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ERDŐS PROBLEM 597 AND FINITE-TARGET INDEPENDENCE

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Abstract. We study the graph-target partition relation $\omega_1^2 \to (\omega_1 \cdot \omega, H)^2$ from Erdős problem 597. In ZFC we construct a connected bipartite graph of cardinality $\aleph_1$ and diameter at most three, containing no $K_{\omega,\omega}$, for which this relation fails. We also construct a fixed finite K_4-free graph H_0 for which the relation fails under the continuum hypothesis. Under Martin's axiom at $\aleph_1$, it holds for every finite graph. Consequently the universal finite-target assertion, and the assertion for H_0 itself, are independent of ZFC, relative to the consistency of ZFC.

1. Introduction and statements

All graphs are simple and undirected. A copy of a graph H in a graph G is an injective map $f\colon V(H) \to V(G)$ satisfying

$$uv \in E(H) \quad \Longrightarrow \quad f(u)f(v) \in E(G).$$

Additional edges among the image vertices are permitted. An order on the host imposes no order requirement on this copy.

For ordinals α, β and a graph H, the relation $\alpha \to (\beta, H)^2$ means that every red-blue coloring of $[\alpha]^2$ has a red homogeneous set of inherited order type β or a blue copy of H. We write

$$\Omega = \omega_1^2, \qquad B = \omega_1 \cdot \omega.$$

We identify Ω with $\omega_1 \times \omega_1$ in row-first lexicographic order, so (a, b) has ordinal position $\omega_1 a + b$. Equivalently, the blue pairs define a graph on Ω, and a red homogeneous set is an independent set in this graph.

Erdős asked whether $\Omega \to (B, H)^2$ holds for every graph H of cardinality at most $\aleph_1$ that contains neither K_4 nor $K_{\omega,\omega}$, and whether it holds when H is finite [4]. Here $K_{\omega,\omega}$ denotes a complete bipartite graph with both vertex classes countably infinite.

Theorem 1.1. *There is a graph H_∞ of cardinality $\aleph_1$ such that*

(i) *H_∞ is connected and bipartite, with diameter at most three.*
(ii) *H_∞ contains neither K_4 nor $K_{\omega,\omega}$.*
(iii) *$\Omega \not\to (B, H_\infty)^2$.*

These conclusions hold in ZFC.

Theorem 1.2. *There is a fixed finite K_4-free graph H_0 such that* CH *implies*

$$\Omega \not\to (B, H_0)^2. \tag{1.1}$$

Under $\mathrm{MA}_{\aleph_1}$*, for every finite graph H,*

$$\Omega \to (B, H)^2. \tag{1.2}$$

Corollary 1.3. *Assume that* ZFC *is consistent. Then each of the following assertions is independent of* ZFC.

(i) *$\Omega \to (B, H)^2$ for every finite K_4-free graph H.*
(ii) *$\Omega \to (B, H_0)^2$ for the graph H_0 in Theorem 1.2.*

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The negative construction uses the nested-interval method of Baumgartner–Hajnal [2] and Chen–Garti–Weinert [3]. The finite passage from an ordered obstruction to an unordered target is an instance of the Nešetřil–Rödl ordering property [7]. We give its sparse-hypergraph proof in the form needed here. The positive partition theorem is due to Baumgartner [1]. Related nonuniversality methods occur in Shelah [8], and Li [6] studies their application to the unrestricted graph-target question.

2. Uncountable rows in an ordinal block

For $S \subseteq \Omega$ and $a < \omega_1$, put

$$S_a = \{b < \omega_1 : (a, b) \in S\}.$$

We call a an uncountable row of S when S_a is uncountable. Every uncountable subset of ω_1 is unbounded and has order type ω_1.

Lemma 2.1. *If $S \subseteq \Omega$ has only finitely many uncountable rows, then $\operatorname{otp}(S) < B$. In particular, a subset of order type B has infinitely many uncountable rows.*

Proof. Suppose the uncountable rows are $a_1 < \cdots < a_k$. Each interval of rows before a_1 or between successive a_i contains only countably many rows, since its indices are bounded below ω_1. All its slices are countable, so its intersection with S has countable order type. Each slice S_{a_i} has order type ω_1.

