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FREE FERMIONS FROM INTEGER ARITHMETIC: A GALOIS-THEORETIC TEST FOR FREE-FERMIONIC SPECTRA

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Abstract. The spectrum of a free-fermionic operator is built from a few single-particle quantities. For operators with rational entries this confines, as is known, the Galois group of the characteristic polynomial of each sector to a hyperoctahedral group, up to elementary abelian 2-groups. We turn this into a test for free-fermionic spectra that uses only integer arithmetic: Frobenius cycle types modulo primes can prove that a spectrum is not free-fermionic, and an exact recovery of the single-particle data, certified by power sums, confirms it when it is. As a worked example, the test recovers Onsager's dispersion relation from the characteristic polynomial of the Ising transfer matrix. As applications, we prove that the periodic Fendley chain with generic couplings admits no free-fermion solution of the Chapman–Elman–Mann form for a range of sizes, bearing on whether a simplicial clique is necessary for free-fermion solvability, and we detect hidden tensor-product and matchgate structure in any basis. The Galois group grades spectral structure: hyperoctahedral for free fermions, symmetric on subsets with particle-number conservation, apparently imprimitive for the partially decoupled Bethe equations of the exclusion process, and alternating within sectors for the interacting models tested. Conversely, for operators with simple spectrum in each sector, a Galois action through the hyperoctahedral group on sign vectors characterizes exactly the polynomials, with coefficients in the base field, in operators with free-fermionic spectra.

1. Introduction

A free-fermionic model is exactly solvable because its many-body spectrum is built from a few single-particle energies. For the two-dimensional Ising model this is the content of Kaufman's spinor solution [36] of Onsager's model [46]: every eigenvalue of the row transfer matrix on a ring of L sites has the form

$$\lambda_s = \kappa \prod_{k=1}^{L} e^{s_k \gamma_k/2}, \qquad s \in \{\pm 1\}^L, \tag{1}$$

with a fixed parity of s in each spin-flip sector. The same structure governs matchgate circuits, which are classically simulable for this reason [57, 56, 31], and Hamiltonians that are free only after a nonlocal change of variables, the free fermions in disguise of Fendley [16], characterized graph-theoretically by Chapman, Flammia, Elman, and Mann [10, 12, 9].

Arranged by s, the spectrum (1) is the rank-one tensor $\kappa\,(u_1, u_1^{-1}) \otimes \cdots \otimes (u_L, u_L^{-1})$, with $u_k = e^{\gamma_k/2}$, a point of the Segre variety $(\mathbb{P}^1)^L$, and its eigenvalues satisfy $\lambda_s \lambda_t = \lambda_{s'} \lambda_{t'}$ whenever $s + t = s' + t'$. For an operator with rational entries every automorphism of $\overline{\mathbb{Q}}$ preserves these relations, so the Galois group of the characteristic polynomial is confined to the symmetries of the hypercube: for the square of the transfer matrix in a spin-flip sector it is a subgroup of the hyperoctahedral group $W(B_L)$, of order $2^L L!$, and for the transfer matrix itself, whose eigenvalues involve the half-angles $\gamma_k/2$ and a possibly irrational κ, an extension of such a

subgroup by an elementary abelian 2-group (Proposition 2.4); a generic matrix of size 2^L has the symmetric group S_{2^L}. This confinement is a special case of a theorem of Jouve, Kowalski, and Zywina [30]: for an element g of a reductive group, the Galois group of the characteristic polynomial of $\rho(g)$ lies in the Weyl group extended by the Galois action on characters, and acts through the weights of ρ. Free fermions are the case of $\mathrm{Spin}(2L)$ in its spin representation, whose weights are the sign vectors $(\pm\frac{1}{2},\dots,\pm\frac{1}{2})$. With particle-number conservation the hidden group is $U(L)$ acting on the N-particle space $\Lambda^N\mathbb{C}^L$, and its Weyl group S_L acts on N-element subsets.

This paper uses the confinement as a test that reads off the hidden group through its Weyl group, with free fermions as the main case. The Galois group of an integer polynomial is accessible through its factorizations modulo primes: by Chebotarev's density theorem [53], the degrees of the irreducible factors modulo p are cycle types of Frobenius elements, which equidistribute in the group. A cycle length incompatible with the allowed group orders, or a certificate that the group contains an alternating group A_d with $d \geq 5$ and $d > L$, rules out free-fermionic structure. The test needs no fermionic representation, works in any basis, and costs a few factorizations of integer polynomials. It concerns the spectrum only: it detects free-fermionic spectral structure, not fermionic dynamics, and operators whose spectra are trivially of free form, such as commuting stabilizer Hamiltonians, pass it.

Results. Section 2 proves the hyperoctahedral lemma and derives rigorous criteria for non-freeness: for transfer matrices and their squares, for complex and orthogonal operators, for number-conserving free fermions and free parafermions, and for Pauli Hamiltonians with symmetry sectors. Section 3 gives a three-step procedure (screen, recover, certify) and works it out for the Ising transfer matrix, recovering Onsager's dispersion relation from the integer characteristic polynomial and certifying the Kaufman form exactly for strips of width 3 to 7. Section 4 contains the applications. The first concerns claw-free frustration graphs without a simplicial clique, which lie outside the free-fermion solutions of Chapman, Elman, and Mann [9]: we prove that for generic couplings the periodic Fendley chain and further cycle powers admit no free-fermion solution of their form at a range of sizes (Theorem 4.1), and we certify 23 graphs on at most 9 vertices, 176 on 10, and 29 on 11. This complements exact free solutions reported for other graphs of this kind, including the periodic Fendley chain on 8 and 9 sites [2]. The second recovers hidden group structure, tensor factors and free-fermion modes, from the multiplicative relations among eigenvalues. The third concerns quantum circuits (Section 4.3). Section 5 shows how the Galois group grades spectral structure: interacting integrable models have alternating groups inside their symmetry sectors, in agreement with recent work on Bethe equations [6, 40], while the totally asymmetric exclusion process, whose Bethe equations partially decouple, appears to have imprimitive groups of intermediate size. It also proves a converse: a Galois action through sign vectors characterizes exactly the polynomials in operators with spectra of free form, so the Galois action sees free form up to such a polynomial and nothing finer (Proposition 5.1). Appendix A validates the test on free-fermion models from the literature: in every family the points flagged free are known free-fermion points or have roots-of-unity spectra, except for three-dimensional Ising bars with a Clifford background, which it explains through the Lie algebra of a Pauli orbit. At one of these points the hidden group is symplectic, which the screen does not exclude, although the Galois action on the distinct eigenvalues of a power of the transfer matrix in a sector is not through sign vectors; Table 6 summarizes the families. Appendix B recovers the single-particle spectra of further known models (Table 7), and Section 6 lists open questions.

The arithmetic of spectra has been used before for particular models, for example the Galois action on the spectra of small Heisenberg rings [44] and integer factorization as a signature of Hilbert-space fragmentation [48]. The use of hyperoctahedral Galois groups as a test for free-fermionic structure, the certificates of non-freeness, and the three-step procedure appear to be new.

2. The hyperoctahedral lemma and certificates of non-freeness

Definition 2.1. Let $\chi \in \mathbb{Q}[\mu]$. We say χ has *free form of rank* L if there are $\kappa \in \mathbb{Q}^\times$, a squarefree reciprocal polynomial $P \in \mathbb{Q}[u]$ of degree $2L$ with roots $u_1^{\pm 1}, \dots, u_L^{\pm 1}$, none equal to ± 1, and a set S of sign vectors, either all of $\{\pm 1\}^L$ or its even-parity half, such that

$$\chi(\mu) = \prod_{s \in S} \Big(\mu - \kappa \prod_{k=1}^{L} u_k^{s_k} \Big).$$

The arguments below use only the splitting field N of P. For uniform couplings P need not be squarefree ($\gamma_q = \gamma_{-q}$ below), and χ is then not literally of free form of lower rank; but N is the splitting field of the squarefree part of P with the factors $u \mp 1$ removed, so parts (1) and (2) of Lemma 2.2 and Corollary 2.3 hold with the rank $r \le L$ of that part. For the Ising transfer matrix V, χ is the characteristic polynomial of V^2 on a spin-flip sector, P is the characteristic polynomial of the single-particle matrix, and $u_k = e^{\gamma_k}$ [36]. For a free-fermion Hamiltonian the form is additive, $E_s = e_0 + \sum_k s_k \varepsilon_k$, and the arguments below apply with sums in place of products.

Let $W(B_L) = C_2 \wr S_L$ act on $\{\pm 1\}^L$ by $(\pi, \epsilon) \cdot s = (\epsilon_{\pi^{-1}(k)} s_{\pi^{-1}(k)})_k$, and let $W(D_L)$ be the subgroup with $\prod_k \epsilon_k = 1$.

Lemma 2.2. *Let χ have free form of rank L (Definition 2.1), with the roots $\mu_s = \kappa \prod_k u_k^{s_k}$, $s \in S$, pairwise distinct, and let N be the splitting field of P. Then:*

1. *the splitting field of χ is contained in N, so $\mathrm{Gal}(\chi)$ is a quotient of $\mathrm{Gal}(N/\mathbb{Q})$;*
2. *$\mathrm{Gal}(N/\mathbb{Q})$ embeds in $W(B_L)$ via $\sigma(u_k) = u_{\pi(k)}^{\epsilon_k}$;*
3. *$\mathrm{Gal}(N/\mathbb{Q})$ acts on the roots of χ through sign vectors: $\sigma(\mu_s) = \mu_{(\pi,\epsilon)\cdot s}$;*
4. *if S is the even half and no μ_s with s even equals $\kappa \prod_k u_k^{t_k}$ with t odd, the image lies in $W(D_L)$.*

In particular every cycle of a Frobenius element at a prime of good reduction has length at most the largest order $\mathrm{maxord}(B_L)$ of an element of $W(B_L)$; if $S = \{\pm 1\}^L$, or under the hypothesis of (4), its cycle type is that of an element of $W(B_L)$, respectively $W(D_L)$, acting on S.

Proof. Each μ_s is a monomial in the roots of P, which gives (1). An automorphism σ permutes the roots of P and commutes with $u \mapsto u^{-1}$, so it permutes the L pairs $\{u_k, u_k^{-1}\}$, which are distinct because P is squarefree with no root ± 1. Recording the permutation π of the pairs and whether u_k goes to $u_{\pi(k)}$ or to its inverse gives an injective homomorphism into $W(B_L)$, which is (2). Then $\sigma(\mu_s) = \kappa \prod_k u_{\pi(k)}^{\epsilon_k s_k} = \kappa \prod_j u_j^{\epsilon_{\pi^{-1}(j)} s_{\pi^{-1}(j)}} = \mu_{(\pi,\epsilon)\cdot s}$, which is (3). For (4), σ maps roots of χ to roots of χ, and (π, ϵ) changes the parity of s by $\prod_k \epsilon_k$. □

Lemma 2.2 is the elementary core of a general statement. For a reductive group G over $\mathbb{Q}$ with a faithful representation ρ, Jouve, Kowalski, and Zywina show that the Galois group of the splitting field of $\det(T - \rho(g))$ is a subquotient of the group generated by the Weyl group and the Galois action on the characters of a maximal torus, that the eigenvalues are the values of the weights of ρ at g, and that Galois acts through its action on the weights [30, Prop. 2.1, Lemma 2.4]; random elements attain the whole group [30, Thm. 1.1] (see also [24]).

