with a single matrix entry.

For the bound Cp used below, the Fourier symbol has modulus at most one and the kernel satisfies $|\nabla K(x)| \le C|x|^{-3}$ off the origin. Indeed the periodic Green function differs from the logarithmic fundamental solution by a smooth function near zero. For λ above the mean of $|f|$, use the dyadic decomposition at height λ. Then the good part has squared L^2 norm at most $C\lambda\|f\|_1$ and the bad squares have total area at most $\|f\|_1/\lambda$. For each mean-zero bad part, subtract the kernel at the center of its square. Outside the tripled square of side comparable to h, the integrated kernel difference is bounded by $C\int_h^\infty h r^{-2}\,\mathrm{d}r$. Together with the L^2 bound on the good part this gives weak type $(1, 1)$. For smaller λ the weak bound follows from the area of the torus. Interpolating with L^2 gives norm $C/(q - 1)$ for $1 < q \le 4/3$, and self-adjointness gives Cp for $p \ge 4$. Interpolation covers $2 \le p \le 4$.

Area preservation gives the same bound in material L^p. For an independent derivative, differentiate $D(v\circ X)=HF$. The difference between $D\partial^\alpha(v\circ X)F^{-1}$ and $\partial^\alpha H$ is a sum of $(\partial^{\alpha-\eta}H)(\partial^\eta F)F^{-1}$, $0<\eta\le\alpha$, with the Leibniz coefficients. The first matrix is a pulled-back gradient, and its trace and skew part are controlled by $\partial^\alpha(d,c)$ and this difference. At radius $2s$ the commutator therefore costs $CpM[F]_{2s}\le 1/2$ and can be absorbed. On a disk of radius h, rescaled Sobolev embedding for $p\ge 4$ gives $\|g\|_\infty\le Ch^{-2/p}(\|g\|_p+h\|\nabla g\|_p)$, uniformly in p. Taking $h=s/(C_\sigma R)$ absorbs the physical derivative in the loss from $2s$ to s, while $h^{-2/p}$ is bounded. □

Corollary 6.2. *If $\|F^{\pm1}\|_r$, $\|D\theta_0\|_r$ and $\|\omega(0)\|_r$ are at most M, then, for the exponent p of Lemma 6.1, $\nabla u,\nabla^2 p,F_t,F_{tt}$ have norms bounded by $[C(1+T)pM]^C$ at radius $r/[C(1+T)pM]^C$.*

Proof. Use $\nabla\theta=F^{-T}D\theta_0$ and integrate $\dot\omega=\partial_1\theta$. Apply the lemma first to u, then to ∇p using $\Delta p=\partial_2\theta-\operatorname{Tr}((\nabla u)^2)$. Finally $F_t=(\nabla u)F$ and $F_{tt}=(\mathbf{e}_2\otimes\nabla\theta-\nabla^2 p)F$. □

6.2. **Expansion in the new angle.** Use a stream function $N^{-2}\Sigma$, a temperature increment $N^{-1}\Theta$, and an additional solenoidal Eulerian velocity $N^{-2}v$ independent of z. In the profile formulas below, v and its coefficients v_l inside expressions in parent labels stand for $v\circ X$ and $v_l\circ X$, respectively. Physical curls are taken before pullback. The total velocity increment in parent labels is $N^{-1}FV$, where

$$V=n^\perp\Sigma_z+N^{-1}\widetilde V,\qquad \widetilde V=D^\perp\Sigma+F^{-1}v. \tag{6.1}$$

Writing $C_X=F^{-1}F^{-T}$, its vorticity is

$$\Omega=|k|^2\Sigma_{zz}+N^{-1}\{2C_Xn\cdot D\Sigma_z+[D\cdot(C_Xn)]\Sigma_z\}+N^{-2}\{D\cdot(C_XD\Sigma)+\operatorname{curl}v\}. \tag{6.2}$$

Here the curl of v is its physical curl pulled back by X. Since $n\cdot n^\perp=0$, relative transport is $N^{-1}\mathcal T$, with $\mathcal T=V\cdot D+(n\cdot\widetilde V)\partial_z$. Thus the two scalar equations are

$$\begin{aligned}\Omega_t-k_1\Theta_z&=N^{-1}\big((F^{-1}\mathbf{e}_1)\cdot D\Theta-\mathcal T\Omega-V\cdot D\omega\big),\\ \Theta_t+V\cdot D\theta_0&=-N^{-1}\mathcal T\Theta.\end{aligned} \tag{6.3}$$

Use the expansions

$$\Sigma=\sum_{l\ge0}N^{-l}\sigma_l,\quad \Theta=\sum_{l\ge0}N^{-l}(\vartheta_l+N^{-1}h_l),\quad v=\sum_{l\ge0}N^{-l}v_l.$$

Write $\langle f\rangle=(2\pi)^{-1}\int_0^{2\pi}f\,\mathrm dz$ and $Qf=f-\langle f\rangle$. The profiles σ_l,ϑ_l have mean zero and h_l,v_l are independent of z. For a formal series f let $[f]_l$ denote the coefficient of N^{-l}, zero for $l<0$. Set $Y_l=\partial_z^{-1}Q[\Omega]_l$, using the primitive with zero mean.

Lemma 6.3 (Coefficient equations)**.** *The coefficients are determined in the following order.*

(a) *If* (B_1, A_1) *solves* (3.3) *with initial values* $(1, 0)$*, then* $\vartheta_0 = \epsilon\chi B_1\psi$*,* $\sigma_0 = \epsilon\chi A_1\Psi/|k|^2$*, and* $Y_0 = \epsilon\chi A_1\psi$*, where* $\Psi' = \psi$*.*

(b) *For* $l \geq 1$*, the pair* (ϑ_l, Y_l) *has zero initial data. With negative-index profiles set to zero, the vorticity formula gives*

$$|k|^2\partial_z\sigma_l = Y_l - 2C_X n \cdot D\sigma_{l-1} - [D \cdot (C_X n)]\sigma_{l-1} - \partial_z^{-1} D \cdot (C_X D\sigma_{l-2}).$$

Substituting this expression in the first of the following equations gives a closed system for the pair.

$$(\vartheta_l)_t = \beta\partial_z\sigma_l - Q[\widetilde{V} \cdot D\theta_0 + \mathcal{T}\Theta]_{l-1},$$
$$(Y_l)_t = k_1\vartheta_l + \partial_z^{-1}Q[(F^{-1}\mathbf{e}_1) \cdot D\Theta - \mathcal{T}\Omega - V \cdot D\omega]_{l-1}.$$

(c) *At order* N^{-2-l} *the means* v_l, h_l *solve* (4.5) *with zero initial temperature variation and zero displacement at both ends. Their forces in parent labels are*

$$-D \cdot \langle [FV \otimes V]_l \rangle, \qquad -D \cdot \langle [V\Theta]_l \rangle,$$

with divergence taken row by row in the first expression.

(d) *All* $\sigma_l(0)$ *and* $h_l(0)$ *vanish. The only initial temperature increment is* $N^{-1}\epsilon\chi\psi$*. The velocity at time zero changes by* $\sum_l N^{-2-l}v_l(0)$*.*

Proof. The principal terms of (6.3) are $(|k|^2\Sigma_{zz})_t = k_1\Theta_z$ and $\Theta_t = \beta\Sigma_z$. At order l, applying Q and the zero-mean primitive gives the stated forced system. A mean in its source has index at most $l-1$ since it enters (6.1) or Θ with an extra N^{-1}. Averaging the conservative momentum and temperature equations eliminates fast derivatives and gives the two fluxes by the Piola identity. Their linearized equations are inverted by (4.6). Induction in (6.2) at $t = 0$ now gives $\sigma_l(0) = 0$. □

6.3. **Amplitude and deformation estimates.** In $\|f\|_{s;W}$ divide each independent derivative by $W(t, y) = \epsilon B_1(t, y) > 0$ before taking its supremum on $\operatorname{supp}\chi$. The restriction is used only for functions supported there.

Lemma 6.4 (Weighted propagation)**.** *Under* (3.11)*, let* C_* *be the bound in Lemma 3.3. If* $[\beta/|k|^2]_s + [k_1]_s \leq (4C_*T)^{-1}$*, the solution with zero data of* $(\vartheta, Y)_t = (\beta Y/|k|^2, k_1\vartheta) + S$ *satisfies* $\|(\vartheta, Y)\|_{s;W} \leq 2C_*T\|S\|_{s;W}$*. The solution with initial data* $(1, 0)$ *satisfies* $\|(B_1, A_1)\|_{s;B_1} \leq 2C_*$ *and* $|\partial_i \log B_1| \leq 2C_*/s$*.*

Proof. The propagator from s_0 to t has norm at most $C_*B_1(t)/B_1(s_0)$, so Duhamel's formula gives the pointwise bound. Apply it to the independent derivatives and absorb the positive-order coefficient terms by Lemma 5.1. The derivative-free initial value contributes at most C_* to the estimate for unit data. □

In every oscillatory source at least one factor is oscillatory. It retains the weight W, and all other factors can be estimated in the ordinary norm whenever W is bounded. Thus the repeated use of Duhamel's formula costs only the single growth factor already present in the wave.

