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RATIONAL POINTS ON BAXTER'S CURVE AND ROOTS OF UNITY IN THE EIGHT-VERTEX MODEL

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Abstract. The Boltzmann weights of the symmetric eight-vertex model with fixed invariants Δ and Γ form a curve of genus one, the intersection of two quadrics in $\mathbb{P}^3$. When Δ and Γ are rational, this curve is an elliptic curve over $\mathbb{Q}$ with at least eight rational points and full rational 2-torsion, and the crossing parameter of Baxter's elliptic parametrization corresponds to a rational point on it. By Mazur's theorem, if this point is a torsion point (an elliptic root of unity), its order is 3, 4, 6, or 8. We give the loci of each exact order explicitly; over $\mathbb{C}$ they lie in finite-order conditions found by Boukraa and Maillard. Consequently the finite-order loci of orders 5, 7, 10, 12, and 20 contain no rational nondegenerate (Δ, Γ), and the order-8 case, which occurs with rational weights, has no rational counterpart in the six-vertex limit. Away from these loci the crossing point has infinite order, so the curve has positive rank, and its multiples produce infinite families of commuting inhomogeneous transfer matrices with rational weights.

1. Introduction

The symmetric (zero-field) eight-vertex model has Boltzmann weights a, b, c, d, and Baxter's solution [2] rests on the fact that transfer matrices whose weights share the two invariants

$$\Delta = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}, \qquad \Gamma = \frac{ab - cd}{ab + cd}$$

commute. In terms of

$$J_x = ab + cd, \qquad J_y = ab - cd, \qquad J_z = \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2),$$

one has $\Delta = J_z/J_x$ and $\Gamma = J_y/J_x$, and the transfer matrix commutes with the XYZ Hamiltonian with couplings proportional to (J_x, J_y, J_z) [15]. Fixing Δ and Γ imposes two quadratic conditions on $(a : b : c : d) \in \mathbb{P}^3$, and the weights range over the intersection of two quadrics, a curve of genus one (Clebsch's biquadratic in the language of [9]). Baxter parametrizes it by elliptic functions of a spectral parameter u, with a crossing parameter fixed by Δ and Γ [2].

The case in which the crossing parameter is a rational multiple of a period, the elliptic root of unity, is special: the transfer matrix acquires additional degeneracies [6], and one such line, $J_xJ_y + J_xJ_z + J_yJ_z = 0$, carries a hidden lattice supersymmetry and a combinatorial structure [14, 12, 4, 8, 13]. Boukraa and Maillard [5] described varieties in $(J_x : J_y : J_z)$ on which the shift by the crossing parameter has finite order, including orders 3, 4, 6, 8, 10, 12, and 20 for the crossing point used here; Maillard and Boukraa [10] give the conditions for orders dividing 4, 6, and 10, and conditions of orders 8, 12, and 20, in closed form and recall the Mordell–Weil theorem for the group of rational points of the curve, without a bound on torsion.

This note considers rational Δ and Γ, as for any rational weights with $ab+cd \neq 0$. Then the curve is an elliptic curve over $\mathbb{Q}$ (Section 2): it has at least eight rational points, full rational 2-torsion, and a rational point corresponding to the crossing parameter. Mazur's theorem [11] on rational torsion then restricts the possible roots of unity to orders 3, 4, 6, and 8 (Theorem 3.1), whose loci we give explicitly; the free-fermion line $\Delta = 0$ [7] is a component of the order-4 locus and the supersymmetric line is an order-6 case. Off these loci the crossing point has infinite order, which gives infinitely many rational points and infinite commuting families of rational inhomogeneous transfer matrices (Section 4), a precise form of the observation in [9] that rational points of the curve come in infinite families.

Code that reproduces every computation in this note is available at `https://github.com/shankarsivarajan/baxter-curve-torsion`.