For $\mathrm{Spin}(2L) \times \mathbb{G}_m$ in the spin representation this gives Lemma 2.2. Galois groups and multiplicative relations among eigenvalues are related in [39, 52].

Corollary 2.3 (non-freeness certificate)**.** *If an irreducible factor of χ of degree $d \geq 5$ has Galois group containing A_d and $d > L$, then χ divides no polynomial of free form of rank at most L.*

Proof. Parts (1) and (2) of Lemma 2.2 do not require the μ_s to be distinct: the splitting field of every divisor of a polynomial of free form of rank L lies in N, so the Galois group of every factor is a quotient of a subgroup of $W(B_L) = C_2^L \rtimes S_L$. Since the factor is irreducible of degree d, its Galois group is a subgroup of S_d; containing A_d, it is A_d or S_d, and for $d \geq 5$ it has the simple group A_d as a composition factor, so A_d is a composition factor of a subgroup of $W(B_L)$. A subgroup H of $W(B_L)$ has the abelian normal subgroup $H \cap C_2^L$ with quotient in S_L, so A_d is a composition factor of a subgroup of S_L, and $d!/2 \leq L!$. This forces $d \leq L$. □

The same argument applies to every bound below: in each case the nonabelian composition factors of the relevant Galois group are those of subgroups of S_L (of S_α or S_t in Proposition 2.9), so the order bounds stated there are needed only for element orders, and an alternating group A_d, $d \geq 5$, is excluded as soon as $d > L$ (respectively $d > \alpha$ and $d > t$). In each application the rank is fixed in advance by the size of the system: for a transfer matrix on L sites, or a circuit on L qubits, it is L, the number of Majorana pairs, and for a Hamiltonian it is the number of modes; certificates exclude free form of this rank. The bounds below apply in the same way to sectors and factors: they constrain every polynomial whose roots are of the given form, not only the full characteristic polynomial. The operator itself, not its square, is usually given, and its entries may be complex.

Proposition 2.4. *Suppose every eigenvalue of a rational matrix is of the form $\lambda_s = \kappa \prod_k v_k^{s_k}$ for some $s \in \{\pm 1\}^L$, with $\kappa^2 \in \mathbb{Q}$ and $v_k^{\pm 2}$ the roots of a rational reciprocal polynomial of degree $2L$. Then all eigenvalues lie in a Galois extension of $\mathbb{Q}$ of degree at most $G_L = 2^{2L+1}L!$, in whose Galois group every element has order dividing $2a$ for the order a of some element of $W(B_L)$. If instead the matrix is the realification of a matrix over $\mathbb{Q}(i)$ whose eigenvalues have this form with $\kappa^2 \in \mathbb{Q}(i)$ and $v_k^{\pm 2}$ the roots of a reciprocal polynomial over $\mathbb{Q}(i)$, then all eigenvalues lie in a Galois extension of $\mathbb{Q}$ of degree at most $2G_L^2$ in whose Galois group every element has order dividing $4\,\mathrm{lcm}(a, b)$ for the orders a, b of some elements of $W(B_L)$. Consequently a factor of degree $d \geq 5$ with Galois group containing A_d and $d > L$, or a Frobenius cycle whose length divides no such $2a$ (respectively no such $4\,\mathrm{lcm}(a, b)$; for instance, for a single set in the first case, a cycle longer than $2\,\mathrm{maxord}(B_L)$), proves that the eigenvalues are not all of this form.*

Proof. Let N be the splitting field of the reciprocal polynomial, of degree at most $2^L L!$. The set $\{v_k^{\pm 2}\}$ is Galois stable, so $N(v_1, \ldots, v_L, \kappa)$ is Galois over $\mathbb{Q}$ of degree at most $2^{L+1} \cdot 2^L L!$, and the kernel of the restriction map from its Galois group onto $\mathrm{Gal}(N/\mathbb{Q})$ is an elementary abelian 2-group; if the image of g has order a, then $g^{2a} = 1$. In the second case the same construction over $\mathbb{Q}(i)$ gives a Galois extension K of $\mathbb{Q}(i)$ of degree at most G_L containing the eigenvalues of the complex matrix. The eigenvalues of its realification are these and their complex conjugates, which lie in $K\bar{K}$, a Galois extension of $\mathbb{Q}$ of degree at most $2G_L^2$. The group $\mathrm{Gal}(K\bar{K}/\mathbb{Q}(i))$ embeds in $\mathrm{Gal}(K/\mathbb{Q}(i)) \times \mathrm{Gal}(\bar{K}/\mathbb{Q}(i))$, so its elements have orders dividing $\mathrm{lcm}(2a, 2b)$, and every element of $\mathrm{Gal}(K\bar{K}/\mathbb{Q})$ has its square in this subgroup. In both cases the nonabelian composition factors are those of subgroups of S_L, and A_d with $d \geq 5$ is one only if $d \leq L$, as in Corollary 2.3. □

Remark 2.5 (two sectors). Suppose the eigenvalues form two sets as in Proposition 2.4, each with its own rational data $\kappa_i^2 \in \mathbb{Q}$ and $P_i \in \mathbb{Q}[u]$ (respectively over $\mathbb{Q}(i)$), as for the two parity sectors of a transfer matrix, with no assumption that the sectors are defined over $\mathbb{Q}$. The argument of the proof, applied to the compositum of the two fields, gives element orders dividing $2\,\mathrm{lcm}(a,b)$, respectively $4\,\mathrm{lcm}(a,b,c,d)$, for orders a,b,c,d of elements of $W(B_L)$, and the composition-factor criterion is unchanged. Every cycle-length certificate in this paper also satisfies this stronger condition.

Remark 2.6 (orthogonal operators). If $M = cU$ with U real orthogonal and $c^2 \in \mathbb{Q}$, then after removal of the factor $x^2 - c^2$ (or of the factors $x \mp c$ if $c \in \mathbb{Q}$) the characteristic polynomial is $f(x) = x^m h(x + c^2/x)$, and the Galois group of f is imprimitive with blocks $\{x, c^2/x\}$, so it never contains $A_{\deg f}$. Proposition 2.4 applies instead to the half-degree polynomial h, whose splitting field lies in that of f: a factor of h of degree $d \geq 5$ containing A_d with $d > L$ certifies non-freeness.

Proposition 2.7 (number-conserving free fermions)**.** *Suppose every eigenvalue of a rational matrix is $E_A = \sum_{k\in A} \varepsilon_k$ for an N-element subset A of $\{1,\dots,L\}$, where the ε_k are the distinct roots of a rational polynomial Q of degree L. Then the Galois group of the characteristic polynomial is a quotient of $\mathrm{Gal}(Q) \leq S_L$. If the eigenvalues are the E_A for all A and are pairwise distinct, the Galois group acts on them through the action of S_L on N-element subsets.*

Proof. Each E_A lies in the splitting field of Q. An automorphism σ permutes the roots of Q by a permutation π, and $\sigma(E_A) = E_{\pi(A)}$. □

This is the case of $U(L)$ acting on $\Lambda^N \mathbb{C}^L$ in [30, Lemma 2.4], whose Weyl group is S_L. As in Corollary 2.3, a factor of degree $d \geq 5$ whose Galois group contains A_d with $d > L$ excludes this form; Section 5 confirms the action on subsets for tight-binding chains.

Remark 2.8 (free parafermions). For free parafermions with $\mathbb{Z}_3$ symmetry [15] the energies are $E = e_0 + \sum_k \omega^{a_k}\varepsilon_k$, $a \in \{0,1,2\}^L$, $\omega = e^{2\pi i/3}$. If $e_0 \in \mathbb{Q}(\omega)$ and the ε_k^3 are the roots of a polynomial of degree L over $\mathbb{Q}(\omega)$, then, as in Proposition 2.4, every energy lies in the field generated over $\mathbb{Q}(\omega)$ by the cube roots of the roots of this polynomial and of its complex conjugate, a Galois extension of $\mathbb{Q}$ of degree at most $2(3^L L!)^2$ whose nonabelian composition factors are those of subgroups of S_L; a factor of degree $d \geq 5$ whose Galois group contains A_d with $d > L$ excludes this form.

Pauli Hamiltonians. Let $H = \sum_j b_j P_j$ be a sum of Pauli strings with frustration graph G (an edge between anticommuting strings). In the free-fermion solutions of Chapman, Elman, and Mann [9, Sec. I.A, Result 1], each symmetry sector, a joint eigenspace of commuting conserved charges, carries an additive free spectrum $\sum_{j=1}^{\alpha} s_j\varepsilon_j$ with $\alpha = \alpha(G)$ modes, the independence number of G. Every Pauli realization of G is a representation of the twisted group algebra of G; its irreducible modules, which we call central sectors, are labeled by the eigenvalues of central Pauli products, are defined over $\mathbb{Q}(i)$, and have dimension $D = 2^{r/2}$ with r the rank of the anticommutation form. For rational couplings the restriction of H to a central sector is a Hermitian matrix over $\mathbb{Q}(i)$, so its characteristic polynomial lies in $\mathbb{Q}[E]$. We say that a central sector has the *Chapman–Elman–Mann form* if it is the direct sum of the joint eigenspaces J of commuting operators with entries in $\mathbb{Q}(i)$ that lie in the algebra generated by the terms of H and commute with H, as the conserved charges of [9] do, and if on each J the spectrum is the multiset of the numbers $\sum_j s_j\varepsilon_{J,j}$, with no constant term, over all $s \in \{\pm 1\}^\alpha$ or over a parity half, each s counted at least once, where $\pm\varepsilon_{J,j}$ are the roots of a polynomial whose coefficients lie in the field generated over $\mathbb{Q}(i)$ by the eigenvalues labeling J. Allowing parity halves only weakens the criterion below; each J has at least $2^{\alpha-1}$ states.