Lemma 6.5 (Change of labels)**.** *Use a common radius $s \le 1$ and suppose $K \ge 1$ bounds $T, s^{-1}, |n|, \|F\|_s$ and the norms of*

$$V, \qquad \widetilde{V}/R, \qquad \mu^{-1}\big(V - n^{\perp}\epsilon\chi A_1\psi/|k|^2\big).$$

If $\mu K^4 \le c_\sigma$, the change of labels has size $O(TK/N)$, and on a fixed fraction of the radius

$$D_+X_+ = F\big(I + c\psi'(z)n^{\perp}\otimes n\big) + O(\mu K^C), \qquad c = \epsilon\chi\int_0^t A_1/|k|^2\,\mathrm{d}s.$$

All leading quantities are evaluated at the original labels.

Proof. Write $X_+ = X(t, a+\Delta a)$, with the translated phases implicit. The relative label solves $\Delta a(0) = 0$ and

$$\partial_t\Delta a = N^{-1}V\big(t, a+\Delta a, (z_i + M_i\cdot\Delta a)_{i\le j}, z + Nn\cdot\Delta a\big).$$

The velocities in independent variables are therefore V/N, $M_i\cdot V/N$ and $n\cdot\widetilde{V}/N$. The sum of their sizes divided by s, integrated up to T, is at most $T\mu K(1+|n|)/s \le 2\mu K^4$. For a real periodic shift h of the independent variables, the chain rule gives

$$\|g\circ(I+h)\|_u \le \|g\|_s \quad \text{if} \quad u + \sum_k [h_k]_u \le s.$$

Indeed for a partition of m derivatives into q blocks of sizes $m_1,\dots,m_q$, $q!\prod m_k! \le m!$. After division by $(m!)^\sigma$ each term is bounded by the corresponding coefficient in ordinary power-series composition. An identity derivative costs u and picks one coordinate. The shift derivatives cost $\sum_k[h_k]_u$. The zeroth-order bound holds for every real shift. On $u = s/4$, integration along the segment between shifts of norm at most $s/8$ therefore gives a Lipschitz bound $C_\sigma s^{-1}\|g\|_s\sum_k\|h_k - \tilde h_k\|_u$, using Lemma 5.1 for ∇g on half the radius. The integrated independent velocity has sum of norms at most $2s\mu K^4$. Hence its integral equation is a contraction on the ball of shifts of norm $s/8$ for sufficiently small μK^4. Each component has at most its own integrated velocity norm there. This proves the composition bound uniformly in the number of phases and in the terminal time, also for the backward flow.

Finally $D_+\Delta a = D\Delta a + N\partial_z\Delta a\otimes n$. The old derivative costs $O(R/N)$ times the radius factors. Differentiating $N\Delta a = \int V$ and replacing the moving variables by the initial ones gives $N\partial_z\Delta a = c\psi' n^{\perp} + O(\mu K^C)$. The chain rule gives the claimed deformation. By (3.3), $c = (B(t) - B(0))/\beta$ when $\beta \ne 0$. □

6.4. **The uniform estimate for one wave.**

Proposition 6.6 (One periodic wave)**.** *Fix $m \ge 5$ and $\sigma > 1$. For $T \le T_0$, assume the following on every parent fiber for some $M \ge 2 + T_0 + T^{-1}$ and $0 < r \le 1$.*

(i) $\|F^{\pm1}\|_r + \|D\theta_0\|_r + \|\omega(0)\|_r \le M$.

(ii) *$Nn \in \mathbb{Z}^2$, $|n| + |n|^{-1} + \sup |k|^{-1} \le M$, ψ is a profile of Lemma 4.3 of degree at most M, and χ is an even product cutoff of Lemma 4.4 of width at least r/M. On its support $\epsilon(|k|B_1 + |A_1|) \le M$.*

(iii) *The parent satisfies (4.1), and the updates (3.6) satisfy it with upper bound $1/8 - 1/M$.*

Write $d = j + 3$ and set

$$H = 2 + j + M + \log(2 + NR/r), \qquad U_l = [CHd(l+3)/r]^{Cd}.$$

There are fixed constants $C \ge 2$, $C_0 \ge 4C + 4$, depending only on m, σ, for which the following holds whenever $J \ge C_0$ and

$$\mu U_{2J+4}^{C_0} \le \tfrac{1}{2}, \qquad N^{C_0}(H/r)^{C_0}\mu^{J/4} \le H^{-C_0}. \tag{6.4}$$

(a) *There is an exact flow on each fiber with initial temperature $\theta_0 + N^{-1}\epsilon\chi\psi$. It is odd on the untranslated fiber if the parent velocity and temperature are odd.*

(b) *At radius r/H^C, the material norms of $F_+^{\pm 1}$, $\nabla u_+, \nabla\theta_+, \nabla^2 p_+$, and the initial norms of $D_+\theta_+(0), \omega_+(0)$ are at most H^C.*

(c) *At the same radius the errors in the four updates (3.6) are at most μU_1^C.*

(d) *Let X^a be the flow of the profiles through order J, with the translation restoring their fiber mean acceleration. For $q = 0, 1, 2$,*

$$\|\partial_t^q (X_+ - X^a)\|_{r/H^C} \le [N(H/r)]^C \mu^{J/4}. \tag{6.5}$$

Writing b_θ for the fiber mean of $N^{-1}\epsilon\chi\psi$, one has

$$u_+(0) - u(0) = \sum_{l=0}^{J} N^{-2-l} v_l(0) - \tfrac{1}{2} T b_\theta \mathbf{e}_2 + e^u.$$

Only the old angles occur in the sum. The error satisfies

$$\|e^u\|_{r/H^C} \le N^C (H/r)^C \mu^{J/4}.$$

On the odd fiber $b_\theta = 0$.

Proof. On each parent fiber pass to the frame $h(t) := t\overline{u(0)} + t^2\overline{\theta_0}\mathbf{e}_2/2$. The fields $u(t, x + h) - h'$ and $\theta(t, x + h) - \overline{\theta_0}$ solve the same equations and have zero mean. Their flow $X - h$ has zero mean displacement from the identity. The shifts have zero physical derivative, so all four increments are invariant. We restore these means at the end.

Corollary 6.2 and Lemma 5.1 bound the parent coefficients and $|k|^{-2}$ by H^C at radius r/H^C. Coercivity follows from Lemma 4.1. Decreasing the radius by a fixed power of H gives the coefficient smallness in Lemmas 5.3 and 6.4. The cutoff and profile have bounded norms at radii cr/M and c/M. Write $s \ge r/H^C$ for the common radius, increasing the fixed exponent C to allow subsequent fixed reductions.

Coefficient bounds. Divide $(\sigma_l, \vartheta_l, Y_l)$ by R^l and (v_l, h_l) by R^{l+1}. In the resulting recurrence the expansion parameter is $\mu = R/N$ and every slow

derivative of a profile is D/R. For example the $N^{-2}\operatorname{curl} v$ term in (6.2) uses one slow derivative of a mean coefficient. The divergence of each angular mean flux uses one more slow derivative than the corresponding product.

Assign degree $2l+1$ to the normalized oscillatory profiles and $2l+2$ to the normalized means. In the two fluxes at order l the degrees of the factors add to at most $2l+2$. In the oscillatory source the potentially largest sum is the product of a mean v_a and a fast derivative at order b, where $a+b=l-1$. Its degree is $(2a+2)+(2b+1)=2l+1$. The other products have no larger degree. Linear lower-order inputs have strictly smaller degree. The algebraic recovery of σ_l from Y_l and the solve for each pair are counted as fixed operations at the same node. In a tree obtained by substitution the number of leaves plus the number of unary nodes is at most the root degree, since a unary node loses at least one degree and a binary node distributes its degree between its children. There is one fewer binary node than leaves. Thus a term of order l uses at most $C(l+1)$ operations. The choices of input orders and summands at each node give at most $[C(l+3)]^{C(l+1)}$ terms.