2. The curve

For $\Delta, \Gamma \in \mathbb{C}$ let $E = E_{\Delta,\Gamma} \subset \mathbb{P}^3$ be defined by

$$Q_1 = a^2 + b^2 - c^2 - d^2 - 2\Delta(ab + cd) = 0, \qquad Q_2 = (ab - cd) - \Gamma(ab + cd) = 0.$$

Lemma 2.1. *$E_{\Delta,\Gamma}$ is a smooth curve of genus one if and only if*

$$(\Delta^2 - 1)(\Gamma^2 - 1)(\Delta^2 - \Gamma^2) \neq 0.$$

Proof. In the coordinates $p = a + b$, $m = a - b$, $P = c + d$, $M = c - d$,

$$2Q_1 = (1 - \Delta)p^2 + (1 + \Delta)m^2 - (1 + \Delta)P^2 - (1 - \Delta)M^2,$$
$$4Q_2 = (1 - \Gamma)p^2 - (1 - \Gamma)m^2 - (1 + \Gamma)P^2 + (1 + \Gamma)M^2.$$

An intersection $\sum_i \alpha_i x_i^2 = \sum_i \beta_i x_i^2 = 0$ of two diagonal quadrics in $\mathbb{P}^3$ is a smooth curve of genus one exactly when the four points $(\alpha_i : \beta_i) \in \mathbb{P}^1$ are distinct. Here they are $(1-\Delta : 1-\Gamma)$, $(1+\Delta : \Gamma-1)$, $(1+\Delta : 1+\Gamma)$, $(1-\Delta : -1-\Gamma)$, and the six coincidences occur exactly when $\Gamma = 1$, $\Delta = \Gamma$, $\Delta = 1$, $\Delta = -1$, $\Delta = -\Gamma$, $\Gamma = -1$. □

From now on $(\Delta^2 - 1)(\Gamma^2 - 1)(\Delta^2 - \Gamma^2) \neq 0$. The eight points

$$(1 : 0 : \pm 1 : 0), \quad (0 : 1 : \pm 1 : 0), \quad (1 : 0 : 0 : \pm 1), \quad (0 : 1 : 0 : \pm 1)$$

lie on $E_{\Delta,\Gamma}$ for all Δ, Γ and are distinct. We take $O = (1 : 0 : 1 : 0)$ as the origin, write $P_c = (0 : 1 : 1 : 0)$, and call

$$c = [P_c] - [O] \in E(\mathbb{C})$$

the *crossing point*; Lemma 2.4 explains the name.

Lemma 2.2. *The maps $\iota(a, b, c, d) = (-a, -b, c, d)$ and $\kappa(a, b, c, d) = (b, a, d, c)$ are automorphisms of E without fixed points; hence they are translations by distinct nonzero points $\tau = [(1 : 0 : -1 : 0)] - [O]$ and $\tau' = [(0 : 1 : 0 : 1)] - [O]$ of order two, and (identifying each $P \in E$ with $[P] - [O]$)*

$$E[2] = \{O,\ (1 : 0 : -1 : 0),\ (0 : 1 : 0 : 1),\ (0 : 1 : 0 : -1)\}.$$

If $\Delta, \Gamma \in \mathbb{Q}$, then E is an elliptic curve over $\mathbb{Q}$ with $E[2] \subset E(\mathbb{Q})$. The eight points above are $E[2]$ and $P_c + E[2]$; in particular $c \notin E[2]$.

Proof. Both maps preserve Q_1 and Q_2. A fixed point of ι in $\mathbb{P}^3$ has $a = b = 0$ or $c = d = 0$; on E the first gives $cd = 0$ from Q_2 and then $c = d = 0$ from Q_1, and the second similarly forces $a = b = 0$, so there are none. A fixed point of κ is $(0 : 0 : 1 : \pm 1)$, which is not on E because there $Q_2 = \mp(1+\Gamma) \neq 0$, or $(1 : \pm 1 : t : \pm t)$, and substituting into Q_1 and Q_2 requires $(1-\Delta)/(1+\Delta) = (1-\Gamma)/(1+\Gamma)$ or $(1+\Delta)/(1-\Delta) = (1-\Gamma)/(1+\Gamma)$, that is, $\Delta = \pm\Gamma$, which is

