Failure of the Li–Liu spectral hypothesis R under Dirichlet GRH

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Abstract

Under Dirichlet GRH, the Li–Liu spectral hypothesis R fails. For every fixed $D > 0$ and $0 < \varepsilon < 1$, each sufficiently large interval $[X, 2X]$ contains at least $X \exp(-(\log X)^{\varepsilon})$ even integers N for which $\sqrt{N}\,\|T_N\| > (\log N)^D$. We construct these integers unconditionally and use GRH only at the modulus N to obtain the norm bound. Each has at least $(\log N)^D$ representations as a sum of two distinct primes. The proof isolates a matrix entry in which the continuous model cancels and the two arithmetic terms have different Buchstab densities.

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1 Introduction

Li and Liu [5] attach a compact self-adjoint operator T_N to each even $N > e^{10}$ satisfying GRH(N), the Riemann hypothesis for the primitive Dirichlet L-functions of conductor dividing N, including the Riemann zeta function. Their hypothesis R places every zero of an auxiliary spectral product on the line $\Re s = 1/2$ for every sufficiently large such N. By their Theorem 6.1, this is equivalent at N to

$$\sqrt{N}\,\|T_N\| \le (\log N)^{3/2}\sqrt{\mathfrak{S}(N)}, \qquad \mathfrak{S}(N) = 2\prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{\substack{p\mid N\\ p>2}}\frac{p-1}{p-2}. \tag{1}$$

Theorem 1.1. *Fix $D > 0$ and $0 < \varepsilon < 1$. There exists $C = C(D, \varepsilon)$ such that, for every sufficiently large X, at least $X\exp(-(\log X)^{\varepsilon})$ even integers $N \in [X, 2X]$ have both of the following properties.*

1. *N has at least $(\log N)^D$ distinct representations $N = p + q$ by primes $p < q$ with $p \le (\log N)^C$.*
2. *If* GRH(N) *holds, then $\sqrt{N}\,\|T_N\| > (\log N)^D$.*

Since $\mathfrak{S}(N) \ll N/\varphi(N) \ll \log\log N$, taking $D = 2$ contradicts (1) under global Dirichlet GRH. The first conclusion holds unconditionally on the same set of moduli.

Maier's matrix method detects fluctuations of primes in short intervals [6]. Granville and Soundararajan developed the connection between intervals and progressions for general arithmetic sequences [3], and Pandey and Woo obtained related results for systems of prime linear forms [7]. Here the shifts $N - m$ and $N + m$ occur in the same matrix entry. Their continuous contributions agree, while their progression averages sample Buchstab's function at two consecutive arguments.

2 Two opposite shifts

Write Λ for the von Mangoldt function. For a smooth real weight b supported in $(0, N/10)$, set

$$B_N(b) = \sum_{\substack{1\le m<N\\ (m,N)=1}} \Lambda(m)b(m)\{\Lambda(N-m) - \Lambda(N+m)\}.$$

Lemma 2.1. *Assume* GRH(N). *Suppose* $1 \le H \le M < N/20$ *and* b *is supported in* $[M, 2M]$ *and in an interval of length* H. *If* $\|b\|_\infty + H\|b'\|_\infty \le 1$, *then*

$$|B_N(b)| \ll M\sqrt{N/H}\,\|T_N\|.$$

Proof. Put $\alpha = 1/\log N$, $\beta = 1/8$. For $f \in C_c^\infty(0,\infty)$ write $\Phi_f(u) = \sqrt{u}\,f(u)$ and $\widehat{\Phi}_f(s) = \int_0^\infty \Phi_f(u)u^{s-1}\,\mathrm{d}u$. On the Hardy strip $\alpha < \Re s < \beta$, let v_f represent the functional

$$h \longmapsto \frac{1}{2\pi i}\int_{(\sigma)} N^s(1+s)\widehat{\Phi}_f(s)h(s)\,\mathrm{d}s, \qquad \alpha < \sigma \le 1/9.$$

With arc length on the two boundary lines, the norm formula of [5, Prop. 7.4] is

$$\|v_f\|^2 = \frac{1}{2\pi}\int_0^\infty \frac{(Nu)^{2\alpha}|f(u)/2 - uf'(u)|^2}{1+(Nu)^{-2(\beta-\alpha)}}\,\mathrm{d}u. \tag{2}$$

Indeed, writing $h(s) = \int_{\mathbb{R}} e^{sx}A(x)\,\mathrm{d}x$, the functional pairs $A(x)$ with $(\Phi_f - u\Phi_f')(e^{-x}/N)$. Parseval on the two boundary lines gives (2) for these supports.