The part of S after row a_k has order type at most ω_1. Indeed, every element of that part has only countably many predecessors there, since the preceding row indices form a countable set and every corresponding slice is countable. A well-order in which each element has countably many predecessors has order type at most ω_1. Adding these pieces gives

$$\operatorname{otp}(S) \le \omega_1 \cdot (k + 1) < \omega_1 \cdot \omega.$$

When $k = 0$, the same predecessor argument applies to all of S. □

3. An uncountable target omitted in ZFC

Define a graph G_* on $\omega_1 \times \omega_1$ by

$$(a, b) \sim (r, s) \iff a < r < s < b \text{ or } r < a < b < s. \tag{3.1}$$

Lemma 3.1. *The graph G_* has no independent set of order type B. Every nonempty induced subgraph of G_* has a vertex of at most countable degree.*

Proof. If S has order type B, Lemma 2.1 gives two uncountable rows $a < r$. Choose $s \in S_r$ with $s > r$, and then $b \in S_a$ with $b > s$. The vertices $(a, b), (r, s)$ are adjacent by (3.1).

Now let S be any nonempty vertex set. Choose $(a, b) \in S$ in the least row occupied by S. Every neighbor (r, s) of (a, b) in S satisfies $a < r < s < b$. Both its coordinates lie in the countable ordinal b, so there are at most countably many such neighbors. □

Let T be the graph on $\omega_1^{<\omega}$ in which two finite sequences are adjacent when one is obtained from the other by appending a single ordinal. Its cardinality is $\aleph_1$, since

$$|\omega_1^{<\omega}| = \sum_{n<\omega} \aleph_1^n = \aleph_1.$$

Every vertex has $\aleph_1$ immediate extensions. The graph is a tree. To see that it has no cycle, choose a vertex of maximum sequence length on a purported finite cycle. Its two cycle neighbors would both be its unique immediate predecessor, which is impossible. Thus T contains neither K_4 nor $K_{\omega,\omega}$, since each of those graphs contains a finite cycle.

Lemma 3.2. *There is no copy of T in G_*.*

Proof. Under an injective edge-preserving map, the $\aleph_1$ immediate extensions of every vertex of T would give $\aleph_1$ distinct neighbors of its image. Consequently the subgraph induced on the image would have minimum degree at least $\aleph_1$, contrary to Lemma 3.1. □

Proof of Theorem 1.1. The blue graph G_* and its red complement already witness $\Omega \not\to (B,T)^2$.

Partition the vertices of T into A_0, A_1 according to the parity of sequence length. Add vertices h_0, h_1, place h_i in the side A_i, and join h_i to every vertex in the opposite side, including h_{1-i}. Call the resulting graph H_∞. It is bipartite and has cardinality $\aleph_1$. Two old vertices in the same side are joined through the opposite hub by a path of length two. Two old vertices in opposite sides are joined through the two hubs by a path of length three. A pair involving a hub has distance at most two. Hence H_∞ is connected and has diameter at most three. Bipartiteness implies that it is K_4-free.

If H_∞ contained a $K_{\omega,\omega}$, removing the two hubs would leave two infinite vertex classes with every cross-edge present in T, a contradiction. Finally, any copy of H_∞ in G_* would restrict to a copy of T. Lemmas 3.1 and 3.2 therefore give the required negative partition relation for H_∞. □

4. A sparse nested graph under CH

Assume CH. For every $t < \omega_1$, fix an injection

$$e_t \colon t \longrightarrow \mathbb{N}.$$

A *pattern* is a countable set P of vertices (r,s) with $r < s$ such that P occupies infinitely many rows and contains infinitely many vertices in each row that it occupies. Since

$$(\aleph_1)^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_1,$$

we may enumerate all patterns as $(P_\nu)_{\nu<\omega_1}$, allowing repetitions.