Proposition 2.9. *Suppose a central sector of dimension D has the Chapman–Elman–Mann form. If a factor of degree $d \geq 5$ of its characteristic polynomial, in E or in E^2, has Galois group containing A_d, then $d \leq \alpha$ or $d\,2^{\alpha-1} \leq D$. If α is even, the set of eigenvalues of the sector is symmetric under $E \mapsto -E$.*

Proof. Work over $\mathbb{Q}(i)$. The Galois group of the factor over $\mathbb{Q}(i)$ is a subgroup of index at most 2 of its group over $\mathbb{Q}$, so it still contains A_d, and the factor remains irreducible over $\mathbb{Q}(i)$. Let E be an eigenvalue in the subspace J such that E or E^2 is a root of the factor, let F be the field generated over $\mathbb{Q}(i)$ by the labels of J, and let $t = [F : \mathbb{Q}(i)]$. An automorphism of $\overline{\mathbb{Q}}$ over $\mathbb{Q}(i)$ fixes the central sector and maps J to the joint eigenspace with the conjugate labels; the t embeddings of F give t distinct joint eigenspaces, each of dimension at least $2^{\alpha-1}$, so $t\,2^{\alpha-1} \leq D$. The splitting field over F of the polynomial with roots $\pm\varepsilon_{J,j}$ contains E and E^2, and as in Lemma 2.2 its Galois group is isomorphic to a subgroup of $W(B_\alpha)$. (Repeated or vanishing $\varepsilon_{J,j}$ only shrink this group.) Let M be this splitting field; its Galois closure Ω over $\mathbb{Q}(i)$ is the compositum of its images $\tau(M)$ under the t embeddings τ of F. The group $\mathrm{Gal}(\Omega/\mathbb{Q}(i))$ permutes the embeddings by composition, with image in S_t; its kernel K fixes every $\tau(F)$ pointwise, so it maps each $\tau(M)$, which is Galois over $\tau(F)$, into itself, and K embeds in $\prod_\tau \mathrm{Gal}(\tau(M)/\tau(F)) \cong \mathrm{Gal}(M/F)^t$. The Galois group of the factor over $\mathbb{Q}(i)$ is a quotient of $\mathrm{Gal}(\Omega/\mathbb{Q}(i))$ and contains the simple group A_d, so A_d is a composition factor of K or of a subgroup of S_t, hence of a subgroup of $W(B_\alpha)$ or of S_t. As in Corollary 2.3, $d \leq \alpha$ or $d \leq t \leq D/2^{\alpha-1}$. For even α the map $s \mapsto -s$ preserves parity, so the set of eigenvalues on each J, and hence on the sector, is symmetric. □

We therefore certify non-freeness of the Chapman–Elman–Mann form when

$$d > \alpha \qquad \text{and} \qquad d\,2^{\alpha-1} > D,$$

or, for even α, when the set of eigenvalues of a central sector is not symmetric under $E \mapsto -E$, that is, when its squarefree part χ_{red} does not satisfy $\chi_{\mathrm{red}}(-E) = \pm\chi_{\mathrm{red}}(E)$.

Certifying alternating groups. A primitive permutation group of degree d containing a cycle of prime length $p \leq d-3$ contains A_d, by a theorem of Jordan [11, Thm. 3.3E]. A transitive group containing a p-cycle c with $p > d/2$ is primitive. Indeed, c permutes the blocks of a block system in orbits of size 1 or p. A block fixed by c that contains a point moved by c contains all p such points, so it has more than $d/2$ points and is everything; otherwise the moved points lie in at least $p > d/2$ distinct blocks, which are then singletons. A Frobenius element with a single cycle of prime length p, $d/2 < p < d-2$, and all other cycle lengths prime to p has a power that is a p-cycle, so its observation certifies A_d in the Galois group of an irreducible factor of degree d. The range $d/2 < p < d-2$ contains a prime only for $d \geq 8$; smaller factors are certified only through cycle lengths.

3. Screen, recover, certify

Let M be a square integer matrix and χ its characteristic polynomial, restricted to a symmetry sector when one is known.

(1) *Screen.* Remove zero roots and repeated factors, replace $f(x) = g(x^k)$ by g when all exponents are divisible by k, and factor over $\mathbb{Q}$. For each factor of degree at least 4, factor modulo 80 to 400 primes of good reduction, record the cycle types, and test the cycle lengths and Jordan's criterion. If a criterion of Section 2 applies, stop: the spectrum is not free-fermionic, with proof. Otherwise the point is a *free candidate*.

(2) *Recover.* Compute the roots to high precision. Within a sector, the ratios μ_s/μ_t for s, t differing in two positions i, j are the most frequent nontrivial ratios, of the form $u_i^{\pm 2}u_j^{\pm 2}$; for $S = \{\pm 1\}^L$ the ratios $u_i^{\pm 2}$ for one position are more frequent still and give the u_i directly. Choosing $r_{ij} = u_i^2 u_j^2$ consistently among them gives $u_i^4 = r_{ij}r_{ik}/r_{jk}$, and the remaining u_k are peeled off one at a time. Accept the labeling only if it regenerates the whole spectrum, then recognize the coefficients of P and κ as rationals.

(3) *Certify.* Compare exact power sums. For $S = \{\pm 1\}^L$, $\sum_s \mu_s^m = \kappa^m \prod_k (u_k^m + u_k^{-m}) = \kappa^m \det(2T_m(C))$, where C is the companion matrix of the polynomial with roots $c_k = (u_k + u_k^{-1})/2$ and T_m is the Chebyshev polynomial. For the even half,

$$\sum_{s \text{ even}} \mu_s^m = \frac{\kappa^m}{2}\Big(\det(2T_m(C)) + \sigma \det(2U_{m-1}(C))\Big), \qquad \sigma^2 = \det(C^2 - I),$$

with U_{m-1} the Chebyshev polynomial of the second kind; the sign of σ is fixed by the case $m = 1$. Equality with the exact traces $\operatorname{Tr} M^m$ for $m = 1, \ldots, \deg \chi$ gives χ exactly, by Newton's identities.

A certificate in step 1 proves that the spectrum at a given rational point is not of free form; steps 2 and 3, when they succeed, prove that it is. If neither applies, the procedure is inconclusive at that point. Step 1 alone has false positives: rational or cyclotomic spectra (finite-order and Clifford operators), accidental factorizations at special points, and sectors too small for any group to be excluded. Commuting stabilizer Hamiltonians such as the toric code [37] have integer spectra that are sums of independent signed couplings with parity constraints, and are therefore trivially of free form: the procedure concerns free-fermionic structure of the spectrum, not whether the dynamics is that of nontrivially coupled fermions. Step 2 is unambiguous for a simple spectrum; with repeated pairs, as for the uniform couplings of Section 3.1, ratios coincide, the labeling is accepted only if it regenerates the spectrum, and step 3 certifies the result. Free form for generic parameters means free form over the function field $\mathbb{Q}(b)$ of the parameters b, with all data, κ, P, and the operators defining sectors, in $\mathbb{Q}(b)$ or $\mathbb{Q}(i)(b)$; since $\mathbb{Q}(i)(b^*) \cong \mathbb{Q}(i)(b)$ for algebraically independent values b^*, it is equivalent to free form at any such values, with data in $\mathbb{Q}(i)(b^*)$. If a family has free form for generic parameters, then at a rational parameter point the Galois group of each factor is a quotient of a subgroup of the generic Galois group (of a decomposition group at a prime of the integral closure above the maximal ideal of the point), and every bound above passes to such subquotients, since it bounds a composition factor or the order of an element [7, Ch. V, §2, Thm. 2]. Symmetry of the set of eigenvalues under $E \mapsto -E$ also passes to specializations, since the eigenvalues at the point are the specializations of the generic ones. A certificate at one rational point therefore excludes free form for generic parameters.

3.1. **Worked example: the Ising transfer matrix.** Take the zero-field Ising model on a ring of L sites with bond weight $x^{1+ss'}$ for an integer x, so that $e^K = x$ and the row transfer matrix $V = V_L$ has integer entries. The global spin flip commutes with V; in the basis of configurations with $s_1 = +1$, the sector matrices $V_\pm(r, r') = V(r, r') \pm V(r, -r')$ are integer matrices of size 2^{L-1}.

Recovery. Step 2, applied to the roots of the sector polynomials at 1200 bits, returns for $x = 2$ and $L = 4, \ldots, 7$

$$\cosh \gamma_q = \frac{289}{120} - \cos q,$$

with $q = (2k+1)\pi/L$ in the flip-even sector and $q = 2\pi k/L$ in the flip-odd sector, and $\kappa^2 = x^{4L}(x^2 - x^{-2})^L$. For $L = 5, 6, 7$ the recovered single-particle polynomials agree exactly

with $(-1)^L 2^{1-L}\big(T_L(\frac{289}{120}-c)\pm 1\big)$, with the upper sign in the flip-even sector; for $L=7$ in the flip-even sector the polynomial in $c=\cosh\gamma$ is

$$\frac{(120c-409)(1728000c^3-11620800c^2+25041960c-17261509)^2}{358318080000000}.$$

For $L=4$ the labeling of the flip-odd sector is ambiguous: $\gamma_i \mapsto \frac{1}{2}\sum_j \gamma_j - \gamma_i$ preserves the multiset of sums $\sum_k s_k\gamma_k$ over even s, a reflection of triality for D_4. The result is Onsager's dispersion relation, $\cosh\gamma_q = \cosh 2K^*\cosh 2K - \sinh 2K^*\sinh 2K\cos q$ with $\tanh K^* = e^{-2K}$ [46, 36], obtained without a fermionic or spinor construction. For a disordered and frustrated strip ($L=6$, couplings random in sign and size across the row and constant along it), the single-particle polynomial in c of each sector is an irreducible rational sextic; the two sextics differ only in their constant terms, and the Galois group of each is S_6, so the Galois group of each sector polynomial lies in $C_2^6 \rtimes S_6 = W(B_6)$.

Certificates. For $x=2,3$ and $L=3,\dots,7$, step 3 certifies in both sectors

$$\det(\mu - V_\pm^2) = \prod_{s\ \text{even}} \Big(\mu - \kappa^2 \prod_q e^{s_q\gamma_q}\Big),$$

with $\cosh\gamma_q = A - \cos q$, $A = \cosh^2 2K/\sinh 2K$, antiperiodic momenta in the flip-even sector and periodic momenta in the flip-odd sector. All products over momenta are determinants of Chebyshev polynomials in the companion matrix of $P(c) = (-1)^L 2^{1-L}\big(T_L(A-c)\pm 1\big)$, upper sign in the flip-even sector, so no algebraic numbers appear. The quantity $\sigma = \prod_q \sinh\gamma_q$ is rational: with $A\pm 1 = \cosh\theta_\pm$ one has $e^{\theta_\pm} = e^{2K\pm 2K^*}$ (for $x=2$, $20/3$ and $12/5$), and $\prod_q (c_q^2-1)$ is a square; for $L=3$ and $x=2$, $\sigma = 14874031/1728000$ and $12780719/1728000$ in the two sectors. For $x>1$ the matrix V_L is similar to a positive definite matrix, so its spectrum is determined by that of V_L^2. Since the spectrum of V_L determines $\operatorname{Tr} V_L^M$ for all M and $\operatorname{Tr} V_L^M$ is symmetric in L and M on the torus, these certificates give the exact partition function of every $L\times M$ torus with $3\le \min(L,M)\le 7$ at these couplings, and for $3\le L\le 7$ its restriction to each boundary sector in the direction of length L. A proof valid for all L requires an identity uniform in L, supplied in the known proofs by the spinor representation [36] or a determinant formula for the partition function.