Take $f(u) = b(Nu)$ and fix $g \in C_c^\infty(4/5, 6/5)$ equal to one on $[9/10, 11/10]$. We claim that

$$\langle T_N v_g, v_f\rangle = -B_N(b)/N. \tag{3}$$

To evaluate the entry, apply the two functionals to the reproducing kernel of [5, Prop. 5.4]. The factors $1+s$ cancel its denominators. In the character term, move each contour to real part one and expand the logarithmic derivatives. The odd characters are nonprincipal, and GRH(N) makes the shift pole-free. Mellin inversion gives $N^{-1}f(m/N)g(n/N)$. Orthogonality on units gives the equal residue class with a plus sign and the opposite class with a minus sign. The supports select $n = N+m$ and $n = N-m$, respectively, and exclude the diagonal.

For the continuous term extend f, g by zero and put

$$J(y) = \int_0^\infty [f(u)g(u+y) + g(u)f(u+y) - f(u)g(y-u)]\,\mathrm{d}u.$$

The three beta integrals in the Li–Liu kernel give this expression under Mellin inversion [5, Lemma 7.2]. For example, the first integral of the powers is $\int_0^\infty u^{-1/2-s}(u+y)^{-1/2-t}\,\mathrm{d}u = y^{-s-t}\mathrm{B}(1/2-s, s+t)$. Interchanging s, t gives the second, and integration over $0 < u < y$ gives the third. Also $\int_0^\infty J(y)\,\mathrm{d}y = 0$, since the first two terms partition the double integral of $f(u)g(v)$, which the third subtracts. The Mellin transform of J therefore cancels the pole at $w = 1$ of $\sum_{k\ge1}\mathfrak{S}(kN)k^{-w}$. Moving the $w = s+t$ contour to $\Re w = 3/2$ and expanding gives $\sum_{k\ge1}\mathfrak{S}(kN)J(k)$. The tests are smooth with compact support, so the Mellin integrals converge absolutely on the shifted lines. Only $k = 1$ can contribute, and $g(1+u) = g(1-u) = 1$ on the support of f. Thus $J(1) = 0$, proving (3).

Finally (2) gives $\|v_g\| \ll 1$. The support of f has length at most H/N, and there $Nu \le 2M$ and $|uf'(u)| \le 2M/H$. Hence $\|v_f\| \ll M/\sqrt{NH}$. Apply Cauchy–Schwarz to (3). □

3 The progression average

Put $L = \log X$, $d = \varepsilon/3$, $z = L^d$, and $P = \prod_{p\le z} p$. We use an even divisor Q of P with two properties. First, for an absolute $c > 0$,

$$\sum_{\substack{n\le y\\ n\equiv r \bmod Q}} \Lambda(n) = \frac{y}{\varphi(Q)} + O(Xe^{-c\sqrt{L}}) \qquad (X/2 \le y \le 3X,\ (r,Q) = 1). \tag{4}$$

Second, for fixed $v > 4$, $c_0 > 0$, and $W \in C_c^\infty(0,\infty)$,

$$\sum_{\substack{n\ge 1\\(n,Q)=1}} W\Big(\frac{n}{c_0 z^v}\Big) = \frac{c_0 z^v}{\log z}\left(\omega(v)\int_0^\infty W(t)\,\mathrm{d}t + o(1)\right), \qquad \frac{Q}{\varphi(Q)} \asymp \log z, \tag{5}$$

where ω is Buchstab's function.

To choose Q, use the exceptional prime of Ford, Green, Konyagin, Maynard and Tao [2, Cor. 6 and (7.2)]. It is a number ℓ equal to one or a prime $\gg \log\log X$. Take $Q = P/(P,\ell)$. Since $\log Q \ll z = o(\sqrt{L})$, their estimate (7.2) applies to each conductor $r > 1$ dividing Q. Summing its primitive character bounds over $r \mid Q$ costs at most $Q/\varphi(Q) \ll \log z$, by $\sum_{r\mid Q}\varphi(r) = Q$. Orthogonality and the ordinary prime number theorem give (4), absorbing this factor in the exponential error. Passing from a primitive character to its induced character removes only prime powers above primes dividing Q, of total weight $O(\log Q \log X)$.

Mertens' theorem gives the product estimate in (5). For $Q = P$, the weighted estimate is partial summation from Buchstab's asymptotic [1, (1.3)]. If a prime ℓ is removed, write $\Phi(Y,z) = \#\{1 \le n \le Y : (n,P) = 1\}$. For $Y \asymp z^v$, the extra integers up to Y number at most

$$\sum_{j\ge 1}\Phi(Y/\ell^j, z) \ll \frac{Y}{\ell\log z} + \sqrt{Y} = o(Y/\log z).$$

For $\ell^j \le \sqrt{Y}$ use Buchstab's bound $\Phi(t,z) \ll t/\log z$ for $t \ge z^2$, and sum the remaining geometric tail trivially. This proves (5).