For $x = (a,b)$ we construct an outgoing neighborhood $\Gamma(x)$. The construction proceeds by recursion on b. If $a \geq b$, set $\Gamma(x) = \varnothing$. If $a < b$, call an index ν *eligible for* x when

$$\nu < b \quad \text{and} \quad a < r < s < b \quad \text{for every } (r,s) \in P_\nu. \tag{4.1}$$

The eligible indices form a countable set. List them without repetition in a sequence of length at most ω. At each step, choose one point (r,s) from the current eligible pattern subject to the following rules.

(i) Its row has not previously been selected into $\Gamma(x)$.

(ii) For every previously selected row $t > r$,

$$e_t(r) > e_t(a), \qquad e_t(r) > e_t(u) \quad \text{for every previously selected row } u < t. \tag{4.2}$$

(iii) For every previously selected point (u,v) with $u < r$,

$$(r,s) \notin \Gamma(u,v). \tag{4.3}$$

Let G be the undirected graph whose edges are the pairs xy with $y \in \Gamma(x)$.

Lemma 4.1. *Every selection step is possible. The resulting graph satisfies*

(i) *if $(r,s) \in \Gamma(a,b)$, then $a < r < s < b$.*

(ii) *$\Gamma(a,b)$ contains at most one vertex in each row.*

(iii) *$\Gamma(a,b)$ meets every pattern eligible for (a,b).*

Proof. There are only finitely many previous selections at a given step. Rule (i) excludes finitely many rows. For a fixed previously selected row t, let M_t be the maximum of $e_t(a)$ and the finitely many values $e_t(u)$ with $u < t$ previously selected. Injectivity of e_t implies that

$$\{r < t : e_t(r) \leq M_t\}$$

is finite. Thus rule (ii) also excludes only finitely many rows. The current pattern has infinitely many rows, so an admissible row r can be chosen.

At this row, each set $\Gamma(u,v)$ occurring in rule (iii) is already defined, since $v < b$ by (4.1). By induction it contains at most one point in row r. Consequently rule (iii) excludes finitely many points from an infinite slice of the pattern. A permissible (r,s) exists. The three asserted properties follow from eligibility, rule (i), and the processing of every eligible index. □

If vertices x, y lie in increasing rows, then

$$xy \in E(G) \quad \Longleftrightarrow \quad y \in \Gamma(x). \tag{4.4}$$

In particular, G has no same-row edges.

Lemma 4.2. *The graph G has no independent set of order type B.*

Proof. Let $S \subseteq \Omega$ have order type B. By Lemma 2.1, choose an uncountable row a of S with infinitely many uncountable rows of S above it. Choose a countably infinite collection of those later rows, and in each chosen row r choose infinitely many points $(r, s) \in S$ with $s > r$. Their union is a pattern $P_\nu \subseteq S$.

The coordinates of P_ν, together with ν, are bounded below ω_1. Since S_a is unbounded, choose $b \in S_a$ strictly above all these ordinals. Then P_ν is eligible for (a, b). Lemma 4.1 supplies a point of $\Gamma(a, b) \cap P_\nu$, giving an edge of G with both ends in S. □

5. The ordered six-vertex obstruction

For $y \in \Gamma(x)$, let $\tau_x(y)$ be its selection index in the construction of $\Gamma(x)$. The uniqueness of its selected row makes this index well defined.

Lemma 5.1 (Selection reversal)**.** *Suppose $u, v \in \Gamma(x)$, the row of u is smaller than the row of v, and $uv \in E(G)$. Then $\tau_x(v) < \tau_x(u)$.*

Proof. Equation (4.4) gives $v \in \Gamma(u)$. If u had been selected before v in $\Gamma(x)$, rule (4.3) would have excluded v. The two selection indices are distinct, so the stated inequality follows. □