Allocate a loss $s/[C(l+1)]$ per operation, with total loss at most $s/2$ along every branch. By Lemma 5.1, a derivative D/R or ∂_z costs $[C(l+1)/s]^{\sigma}$. A product, angular mean or normalized fast primitive costs a fixed factor. For a nonzero oscillatory source at least one input carries $W=\epsilon B_1$ on the cutoff support. Estimate the remaining inputs in the ordinary norm, using $\sup_{\operatorname{supp}\chi} W\le M^2$. Lemma 6.4 then preserves the weight. For a mean solve use (4.6) and Lemma 5.3 in fiber L^2. Integrating in the fiber parameter also bounds the full L^2 norm for normalized measure in the independent variables. Lemma 5.2(c) gives the supremum estimate at the allocated loss. Thus the cost of any one operation is at most $[CH^C d(l+3)/r]^{Cd}$. Multiplying the costs and increasing the fixed constant in U_l gives

$$\|(\sigma_l,\vartheta_l,Y_l)\|_{s/2;W}\le R^l U_l^{l+1},\quad \|(v_l,h_l)\|_{s/2}\le R^{l+1}U_l^{l+1}. \tag{6.6}$$

The same estimates hold for one material time derivative. For $v_l\circ X=Fw_t$ use the C_tL^2 force estimate for w_{tt} in Lemma 5.3, since all mean forces are continuous in time. Differentiating the oscillatory amplitude equations or the algebraic recovery costs a fixed number of additional operations.

The residual and its mean. Truncate all four series at J. The resulting velocity is exactly solenoidal. The nonzero angular modes of the vorticity and scalar defects vanish through order J by Lemma 6.3. For the zero angular mode, average the momentum and temperature equations before taking curl. The only nonlinear sources left are the two flux divergences of that lemma. The lower-order mean momentum defect is a slow pressure gradient and has zero curl. Angular averaging commutes with physical curl on the mean since F is independent of z and an averaged ∂_z vanishes. This proves cancellation in both scalar defects, including the angular mean.

Each surviving product has order $J<l\le 2J+4$ in the normalized expansion, up to the fixed derivative that takes curl. Including the final product and derivatives in the tree gives $\mu^l U_l^{l+1}\le \mu^{(3l-1)/4}$ by (6.4). Summing from

$l = J+1$ gives $O(\mu^{3J/4})$, apart from fixed physical derivatives. For estimates after adding the new angle these derivatives use $R_+ = R+N|n| \le N(1+M)$. At a fixed fraction of the radius each therefore costs $N(H/r)^C$.

On every physical torus the scalar mean force for h_l is a divergence, so $\overline{h_l} = 0$ and $\overline{v_l}$ is constant in time. As $Y\circ X^{-1}$ is solenoidal, $\int(\nabla u)(Y\circ X^{-1})\,\mathrm{d}x = 0$. Integrating $Y_t = v_l\circ X + (\nabla u\circ X)Y$ between its zero endpoint values gives $\overline{v_l} = 0$. The stream-function velocity has zero physical mean as well. Write $b_\theta = \overline{N^{-1}\epsilon\chi\psi}$, where the overline is taken on each physical fiber. Translate the approximate path by $b_\theta t(t-T)\mathbf{e}_2/2$, its Eulerian fields by the same translation, and add the translation velocity to the velocity field. This fixes both endpoints and gives the prescribed mean acceleration. Fiber averaging commutes with independent derivatives, so $\|b_\theta\| \le N^{-1}H^C$ on this radius. The shift is constant on each physical fiber and vanishes on the odd fiber. Thus D_+ annihilates the shift and its time derivatives.

Choose p^a with zero mean by $\Delta p^a = \partial_2\theta^a - \mathrm{Tr}((\nabla u^a)^2)$. Let S be the scalar transport defect and E the momentum defect using this pressure and θ^a. Then $\operatorname{div} E = 0$, $\operatorname{curl} E$ is the vorticity defect, and, in the centered frame, $\overline{E} = (b_\theta - \overline{\theta^a})\mathbf{e}_2 = -\mathbf{e}_2\int_0^t \overline{S}\,\mathrm{d}s$. Indeed $\overline{\theta^a(0)} = b_\theta$ and $\mathrm{d}\overline{\theta^a}/\mathrm{d}t = \overline{S}$ by incompressibility. Before translating, apply Lemma 6.1 in parent coordinates, with R_+, to recover the gradient of the mean-free defect. Translating the reference path along with the field preserves this pullback. On each fiber, Poincaré's inequality recovers the mean-free value from its label gradient. It also applies to each independent derivative, which commutes with the fiber mean.

For the change from parent to approximate particle labels, (6.6) gives $V = n^\perp\partial_z\sigma_0 + O(\mu U_1^C)$ and $\|\widetilde{V}/R\| \le U_1^C$ on a fixed smaller radius. Indeed order one costs μU_1^C, and the sum for $l \ge 2$ is bounded by a geometric series using $U_l \le \mu^{-1/4}$. Lemma 6.5 applies with $K \le U_1^C$ by the first inequality of (6.4). Composition on a fixed smaller radius is bounded there by the majorant in that lemma. Transport of the scalar defect gives $\theta^a\circ X^a - \theta^a(0) = \int_0^t S\circ X^a\,\mathrm{d}s$. Together these estimates yield

$$|\mathcal{R}(X^a, p^a\circ X^a)|_{s',m} \le [N(H/r)]^C\mu^{J/2}, \qquad s' \ge r/H^C.$$

The fixed number of physical derivatives in the residual costs $[N(H/r)]^C$.

Principal fields and correction. One fast derivative cancels N^{-1} in the leading velocity and temperature, whereas a slow derivative leaves R/N. This yields the two gradient updates with error $O(\mu U_1^C)$, and Lemma 6.5 gives the update of F. To compare pressures, subtract from p^a the translated parent pressure and the scalar trial potential with expression $N^{-2}q\Psi(z)/|k|^2$ in parent coordinates. Its Hessian has principal part $q\psi'(z)\widehat{k}\otimes\widehat{k}$. Every other term has at least one slow derivative in place of a fast derivative and is $O(\mu U_1^C)$. In the pressure equation the term quadratic in the leading wave gradient vanishes since $(\widehat{k}^\perp\otimes\widehat{k})^2 = 0$. The cross term with ∇u gives the second term in q in (3.4). Thus the Laplacian of the difference of the three

potentials is $O(\mu U_1^C)$. Its gradient has zero curl, so Lemma 6.1 bounds the full Hessian difference, allowing the logarithmic cost in the exponent of U_1. The label changes have the same error. The inverse deformation is controlled by its adjugate in two dimensions.

The error in the budget (4.1) is at most $(T_0^2+T_0)O(\mu U_1^C)$. Since $M \le H$, $T_0^2+T_0 \le 2H^2$ and $\mu U_1^C \le (2U_{2J+4}^{C_0-C})^{-1}$, it is smaller than the margin $1/M$. Lemma 4.1 therefore applies to X^a. The weighted amplitude estimate, the pointwise amplitude bounds on the cutoff support, and the cutoff and profile estimates give H^C bounds for the leading fields at the common reduced radius. The errors there are bounded by $\mu U_1^C \le 1$. Together these bounds control $F^a, (F^a)^{-1}, F_t^a = (\nabla u^a)F^a$ and the blocks in (5.2) by H^C at radius r/H^C. The same polynomial bounds control the reciprocal coercivity constant and the inverse of $(F^a)^T F^a$. Reducing the radius by a further power of H gives the coefficient smallness in Lemma 5.3.

The time derivatives of the two upper blocks are controlled since

$$F_{tt}^a = \mathbf{e}_2 \otimes D_+\theta_+(0) - \Pi^a F^a + D_+\mathcal{R}(X^a, p^a \circ X^a).$$

The last term is bounded by $[N(H/r)]^C \mu^{J/2}$ and hence by one under (6.4). In every div–curl estimate one can take $p \le CH$ since $R_+ \le N(1+M)$ and the input radius is r/H^C. Thus each of the finitely many radius reductions outside the trees costs a fixed power of H.

In the centered frame the physical mean of $X^a - a$ equals $b_\theta t(t-T)\mathbf{e}_2/2$, with the corresponding expressions for its time derivatives. The material pressure and pulled-back pressure gradient have zero fiber mean because X^a preserves volume. Their physical label derivatives are $(F^a)^T(\nabla p^a \circ X^a)$ and $\Pi^a F^a$, respectively. Apply the fiber Poincaré inequality also to each independent derivative. The bounds for F^a, its two time derivatives and $D_+\theta_+(0)$ therefore give all the zeroth-order terms of the Newton norm. On a fixed smaller radius each of the m physical derivatives costs at most $R_+(H/r)^C$. Thus the parameter P of Proposition 5.4 is at most $[N(H/r)]^C$. The corrected gradients and pressure Hessian are recovered from $F_t F^{-1}$, $F^{-T}D_+\theta_+(0)$ and $D_+(F^{-T}D_+p)F^{-1}$ with p material. Their differences from the corresponding approximate quantities cost a fixed polynomial in P times the Newton correction, using $m \ge 5$ and the transported scalar defect above. Increasing C_0 in (6.4) absorbs this polynomial and makes each difference $O(\mu)$. This proves (a)–(c).