excluded. An automorphism of a curve of genus one without fixed points is a translation, and $\iota^2 = \kappa^2 = 1$, so ι and κ are translations by the points of order two $[\iota(O)] - [O]$ and $[\kappa(O)] - [O]$, which differ because $\iota(O) = (1:0:-1:0) \neq (0:1:0:1) = \kappa(O)$; $\iota\kappa(O) = (0:1:0:-1)$. The images of P_c under $\iota, \kappa, \iota\kappa$ are $(0:1:-1:0)$, $(1:0:0:1)$, $(1:0:0:-1)$, and $P_c \notin E[2]$ because the eight points are distinct. □

Remark 2.3. With $t_1 = \frac{\Delta-1}{\Gamma-1}$, $t_2 = \frac{\Delta+1}{\Gamma-1}$, $t_3 = \frac{\Delta+1}{\Gamma+1}$, $t_4 = \frac{\Delta-1}{\Gamma+1}$, the curve E is isomorphic over $\mathbb{Q}(\Delta, \Gamma)$, and over $\mathbb{Q}$ when $\Delta, \Gamma \in \mathbb{Q}$, to $y^2 = (x-e_1)(x-e_2)(x-e_3)$, $e_i = -\prod_{j\in\{1,2,3\},\, j\neq i}(t_4 - t_j)$, that is,

$$e_1 = \frac{-4(\Delta+\Gamma)}{(\Gamma-1)(\Gamma+1)^2}, \quad e_2 = \frac{-4(\Delta-1)}{(\Gamma-1)(\Gamma+1)^2}, \quad e_3 = \frac{-4(\Delta-1)(\Delta+\Gamma)}{(\Gamma-1)^2(\Gamma+1)^2}.$$

Indeed, E has the rational point O, so it is isomorphic over $\mathbb{Q}(\Delta, \Gamma)$ (and over $\mathbb{Q}$ when $\Delta, \Gamma \in \mathbb{Q}$) to its Jacobian, which is the Jacobian of the double cover $y^2 = \det(\mu A + B)$, where A and B are the Gram matrices of $2Q_1$ and $4Q_2$ in the coordinates of Lemma 2.1 [1, Sec. 3.3 and Thm. 3.1]. This double cover has the rational points over the four rational roots $-1/t_i$ of $\det(\mu A + B)$; sending one of them to infinity and scaling $(x, y) \mapsto (c_3 x, c_3 y)$ by the leading coefficient c_3 gives a monic cubic model, which becomes the model above under the change of coordinates $(x, y) \mapsto \big(x/((\Gamma-1)(\Gamma+1))^2,\, y/((\Gamma-1)(\Gamma+1))^3\big)$ and a translation in x. Its Legendre parameter is $(\Delta^2 - \Gamma^2)/(1-\Gamma^2)$ and

$$j(E) = \frac{256\,(\Delta^4 - \Delta^2\Gamma^2 - \Delta^2 + \Gamma^4 - \Gamma^2 + 1)^3}{(\Delta^2-1)^2(\Gamma^2-1)^2(\Delta^2-\Gamma^2)^2}.$$

Lemma 2.4. *Let $k^2 \notin \{0, 1\}$, let* sn *be the Jacobi function of modulus k, $\Lambda = 4K\mathbb{Z} + 2iK'\mathbb{Z}$ its period lattice, with $K = K(k)$ and $K' = K(\sqrt{1-k^2})$ the complete elliptic integrals, and $\lambda \in \mathbb{C} \setminus \frac{1}{2}\Lambda$ with $1 + k\,\mathrm{sn}^2\,\lambda \neq 0$. The weights*