4. Applications

4.1. **Claw-free frustration graphs without a simplicial clique.** Chapman and Flammia, and Elman, Chapman, and Flammia, showed that claw-free frustration graphs satisfying further conditions give free-fermion solutions [10, 12]. Chapman, Elman, and Mann proved that a claw-free graph with a simplicial clique, a clique K such that for every $j\in K$ the neighborhood of j induces a clique in $G\setminus K$, admits a free-fermion solution, with energies given by the roots of independence polynomials [9, Sec. I.A, Result 1]; whether a simplicial clique is necessary is open [9, Sec. IX] (see also [50]). The spectrum depends on the realization only through the central sector [9, Sec. VIII], so the central sectors of the realization with one qubit per vertex cover all realizations. Graphs that become simplicial after twin collapse are free [49] and are excluded. Exact free solutions of the Chapman–Elman–Mann form for 107 graphs outside these classes, among them C_8^2 and C_9^2, are reported in [2].

Small graphs. There are $14, 50, 191, 881, 4494$ connected claw-free graphs on $5,\dots,9$ vertices, of which $0, 0, 2, 13, 186$ have no simplicial clique; all of these contain even holes, and 173 remain after twin collapse. Each was tested in every central sector for three random choices of integer couplings with Proposition 2.9 (Table 1). No control with a simplicial clique was ever certified, and none of the 106 graphs of [2] with exact free certificates that lie in our sets on at most 10

vertices is certified. Among the extensions by one vertex of the undecided graphs on 9 vertices, 176 of the 2624 graphs on 10 vertices are certified, 799 have a factor containing A_d to which Proposition 2.9 does not apply, 192 are consistent with free form, and 1457 give the test no power; in a sample of 800 graphs on 11 vertices with $\alpha \geq 3$, 29 are certified, 582 are consistent, and 189 give the test no power. Here a graph is consistent if some factor has degree $d \geq 5$ with $d > \alpha$ and $d\,2^{\alpha-1} > D$, so that Proposition 2.9 could apply, but no certificate is found; otherwise the test has no power beyond the symmetry test for even α, which these graphs pass. The classes are assigned in the order certified, A_d uncertified, consistent, no power. A graph counts as certified if some choice of couplings certifies it; of the 192 consistent graphs, 190 are consistent at all three choices. Numerically, none of the 190 graphs on 10 vertices consistent at all three choices, nor any of the 120 sampled on 11, has a sector spectrum that is a full additive hypercube or a union of two full 3-mode hypercubes; the free forms not excluded for them involve parity halves, which at these sizes impose almost no constraint.

class	graphs	certified	A_d, uncertified	consistent	no power
no simplicial clique, $n = 7$	1	1	0	0	0
no simplicial clique, $n = 8$	9	2	3	0	4
no simplicial clique, $n = 9$	163	20	48	6	89
simplicial clique (controls)	30	0	0	5	25
simplicial after twin collapse (controls)	28	0	0	0	28
with a claw (controls)	20	13	0	0	7

TABLE 1. Claw-free frustration graphs without a simplicial clique, $n \leq 9$, and controls. "A_d, uncertified": some factor has Galois group containing A_d, but $d\,2^{\alpha-1} > D$ fails, so Proposition 2.9 does not apply. "Consistent" and "no power" are defined in the text.

Cycle powers. The square C_M^2 of the cycle is the frustration graph of Fendley's chain [16] with periodic boundary conditions, whose spectrum Zheng, Pozsgay, and Chen obtain from functional relations of a commuting family of transfer matrices, with free, nested free, or interacting spectra depending on the couplings [60] (see also [21]). Cycle powers C_M^k are claw-free for every k. For each of three random choices of integer couplings, every central sector is certified by Proposition 2.9: by a factor whose Galois group contains the alternating group (for C_{17}^2, of degree 128), or, for C_M^4 with $M = 11, 13, 14$, by the asymmetry of the spectrum. By the remark on specialization in Section 3, this proves the following.

Theorem 4.1. *For generic couplings, in particular for algebraically independent couplings, no Pauli realization of the frustration graph C_M^2 of the periodic Fendley chain has a free-fermion solution of the Chapman–Elman–Mann form for $M = 7, 10, 11$ and $13 \leq M \leq 19$; the same holds for C_M^3 with $13 \leq M \leq 18$ and for C_M^4 with $M \in \{11, 13, 14, 15, 17, 18\}$.*

For C_M^2 with $M = 8, 9, 12$ the sector spectra are degenerate and small, and the test has no power; for $M = 8, 9$ exact free solutions are reported in [2], which also finds numerically that C_7^2 and C_{10}^2 have none.

The undecided graphs are mostly those with $\alpha = 2$. For them, if the spectrum is symmetric under $E \mapsto -E$, the factors of the polynomial in E^2 have degree at most $D/2$, and a factor in E is either even, so imprimitive, or paired with its reflection, so of degree at most $D/2$. Then either $d\,2^{\alpha-1} > D$ fails or A_d cannot occur, and deciding these graphs requires a test of free form that is not vacuous on parity halves.

4.2. **Hidden group structure from multiplicative relations.** Galois automorphisms preserve every multiplicative relation $\prod_i \lambda_i^{n_i} = 1$ among eigenvalues [39, 52]. The relation lattice and the configuration of weights it induces identify the algebraic group in which an operator lives: a hypercube or half-cube for free fermions, a grid $\{1, \ldots, a\} \times \{1, \ldots, b\}$ for a tensor product $A \otimes B$, a cross-polytope for a generic orthogonal or symplectic matrix, and no relations for a generic matrix. We compute the relations by lattice reduction on the logarithms of the eigenvalues at 2000 to 4000 bits, keep relations with residual at most $2^{-p/2}$ at a precision of p bits (every accepted relation has residual at most 2^{-1997}), and identify the configuration against a library of these shapes; the relations are verified numerically, not exactly. Table 2 shows that the factor structure is recovered from the characteristic polynomial alone, even after conjugation by a random unimodular matrix: the number of free-fermion modes, the sizes of hidden tensor factors, and mixtures of the two. Spectral methods cannot distinguish free fermions from a product of independent qubits, which give the same cube.

operator	expected rank	found	configuration
$A \otimes B$: 2×4, 3×3, 2×3, 4×4, $2 \times 2 \times 2$	$\sum_i a_i - r + 1$	$5, 5, 4, 7, 4$	grids (cube)
generic, size $6, 9, 16, 32$	full	full	no relations
Ising, zero field, $L = 4, \ldots, 7$	$L + 1$	$5, 6, 7, 8$	half-cube D_L
Ising, real field, $L = 5$	32	32	no relations
matchgate circuits, $n = 5, 6$	modes $+ 1$	$5, 7$	cube, D_6
XXZ, eight-vertex (integrable), $L = 6$	full	full	no relations
Ising ($L = 4, 5$) $\otimes$ generic 3×3	$L + 3$	$7, 8$	simplex $\times$ half-cube

TABLE 2. Rank of the multiplicative group generated by the distinct eigenvalues and the recovered configuration. For a tensor product of r factors of sizes $a_1, \ldots, a_r$ the expected rank is $\sum_i a_i - r + 1$; "full" means equal to the number of distinct eigenvalues. Circuit matrices are integer multiples of rational orthogonal matrices, and the scale adds one to the rank; for odd n, matchgate circuits have a zero mode that lowers it by one. The Ising entries use random couplings.

4.3. **Quantum circuits.** Circuits of real orthogonal two-qubit gates with rational entries (rational points on the circle) have rational unitaries. Nearest-neighbor matchgates $G(A, B)$, acting as A on $\mathrm{span}\{|00\rangle, |11\rangle\}$ and as B on $\mathrm{span}\{|01\rangle, |10\rangle\}$ with $\det A = \det B$, generate free-fermionic circuits [57, 56, 31]. On 5, 6, and 7 qubits, brickworks of matchgates are free, with longest cycles $4, 12, 12$, even after conjugation by random local rotations or, on 5 and 6 qubits, by a random rational orthogonal matrix of full size, since the test depends only on the spectrum. Next-nearest-neighbor matchgates routed through fermionic swaps and Floquet matchgate circuits are free as well. The screen flags 25 circuits: on each of 5, 6, and 7 qubits, brickworks of generic gates (longest cycles $30, 62, 126$), a single generic gate repeated as a Floquet circuit, and matchgate circuits with one generic gate, with one SWAP, with next-nearest-neighbor matchgates without fermionic swaps, with gates $G(A, B)$ having $\det A = -\det B$, with a single-qubit kick, or with alternate layers conjugated by a generic single-qubit rotation; together with matchgates on the edges of a 2×3 grid. By Remark 2.6 all 25 are certified not free: each has a factor of the half-degree polynomial whose Galois group contains A_d with $d > n$. For periodic brickworks at even n, each parity sector is free with its own modes, as for the Ising sectors of Section 3.1, and a hypercube labeling of each sector is recovered. Clifford-type circuits have

roots-of-unity spectra and pass the screen; their spectra are highly degenerate, which is why step 1 alone is not a proof of freeness.

5. What the Galois group sees

The Galois groups of sector factors grade the algebraic structure of a spectrum (Table 3). The bounds in the first two rows are Lemma 2.2 and Proposition 2.7, and equality with the full group is observed through Frobenius statistics; the last row is certified in every case listed; the third is observed.

spectral structure	Galois group of a sector factor	examples
free fermions	in $W(B_L)$ or $W(D_L)$	Ising strips, matchgate circuits, Fendley's chains
free fermions, particle number conserved	in S_L on N-element subsets	tight-binding chains, Alexander braid representations
partially decoupled Bethe equations	imprimitive (observed), blocks over real cyclotomic fields	totally asymmetric exclusion
interacting, integrable or not	contains A_d	XXZ, eight-vertex, symmetric exclusion, M_1, Baxter–Wu

Table 3. Galois groups of factors of sector polynomials.

Number conservation. For tight-binding chains with random integer hoppings and on-site energies, with $(L, N) = (8, 3)$, $(9, 4)$, and $(10, 3)$, the N-particle sector polynomial is irreducible of degree $\binom{L}{N}$, the longest Frobenius cycle equals Landau's function $g(L) = 15, 20, 30$, and over 1500 primes the frequencies of cycle types agree with the class distribution of S_L acting on N-element subsets within total variation 0.034, 0.057, 0.045, with no Frobenius element outside the group. The Alexander braid representations of Appendix A show the same signature.