The Newton bound in $C_t H^m$ gives (6.5) by Sobolev embedding on each fiber and for each independent derivative. On half the radius one more physical derivative costs $R_+(H/r)^C$. All oscillatory velocity coefficients vanish at time zero by Lemma 6.3, leaving the stated means and $e^u = \partial_t(X_+ - X^a)(0)$. □

7. The periodic cascade

We prove the return and earlier-time estimates in Proposition 2.2(a)–(b). Take the base shear of Lemma 4.5 and the gradient magnitudes in (2.2). For

the successive wave frequencies take

$$\log N_* = (\log G_0)^2, \qquad N_i = \lceil N_*^{(6/5)^i} \rceil \quad (i \geq 1). \tag{7.1}$$

Scale time at the jth return by

$$\gamma_j(t) := \delta_j \sqrt{G_j \, \nu_j \cdot \mathbf{e}_2}\,(t - t_j), \qquad I_j := [t_j, T_j], \qquad \gamma_j(T_j) = 3. \tag{7.2}$$

Proposition 3.5 places the next return near $\gamma_j = 2$.

7.1. **Induction hypotheses.** Stage j consists of the base shear and j periodic waves. Its flow and deformation are X_j and F_j. The independent phases z_i restrict on a translated fiber to $M_i \cdot a + \zeta_i$, where $M_i = N_i n_i \in \mathbb{Z}^2$. The physical solution has $\zeta = 0$. At time zero the $(j+1)$st stage adds the temperature

$$N_{j+1}^{-1} \epsilon_{j+1}\, \chi_{j+1}(a, z_1, \ldots, z_j)\, \psi_{j+1}(z_{j+1}). \tag{7.3}$$

Here χ_{j+1} is even, equals one near zero and is supported where the earlier cutoffs equal one. The odd profile ψ_{j+1} is given by Lemma 4.3. The material covector n_{j+1} is chosen so that $F_j(t_j, 0)^{-T} n_{j+1}$ is nearly horizontal. Set $R_j = 1 + \sum_{i \leq j} |M_i|$ and use expansion order $J_* + i$ at stage i.

For the following suprema take all times in $[0, T_j]$, particles and phase translations. Write $K_j = 1 + \sup(|F_j| + |F_j^{-1}|)$ and $W_j = 1 + \sup(|\nabla u_j| + |\nabla^2 p_j|^{1/2})$. The constants and polynomial bounds in the following hypotheses are uniform in j.

(H1) *Central trajectory.* At the origin the wave amplitudes on parent $i-1$ satisfy $A_i(t_i) = 0$, $(B_i k_i)(t_i) = -G_i \nu_i$, $1.9 \leq \gamma_{i-1}(t_i) \leq 2.1$ and $2/5 \leq \delta_i / \delta_{i-1} \leq 3/5$ for $1 \leq i \leq j$. The invariants $\beta_i = -\nabla \theta_{i-1}(0,0) \cdot n_i^{\perp}$ satisfy $G_{i-1}/2 \leq \beta_i \leq 2G_{i-1}$ and $c \leq |n_i| \leq C\beta_i$. On I_i the parent gradients at the origin satisfy $|\nabla u_{i-1}| \leq C\sqrt{G_{i-1}}$ and $|\nabla \theta_{i-1}| \leq C G_{i-1}$.

(H2) *Uniform bounds.* Stage j is exact on $[0, T_{j-1}]$, with $T_{-1} = T_0$, and is odd on the zero translation. On $[0, T_j]$,

$$\sup |\nabla \theta_j| \leq C G_j, \qquad b_j \leq 1/16 + \sum_{i=0}^{j-1} 2^{-i-6}.$$

Each of K_j and W_j is bounded by a fixed power of G_j. At a radius ρ_j whose reciprocal is polynomial in G_j, the independent-variable norms of $F_j^{\pm 1}, \nabla u_j, \nabla \theta_j, \nabla^2 p_j$ in material coordinates and of $\nabla \theta_j(0), \omega_j(0)$ are also polynomial in G_j.

(H3) *Approximation error.* For $1 \leq i \leq j$, on $[0, T_{i-1}]$ the differences between $F_i, \nabla u_i, \nabla \theta_i, \nabla^2 p_i$ and their respective leading expressions in (3.6) have norm at most $N_i^{-1/8}$ at radius ρ_i.

The base shear satisfies (H1)–(H3) with $j = 0$. Shrink the base radius once so that the positive-order parts of its stored Gevrey norms are at most one, by Lemma 5.1. We impose the same normalization after each step.

Suppose these hypotheses hold through stage j. We also carry the following estimate on the support of each wave cutoff. For the wave added to stage j the required bound is

$$\sup_{t\le t_j+G_j^{-1/2}} |Bk| \le CG_{j+1}e^{-c/\sqrt{\delta_j}}, \qquad \sup_{t\le T_j} |Bk| \le CG_{j+1}. \tag{7.4}$$

Assuming the corresponding bounds at earlier stages, we construct (7.3), obtain (7.4), and verify (H1)–(H3) for stage $j+1$ with the same constants. All choices of constants and of the initial expansion order J_* are uniform in j.

7.2. **The next return.** Write $N = N_{j+1}$ and $R = R_j$. Three comparisons govern the induction. For fixed C the scale definitions give

$$\begin{gathered}(j+3)^C \log G_{j+1} = o(\log N), \qquad \log N = o(\delta_j^{-1/2}),\\ (G_{j-1}/G_j)^{1/2} + N_j^{-1/8} \le \delta_j^{20} \qquad (j \ge 1).\end{gathered} \tag{7.5}$$

Indeed the exponential bases in the first two comparisons are $11/10 < 6/5 < \sqrt{5/3}$. For the third use $\delta_j \ge (2/5)^j\delta_0$ and $G_j/G_{j-1} = G_{j-1}^{1/10}$. The little-oh bounds hold uniformly as $\delta_0 \downarrow 0$, and also as $j \to \infty$ with δ_0 fixed. By the covector bounds in (H1), $R_j = N_j^{1+o(1)}$ for $j \ge 1$, while $R_0 = 1$. In particular $R/N \le N^{-1/6+o(1)}$.

Round $NF_j(t_j,0)^T\mathbf{e}_1$ to the nearest lattice point M_{j+1} and set $n = M_{j+1}/N$. By (7.5) and the polynomial bound on K_j, the rounding error satisfies $|k(t_j,0) - \mathbf{e}_1| \le CK_j/N \le \delta_j^{10}$. For $j = 0$ the central shear is exact. For $j \ge 1$ apply Lemma 3.4 to wave j with $\eta \lesssim (G_{j-1}/G_j)^{1/2}$. The normalized error from the shear on I_j is at most $C\eta/\delta_j^2 + N_j^{-1/8} \le \delta_j^{10}$. The history before t_j is controlled by (4.1), since

$$G_j^{-1/2}\left(|\nabla u_j(0,0)| + \int_0^{T_j} \lambda_{\max}^+(\nabla^2 p_j(t,0))\,\mathrm{d}t\right) \le \frac{2b_j}{T_0\sqrt{G_j}} \le \delta_j.$$

Thus Proposition 3.5 gives the next return t_{j+1}, with $\gamma_j(t_{j+1}) = 2 + o(1)$ and $\delta_{j+1}/\delta_j = 1/2 + o(1)$. Since $\gamma_j(t_{j+1}) \le 2.1$, the return leaves at least $0.3|I_j|$ in the current window, while $|I_{j+1}|/|I_j| = O((G_j/G_{j+1})^{1/2})$. Hence $I_{j+1} \subset I_j$. For the unit incident solution write $d_1 = |k|B_1$ and take $\epsilon_{j+1} = G_{j+1}/d_1(t_{j+1},0)$. This fixes $Bk = -G_{j+1}\nu_{j+1}$ at the return. At the origin $\epsilon_{j+1}d_1 \lesssim G_{j+1}$ throughout the window, with the extra factor $e^{-c\delta_j^{-1/2}}$ before $t_j + G_j^{-1/2}$.

