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ERDŐS PROBLEM 706 AND UNIFORM MULTISCALE REFINEMENTS

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Abstract. We present a detailed, attributed exposition of the OpenAI–Galán argument giving exponential growth in Erdős problem 706. The specialized character-rigidity proof is included, together with its compact-tower and radial arguments, the transfer from arbitrary colorings to weak measurable colorings for algebraic distances, and Galán's deduction using Bukh's separated-distance theorem. This yields finite planar graphs with r forbidden distances and chromatic number greater than 4.0485^r. The full negative answer to problem 706 is credited to that prior work. We then prove a uniform product estimate for arbitrary nonempty compact distance clusters in a fixed normalization band, using one separation ratio for every number and every choice of clusters. For a fixed finite weighted unit-distance graph of independence ratio at most a, we also obtain all-length exponential bounds of every base $b < 1/a$ with sufficient separation $O((1-ab)^{-5/2})$. The latter results are presented as refinements, with their relationship to earlier Fourier, complete-positivity, and finite-graph methods made explicit.

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1. The problem, attribution, and organization

For a finite set $P \subset \mathbb{R}^2$ and a finite set $D \subset (0,\infty)$, let $G(P,D)$ join distinct points $x, y \in P$ when $|x-y| \in D$. Put

$$L(r) = \sup_{\substack{P\subset\mathbb{R}^2 \text{ finite} \\ D\subset(0,\infty),\ |D|=r}} \chi(G(P,D)), \qquad r \geq 1.$$

Erdős problem 706 asks whether the chromatic numbers of these graphs have a polynomial upper bound in the number r of permitted edge lengths. We use the formulation recorded in [12, 11].

The argument in Part I establishes

$$4.0485^r < L(r) \leq 7^r \qquad (r \geq 1). \tag{1}$$

Its ingredients have different authors. OpenAI proved the rigidity theorem for rotation-invariant laws on the character group of the algebraic plane and constructed the continuous-factor transfer for the unit distance [19]. Sergi Galán extended the transfer to algebraic forbidden distances and used Bukh's product limit to deduce the exponential growth in (1) [14, 4]. The numerical base uses the published one-distance density bound of Ambrus, Csiszárik, Matolcsi, Varga and

Zsámboki [1]. The elementary bound $m(\{1\}) \le 1/2$ already gives $2^r \le L(r)$ by the same argument.

1.1. **Scope of the full proof.** For a measurable $A \subset \mathbb{R}^2$, write

$$\overline{\delta}(A) = \limsup_{L\to\infty} \frac{|A \cap [-L, L]^2|}{4L^2}, \qquad m(D) = \sup_{A \text{ avoids } D} \overline{\delta}(A),$$

where avoidance means $|x - y| \notin D$ for every $x, y \in A$. For arbitrary colorings there is initially no measurable color class to which one can apply this density bound. The transfer theorem in Section 5 supplies a weak measurable coloring and then removes all forbidden-distance pairs by taking density points. It proves

$$\chi(\mathbb{R}^2, D) \ge 1/m(D) \qquad (D \text{ finite, positive, and algebraic}).$$

The character-rigidity theorem that enables this transfer is proved in Sections 2–4. In particular, the relative-singularity argument, the compact tower, the conditional multiple averages, and the number-field radial argument are included. The final passage to finite graphs is proved in Section 6. Standard spectral theory, disintegration, and ergodic decomposition are used with their usual hypotheses. Bukh's product theorem remains an explicitly cited published input.

1.2. **Attribution of the components.**

Mathematical component	Attribution
Character rigidity, relative singularity, compact towers, conditional averages, and radial identities	OpenAI [19]
Continuous-factor transfer for the unit distance	OpenAI [19]
Algebraic-distance extension and exponential solution of problem 706	Sergi Galán [14]
Separated-scale density product limit	Bukh [4]
The bound $m(\{1\}) \le 0.2470$	Ambrus et al. [1]
Uniform compact-cluster estimate and graph-dependent separation bounds	Refinements developed in Part II

The complete solution of problem 706 is prior work in the attribution above. Our contribution is the exposition of that proof chain and the refinements in Part II. The proof of the prior result has been rewritten with explicit hypotheses and intermediate estimates. The original sources retain credit for the mathematical ideas.

1.3. **The additional uniformity.** For $M \ge 1$, let $\mathcal{K}_M$ be the nonempty compact subsets of $[1, M]$. Theorem 7.1 has the order of quantifiers

$$\forall M \ge 1 \ \forall \varepsilon \in (0, 1/2) \ \exists Q > M \quad \forall r \ge 1 \quad \forall K_1, \dots, K_r \in \mathcal{K}_M.$$

When $t_{i+1}/t_i \ge Q$, it gives

$$\prod_i m(K_i) \le m\left(\bigcup_i t_i K_i\right) < \prod_i (m(K_i) + \varepsilon).$$

Thus one ratio is chosen before both the length and the cluster shapes. The proof uses the complete-positivity description of independence density [10], finite certificate extraction [10, 7], and a uniform finite cover of $\mathcal{K}_M$. Theorem 7.3 supplies an effective companion from a finite weighted graph, with separation $O((1 - ab)^{-5/2})$ for each fixed weighted realization. It uses established weighted quadratic inequalities [15, 17].

Earlier separated-scale arguments of Bukh, de Oliveira Filho and Vallentin, DeCorte–de Oliveira Filho–Vallentin, and Castro-Silva are discussed at the start of Part II [4, 18, 10, 5]. Galán's explicit two-distance estimates are an additional quantitative predecessor [14]. The exact all-length and all-cluster uniformity is the refinement asserted here. Historical priority for that refinement remains provisional because the full texts of [20] and [21] were unavailable in the literature audit. Their bibliographic records and later descriptions were checked.

Part 1. The OpenAI–Galán argument for problem 706

2. Character laws and relative singularity

The results and the proof strategy in this section are due to OpenAI [19]. We give their spectral argument in full, including the conditional-measure constructions and the order of the finite choices in the relative-singularity argument. The additional explanations concern versions of conditional probabilities and quantitative estimates.

2.1. **The algebraic plane and its character group.** Identify the real plane with $\mathbb{C}$, and put

$$F = \overline{\mathbb{Q}} \cap \mathbb{R}, \qquad E = F(i), \qquad K = \{u \in E : u\overline{u} = 1\}.$$

Both the additive group E and the multiplicative group K will carry the discrete topology. Write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The compact character group of the discrete additive group E is

$$\mathbb{D} = \widehat{E_{\mathrm{disc}}} = \{d : E \to \mathbb{T} : d(z+w) = d(z)d(w)\}.$$

Its operation is written additively, so

$$(d + d')(z) = d(z)d'(z), \qquad (-d)(z) = \overline{d(z)}.$$

For $a, z \in E$, define

$$(ad)(z) = d(az), \qquad \psi_z(d) = d(z).$$

In particular, K acts on $\mathbb{D}$ by continuous automorphisms.

The ordinary continuous characters of the real plane give the subgroup

$$C = j(\mathbb{R}^2), \qquad j(\xi)(z) = \exp(i\xi \cdot z), \tag{2}$$

where z is viewed as a real coordinate vector in the dot product.

Lemma 2.1 (OpenAI [19])**.** *The group K is countable and divisible, and $\mathbb{D}$ is compact metrizable. The map j is a continuous injection. Its image C is a Borel subgroup, and $j^{-1} : C \to \mathbb{R}^2$ is Borel. Every nonzero scalar $a \in E$ maps C onto itself. On the frequency parameters this action is the transpose of the real-linear map of multiplication by a, which is orthogonal when $a \in K$.*

Proof. The fields F and E are countable. If n is a positive integer and $u \in K$, a complex root $v^n = u$ is algebraic and has modulus one. Its real and imaginary parts belong to F, so $v \in K$. Inversion handles negative powers. Every nonzero power map on K is therefore surjective.

The group $\mathbb{D}$ is a closed subspace of the countable product $\mathbb{T}^E$, defined by the character equations. It is consequently compact metrizable. Continuity of j follows coordinate by coordinate. If $j(\xi) = j(\eta)$, the continuous plane character with frequency $\xi - \eta$ is trivial on the dense subgroup $\mathbb{Q} + i\mathbb{Q} \subset E$. It is trivial on the whole plane, so $\xi = \eta$.

For each positive integer n, the restriction of j to the closed ball $\overline{B}(0, n) \subset \mathbb{R}^2$ is a homeomorphism onto its compact image. Hence

$$C = \bigcup_{n \geq 1} j(\overline{B}(0, n))$$

is Borel. The inverse is continuous on each of these compact images, which proves its Borel measurability.

Let M_a be the real matrix of multiplication by a. Directly from the definitions,

$$a\, j(\xi) = j(M_a^{\mathsf{T}} \xi).$$

For $a \neq 0$, the inverse scalar a^{-1} belongs to E. This proves the assertion about C. If $|a| = 1$, the matrix M_a, and therefore its transpose, is orthogonal. □

Definition 2.2 (OpenAI [19])**.** A K-invariant Borel probability measure ν on $\mathbb{D}$ is *wild* if $\nu(C) = 0$. Its Fourier coefficients are

$$f_\nu(z) = \int_{\mathbb{D}} \psi_z(d)\; d\nu(d), \qquad z \in E.$$

The theorem whose proof is developed here and in the following compact-factor arguments is the following result of OpenAI.

Theorem 2.3 (Haar rigidity, OpenAI [19])**.** *Every wild probability measure on $\mathbb{D}$ is the normalized Haar measure $m_{\mathbb{D}}$. Equivalently, if a K-invariant probability measure ν satisfies $\nu(C) = 0$, then*

$$f_\nu(z) = 0 \qquad (z \in E \setminus \{0\}).$$

The equivalence uses uniqueness of finite measures from their Fourier coefficients on $\mathbb{D}$. Indeed, the complex linear span of the coordinate characters ψ_z is a unital conjugation-closed algebra separating points, so Stone–Weierstrass gives uniform density in the continuous functions. Haar measure has coefficient zero at every nonzero z. To see that ψ_z is nontrivial there, choose an ordinary plane frequency ξ with $\exp(i\xi \cdot z) \neq 1$.

2.2. **Cosets and line-continuity events.**

Lemma 2.4 (OpenAI [19])**.** *Let ν be wild. Every coset of C has ν-measure zero. The difference of two independent samples of law ν also gives C measure zero. For every $a \in E \setminus \{0\}$, the set*

$$L_a = \{d \in \mathbb{D} : t \mapsto d(at) \text{ is continuous on } F \text{ in its real topology}\}$$

is Borel and has ν-measure zero.

Proof. Suppose $\nu(d + C) > 0$. Distinct cosets in its K-orbit are disjoint and have equal positive mass, so this orbit is finite. The action on the orbit has a finite abelian image A, since K is abelian. Divisibility of K makes every power map on A surjective. Taking the power $|A|$ shows A is trivial. Thus $(u-1)d \in C$ for every $u \in K$. Choose $u \neq 1$. Multiplication by $(u-1)^{-1} \in E$ preserves C, so $d \in C$, contradicting $\nu(C) = 0$.

The relation $d' - d \in C$ is Borel. Its sections are cosets of C of measure zero. Integrating the sections proves the independent difference assertion.

Restriction to aF is a continuous map

$$R_a : \mathbb{D} \to \widehat{F_{\mathrm{disc}}}, \qquad R_a(d)(t) = d(at).$$

A continuous character on F is uniformly continuous and extends uniquely to a continuous character on $\mathbb{R}$. It therefore has the form $t \mapsto e^{i\lambda t}$. To obtain this form, choose a continuous argument near zero. The character equation makes that argument additive on a smaller interval, hence linear by continuity. Subdividing an arbitrary real input extends the formula to all of $\mathbb{R}$. Conversely each such character restricts continuously to F. The map $\lambda \mapsto (t \mapsto e^{i\lambda t})$ is continuous and injective into $\widehat{F_{\mathrm{disc}}}$. Compact exhaustion, as in Lemma 2.1, makes its image Borel. Consequently L_a is Borel.

If aF and bF are nonparallel, a, b form an F-basis of E. A character continuous on both lines satisfies

$$d(at + bs) = e^{i(\lambda t+\eta s)} \qquad (t, s \in F)$$

for some real λ, η. The corresponding real-linear coordinate change is invertible, so this character belongs to C. Hence $L_a \cap L_b \subset C$.

There are infinitely many distinct unoriented lines uaF, $u \in K$. For example, the elements $(1+in)/(1-in)$, for positive integers n, have distinct arguments modulo π. The corresponding events all have the same ν-measure by invariance. Their pairwise intersections have measure zero. Summing their measures over an arbitrarily large finite collection forces the common measure to be zero. □

2.3. **Integration along an algebraic line.** Fix a positive normalized translation-invariant mean $\mathfrak{m}$ on the bounded functions on the discrete group $F/\mathbb{Z}$. Such a mean exists because the group is abelian [2, Theorem G.2.1]. For a bounded function $h : F \to \mathbb{C}$ supported in a bounded interval, define

$$I(h) = \mathfrak{m}\left(t + \mathbb{Z} \longmapsto \sum_{k\in\mathbb{Z}} h(t + k)\right). \tag{3}$$

Lemma 2.5 (Invariant line integration, OpenAI [19])**.** *The functional I is well defined, positive, linear, and invariant under translations by F. For $\phi \in C_c(\mathbb{R})$,*

$$I(\phi|_F) = \int_{\mathbb{R}} \phi(t)\ dt.$$

If $V : F \to \mathcal{H}$ is bounded and supported in a bounded interval, where $\mathcal{H}$ is a Hilbert space, a weak integral $I(V)$ is well defined and satisfies

$$\|I(V)\|_{\mathcal{H}} \le I(\|V(t)\|_{\mathcal{H}}). \tag{4}$$

Bounded linear operators commute with this weak integral.

Proof. Changing a representative of $t + \mathbb{Z}$ merely reindexes the sum in (3). If h is supported in $[-R, R]$, at most $2R + 2$ summands are nonzero at any point. The periodization is therefore bounded. Positivity and linearity follow from the mean. Periodizing $h(\cdot + s)$ translates the periodization by $s + \mathbb{Z}$, so I is translation invariant.

For a nonzero integer n, the ordinary circle character $t + \mathbb{Z} \mapsto e^{2\pi int}$ has mean zero. Indeed a rational translation can multiply it by a scalar different from one. The constant character has mean one. Uniform approximation by trigonometric polynomials shows that $\mathfrak{m}$ agrees with circle integration on continuous periodic functions restricted to F. The periodization of $\phi \in C_c(\mathbb{R})$ is continuous and periodic. Thus

$$I(\phi|_F) = \int_0^1 \sum_{k\in\mathbb{Z}} \phi(t + k)\ dt = \int_{\mathbb{R}} \phi(t)\ dt.$$

For the Hilbert-valued assertion, use inner products linear in the first variable. Positivity of I gives, for every $v \in \mathcal{H}$,

$$|I(\langle V(t), v\rangle)| \le I(|\langle V(t), v\rangle|) \le \|v\|_{\mathcal{H}} I(\|V(t)\|_{\mathcal{H}}).$$

Hilbert-space duality gives the vector $I(V)$ and (4). Scalar pairing also proves commutation with bounded linear operators and translation invariance for these integrals. □

This construction applies to all the bounded functions in question. Ordinary real-variable measurability is unnecessary.

Lemma 2.6 (Vanishing line average, OpenAI [19])**.** *If ν is wild, then*

$$I\big(\phi(t)f_\nu(at)\big) = 0 \qquad (a \in E \setminus \{0\},\ \phi \in C_c(\mathbb{R})). \tag{5}$$

Proof. On $L^2(\nu)$, let $T_t v = \psi_{at} v$, $t \in F$. We first show that the only vector whose orbit is norm-continuous in the real topology on F is zero. Suppose v is such a vector and put

$$c_v(t) = \int_{\mathbb{D}} |v|^2 \psi_{at}\, d\nu.$$

This function is positive definite. Moreover,

$$|c_v(t+s) - c_v(t)| \le \|v\|_2 \|T_s v - v\|_2,$$

so it is uniformly continuous on F. It extends to a continuous function on $\mathbb{R}$. Approximating any finite list of real arguments by elements of F shows that the extension remains positive definite.

Bochner's theorem on $\mathbb{R}$ gives a finite positive frequency measure representing this extension [2, Theorem D.2.2]. Push that measure to $\widehat{F_{\mathrm{disc}}}$ using $\lambda \mapsto (t \mapsto e^{i\lambda t})$. Its Fourier coefficients agree with those of $R_{a*}(|v|^2\nu)$. Coordinate characters separate points, and their complex linear span is a self-conjugate algebra, so Stone–Weierstrass and uniqueness of finite measures identify these two measures. The first is concentrated on the Borel set of continuous characters. Consequently $|v|^2\nu$ is concentrated on L_a. Lemma 2.4 gives $\nu(L_a) = 0$, so $v = 0$.

Now form the weak integral

$$V_\phi = I\big(\phi(t) T_t 1\big) \in L^2(\nu).$$

Commutation with T_s and translation invariance give

$$T_s V_\phi - V_\phi = I\big((\phi(t-s) - \phi(t)) T_t 1\big).$$

Since $\|T_t 1\|_2 = 1$, Lemma 2.5 yields

$$\|T_s V_\phi - V_\phi\|_2 \le I(|\phi(\cdot - s) - \phi|) = \int_{\mathbb{R}} |\phi(t-s) - \phi(t)|\, dt \longrightarrow 0$$

as $s \in F$ tends to zero. Thus $V_\phi = 0$ by the preceding argument. Pairing with the constant function proves (5). □

2.4. **A measurable center without moment assumptions.** The subgroup C need not be closed. A probability measure on one of its cosets nevertheless allows an equivariant choice of a center.

Lemma 2.7 (Measurable center, OpenAI [19])**.** *For every Borel probability measure λ on $\mathbb{R}^2$, the function*

$$J_\lambda(s) = \int_{\mathbb{R}^2} \Big(\sqrt{1 + |s-t|^2} - \sqrt{1 + |t|^2}\Big)\ d\lambda(t) \tag{6}$$

has a unique minimizer $L(\lambda)$. The map L is Borel on the space of probability measures with its usual weak Borel structure. It is equivariant under translations and orthogonal maps.

Proof. The function $g(u) = \sqrt{1 + |u|^2}$ is 1-Lipschitz, so the integrand in (6) has absolute value at most $|s|$. The integral is finite for every s, without any moment assumption, and is Lipschitz in s. The Hessian of g is positive definite. Its eigenvalues are $(1 + |u|^2)^{-1/2}$ and $(1 + |u|^2)^{-3/2}$. Hence J_λ is strictly convex.

Choose $R > 0$ with $p = \lambda(\overline{B}(0, R)) > 1/2$. On this ball the integrand is at least $|s| - R - \sqrt{1 + R^2}$, and on its complement it is at least $-|s|$. It follows that

$$J_\lambda(s) \ge (2p - 1)|s| - p(R + \sqrt{1 + R^2}) \longrightarrow \infty.$$

Continuity and coercivity give a minimum, and strict convexity gives uniqueness.

For fixed s, the integrand is bounded and continuous in t. Thus $\lambda \mapsto J_\lambda(s)$ is Borel. By continuity in s, the minimum equals the infimum over $\mathbb{Q}^2$. Enumerate $\mathbb{Q}^2$ and let $q_n(\lambda)$ be its first element with objective value less than this infimum plus $1/n$. The maps q_n are Borel. For fixed λ, their tail lies in a compact sublevel set. Every cluster point is a minimizer, so the entire sequence converges to $L(\lambda)$. This proves that L is Borel.