$$w(u) = \big(\mathrm{sn}(\lambda - u) : \mathrm{sn}\,u : \mathrm{sn}\,\lambda : k\,\mathrm{sn}\,\lambda\,\mathrm{sn}\,u\,\mathrm{sn}(\lambda-u)\big)$$

satisfy $Q_1 = Q_2 = 0$ for

$$\Delta = -\frac{\mathrm{cn}\,\lambda\,\mathrm{dn}\,\lambda}{1 + k\,\mathrm{sn}^2\,\lambda}, \qquad \Gamma = \frac{1 - k\,\mathrm{sn}^2\,\lambda}{1 + k\,\mathrm{sn}^2\,\lambda},$$

independently of u; conversely, every (Δ, Γ) satisfying the condition of Lemma 2.1 arises from some such k and λ. If Δ and Γ satisfy that condition, then $u \mapsto w(u)$ is an isomorphism $\mathbb{C}/\Lambda \to E_{\Delta,\Gamma}$ of groups with $w(0) = O$ and $w(\lambda) = P_c$. Thus c corresponds to the crossing parameter λ, and c has order N in E if and only if λ has order N in $\mathbb{C}/\Lambda$.

Proof. For Q_2, the weights give $ab - cd = \mathrm{sn}\,u\,\mathrm{sn}(\lambda-u)(1 - k\,\mathrm{sn}^2\,\lambda)$ and $ab + cd = \mathrm{sn}\,u\,\mathrm{sn}(\lambda - u)(1 + k\,\mathrm{sn}^2\,\lambda)$. For Q_1, substitute the addition formula

$$\mathrm{sn}(\lambda - u) = \frac{\mathrm{sn}\,\lambda\,\mathrm{cn}\,u\,\mathrm{dn}\,u - \mathrm{sn}\,u\,\mathrm{cn}\,\lambda\,\mathrm{dn}\,\lambda}{1 - k^2\,\mathrm{sn}^2\,\lambda\,\mathrm{sn}^2\,u};$$

the numerator of Q_1 then reduces to zero modulo $\mathrm{cn}^2 = 1 - \mathrm{sn}^2$ and $\mathrm{dn}^2 = 1 - k^2\,\mathrm{sn}^2$, applied to u and to λ. This is a form of Baxter's parametrization [2, 3]. From $\mathrm{sn}(u + 2K) = -\mathrm{sn}\,u$ and $\mathrm{sn}(u + iK') = 1/(k\,\mathrm{sn}\,u)$ one gets $w(u + 2K) = \iota w(u)$ and $w(u + iK') = \kappa w(u)$. The map w is a nonconstant holomorphic map between curves of genus one taking 0 to O, hence a homomorphism, and at a point where all four coordinates are finite, $w(u) = O$ forces $\mathrm{sn}\,u = 0$, so $u \in \{0, 2K\} + \Lambda$, while $w(2K) = \iota(O) \neq O$. At a pole of $\mathrm{sn}\,u$ one has $w(u) = \kappa w(u - iK')$, which equals O only if $w(u - iK') = \kappa(O) = (0:1:0:1)$, impossible since its third coordinate

$\operatorname{sn}\lambda$ is nonzero; poles of $\operatorname{sn}(\lambda-u)$ are treated in the same way. So the kernel is trivial and w is an isomorphism.

Conversely, every (Δ,Γ) with $(\Delta^2-1)(\Gamma^2-1)(\Delta^2-\Gamma^2)\neq 0$ arises in this way. With $g=(1-\Gamma)/(1+\Gamma)$, the formulas require $k\operatorname{sn}^2\lambda=g$ and $gk^2-\big(1+g^2-(1+g)^2\Delta^2\big)k+g=0$; the roots k and $1/k$ of this quadratic equal 1 only if $\Delta=\pm\Gamma$ and -1 only if $\Delta^2=1$, and replacing λ by $\lambda+2K$ changes the sign of Δ. Since $g\neq 0,\infty$, we have $\operatorname{sn}\lambda\neq 0,\infty$, so $\lambda\notin\frac{1}{2}\Lambda$, and $1+k\operatorname{sn}^2\lambda=1+g=2/(1+\Gamma)\neq 0$. □