Lemma 7.1 (Incident covectors)**.** *At the next stage, $G_j/2 \le \beta_{j+1} \le 2G_j$ and $c \le |n_{j+1}| \le C\beta_{j+1}$.*

Proof. For $n_1 = \mathbf{e}_1$ we have $\beta_1 = G_0\nu_0\cdot\mathbf{e}_2$. For $j \ge 1$, expand $F_{j-1}(t_j,0)^T\mathbf{e}_1$ in the basis $n_j, \nabla\theta_{j-1}(0,0)$. The determinant of this basis is $-\beta_j$. Since $\det F_{j-1} = 1$, the two numerators in the basis expansion are $-\partial_2\theta_{j-1}(t_j,0)$

and $k_{j,2}(t_j,0)$ when the denominator is β_j. The deformation update (3.6) and rounding now give

$$n_{j+1} = \big(G_j\nu_{j,2} - \partial_2\theta_{j-1}(t_j,0)\big)\frac{n_j}{\beta_j} + k_{j,2}(t_j,0)\frac{\nabla\theta_j(0,0)}{\beta_j} + e_j,$$

where $|e_j| \lesssim K_{j-1}N_j^{-1/8} + N_{j+1}^{-1}$. Here we used $-B_j(t_j)k_{j,2}(t_j,0) = G_j\nu_{j,2}$ and $\nabla\theta_j(0,0) = \nabla\theta_{j-1}(0,0) + B_j(0)n_j$. Taking the negative determinant with $\nabla\theta_j(0,0)$ yields

$$\beta_{j+1} = G_j\nu_{j,2} - \partial_2\theta_{j-1}(t_j,0) + O(G_0|e_j|),$$

since the early bounds give $|\nabla\theta_j(0,0)| \le 3G_0$. The old gradient is $O(G_{j-1})$, so the claimed bounds for β_{j+1} and the lower bound for $|n_{j+1}|$ follow. In the recurrence for n_{j+1}/β_{j+1} the multiplier of n_j/β_j is $1 + O(G_0|e_j|/G_j)$. The additive error is at most

$$C\frac{G_0}{\delta_{j-1}G_{j-1}G_j} + C\frac{|e_j|}{G_j},$$

because $|k_{j,2}(t_j,0)| \lesssim \delta_{j-1}^{-1}$. Both errors are summable by (7.5), which proves the upper bound. □

7.3. **Localization and the pressure bound.** Let $\sigma_j = 2^{-j-9}T_0^{-2}$. Take the product cutoff of Lemma 4.4 in $a, z_1, \dots, z_j$ with width

$$\ell = \frac{\rho_j \min(1,\sigma_j)}{(j+2)G_{j+1}^C}.$$

Reduce each new radius below one fourth of its cutoff width. The supports are then nested inside the regions where all earlier cutoffs are one. Use the stored parent fields at ρ_j. Lemma 5.1 bounds $k = F_j^{-T}n$, $|k|^{-2}$ and the coefficients in (3.3) on a radius ρ_j/G_{j+1}^C, using the real lower bound $|k| \ge |n|/K_j$. Lemma 6.4 then bounds the independent derivatives of $\log B_1$ and the parent coefficients by $G_{j+1}^{C'}/\rho_j$. The relative derivative bound for d_1 first gives $d_1(t,\cdot)/d_1(t,0) \in [1/2, 2]$ throughout the support. The central amplitude estimate then gives $\rho_j \sup|\partial_{a,z}q^\circ| \le G_{j+1}^C$ on this support, where q° is the pressure increment for $(\epsilon_{j+1}A_1, \epsilon_{j+1}B_1)$ before cutoff. Taking C larger gives $|q^\circ(t,\cdot) - q^\circ(t,0)| \le \sigma_j$, and the local invariant $\beta = -\nabla\theta_j(0)\cdot n^\perp$ satisfies $\beta_{j+1}/2 \le \beta \le 2\beta_{j+1}$. This gives both amplitude bounds in (7.4).

Write $d = |k|B$ for the cut amplitude. On its positive set $|k| \gtrsim 1$ by the invariant and $|k| \lesssim K_j\beta_{j+1}$ by Lemma 7.1. From (3.10) the logarithmic slope $h = d'/d$ solves $h' + h^2 \le CW_j^2$ and $|h(0)| \le W_j$. Lemma 3.3 bounds its negative part by $C(|\nabla u_j(0)| + T_j \sup\lambda_{\max}^+) \lesssim W_j$, using both $\sup\lambda_{\max}^+ \le W_j^2$ and $T_0^2 \sup\lambda_{\max}^+ \le 1/8$. Consequently

$$|A| \lesssim dK_jW_j, \qquad |q| \lesssim dK_jW_j^2, \qquad |c|\,|n|^2 \lesssim G_jG_{j+1}. \tag{7.6}$$

The first inequality follows from $A = |k|^2(d/|k|)'/\beta$ and the last from $\beta c = B(t) - B(0)$. The same amplitude estimates hold without the cutoff on its support.

Use the Fejér profile of Lemma 4.3 of degree $D_j = \lceil G_{j+1}^C/\sigma_j \rceil$. Before $t_j + G_j^{-1/2}$ the gain bounds $|q^\circ| \le \sigma_j$. Afterwards $q^\circ(t,0) \le 0$ by Proposition 3.5, so $q = \chi q^\circ \le \sigma_j$. Since $-1/D_j \le \psi' \le 1$, the positive part of $\psi' q$ is at most $\sigma_j + |q|/D_j \le 2\sigma_j$. The leading pressure update therefore adds at most $2T_0^2\sigma_j$ to (4.1). Its leading velocity gradient at time zero equals the parent's, since $A(0) = 0$. This verifies the pressure margin needed for Proposition 6.6 before constructing the new stage.

7.4. **Closing the induction.** Apply Proposition 6.6 with $m = 5$, $r = \rho_j$ and $T = T_j$. On every fiber $|F_j^{-T} n| \ge |n|/\|F_j\|_\infty \ge c/K_j$ by Lemma 7.1. The preceding bounds, r/ℓ, the degree D_j and the inverse pressure margin admit $M \le G_{j+1}^C$. Hence $H \le G_{j+1}^C$ and, for $J = J_* + j + 1$,

$$\log U_{2J+4} \lesssim (j+3)(\log G_{j+1} + \log(J+3)) = o(\log N).$$

Since $\mu = R/N \le N^{-1/6+o(1)}$, the first inequality of (6.4) follows. To check the second, take δ_0 sufficiently small that $\mu \le N^{-1/7}$ uniformly. The prefactors $N^{C_0}(H/r)^{C_0}H^{C_0}$ are bounded by N^{C_1} for a fixed C_1. Choosing $J_* > 28(C_1+3)$ gives

$$N^{C_1}\mu^{J/4} \le N^{-3-(j+1)/28}.$$

The four approximation errors are therefore at most $\mu U_1^C \le N^{-1/6+o(1)} \le N^{-1/8}$. Since $A(0) = 0$, the leading velocity gradient at time zero equals the parent's. The exact budget thus increases by at most $2T_0^2\sigma_j + (T_0^2 + T_0)N^{-1/8} \le 2^{-j-6}$. This pressure bound holds throughout the construction interval $[0, T_j]$ for stage $j+1$, including the part beyond T_{j+1}.

From (3.6) and (7.6),

$$K_{j+1} \le (1 + CG_jG_{j+1})K_j + 1, \qquad W_{j+1} \le C(1+G_{j+1})(1+K_j)W_j + 1,$$

and $\sup|\nabla\theta_{j+1}| \le \sup|\nabla\theta_j| + CG_{j+1} + 1$. Taking logarithms in the first recurrence and using $\sum_{i=0}^{j} \log G_i \le 11 \log G_j$ gives $\log K_j \lesssim \log G_j$. Substitution in the second recurrence and the same summation give $\log W_j \lesssim \log G_j$. In particular the pressure Hessian is polynomially bounded in G_j at each stage. To obtain the same kind of bound for the full Gevrey norms, reduce the positive-order part to at most one by shrinking the radius below r/H^C and $\ell/4$, using Lemma 5.1(b). Take ρ_{j+1} to be the smaller of $\ell/4$ and $c\rho_j H^{-2C}$, increasing the fixed C if necessary. The positive-order norms are then at most one. Thus the norms used in choosing M at the next step depend polynomially on the real bounds just proved, with an exponent independent of the radius. Write $L_j = \log(1/\rho_j)$ and $g_j = \log G_j$. Since $\rho_j/\ell \le [2^j G_{j+1}(1+T_0^2)]^C$ and $\log\log(2+NR) = O(1 + j + \log\log G_0)$, the

stated choice gives

$$L_{j+1} \le L_j + C\big(g_{j+1} + j + 1 + \log(1 + L_j)\big) \le \tfrac{21}{20} L_j + C' g_{j+1}.$$

Here $C \log(1+x) \le x/20 + C''$ for $x \ge 0$. Since $g_{j+1} = 11 g_j/10$, division by g_{j+1} gives $L_{j+1}/g_{j+1} \le (21/22) L_j/g_j + C'$. Fix the exponents for K_j and W_j from their real recurrences first. With those exponents fixed, a bound for L_j/g_j larger than both L_0/g_0 and $22C'$ is preserved at every step. This gives a uniform exponent for the reciprocal radius.