If λ' is the law of $t - r$ for t of law λ, an identity between bounded integrands gives

$$J_{\lambda'}(s) = J_\lambda(s + r) - J_\lambda(r).$$

Therefore $L(\lambda') = L(\lambda) - r$. For an orthogonal map O,

$$J_{O_*\lambda}(Os) = J_\lambda(s),$$

which gives $L(O_*\lambda) = OL(\lambda)$. □

2.5. **Conditional measures and compact approximation.** All systems in the rest of this section are standard probability spaces with probability-preserving actions of the countable group K. Factors are countably generated modulo null sets. We use standard Borel models for them, identify factor functions with their pullbacks, and write $U_k H(x) = H(kx)$. Countability allows all group laws and equivariance identities used for these models to hold on common invariant conull sets.

For an equivariant factor map $\pi : X \to Y$, disintegration gives a measurable kernel $y \mapsto \mu_y$, concentrated on $\pi^{-1}(y)$ for almost every y, with

$$\mu = \int_Y \mu_y \, d\mu_Y(y).$$

We use existence and almost-everywhere uniqueness of disintegration in this standard Borel setting [22, Theorem 4 and Remark 5]. Uniqueness implies

$$\mu_{ky} = k_*\mu_y$$

for almost every y, for each $k \in K$. Equality here is equality of probability measures, obtained by testing a countable generating algebra. Countability of K gives a common invariant conull set. On this set the equality holds for all Borel tests at once.

We also use the composition rule for conditional laws. If $W \subset Y \subset X$, first conditioning on Y and then integrating over Y conditioned on W gives the conditional law over W. This follows from the tower property, with equality of probability measures fixed on a common conull set by a countable generating algebra.

Definition 2.8 (Compact approximation property, OpenAI [19])**.** Let $(Y_\alpha)_{\alpha \le \theta}$ be a tower of factors of an ergodic system, where θ is a countable ordinal. The initial factor Y_0 is trivial. At a limit ordinal, take the factor generated by all earlier factors. At every successor, with $Y = Y_{\alpha+1}$ and $W = Y_\alpha$, require that for every bounded complex-valued H on Y and every $\varepsilon > 0$, there are bounded functions $H_1, \dots, H_q$ on Y and a finite B such that, for every $k \in K$, there are $b_1^k, \dots, b_q^k \in L^\infty(W)$ satisfying

$$\left\| U_k H - \sum_{r=1}^{q} b_r^k H_r \right\|_{L^2(Y)} < \varepsilon, \qquad \|b_r^k\|_\infty \le B. \tag{7}$$

The functions H_r and the bound B are chosen before k.

The compact-factor construction later supplies a tower with precisely this property. The next lemma uses only (7) and the preceding spectral results.

Lemma 2.9 (Relative singularity, OpenAI [19])**.** *Let X be an ergodic standard probability-preserving K-system, and let $d : X \to \mathbb{D}$ be an equivariant measurable map with wild law. For every tower from Definition 2.8 and every factor Y in that tower, let ν_Y be the law of*

$$d(x) - d(x')$$

for two conditionally independent samples of X over Y. Then $\nu_Y(C) = 0$.

Proof. All the relative difference laws are K-invariant by equivariance of their conditional kernels. We prove the assertion by transfinite induction in a universal form. At each ordinal it is asserted for all ambient ergodic standard systems, all equivariant maps with wild law, and all towers satisfying (7). This permits the earlier induction assertion to be used on a new factor system. At the trivial factor, Lemma 2.4 proves the assertion.

Fix a noninitial ordinal, assume this universal assertion at every earlier ordinal, and suppose it fails at its stage Y in some X. Let κ_y denote the conditional law of d over Y.
Finite cosets and equivariant centers. The function

$$r(y, z) = \kappa_y(z + C) = \int_{\mathbb{D}} \mathbf{1}_C(d' - z)\ d\kappa_y(d')$$

is Borel in (y, z) by Borel parameter integration. Kernel equivariance gives $r(ky, kz) = r(y, z)$ for every z on a common conull set of y. Thus $r(y(x), d(x))$ is an invariant function on X, with integral $\nu_Y(C) > 0$. Ergodicity makes it a constant $a > 0$ almost surely.

For almost every y, therefore, κ_y-almost every z belongs to a coset of mass a. Any positive-mass coset meets this full-measure set and consequently has mass a. There are finitely many such cosets and they exhaust the measure. Their number m satisfies $ma = 1$, so it is independent of y. Also $\kappa_y(C) = 0$ almost surely because the marginal law of d is wild.

On the Borel set $r(y, z) > 0$, define a probability kernel on $\mathbb{R}^2$ by

$$\lambda_{y,z}(A) = \frac{1}{r(y, z)} \int_{\mathbb{D}} \mathbf{1}_C(d' - z)\, \mathbf{1}_A\big(j^{-1}(d' - z)\big)\ d\kappa_y(d').$$

The integrand is set to zero off $d' - z \in C$. Lemma 2.1 and parameter integration make this a Borel kernel. Set

$$b(y, z) = z + j\big(L(\lambda_{y,z})\big).$$

If z is replaced by $z + j(r_0)$, the kernel is translated by $-r_0$. Lemma 2.7 shows that $b(y, z)$ is unchanged. Thus it is constant on each positive-mass coset and lies in that coset. Let O_k be the orthogonal action of k on frequency parameters. Then $\lambda_{ky,kz} = (O_k)_*\lambda_{y,z}$, which gives

$$b(ky, kz) = k\, b(y, z).$$

Pull this function back along $x \mapsto (y(x), d(x))$, which lies in its domain almost surely, and extend it arbitrarily on the null complement. Let Z be the factor generated by Y and this map b. It is standard and ergodic. Over Y, the conditional law β_y of b is uniform on the m distinct centers of the positive-mass cosets. Since $b(x) \in d(x) + C$, its marginal law is wild. It is equivariant by construction.
Encoding the finite phase lists. For each y, take the unordered list of m values of ψ_1 at the centers, with repetitions when phases coincide. Encode the list by its elementary symmetric functions, giving $H(y) \in \mathbb{C}^m$. This is a bounded measurable vector. Indeed, its power sums are

$$m \int_{\mathbb{D}} \psi_1(b)^n\ d\beta_y(b), \qquad 1 \le n \le m,$$

and Newton's identities recover the elementary symmetric functions.

The encoding is a continuous injection of the compact space $\mathbb{T}^m/S_m$. Give this space the matching metric

$$d_{\mathrm{match}}([z_1,\dots,z_m],[z'_1,\dots,z'_m]) = \min_{\pi\in S_m}\max_r |z_r - z'_{\pi(r)}|.$$

Its inverse encoding is uniformly continuous. Equivariance of β_y shows that U_kH encodes the list at frequency k.

Choose $\rho > 0$ such that the closed arc $\{|z-1| \le 5\rho\}$ has circle Haar probability less than $1/(16m^2)$. There are $N \ge 1$ and $c > 0$ such that every probability ω on $\mathbb{T}$ satisfying

$$\left|\int z^n \, d\omega(z)\right| < c \qquad (1 \le n \le N)$$

also satisfies

$$\omega\{|z-1| \le 4\rho\} < \frac{1}{4m^2}. \tag{8}$$

Here is a quantitative construction. Take a continuous majorant of the smaller arc, valued in $[0,1]$, which vanishes outside the larger arc. Approximate it uniformly within $1/(64m^2)$ by a real trigonometric polynomial of degree at most N. Its constant coefficient is less than $5/(64m^2)$. Choose c so that the sum of absolute values of the nonconstant coefficients, multiplied by c, is less than $1/(32m^2)$. Negative moments are conjugates of positive ones. The integral of the majorant against ω is then less than $1/(8m^2)$, proving (8).

Choose $s > 0$ such that encoding vectors at distance less than s have phase lists matching within ρ. Put

$$\tau = s/100, \qquad \gamma > 0, \qquad \frac{2\gamma^2}{\tau^2} < 10^{-4}. \tag{9}$$

An approximation over an earlier factor. There is an earlier factor W and fixed bounded vector-valued functions $H_1,\dots,H_q$ on Y such that, for every $k \in K$,

$$\left\| U_kH - \sum_{r=1}^{q} b_r^k H_r \right\|_2 < \gamma, \qquad b_r^k \in L^\infty(W), \quad \|b_r^k\|_\infty \le B, \tag{10}$$

for one fixed finite B. At a successor stage, apply (7) to each coordinate and combine the functions, taking coordinate errors whose squares sum to less than γ^2. At a limit stage, first approximate H within $\gamma/2$ by a bounded vector on an earlier successor factor. Such approximation follows from convergence in L^2 of conditional expectations onto an increasing cofinal family of earlier factors. The approximation error is unchanged by rotation. Apply (7) at that successor with error below $\gamma/2$, and let W be its predecessor. In either case all data in (10) are fixed before k is chosen.

The tower through W is also a tower of factors of Z. Its measures, joins, and approximation properties are the same. The earlier universal induction assertion, applied in Z to b, therefore says that the relative difference law of b over W is wild. Its Fourier coefficient at $a \in E$ is

$$p(a) = \|\mathbb{E}_Z(\psi_a(b) \mid W)\|_2^2 \ge 0, \qquad p(0) = 1. \tag{11}$$

This follows by conditional independence. Lemma 2.6 applies to p.

Finite bins before a small algebraic parameter. The two coefficient arrays for two instances of (10) lie in a fixed bounded subset of $\mathbb{C}^{2q}$. Partition it into finitely many Borel bins, say L bins, with coordinate diameters smaller than

$$\frac{\tau}{1 + \sum_{r=1}^{q} \|H_r\|_\infty}.$$

At any fiber where the essential bounds hold, two pairs of arrays in one bin give approximating vectors differing pointwise by at most τ in each of the two instances. Choose an integer $M \geq 2$ satisfying

$$\frac{0.9M}{L} > 10m^2. \tag{12}$$

All approximation data and this number of tests have now been fixed.

Set $Q = N\binom{M}{2}$, and choose

$$\varepsilon_0 = \min\left\{\frac{1}{2}, \frac{c^2}{200Q}\right\}.$$

There is $t \in F$, $0 < |t| < 1/M$, such that

$$p(n(j-l)t) < \varepsilon_0 \qquad (1 \leq n \leq N,\ 1 \leq j < l \leq M). \tag{13}$$

To prove this, take a nonnegative $\phi \in C_c((-1/M, 1/M))$ with positive integral. For every indicated nonzero integer $n(j-l)$, Lemma 2.6 gives

$$I\big(\phi(t)p(n(j-l)t)\big) = 0.$$

The finite sum of these nonnegative tests also has zero weighted integral. If it were at least ε_0 everywhere $\phi > 0$, positivity would contradict $I(\phi) = \int \phi > 0$. Some such t makes the sum less than ε_0, and hence satisfies (13). It is nonzero because the sum at zero is $Q \geq 1$.

Put $x_j = jt$. By Markov's inequality and (11), the probability that any one of

$$\Big|\mathbb{E}_Z\big(\psi_{n(x_j - x_l)}(b) \mid W = w\big)\Big| < c \qquad (1 \leq n \leq N,\ j < l)$$

fails is at most $p(n(x_j - x_l))/c^2$. A union bound and (13) make the probability of any failure less than $1/200$. Apply (8) to the conditional law of $\psi_{x_j - x_l}(b)$. On a set of W-measure greater than 0.995, all bounds

$$\mathbb{P}_Z\big(|\psi_{x_j}(b) - \psi_{x_l}(b)| \leq 4\rho \mid W = w\big) < \frac{1}{4m^2}, \qquad j < l, \tag{14}$$

hold simultaneously.

Choosing one fiber and one bin. Every $|x_j| < 1$ is a sum of two algebraic unit directions. Define

$$k_j = \frac{x_j}{2} + i\sqrt{1 - \frac{x_j^2}{4}}, \qquad k_j' = \frac{x_j}{2} - i\sqrt{1 - \frac{x_j^2}{4}}.$$

Their real and imaginary parts belong to F, and $k_j, k_j' \in K$, with $k_j + k_j' = x_j$. Use (10) for all k_j, k_j'. Let $e_j^+(w)$ and $e_j^-(w)$ be the corresponding conditional squared errors over $W = w$. Each has integral less than γ^2. Call j good at w when both errors are less than τ^2. If at least $0.1M$ indices fail to be good, then

$$\frac{1}{M}\sum_{j=1}^{M}(e_j^+(w) + e_j^-(w)) \geq 0.1\tau^2.$$

Markov's inequality and (9) bound the measure of these fibers by

$$\frac{20\gamma^2}{\tau^2} < 0.001.$$

Thus more than $0.9M$ indices are good on a set of fibers of measure greater than 0.999.

Choose a fiber w in this set and in the set where all (14) hold. Discard the null sets for the finitely many conditional identities, coefficient bounds, and essential bounds used below. The intersection still has positive measure. Bin the coefficient pairs at this fiber. By (12), one bin

contains a set J of good indices with $|J| > 10m^2$. Fix $j_0 \in J$. For each $j \in J$, the conditional L^2 distance from $U_{k_j}H$ to $U_{k_{j_0}}H$ is less than 3τ. The two approximation errors each contribute less than τ, and the bin contributes at most τ. The same holds with k_j, k_{j_0} replaced by k'_j, k'_{j_0}.

For uniform $j \in J$ and y sampled over $W = w$, Markov's inequality shows that the probability that either encoding distance is at least s is at most

$$\frac{18\tau^2}{s^2} = 0.0018.$$

Let $q(y)$ be the fraction of $j \in J$ for which both phase lists match their j_0 lists within ρ. The matching metric is measurable, and the choice of s gives

$$\mathbb{E}(q \mid W = w) \geq 1 - 0.0018 > 0.95. \tag{15}$$

The collision contradiction. Fix such a y, and let b_0 be any center in the finite support of β_y. For every matching j, each of $\psi_{k_j}(b_0)$ and $\psi_{k'_j}(b_0)$ is within ρ of an entry in its representative list for j_0. It follows that

$$\psi_{x_j}(b_0) = \psi_{k_j}(b_0)\psi_{k'_j}(b_0)$$

is within 2ρ of one of at most m^2 products of entries from those two lists. The two matchings may select different centers, and all m^2 products are allowed.

Assign the $q(y)|J|$ matching indices to such products. If their class sizes are $n_1, \ldots, n_{m^2}$, with zero classes included as necessary, then

$$\sum_r n_r^2 \geq \frac{q(y)^2|J|^2}{m^2}.$$

These count ordered pairs in common classes, including diagonals. Removing diagonals loses at most $|J|$ pairs. Hence independent uniform $j, l \in J$ satisfy

$$\mathbb{P}_{j,l}\big(j \neq l,\ |\psi_{x_j}(b_0) - \psi_{x_l}(b_0)| \leq 4\rho\big) \geq \frac{q(y)^2}{m^2} - \frac{1}{|J|}.$$

This bound holds for every b_0 in the finite support. Integrate first over β_y, then over the conditional law of y given w. The composition rule for conditional laws identifies this with conditioning in Z over $W = w$. Jensen's inequality and (15) give a lower bound strictly larger than

$$\frac{0.95^2}{m^2} - \frac{1}{|J|} > \frac{0.9025 - 0.1}{m^2} > \frac{1}{4m^2}.$$

On the other hand, every distinct pair in J obeys (14). Averaging those simultaneous bounds over the ordered pairs in $J \times J$, with the diagonal excluded, gives an upper bound less than $1/(4m^2)$. The choices of J and j_0 after fixing w are legitimate because all original pair bounds already hold at that fiber. This is the required contradiction.

The induction step holds for every ambient system and every tower at the current ordinal. Universal transfinite induction therefore proves the lemma for all countable towers. □

3. Compact towers and conditional averages

This section develops the compact-factor construction and the conditional multiple-average argument of OpenAI [19]. Their role in the rigidity proof is to produce a factor that satisfies the approximation hypothesis of Lemma 2.9, while retaining enough mixing above that factor to separate conditional correlations. The underlying structure theory originates with Furstenberg and Zimmer. The finite-dimensional fiber argument below is closely related to [13, Lemmas 6.6

and 7.2], and the module formulation is discussed, with corrections, in [16]. We give the details needed here.

All actions in this section are probability-preserving actions of the discrete group K. As established in the spectral preliminaries, K is countable and abelian, and each integer power homomorphism

$$K \longrightarrow K, \qquad u \longmapsto u^d, \qquad d \in \mathbb{Z} \setminus \{0\},$$

is surjective. The operator convention is $U_k f(x) = f(kx)$. Factors and functions are identified modulo null sets.

3.1. **Conditional products.** Suppose that $\pi : X \to Y$ is a factor map between standard probability spaces, with measures μ and ν. Disintegration gives a measurable family of probabilities μ_y, concentrated on $\pi^{-1}(y)$, such that

$$\mu = \int_Y \mu_y \, d\nu(y).$$

The standard Borel disintegration theorem and uniqueness of its kernel are sufficient here [22, Theorem 4 and Remark 5]. Indeed, the conditional integration identity for a countable family of Borel sets separating the points of Y gives the asserted concentration on a fiber. For an equivariant map, uniqueness gives $k_*\mu_y = \mu_{ky}$ for almost every y, for each k. Intersecting these conull sets over the countable group, and then over their translates, makes the kernel equivariant on a common invariant conull set.

The relative square $\mathcal{R} = X \times_Y X$ carries the probability

$$\mu_{\mathcal{R}} = \int_Y \mu_y \otimes \mu_y \, d\nu(y) \tag{16}$$

and the diagonal K-action. Its two coordinate marginals are μ. We write $P = \mathbb{E}(\cdot \mid Y)$ on X, and $P_{\mathcal{R}}$ for conditional expectation from $\mathcal{R}$ onto Y. For bounded a, b on X, set $(a \otimes b)(x, x') = a(x)b(x')$. The defining conditional independence then says

$$P_{\mathcal{R}}(a \otimes b) = (Pa)(Pb). \tag{17}$$

Conditional expectation commutes with every U_k, is a contraction in both L^2 and L^∞, and satisfies $P(cf) = cPf$ for bounded Y-measurable c.

3.2. **Construction of the tower.**

Lemma 3.1 (Compact tower, OpenAI [19])**.** *For an ergodic probability-preserving action of K on a standard space X, one can find a countable ordinal η and a tower of invariant factors $(Y_\alpha)_{\alpha \le \eta}$ satisfying the following conditions.*

1. *Y_0 is trivial. At each limit ordinal, the factor is generated by the preceding factors.*
2. *For every $\alpha < \eta$, every $H \in L^\infty(Y_{\alpha+1})$, and every $\varepsilon > 0$, there are bounded functions $H_1, \dots, H_q$ on $Y_{\alpha+1}$ and $B < \infty$ such that for each $k \in K$ one can choose $b_1^k, \dots, b_q^k \in L^\infty(Y_\alpha)$ with*

$$\left\| U_k H - \sum_{r=1}^{q} b_r^k H_r \right\|_2 < \varepsilon, \qquad \|b_r^k\|_\infty \le B. \tag{18}$$

   The functions H_r and the bound B are independent of k.
3. *$X \times_{Y_\eta} X$ is ergodic.*

Thus every successor has precisely the approximation property (7)*.*

Proof. We first show how to enlarge an invariant factor Y whenever $X \times_Y X$ is not ergodic.

An invariant kernel. Choose a bounded nonconstant invariant function L on $X \times_Y X$, for example the indicator of a nontrivial invariant set. The conditional expectation of L onto either coordinate is invariant on X. Ergodicity makes both expectations equal to $\int L \, d\mu_{\mathcal{R}}$. Subtracting this constant preserves nonzeroness and invariance and gives

$$\int L(x, x') \, d\mu_{\pi(x)}(x') = 0, \qquad \int L(x, x') \, d\mu_{\pi(x')}(x) = 0$$

almost everywhere in the respective coordinates.

On $\mathcal{H}_y = L^2(\mu_y)$, define

$$(A_y f)(x) = \int L(x, x') f(x') \, d\mu_y(x').$$

Put $C = \|L\|_\infty$. Each A_y is Hilbert–Schmidt and

$$\|A_y\|_{\mathrm{HS}}^2 \le C^2, \qquad A_y 1 = A_y^* 1 = 0. \tag{19}$$

The kernel is nonzero on a positive-measure set of fibers, so the same is true of the operator field. Kernel invariance identifies the operators at y and ky by unitary conjugacy. More explicitly, pullback by $x \mapsto kx$ maps $\mathcal{H}_{ky}$ unitarily onto $\mathcal{H}_y$ and intertwines A_{ky} with A_y.

