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A SET MAPPING WITH SMALL INTERSECTIONS AND NO LARGE FREE SET

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Abstract. Let μ be an uncountable cardinal of countable cofinality and suppose that $2^\mu = \mu^+$. We construct a self-avoiding set mapping on μ^+ whose values have cardinality at most μ, whose distinct values have intersection of cardinality less than μ, and which has no free set of cardinality μ^+. The construction combines a countable neighborhood base with an enumeration of the μ-sized subsets and a schedule having fewer than μ requirements at each bounded depth. In particular, under the generalized continuum hypothesis, it gives a negative answer to the free-set question of Erdős and Hajnal at $\aleph_{\omega+1}$.

1. Introduction and main result

Let S be a set. A mapping $F : S \to \mathcal{P}(S)$ is *self-avoiding* if $x \notin F(x)$ for every $x \in S$. A subset $H \subseteq S$ is *free* for F if

$$x, y \in H, \quad x \neq y \quad \Longrightarrow \quad y \notin F(x).$$

The question considered here asks whether small pairwise intersections of the values of a set mapping force a large free set. In the case raised by Erdős and Hajnal [1, Problem 35], one assumes the generalized continuum hypothesis, takes $\mu = \aleph_\omega$ and $\kappa = \mu^+$, and requires

$$f : \kappa \longrightarrow [\kappa]^{\leq \mu}, \qquad |f(\alpha) \cap f(\beta)| < \mu \quad (\alpha \neq \beta).$$

We give a counterexample under the weaker cardinal-arithmetic assumption $2^\mu = \mu^+$. Throughout, we work in ZFC, identify cardinals with their initial ordinals, and write $[S]^\mu$ for the family of subsets of S of cardinality μ.

Theorem 1.1. *Let μ be an uncountable cardinal with $\mathrm{cf}(\mu) = \aleph_0$ and $2^\mu = \mu^+$. Put $\kappa = \mu^+$. There is a self-avoiding mapping $f : \kappa \to [\kappa]^{\leq \mu}$ such that*

$$|f(\alpha) \cap f(\beta)| < \mu \qquad \text{for all distinct } \alpha, \beta < \kappa,$$

but no subset of κ of cardinality κ is free for f.

The intersection bound is pointwise: the construction does not require a single cardinal below μ to bound all pairwise intersections. This distinction is essential at a singular cardinal.

The use of values concentrated near their indexing points is motivated by Hechler's construction discussed in [1, p. 279]. All ingredients needed for the construction below are proved here.

2. Proof of the main theorem

Proof of Theorem 1.1. Fix an increasing sequence $\langle \mu_n : n < \omega \rangle$ of infinite cardinals below μ with supremum μ. Put $\kappa = \mu^+$.

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2.1. **The sequence space and its enumerations.** Let $X = \mu^\omega$, with the product topology obtained by giving μ the discrete topology. For $x \in X$ and $n < \omega$, define

$$U_n(x) = \{y \in X : y \restriction n = x \restriction n\}. \tag{2.1}$$

These sets form a decreasing neighborhood base at x. All closures below are taken in X. If $x \neq y$, then $U_N(x)$ and $U_N(y)$ are disjoint for some finite N. The collection $\mathcal{B}$ of all finite-prefix cylinders is a base of cardinality μ, since there are μ finite sequences from an infinite alphabet of size μ.

We first verify that $|X| = \kappa$. If $|X| \leq \mu$, list its points as $\langle z_\xi : \xi < \mu\rangle$, allowing repetitions. For each n, choose

$$z(n) \in \mu \setminus \{z_\xi(n) : \xi < \mu_n\}.$$

This choice is possible because $\mu_n < \mu$. For every $\xi < \mu$, some n satisfies $\xi < \mu_n$, and then $z(n) \neq z_\xi(n)$. Thus z is absent from the list, a contradiction. On the other hand,

$$|X| = \mu^{\aleph_0} \leq (2^\mu)^{\aleph_0} = 2^{\mu\cdot\aleph_0} = 2^\mu = \kappa. \tag{2.2}$$

Since κ is the successor of μ, these inequalities give $|X| = \kappa$. Fix a bijective enumeration $X = \{x_\alpha : \alpha < \kappa\}$.

The family $[X]^\mu$ also has cardinality κ. Indeed,

$$|[X]^\mu| \leq \kappa^\mu = (2^\mu)^\mu = 2^{\mu\cdot\mu} = 2^\mu = \kappa. \tag{2.3}$$

For the reverse inequality, fix $B \in [X]^\mu$. The sets $B \cup \{x\}$, for $x \in X \setminus B$, are κ distinct members of $[X]^\mu$. We may therefore enumerate this family as $\langle A_\xi : \xi < \kappa\rangle$.

2.2. **Scheduling the requirements.** For each $\alpha < \kappa$, choose an injection $h_\alpha : \alpha \to \mu$; this is possible because $|\alpha| \leq \mu$. For $\xi < \alpha$, put

$$d_\alpha(\xi) = \min\{n < \omega : h_\alpha(\xi) < \mu_n\}. \tag{2.4}$$

For every finite N, injectivity of h_α and monotonicity of the sequence $\langle \mu_n\rangle$ give

$$|\{\xi < \alpha : d_\alpha(\xi) < N\}| \leq \mu_N < \mu. \tag{2.5}$$

In fact, $d_\alpha(\xi) < N$ implies $h_\alpha(\xi) < \mu_N$. For $N = 0$ the set on the left is empty.