On $I_{j+1} \subset I_j$, Lemma 3.4 bounds the preceding wave contributions by $C_0\sqrt{G_j}$ and $4G_j$. The old contributions are smaller by factors $(G_{j-1}/G_j)^{1/2}$ and G_{j-1}/G_j, respectively, so the constants in (H1) can be fixed once. At t_{j+1} the returning wave on the parent has temperature gradient $-G_{j+1}\nu_{j+1}$ and zero velocity gradient. Adding the parent gradients and the error in (H3) gives (2.4) for the exact solution. The even cutoff and odd profile preserve oddness on the zero translation. This closes the induction and proves Proposition 2.2(a).

7.5. **Effect on the preceding interval.**

Proof of Proposition 2.2(b). On $[0, t_j]$, (7.4) and (7.6) bound the two leading gradient increments in (3.6) by $G_{j+1}^{C} e^{-c/\sqrt{\delta_j}}$ on the support of the new wave. They vanish off this support. The right-hand side is at most $N_{j+1}^{-1/8}$ by (7.5). Comparison at equal particle labels adds at most another $N_{j+1}^{-1/8}$ by (H3). Formula (7.1) gives $2N_{j+1}^{-1/8} \le 2^{-j-1}$ for every j once δ_0 is sufficiently small. This is (2.5). □

8. Smoothness of the initial data

We prove Proposition 2.2(c) using the initial velocity selected by the correction with prescribed temperature and endpoints. Estimate (6.6) bounds its first angular mean increment by $N_{j+1}^{-2} R_j N_{j+1}^{o(1)}$. Estimating its C^r norm directly introduces a further factor $R_j^r N_{j+1}^{o(1)}$. We estimate its dependence on a fixed number of recent angles before taking derivatives on the physical torus.

8.1. **The transverse Green function.**

Lemma 8.1. *Assume (3.11). Use the solution with $(B_1, A_1)(0) = (1, 0)$ and put $d := |k| B_1$ and $I(t) := \int_0^t d(s)^{-2}\,\mathrm{d}s$. For $(|k|^2 y')' - \beta k_1 y = f$ with $y(0) = y(T) = 0$, the Green function is*

$$G(t,s) = -B_1(t) B_1(s) \frac{I(\min(s,t))[I(T) - I(\max(s,t))]}{I(T)}.$$

In particular

$$|k(0)|^2 y'(0) = -\int_0^T B_1(s) \frac{I(T) - I(s)}{I(T)} f(s)\,\mathrm{d}s.$$

The kernel also satisfies

$$|G(t,s)| + |\partial_t G(t,s)| \le C\,\frac{d(\min\{s,t\})}{d(\max\{s,t\})}.$$

On the diagonal take either one-sided derivative. The constant is polynomial in $1+T+T^{-1}$, $\sup(|k|+|k|^{-1})$, $\sup|d'/d|$ *and* $\sup|B_1'/B_1|$.

Proof. Putting $y = B_1 u$ gives $(d^2u')' = B_1 f$. Two integrations with the endpoint conditions give the formulas. Lemma 3.3 yields $I(T) - I(s) \le 4T/d(s)^2$. Integrating near zero gives $I(T)^{-1} \lesssim d(0)^2(T^{-1} + \sup|d'/d|)$. Together with $I(t)/I(T) \le 1$ these imply the bound for G. When $t < s$, differentiating $I(t)$ costs the additional bounded factor $d(0)^2/d(t)^2$ after extracting $d(t)/d(s)$. When $t > s$, differentiating the tail gives at most $B_1(s)B_1(t)d(t)^{-2}$. All other terms are bounded by $|B_1'(t)/B_1(t)|\,|G(t,s)|$. □

Let $d_h(t,\xi)$ be the unit-data amplitude for wave h at the older labels ξ. On $\operatorname{supp}\chi_h$ use the time weight

$$w_h(t) = e^{-c/\sqrt{\delta_{h-1}}} + \min\{1, d_h(t,0)/d_h(t_h,0)\}.$$

The early estimate gives $w_h(0) \le 2e^{-c/\sqrt{\delta_{h-1}}}$ for a common $c > 0$. Comparison and localization yield

$$w_h(s) \lesssim w_h(t) \quad (s \le t), \qquad \frac{d_h(t,\xi)}{d_h(s,\xi)} w_h(s) \lesssim w_h(t) \quad (t \le s).$$

For $k > h$ we also have $w_k \lesssim w_h$ on the common window. Use the early bound for wave k up to t_h and $d_h(t,0) \ge d_h(t_h,0)/2$ thereafter.

For a forcing bounded by w_h, split the integral at $s = t$. Use the first weight inequality for $s \le t$ and the second for $s \ge t$. Lemma 8.1 gives $|y| + |y'| \lesssim w_h$ on the same support. The equation gives the bound for y'' with an additional factor polynomial in the coefficients $|k|^{-2}$, $(|k|^2)'$ and βk_1. For the label derivatives write $p_0 = |k|^2$, $b_0 = \beta k_1$, and $L = p_0\partial_t^2 + p_0'\partial_t - b_0$. Each $\partial^\alpha y$ has zero endpoint values and solves

$$L\partial^\alpha y = \partial^\alpha f - \sum_{0<\eta\le\alpha} \binom{\alpha}{\eta} \big((\partial^\eta p_0)\partial^{\alpha-\eta} y'' + (\partial^\eta p_0')\partial^{\alpha-\eta} y' - (\partial^\eta b_0)\partial^{\alpha-\eta} y\big).$$

Use the scalar inverse bound for the triple (y, y', y'') with the time weight w_h. By Lemma 5.1(b), the radius can be reduced so the sum of the three positive-order coefficient norms is smaller than half the reciprocal of this bound. Summing with the Gevrey weights and absorbing the commutators gives the same time weight for all three fields.

8.2. **One angular average.** Write $z = z_i$, $n = n_i$, $l = n^\perp$, and let D and F, Π refer to the parent. With $D_+ = D + N_i n\partial_z$, the leading matrices are

$$F_* = F(I + c\psi' l \otimes n), \qquad \Pi_* = \Pi + q\psi'\widehat{k} \otimes \widehat{k}.$$

A leading solenoidal displacement has the form $V = V_0 + lw$, with $D\cdot V_0 = 0$, V_0 independent of z, and $\langle w\rangle = 0$, where brackets denote the z-average.

Lemma 8.2. *Set $W = w + c\psi'(n\cdot V_0)$. The averaged Dirichlet form (4.3) is the parent form on V_0 plus the transverse form*

$$\int_0^T \left\langle |k|^2 W_t^2 + \beta k_1 W^2 \right\rangle \mathrm{d}t$$

along each trajectory.

Proof. Since $Fl = k^\perp$, we have $F_*V = FV_0 + k^\perp W$ with $\langle W\rangle = 0$. All cross terms average to zero, as does the pressure increment because $\widehat{k}\cdot F_*V = (n\cdot V_0)/|k|$ is independent of z. Integrate by parts using $W(0) = W(T) = 0$ and $(k^\perp)_{tt} + \Pi k^\perp = -\beta \mathbf{e}_2$. □

For a material force $E = F_i^T f$, substitution of $w = W - c\psi'(n\cdot V_0)$ gives the forces in the two forms:

$$\langle E\rangle - cn\langle \psi'(l\cdot E)\rangle, \qquad l\cdot E - \langle l\cdot E\rangle.$$

8.3. **A finite number of recent angles.** Fix a depth m and consider stages $j-m < i \le j$, with $N = N_{j+1}$. For fixed derivative order, depth and target accuracy we write C_j for bounds with $\log(2+C_j) \le C(j+3)^C \log G_{j+1}$. Thus $C_j = N^{o(1)}$ by (7.5). Finite products, a fixed number of independent derivatives, and the embedding of Lemma 5.2 at radius $\rho \ge C_j^{-1}$ have this cost. Constants denoted by C_j may increase from one estimate to the next within this class. The bounds (7.6) and the slope estimate for d_i bound the constant in Lemma 8.1 by C_j on each cutoff support. Let P_h denote normalized averaging in z_h. For a field f at stage i, its h-component and its part independent of the retained angles are

$$f_h = (I - P_h)P_{h+1}\cdots P_i f, \qquad f_< = P_{j-m+1}\cdots P_i f.$$

An empty product is the identity. Hence $f = f_< + \sum_{h=j-m+1}^{i} f_h$.