Measurability of a finite-dimensional spectral space. For $\delta > 0$, consider

$$Q_y = \mathbf{1}_{(\delta,\infty)}(A_y^* A_y).$$

We justify the required measurability before choosing δ. Take a countable algebra of Borel sets generating X, and enumerate the associated simple functions with complex rational coefficients. Their restrictions are dense in every $\mathcal{H}_y$. Parameter integration against L and its adjoint kernel shows that applying A_y, A_y^*, or any polynomial in $A_y^* A_y$ to these sections gives measurable sections.

All spectra of $A_y^* A_y$ lie in $[0, C^2]$. On this interval the continuous functions

$$h_n(t) = \min\{1, \max\{0, n(t - \delta)\}\}$$

converge pointwise to $\mathbf{1}_{(\delta,\infty)}(t)$, including the value zero at $t = \delta$. Choose polynomials p_n with $\sup_{[0,C^2]} |p_n - h_n| \to 0$. They may be chosen with a common supremum bound. The spectral theorem on each fiber gives

$$p_n(A_y^* A_y) s_y \longrightarrow Q_y s_y \quad \text{in } \mathcal{H}_y$$

for every test section s from the countable family. The common bound and dominated convergence show that the measurable sections on the left are Cauchy in global $L^2(X)$. A subsequence converges in fiber norm for almost every y, identifying the global limit with $Q_y s_y$. Hence every projected test section is measurable. Their Gram matrices show that the rank of Q_y is measurable.

For each fiber with $A_y \neq 0$, some positive spectral cutoff has nonzero range. By countability of the cutoffs $1/n$, one can now fix $\delta > 0$ for which $Q_y \neq 0$ on a set of positive measure. The trace bound gives

$$\delta \operatorname{rank} Q_y \le \operatorname{tr}(A_y^* A_y) \le C^2.$$

Unitary conjugacy makes this integer-valued rank invariant on Y. Every factor of the ergodic system X is ergodic, so $\operatorname{rank} Q_y = d$ almost everywhere for one integer $d \ge 1$. By (19), the range is perpendicular to the fiber constants.

The projected test sections span a dense subspace of each range. Select the first available independent section at each step and apply Gram–Schmidt in the fibers. The selection tests

and normalizing factors are measurable because they are expressed by the Gram matrices. This constructs measurable sections $e_1, \dots, e_d$ satisfying

$$\int e_i \overline{e_j}\, d\mu_y = \delta_{ij}, \qquad \int e_i\, d\mu_y = 0. \tag{20}$$

They belong to global $L^2(X)$, since their fiber norms are one.

Boundedness and the new factor. For each k, the functions $U_k e_1, \dots, U_k e_d$ form another orthonormal basis of the same fiber range. Thus

$$U_k e_i = \sum_{j=1}^{d} m_{ij}^k e_j$$

with a measurable unitary matrix $M^k = (m_{ij}^k)$ over Y. In particular $|m_{ij}^k| \le 1$ almost everywhere. These identities hold for every k on a common conull set, because K is countable. They imply that

$$x \longmapsto \sum_{i=1}^{d} |e_i(x)|^2$$

is K-invariant. Its integral is d, and ergodicity therefore gives

$$\sum_{i=1}^{d} |e_i(x)|^2 = d \quad \text{almost everywhere.} \tag{21}$$

In particular each e_i is bounded by $\sqrt{d}$.

Let Y' be the factor generated by $Y, e_1, \dots, e_d$. The transformation formula above, also applied to k^{-1}, makes Y' invariant. It is strict over Y. Indeed, a Y-measurable e_i would equal its conditional mean, which is zero by (20), although its conditional squared norm is one.

Uniform approximation over Y. Finite polynomials in $e_1, \dots, e_d$ and their conjugates, with coefficients in $L^\infty(Y)$, form a bounded unital algebra closed under conjugation and generating Y'. The functional monotone class theorem, followed by L^2 approximation, shows that this algebra is dense in $L^2(Y')$.

Choose a polynomial p with $\|H - p\|_2 < \varepsilon$. Write its terms as

$$p = \sum_{a=1}^{m} c_a M_a(e_1, \dots, e_d, \overline{e}_1, \dots, \overline{e}_d), \qquad c_a \in L^\infty(Y),$$

where M_a is a monomial of total degree D_a. Substituting the unitary basis transformations into $U_k p$ gives only monomials of degrees at most $\max_a D_a$. The a-th original monomial contributes at most d^{D_a} terms. Every resulting coefficient is a Y-measurable function, and after collecting equal monomials its supremum norm is at most

$$B = \sum_{a=1}^{m} d^{D_a} \|c_a\|_\infty.$$

This estimate holds for every k, since unitary matrix entries have modulus at most one and $\|U_k c_a\|_\infty = \|c_a\|_\infty$. Taking the finitely many possible monomials as H_r gives $U_k p = \sum_r b_r^k H_r$ with the required coefficient bound. Finally, $\|U_k H - U_k p\|_2 = \|H - p\|_2 < \varepsilon$. This proves (18) for Y' over Y.

The argument uses bounded coefficients for the transformations of the chosen basis and of a fixed polynomial. For precision, the algebraic module $\sum_i L^\infty(Y) e_i$ can have an L^2 closure containing vectors with merely $L^2(Y)$ coefficients. The construction above takes its basis from the measurable fiber ranges and establishes all coefficient bounds directly.

Iteration and termination. Begin with the trivial factor. Whenever the relative square over the current factor is nonergodic, apply the preceding construction. At a countable limit ordinal take the generated factor. There are countably many preceding factors and each is countably generated modulo null sets, so their join has the same property and admits a standard probability model.

If the process continued through every countable ordinal, then for each $\alpha < \omega_1$ there would be a unit vector

$$f_\alpha \in L^2(Y_{\alpha+1}) \quad \text{perpendicular to } L^2(Y_\alpha).$$

One of the newly adjoined e_i supplies such a vector. For $\alpha < \beta$, nestedness places f_α in $L^2(Y_\beta)$, so f_α and f_β are orthogonal. This would be an uncountable orthonormal family in the separable space $L^2(X)$. Consequently the procedure stops at a countable ordinal η. At that stage the relative square is ergodic, while every earlier successor satisfies (18). □

If $d : X \to \mathbb{D}$ is equivariant and its law gives C measure zero, Lemma 2.9 now applies to every factor in this tower. Its hypotheses are exactly the countable construction from the trivial factor, the generated joins at limit stages, and the uniform approximation (18). It follows that the law of $d(x) - d(x')$ for independent conditional samples over any Y_α also gives C measure zero.

3.3. **Averages along the discrete rotation group.** A sequence of nonempty finite sets (Φ_N) in K is a Følner sequence if

$$\frac{|k\Phi_N \triangle \Phi_N|}{|\Phi_N|} \longrightarrow 0 \qquad \text{for every } k \in K. \tag{22}$$

Such a sequence exists. To see this, enumerate K and at step N consider the subgroup generated by its first N elements. It is a product of a finite abelian group and a finitely generated free abelian group. Take the whole finite factor and a sufficiently large box in the free factor so that translations by the first N elements have relative boundary less than $1/N$. These finite sets satisfy (22). For any family b_u in a vector space, write

$$\mathrm{Av}_{u\in\Phi}\, b_u = \frac{1}{|\Phi|}\sum_{u\in\Phi} b_u.$$

Lemma 3.2 (Mean averages)**.** *Let V be a unitary representation of K on a Hilbert space. For every Følner sequence, the operators $\mathrm{Av}_{u\in\Phi_N} V_u$ converge strongly to orthogonal projection onto the invariant vectors.*

Proof. The orthogonal complement of the invariant vectors is the closed linear span of the vectors $V_k v - v$. Indeed, a vector perpendicular to every such difference is fixed by every V_k^*, hence by every V_k. On an individual difference, commutativity and (22) give

$$\|\mathrm{Av}_{u\in\Phi_N} V_u(V_k v - v)\| \le \frac{|k\Phi_N \triangle \Phi_N|}{|\Phi_N|}\|v\| \longrightarrow 0.$$

The averaging operators have norm at most one. Density proves convergence to zero on the stated orthogonal complement, and they act as the identity on invariant vectors. □

For every nonzero integer d, the operators $u \mapsto U_{u^d}$ also form a unitary representation of K. Surjectivity of $u \mapsto u^d$ shows that its invariant vectors are the invariant vectors of the original action. In particular, on any ergodic K-system,

$$\mathrm{Av}_{u\in\Phi_N} U_{u^d} f \longrightarrow \int f \quad \text{in } L^2 \qquad (d \neq 0). \tag{23}$$

This applies the mean theorem to a representation indexed by the original averaging sets Φ_N.

Lemma 3.3 (Conditional multiple averages, OpenAI [19])**.** *Let X be an ergodic standard probability-preserving K-system, let Y be an invariant factor, and suppose that $X \times_Y X$ is ergodic. Put $P = \mathbb{E}(\cdot \mid Y)$. For $l \geq 1$, pairwise distinct integers $n_1, \dots, n_l$, bounded functions $v_1, \dots, v_l$, and every Følner sequence in K,*

$$\mathrm{Av}_{u \in \Phi_N} \left\| P\left(\prod_{j=1}^{l} U_{u^{n_j}} v_j \right) - \prod_{j=1}^{l} U_{u^{n_j}} P v_j \right\|_2^2 \longrightarrow 0. \tag{24}$$

Proof. Keep the notation $\mathcal{R} = X \times_Y X$ from (16). We first establish a two-function estimate on X, transfer that estimate to $\mathcal{R}$, and then prove the multiple-average assertion.
A conditional two-function estimate. For bounded v, w and $d \in \mathbb{Z} \setminus \{0\}$, we claim that

$$\mathrm{Av}_{u \in \Phi_N} \| P(v U_{u^d} w) - P v \, U_{u^d} P w \|_2^2 \longrightarrow 0. \tag{25}$$

Let $v^\circ = v - Pv$ and $w^\circ = w - Pw$. The conditional module identity and commutation with the action show that the difference inside the norm is $P(v^\circ U_{u^d} w^\circ)$. By conditional independence,

$$\| P(v^\circ U_{u^d} w^\circ) \|_2^2 = \int_{\mathcal{R}} (v^\circ \otimes \overline{v^\circ}) \, U_{u^d} (w^\circ \otimes \overline{w^\circ}) \, d\mu_{\mathcal{R}}. \tag{26}$$

The second tensor has integral $\int_Y |Pw^\circ|^2 \, d\nu = 0$. The action on $\mathcal{R}$ is ergodic by hypothesis. Apply (23) there and pair the resulting L^2 convergence with $v^\circ \otimes \overline{v^\circ}$. This proves (25) along every Følner sequence.

The same estimate holds for bounded functions on $\mathcal{R}$, with $P_{\mathcal{R}}$ in place of P. For elementary tensors $v = a_1 \otimes a_2$ and $w = b_1 \otimes b_2$, formula (17) gives

$$P_{\mathcal{R}}(v U_{u^d} w) = P(a_1 U_{u^d} b_1) P(a_2 U_{u^d} b_2).$$

If the two factors on the right are A_1, A_2, and their respective factor approximations are B_1, B_2, use

$$A_1 A_2 - B_1 B_2 = (A_1 - B_1) A_2 + B_1 (A_2 - B_2).$$

Every companion factor is uniformly bounded. The estimate on X therefore proves the assertion for elementary tensors, and bilinearity proves it for finite sums.

For completeness, finite sums of bounded tensors approximate every bounded function on $\mathcal{R}$ in L^2, with a common supremum bound. Choose increasing finite partitions generated by countable Borel generators in each coordinate. Conditional expectations onto their rectangular grids converge in L^2, preserve the supremum bound, and are finite sums of coordinate tensors restricted to $\mathcal{R}$. If $D_u(v, w)$ denotes the pair difference for either of the conditional expectation operators, contraction gives

$$\| D_u(v, w) - D_u(v', w) \|_2 \leq 2 \|w\|_\infty \|v - v'\|_2.$$

The analogous estimate for changing w has right side $2\|v'\|_\infty \|w - w'\|_2$. These bounds are uniform in u. Approximating the two arguments successively extends the pair estimate to all bounded functions on $\mathcal{R}$.
A vector assertion with one centered function. Let Z denote either X or $\mathcal{R}$, let λ be its probability measure, and let $Q = \mathbb{E}_Z(\cdot \mid Y)$. Both systems are ergodic, and both now satisfy the pair estimate. We prove that, whenever the integers $n_1, \dots, n_l$ are pairwise distinct and nonzero and

$Qv_{j_*} = 0$ for some j_*,

$$\left\| \mathrm{Av}_{u\in\Phi_N} \prod_{j=1}^{l} U_{u^{n_j}} v_j \right\|_2 \longrightarrow 0 \tag{27}$$

for every Følner sequence. Rescaling reduces to $\|v_j\|_\infty \le 1$. For $l = 1$, the integral of v_1 is zero, so (23) proves the assertion.

Assume the assertion for $l - 1$. Put

$$H_u = \prod_{j=1}^{l} U_{u^{n_j}} v_j, \qquad b_{j,h} = (U_{h^{n_j}} v_j)\overline{v_j}.$$

For each fixed h, commutativity and invariance of λ give

$$\begin{aligned} \langle H_{hu}, H_u \rangle &= \int_Z \prod_{j=1}^{l} U_{u^{n_j}} b_{j,h}\, d\lambda \\ &= \int_Z b_{l,h} \prod_{j<l} U_{u^{n_j - n_l}} b_{j,h}\, d\lambda. \end{aligned} \tag{28}$$

The $l - 1$ exponents in the second line are distinct and nonzero. For each shifted factor, write $b_{j,h} = Qb_{j,h} + (b_{j,h} - Qb_{j,h})$. After averaging in u, every expanded term containing a centered shifted factor tends to zero in L^2 by induction. Multiplication by the fixed bounded function $b_{l,h}$ then gives zero in the scalar integral. In the remaining term the shifted product is Y-measurable, so $b_{l,h}$ can also be replaced by $Qb_{l,h}$.

All the resulting conditional factors have modulus at most one. If $j_* = l$, bound its unshifted factor in L^1. If $j_* < l$, bound its shifted factor in L^1 and use measure preservation. Since the underlying measure is a probability, the L^1 norm is at most the L^2 norm in either case. Hence

$$\limsup_{N\to\infty} |\mathrm{Av}_{u\in\Phi_N} \langle H_{hu}, H_u \rangle| \le a(h), \qquad a(h) = \|Qb_{j_*,h}\|_2. \tag{29}$$

This argument retains the distinguished centered index even when its exponent is the one removed in (28).

Apply the pair estimate on Z to $\overline{v_{j_*}}, v_{j_*}$ and the nonzero exponent n_{j_*}. Both conditional means vanish. The estimate therefore states that the average of $a(h)^2$ tends to zero along every Følner sequence. Cauchy–Schwarz for the finite average gives the same conclusion for $a(h)$.

We will need the resulting uniformity over translates. For any chosen Følner sequence (Ψ_M),

$$\sup_{t\in K} \mathrm{Av}_{h\in\Psi_M} a(ht) \longrightarrow 0. \tag{30}$$

Otherwise there would be $\varepsilon > 0$, increasing indices M_i, and $t_i \in K$ for which the averages over $\Psi_{M_i} t_i$ exceed ε. Those translated sets are themselves a Følner sequence, since

$$\frac{|k(\Psi_{M_i} t_i) \triangle (\Psi_{M_i} t_i)|}{|\Psi_{M_i} t_i|} = \frac{|k\Psi_{M_i} \triangle \Psi_{M_i}|}{|\Psi_{M_i}|} \longrightarrow 0$$

for each k. This contradicts the vanishing already obtained for every Følner sequence.

The Hilbert-space shift estimate. Fix a finite nonempty subset $S \subset K$. Because $\|H_u\|_2 \le 1$, the difference between $\mathrm{Av}_{u\in\Phi_N} H_u$ and $\mathrm{Av}_{u\in\Phi_N} \mathrm{Av}_{h\in S} H_{hu}$ has norm at most

$$\frac{1}{|S|} \sum_{h\in S} \frac{|h\Phi_N \triangle \Phi_N|}{|\Phi_N|},$$

which tends to zero. Jensen's inequality in $L^2(Z)$, followed by expansion of the finite square, yields

$$\limsup_{N\to\infty} \|\mathrm{Av}_{u\in\Phi_N} H_u\|_2^2 \le \frac{1}{|S|^2} \sum_{h,h'\in S} \limsup_{N\to\infty} |\mathrm{Av}_{u\in\Phi_N} \langle H_{hu}, H_{h'u}\rangle|$$

$$\le \frac{1}{|S|^2} \sum_{h,h'\in S} a(h(h')^{-1}). \tag{31}$$

For the last inequality substitute $u' = h'u$. The set of new averaging indices is $h'\Phi_N$, again a Følner sequence, so (29) applies.

Now choose $S = \Psi_M$. The last line of (31) is at most $\sup_t \mathrm{Av}_{h\in\Psi_M} a(ht)$, which tends to zero by (30). Here the limit superior in N is taken with S fixed, and only afterwards is M allowed to grow. This proves (27) and completes the induction on both X and $\mathcal{R}$.

Return to conditional averages. The integers in the lemma may include zero. Expand every v_j as $Pv_j + (v_j - Pv_j)$. Consider one expanded product

$$W_u = \prod_{j=1}^{l} U_{u^{n_j}} w_j$$

with $Pw_j = 0$ for at least one index. A further use of conditional independence gives

$$\|PW_u\|_2^2 = \int_{\mathcal{R}} \prod_{j=1}^{l} U_{u^{n_j}} (w_j \otimes \overline{w_j})\, d\mu_{\mathcal{R}}. \tag{32}$$

For the centered index,

$$P_{\mathcal{R}}(w_j \otimes \overline{w_j}) = |Pw_j|^2 = 0.$$

Choose one integer c such that $n_j + c \neq 0$ for every j. Applying U_{u^c} to the whole integrand in (32) leaves its scalar integral unchanged and replaces each exponent n_j by $n_j + c$. They remain distinct, and are now all nonzero. The vector assertion (27) on $\mathcal{R}$ therefore makes the averaged integrand converge to zero in L^2. Its scalar integral tends to zero as well. Thus

$$\mathrm{Av}_{u\in\Phi_N} \|PW_u\|_2^2 \longrightarrow 0.$$

The expansion contains finitely many such centered products. For q of them, the squared norm of their sum is at most q times the sum of their squared norms, so their total projected contribution has vanishing averaged squared norm. The remaining product is already Y-measurable and equals $\prod_j U_{u^{n_j}} Pv_j$. This establishes (24). □

4. Radial identities and completion of character rigidity

The radial identities, the construction inside a number field, and the argument with bounded residues in this section follow OpenAI [19].

4.1. **Conditional Fourier data.** Retain the groups

$$F = \overline{\mathbb{Q}} \cap \mathbb{R}, \qquad E = F(i), \qquad K = \{u \in E : |u| = 1\}, \qquad \mathbb{D} = \widehat{E_{\mathrm{disc}}},$$

and the Borel subgroup $C \subset \mathbb{D}$ of ordinary continuous characters of the plane. For $z \in E$ write $\psi_z(d) = d(z)$. The action of $u \in K$ on $\mathbb{D}$ is given by $(ud)(z) = d(uz)$, and its action on functions is $U_u H(d) = H(ud)$.