Lemma 5.2 (Triangle comparison)**.** *If u, v, w form a triangle of G in rows $r_u < r_v < r_w$, then*

$$e_{r_w}(r_u) < e_{r_w}(r_v). \tag{5.1}$$

Proof. Both v, w lie in $\Gamma(u)$. Lemma 5.1 places w before v in that neighborhood. Rule (4.2), applied when v is selected and with $t = r_w$, gives (5.1). □

Let F be the complete tripartite graph with parts

$$\{x, p\}, \qquad \{y, z\}, \qquad \{q, w\},$$

equipped with the order

$$x < y < p < q < z < w. \tag{5.2}$$

Proposition 5.3. *There is no edge-preserving copy of F in G whose six image rows are strictly increasing in the order (5.2).*

Proof. Suppose such a copy exists, and retain the labels of F for its image vertices. Write r_v for the row of an image vertex v. The vertices y, q, z, w lie in $\Gamma(x)$. Applying Lemma 5.1 to the edges zw, qz, yq gives

$$\tau_x(w) < \tau_x(z) < \tau_x(q) < \tau_x(y).$$

At the selection of y, the row r_w has already been selected, as has $r_z < r_w$. Rule (4.2) therefore gives

$$e_{r_w}(r_z) < e_{r_w}(r_y). \tag{5.3}$$

The triples y, p, w and p, z, w are triangles of F. Lemma 5.2 gives

$$e_{r_w}(r_y) < e_{r_w}(r_p) < e_{r_w}(r_z),$$

contradicting (5.3). □

Proposition 5.4. *The graph G is K_4-free.*

Proof. A K_4 would occupy four distinct rows $r_a < r_b < r_c < r_d$. Its last three vertices lie in $\Gamma(a)$. Selection reversal gives $\tau_a(d) < \tau_a(c) < \tau_a(b)$. At the selection of b, rule (4.2) implies $e_{r_d}(r_c) < e_{r_d}(r_b)$. The triangle b, c, d gives the opposite strict inequality by Lemma 5.2. □

6. A finite unordered target

The following specialization of the Nešetřil–Rödl ordering property [7] removes the prescribed order on F.

Lemma 6.1. *There is a finite K_4-free graph H_0 such that every linear order on $V(H_0)$ contains an order-preserving, edge-preserving copy of the ordered graph F from (5.2).*

Proof. For a sufficiently large integer n, choose a random 6-uniform hypergraph on $[n]$, including each six-element set independently with probability $p = n^{-19/4}$. Let X count its hyperedges. Let Y count pairs of distinct hyperedges with an intersection of size at least two. Let Z count triples of distinct hyperedges E_1, E_2, E_3 admitting distinct vertices

$$v_{12} \in E_1 \cap E_2, \qquad v_{23} \in E_2 \cap E_3, \qquad v_{31} \in E_3 \cap E_1.$$

There are at most n^{10} possible pairs counted by Y. Indeed, choose two common vertices and at most four further vertices for each of the two hyperedges. There are at most n^{15} possible triples counted by Z, by choosing the three intersection witnesses and at most four further vertices for each hyperedge. Thus

$$\begin{aligned}\mathbb{E}(X - Y - Z) &\geq \binom{n}{6} p - n^{10}p^2 - n^{15}p^3 \\ &= \left(\frac{1}{720} + o(1)\right) n^{5/4} - n^{1/2} - n^{3/4}.\end{aligned}$$

There is an absolute $c > 0$ such that, for all sufficiently large n, some outcome satisfies $X - Y - Z \geq cn^{5/4}$. From this outcome, delete one hyperedge from each originally counted bad pair or triple. Repeated deletions of the same hyperedge are counted once. The resulting hypergraph $\mathcal{H}$ has

$$m = |\mathcal{H}| \geq cn^{5/4}, \tag{6.1}$$

distinct hyperedges intersect in at most one vertex, and no triple of hyperedges has three distinct intersection witnesses as above.

Independently for each $E \in \mathcal{H}$, choose a uniformly random bijection from the six labels of F onto E, and place the corresponding edges of F on E. Let H_0 be the union of all these graph edges. Every graph edge has a unique supporting hyperedge, since two hyperedges meet in at most one vertex.