Proof of Theorem 1.1. Choose a fixed $u > 4$ so large that

$$\omega(u) > \omega(u+1), \qquad d(u/2-1) > D+2. \tag{6}$$

Such u occur arbitrarily far out. If $\omega(t) \le \omega(t+1)$ eventually, the equation $t\omega'(t) = \omega(t-1) - \omega(t)$ makes ω nonincreasing on a tail. It must then be constant there. The same equation propagates a constant tail backwards to $(1,2)$, where $\omega(t) = 1/t$.

Set $H = z^u$ and $M = zH$. Fix nonnegative $w, v \in C_c^\infty(1,2)$ of integral one. For each integer a put

$$b_a(m) = w(m/M)v((m-a)/H), \qquad S_a = \sum_{m\ \text{odd}}\Lambda(m)b_a(m)\{\mathbf{1}_{(a-m,Q)=1} - \mathbf{1}_{(a+m,Q)=1}\}.$$

Summing the first term over a separates the m-sum from the sum over $h = m - a$. The prime number theorem and (5) at H give $(MH/\log z)(\omega(u) + o(1))$. For the second term put $t = a + m$. Uniformly on its support $2M - 2H \le t \le 4M$,

$$\sum_{m\ \text{odd}}\Lambda(m)w(m/M)v((2m-t)/H) = \frac{H}{2}w(t/(2M)) + o(H).$$

Indeed $H = M^{u/(u+1)} > M^{4/5}$. Huxley's prime number theorem in short intervals [4] gives $\psi(y+s) - \psi(y) = s + o(H)$ uniformly for $y \asymp M$, $0 \le s \le 2H$: use his asymptotic for $s \ge M^{2/3}$ and the trivial bound below $M^{2/3}$. Partial summation against the smooth weight gives the formula, since $H/M \to 0$ and the powers of two cost $O(\log M)$. There are $O(M/\log z)$ admissible t by (5). Summing over $(t,Q) = 1$ and applying (5) at $2M$ gives the second term $(MH/\log z)(\omega(u+1) + o(1))$. Thus

$$\sum_a S_a = \frac{MH}{\log z}\{\omega(u) - \omega(u+1) + o(1)\}.$$

Only $O(M)$ values of a occur, and odd a contribute zero because $2 \mid Q$. Fix an even a with $S_a \gg H/\log z$.

Let $\mathcal{A} = \{N \in [X, 2X] : N \equiv a \bmod Q\}$, so $K = |\mathcal{A}| = X/Q + O(1)$. For odd $m = p^j$ in the support of b_a, (4) gives

$$\sum_{\substack{N \in \mathcal{A} \\ (m,N)=1}} \Lambda(N \pm m) = \frac{X}{\varphi(Q)} \mathbf{1}_{(a \pm m, Q)=1} + O(Xe^{-c\sqrt{L}}).$$

In a nonunit class every nonzero term is a power of a single prime dividing $(a \pm m, Q)$. Terms removed by $p \mid N$ are powers of p. Each costs $O(\log X)$ in total. Since $\sum_m \Lambda(m) b_a(m) \ll H$ by the short-interval estimate, we obtain

$$\frac{1}{K} \sum_{N \in \mathcal{A}} B_N(b_a) = \frac{X}{K\varphi(Q)} S_a + o(H) \gg H.$$

Here $Qe^{-c\sqrt{L}} = o(1)$. Pointwise $|B_N(b_a)| \ll H \log X$, so $B_N(b_a) \gg H$ for $\gg X/(Q \log X)$ members of $\mathcal{A}$, with a fixed positive lower-bound constant.

Discard any such N within $2M$ of a proper prime power in $[X/2, 3X]$. This removes $O(M\sqrt{X} \log X)$ integers. Since $\log Q \ll L^d$ and $d < \varepsilon$, at least $X \exp(-L^\varepsilon)$ moduli remain. For each of them the sum with $\Lambda(N - m)$ alone has weight $\gg H$ and every positive term has $N - m$ prime. Proper prime powers $m \le 2M$ contribute $O(\sqrt{M} \log^2(2M) \log X) = o(H)$ by (6). Each remaining pair contributes $O(\log(2M) \log X)$, so there are at least $(\log N)^D$ distinct prime pairs with smaller prime at most $2M \le (\log N)^{d(u+1)+1}$.

Finally $\|b_a\|_\infty + H\|b_a'\|_\infty \ll 1$ uniformly in a, X. At each of these moduli satisfying GRH(N), Lemma 2.1 and (6) give

$$\sqrt{N}\, \|T_N\| \gg \frac{H^{3/2}}{M} = (\log X)^{d(u/2-1)} > (\log N)^D.$$

□

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