Let μ be an ergodic K-invariant Borel probability on $\mathbb{D}$ such that $\mu(C) = 0$. Since E and K are countable and $\mathbb{D}$ is compact metrizable, $(\mathbb{D}, \mu)$ is a standard probability-preserving K-system. Apply Lemma 3.1 to this system and denote the terminal factor by Y, with probability

measure η. The relative product $\mathbb{D} \times_Y \mathbb{D}$ is ergodic. The identity map $\mathrm{id}_{\mathbb{D}}$ from $(\mathbb{D}, \mu)$ to $\mathbb{D}$ is equivariant and has wild law, so Lemma 2.9 applies to the same tower. Consequently the law ν_Y of $d - d'$, for two conditionally independent points over Y, satisfies $\nu_Y(C) = 0$. Equivariance of the conditional measures also makes ν_Y K-invariant.

Let μ_y denote conditional probability over $y \in Y$, and set

$$P = \mathbb{E}(\cdot \mid Y), \qquad g_z = P\psi_z, \qquad m(z) = \|g_z\|_2^2 \quad (z \in E). \tag{33}$$

Conditional independence gives

$$\begin{aligned} \int_{\mathbb{D}} \psi_z(d)\, d\nu_Y(d) &= \int_Y \left(\int_{\mathbb{D}} \psi_z(d)\, d\mu_y(d) \right) \overline{\left(\int_{\mathbb{D}} \psi_z(d')\, d\mu_y(d') \right)}\, d\eta(y) \\ &= \int_Y |g_z|^2\, d\eta = m(z). \end{aligned} \tag{34}$$

In particular, $0 \le m(z) \le 1$ and $m(0) = 1$. Conditional expectation commutes with the action on this invariant factor. Hence

$$g_{uz} = U_u g_z, \qquad m(uz) = m(z), \qquad g_{-z} = \overline{g_z}, \qquad \|g_z\|_\infty \le 1.$$

For every nonzero $z \in E$, its modulus belongs to $F_{>0}$ and $z/|z| \in K$. Thus $m(z)$ depends only on $|z|$. We also use $m(r)$ for this function on $F_{\ge 0}$, and put

$$w(r) = -\log m(r) \in [0, \infty], \qquad -\log 0 = \infty.$$

Then $w(0) = 0$.

Write I for the positive functional in Lemma 2.6. Its domain contains every bounded, boundedly supported function on F, and $I(\phi|_F) = \int_{\mathbb{R}} \phi(t)\, dt$ for $\phi \in C_c(\mathbb{R})$. Applying that lemma to ν_Y, with the nonzero direction $1 \in E$, and using (34), gives

$$I\big(t \longmapsto \phi(t) m(t)\big) = 0 \qquad \big(\phi \in C_c(\mathbb{R})\big). \tag{35}$$

For a finite nonempty set $\Phi \subset K$, use the notation

$$\mathrm{Avg}_{u \in \Phi}\, b_u = \frac{1}{|\Phi|} \sum_{u \in \Phi} b_u.$$

We will use mean ergodicity along every Følner sequence. For a unitary representation V of K, the orthogonal complement of its invariant vectors is the closed span of vectors $V_h f - f$. Indeed, orthogonality to all these differences is equivalent to invariance under each V_h. The average of such a difference has norm at most

$$\frac{|h\Phi \mathbin{\triangle} \Phi|}{|\Phi|} \|f\|.$$

This tends to zero along a Følner sequence, and the averaging operators have norm at most one. Approximation by finite linear combinations of differences proves convergence to the invariant projection. For each nonzero integer q, the power map $u \mapsto u^q$ is onto K. The representation $u \mapsto U_{u^q}$ therefore has the same invariant vectors as $u \mapsto U_u$. The factor Y is ergodic because the original system is ergodic, so its invariant functions are constant.

4.2. **The two radial identities.**

Lemma 4.1. *For all $A, B \in F_{\ge 0}$,*

$$w(|A - B|) \le w(A) + w(B). \tag{36}$$

For every $r \in F_{>0}$ and every Følner sequence (Φ_n) in K,

$$\mathrm{Avg}_{u \in \Phi_n}\, m\big(2r |\operatorname{Re} u|\big) \longrightarrow m(r)^2. \tag{37}$$

Proof. If v_u and h_u have L^2 norm at most one, then

$$\big|\|v_u\|_2^2 - \|h_u\|_2^2\big| \le 2\|v_u - h_u\|_2. \tag{38}$$

If the average of $\|v_u - h_u\|_2^2$ tends to zero, Cauchy–Schwarz in the finite average shows that the average of the left side of (38) tends to zero as well.

Suppose $A, B > 0$ and write $L = \sqrt{AB} \in F$. For every $u \in K$,

$$|A - Bu^2|^2 = (A-B)^2 + 4AB(\operatorname{Im} u)^2 = |-L + (A-B)u + Lu^2|^2. \tag{39}$$

The first equality follows from $\operatorname{Re}(u^2) = 1 - 2(\operatorname{Im} u)^2$. For the second, factor the last expression inside the modulus as

$$u\big((A-B) + L(u - u^{-1})\big) = u\big((A-B) + 2iL\operatorname{Im} u\big).$$

The character on the left of (39) factors as

$$\psi_{A-Bu^2} = \psi_A \, U_{u^2} \psi_{-B}.$$

Apply Lemma 3.3 with exponents $0, 2$ and the bounded functions ψ_A, ψ_{-B}. Its hypotheses hold because $\mathbb{D} \times_Y \mathbb{D}$ is ergodic. Along any Følner sequence,

$$\operatorname{Avg}_{u\in\Phi_n} \big\|P\psi_{A-Bu^2} - g_A U_{u^2} g_{-B}\big\|_2^2 \longrightarrow 0.$$

Both functions in the difference have norm at most one, so (38) transfers this to squared norms. Moreover,

$$\big\|g_A U_{u^2} g_{-B}\big\|_2^2 = \int_Y |g_A|^2 U_{u^2} |g_{-B}|^2 \, d\eta.$$

Mean ergodicity on Y for the representation $u \mapsto U_{u^2}$ therefore yields

$$\operatorname{Avg}_{u\in\Phi_n} m(A - Bu^2) \longrightarrow \left(\int_Y |g_A|^2 \, d\eta\right)\left(\int_Y |g_{-B}|^2 \, d\eta\right) = m(A)m(B). \tag{40}$$

For the other character in (39), apply Lemma 3.3 with exponents $0, 1, 2$. The resulting approximant to $P\psi_{-L+(A-B)u+Lu^2}$ is

$$h_u = g_{-L} \, U_u g_{A-B} \, U_{u^2} g_L,$$

and its averaged squared L^2 error tends to zero. Its two outer factors have absolute value at most one, whence

$$\|h_u\|_2^2 \le \|U_u g_{A-B}\|_2^2 = m(|A-B|).$$

Radiality and (39) identify the squared norms of the two projected characters. Apply (38) once more and use (40) to obtain

$$m(A)m(B) \le m(|A-B|).$$

If $m(A)$ and $m(B)$ are positive, the last quantity is positive too, and taking negative logarithms proves (36). If either factor vanishes, its w-value is infinite and the same inequality holds in the extended-real order. The computation also permits $A = B$, when the middle character in the second application is $\psi_0 = 1$. If $A = 0$ or $B = 0$, (36) is an equality because $w(0) = 0$.

For (37), use $u^{-1} = \overline{u}$ and

$$ru + ru^{-1} = 2r \operatorname{Re} u.$$

Lemma 3.3, with exponents $1, -1$ and the two functions ψ_r, ψ_r, gives

$$\operatorname{Avg}_{u\in\Phi_n} \big\|P\psi_{ru+ru^{-1}} - U_u g_r \, U_{u^{-1}} g_r\big\|_2^2 \longrightarrow 0.$$

The squared norm of the approximant is

$$\begin{aligned}\left\|U_u g_r\, U_{u^{-1}} g_r\right\|_2^2 &= \int_Y U_u|g_r|^2\, U_{u^{-1}}|g_r|^2\, d\eta \\ &= \int_Y |g_r|^2\, U_{u^{-2}}|g_r|^2\, d\eta.\end{aligned}$$

Mean ergodicity for $u \mapsto U_{u^{-2}}$ makes its average converge to $m(r)^2$. Applying (38) and radiality proves (37). All uses of Lemma 3.3 and mean ergodicity hold along every Følner sequence, which proves the stated quantifier. □

4.3. **Bases of unit directions in a number field.**

Lemma 4.2. *Let $N_0 \subset F$ be a number field, put $N = N_0(i)$, and write $d_0 = [N : \mathbb{Q}]$. For every positive integer L there is a finite set $S \subset K \cap N$ with $|S| = L$ such that every subset of S with at most d_0 elements is linearly independent over $\mathbb{Q}$. In particular, every d_0 distinct elements form a $\mathbb{Q}$-basis of N.*

Proof. For $t \in N_0$ consider

$$u(t) = \frac{1 + it}{1 - it}.$$

Its denominator is nonzero, since t is real. Complex conjugation interchanges numerator and denominator, so $|u(t)| = 1$, and $u(t) \in K \cap N$.

Choose $\mathbb{Q}$-bases of N_0 and N. Expressing t in the first basis identifies its parameters with a point of $\mathbb{Q}^{[N_0:\mathbb{Q}]}$. Multiplication by $1 - it$ on N has a matrix with entries affine in these parameters and rational coefficients. The inverse matrix expresses the coordinates of $u(t)$ as rational functions over $\mathbb{Q}$. At every rational parameter point, this matrix is invertible because $1 - it$ is a nonzero element of the field N.

Suppose a nonzero $\mathbb{Q}$-linear functional $\ell : N \to \mathbb{Q}$ annihilates all the $u(t)$, and fix $a \in N_0$. As a function of a rational variable h, $\ell(u(ha))$ is a rational function over $\mathbb{Q}$ that vanishes for every $h \in \mathbb{Q}$. Its denominator is nonzero at these points, so its numerator has infinitely many roots and the rational function is identically zero. The formal expansion in $N[[h]]$ is

$$u(ha) = 1 + 2iha - 2h^2a^2 + O(h^3).$$

Its constant, linear, and quadratic coefficients imply

$$\ell(1) = 0, \qquad \ell(ia) = 0, \qquad \ell(a^2) = 0.$$

This holds for every $a \in N_0$. Every element of N_0 is a rational linear combination of squares, since

$$4a = (a + 1)^2 - (a - 1)^2.$$

Thus ℓ vanishes on both N_0 and iN_0. These two subspaces span N, contradicting $\ell \neq 0$.

Now let $W_1, \ldots, W_b$ be finitely many proper rational linear subspaces of N. For each one choose a nonzero rational linear functional ℓ_j that vanishes on W_j. The rational function $\ell_j(u(t))$ is not identically zero by the preceding paragraph, so it has a nonzero numerator polynomial in the rational coordinates of t. The product of these finitely many numerator polynomials is a nonzero polynomial over $\mathbb{Q}$. Such a polynomial has a rational point where it is nonzero, as follows by induction on the number of variables from the corresponding assertion for one variable. At that parameter point all denominators are nonzero as well. Consequently some $u(t)$ avoids every W_j.

Construct S successively. At each step exclude the spans of all subsets of at most $d_0 - 1$ previously selected points. There are finitely many such spans, and each has dimension less

than d_0. The preceding avoidance property supplies another unit direction outside all of them. Every subset with at most d_0 elements remains independent after this addition. Repeating the procedure gives a set of any prescribed finite size. □

4.4. **Integer combinations below a fixed cap.**

Lemma 4.3. *Suppose $w : F_{\geq 0} \to [0,\infty]$ satisfies $w(0) = 0$ and (36). Let $p_1, \dots, p_q \in F_{\geq 0}$ and $H \in [0,\infty)$ satisfy*

$$p = \max_{1\leq j\leq q} p_j > 0, \qquad w(p_j) \leq H.$$

For an integer $M \geq 0$, suppose integers $a_1, \dots, a_q$ obey

$$z = \sum_{j=1}^{q} a_j p_j \in [0,p), \qquad \sum_{j=1}^{q} |a_j| \leq M,$$

then $w(z) \leq 4MH$.

Proof. For $x \in F$ let

$$\rho_p(x) = x - p\lfloor x/p \rfloor \in F \cap [0,p).$$

This is the unique representative of x modulo $p\mathbb{Z}$ in $[0,p)$, so $\rho_p(x+y) = \rho_p(\rho_p(x) + \rho_p(y))$. Since p equals one of the p_j, we have $w(p) \leq H$. If $0 < p_j < p$, the residues of p_j and $-p_j$ are respectively p_j and $p - p_j$, and

$$w(p - p_j) \leq w(p) + w(p_j) \leq 2H.$$

If $p_j = 0$ or $p_j = p$, both signed residues are zero. Thus every signed input p_j or $-p_j$ has a residue with w-value at most $2H$.

Consider two current residues $a, b \in F \cap [0,p)$. If $a + b < p$, the quantities in the following computation are nonnegative, and three applications of (36) give

$$\begin{aligned} w(a+b) &= w\big(p - ((p-a) - b)\big) \\ &\leq w(p) + w((p-a) - b) \\ &\leq w(p) + w(p-a) + w(b) \\ &\leq 2w(p) + w(a) + w(b). \end{aligned}$$

If $a + b > p$, then $\rho_p(a+b) = a - (p-b) > 0$ and

$$w(\rho_p(a+b)) \leq w(a) + w(p-b) \leq w(a) + w(p) + w(b).$$

If $a + b = p$, the new residue is zero and its w-value is zero. In all three cases,

$$w(\rho_p(a+b)) \leq w(a) + w(b) + 2H.$$

Adding the residue of one signed input therefore increases the current bound by at most $4H$.

Expand the integer combination into $\sum_j |a_j|$ signed inputs, starting with residue zero. After all inputs the resulting residue has w-value at most $4H \sum_j |a_j| \leq 4MH$. The assumed range $0 \leq z < p$ makes this final residue equal to z. The same reasoning includes zero inputs, zero coefficients, and the case $H = 0$. □

4.5. **A uniform interval bound and the final contradiction.**

Proof of Theorem 2.3. First suppose that μ is ergodic and wild. Use the factor Y and the functions m, w constructed in Subsection 4.1. Assume that $m(r) > 0$ for some fixed $r \in F_{>0}$. Choose, for example,

$$H = 1 - 2\log m(r) > 0, \qquad e^{-H} = e^{-1}m(r)^2 < m(r)^2,$$

and define

$$A_0 = \{u \in K : w(2r|\operatorname{Re} u|) \le H\}.$$

There are finitely many $q_1, \dots, q_J \in K$ such that

$$\text{for every } v \in K, \text{ some } q_j v \text{ belongs to } A_0. \tag{41}$$

To prove this, suppose that for every finite set $T \subset K$ there is $v \in K$ with $Tv \cap A_0 = \varnothing$. Fix a Følner sequence (Φ_n) in the countable abelian group K and choose such a v_n for $T = \Phi_n$. The sets $\Phi_n v_n$ are also a Følner sequence, since

$$\frac{|h(\Phi_n v_n) \triangle (\Phi_n v_n)|}{|\Phi_n v_n|} = \frac{|h\Phi_n \triangle \Phi_n|}{|\Phi_n|} \longrightarrow 0 \qquad (h \in K).$$

For every u in these sets, $w(2r|\operatorname{Re} u|) > H$, and hence $m(2r|\operatorname{Re} u|) < e^{-H}$. Their averages are at most e^{-H}, contradicting (37) along this translated sequence. Thus some finite $T = \{q_1, \dots, q_J\}$ has the property that Tv meets A_0 for every v, proving (41).

Choose the number field

$$N_0 = \mathbb{Q}\big(\operatorname{Re} q_1, \operatorname{Im} q_1, \dots, \operatorname{Re} q_J, \operatorname{Im} q_J\big) \subset F, \qquad N = N_0(i), \qquad d_0 = [N : \mathbb{Q}].$$

There are only finitely many generators and each is algebraic, so N_0 is a finite extension of $\mathbb{Q}$. Every q_j belongs to N. Apply Lemma 4.2 to choose a finite $S \subset K \cap N$ with

$$|S| = J(d_0 - 1) + 1$$

such that every d_0 distinct elements of S form a rational basis of N. Consider the finite family of bases

$$\mathcal{B} = \{\{q_j s : s \in T\} : 1 \le j \le J,\ T \subset S,\ |T| = d_0\}. \tag{42}$$

Each member is a basis because multiplication by $q_j \neq 0$ is an invertible $\mathbb{Q}$-linear map on N.

For any $v \in K$, apply (41) to every sv with $s \in S$. Assign to each s one index j for which $q_j sv \in A_0$. Since $|S| > J(d_0 - 1)$, some index occurs at least d_0 times. Choosing d_0 of those points produces a basis $B \in \mathcal{B}$ satisfying

$$kv \in A_0 \qquad (k \in B). \tag{43}$$

Thus the basis may depend on v, while the family $\mathcal{B}$ has already been fixed.

For each $B \in \mathcal{B}$ write the rational basis expansions

$$1 = \sum_{k \in B} b_{B,k} k, \qquad i = \sum_{k \in B} c_{B,k} k.$$

Take a positive integer D_0 divisible by the denominators of all the $b_{B,k}$, over every $B \in \mathcal{B}$. Set $a_{B,k} = D_0 b_{B,k} \in \mathbb{Z}$ and

$$M_0 = \max_{B \in \mathcal{B}} \sum_{k \in B} |a_{B,k}|.$$

This is a finite positive integer, and every candidate basis obeys

$$D_0\, 1 = \sum_{k \in B} a_{B,k} k, \qquad \sum_{k \in B} |a_{B,k}| \le M_0. \tag{44}$$

For the second uniform choice put

$$\beta_B = \sum_{k\in B} |b_{B,k}|, \qquad \gamma_B = \sum_{k\in B} |c_{B,k}|, \qquad c_0 = \min\left\{1, \min_{B\in\mathcal{B}} \frac{1}{\sqrt{\beta_B^2+\gamma_B^2}}\right\} > 0.$$

Both basis expansions represent nonzero elements, so the denominators in this expression are positive. For any complex v with $|v| = 1$ let $f_B(v) = \max_{k\in B} |\operatorname{Re}(kv)|$. Taking real parts in the two expansions after multiplying by v gives

$$|\operatorname{Re} v| \le \beta_B f_B(v), \qquad |\operatorname{Im} v| = |\operatorname{Re}(iv)| \le \gamma_B f_B(v).$$

Since $|v|^2 = 1$, it follows that

$$\max_{k\in B} |\operatorname{Re}(kv)| \ge \frac{1}{\sqrt{\beta_B^2+\gamma_B^2}} \ge c_0 \qquad (B\in\mathcal{B},\ |v|=1). \tag{45}$$

The constants D_0, M_0, c_0 have all been fixed before the next choice of v. The elements k of each candidate basis lie in the fixed field N, and kv lies in K for every $v \in K$.

Now let $v \in K$ satisfy $|\operatorname{Re} v| < c_0/D_0$, and choose a basis $B \in \mathcal{B}$ satisfying (43). For $k \in B$ put

$$p_k = 2r|\operatorname{Re}(kv)|, \qquad p = \max_{k\in B} p_k.$$

These numbers belong to $F_{\ge 0}$. Goodness of the basis and (45) give

$$w(p_k) \le H \quad (k\in B), \qquad p \ge 2rc_0 > 0. \tag{46}$$

Multiply (44) by v, take real parts, and then multiply by $2r$. This gives

$$2rD_0 \operatorname{Re} v = \sum_{k\in B} a_{B,k}\, 2r \operatorname{Re}(kv).$$

Choose $\varepsilon_0, \varepsilon_k \in \{1,-1\}$ so that

$$\varepsilon_0 \operatorname{Re} v = |\operatorname{Re} v|, \qquad \operatorname{Re}(kv) = \varepsilon_k |\operatorname{Re}(kv)|.$$

At a zero real part take the corresponding sign to be 1. Then the nonnegative number

$$z = 2rD_0|\operatorname{Re} v| = \sum_{k\in B} (\varepsilon_0 a_{B,k}\varepsilon_k) p_k$$

is an integer combination whose coefficient absolute-value sum is at most M_0. Also

$$0 \le z < 2rc_0 \le p.$$

Lemma 4.3, with the data in (46), now proves

$$w\big(2rD_0|\operatorname{Re} v|\big) \le 4M_0H \qquad \left(v\in K,\ |\operatorname{Re} v| < \frac{c_0}{D_0}\right). \tag{47}$$

This includes $\operatorname{Re} v = 0$ and any zero values among the individual p_k.