Every triangle of H_0 lies in one supporting hyperedge. If two of its edges have the same support E, all three triangle vertices belong to E, and the support of its third edge must also be E. If the three supports are distinct, the three triangle vertices are distinct intersection witnesses, contrary to the construction of $\mathcal{H}$. If H_0 contained a K_4, two triangles sharing an edge would have the same supporting hyperedge. All four vertices would lie in that hyperedge, and uniqueness of support would place all six edges in its copy of F. This contradicts the fact that $F = K_{2,2,2}$ is K_4-free. Therefore every choice of the label bijections produces a K_4-free graph H_0.

Fix a linear order on $[n]$. For any $E \in \mathcal{H}$, the probability that its labels occur in precisely the order (5.2) is $1/6!$. These events are independent over the hyperedges, since their label bijections were chosen independently. The probability that at least one of the $n!$ linear orders has no successful hyperedge is at most

$$n! \left(1 - \frac{1}{6!}\right)^m \leq \exp\left(n \log n - \frac{m}{720}\right) < 1 \tag{6.2}$$

for sufficiently large n, by (6.1). Some choice of the label bijections therefore works for every linear order on $[n]$. Fix that choice to obtain H_0. □

Proposition 6.2. *Under* CH*, the graph G from Section 4 contains no copy of H_0.*

Proof. Suppose $f\colon V(H_0) \to V(G)$ is an injective edge-preserving map. Order $V(H_0)$ by the rows of its images under f, breaking ties by any fixed linear order. Lemma 6.1 gives an ordered copy of F for this linear order. Its six image rows are nondecreasing in the order x, y, p, q, z, w. Each consecutive pair in this list is an edge of F. Since G has no same-row edges, each consecutive pair of image rows is strictly increasing. Proposition 5.3 now gives a contradiction. □

Color the edges of G blue and all other pairs red. Lemma 4.2 and Proposition 6.2 show that this coloring has neither a red set of order type B nor a blue copy of H_0. The finite graph H_0 was fixed in Lemma 6.1, before the assumption of CH. This proves (1.1).

7. The positive relation and independence

Recall that $\mathrm{MA}_{\aleph_1}$ states that for every nonempty partially ordered set satisfying the countable chain condition, every family of at most $\aleph_1$ dense subsets is met by a filter. We use the following partition theorem.

Theorem 7.1 (Baumgartner [1, Theorem 3.5(a)])**.** *Assume* $\mathrm{MA}_{\aleph_1}$*. Then*

$$\omega_1 \cdot \omega \to (\omega_1 \cdot \omega, n)^2 \qquad \text{for every finite } n.$$

Here the second alternative is a blue complete graph on n vertices.

Proof of (1.2) *and Theorem 1.2.* Let H be a finite graph with n vertices. For $n = 0$, every host contains the empty copy, so the assertion is immediate. For $n > 0$, restrict an arbitrary coloring of $[\Omega]^2$ to its initial segment of type B. Theorem 7.1 gives a red set of order type B or a blue K_n. Every n-vertex graph has an edge-preserving copy in K_n, proving (1.2). Together with the construction in Section 6, this proves Theorem 1.2. □

Proof of Corollary 1.3. Let A be the assertion that $\Omega \to (B, H)^2$ for every finite K_4-free graph H, and let A_0 be $\Omega \to (B, H_0)^2$. Theorem 1.2 proves

$$\mathrm{CH} \implies \neg\mathsf{A} \wedge \neg\mathsf{A}_0, \qquad \mathrm{MA}_{\aleph_1} \implies \mathsf{A} \wedge \mathsf{A}_0.$$

Gödel's relative consistency theorem gives a model of $\mathrm{ZFC} + \mathrm{CH}$ whenever ZFC is consistent [5]. The relative consistency theorem of Solovay and Tennenbaum gives a model of $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1}$ under the same assumption [9]. Thus each of A and A_0, as well as each of their negations, is consistent with ZFC relative to its consistency. □

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