3. Roots of unity

Theorem 3.1. *Let $\Delta,\Gamma\in\mathbb{Q}$ with $(\Delta^2-1)(\Gamma^2-1)(\Delta^2-\Gamma^2)\neq 0$.*

1. *$E_{\Delta,\Gamma}(\mathbb{Q})_{\mathrm{tors}}\cong\mathbb{Z}/2\times\mathbb{Z}/2\nu$ with $\nu\in\{1,2,3,4\}$.*
2. *If the crossing point c has finite order N, then $N\in\{3,4,6,8\}$.*
3. *c has order*
   - *3 if and only if $\Delta-\Gamma+\Delta\Gamma=0$;*
   - *4 if and only if $\Delta\Gamma=0$;*
   - *6 if and only if $\Delta+\Gamma+\Delta\Gamma=0$, $\Delta-\Gamma-\Delta\Gamma=0$, or $\Delta+\Gamma-\Delta\Gamma=0$;*
   - *8 if and only if $\Delta^2+\Gamma^2=\Delta^2\Gamma^2$ or $\Gamma^2-\Delta^2=\pm\Delta^2\Gamma^2$.*
4. *Otherwise c has infinite order, and $E_{\Delta,\Gamma}(\mathbb{Q})$ has positive rank.*

Proof. (1) By Lemma 2.2, E is an elliptic curve over $\mathbb{Q}$ with full rational 2-torsion, and Mazur's theorem [11, Thm. 8] leaves exactly the groups $\mathbb{Z}/2\times\mathbb{Z}/2\nu$, $1\leq\nu\leq 4$, whose elements have orders $1,2,3,4,6,8$. (2) follows from (1) and $c\notin E[2]$ (Lemma 2.2).

(3) Let $U\subset\mathbb{C}^2$ be the complement of the curve $(\Delta^2-1)(\Gamma^2-1)(\Delta^2-\Gamma^2)=0$. Over U the curves $E_{\Delta,\Gamma}$ form an elliptic curve with zero section O and a section c, so for each n the set $Z_n=\{nc=O\}\subset U$ is the pullback of the zero section by the section nc, a divisor on U or empty, since $nc\neq O$ at the generic point. We computed nc exactly (the code cited in the introduction reproduces this computation) over the function field $\mathbb{Q}(\Delta,\Gamma)$ for $n\leq 8$, by the chord construction: the plane through three points of E meets it in a fourth point, and translation by c is $x\mapsto F(F(x,A,B),C,B)$, where F denotes the residual point, B is an auxiliary point among the eight, and (A,C) is one of the pairs (P_c,O), $(\iota P_c,\iota O)$, $(\kappa P_c,\kappa O)$, $(\iota\kappa P_c,\iota\kappa O)$, all of which satisfy $[A]-[C]=c$. The resulting coordinates of nc are polynomials in Δ,Γ with no common factor, and $nc=O$ along a curve exactly where the greatest common divisor of the coordinates x_1, x_3, and x_0-x_2 of the point $nc=(x_0:x_1:x_2:x_3)$ vanishes. Its irreducible factors in U are: none for $n=1,2$; $\Delta\Gamma+\Delta-\Gamma$ for $n=3$; Δ and Γ for $n=4$; the four sign patterns $\Delta\Gamma\pm\Delta\pm\Gamma$ for $n=6$; and Δ, Γ, $\Delta^2\Gamma^2-\Delta^2-\Gamma^2$, $\Delta^2\Gamma^2\mp(\Delta^2-\Gamma^2)$ for $n=8$. Since Z_n is a divisor, it is the union of these curves; distinct loci meet only outside U, so on each locus the order is exactly as stated.