Interacting integrable models. Integrable models that are not free behave like generic ones inside their symmetry sectors (Table 4). In inhomogeneous eight-vertex models whose columns share Baxter's invariants $\Delta = (a^2 + b^2 - c^2 - d^2)/(2(ab + cd))$ and $\Gamma = (ab - cd)/(ab + cd)$, so that the transfer matrices commute [13], and in the six-vertex model, the sector factors have Galois groups containing alternating groups of full degree, while free-fermion columns, $a^2 + b^2 = c^2 + d^2$ [14], give small groups.

columns	factor degrees	largest factor
free fermion, all different	4 ($\times$4), 12 ($\times$4)	longest cycle 4
integrable, fixed (Δ, Γ)	12 or 16	contains A_{12}, A_{16}
six-vertex, fixed Δ	$1, 6, 10, 15$	contains A_{15}
generic	16 ($\times$4)	contains A_{16}

Table 4. Inhomogeneous eight-vertex transfer matrices, $L = 6$, 200 primes per factor; in the integrable rows, $\Delta \neq 0$ and, except in the six-vertex row ($\Gamma = 1$), the parameters lie away from the degenerate values $\Delta = \pm 1$, $\Gamma = \pm 1$, $\Delta = \pm\Gamma$. Containment of A_{12}, A_{15}, A_{16} certifies non-freeness by Proposition 2.4, since $12, 15, 16 > 6$.

The lattice supersymmetric M_1 model [45, 18], whose chain is equivalent to the XXZ chain at $\Delta = -1/2$, shows its supersymmetry exactly in the characteristic polynomial: every nonzero factor of the sector with f fermions divides the polynomial of a sector with $f \pm 1$ fermions, and the weighted count of zero modes equals the Witten index. Its sectors on chains, ladders, a triangular strip, a 3×4 grid, and trees nevertheless contain alternating groups, for example A_{44} and A_{76} on an open chain of 12 sites. These results agree with the large Galois groups of Bethe equations for the Heisenberg chain [6] and with the theorem that the Galois group of the Bethe equations of the GL_d Gaudin model with one-box weights at all sites contains the alternating group whenever the target shape is neither a hook nor self-transpose [40, Thm. C]. The symmetric exclusion process on 14 sites with 7 particles gives A_{32}, A_{27}, and A_{20} in the sector of momentum 0.

The totally asymmetric exclusion process. On a ring of L sites with N particles, the Bethe equations of this process decouple, given one collective parameter [26, 25], and its transition probabilities are determinants [51]; the generator has integer entries. Over $\mathbb{Q}$ the N-particle sector splits into norms of momentum sectors, so we use the rational sectors of momentum 0 and π. In every case examined these contain factors whose Galois groups contain alternating groups far above the free bounds, for example A_{63} for $(L, N) = (14, 7)$, A_{80} for $(15, 5)$, and A_{131} for $(13, 6)$, which proves that no sector spectrum is of free form. Other factors are not alternating. For $(L, N) = (14, 7)$ the sector of momentum 0 has a factor of degree 84 with no Jordan certificate over 600 primes, whose Frobenius cycle types are compatible only with 3 blocks of 28; among 400 further primes, at each of the 267 primes inert in $\mathbb{Q}(\cos 2\pi/7)$ every cycle length is divisible by 3, the pattern of a norm from $\mathbb{Q}(\cos 2\pi/7)$. For $(15, 5)$, factors of degrees 90 and 30 are compatible only with 2 blocks and with norms from $\mathbb{Q}(\sqrt{5})$, at the 203 inert primes among 400. In both cases the block field is $\mathbb{Q}(\zeta_g)^+$ with $g = \gcd(L, N)$. No blocks occur for $(13, 4)$, $(15, 3)$, $(16, 4)$, and $(12, 6)$, where $\mathbb{Q}(\zeta_g)^+ = \mathbb{Q}$ and all large factors contain alternating groups, nor for $(10, 5)$, where the sector of momentum 0 has factors of degree 15, containing A_{15}, and of degree 5. These block structures are observed, not proved. Golinelli and Mallick explain high-order degeneracies of the spectrum of this process by a hidden symmetry of its Bethe ansatz [22] (see also [23]); whether that symmetry accounts for these block structures is open.

Integrability organizes the spectrum into sectors and commuting families. Free fermions in addition make every eigenvalue a monomial in a few algebraic numbers, which confines the Galois group to a Weyl group; partial decoupling of Bethe equations appears to leave an intermediate, imprimitive trace; in the other interacting models tested no trace remains. The test is therefore a test for free-fermionic spectral structure, in any basis and in disguise, and not for integrability. Like any test through Weyl groups, it cannot distinguish types B_L and C_L, whose Weyl groups coincide: a symplectic hidden group, with eigenvalues of the form $c\,\mu_j^{\pm 1}$, can pass the screen. Appendix A gives an example, which the relation lattice of Section 4.2 separates and whose Galois action is not through sign vectors.

The converse. A Galois action through sign vectors does not force free form, but it forces it up to a polynomial in the operator, and the Galois action can see nothing finer. We say that χ has free form over a field F if Definition 2.1 holds with F in place of $\mathbb{Q}$; parts (1) to (4) of Lemma 2.2 hold over F with the same proof.

Proposition 5.1 (converse up to polynomials)**.** *Let F be a field of characteristic 0, for example $\mathbb{Q}$, $\mathbb{Q}(i)$, or a field $\mathbb{Q}(t_1, \ldots, t_m)$ of parameters, and let $\chi \in F[\mu]$ have pairwise distinct roots μ_s indexed by a set S, all of $\{\pm 1\}^L$ or its even-parity half, with splitting field K. Suppose there is a homomorphism $\rho\colon \mathrm{Gal}(K/F) \to W(B_L)$ with $\sigma(\mu_s) = \mu_{\rho(\sigma)\cdot s}$ for all σ and s. Then there are a polynomial $\prod_{s \in S}(\mu - \lambda_s)$ of free form of rank L over F with $\kappa = 1$ and pairwise distinct*

roots $\lambda_s = \prod_k u_k^{s_k}$, $u_k \in K$, *and polynomials* $h, \tilde{h} \in F[x]$ *with* $\mu_s = h(\lambda_s)$ *and* $\lambda_s = \tilde{h}(\mu_s)$ *for all* $s \in S$, *such that* $\sigma(u_k) = u_{\pi(k)}^{\epsilon_k}$ *for* $\rho(\sigma) = (\pi, \epsilon)$. *Conversely, if* $\mu_s = h(\lambda_s)$ *with* $h \in F[x]$ *for the roots* λ_s *of a polynomial of free form over* F, *and the* μ_s *are pairwise distinct, then* $\mathrm{Gal}(\overline{F}/F)$ *permutes the* μ_s *through sign vectors, and so does* $\mathrm{Gal}(K/F)$ *for the splitting field* K *of* $\prod_s(\mu - \mu_s)$. *The same holds for the additive form, with* $\lambda_s = \sum_k s_k \varepsilon_k$.

Proof. The converse follows from Lemma 2.2(3), applied to the restriction of $\sigma \in \mathrm{Gal}(\overline{F}/F)$ to the splitting field of P: $\sigma(\mu_s) = h(\sigma(\lambda_s)) = h(\lambda_{\rho(\sigma)\cdot s}) = \mu_{\rho(\sigma)\cdot s}$. If S is the even half, $\rho(\sigma)$ preserves S: otherwise σ would map the values $\kappa \prod_k u_k^{s_k}$ with s even onto those with s odd, although the two sums differ by $\kappa \prod_k (u_k - u_k^{-1}) \neq 0$ (in the additive case the sums of L-th powers differ by $2^L L! \prod_k \varepsilon_k \neq 0$). The action of $\mathrm{Gal}(\overline{F}/F)$ on the μ_s factors through $\mathrm{Gal}(K/F)$, and its image in the permutations of S lifts to a homomorphism into $W(B_L)$: the action on S is faithful except for the even half with $L = 2$, where the image has order at most 2 and lifts through the element $(\mathrm{id}, (-1, -1))$. For the main statement let $G = \mathrm{Gal}(K/F)$ act on the signed indices $\pm 1, \ldots, \pm L$ through ρ, where (π, ϵ) sends k to $\epsilon_k \pi(k)$ and $-k$ to $-\epsilon_k \pi(k)$, and choose a representative k_0 of each orbit O, with stabilizer H, taking only one of O and $-O$ when they differ. If $-k_0 \in O$, let $H' \supset H$ be the stabilizer of $\{\pm k_0\}$, of index 2 over H, choose $\tau \in H' \setminus H$, and put $u_{k_0} = b/\tau(b)$ for $b \in K^H$, $b \neq 0$; since $\tau^2 \in H$, $\tau(u_{k_0}) = u_{k_0}^{-1}$. If $-k_0 \notin O$, put $u_{k_0} = b$ and $u_{-j} = u_j^{-1}$ for $j \in O$. Then $u_{\sigma(k_0)} = \sigma(u_{k_0})$ is well defined because u_{k_0} is fixed by H (b is, and H is normal in H', of index 2), $u_{-k} = u_k^{-1}$ for every signed index k, and $\sigma(u_k) = u_{\pi(k)}^{\epsilon_k}$; hence $\sigma(\lambda_s) = \lambda_{\rho(\sigma)\cdot s}$ and the roots $u_k^{\pm 1}$ of $P(u) = \prod_k (u - u_k)(u - u_k^{-1})$ form a Galois-stable set, so $P \in F[u]$. The conjugates $\sigma(b)$, for σ in a set of representatives of G/H, are the coordinates of b for an isomorphism $K^H \otimes_F \overline{F} \cong \overline{F}^{|O|}$, so as b varies they are independent coordinates y_j, $j \in O$, with $u_j = y_j/y_{-j}$ or $u_j = y_j$; the coordinates for different orbits are independent as well. Each of λ_s/λ_t for $s \neq t$, u_k^2, and $u_j u_k^{\pm 1}$ for $j \neq \pm k$ is then a nonconstant Laurent monomial in independent coordinates, and since F is infinite we may choose the b outside the finitely many hypersurfaces on which one of these equals 1. Then the λ_s are pairwise distinct and P is squarefree, reciprocal, and has no root ± 1. Finally the interpolation polynomial $h(x) = \sum_s \mu_s \prod_{t \neq s} (x - \lambda_t)/(\lambda_s - \lambda_t)$ satisfies $h(\lambda_s) = \mu_s$, and each $\sigma \in G$ permutes its summands, so $h \in F[x]$; the same argument with the roles exchanged gives $\tilde{h}$. In the additive case put $\varepsilon_{k_0} = b - \tau(b)$ or $\varepsilon_{k_0} = b$, respectively, and $\varepsilon_{-k} = -\varepsilon_k$; the conditions become the nonvanishing of the nonzero linear forms $\lambda_s - \lambda_t$, ε_k, and $\varepsilon_j \pm \varepsilon_k$ in the coordinates. □