Take any $s \in F$ with $0 < s < 2rc_0$ and set

$$t = \frac{s}{2rD_0} \in F.$$

Because $D_0 \ge 1$ and $c_0 \le 1$, we have $0 < t < c_0/D_0 \le 1$. The number

$$v = t + i\sqrt{1-t^2}$$

belongs to K, since its coordinates are real algebraic numbers and its modulus is one. Its real part is t, so (47) yields the uniform bound

$$m(s) \geq e^{-4M_0H} > 0 \qquad \big(s \in F \cap (0, 2rc_0)\big). \tag{48}$$

Choose a nonnegative $\phi \in C_c(\mathbb{R})$ with support contained in $(0, 2rc_0)$ and with positive integral. The functions $\phi|_F$ and $t \mapsto \phi(t)m(t)$ belong to the domain of I. By (48) and positivity,

$$I\big(t \longmapsto \phi(t)m(t)\big) \geq e^{-4M_0H} I(\phi|_F) = e^{-4M_0H} \int_{\mathbb{R}} \phi(t)\, dt > 0.$$

This contradicts (35). The functional acts on all the bounded, boundedly supported functions on F used here, so the argument applies directly to m with its original algebraic domain.

It follows that $m(r) = 0$ for every $r \in F_{>0}$. Radiality then gives $m(z) = 0$ for every nonzero $z \in E$. For such z,

$$\left| \int_{\mathbb{D}} \psi_z \, d\mu \right| = \left| \int_Y g_z \, d\eta \right| \leq \|g_z\|_2 = 0.$$

Let $m_{\mathbb{D}}$ be Haar probability on the compact abelian group $\mathbb{D}$. For $z \neq 0$, an ordinary continuous character $j(\xi) \in \mathbb{D}$ can be chosen with $\psi_z(j(\xi)) = e^{i\xi \cdot z} \neq 1$. Thus ψ_z is a nontrivial character of $\mathbb{D}$, and translation invariance of $m_{\mathbb{D}}$ forces $\int \psi_z \, dm_{\mathbb{D}} = 0$. The characters ψ_z contain the constants, are closed under multiplication and conjugation, and separate points of $\mathbb{D}$. The Stone–Weierstrass theorem makes their complex linear span dense in the continuous functions on this compact space. The two probabilities have the same integral on this dense span, and hence on every continuous function. Uniqueness of Borel probability measures on a compact metrizable space gives $\mu = m_{\mathbb{D}}$.

Finally let μ be any wild K-invariant probability. The Farrell–Varadarajan ergodic decomposition theorem applies to the probability-preserving action of the countable group K on the standard Borel space $\mathbb{D}$ [6, Theorem 3.1]. It writes

$$\mu = \int \mu_\omega \, d\lambda(\omega),$$

where almost every μ_ω is K-invariant and ergodic. The set C is Borel and K-invariant, so

$$0 = \mu(C) = \int \mu_\omega(C) \, d\lambda(\omega).$$

Nonnegativity gives $\mu_\omega(C) = 0$ for almost every ω. The ergodic case proved above identifies each of these μ_ω with $m_{\mathbb{D}}$. Integrating this equality gives $\mu = m_{\mathbb{D}}$, which proves the theorem. □

5. Transfer to ordinary colorings

The proper-to-weak transfer for the unit distance, including the continuous-factor construction below, is due to OpenAI [19]. Sergi Galán extended the argument to algebraic forbidden distances and combined it with the separated-distance theorem of Bukh to answer Erdős's exponential-growth question [14]. We give the finite-distance transfer and that deduction in full, using the rigidity theorem proved above.

Retain the notation

$$F = \overline{\mathbb{Q}} \cap \mathbb{R}, \qquad E = F(i), \qquad K = \{u \in E : u\overline{u} = 1\}, \qquad \mathbb{D} = \widehat{E_{\mathrm{disc}}}.$$

The group E is additive and K is multiplicative. For $\chi \in \mathbb{D}$ and $z \in E$, write

$$\psi_z(\chi) = \chi(z), \qquad (u\chi)(z) = \chi(uz), \qquad C = j(\mathbb{R}^2), \quad j(\xi)(z) = e^{i\xi \cdot z}.$$

The ordinary-character subgroup C is Borel in $\mathbb{D}$. Indeed, j is continuous and injective, and its image is the countable union of the compact images of closed frequency balls. Its inverse on C

is Borel, since each compact restriction is a homeomorphism onto its image. Theorem 2.3 states that a K-invariant Borel probability measure on $\mathbb{D}$ assigning zero mass to C is Haar probability.

For $X \subseteq \mathbb{R}^2$ and $D \subset (0,\infty)$, let $G(X,D)$ have vertex set X, with an edge between distinct x, y precisely when $|x-y| \in D$. Write $\chi(\mathbb{R}^2, D) = \chi(G(\mathbb{R}^2, D))$. Let σ denote normalized arc length on S^1. A measurable set $A \subseteq \mathbb{R}^2$ *weakly avoids* D if

$$\int_{B(0,R)} \int_{S^1} 1_A(x) 1_A(x+du)\, d\sigma(u)\, dx = 0 \qquad (d \in D,\ R > 0). \tag{49}$$

A weak measurable k-coloring is a measurable partition into k classes satisfying (49). As throughout the paper, $m(D)$ uses the upper densities of measurable sets that avoid every distance in D pointwise.

Theorem 5.1 (OpenAI and Galán)**.** *Let $D \subset F \cap (0,\infty)$ be finite and nonempty, and let $k \geq 1$ be an integer. The following assertions are equivalent.*

1. *$G(E,D)$ has a proper k-coloring.*
2. *$G(\mathbb{R}^2, D)$ has a proper k-coloring.*
3. *There is a weak measurable k-coloring of $\mathbb{R}^2$ avoiding D.*

Consequently,

$$\chi(\mathbb{R}^2, D) \geq \frac{1}{m(D)}. \tag{50}$$

5.1. **Continuous vectors form a positive factor.** In the next lemma, the acting group E initially has the discrete topology. Continuity of a vector refers to the usual subspace topology on $E \subset \mathbb{R}^2$. We use the inner product $\langle v, w\rangle = \int \overline{v} w$, which is linear in the second variable.

Lemma 5.2. *Suppose a standard probability space (Ω, μ) carries a measure-preserving action of E, with Koopman operators T_z. The subspace*

$$\mathcal{H}_c = \left\{ v \in L^2(\mu) : \lim_{\substack{z \to 0 \\ z \in E}} \|T_z v - v\|_2 = 0 \right\}$$

is $L^2(\mathcal{F}_c)$ for a completed sub-σ-algebra $\mathcal{F}_c$. Its orthogonal projection is conditional expectation onto $\mathcal{F}_c$. It is also the range of the spectral projection on $C \subset \mathbb{D}$. On $\mathcal{H}_c$, the translations extend uniquely to a strongly continuous unitary representation $(T_x)_{x \in \mathbb{R}^2}$ that preserves nonnegative functions and fixes the constant one. If unitary Koopman operators V_u, $u \in K$, satisfy

$$V_u T_z V_u^{-1} = T_{uz},$$

then the orthogonal projection onto $\mathcal{H}_c$ commutes with every V_u.

Proof. The subspace $\mathcal{H}_c$ is closed, because

$$\|T_z v - v\|_2 \leq 2\|v - w\|_2 + \|T_z w - w\|_2.$$

It contains constants, is closed under complex conjugation, and is invariant under T_z and T_z^{-1}.

The spectral theorem for the discrete abelian group E gives a projection-valued measure P on $\mathbb{D}$. For $v \in L^2(\mu)$, its finite positive scalar measure $\lambda_v(A) = \|\mathsf{P}(A)v\|_2^2$ satisfies

$$\langle v, T_z v\rangle = \int_{\mathbb{D}} \psi_z(\chi)\, d\lambda_v(\chi), \qquad \|T_z v - v\|_2^2 = \int_{\mathbb{D}} |\psi_z(\chi) - 1|^2\, d\lambda_v(\chi). \tag{51}$$

The applicable spectral theorem and Bochner's theorem are stated in [2, Theorems D.3.1 and D.2.2]. If $\lambda_v(\mathbb{D} \setminus C) = 0$, the second identity and dominated convergence imply $v \in \mathcal{H}_c$.

Conversely, suppose $v \in \mathcal{H}_c$. The function $F_v(z) = \langle v, T_z v\rangle$ is uniformly continuous on E, since

$$|F_v(z+h) - F_v(z)| \le \|v\|_2 \|T_h v - v\|_2.$$

It extends uniquely to a continuous function on $\mathbb{R}^2$. For any finite list of points in $\mathbb{R}^2$, approximation by points of E passes the positive-definite quadratic inequality to the limit. The extension is therefore positive definite. Bochner's theorem gives a finite positive measure ρ_v on $\mathbb{R}^2$ whose Fourier transform is this extension. The pushforward $j_*\rho_v$ and λ_v have identical integrals against every ψ_z. The complex linear span of the functions ψ_z is a unital self-adjoint algebra separating points of $\mathbb{D}$. Stone–Weierstrass and uniqueness of finite measures on a compact space give

$$\lambda_v = j_*\rho_v.$$

Thus λ_v is carried by C. This proves that $\mathcal{H}_c$ is the range of $\mathsf{P}(C)$.

We next establish the factor assertion, which will preserve the color probabilities. If bounded v, w belong to $\mathcal{H}_c$, then

$$\|T_z(vw) - vw\|_2 \le \|v\|_\infty \|T_z w - w\|_2 + \|w\|_\infty \|T_z v - v\|_2.$$

Hence the bounded members form a unital self-adjoint algebra $\mathcal{A}$. For a Lipschitz map $\Phi : \mathbb{C} \to \mathbb{C}$, the Koopman property also gives

$$\|T_z\Phi(v) - \Phi(v)\|_2 \le \mathrm{Lip}(\Phi)\|T_z v - v\|_2.$$

In particular, projection of v onto a closed complex disk produces bounded members of $\mathcal{H}_c$ converging to v in L^2. Thus the L^2 closure of $\mathcal{A}$ is $\mathcal{H}_c$.

Let $\mathcal{F}_c$ be the completed σ-algebra generated by $\mathcal{A}$. For any finite family from $\mathcal{A}$, polynomials in their coordinates and conjugates approximate continuous functions on the closure of their joint range. The joint range is bounded, and its closure is compact. By regularity of the pushforward probability, these continuous functions are dense in the L^2 space of the joint distribution. The monotone-class theorem then shows that finite-coordinate functions are dense in $L^2(\mathcal{F}_c)$. Consequently $L^2(\mathcal{F}_c) \subseteq \mathcal{H}_c$. The reverse inclusion follows from the bounded truncations just constructed, so equality holds. The orthogonal projection is therefore conditional expectation. In particular, it sends nonnegative functions to nonnegative functions and fixes one. For completeness, if the conditional expectation of $f \ge 0$ were negative on a positive-measure $\mathcal{F}_c$-measurable set, its integral on that set would be negative, contradicting its defining equality with the integral of f.

For $x \in \mathbb{R}^2$, choose $z_n \in E$ with $z_n \to x$. For $v \in \mathcal{H}_c$,

$$\|T_{z_n} v - T_{z_m} v\|_2 = \|T_{z_n - z_m} v - v\|_2 \longrightarrow 0.$$

Define $T_x v$ to be the limit. The same identity gives independence of the approximating sequence and strong continuity in x. Approximating sums and negatives gives the group law, inverses, and unitarity. The positive cone of L^2 is closed, so these extensions preserve positivity and fix one.

Finally, covariance implies

$$\|T_z V_u v - V_u v\|_2 = \|T_{u^{-1}z} v - v\|_2 \longrightarrow 0.$$

Thus V_u and V_u^{-1} preserve $\mathcal{H}_c$. The orthogonal projection onto this subspace commutes with V_u. □

5.2. **An invariant probability on proper labelings.** We give the averaging argument explicitly. The elementary amenability facts used here can also be found in [2, Appendix G].

Lemma 5.3. *If $G(E,D)$ has a proper k-coloring, there is a standard probability space of proper labelings on which translations by E and rotations by K act measure preservingly. The origin-label indicators $f_1,\dots,f_k$ satisfy*

$$0 \le f_i \le 1, \qquad \sum_i f_i = 1, \qquad V_u f_i = f_i, \qquad \langle f_i, T_z f_i\rangle = 0 \quad (z \in dK,\ d \in D). \tag{52}$$

Proof. The proper labelings form a nonempty closed subset Ω of $\{1,\dots,k\}^E$. Each edge imposes one closed cylinder condition. Since E is countable, Ω is compact metrizable. The group $\Gamma = E \rtimes K$, with multiplication

$$(z,u)(w,v) = (z + uw, uv),$$

acts continuously by

$$((z,u)\omega)(x) = \omega\big(u^{-1}(x-z)\big).$$

This action preserves the proper-labeling conditions.

We first construct finite left Følner sets in Γ. An abelian group admits a finite set approximately invariant under any prescribed finite set of translations. Indeed, the prescribed elements generate a finitely generated abelian group, which is a product of a finite group and finitely many infinite cyclic groups. Products of the finite group with sufficiently large integer boxes have the required property.

Given a finite $S \subset \Gamma$ and $\eta > 0$, choose a finite nonempty $B \subset K$ such that

$$|uB \triangle B| \le \eta |B| \qquad ((z,u) \in S).$$

Choose a finite nonempty $A \subset E$ such that

$$|(A + v^{-1}z) \triangle A| \le \eta |A|$$

for every $(z,u) \in S$ and every $v \in B \cup uB$. Only finitely many additive translations occur here. Put

$$F_S = \{(va, v) : v \in B,\ a \in A\}.$$

Left multiplication by (z,u) sends the fiber above v to the fiber above uv, with additive coordinate translated by $(uv)^{-1}z$. Fibers with rotation coordinate in $B \triangle uB$ contribute at most $\eta|A||B|$ to the symmetric difference. The common fibers contribute at most the same quantity. Hence

$$|(z,u)F_S \triangle F_S| \le 2\eta |F_S| \qquad ((z,u) \in S).$$

Enumerating Γ and taking increasing finite S and $\eta \downarrow 0$ gives finite nonempty sets F_n with $|gF_n \triangle F_n|/|F_n| \to 0$ for every $g \in \Gamma$.

Fix one proper labeling ω_0, and let

$$\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{g\omega_0}.$$

Compactness of the probability measures on Ω gives a weakly convergent subsequence with limit μ. For continuous h on Ω and fixed $g \in \Gamma$, the difference between the integrals of h and $h \circ g$ against μ_n is bounded by

$$2\|h\|_\infty \frac{|gF_n \triangle F_n|}{|F_n|}.$$

It tends to zero, so μ is Γ-invariant.

Use the Koopman convention $U_g f(\omega) = f(g^{-1}\omega)$, and write $T_z = U_{(z,1)}$ and $V_u = U_{(0,u)}$. For $f_i(\omega) = 1_{\{\omega(0)=i\}}$, the function $T_z f_i$ indicates that $\omega(z) = i$. Rotations fix the origin, and properness forbids equal labels at 0 and every $z \in dK$. This proves (52) and the covariance used in Lemma 5.2. □

5.3. **Rigidity removes the residual correlations.** Apply Lemma 5.2 to the invariant labeling law, and write

$$p_i = \mathsf{P}(C) f_i = \mathbb{E}(f_i \mid \mathcal{F}_c), \qquad b_i = f_i - p_i.$$

Then p_i, b_i are real and rotation invariant, and

$$0 \le p_i \le 1, \qquad \sum_i p_i = 1. \tag{53}$$

For the scalar spectral measure of b_i, covariance gives

$$\int \psi_{uz}\, d\lambda_{b_i} = \langle b_i, T_{uz} b_i \rangle = \langle b_i, T_z b_i \rangle = \int \psi_z\, d\lambda_{b_i}.$$

Fourier uniqueness on $\mathbb{D}$ shows that λ_{b_i} is K-invariant. Since $\mathsf{P}(C) b_i = 0$, it assigns zero mass to C. If its mass is positive, normalize it and apply Theorem 2.3. Including the zero-mass case, the result is

$$\lambda_{b_i} = \|b_i\|_2^2 m_{\mathbb{D}}, \qquad \langle b_i, T_z b_i \rangle = 0 \quad (z \in E \setminus \{0\}), \tag{54}$$

where $m_{\mathbb{D}}$ is Haar probability. The last equality follows from Haar orthogonality. In fact, for $z \neq 0$ one can choose an ordinary character χ_0 with $\chi_0(z) \neq 1$. Translation of Haar measure by χ_0 then multiplies its zth Fourier coefficient by $\chi_0(z)$ while preserving that coefficient, so it is zero.

Both $\mathcal{H}_c$ and its orthogonal complement reduce every T_z. Thus the cross terms in $\langle f_i, T_z f_i \rangle$ vanish. Equations (52) and (54) imply

$$\langle p_i, T_z p_i \rangle = 0 \qquad (z \in dK,\ d \in D).$$

The algebraic unit directions are dense in S^1. For example,

$$\frac{1 - t^2 + 2it}{1 + t^2} \in K \qquad (t \in \mathbb{Q})$$

forms a dense subset as rational t varies. Since $d \in F$, the set dK consists of algebraic points and is dense in the circle of radius d. Strong continuity consequently gives

$$\langle p_i, T_v p_i \rangle = 0 \qquad (v \in \mathbb{R}^2,\ |v| \in D). \tag{55}$$

Unitarity and the group law give, for every $x \in \mathbb{R}^2$,

$$\langle T_x p_i, T_{x+v} p_i \rangle = 0 \qquad (|v| \in D). \tag{56}$$

5.4. **Jointly measurable fields and a single sample.** We now realize the continuous L^2-valued maps by measurable fields and keep track of both translated endpoints.

Choose measurable rounding maps $q_n : \mathbb{R}^2 \to \mathbb{Q}^2 \subset E$ with countable range and $|q_n(x) - x| \le 2^{-n}$. For each rational point in the union of these ranges, choose Borel representatives of all $T_q p_i$ simultaneously. They may be chosen with values in $[0,1]$ and sum one everywhere, by modifying a common null set for this countable family. Let $H_i^{(n)}(x, \omega)$ be the representative at $q_n(x)$. These functions are jointly measurable and take values in the probability simplex. Moreover,

$$\sup_x \|H_i^{(n)}(x, \cdot) - T_x p_i\|_2 \le \sup_{|h| \le 2^{-n}} \|T_h p_i - p_i\|_2 \longrightarrow 0.$$

Pass to a subsequence for which this supremum is at most 2^{-n}, simultaneously for the finitely many i. On each bounded spatial domain, the successive differences then have summable product-space L^2 norms and hence summable L^1 norms. Their pointwise absolute sums are finite almost everywhere, by Tonelli. Consequently $H_i^{(n)}$ converges almost everywhere on $\mathbb{R}^2 \times \Omega$. Define H_i by this limit where it exists, and assign the fixed simplex vector $(1, 0, \ldots, 0)$ elsewhere. Then the H_i are jointly measurable, belong to $[0, 1]$, and sum to one everywhere. For almost every x, bounded convergence in ω and the displayed L^2 convergence imply

$$H_i(x, \cdot) = T_x p_i \quad \text{in } L^2(\mu). \tag{57}$$

Let $N \subset \mathbb{R}^2$ be a common null set outside which (57) holds for all i. For each $d \in D$ and positive integer m,

$$\int_{B(0,m)} \int_{S^1} \big(1_N(x) + 1_N(x + du)\big)\, d\sigma(u)\, dx = 0. \tag{58}$$

For the second term, fix u, use translation invariance of Lebesgue measure, and then integrate in u. Thus (56) applies to the selected representatives for almost every pair (x, u). Since the fields are real and nonnegative, Tonelli gives

$$\int_\Omega \int_{B(0,m)} \int_{S^1} H_i(x, \omega) H_i(x + du, \omega)\, d\sigma(u)\, dx\, d\mu(\omega) = 0.$$

There are only finitely many i, d and countably many m. Choose one ω for which all the inner integrals vanish. Assign x the least index i with $H_i(x, \omega) > 0$. This gives a measurable coloring, because the fields sum to one. For each i, d, m, the positive-product locus has zero $dx\, d\sigma(u)$ measure. A pair assigned color i has positive H_i at both endpoints. The corresponding color class therefore satisfies (49). This proves the implication from assertion 1 to assertion 3 of Theorem 5.1.