(4) If c has finite order, it is one of the four orders in (3) by (2), so off the loci c has infinite order, and $E(\mathbb{Q})\supset\mathbb{Z}c$ has positive rank. □

In terms of the couplings, the loci of Theorem 3.1(3) are $-J_xJ_y+J_xJ_z+J_yJ_z=0$ (order 3); $J_yJ_z=0$ (order 4); $\pm J_xJ_y\pm J_xJ_z+J_yJ_z=0$ with the other three sign patterns (order 6); and $J_x^2(J_y^2+J_z^2)=J_y^2J_z^2$, $J_x^2(J_y^2-J_z^2)=\pm J_y^2J_z^2$ (order 8). In projective couplings, $J_x=0$, where Δ and Γ are infinite, also gives order 4: the proof of Lemma 2.4 gives $(J_x:J_y:J_z)=(1+k\operatorname{sn}^2\lambda:1-k\operatorname{sn}^2\lambda:-\operatorname{cn}\lambda\operatorname{dn}\lambda)$ along $w(u)$ with no hypothesis on $1+k\operatorname{sn}^2\lambda$, so $J_x=0$ means $k\operatorname{sn}^2\lambda=-1$, so $\lambda\equiv\pm iK'/2$ modulo $2K$ and $2iK'$, a point of order 4. The projective locus of order 4 is therefore $J_xJ_yJ_z=0$.

Over $\mathbb{C}$ these loci are known. Maillard and Boukraa give in closed form the conditions for orders dividing 4, namely $J_xJ_yJ_z=0$, and dividing 6, the product of the four sign patterns

above, which contains the loci of orders 3 and 6, together with a condition for order dividing 10, one of whose four factors (their t_2) is the curve of order 5 in Remark 3.2 [10, Sec. 5.2.2, eqs. (54)–(56)]; they also give conditions of orders 12 and 20 [10, eq. (48), the display after eq. (50), and eq. (53)]. The three loci of order 8 are the zero sets of their doubling polynomials $J_x^{(2)}$, $J_y^{(2)}$, $J_z^{(2)}$, that is, the preimages of the three components of the order-4 locus under their doubling map, the points whose double has order 4. They state $J_z^{(2)} = 0$ as a condition of order 8 [10, Sec. 5.2.1], [5, Sec. VI.A] (numbering as in arXiv:hep-th/0003212). Boukraa and Maillard also assign exact orders to individual components, among them order 3 to the locus $-J_xJ_y + J_xJ_z + J_yJ_z = 0$ [5, App. B]. New here are the proof that, in the domain $(\Delta^2-1)(\Gamma^2-1)(\Delta^2-\Gamma^2) \neq 0$, the curves of Theorem 3.1(3) and the curves of orders 5 and 7 in Remark 3.2 are all the components of $\{nc = O\}$ for $n \leq 8$, with exactly the stated orders, the locus of order 7, the correction of one order from 5 to 10 (Remark 3.2), and the statements over $\mathbb{Q}$. The order-6 locus $J_xJ_y + J_xJ_z + J_yJ_z = 0$ is the supersymmetric line [14, 12, 4, 8, 13].

Remark 3.2. The same computation gives, over $\mathbb{C}$, the locus of order 5, a curve of degree 3 in each of Δ and Γ,

$$\Delta^3\Gamma^3 - \Delta^3\Gamma^2 - \Delta^3\Gamma + \Delta^3 + \Delta^2\Gamma^3 + 2\Delta^2\Gamma^2 + \Delta^2\Gamma - \Delta\Gamma^3 - \Delta\Gamma^2 - \Gamma^3 = 0,$$

and the locus of order 7, of degree 6 in each variable,

$$\begin{aligned}(\Gamma-1)^2(\Gamma+1)^4\Delta^6 - 2\Gamma(\Gamma-1)^2(\Gamma+1)(\Gamma^2+1)\Delta^5 - \Gamma^2(\Gamma-1)^2(\Gamma+1)^2\Delta^4 \\ + 4\Gamma^3(\Gamma+1)(\Gamma^2-\Gamma+1)\Delta^3 - \Gamma^4(\Gamma+1)^2\Delta^2 - 2\Gamma^5(\Gamma+1)\Delta + \Gamma^6 = 0.\end{aligned}$$