For a positive time weight w, write

$$\|f\|_{\rho;w} = \sum_{n\ge 0} \frac{\rho^n}{(n!)^\sigma} \max_{|\alpha|=n} \sup_{t,a,z} \frac{|\partial_{a,z}^\alpha f(t,a,z)|}{w(t)}.$$

Write $D_i = \nabla_a + \sum_{h\le i} M_h \partial_{z_h}$. We say that E has the *component bounds* at level i if, for every $b > 0$ and all sufficiently large j, on some radius $\rho \ge C_j^{-1}$ it decomposes as $E = f + e$ with

$$\begin{gathered} \|e\|_\rho \le N^{-b}, \qquad \|f_<\|_\rho \le C_j, \\ \operatorname{supp} f_h \subset \operatorname{supp} \chi_h, \qquad \|f_h\|_{\rho;w_h} \le C_j \quad (j-m < h \le i). \end{gathered} \tag{8.1}$$

All representatives are continuous in time with their indicated spatial derivatives. For a varying endpoint T the radius and the bounds are uniform over the stated interval of T. They may depend on the fixed depth and accuracy. A fixed number of products, derivatives on smaller radii or radius reductions

keep ρ^{-1} and the constants within the stated C_j bound. An unweighted error of every inverse power of N retains that property after multiplication by any fixed power of N.

Proposition 8.3 (Endpoint estimate for the last m phases)**.** *Let $Y = F_i V$ solve (4.2) on stage i with force $F_i^{-T}E$ and zero endpoint values, so $D_i \cdot V = 0$. If E is continuous in time and has the component bounds at level i, then V, V_t, V_{tt} have the same bounds. The estimates are uniform for $T \in [t_i, T_{i-1}]$. In particular the h-component of $V_t(0)$ is bounded on the same smaller radius by $2C_j e^{-c/\sqrt{\delta_{h-1}}} + N^{-b}$ for every fixed b.*

A field supported in $\operatorname{supp}\chi_i$ and bounded by C_j in the w_i norm can be multiplied by any field at level i of unweighted norm C_j. The average of the product in z_i has the component bounds at level $i-1$. Indeed, for $h < i$ the projections defining the h-component integrate only variables $z_h, \dots, z_i$. Membership in $\operatorname{supp}\chi_h$ depends only on the unaffected variables. Nesting of the cutoffs therefore preserves this support under the projections, and the weighted norm bound follows from $w_i \lesssim w_h$. The remaining, older component has unweighted norm $O(C_j)$. Consequently the families with these bounds form an algebra. They are closed under independent differentiation on a smaller radius, D/R_{i-1} when D is the parent derivative, and integration from zero since $w_h(s) \lesssim w_h(t)$ for $s \le t$. For a pure h-component the primitive in z_h of mean zero costs at most 2π and preserves its support.

Lemma 8.4 (Deformation of a periodic wave)**.** *Write $R = R_{i-1}$, $\mu = R/N_i$ and $z = z_i$. Suppose the parent matrices F, F_t, F^{-1} have the component bounds and, at each fixed order l, so do the profiles $R^{-l}\sigma_l$ and $R^{-l-1}v_l$ of wave i. Then*

$$\frac{F_i - F_*}{\mu}, \qquad \frac{F_{i,t} - F_{*,t}}{\mu}$$

have the component bounds on the construction interval $[0, T_{i-1}]$.

Proof. First work with the approximate flow X^a before its Newton correction. In parent labels its relative velocity is $N_i^{-1}V$. Set $V_f = n^\perp \partial_z \sigma_0$ and write $V = V_f + \mu H$. By (6.1) the coefficient of μ^l in H is

$$n^\perp \partial_z \frac{\sigma_{l+1}}{R^{l+1}} + \frac{D^\perp}{R}\frac{\sigma_l}{R^l} + F^{-1}\frac{v_l}{R^{l+1}},$$

where $\sigma_{J_i+1} = 0$ and $J_i = J_* + i$ is the truncation order. Each fixed coefficient has the component bounds. With $U = U_{2J_i+4}$ at this stage, (6.6) bounds the tail after order p, on half the radius, by $C_j \sum_{l>p} \mu^l U^{l+2} \le 2C_j \mu^{p+1} U^{p+3}$. For fixed p, $U^{p+3} \le C_j$ by Section 7.4. Thus H has the component bounds to arbitrary accuracy.

Let $S_t g$ be the value of a field g at the relative particle labels at time t. The velocity of these labels in the independent variables is $\mu b(t)$, where

$$b = \big(V/R, (M_h \cdot V/R)_{h<i}, n \cdot H\big), \qquad L_s g = b(s) \cdot \nabla_{a, z_1, \dots, z_i} g.$$

Indeed $n \cdot V_f = 0$. The sum of the component norms of b is at most C_j, since $\sum_{h<i} |M_h|/R \le 1$. Lemma 6.5 bounds composition with this flow on a fixed fraction of the radius. The smallness it needs holds at every radius $r \ge C_j^{-1}$ used below since $\mu C_j \to 0$.

Freeze g at time t. Integration along the relative flow gives $S_t g = g + \mu \int_0^t S_s L_s g \, \mathrm{d}s$. Repeated substitution gives, at order $k < p$, the term

$$\mu^k \int_{0<s_k<\cdots<s_1<t} L_{s_k} \cdots L_{s_1} g \, \mathrm{d}s_k \cdots \mathrm{d}s_1. \tag{8.2}$$

The zeroth term is g. The remainder has the same integrand at $k = p$ with S_{s_p} acting on it. For fixed p, allot $r/(2p)$ to each derivative and use the composition bound on a quarter of the remaining radius. Its unweighted norm is at most $C_j \mu^p \|g\|_r$. The reciprocal of the final radius is at most C_j. In every finite term the derivatives and products are evaluated at the original labels. Since $w_h(s_k) \lesssim w_h(t)$, these terms inherit the component bounds of g. By taking p large enough before dividing, $(S_t - I)g/\mu$ has the component bounds. For a time-dependent g with g_t in the same class, the derivative of this quotient is

$$(S_t - I)g_t/\mu + S_t(L_t g).$$

Scale the relative displacement by setting $Y := N_i \Delta a$. Its leading term is $Y_0 := \int_0^t V_f(s) \, \mathrm{d}s = c\psi n^\perp$. Then $Y_t = S_t V(t)$ before the translation restoring the mean. For $Z = (Y - Y_0)/\mu$ we have

$$Z(0) = 0, \qquad Z_t = S_t H + (S_t - I)V_f/\mu.$$

This gives the component bounds for Z, Z_t, Y, Y_t. Use $\bar{D} = D/R$. The exact chain rule for the approximate flow is $F^a = (S_t F)(I + \partial_z Y \otimes n + \mu \bar{D} Y)$. Subtracting $F_* = F(I + \partial_z Y_0 \otimes n)$ gives

$$\frac{F^a - F_*}{\mu} = \frac{(S_t - I)F}{\mu}(I + \partial_z Y_0 \otimes n) + (S_t F)(\partial_z Z \otimes n + \bar{D} Y).$$

Every factor and its first time derivative now have the component bounds. The translations restoring the physical means depend only on time and the fiber parameter. Their D_i derivatives vanish.

To pass to the exact flow, apply D_i/μ to the material path estimate (6.5) for $q = 0, 1$. On half the radius this costs at most $R_i C_j/\mu = N_i^{7/6+o(1)}$. Since $\log N_i \ge c_m \log N$ and $J_i \to \infty$, the resulting unweighted error is at most N^{-b} for every fixed b and all large j. This gives both assertions for the exact deformation. □

Proof of Proposition 8.3. Induction on the number of phases. At level i we prove the component bounds for $F_i, F_{i,t}, F_i^{-1}$ and $D_i \theta_{i,0}$ on $[0, T_{i-1}]$, and for the endpoint inverse on every $[0, T]$ with $T \in [t_i, T_{i-1}]$. At $i = j - m$ there are no retained angles. The unweighted coefficient bounds on every full construction interval follow from Proposition 6.6(b) and $F_t = (\nabla u \circ X)F$. On a smaller radius the ordinary endpoint inverse at each stage follows from

Lemmas 4.1, 5.3 and 5.2, using $F_{tt} = (\mathbf{e}_2 \otimes \nabla\theta - \Pi)F$ for the time derivatives of the mass blocks. The pressure budget holds on the full construction interval by Section 7.4. These estimates cost C_j, uniformly for $T \geq t_1$.

Suppose the two conclusions hold at level $i-1$ and put $R = R_{i-1}$. Construct the means of wave i on its actual interval $[0, T_{i-1}]$. At each fixed order l, (6.6) gives the w_i bound on the newest cutoff for $(\sigma_l, \vartheta_l, Y_l)/R^l$ and their first time derivatives. In fact their norm with weight $\epsilon_i B_{1,i}$ is at most $U_l^{l+1} \leq C_j$, and on the cutoff the weight is at most $C_j w_i$ by localization and (7.4). The profiles vanish outside this cutoff by the oscillatory coefficient equations with zero data there.