5.5. **Density points and finite-color compactness.**

Lemma 5.4. *If a measurable set A weakly avoids D, then its set A^* of Lebesgue density-one points avoids D pointwise and satisfies $|A \triangle A^*| = 0$. For a measurable partition into finitely many classes, their density-one sets are disjoint and have conull union.*

Proof. Suppose $x, y \in A^*$ with $v = y - x$ and $|v| = d \in D$. For small $h > 0$, take X uniformly in $B(x, h)$ and, independently, take U uniformly on the positive-length arc

$$\{u \in S^1 : |du - v| < h\}.$$

Density one at x gives $\mathbb{P}(X \notin A) = o(1)$ as $h \downarrow 0$. The distribution of $X + dU$ is a mixture of translates of the uniform radius-h disk distribution. It has density at most $1/(\pi h^2)$ and is supported in $B(y, 2h)$. Thus

$$\mathbb{P}(X + dU \notin A) \le \frac{|B(y, 2h) \setminus A|}{\pi h^2} = o(1).$$

For sufficiently small h, both points belong to A with positive probability. The law of (X, U) is absolutely continuous with respect to planar Lebesgue measure times arc length, and its first coordinate has bounded support. This contradicts (49). Hence A^* avoids every $d \in D$.

The set A^* is measurable, as follows by expressing the density limit using measurable local averages. Lebesgue differentiation gives $|A \triangle A^*| = 0$. Two disjoint sets cannot both have density one at the same point, so the density-one sets of a finite partition are disjoint. Their union is conull by the same differentiation theorem. □

Completion of the proof of Theorem 5.1. Assertion 2 implies assertion 1 by restriction, and the preceding construction proves that assertion 1 implies assertion 3. Assume assertion 3 and let $A_1^*, \dots, A_k^*$ be the sets from Lemma 5.4. Write $B = \bigcup_i A_i^*$, a conull subset of the plane. For every finite $P \subset \mathbb{R}^2$, the set

$$\bigcap_{x \in P} (B - x)$$

is conull and nonempty. A translation parameter in this intersection sends P into B. The classes A_i^* then properly color the translated distance graph and hence $G(P, D)$.

For completeness, consider the compact space $\{1, \dots, k\}^{\mathbb{R}^2}$. Each edge of $G(\mathbb{R}^2, D)$ imposes a closed condition on two coordinates. Every finite collection of these conditions is satisfiable by the preceding finite-configuration argument. The finite intersection property therefore gives a simultaneous proper coloring. This is the finite-color compactness principle of de Bruijn and Erdős [8, Theorem 1], and proves assertion 2.

The sets A_i^* partition the plane up to a null set. For every positive integer n, at least one has relative area at least $1/k$ in $[-n, n]^2$. One fixed index occurs for infinitely many n, so that class has upper density at least $1/k$. It avoids D pointwise, giving $m(D) \geq 1/k$. Applying this to any finite proper coloring proves (50). □

6. Exponential growth for finite distance graphs

Define

$$L(r) = \sup\{\chi(G(P, D)) : P \subset \mathbb{R}^2 \text{ finite},\ D \subset (0, \infty),\ |D| = r\}, \qquad r \geq 1.$$

The graph here includes every edge whose length lies in D. The question in Erdős 706 asks whether this quantity admits a polynomial bound in r. The following argument is Galán's application of the preceding transfer and Bukh's product limit [14, 4].

6.1. **An elementary upper bound.**

Lemma 6.1. *For every finite set D of positive real distances,*

$$\chi(\mathbb{R}^2, D) \leq 7^{|D|}.$$

Proof. We first give a proper seven-coloring for the unit distance. Tile the plane by regular hexagonal Voronoi cells of a triangular lattice generated by vectors a, b of equal length $\sqrt{3}s$ and angle 60 degrees, where $s = 2/5$. Each closed cell has circumradius s and diameter $2s < 1$. The sublattice generated by $2a + b$ and $-a + 3b$ has index seven. For integers m, n, the squared length of $m(2a + b) + n(-a + 3b)$ is

$$7|a|^2(m^2 + mn + n^2).$$

Thus distinct centers in the same sublattice coset are separated by at least $\sqrt{21}s$. Color each cell by the coset of its center. Points in different cells of the same color are separated by at least

$$(\sqrt{21} - 2)s > 1,$$

because $\sqrt{21} > 9/2$. Assign each boundary point to one of its incident cells. The strict bounds remain valid, proving the seven-coloring.

For $D = \{d_1, \dots, d_r\}$, take the product of the seven-colorings obtained by scaling this construction by each d_i. If two points have distance d_i, their ith coordinates differ. The resulting coloring has at most 7^r colors. □

6.2. **Integer power lists and the limiting density.** Put $m_1 = m(\{1\})$. One has $m_1 > 0$, for example from sufficiently small squares on $2\mathbb{Z}^2$. Also

$$m_1 \le \frac{1}{2}. \tag{59}$$

Indeed, a unit-distance-avoiding set A and its translate by a unit vector are disjoint. Their areas in $[-R, R]^2$ differ by $O(R)$, since a fixed translation changes the square by a region of area $O(R)$. Dividing the disjointness inequality by $4R^2$ and taking the upper limit proves (59).

Bukh's separated-distance product theorem, applied to the fixed list of r singleton distance sets, gives

$$m(D_{r,q}) \longrightarrow m_1^r \quad \text{as } q \to \infty, \qquad D_{r,q} = \{1, q, q^2, \dots, q^{r-1}\}. \tag{60}$$

This is [4, Theorem 1]. The parameter r is fixed when the limit is taken. The limit remains valid along integers $q \ge 2$, when $D_{r,q}$ has exactly r distinct positive algebraic distances. For $r = 1$, the statement is the identity $m(D_{1,q}) = m_1$.

Let

$$n_r = \lceil m_1^{-r} \rceil.$$

By (60), for every sufficiently large integer q,

$$\frac{1}{m(D_{r,q})} > n_r - 1. \tag{61}$$

This also holds when m_1^{-r} is an integer, because $n_r - 1$ is strictly smaller than that limit. If $G(E, D_{r,q})$ admitted a proper $(n_r - 1)$-coloring, Theorem 5.1 would give $m(D_{r,q}) \ge 1/(n_r - 1)$, contradicting (61). Here $n_r \ge 2$ by (59). Finite-color compactness therefore supplies a finite $P \subset E$ with

$$\chi(G(P, D_{r,q})) \ge n_r \ge m_1^{-r}. \tag{62}$$

Equivalently, apply the finite intersection property to the proper-labeling constraints on the countable vertex set E. If every finite induced graph were $(n_r - 1)$-colorable, those constraints would give such a coloring of all of E.

The elementary input (59) already implies

$$L(r) \ge 2^r. \tag{63}$$

Thus this exponential lower bound requires no numerical optimization of the unit-distance density.

6.3. **The numerical bound and Erdős 706.** The theorem of Ambrus, Csiszárik, Matolcsi, Varga and Zsámboki gives

$$m_1 \le 0.2470 = \frac{247}{1000} \tag{64}$$

for measurable planar sets avoiding the unit distance [1]. This is the published density bound used for the numerical conclusion below. The exact comparison

$$\frac{1000}{247} - 4.0485 = \frac{1000}{247} - \frac{8097}{2000} = \frac{41}{494000} > 0$$

makes the strict numerical inequality independent of decimal rounding.

Theorem 6.2 (Galán, resolving Erdős 706)**.** *For every integer $r \ge 1$, there are an integer $q \ge 2$ and a finite set of algebraic planar points $P \subset E$ such that*

$$\chi\big(G(P, \{1, q, \dots, q^{r-1}\})\big) \ge \lceil m_1^{-r} \rceil > 4.0485^r.$$

Consequently

$$4.0485^r < L(r) \le 7^r \qquad (r \ge 1). \tag{65}$$

Using only the elementary density bound $m_1 \le 1/2$, one obtains the numerical-free bound $2^r \le L(r) \le 7^r$. In particular, for every $A > 0$ and every $c \ge 0$, all sufficiently large r admit a finite distance graph with r forbidden distances and chromatic number greater than Ar^c.

Proof. Equation (62) gives the finite graph, with integer power distances. By (64),

$$m_1^{-r} \ge (1000/247)^r > 4.0485^r.$$

Lemma 6.1 gives the upper bound in (65). The numerical-free statement follows from (63). Finally, $2^r/(Ar^c) \to \infty$, since $r \log 2 - c \log r - \log A \to \infty$. This proves the stated failure of every polynomial upper bound. □

Part 2. Uniform cluster bounds and quantitative amplification

7. Uniform refinements and their predecessors

We retain the independence density $m(D)$ defined in Section 1. Dilation gives $m(tD) = m(D)$ for every $t > 0$.

Density estimates for sets with forbidden distances have a long history. Early work includes Székely [20] and Székely and Wormald [21]. Bukh proved a separated-scale product limit for a fixed finite family of finite distance sets, and more generally for a fixed family of supersaturable properties [4, Theorems 1 and 8]. The corresponding translation-product lower bound appears in [4, Lemma 6]. Castro-Silva obtained product limits for fixed families of admissible configurations, together with continuity results for their independence densities [5, Theorems 4 and 5].

Our first result allows both the length of the family and all of its cluster shapes to vary after a single separation ratio has been chosen. Fix $M \ge 1$, and let $\mathcal{K}_M$ be the Hausdorff metric space of nonempty compact subsets of $[1, M]$. In particular, cluster cardinalities and internal spacings are unrestricted.

Theorem 7.1. *Let $M \ge 1$ and $0 < \varepsilon < 1/2$. There is $Q = Q(M, \varepsilon) > M$ such that, for every integer $r \ge 1$, every $K_1, \dots, K_r \in \mathcal{K}_M$, and every sequence of positive scales satisfying*

$$\frac{t_{i+1}}{t_i} \ge Q \qquad (1 \le i < r),$$

one has

$$\prod_{i=1}^{r} m(K_i) \;\le\; m\left(\bigcup_{i=1}^{r} t_i K_i\right) \;\le\; \frac{1-\kappa}{\displaystyle\prod_{i=1}^{r}(m(K_i)+\varepsilon)^{-1} - \kappa} \;<\; \prod_{i=1}^{r}(m(K_i)+\varepsilon), \qquad \kappa = \frac{\varepsilon}{4}. \tag{66}$$

All the products in this theorem are positive. Indeed, the elementary uniform estimate

$$a_M := \frac{1}{16(M+1)^2} \le m(K) \le \frac{1}{2} \qquad (K \in \mathcal{K}_M) \tag{67}$$

will be proved below. For $Q > M$, define

$$\mathcal{R}_M(Q) = \sup\left\{ \left(\frac{m(\bigcup_{i=1}^r t_i K_i)}{\prod_{i=1}^r m(K_i)} \right)^{1/r} : \begin{array}{l} r \ge 1,\ K_i \in \mathcal{K}_M,\ t_i > 0, \\ t_{i+1}/t_i \ge Q\ (1 \le i < r) \end{array} \right\}.$$

Corollary 7.2. *For every $M \geq 1$,*

$$\lim_{Q\to\infty} \mathcal{R}_M(Q) = 1.$$

Equivalently, the supremum of

$$\frac{1}{r}\log\frac{m(\bigcup_{i=1}^r t_i K_i)}{\prod_{i=1}^r m(K_i)}$$

over the same families tends to zero as Q tends to infinity.

The separation in Theorem 7.1 comes from a finite cover of $\mathcal{K}_M$ by certificate neighborhoods. We give an explicit expression in the resulting finite certificate data. A second construction gives numerical separations directly from a finite weighted graph. For a finite graph $G = (V, E)$ and weights $p_i \geq 0$ satisfying $\sum_{i\in V} p_i = 1$, write

$$\alpha(G, p) = \max\left\{\sum_{i\in I} p_i : I \subseteq V \text{ is independent in } G\right\}.$$

A unit-distance realization of G consists of distinct points $u_i \in \mathbb{R}^2$ such that $|u_i - u_j| = 1$ whenever $\{i, j\} \in E$.

Theorem 7.3. *Suppose that a finite graph G has a unit-distance realization and weights $p_i \geq 0$ with $\sum_i p_i = 1$ and $\alpha(G, p) \leq a < 1$. For every $1 < b < 1/a$, there are an integer $Q > 1$ and $0 < \kappa \leq 1/2$ such that, for every $r \geq 1$ and every positive distance list $d_1 < \cdots < d_r$ with $d_{i+1}/d_i \geq Q$,*

$$m(\{d_1, \ldots, d_r\}) \leq \frac{1-\kappa}{b^r - \kappa} < b^{-r}. \tag{68}$$

The constants are given by finite expressions in the realization, the weights, a, and b. For a fixed such weighted realization, they can be chosen so that

$$Q = O\big((1-ab)^{-5/2}\big) \qquad \text{as } b \uparrow 1/a. \tag{69}$$

The proof uses the weighted Motzkin–Straus inequality. This inequality is established prior work, and its precise weighted form is given by Laurent and Vargas [17, Section 4], with the underlying characterization attributed there to Gibbons, Hearn, Pardalos and Ramana [15]. We include its short mass-moving proof to fix the normalization. We then smooth its copositive kernel and estimate the smoothing error and Fourier tail. For the graph of Bellitto, Pêcher and Sédillot [3], these estimates yield, for example,

$$\frac{d_{i+1}}{d_i} \geq 10^{34} \quad \Longrightarrow \quad m(\{d_1, \ldots, d_r\}) \leq \frac{1-10^{-7}}{3.89911^r - 10^{-7}}.$$

Four explicit choices are recorded in Section 13.

Our starting point for general compact clusters is the exact completely positive formulation of independence density due to DeCorte, de Oliveira Filho and Vallentin [10, Theorem 6.3 and Lemma 10.1]. Their Section 10 develops finite thick constraints and separated-distance estimates. The finite-distance extraction argument is also recorded by Davies [7, Appendix A, Lemma A.1], following that compactness and duality framework. Geometrically weighted Fourier certificates already appear in de Oliveira Filho and Vallentin [18, Section 5]. Here finite certificate extraction is followed by a Hausdorff-compactness argument and a variable-weight estimate whose margin is independent of the number of clusters. The effective graph construction uses a quadratic smoothing error, a decay coefficient of order $s^{-1/2}$, and a fixed initial frequency threshold.

8. Density, radial spectra, and finite constraints

8.1. **Periodic approximation and products.**

Lemma 8.1. *If $D \subset (0,\infty)$ is bounded, $m(D)$ is the supremum of the densities of periodic measurable sets avoiding D. The same supremum is obtained if the upper-density definition is also allowed to optimize the centers of the averaging squares.*

Proof. Choose $R > \sup D$. For a D-avoiding set A, take restrictions $B_L = A \cap [-L, L]^2$ along a sequence realizing its upper density, and form

$$A_L = B_L + (2L + R)\mathbb{Z}^2.$$

Points in different translates of B_L are at distance at least R. Thus A_L avoids D, and its density is

$$\frac{|B_L|}{(2L+R)^2},$$

which approaches $\overline{\delta}(A)$ along that sequence. If centers are optimized as well, translate each witnessing restriction before periodizing it. Periodic sets have the same density under either averaging convention, proving the last assertion. □

Lemma 8.2. *For bounded sets $D_1, \dots, D_r \subset (0,\infty)$,*

$$m\left(\bigcup_{i=1}^{r} D_i\right) \geq \prod_{i=1}^{r} m(D_i).$$

Proof. Choose periodic sets A_i avoiding D_i, with densities δ_i arbitrarily close to $m(D_i)$. Let X_i be independent uniform translations in fundamental domains for their respective periods. For every fixed square S, Fubini's theorem gives

$$\mathbb{E}\frac{|S \cap \bigcap_i (A_i + X_i)|}{|S|} = \prod_i \delta_i.$$

The random variables on the left lie in $[0,1]$. Reverse Fatou along the squares $[-n,n]^2$ gives a choice of translations for which the upper density of the intersection is at least $\prod_i \delta_i$. This intersection avoids every D_i. Letting the δ_i approach their suprema proves the claim. This argument also accommodates incommensurate period lattices. □

To prove (67), place copies of $[0,1/4]^2$ on the lattice $(M+1)\mathbb{Z}^2$. The diameter of each copy is below one, and points in different copies are at distance at least $M + 3/4$. The resulting density is a_M. For the upper bound, choose $d \in K$. If A avoids K, then A and $A + de_1$ are disjoint. Their areas in a square of side $2L$ differ by $O(dL + d^2)$, so averaging and taking the upper limit gives $2\overline{\delta}(A) \leq 1$.

8.2. **The radial spectral measure.**

We use the Fourier phase $e^{-i\xi\cdot x}$. Let $\Lambda \subset \mathbb{R}^2$ be a full-rank lattice, and equip the torus $\mathbb{R}^2/\Lambda$ with probability Haar measure. Its dual frequencies are

$$\Lambda^* = \{\xi \in \mathbb{R}^2 : \xi \cdot \lambda \in 2\pi\mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

For real $f \in L^2(\mathbb{R}^2/\Lambda)$, set

$$u = \int f, \qquad v = \int f^2, \qquad C_f(x) = \int f(y) f(y+x)\, dy.$$

Parseval's identity and L^2 continuity of translation give

$$C_f(x) = \sum_{\xi \in \Lambda^*} |\widehat{f}(\xi)|^2 e^{i\xi\cdot x}.$$

The series is absolutely and uniformly convergent. Consequently its rotational mean has the representation

$$p_f(d) := \frac{1}{2\pi}\int_0^{2\pi} C_f(d\omega_\theta)\,d\theta = \int_0^\infty J_0(ds)\,d\mu_f(s), \qquad \mu_f = \sum_{\xi\in\Lambda^*} |\widehat{f}(\xi)|^2\delta_{|\xi|}, \tag{70}$$

where $\omega_\theta = (\cos\theta, \sin\theta)$. The measure μ_f has total mass v and zero atom u^2. If $f = 1_A$ for a periodic set of positive density δ, then

$$\nu_A := \delta^{-1}\mu_{1_A} \quad \text{is a probability measure with} \quad \nu_A(\{0\}) = \delta. \tag{71}$$

If A avoids d, then

$$\int J_0(ds)\,d\nu_A(s) = 0. \tag{72}$$

For $f \geq 0$, one also has $p_f(d) \geq 0$ for every d.

8.3. **Bessel estimates and thick copositive kernels.** We will use the following elementary bounds throughout.