2.3. **The set mapping.** For each $\alpha < \kappa$, consider the indices

$$I_\alpha = \{\xi < \alpha : x_\alpha \in \overline{A_\xi \setminus \{x_\alpha\}}\}.$$

For every $\xi \in I_\alpha$, choose a point

$$y_{\alpha,\xi} \in (A_\xi \setminus \{x_\alpha\}) \cap U_{d_\alpha(\xi)}(x_\alpha).$$

The defining closure condition makes this intersection nonempty. Define

$$F(x_\alpha) = \{y_{\alpha,\xi} : \xi \in I_\alpha\}. \tag{2.6}$$

Every chosen point differs from x_α, so $x_\alpha \notin F(x_\alpha)$. Moreover, $|F(x_\alpha)| \leq |I_\alpha| \leq |\alpha| \leq \mu$. By (2.5), for every $\alpha < \kappa$ and every finite N,

$$|F(x_\alpha) \setminus U_N(x_\alpha)| \leq \mu_N < \mu. \tag{2.7}$$

Indeed, if $d_\alpha(\xi) \geq N$, then the selected point belongs to $U_N(x_\alpha)$. Only indices of depth less than N can contribute points outside that neighborhood. Repeated choices do not increase their number.

2.4. **Pairwise intersections.** Let $\alpha \neq \beta$. Choose a finite N such that $U_N(x_\alpha) \cap U_N(x_\beta) = \varnothing$. Every common point of the two values of F lies outside at least one of these neighborhoods. Hence

$$F(x_\alpha) \cap F(x_\beta) \subseteq \big(F(x_\alpha) \setminus U_N(x_\alpha)\big) \cup \big(F(x_\beta) \setminus U_N(x_\beta)\big).$$

Applying (2.7), and using that μ_N is infinite, we obtain

$$|F(x_\alpha) \cap F(x_\beta)| \leq \mu_N + \mu_N = \mu_N < \mu. \tag{2.8}$$

Only two exceptional sets are combined here. No estimate for a countable union of sets of size less than the singular cardinal μ is used. The depth N may depend on the pair α, β.

2.5. **No large free set.** Let $H \subseteq X$ have cardinality κ. For each basic open set $B \in \mathcal{B}$ that meets H, choose one point of $B \cap H$, and let D be the set of chosen points. Then $D \subseteq H$, $|D| \leq \mu$, and $H \subseteq \mathrm{cl}(D)$. Indeed, every basic neighborhood of a point of H contains one of the chosen representatives.

Enlarge D inside H to a set A of cardinality exactly μ. Then $H \subseteq \mathrm{cl}(A)$. Write $A = A_\xi$. Since both A and $\{x_\alpha : \alpha \leq \xi\}$ have cardinality at most μ, their union cannot exhaust H. Choose

$$x_\alpha \in H \setminus A \qquad \text{with} \qquad \alpha > \xi.$$

Because $x_\alpha \notin A$ and $x_\alpha \in \mathrm{cl}(A)$, we have

$$x_\alpha \in \overline{A_\xi \setminus \{x_\alpha\}}.$$

Thus $\xi \in I_\alpha$, and the construction selected a point

$$y_{\alpha,\xi} \in F(x_\alpha) \cap A \subseteq F(x_\alpha) \cap H.$$

This point is distinct from x_α. Hence H is not free for F. Since H was arbitrary, F has no free subset of cardinality κ.

2.6. **Transfer to the ordinal domain.** Use the fixed bijection $\alpha \mapsto x_\alpha$ to define

$$f(\alpha) = \{\beta < \kappa : x_\beta \in F(x_\alpha)\}. \tag{2.9}$$

The bijection preserves self-avoidance, the cardinalities of values and their intersections, and freeness. Therefore f has all the asserted properties. □

3. The Erdős–Hajnal question

Corollary 3.1. *Assume the generalized continuum hypothesis, and put $\mu = \aleph_\omega$ and $\kappa = \aleph_{\omega+1}$. There is a self-avoiding mapping $f : \kappa \to [\kappa]^{\leq \mu}$ whose distinct values have intersection of cardinality less than μ, but which has no free set of cardinality κ.*

Proof. Take $\mu = \aleph_\omega$. This cardinal has cofinality $\aleph_0$, and GCH gives $2^\mu = \mu^+ = \aleph_{\omega+1}$. Theorem 1.1 applies. □

Identifying $\aleph_{\omega+1}$ with its initial ordinal $\omega_{\omega+1}$ gives exactly the domain in the Erdős–Hajnal question. Thus the stated pointwise intersection condition does not force a free set of the required size.

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References

[1] P. Erdős and A. Hajnal, *Unsolved and solved problems in set theory*, in *Proceedings of the Tarski Symposium*, Proceedings of Symposia in Pure Mathematics **25**, American Mathematical Society, Providence, RI, 1974, pp. 269–287.