For an operator T over F whose characteristic polynomial on a sector is χ, the operator $T_0 = \tilde{h}(T)$ on that sector is a polynomial in T with spectrum of free form, and $T = h(T_0)$, since T is diagonalizable on the sector and $h(\tilde{h}(x)) - x$ vanishes at its eigenvalues. The proposition is a case of Tschirnhausen equivalence, an isomorphism of the algebras $F[x]/(\chi)$ and $F[x]/(\chi_0)$ for the polynomial χ_0 of free form; its content is that a polynomial of free form with the prescribed Galois action exists. For a single matrix with a small Galois group the hypothesis holds for many labelings and the conclusion is weak, which is why the converse of Lemma 2.2 fails for individual matrices. The polynomial h cannot be omitted. Let $L \geq 3$, let $u_1, \ldots, u_L$ be indeterminates, let F be the field of rational functions in them invariant under $W(B_L)$, acting by $u_k \mapsto u_{\pi(k)}^{\epsilon_k}$, and let $\chi = \prod_s (\mu - \lambda_s)$ over $S = \{\pm 1\}^L$, which has free form over F. Then $\chi(\mu - 1)$, with roots $1 + \lambda_s$, has the same Galois action, but it has no free form of any rank. The roots of a polynomial of free form with 2^L roots generate a multiplicative group of rank at most $L + 2$, whereas the $1 + \lambda_s$ are irreducible elements of the factorial ring $\mathbb{Q}[u_1^{\pm 1}, \ldots, u_L^{\pm 1}]$,

because each λ_s is a monomial with a primitive exponent vector, and $1+\lambda_s$ and $1+\lambda_t$ are associate only for $t=\pm s$, with $(1+\lambda_{-s})/(1+\lambda_s)=\lambda_{-s}$; so they generate a group of rank $2^{L-1}+L>L+2$. Since F is a field of rational functions (namely $F=\mathbb{Q}(e_1(w),\dots,e_L(w))$ for the elementary symmetric functions of $w_k=u_k+u_k^{-1}$), this also shows that a Galois action through sign vectors at generic parameters does not force free form identically in the parameters. A Galois action through sign vectors, and not merely a Galois group or cycle types compatible with $W(B_L)$ (Appendix A gives a spectrum with the second property but not the first), therefore characterizes exactly the polynomials in operators with spectra of free form. Whether the polynomial can be taken to be a monomial, or affine for a Hamiltonian, is a question about multiplicative or additive relations, which the relation lattice of Section 4.2 addresses.

The analogy with zeros of L-functions is close: there Frobenius elements equidistribute in a monodromy group whose type determines the random-matrix statistics [34]; here the Frobenius statistics of a single characteristic polynomial reflect its Galois group, and in the examples above, at random parameters, they are consistent with the full Weyl group of the hidden group: $W(D_L)$ for free fermions and S_L with particle-number conservation.

6. Outlook

Proposition 5.1 shows that a Galois action through sign vectors, which is finer than the Galois group, determines free form exactly up to a polynomial in the operator, even identically in the parameters. A complete test for free form itself therefore has to combine the Galois action with the relation lattice; we have used the two side by side, and an exact criterion that combines them, with certified rather than numerical relations, remains to be formulated. More generally the method reads off a hidden reductive group through its Weyl group and weights: $\mathrm{Spin}(2L)$ for free fermions, $U(L)$ with particle-number conservation, the reflection group $C_3 \wr S_L$ for free parafermions with $\mathbb{Z}_3$ symmetry, and products of general linear groups for hidden tensor products. The imprimitive groups of the exclusion process suggest a finer invariant that measures partial decoupling of Bethe equations; proving the block structure of Section 5 is open. The limiting factor is size: the screen has no power when all factors have small degree, which leaves part of Section 4.1 undecided.

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Appendix A. Validation on known models

Unless stated otherwise, couplings are integers drawn at random across the row and constant along the transfer direction, which removes translation symmetry. "Free" means that the screen finds no certificate under the criteria of Section 2; "not free" means certified by Corollary 2.3, Proposition 2.4, Remark 2.6, or Remark 2.8. The maximal element orders are $\mathrm{maxord}(B_L)=8,12,12,24,30$ for $L=4,\dots,8$.

A.1. **Ising strips and fields.** In zero field, with couplings random in sign and size so that plaquettes are frustrated, each sector polynomial is irreducible of degree 2^{L-1}, and the longest Frobenius cycle over 150 primes equals $\mathrm{maxord}(B_L)$: 12 at $L=6$ and 24 at $L=7$. Over 3000 primes the frequencies of cycle types agree with the class distribution of $W(D_L)$ acting on the 2^{L-1} even sign vectors, of order $2^{L-1}L!$ (23040 at $L=6$, 322560 at $L=7$), within total variation 0.02 to 0.034, with no Frobenius element outside the group. A real field makes the polynomial irreducible of degree 2^L, with a full 2^L-cycle; at $L=6$ its Galois group contains

A_{64}. At the imaginary field $h = i\pi/2$ the polynomial satisfies $f(x) = g(x^2)$ and g has the same hyperoctahedral image; at $h = i\pi r$ with $r = 1/3, 1/4, 1/6$ the spectrum is not free. The Ising model on the Möbius strip and the Klein bottle [43] is free, as is a field $i\pi/2$ on alternate sites; one non-planar chord per row gives A_{64} at $L = 7$.

A.2. **Eight-vertex models.** Deforming free-fermion columns by $d_i \mapsto d_i + t$, $t \in \{-6, \dots, 6\}$, the test is free exactly at $t = 0$ and certifies non-freeness at the other 12 values; in a two-parameter deformation $(t, u) \in \{-2, \dots, 2\}^2$ it is free exactly at the origin.

A.3. **Crossing and four-spin couplings.** For the square-lattice Ising model with nearest-neighbor couplings, crossing diagonal couplings L_1, L_2, and a four-spin coupling M on each plaquette, the map of Ising models with two- and four-spin couplings to an eight-vertex model [32, 59] and the Fan–Wu condition [14] give, after the nearest-neighbor weights cancel, the free-fermion condition

$$e^{4M} = \frac{\cosh 2(L_1 - L_2)}{\cosh 2(L_1 + L_2)}. \tag{2}$$

We have not found condition (2) stated in this form in [32, 59, 14]. Without four-spin coupling it reduces to $\sinh 2L_1 \sinh 2L_2 = 0$, that is, to $L_1 L_2 = 0$ for real couplings. On 15 integer points of (2), including sign weights (complex couplings), the test reports free; on 18 nearby points off it, it certifies non-freeness. On grids of 2744 points at zero field and at field $i\pi/2$ ($L = 6$), the 2640 points off (2) in each grid are certified not free; the other 104 points are degenerate solutions of (2), with a diagonal coupling or the four-spin weight trivial or a pure sign, and are flagged free, so the nontrivial part of the locus is tested only on the 15 points above. With both diagonal couplings neither trivial nor a pure sign and no four-spin term, all 36 points of a random-column scan ($L = 6$) are certified not free; all 189 such points with uniform couplings at $L = 7$ are certified not free. These null results are consistent with Barahona's theorem that spin-glass problems become hard off planar lattices or in a field [3].

A.4. **Three-dimensional models.** The three-dimensional model of Wang and Hazzard [58] is solvable at imaginary couplings $K_\perp = \beta h = i\pi/4$. With per-link random weights on its locus the test reports free (factors of degree 8, longest cycle $6 \le \mathrm{maxord}(B_4)$, with 4 modes per flux sector); off the locus it certifies non-freeness, with a factor of degree 256 containing A_{256}. Its bond graph is non-planar, but the bonds with non-Clifford weights form a perfect matching within each layer, and all three-dimensional connectivity runs through bonds and fields at $i\pi/4$. In a scan of Ising models on the non-planar bars $G \times \mathbb{Z}$ with cross-sections $G = K_4$, 2×3, prism, and 2×4, with real and Gaussian-integer couplings, all 563 points are certified not free, by alternating groups or by cycle lengths, such as 5, 7, or 9, that divide no element order allowed by Proposition 2.4, even with two sectors (Remark 2.5). With all couplings equal to $i\pi/4$ the transfer matrix C of the bars with cross-sections 2×3 and prism is a Clifford operator, and C^k is a scalar for some k; this is a point of the tractable set of [28]. Generic bonds give a transfer matrix $T = CB$ with B a product of exponentials of Pauli strings, so T^k is a scalar times a product of exponentials of the Pauli strings in the orbit of those of B under conjugation by C. The Lie algebra generated by these strings is, by [1], a direct sum of free-fermionic, orthogonal, symplectic, or unitary algebras or of controlled versions of these (tensored with the group algebra of a group of commuting Pauli strings); T^k lies in the corresponding complexified group, which determines the structure of the spectrum of T^k, and hence that of T up to k-th roots of unity. With one generic bond per period, all 12 tests are flagged free, 6 of them with no factor of degree above 4; with two bonds, 4 of 12 are flagged free, 1 of them without power, and 8 are certified not free; with three or four bonds,

all 24 are certified. The dimension of this Lie algebra separates the flagged from the certified points: at every certified point it is at least 240. At the flagged points it is abelian (7 points without power and 2 others), $\mathfrak{su}(2)^3$ in each sector of the global spin flip (3 points), or $\mathfrak{su}(2)$ tensored with the group algebra of a group of 32 commuting Pauli strings (3 points, one with further central elements), all free-fermionic, except at one point with two bonds, where it is $\mathfrak{sp}(8)$ in each sector of the global spin flip. There each sector of T^k is the sum of four copies of the defining representation of Sp(8), with eigenvalues $c\,\mu_j^{\pm 1}$, $j = 1, \ldots, 4$, which are not of free form. The screen, at rank 6, flags this point free; this is to be expected, since for T^k the group $W(C_4) = W(B_4)$ embeds in $W(B_6)$. The Galois action on the 8 distinct eigenvalues of T^k in a sector is nevertheless not through sign vectors: over $\mathbb{Q}(i)$ they are the roots of an irreducible polynomial of degree 8 with Frobenius elements that are 8-cycles, for example at the prime of $\mathbb{Q}(i)$ above 1153 containing $i - 140$ in the sector even under the spin flip and at its conjugate in the odd sector, whereas $W(B_3)$ and $W(D_4)$, the groups that act on 8 sign vectors, have no element of order 8; over 3000 split primes the frequencies of cycle types agree with the cycle-type distribution of $W(C_4)$, of order 384, acting on the 8 weights $\pm e_j$, within total variation 0.023. The relation lattice of Section 4.2, applied to the transfer matrix itself, finds on its two factors of degree 16 a cross-polytope, as for an orthogonal or symplectic matrix, not a cube.

A.5. **Calibrations.** Table 5 lists further models known to be free or not.

model	size	expected	result
transverse-field Ising chain	$N = 5, \ldots, 8$	free	longest cycles $12, 12, 24, 30$
same, plus longitudinal field	$N = 5, \ldots, 8$	not free	full cycles, A_{2^N}
Fendley's chain	$S = 2, \ldots, 5$	free	longest cycles $4, 4, 6, 12$
$\sum b_m Z_m Z_{m+1} X_{m+2}$ [16]	modes		
free parafermions [15]	$L = 3, 4$	free	$f(x) = g(x^3)$, within $C_3 \wr S_L$
parafermions plus $\tau^\dagger$ terms	$L = 3, 4$	not free	A_9, A_{27}

TABLE 5. Calibrations. For Hamiltonians the energies are additive and the bound is maxord(B_S) in the number S of modes. Fendley's chain has no Jordan–Wigner map to free fermions and its frustration graph contains no claw; the control with a longitudinal field has a claw [10, 12].