Lemma 8.3. *For $t > 0$,*

$$|J_0(t)| \leq \min\left\{1, \frac{4}{5\sqrt{t}}\right\}, \qquad |J_1(t)| \leq 1, \qquad 0 \leq 1 - J_0(t) \leq \frac{t^2}{4}. \tag{73}$$

Proof. The integral representations of J_0 and J_1 give their bounds by one. The representation $J_0(t) = \mathbb{E}\cos(t\cos\Theta)$ for uniform Θ also gives $1 - J_0(t) \leq t^2\mathbb{E}\cos^2\Theta/2 = t^2/4$. For the decay estimate, put $w(t) = \sqrt{t}J_0(t)$ and $q(t) = 1{+}1/(4t^2)$. Bessel's differential equation gives $w'' + qw = 0$. Thus

$$S(t) = w(t)^2 + \frac{w'(t)^2}{q(t)}, \qquad S'(t) = -\frac{q'(t)w'(t)^2}{q(t)^2} \geq 0.$$

The usual large-argument expansions of J_0 and its derivative yield $S(t) \to 2/\pi$ as $t \to \infty$. It follows that $tJ_0(t)^2 \leq 2/\pi < 16/25$. The differential equation, integral representations, and asymptotic expansions used here are recorded in [9, Sections 10.2, 10.9 and 10.17]. □

A real symmetric matrix $Z = (Z_{ij})$ is copositive if $x^{\mathsf{T}}Zx \geq 0$ for every vector x with nonnegative entries. Given a nonempty finite set $U \subset \mathbb{R}^2$, a copositive matrix Z indexed by U, and $\rho > 0$ small enough that the balls $B(u,\rho)$ are disjoint, define

$$r(s) = \sum_{u,v\in U} Z_{uv}\int_{B(u,\rho)}\int_{B(v,\rho)} J_0(s|x-y|)\,dy\,dx. \tag{74}$$

We call these functions thick copositive kernels. For a singleton U, any $\rho > 0$ is allowed. They satisfy

$$r(s) = \left|\widehat{1_{B(0,\rho)}}(se_1)\right|^2 \sum_{u,v\in U} Z_{uv}J_0(s|u-v|), \qquad \widehat{1_{B(0,\rho)}}(se_1) = \frac{2\pi\rho J_1(\rho s)}{s}. \tag{75}$$

The value at $s = 0$ is understood by continuity. Rotational averaging proves the factorization. In particular r is continuous, tends to zero at infinity, and

$$r(0) \geq 0, \qquad |r(s)| \leq D_r s^{-2}\ (s > 0), \qquad |r(s) - r(0)| \leq C_r s^2, \tag{76}$$

where one may take

$$D_r = 4\pi^2\rho^2 \sum_{u,v} |Z_{uv}|,$$

$$C_r = \frac{\pi^2\rho^4}{4} \sum_{u,v} |Z_{uv}|(|u-v|+2\rho)^2.$$

The first inequality uses copositivity on the all-ones vector. The other two follow from (75) and Lemma 8.3.

For $f \geq 0$ on a torus and any $t > 0$, copositivity also gives

$$\int r(ts)\, d\mu_f(s) \geq 0. \tag{77}$$

Indeed, for each translation z and angle θ, insert the nonnegative vector

$$\left(\int_{B(u,\rho)} f(z + tR_\theta x)\, dx \right)_{u \in U}$$

into Z, then integrate over z and θ. Fubini and (70) identify the result with the left side of (77). The calculation is justified by Cauchy–Schwarz and the finite measures of the balls.

8.4. **The completely positive input.** A bounded continuous real function F on $\mathbb{R}^2$ is completely positive if, for every finite $U \subset \mathbb{R}^2$, the matrix $(F(u-v))_{u,v\in U}$ is completely positive. A finite matrix is completely positive when it is a sum of matrices xx^{T} with $x \geq 0$. Such an F is positive definite. For radial F with $F(0) = 1$, the radial Bochner representation is

$$F(x) = \int_0^\infty J_0(s|x|)\, d\sigma(s)$$

for a probability measure σ. Its mean over expanding squares is $\sigma(\{0\})$. This last statement follows by first writing J_0 as a circle average of characters and then applying dominated convergence.

We use two results of DeCorte, de Oliveira Filho and Vallentin. For closed $K \subset (0,\infty)$, $m(K)$ equals the supremal mean of continuous completely positive functions F with $F(0) = 1$ and $F(x) = 0$ whenever $|x| \in K$. Rotation averaging allows F to be radial. Moreover, a bounded continuous real F is completely positive exactly when every thick copositive test obtained from (74) is nonnegative with $J_0(s|x-y|)$ replaced by $F(x-y)$. These are [10, Theorem 6.3 and Lemma 10.1]. Lemma 8.1 identifies the density conventions in that characterization with ours.

9. Finite certificates and uniformity in the cluster

9.1. **A finite certificate for a compact cluster.** The finite-distance form of the next lemma is stated in [7, Appendix A, Lemma A.1]. We give a compact-set version of the weak-* compactness and separation argument, retaining escaped spectral mass and the upper-semicontinuous zero-atom objective explicitly.

Lemma 9.1. *Let $K \in \mathcal{K}_M$ and $m(K) < \lambda < 1$. There are finitely many $a_j \in K$, real coefficients z_j, thick copositive kernels r_k, and coefficients $y_k \leq 0$ such that*

$$g(s) = \sum_{j=1}^{n} z_j J_0(a_j s) + \sum_{k=1}^{N} y_k r_k(s) \tag{78}$$

satisfies

$$g(0) \geq 1 - \lambda > 0, \qquad g(s) \geq -\lambda \quad (s \geq 0). \tag{79}$$

Proof. Let $X = [0, \infty]$ be the one-point compactification of the frequency half-line. Every $J_0(as)$ with $a \in K$, and every thick kernel, extends continuously to X with value zero at infinity. Consider probability measures $\mu \in \mathcal{P}(X)$ satisfying

$$\int J_0(as)\, d\mu(s) = 0 \quad (a \in K), \qquad \int r(s)\, d\mu(s) \geq 0 \quad \text{for every thick kernel } r. \tag{80}$$

The maximum of $\mu(\{0\})$ over these measures is $m(K)$. To prove the upper bound, put $\gamma = \mu([0, \infty))$. If $\gamma > 0$, the function

$$F(x) = \gamma^{-1} \int_{[0,\infty)} J_0(s|x|)\, d\mu(s)$$

is continuous, has $F(0) = 1$, and vanishes at distances in K. The thick tests in (80) make it completely positive by the cited characterization. Its mean is $\mu(\{0\})/\gamma$, so

$$\mu(\{0\}) \leq \gamma m(K) \leq m(K).$$

If $\gamma = 0$, the atom at zero vanishes. For the reverse bound, the measures (71) of periodic K-avoiding sets satisfy every constraint and have zero atoms approaching $m(K)$, by Lemma 8.1.

The space $\mathcal{P}(X)$ is weakly compact. Each moment constraint in (80) is closed. The function $\mu \mapsto \mu(\{0\})$ is upper semicontinuous, so its superlevel set at λ is also closed. Since the intersection of all these closed sets is empty, compactness gives finitely many distance constraints and finitely many thick constraints for which no feasible measure has zero atom at least λ.

For these selected constraints, let $\mathcal{C}$ consist of the triples (A, B, c) for which some $\mu \in \mathcal{P}(X)$ satisfies

$$A_j = \int J_0(a_j s)\, d\mu(s), \qquad B_k \leq \int r_k(s)\, d\mu(s), \qquad c \leq \mu(\{0\}).$$

This is a closed convex subset of a finite-dimensional real vector space. For closedness, take a convergent sequence of triples and a weakly convergent subsequence of witnessing measures. Continuity of the moments passes the first two conditions to the limit. For the final condition, upper semicontinuity gives

$$c \leq \limsup_n \mu_n(\{0\}) \leq \mu(\{0\}).$$

The point $(0, 0, \lambda)$ lies outside $\mathcal{C}$. Strict separation therefore gives (α, β, τ) with

$$\tau\lambda > \sup_{(A,B,c)\in\mathcal{C}} (\alpha \cdot A + \beta \cdot B + \tau c). \tag{81}$$

Because $\mathcal{C}$ is downward closed in the B and c coordinates, $\beta_k \geq 0$ and $\tau \geq 0$. The measure δ_∞ places $(0, 0, 0)$ in $\mathcal{C}$. Thus (81) forces $\tau > 0$. Testing the measure δ_s in that inequality yields

$$\lambda - \sum_j \frac{\alpha_j}{\tau} J_0(a_j s) - \sum_k \frac{\beta_k}{\tau} r_k(s) > 1_{\{0\}}(s).$$

Taking $z_j = -\alpha_j/\tau$ and $y_k = -\beta_k/\tau$ proves the lemma. □

9.2. **Hausdorff perturbations.**

Lemma 9.2. *The function $K \mapsto m(K)$ is lower semicontinuous on $\mathcal{K}_M$.*

Proof. Fix $K \in \mathcal{K}_M$ and $\eta > 0$. Restrict a nearly optimal K-avoiding set to a sufficiently large square and then take a compact subset B losing sufficiently little area. Periodizing B with gaps wider than M gives density at least $m(K) - \eta$. The compact set

$$S_B = \{|x - y| : x, y \in B\}$$

is disjoint from K. Thus $\operatorname{dist}(S_B, K) > 0$. Every sufficiently Hausdorff-close $K' \in \mathcal{K}_M$ remains disjoint from S_B. The same periodic set then avoids K', since all cross-copy distances exceed M. Therefore $m(K') \geq m(K) - \eta$ throughout a neighborhood of K. □

Fix $0 < \varepsilon < 1/2$. At each $K \in \mathcal{K}_M$, choose Lemma 9.1 with

$$\lambda_K = m(K) + \frac{\varepsilon}{8},$$

and write g_K for the certificate and $G_K = g_K(0)$. If K' is close to K, choose $a'_j \in K'$ close to each certificate distance $a_j \in K$ and define

$$g_{K,K'}(s) = \sum_j z_j J_0(a'_j s) + \sum_k y_k r_k(s).$$

Its value at zero remains G_K. The map $a \mapsto J_0(a\,\cdot)$ from $[1, M]$ to $C_0([0, \infty))$ is continuous in supremum norm. Indeed, Lemma 8.3 bounds the tail uniformly, and uniform continuity applies on each remaining compact rectangle. After shrinking the neighborhood, this fact and Lemma 9.2 give

$$\|g_{K,K'} - g_K\|_\infty \leq \frac{\varepsilon}{8}, \qquad m(K') \geq m(K) - \frac{\varepsilon}{4}. \tag{82}$$

Set

$$h_{K,K'} = G_K^{-1} g_{K,K'}, \qquad \beta_K = \frac{m(K) + \varepsilon/4}{G_K}, \qquad b(K') = (m(K') + \varepsilon)^{-1}.$$

Then

$$h_{K,K'}(0) = 1, \qquad h_{K,K'} \geq -\beta_K, \qquad (b(K') - 1)\beta_K \leq 1 - \Delta, \qquad \Delta = \frac{\varepsilon}{2}. \tag{83}$$

To verify the last inequality, write $A = m(K)$ and $e = \varepsilon$. Using (82) and $G_K \geq 1 - A - e/8$, the product on its left is at most

$$\frac{(1 - A - 3e/4)(A + e/4)}{(A + 3e/4)(1 - A - e/8)}.$$

All factors are positive. The denominator minus the numerator equals

$$\frac{e}{2} + \frac{Ae}{8} + \frac{3e^2}{32} \geq \frac{e}{2},$$

and the denominator is at most one. This proves (83).

The Hausdorff space $\mathcal{K}_M$ is compact. Choose a finite subcover of the preceding neighborhoods, indexed by ℓ. Write its unperturbed certificates as

$$g_\ell(s) = \sum_j z_{\ell j} J_0(a_{\ell j} s) + \sum_k y_{\ell k} r_{\ell k}(s), \qquad G_\ell = g_\ell(0), \qquad S_\ell = \sum_j |z_{\ell j}|.$$

Only the finitely many distance locations are perturbed when a certificate is assigned to a member of its neighborhood. Define

$$B = \max_{\ell} \frac{(4/5)S_\ell + \sum_k |y_{\ell k}| D_{r_{\ell k}}}{G_\ell}, \tag{84}$$

$$C = \max_{\ell} \frac{(M^2/4)S_\ell + \sum_k |y_{\ell k}| C_{r_{\ell k}}}{G_\ell}, \qquad \eta = \min\left\{1, \sqrt{\frac{\varepsilon}{8C}}\right\}. \tag{85}$$

Here $C > 0$. In fact $y_{\ell k} \le 0$ and $r_{\ell k}(0) \ge 0$ imply

$$\sum_j z_{\ell j} = G_\ell - \sum_k y_{\ell k} r_{\ell k}(0) \ge G_\ell > 0. \tag{86}$$

Every assigned normalized certificate h satisfies

$$|h(s)| \le B s^{-1/2} \quad (s \ge 1), \qquad h(s) \ge 1 - Cs^2 \ge 1 - \Delta/4 \quad (0 \le s \le \eta). \tag{87}$$

These estimates follow from (76) and Lemma 8.3. All constants have been chosen before any family length or cluster list.

10. Variable weights and the product estimate

Lemma 10.1. *Suppose $h_1, \dots, h_r$ satisfy (87), with $h_i(0) = 1$ and $h_i \ge -\beta_i$. Suppose also that*

$$1 < b_i \le B_0, \qquad (b_i - 1)\beta_i \le 1 - \Delta, \qquad 0 < \Delta < 1.$$

Put $P_0 = 1$, $P_i = \prod_{j=1}^{i} b_j$, and $w_i = (b_i - 1)P_{i-1}$. If

$$Q\eta \ge 1, \qquad \sqrt{Q} \ge B_0 + \frac{4(B_0 - 1)BB_0}{\Delta\sqrt{\eta}}, \tag{88}$$

then every positive scale list with $t_{i+1}/t_i \ge Q$ satisfies

$$H(s) := 1 + \sum_{i=1}^{r} w_i h_i(t_i s) \ge \Delta/2 \quad (s \ge 0), \qquad H(0) = P_r. \tag{89}$$

Proof. The identity $w_i = P_i - P_{i-1}$ gives $1 + \sum_{i<j} w_i = P_{j-1}$ and $H(0) = P_r$. For a fixed s, suppose first that no $t_i s$ reaches η. All $h_i(t_i s)$ are positive, so $H(s) \ge 1$. Otherwise let j be the first index with $t_j s \ge \eta$. The constant and earlier summands contribute at least $(1 - \Delta/4)P_{j-1}$. The jth summand contributes at least $-(1 - \Delta)P_{j-1}$. For $k \ge 1$,

$$w_{j+k} \le (B_0 - 1)B_0^k P_{j-1}, \qquad t_{j+k} s \ge Q^k \eta \ge 1.$$

Hence the total possible loss from the later summands is at most

$$(B_0 - 1)BP_{j-1}\eta^{-1/2} \sum_{k \ge 1} (B_0 Q^{-1/2})^k \le \frac{\Delta}{4} P_{j-1}$$

by (88). Combining the three bounds gives $H(s) \ge (\Delta/2)P_{j-1} \ge \Delta/2$. The same estimates cover an empty earlier or later sum. □

Proof of Theorem 7.1. Use the finite cover and constants from Section 9, and put

$$B_0 = (a_M + \varepsilon)^{-1}, \qquad Q = \max\left\{M + 1, \eta^{-1}, \left(B_0 + \frac{8(B_0 - 1)BB_0}{\varepsilon\sqrt{\eta}}\right)^2\right\}.$$

This choice satisfies (88) with $\Delta = \varepsilon/2$. For each K_i, select a neighborhood containing it and the corresponding perturbed normalized certificate h_i. Set $b_i = (m(K_i) + \varepsilon)^{-1}$. Equations (67) and (83) give all the hypotheses of Lemma 10.1.

Let A be a periodic set avoiding $D = \bigcup_i t_i K_i$, of positive density δ. The selected distance locations of h_i belong to K_i. Equations (72) and (77), together with the nonpositive thick coefficients, therefore give

$$\int h_i(t_i s)\, d\nu_A(s) \le 0 \qquad (1 \le i \le r).$$

With $\kappa = \Delta/2 = \varepsilon/4$, integration of (89) now gives

$$1 \ge \int H\, d\nu_A \ge P_r \delta + \kappa(1-\delta).$$

Thus $\delta \le (1-\kappa)/(P_r - \kappa)$. The set D is bounded, so Lemma 8.1 proves the same estimate for $m(D)$. Since $P_r > 1$ and $\kappa > 0$, it is strictly smaller than P_r^{-1}. The lower bound follows from Lemma 8.2 and scale invariance. □

Proof of Corollary 7.2. Theorem 7.1 and (67) imply

$$1 \le \mathcal{R}_M(Q(M,\varepsilon)) \le 1 + \frac{\varepsilon}{a_M}.$$

The function $\mathcal{R}_M$ is nonincreasing. Letting ε decrease to zero gives the result. □

10.1. **A pair-density consequence.** The same finite cover also yields a uniform stability estimate for small pair densities. The torus and pair-density notation is that of (70).

Corollary 10.2. *For fixed M and ε as in Theorem 7.1, there is $L_0 \ge 1$ such that the following holds with the same Q and κ. Let r, K_i, and t_i satisfy the hypotheses of that theorem, and let $f \ge 0$ be nonzero in L^2 of any flat planar torus. Write*

$$u = \int f, \qquad v = \int f^2, \qquad q_i = \sup_{d \in K_i} p_f(t_i d),$$

$$b_i = (m(K_i) + \varepsilon)^{-1}, \qquad P_0 = 1, \qquad P_i = \prod_{j=1}^{i} b_j, \qquad w_i = (b_i - 1)P_{i-1}.$$

Then

$$L_0 \sum_{i=1}^{r} w_i q_i \ge (P_r - \kappa)u^2 - (1-\kappa)v. \tag{90}$$

In particular, if $0 \le \tau < 1/L_0$ and $q_i \le \tau u^2$ for every i, then

$$\frac{u^2}{v} \le \frac{1-\kappa}{(1 - L_0\tau)P_r + L_0\tau - \kappa}.$$

Proof. For the finite cover in Section 9, set

$$L_0 = \max_{\ell} G_\ell^{-1} \sum_j \max\{z_{\ell j}, 0\}.$$

Equation (86) gives $L_0 \ge 1$. By (77) and $p_f \ge 0$, each selected certificate satisfies

$$\int h_i(t_i s)\, d\mu_f(s) \le G_{\ell(i)}^{-1} \sum_j z_{\ell(i)j} p_f(t_i a'_{ij}) \le L_0 q_i.$$

The pointwise lower bound for H, the mass v of μ_f, and its zero atom u^2 give

$$(P_r - \kappa)u^2 + \kappa v \le \int H\, d\mu_f \le v + L_0 \sum_i w_i q_i.$$

This proves (90). The final estimate uses $\sum_i w_i = P_r - 1$ and rearranges. □

11. Effective certificates from weighted graphs

Fix a finite unit-distance realization $(u_i)_{i\in V}$ and weights as in Theorem 7.3. Throughout this section put

$$D_0 = \max_{i,j} |u_i - u_j|, \qquad z = a \sum_{\{i,j\}\in E} (p_i + p_j) = a \sum_i \deg(i) p_i. \tag{91}$$

The assumptions imply $0 < a < 1$ and $z > 0$. Indeed, if the support of p contained no edge, it would be an independent set of weight one.

Lemma 11.1 (Weighted Motzkin–Straus inequality)**.** *For every $x_i \geq 0$,*

$$a \left(\sum_i p_i x_i^2 + \sum_{\{i,j\}\in E} (p_i + p_j) x_i x_j \right) \geq \left(\sum_i p_i x_i \right)^2. \tag{92}$$

Consequently the matrix

$$Z_{ii} = a p_i - p_i^2, \qquad Z_{ij} = \frac{a}{2}(p_i + p_j) 1_{\{i,j\}\in E} - p_i p_j \quad (i \neq j) \tag{93}$$

is copositive.