We claim by induction on l that $(v_l, h_l)/R^{l+1}$ and their first time derivatives have the component bounds at level $i-1$ in parent labels. Normalize each coefficient of V and Θ at order k by R^k. Its oscillatory part has the preceding cutoff and weight estimates, and its mean can only use v_{k-1} or h_{k-1}, because the means enter with an extra N_i^{-1} in (6.1) and in Θ. In the two averaged fluxes of Lemma 6.3(c), each product containing an oscillatory factor at total order l satisfies the averaging estimate just proved. Products involving only means contain strictly lower mean indices. The multiplier F_{i-1} already satisfies the bounds, and the outer derivative is D/R after dividing by R^{l+1}. Thus the two normalized sources f_u, f_θ, written in parent labels, have the component bounds. Set

$$H(t) = \int_0^t f_\theta(s)\,\mathrm{d}s, \qquad E = F_{i-1}^T(f_u + H\mathbf{e}_2).$$

Apply the inverse at level $i-1$ to E, which is allowed since $T_{i-1} \in [t_{i-1}, T_{i-2}]$. For its displacement Z, (4.6) gives respectively $F_{i-1}Z_t$ and $H - D_{i-1}\theta_{i-1,0}\cdot Z$ for the normalized means. One time derivative uses $F_{i-1,t}, Z_{tt}, f_\theta$ and Z_t. This establishes the claim at order l, including $l = 0$, where the mean–mean terms are absent.

Coefficients of the material equation. On $\operatorname{supp}\chi_i$, the leading profile bounds c_t with weight w_i. The amplitude equation bounds c_{tt} and integration bounds c with the same weight. All three vanish off the cutoff. Hence $F_*, F_{*,t}$ have the component bounds, as do $F_i, F_{i,t}$ by Lemma 8.4. The determinant identity gives $F_i^{-1} = \operatorname{adj} F_i$. Direct differentiation of (7.3) gives

$$D_i\theta_{i,0} = D_{i-1}\theta_{i-1,0} + \epsilon_i\chi_i\psi_i' n_i + \mu\epsilon_i(D/R)\chi_i\,\psi_i.$$

Both increments have weight w_i on their cutoff, since the incident amplitude satisfies $\epsilon_i \leq C_j w_i(0) \leq C_j w_i(t)$. This closes the coefficient part of the induction before solving the endpoint problem at level i.

Set $\mathcal{L}_i V = F_i^T((F_i V)_{tt} + \Pi_i F_i V)$ and define $\mathcal{L}_*$ by the same formula with F_*, Π_*. Differentiating the particle equation in the labels gives

$$\mathcal{L}_i V = F_i^T F_i V_{tt} + 2F_i^T F_{i,t} V_t + (F_i^T\mathbf{e}_2)(D_i\theta_{i,0}\cdot V).$$

For $\mathcal{L}_*$ replace F_i by F_* and $D_i\theta_{i,0}$ by $D_{i-1}\theta_{i-1,0} + \psi' B(0)n$. Indeed $F_* = F + \psi' ck^\perp \otimes n$. With $b = c_t k^\perp$,

$$(ck^\perp)_{tt} + \Pi(ck^\perp) = -c\beta \mathbf{e}_2 + b_t + Mb, \qquad b_t + Mb + qk/|k|^2 = B\mathbf{e}_2.$$

Use $B - \beta c = B(0)$ and $(k \otimes k)F_* = k \otimes n$. Hence each coefficient of $(\mathcal{L}_i - \mathcal{L}_*)/\mu$ has the component bounds, by the differentiated incident temperature and Lemma 8.4.

The endpoint inverse. Given $E = f + e$ in (8.1), use f in the split problem of Lemma 8.2. The older force $\langle f \rangle - cn\langle \psi'(l \cdot f)\rangle$ has the component bounds by averaging. The earlier inverse applies since $[t_i, T_{i-1}] \subset [t_{i-1}, T_{i-2}]$. The transverse force $l \cdot (f - \langle f \rangle)$ is supported in $\operatorname{supp} \chi_i$ with weight w_i. Lemma 8.1 and its weighted derivative estimate give these bounds for W, W_t, W_{tt}. Since $w = W - c\psi'(n \cdot V_0)$, they also hold for w, w_t, w_{tt}.

For $V = V_0 + lw$ put $Q := I - P_i$. Test the split equations first against l times a scalar with zero angular mean. This makes $Q(\mathcal{L}_* V - f)$ parallel to n. Testing against a parent solenoidal field makes $P_i(\mathcal{L}_* V - f)$ a parent gradient. Hence $\mathcal{L}_* V - f = Dp_s + n\partial_z p_f$, where $\partial_z p_f = |n|^{-2} n \cdot Q(\mathcal{L}_* V - f)$. Each surviving term either contains a newest coefficient of $\mathcal{L}_*$, one of w, w_t, w_{tt}, or Qf. Consequently its z-primitive p_f has newest support and weight. Only the fast pressure enters the residual. Put $U = V + N_i^{-1} D^\perp \partial_z^{-1} w$. It has zero endpoints, and $D_i \cdot U = 0$ since $n \cdot D^\perp w = -l \cdot Dw$. Direct substitution gives

$$\frac{\mathcal{L}_i U - D_i(p_s + N_i^{-1} p_f) - f}{\mu} = \frac{\mathcal{L}_i - \mathcal{L}_*}{\mu} U + \mathcal{L}_*(D^\perp/R)\partial_z^{-1} w - (D/R)p_f.$$

The right side and U, U_t, U_{tt} have the component bounds. Start with $E_0 = E$. At each step choose $E_k = f_k + e_k$ with $\|e_k\| \le N^{-a}$ and use the preceding construction on f_k, giving U_k and pressure p_k. The next force is $E_{k+1} = (f_k - \mathcal{L}_i U_k + D_i p_k)/\mu$. It has the component bounds by the residual identity. The sum $\sum_{k=0}^{p-1} \mu^k U_k$ has zero endpoint values and is solenoidal. The remaining force modulo a gradient is exactly

$$\sum_{k=0}^{p-1} \mu^k e_k + \mu^p E_p.$$

At fixed depth $\mu \le N^{-\eta_m}$ for some $\eta_m > 0$. Choose $p\eta_m > b+2$ and $a > b+2$, both independent of j. The unweighted inverse at level i costs $C_j = N^{o(1)}$ on a common smaller radius. It supplies the remaining displacement and its first two time derivatives with norms at most N^{-b}. The finite sum has the component bounds on the same interval, which closes the induction. □

8.4. **Summation of the initial data.**

Proof of Proposition 2.2(c). Fix an integer $r \ge 0$. The temperature increment (7.3) has C^r norm at most $\epsilon_{j+1} N_{j+1}^{r+o(1)}$. Since $|n_{j+1}| \gtrsim 1$, the early

bound in (7.4) at time zero and (7.5) bound these norms by N_{j+1}^{-1} for all sufficiently large j.

For the velocity write $N := N_{j+1}$ and choose m with $r(5/6)^{m+1} < 7/6$. On the untranslated fiber the initial increment is the sum of the means $N^{-2-l}v_l(0)$ and the Newton correction. The correction has C^r norm at most $N^{r-3-(j+1)/28+o(1)}$ by Section 7.4, hence at most N^{-1} for large j.

Apply the mean-profile argument in the proof of Proposition 8.3 once more, to wave $j+1$ with parent j and endpoints $0, T_j$. The endpoint inverse needed here is the one already proved at level j. Thus each retained angular component of $v_l(0)/R_j^{l+1}$ has norm at most $C_j e^{-c/\sqrt{\delta_{h-1}}} + N^{-b}$ for every fixed b and each fixed l.

For $j - m < h \le j$ the slowest exponential decay is at $h = j - m + 1$. Since $\delta_{j-m}^{-1/2}/\log N \to \infty$, choosing $b > r + 3$ bounds these components by $O(N^{-1})$ after r physical derivatives for all sufficiently large j.

Only frequencies from stages through $j - m$ enter the remaining component, so each physical derivative costs $R_{j-m}N^{o(1)}$. For fixed l its contribution to the increment is at most

$$N^{-2}R_j(R_j/N)^l R_{j-m}^r N^{o(1)} = N^{-7/6-l/6+r(5/6)^{m+1}+o(1)}.$$

The exponent at $l = 0$ is negative by the choice of m. For the tail, the bound in (6.6) before angular decomposition is

$$N^{-2-l+(5/6)(l+1+r)+o(l+r+1)} \le N^{-l/12}$$

for fixed r and all sufficiently large j, l up to the truncation order. Summing this geometric tail and the finitely many lower orders bounds the full C^r increment by $O_r(N_{j+1}^{-c_r})$ for some $c_r > 0$. By (7.1) this is at most 2^{-j} for all sufficiently large j. Absorbing the finitely many earlier increments in C_r proves the asserted estimate on initial data. □

Data availability. No datasets were generated or analysed in this work.

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