A.6. **Further models.** The three-spin model of Baxter and Wu [5], integrable and not free, is certified not free on uniform strips of widths 7 and 8 (cycles 102 and 66, divisible by the primes 17 and 11, which divide no allowed element order, and A_{15}, A_{24}) and with random columns at widths 6 and 7 (A_{16}, A_{128}). The Kitaev–Heisenberg model [8] on a 2×4 ladder has four factors of degree 64 containing A_{64}; the pure Kitaev ladder has only factors of degree 4, where the test has no power. Among non-Hermitian chains, the transverse-field Ising chain with an imaginary transverse field ($N = 6, 7$, realified) satisfies $f(x) = g(x^2)$ with longest cycles 12 and 24, consistent with free form, while the Ising chain with an imaginary longitudinal field gives A_{64} and A_{128}. A tight-binding chain with a quasiperiodic on-site potential of rational frequency $1/6$ ($L = 9$, $N = 4$) has a factor of degree 126 with longest cycle $20 = g(9)$ and total variation 0.117 from S_9 on 4-element subsets over 300 primes; the chain of Su, Schrieffer, and Heeger [54] on 8 sites has factors too small for the test. For the toric code [37] on a 2×2 torus

with random couplings every factor is linear, and the spectrum is a constant plus the sums over two even half-cubes of the vertex and plaquette couplings.

A.7. **Knot invariants and the Tutte plane.** Braid-group representations on $(\mathbb{C}^2)^{\otimes n}$ from the free-fermion six-vertex R-matrix, which yields the Alexander–Conway polynomial [35], decompose into exterior powers of the reduced Burau matrix with factor degrees $\binom{n-1}{k}$, and their Galois groups lie in S_{n-1} (longest cycles 6 for $n = 6, 7$). Those from the $U_q(\mathfrak{sl}_2)$ R-matrix (Jones polynomial) at rational q contain A_9 ($n = 6$) and A_{14} ($n = 7$) in all 16 long braid words tested. In the semisimple path representations at $q = e^{i\pi/k}$, all eigenvalues are roots of unity exactly for $k \in \{3, 4, 6\}$ (and $q = \pm 1$), and not for $k = 5, 7, 8, 10$; rational $q \neq \pm 1$ never collapses. With Hecke parameter $q^2 = e^{2\pi i/k}$, these are points at which the Jones polynomial is computable in polynomial time [28], and for $k \geq 3$ they are the orders at which the image of the braid group in the Jones representations is finite for every n: it is infinite modulo its center for $k \geq 5$, $k \neq 6, 10$, $n \geq 3$, and for $k = 10$, $n \geq 5$, while the finite images at $k = 3, 4, 6$ (and $k = 10$, $n = 3, 4$) were identified earlier by Jones and others [20, Thm. 2.3 and Sec. 4]. For Potts transfer matrices on planar strips the test is free for $Q = 2$, the Ising hyperbola of the Tutte plane, at widths $W = 6, 7, 8$, with longest cycles $12, 24, 30$ equal to $\mathrm{maxord}(B_W)$, and certifies non-freeness for $Q = 3, 4$.

A.8. **Supersymmetric models and hard squares.** For the hard-core lattice gas at activity $z = -1$, which governs the ground states of the M_1 model [27], the row transfer matrix of hard squares has only roots of unity as eigenvalues for every width and boundary condition tested (periodic widths 6 to 13, open widths $2, 5, 8, 11$), as known [19, 29, 4]; at $z = \pm 2, -\frac{1}{2}, -3, -\frac{1}{3}, 1$ (periodic width 10) it does not, and the groups contain alternating groups. Hard hexagons, the honeycomb lattice, and the king's graph at $z = -1$ give alternating groups.

setting	flagged free	known free or roots-of-unity points
Ising strips, real and imaginary fields	zero field; $h = i\pi/2$	planar zero field [46, 36]; $i\pi/2$
eight-vertex	$a^2 + b^2 = c^2 + d^2$	Fan–Wu [14]
crossing and four-spin couplings	(2)	Fan–Wu via the vertex map
three-dimensional Ising	Wang–Hazzard locus	[58]
braid representations	Alexander; Jones at $k = 3, 4, 6$	free [35]; roots of unity [20]
circuits	matchgates	[57, 31]
number-conserving chains	S_L on subsets	tight-binding chains

TABLE 6. Points flagged free in the main families tested; further models are in Table 5 and the text. Points with roots-of-unity spectra pass the screen without being free-fermionic in any useful sense; the bars with a Clifford background and generic bonds of Appendix A are explained there by the Lie algebra of a Pauli orbit, and one of them is symplectic, not free.

APPENDIX B. RECOVERIES OF KNOWN SINGLE-PARTICLE SPECTRA

Section 3.1 recovers Onsager's dispersion relation; this appendix collects the corresponding recoveries for further free-fermion and free-parafermion models. In each case below, step 2 of the procedure (recovery of the hypercube labeling) was applied to the exact characteristic or

minimal polynomial with no input from the known solution, and the recovered single-particle polynomial, recognized over $\mathbb{Q}$ (or over $\mathbb{Q}(\omega)$), was then compared with the published formula. Where stated, step 3 certified the result exactly: the power sums of the free form, computed from the recovered polynomial alone, agree with the exact traces for every exponent up to the dimension, so the characteristic polynomial is the composed product. Transfer matrices use the multiplicative form; Hamiltonians use the additive form $E_s = e_0 + \sum_k s_k \varepsilon_k$, for which the recovered object is $\prod_k (y - \varepsilon_k^2)$.

model	recovered	agreement with the literature	certificate
dimers, strip width $3 \le L \le 8$	$\prod_k (y - \eta_k^2)$	$\eta_k = \cos \frac{\pi k}{L+1}$, exact [33, 55, 41]	exact
transverse-field Ising chain, open, $4 \le L \le 6$	$\prod_k (y - \varepsilon_k^2)$	exact [42, 47]	exact
XY chain, open, $4 \le L \le 6$	$\prod_k (y - \varepsilon_k^2)$	exact [42]	exact
Kitaev honeycomb, flux-free sector	ε_q	$\lvert 2(J_x e^{iq\cdot n_1} + J_y e^{iq\cdot n_2} + J_z)\rvert$ [38]	numerical
chain in disguise, $8 \le N \le 15$	$\prod_k (y - \varepsilon_k^2)$	exact [16]	exact
free parafermions, $n = 3$, $L = 3, 4$	$\prod_k (y - \varepsilon_k^3)$	exact [15]	exact
Fendley–Pozsgay chains, $3 \le L \le 7$	$\prod_k (y - \varepsilon_k^2)$	exact [17]	none
Ising on the Möbius strip, $4 \le L \le 7$	signed hypercube	numerical [43]	numerical

TABLE 7. Recoveries of known single-particle spectra. "Exact" agreement means a rational identity between the recovered polynomial and the published one; an exact certificate means agreement of all power sums with the exact traces.

Dimers. For the dimer transfer matrix on strips of width L, the recovered numbers $\eta_k = \sinh w_k$ satisfy $\prod_k (y - \eta_k^2) = \prod_{k=1}^{L} (y - \cos^2 \frac{\pi k}{L+1})$ identically, the Chebyshev form of the solution of Kasteleyn, Temperley, and Fisher in Lieb's transfer-matrix formulation [33, 55, 41]; the certificate holds with $\kappa^2 = 1$ for all $m \le 2^L$.

Quantum chains. For open transverse-field Ising and XY chains with random integer couplings, the recovered polynomial is the characteristic polynomial of the Majorana single-particle matrix of Lieb, Schultz, and Mattis [42]; for uniform unit couplings, the critical chain with $J = h = 1$, it gives $\varepsilon_k = 2 \cos \frac{\pi k}{2L+1}$ to 140 digits, as in Pfeuty's solution [47].

Kitaev's honeycomb model. On 2×2 and 3×2 tori, the flux-free sector reproduces the Majorana dispersion of [38] to 10^{-14}, the four loop sectors corresponding to momentum shifts by $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2})$; only the flux sector with all plaquette operators equal to $+1$ has this form.

Free fermions in disguise. For the chain of [16] with $8 \le N \le 15$ sites, the recovered polynomial is $y^S P_M(1/y)$ with P_M given by Fendley's recursion, exactly, in every case, with exact certificates. Here no Jordan–Wigner transformation is available, and the single-particle data are obtained from integer arithmetic alone. The number of levels is $S = \lceil M/3 \rceil = \lfloor N/3 \rfloor$, where $M = N - 2$ is the number of terms, consistent with the example P_4 in [16].

Free fermions beyond Jordan–Wigner. The chains of Fendley and Pozsgay [17] have a frustration graph that contains claws, so the criterion of Chapman, Elman, and Mann does not apply to them directly. For $3 \le L \le 7$ (5 to 9 qubits) the screen has no power, since all factors

have degree at most 4, but recovery gives $r = \lfloor (L+1)/2 \rfloor$ modes and $\prod_k (y - \varepsilon_k^2) = y^r P(1/y)$ with P as in [17, eq. (4.4)], exactly.

Free parafermions. For the three-state chain of [15], energies are sums $\sum_j \omega^{a_j} \varepsilon_j$ with $\omega = e^{2\pi i/3}$; the recovered $\prod_k (y - \varepsilon_k^3)$ agrees exactly with Fendley's polynomial, and the traces in $\mathbb{Z}[\omega]$ certify the spectrum for all $m \le 3^L$. For uniform couplings and $L = 3$, $\varepsilon_k^3 = 4\cos^2 \frac{\pi k}{7}$.

The Möbius strip. The transfer matrix composed with the site reflection has the same moduli hypercube as the open strip; its signs are reproduced to 140 digits by reflection parities $(-1)^{k+1}$ of the modes, ordered by increasing w_k, times the reordering sign $(-1)^{\binom{|A|}{2}}$. Since this labeling depends on the ordering of real numbers, it is not defined over $\mathbb{Q}$, and the identification is numerical; compare [43].

References

[1] G. Aguilar, S. Cichy, J. Eisert, and L. Bittel, *Full classification of Pauli Lie algebras*, preprint (2024), arXiv:2408.00081.

[2] Anonymous, *Which frustration graphs give free fermions? A converse to the even-hole, claw-free theorem, and exact free sectors of the periodic Fendley chain*, preprint, Zenodo (2026), doi:10.5281/zenodo.22994979.

[3] F. Barahona, *On the computational complexity of Ising spin glass models*, J. Phys. A **15** (1982), no. 10, 3241–3253, doi:10.1088/0305-4470/15/10/028.

[4] R. J. Baxter, *Hard squares for $z = -1$*, Ann. Comb. **15** (2011), no. 2, 185–195, doi:10.1007/s00026-011-0089-2.