By Theorem 3.1 these curves, and the finite-order varieties of orders 10, 12, and 20 of [5], contain no rational points in U. For the varieties of orders 12 and 20 (on which $12c = O$ and $20c = O$, respectively, and 8 divides neither 12 nor 20) this uses that they meet the loci of orders 3, 4, and 6 only outside U. The variety that [5] list with order 5, and maintain as such for the Baxter model in the remark closing their Appendix B, is up to sign the factor t_4 of the order-10 condition (56) of [10]. It has order 10, as the computation in the proof of Theorem 3.1, extended to $n = 10$, shows, and its image under $J_x \mapsto -J_x$ is the curve of order 5 above. Thus elliptic roots of unity of these orders cannot be realized with rational weights. Rational points outside U, such as $(0, 0)$ on the curve of order 5, are degenerate.

Remark 3.3. The loci of orders 3, 4, 6 meet the six-vertex line $\Gamma = 1$ (outside U, where the curve degenerates to a pair of conics) at $\Delta = \frac{1}{2}, 0, -\frac{1}{2}$, the trigonometric roots of unity $\Delta = -\cos(2\pi/N)$, which are rational. The order-8 loci meet it only at $\Delta = \pm 1/\sqrt{2}$. So order 8 is an elliptic phenomenon: it occurs with rational weights for the eight-vertex model, for instance with the weights $(2, 4, 1, 2)$, for which $(\Delta, \Gamma) = (\frac{3}{4}, \frac{3}{5})$ lies on $\Gamma^2 - \Delta^2 = -\Delta^2\Gamma^2$, but has no rational counterpart in the six-vertex model.

4. Rational points and commuting families

Transfer matrices of the eight-vertex model on a row of L columns, with the weights of column i given by a point $R_i \in E_{\Delta,\Gamma}$, commute when all columns are shifted by the same spectral parameter; in the parametrization of Lemma 2.4, $R_i = w(u - v_i)$, and the transfer matrices $T(u)$ for fixed inhomogeneities $v_1, \ldots, v_L$ commute. Indeed, the star-triangle relation of the eight-vertex model [3] relates weights at spectral parameters $u - v_i$, $u' - v_i$, and $u - u'$, and depends on u and u' only through these differences, so the argument for the homogeneous model applies column by column. By Lemma 2.4, shifting u by λ replaces each R_i by $R_i + c$ in the group law.

Corollary 4.1. *Let $\Delta, \Gamma \in \mathbb{Q}$ with $(\Delta^2-1)(\Gamma^2-1)(\Delta^2-\Gamma^2) \neq 0$ and off the loci of Theorem 3.1(3), and let $R_1, \dots, R_L \in E_{\Delta,\Gamma}(\mathbb{Q})$. Then the transfer matrices with column weights $R_1 + nc, \dots, R_L + nc$, $n \in \mathbb{Z}$, have rational weights and commute, and their column weights are pairwise distinct for distinct n.*

Proof. The points $R_i + nc$ are rational because c and the R_i are, and distinct for distinct n because c has infinite order (Theorem 3.1(4)); they commute by the remark preceding the corollary. □

Example 4.2. The following weights are positive integers, lie on the indicated curves, and are not among the eight points; on each curve c has infinite order.

(Δ, Γ)	weights (a, b, c, d)
$(-\frac{5}{28}, -\frac{5}{21})$	$(8, 14, 13, 14)$, $(50600, 6678, 10350, 53053)$
$(-\frac{5}{6}, -\frac{25}{27})$	$(3360, 451, 5824, 6765)$
$(-\frac{11}{46}, -\frac{11}{23})$	$(24276960, 132832441, 64664880, 141295279)$

Any of them, shifted by multiples of c, gives an infinite commuting family of rational transfer matrices as in Corollary 4.1.

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