[5] R. J. Baxter and F. Y. Wu, *Exact solution of an Ising model with three-spin interactions on a triangular lattice*, Phys. Rev. Lett. **31** (1973), 1294–1297, doi:10.1103/PhysRevLett.31.1294.

[6] O. R. Bellwood and W. J. Munro, *Algebraic structure of finite-size Bethe solutions in the Heisenberg chain*, preprint (2026), arXiv:2605.05589.

[7] N. Bourbaki, *Commutative Algebra. Chapters 1–7*, Elements of Mathematics, Springer, Berlin, 1989.

[8] J. Chaloupka, G. Jackeli, and G. Khaliullin, *Kitaev–Heisenberg model on a honeycomb lattice: possible exotic phases in iridium oxides $A_2\mathrm{IrO}_3$*, Phys. Rev. Lett. **105** (2010), 027204, doi:10.1103/PhysRevLett.105.027204.

[9] A. Chapman, S. J. Elman, and R. L. Mann, *A unified graph-theoretic framework for free-fermion solvability*, preprint (2023), arXiv:2305.15625.

[10] A. Chapman and S. T. Flammia, *Characterization of solvable spin models via graph invariants*, Quantum **4** (2020), 278, doi:10.22331/q-2020-06-04-278.

[11] J. D. Dixon and B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics, vol. 163, Springer, New York, 1996, doi:10.1007/978-1-4612-0731-3.

[12] S. J. Elman, A. Chapman, and S. T. Flammia, *Free fermions behind the disguise*, Comm. Math. Phys. **388** (2021), no. 2, 969–1003, doi:10.1007/s00220-021-04220-w.

[13] K. Fabricius and B. M. McCoy, *New developments in the eight vertex model*, J. Stat. Phys. **111** (2003), no. 1–2, 323–337, doi:10.1023/A:1022213209641.

[14] C. Fan and F. Y. Wu, *General lattice model of phase transitions*, Phys. Rev. B **2** (1970), no. 3, 723–733, doi:10.1103/PhysRevB.2.723.

[15] P. Fendley, *Free parafermions*, J. Phys. A **47** (2014), no. 7, 075001, doi:10.1088/1751-8113/47/7/075001.

[16] P. Fendley, *Free fermions in disguise*, J. Phys. A **52** (2019), 335002, doi:10.1088/1751-8121/ab305d.

[17] P. Fendley and B. Pozsgay, *Free fermions beyond Jordan and Wigner*, SciPost Phys. **16** (2024), 102, doi:10.21468/SciPostPhys.16.4.102.

[18] P. Fendley, K. Schoutens, and J. de Boer, *Lattice models with $\mathcal{N} = 2$ supersymmetry*, Phys. Rev. Lett. **90** (2003), 120402, doi:10.1103/PhysRevLett.90.120402.

[19] P. Fendley, K. Schoutens, and B. van Eerten, *Hard squares with negative activity*, J. Phys. A **38** (2005), no. 2, 315–322, doi:10.1088/0305-4470/38/2/002.

[20] M. H. Freedman, M. J. Larsen, and Z. Wang, *The two-eigenvalue problem and density of Jones representation of braid groups*, Comm. Math. Phys. **228** (2002), no. 1, 177–199, doi:10.1007/s002200200636.

[21] K. Fukai, B. Pozsgay, and I. Vona, *Solving models with generalized free fermions II: path-product expansion and conserved charges*, preprint (2026), arXiv:2605.31453.

[22] O. Golinelli and K. Mallick, *Spectral degeneracies in the totally asymmetric exclusion process*, J. Stat. Phys. **120** (2005), 779–798, doi:10.1007/s10955-005-6972-7.
[23] O. Golinelli and K. Mallick, *The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics*, J. Phys. A **39** (2006), 12679–12705, doi:10.1088/0305-4470/39/41/S03.
[24] A. Gorodnik and A. Nevo, *Splitting fields of elements in arithmetic groups*, Math. Res. Lett. **18** (2011), no. 6, 1281–1288, doi:10.4310/MRL.2011.v18.n6.a16.
[25] L.-H. Gwa and H. Spohn, *Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation*, Phys. Rev. A **46** (1992), 844–854, doi:10.1103/PhysRevA.46.844.
[26] L.-H. Gwa and H. Spohn, *Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian*, Phys. Rev. Lett. **68** (1992), 725–728, doi:10.1103/PhysRevLett.68.725.
[27] L. Huijse and K. Schoutens, *Supersymmetry, lattice fermions, independence complexes and cohomology theory*, Adv. Theor. Math. Phys. **14** (2010), no. 2, 643–694, doi:10.4310/ATMP.2010.v14.n2.a8.
[28] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 1, 35–53, doi:10.1017/S0305004100068936.
[29] J. Jonsson, *Hard squares with negative activity and rhombus tilings of the plane*, Electron. J. Combin. **13** (2006), no. 1, R67, doi:10.37236/1093.
[30] F. Jouve, E. Kowalski, and D. Zywina, *Splitting fields of characteristic polynomials of random elements in arithmetic groups*, Israel J. Math. **193** (2013), 263–307, doi:10.1007/s11856-012-0117-x.
[31] R. Jozsa and A. Miyake, *Matchgates and classical simulation of quantum circuits*, Proc. R. Soc. A **464** (2008), no. 2100, 3089–3106, doi:10.1098/rspa.2008.0189.
[32] L. P. Kadanoff and F. J. Wegner, *Some critical properties of the eight-vertex model*, Phys. Rev. B **4** (1971), no. 11, 3989–3993, doi:10.1103/PhysRevB.4.3989.
[33] P. W. Kasteleyn, *The statistics of dimers on a lattice*, Physica **27** (1961), no. 12, 1209–1225, doi:10.1016/0031-8914(61)90063-5.
[34] N. M. Katz and P. Sarnak, *Zeroes of zeta functions and symmetry*, Bull. Amer. Math. Soc. **36** (1999), no. 1, 1–26, doi:10.1090/S0273-0979-99-00766-1.
[35] L. H. Kauffman and H. Saleur, *Free fermions and the Alexander–Conway polynomial*, Comm. Math. Phys. **141** (1991), 293–327, doi:10.1007/BF02101508.
[36] B. Kaufman, *Crystal statistics. II. Partition function evaluated by spinor analysis*, Phys. Rev. **76** (1949), 1232–1243, doi:10.1103/PhysRev.76.1232.
[37] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Physics **303** (2003), 2–30, doi:10.1016/S0003-4916(02)00018-0.
[38] A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Physics **321** (2006), no. 1, 2–111, doi:10.1016/j.aop.2005.10.005.
[39] E. Kowalski, *The large sieve, monodromy, and zeta functions of algebraic curves, 2: independence of the zeros*, Int. Math. Res. Not. IMRN **2008** (2008), rnn091, 57 pp., doi:10.1093/imrn/rnn091.
[40] S. Liao and L. Rybnikov, *Maximal transitivity of the cactus group on standard Young tableaux*, preprint (2025), arXiv:2506.16561.
[41] E. H. Lieb, *Solution of the dimer problem by the transfer matrix method*, J. Math. Phys. **8** (1967), no. 12, 2339–2341, doi:10.1063/1.1705163.
[42] E. Lieb, T. Schultz, and D. Mattis, *Two soluble models of an antiferromagnetic chain*, Ann. Physics **16** (1961), no. 3, 407–466, doi:10.1016/0003-4916(61)90115-4.
[43] W. T. Lu and F. Y. Wu, *Ising model on nonorientable surfaces: exact solution for the Möbius strip and the Klein bottle*, Phys. Rev. E **63** (2001), 026107, doi:10.1103/PhysRevE.63.026107.
[44] J. Milewski, G. Banaszak, T. Lulek, M. Łabuz, and R. Stagraczyński, *Galois actions on the eigenproblem of the Heisenberg heptagon*, Open Syst. Inf. Dyn. **19** (2012), no. 2, 1250012, doi:10.1142/S1230161212500126.
[45] H. Nicolai, *Supersymmetry and spin systems*, J. Phys. A **9** (1976), no. 9, 1497–1506, doi:10.1088/0305-4470/9/9/010.
[46] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition*, Phys. Rev. **65** (1944), 117–149, doi:10.1103/PhysRev.65.117.
[47] P. Pfeuty, *The one-dimensional Ising model with a transverse field*, Ann. Physics **57** (1970), no. 1, 79–90, doi:10.1016/0003-4916(70)90270-8.
[48] N. Regnault, S. Liu, and B. A. Bernevig, *Integer characteristic polynomial factorization and Hilbert space fragmentation*, preprint (2022), arXiv:2210.08019.

[49] J. Ruh and S. J. Elman, *Expanding the class of free fermions via twin-collapse methods*, Phys. Rev. A **113** (2026), no. 2, 022456, doi:10.1103/svfv-vdh5.
[50] J. Ruh, S. J. Elman, and A. Chapman, *A graph-theoretic analysis of non-generic free-fermion solvability by Krylov decompositions*, preprint (2026), arXiv:2610.03557.
[51] G. M. Schütz, *Exact solution of the master equation for the asymmetric exclusion process*, J. Stat. Phys. **88** (1997), 427–445, doi:10.1007/BF02508478.
[52] C. J. Smyth, *Additive and multiplicative relations connecting conjugate algebraic numbers*, J. Number Theory **23** (1986), no. 2, 243–254, doi:10.1016/0022-314X(86)90094-6.
[53] P. Stevenhagen and H. W. Lenstra, Jr., *Chebotarëv and his density theorem*, Math. Intelligencer **18** (1996), no. 2, 26–37, doi:10.1007/BF03027290.
[54] W. P. Su, J. R. Schrieffer, and A. J. Heeger, *Solitons in polyacetylene*, Phys. Rev. Lett. **42** (1979), 1698–1701, doi:10.1103/PhysRevLett.42.1698.
[55] H. N. V. Temperley and M. E. Fisher, *Dimer problem in statistical mechanics: an exact result*, Philos. Mag. **6** (1961), no. 68, 1061–1063, doi:10.1080/14786436108243366.
[56] B. M. Terhal and D. P. DiVincenzo, *Classical simulation of noninteracting-fermion quantum circuits*, Phys. Rev. A **65** (2002), 032325, doi:10.1103/PhysRevA.65.032325.
[57] L. G. Valiant, *Quantum circuits that can be simulated classically in polynomial time*, SIAM J. Comput. **31** (2002), no. 4, 1229–1254, doi:10.1137/S0097539700377025.
[58] Z. Wang and K. R. A. Hazzard, *Topological correlations in three-dimensional classical Ising models: an exact solution with a continuous phase transition*, Phys. Rev. Research **5** (2023), 013086, doi:10.1103/PhysRevResearch.5.013086.
[59] F. Y. Wu, *Ising model with four-spin interactions*, Phys. Rev. B **4** (1971), no. 7, 2312–2314, doi:10.1103/PhysRevB.4.2312.
[60] M. Zheng, B. Pozsgay, and S. Chen, *Exact solutions of hidden free fermion models*, preprint (2026), arXiv:2609.14134.