Proof. Denote the quadratic expression in parentheses by $q(x)$. Set every zero-weight coordinate to zero, which preserves $\sum_i p_i x_i$ and can only decrease q. If adjacent positive-weight coordinates x_i, x_j are both positive, keep $S = p_i x_i + p_j x_j$ fixed and write $v = p_i x_i$. Their contribution to q is

$$\frac{v^2}{p_i} + \frac{(S-v)^2}{p_j} + \left(\frac{1}{p_i} + \frac{1}{p_j} \right) v(S - v) = \frac{Sv}{p_i} + \frac{S(S-v)}{p_j}.$$

It is affine for $0 \leq v \leq S$. All terms involving other coordinates are affine as well. An endpoint therefore does not increase q, and removes a positive coordinate. After finitely many such steps, the support is an independent set I. On that support, weighted Cauchy–Schwarz gives

$$\left(\sum_i p_i x_i \right)^2 \leq \left(\sum_{i\in I} p_i \right) \left(\sum_{i\in I} p_i x_i^2 \right) \leq a q(x).$$

This proves (92), and expansion proves the assertion about Z. □

Define

$$P(t) = \sum_{i,j} p_i p_j J_0(t|u_i - u_j|), \qquad R(t) = a + z J_0(t) - P(t). \tag{94}$$

Rotational averaging of $|\sum_i p_i e^{it\omega\cdot u_i}|^2$ gives

$$0 \leq P(t) \leq 1, \qquad P(0) = 1, \qquad P(t) \geq 1 - \frac{D_0^2 t^2}{4}. \tag{95}$$

For a smoothing radius $s > 0$, let

$$\phi_s(t) = \left(\frac{2J_1(st)}{st} \right)^2, \quad \phi_s(0) = 1, \qquad g_s(t) = z J_0(t) - \phi_s(t) R(t). \tag{96}$$

Equivalently,

$$g_s(t) = \phi_s(t)(P(t) - a) + (1 - \phi_s(t)) z J_0(t). \tag{97}$$

Lemma 11.2. *For the spectral probability measure of any periodic set avoiding $d > 0$,*

$$\int g_s(dt)\, d\nu_A(t) \le 0.$$

Proof. Smooth 1_A by convolution with uniform probability on the disk of radius sd. The result is nonnegative. Insert its values at $x + dR_\theta u_i$ into the copositive matrix (93), and integrate over x and θ. The disk Fourier multiplier is $2J_1(sd|\xi|)/(sd|\xi|)$. Its square is $\phi_s(d|\xi|)$, while the angular average of the matrix contribution is $R(d|\xi|)$ by (94). It follows that

$$\int \phi_s(dt) R(dt)\, d\nu_A(t) \ge 0.$$

All Fourier sums are absolutely convergent after the products are integrated, by Parseval and boundedness of the disk multiplier. The forbidden-distance moment (72) then proves the claim. □

Lemma 11.3. *Set*

$$K_0 = \frac{27(4z/5)^4}{1024a^3}, \qquad E_s = K_0 s^2, \tag{98}$$

$$\beta_s = \frac{a + E_s}{1 - a}, \qquad B_s = \frac{(4/5)z + (1 + a + z)\sqrt{2/s}}{1 - a}, \tag{99}$$

$$C_s = \frac{D_0^2 + (1 - a + z)s^2}{4(1 - a)}. \tag{100}$$

The normalized function $h_s = g_s/(1 - a)$ satisfies

$$h_s(0) = 1, \qquad h_s(t) \ge -\beta_s, \qquad |h_s(t)| \le B_s t^{-1/2}\ (t > 0), \qquad h_s(t) \ge 1 - C_s t^2. \tag{101}$$

Proof. The Fourier transform of uniform probability on the radius-s disk gives

$$0 \le \phi_s(t) \le 1, \qquad 1 - \phi_s(t) \le \min\{1, s^2t^2/4\}, \qquad \phi_s(t) \le \min\{1, 4/(s^2t^2)\}. \tag{102}$$

For the middle inequality, take independent uniform disk points X, Y. Then $\phi_s(t) = \mathbb{E}\cos(t(X_1 - Y_1))$, and $\mathbb{E}(X_1 - Y_1)^2 = s^2/2$. The last bound follows from $|J_1| \le 1$.

By (97), $P \ge 0$, and Lemma 8.3,

$$g_s(t) \ge -a + (1 - \phi_s(t))\left(a - \frac{4z}{5\sqrt{t}}\right).$$

The possible deficit below $-a$ is therefore bounded by

$$\frac{s^2}{4} \sup_{t>0} t^2 \left(\frac{4z}{5\sqrt{t}} - a\right)_+.$$

Writing $y = \sqrt{t}$ and $K = 4z/5$, the polynomial to maximize is $Ky^3 - ay^4$. Its positive maximum is $27K^4/(256a^3)$, attained at $y = 3K/(4a)$. This proves $g_s \ge -a - E_s$. Also $g_s(0) = 1 - a$ by (94) and (96).

Since $|R| \le 1 + a + z$, the decay estimate follows from

$$|g_s(t)|\sqrt{t} \le \frac{4}{5}z + (1 + a + z)\sqrt{t}\min\{1, 4/(s^2t^2)\} \le \frac{4}{5}z + (1 + a + z)\sqrt{2/s}.$$

The maximum in the last step occurs at $t = 2/s$. Finally, the identity

$$g_s(t) - (1 - a) = \phi_s(t)(P(t) - 1) + (1 - \phi_s(t))(zJ_0(t) - (1 - a))$$

and (95) give

$$g_s(t) - (1 - a) \ge -\frac{D_0^2 t^2}{4} - \frac{(1 - a + z)s^2t^2}{4}.$$

Dividing by $1-a$ completes the proof. □

12. A two-range estimate and the separation rate

The following estimate uses a fixed initial frequency threshold. This is useful when the lower bound for a certificate approaches its limiting value.

Lemma 12.1. *Suppose*

$$h(0)=1,\quad h(t)\ge -\beta,\quad h(t)\ge 1-Ct^2,\quad |h(t)|\le Bt^{-1/2}\quad (t>0),$$

where $B,C>0$. Fix $b>1$, put $c=b-1$ and $\Delta=1-c\beta>0$, and choose $u>0$ with $Cu^2\le 1/8$. Let $T=(4cB)^2\ge u$ and $Q>\max\{1,b^2\}$. Define

$$\mathcal{L}=\frac{cBb}{\sqrt{u}(\sqrt{Q}-b)},\qquad \mathcal{A}=\frac{cCT^2}{bQ^2-1}. \tag{103}$$

If

$$\mathcal{L}\le 1/8,\qquad \Delta-\mathcal{A}-\mathcal{L}\ge\kappa>0,\qquad \kappa\le 1/2, \tag{104}$$

then, for every $r\ge 1$ and every positive list with $d_{i+1}/d_i\ge Q$,

$$H_r(t)=1+c\sum_{i=1}^{r} b^{i-1}h(d_it)\ge\kappa\quad (t\ge 0),\qquad H_r(0)=b^r. \tag{105}$$

Proof. If all $d_it<u$, then every $h(d_it)\ge 7/8$, so $H_r(t)\ge 1$. Otherwise let j be the first index with $x=d_jt\ge u$. For $1\le k<j$,

$$d_{j-k}t\le xQ^{-k},\qquad d_{j-k}t<uQ^{1-k}.$$

After dividing $H_r(t)$ by b^{j-1}, the constant and the baseline values of earlier summands add to one, since

$$b^{1-j}+c\sum_{k=1}^{j-1}b^{-k}=1.$$

The total earlier deficit is at most

$$cC\min\{x^2,Q^2u^2\}\sum_{k\ge 1}(bQ^2)^{-k}=\frac{cC\min\{x^2,Q^2u^2\}}{bQ^2-1}.$$

The possible loss from later summands is at most

$$\frac{cB}{\sqrt{x}}\sum_{k\ge 1}(b/\sqrt{Q})^k=\frac{cBb}{\sqrt{x}(\sqrt{Q}-b)}.$$

Thus

$$\frac{H_r(t)}{b^{j-1}}\ge 1+ch(x)-\frac{cC\min\{x^2,Q^2u^2\}}{bQ^2-1}-\frac{cBb}{\sqrt{x}(\sqrt{Q}-b)}. \tag{106}$$

If $u\le x\le T$, this is at least $\Delta-\mathcal{A}-\mathcal{L}\ge\kappa$. If $x\ge T$, the first two terms are at least $3/4$. The earlier loss is at most

$$\frac{cCQ^2u^2}{bQ^2-1}\le\frac{cQ^2}{8(bQ^2-1)}<\frac{1}{8},$$

where the strict inequality follows from $Q^2>1$ and $b-c=1$. The later loss is at most $\mathcal{L}\le 1/8$. Hence the right side of (106) is at least $1/2\ge\kappa$. Multiplying by $b^{j-1}\ge 1$ proves the lower bound. At zero, the geometric sum gives $H_r(0)=b^r$. □

Proof of Theorem 7.3. Use h_s from Lemma 11.3 and write $c = b - 1$. Since $b < 1/a$, choose $s > 0$ small enough that

$$\Delta = 1 - c\beta_s = \frac{1 - ab - cK_0 s^2}{1 - a} > 0.$$

Choose $u > 0$ with $C_s u^2 \le 1/8$, and enlarge B_s if needed so that $(4cB_s)^2 \ge u$. As Q tends to infinity, both losses in (103) tend to zero. Thus an integer Q can be chosen to satisfy (104) with $\kappa = \min\{\Delta/2, 1/4\}$. For any periodic set avoiding $d_1, \dots, d_r$, Lemma 11.2 gives $\int h_s(d_i t)\, d\nu_A(t) \le 0$ for every i. Integrating (105) therefore yields

$$1 \ge \int H_r \, d\nu_A \ge b^r \delta + \kappa(1 - \delta).$$

Lemma 8.1 proves (68) for all measurable sets.

For the rate, let $\epsilon = 1 - ab$ decrease to zero and set

$$s = \sqrt{\frac{\epsilon}{2cK_0}}, \qquad C_0 = \frac{D_0^2 + 1 - a + z}{4(1 - a)}, \qquad u = (8C_0)^{-1/2}.$$

For sufficiently small ϵ, one has $s \le 1$, $C_s \le C_0$, and

$$\Delta = \frac{\epsilon}{2(1 - a)} \le \frac{1}{2}.$$

Take $B = \max\{B_s, \sqrt{u}/(4c)\}$ and $T = (4cB)^2$. Choose an integer Q strictly above

$$\max\left\{1, b^2, \sqrt{\frac{1 + 4cC_0T^2/\Delta}{b}}, \left(b + \frac{4cBb}{\sqrt{u}\,\Delta}\right)^2\right\}. \tag{107}$$

Then $\mathcal{A} \le \Delta/4$ and $\mathcal{L} \le \Delta/4 \le 1/8$. Lemma 12.1 applies with $\kappa = \Delta/2$. As $b \uparrow 1/a$, c stays bounded above and away from zero, and

$$s \asymp \epsilon^{1/2}, \qquad B = O(\epsilon^{-1/4}), \qquad T = O(\epsilon^{-1/2}).$$

The third entry of (107) is $O(\epsilon^{-1})$, and the fourth is $O(\epsilon^{-5/2})$. Taking the next integer preserves this order, proving (69). □

13. Explicit weighted-graph examples

Bellitto, Pêcher and Sédillot give a 607-vertex unit-distance graph and nonnegative weights whose maximum independent weight divided by total weight is

$$a = \frac{512933}{1999983}. \tag{108}$$

We use this published finite result from [3, Section 4.4]. Its component ratios average to the whole-graph ratio, with the component total weights as averaging weights. Consequently some positive-weight connected component has independence ratio at most a. Normalize its weights to sum to one. Any two vertices of this component are joined by a simple path of at most 606 unit edges. Thus

$$D_0 \le 606, \qquad z \le 606a =: z_*. \tag{109}$$

Zero-weight vertices may be kept when estimating path length and then omitted from the quadratic form.

For these data, Lemma 11.3 remains valid with the larger error

$$E_s = \frac{27}{2500} a 606^4 s^2$$

and the bounds

$$B_s \le \frac{(4/5)z_* + (1+a+z_*)\sqrt{2/s}}{1-a}, \qquad C_s \le \frac{606^2 + (1-a+z_*)s^2}{4(1-a)}.$$

The following table gives sufficient choices for (68). All entries are exact terminating decimals or integers.

b	s	B	Q	κ
3.89	10^{-6}	320000	10^{24}	10^{-3}
3.899	10^{-7}	1000000	10^{28}	10^{-5}
3.8991	10^{-8}	3200000	10^{32}	10^{-6}
3.89911	10^{-9}	10000000	10^{34}	10^{-7}

Here are rational bounds verifying every condition without a frequency discretization. In all four rows take $C = 125000$ and $u = 1/1000$, so $Cu^2 = 1/8$ and $\sqrt{u} > 3/100$. Upper bounds for $\sqrt{2/s}$ are, in row order,

$$1500, \quad 4500, \quad 15000, \quad 45000.$$

Substitution gives $B_s < B$ and $C_s < C$ in each row. Put

$$c = b - 1, \qquad \Delta = 1 - c\frac{a + E_s}{1-a}, \qquad T = (4cB)^2, \qquad \mathcal{L}^+ = \frac{100cBb}{3(\sqrt{Q} - b)}.$$

Then $\mathcal{L} < \mathcal{L}^+$. Direct rational substitution gives the strict bounds

b	lower bound for Δ	upper bound for $\mathcal{A}$	upper bound for $\mathcal{L}^+$
3.89	$169/10^5$	$2/10^{17}$	$12/10^5$
3.899	$239/10^7$	$2/10^{23}$	$38/10^7$
3.8991	$384/10^8$	$2/10^{29}$	$121/10^9$
3.89911	$543/10^9$	$2/10^{31}$	$38/10^9$

In the last two columns, $\mathcal{A} = cCT^2/(bQ^2 - 1)$ as in (103). Every displayed comparison follows by clearing positive denominators. They give $\Delta - \mathcal{A} - \mathcal{L} > \kappa$ and $\mathcal{L} < 1/8$ in every row. The remaining inequalities $T \ge u$ and $Q > \max\{1, b^2\}$ are immediate from the tabulated values. Lemma 12.1 and the integration argument prove each stated bound for every r.

14. Fixed separation for ordinary finite graphs

The transfer proved in Part I converts the density refinements into statements about arbitrary colorings whenever all forbidden distances are algebraic. The finite graphs in the next result follow by the same compactness step used by Galán. The additional assertion is the choice of a single separation ratio before the length r.

Corollary 14.1. *Let $m_1 = m(\{1\})$ and $1 < b < 1/m_1$. There is an integer $Q \ge 2$ such that for every integer $q \ge Q$ and every $r \ge 1$, there is a finite set $P \subset E$ satisfying*

$$\chi\big(G(P, \{1, q, \dots, q^{r-1}\})\big) > b^r.$$

If a fixed finite weighted unit-distance graph has independence ratio at most $a < 1$, then for $1 < b < 1/a$ one can take the effective Q of Theorem 7.3, with its $O((1-ab)^{-5/2})$ dependence. In particular, $b = 3.89911$ and $Q = 10^{34}$ are sufficient using the published Bellitto–Pêcher–Sédillot input.

Proof. Choose $0 < \varepsilon < 1/2$ so small that $m_1 + \varepsilon < 1/b$. Apply Theorem 7.1 with $M = 1$ and every $K_i = \{1\}$, and increase its separation to an integer $Q \ge 2$. For $q \ge Q$ and $D = \{1, q, \dots, q^{r-1}\}$, that theorem gives $m(D) < b^{-r}$. All distances in D are positive and algebraic, so Theorem 5.1 yields $\chi(\mathbb{R}^2, D) > b^r$ and also $\chi(G(E, D)) > b^r$. If every finite induced graph on E had a proper

$\lfloor b^r \rfloor$-coloring, finite-color compactness would give such a coloring of $G(E, D)$. A finite P with the required chromatic number therefore exists.

For the effective assertion, Theorem 7.3 gives

$$\frac{1}{m(D)} \geq \frac{b^r - \kappa}{1 - \kappa} > b^r,$$

since $0 < \kappa < 1$ and $b^r > 1$. Apply the same transfer and compactness argument. Section 13 gives the stated numerical pair. □

This corollary supplies finite witnesses by compactness. The ratio in its first assertion is existential. Its effective assertion gives an explicit separation bound from a supplied weighted realization and independence-ratio certificate.

Appendix A. Sources, versions, and verification scope

A.1. **Frozen primary sources.** The OpenAI source used for Part I is the September 23, 2026 manuscript *The Euclidean plane is not five-colorable*, frozen at Git commit

`4eaee87548250b2682affda7d05d947c05554bae`.

The spectral, rigidity, and transfer source files are recorded in the accompanying manifest [19]. The Galán source is version 8, October 11, 2026, frozen at Git commit

`cb660f322dce47a773f347d39818ffc46e6ca72c`.

Both are cited as manuscripts. The attribution concerns their mathematical arguments and is separate from prize evaluation.

The source package includes the applicable Apache 2.0 license for the OpenAI repository and the MIT license carrying © 2026 Sergi Galán for his repository. The frozen OpenAI Apache 2.0 license and Galán MIT license retain their terms for their respective upstream materials. This deposit contains the rewritten manuscript. The separate proof and supporting source bundle are available in the pinned repository. The exposition in Sections 2–6 rewrites and expands the arguments of those sources. These changes include the explicit conditional-measure conventions, auxiliary estimates, and a unified notation.

A.2. **Mathematical and numerical checks.** Independent AI-agent reviews checked the rigidity proof, the transfer, and the full finite-graph deduction, and a separate review checked the uniform compact-cluster and graph-amplification arguments. These reviews are supporting audit records. The mathematical evidence is the proofs and the identified published inputs. The separate formal verification described below establishes the negative answer to the original polynomial-bound question with an elementary exponential lower bound.

The base 4.0485 uses the published bound $m_1 \leq 247/1000$. The graph examples in Section 13 use the published 607-vertex weighted-graph result of Bellitto, Pêcher and Sédillot [3]. The underlying large optimization and independent-set computations were not rerun in this work. The displayed separation inequalities were checked in exact rational arithmetic. The proof of the negative answer with base 2 uses no such numerical optimization.

A.3. **Formal verification of the original negative answer.** A separate Lean development, frozen at commit

`331157628c3937f5fd992c1b85c06f146184c465`,

proves the negation of the eventual polynomial upper bound in the original problem. The principal declaration is `Erdos706.erdos_706_negative`. Its statement is also written out independently in `Erdos706.Audit.original_problem_negative`, with the finite point set, the r

positive distances, and the chromatic-number quantifiers explicit. The development establishes

$$2^r \le L(r) \le 7^r \qquad (r \ge 1).$$

It uses the OpenAI rigidity formalization, Galán's algebraic-distance formalization, and an elementary separated-scale spectral certificate. The ordinary-character spectral measure has an atom of mass at least $1/k$ at zero for a proper k-coloring. A positive certificate made from the circular Fourier kernel contradicts this atom bound at $k = 2^r$. Finite-subgraph compactness then gives the finite witnesses required by the original statement. The ratio in this formal construction may depend on r.

The verification used Lean 4.34.1 and Mathlib commit

`d13f23b723b8a846827a245b89c10fc7d3f11612`.

All 56 local modules were rebuilt into an empty output directory, and their source hashes were checked against the frozen proof commit. The axiom reports for the final declarations contain exactly `propext`, `Classical.choice`, and `Quot.sound`. A target-directed replay into an empty Lean kernel accepted all 73,146 transitive declarations for the negative answer, its explicit-statement bridge, the exponential lower bound, and the seven-color upper bound. The accompanying verification files record the commands, hashes, axiom reports, and replay output. A separate attempt to replay the entire imported environment reached its 30-minute resource limit before completion.

This formal result verifies the original negative answer through the base-two argument. The stronger numerical base 4.0485, the compact-cluster theorem, the quantitative amplification theorem, and the fixed-ratio refinements are supported here by their written proofs and cited inputs. Their formalization is outside the scope of the frozen Lean development. The formal source retains the OpenAI and Galán attribution and licenses, with provenance recorded for each imported module. Codex-assisted additions are identified separately in the source documentation.

A.4. **Remaining historical comparison.** The literature audit compared the relevant full arguments in [4, 18, 10, 5, 7, 14] and the weighted form in [17]. The source of the earlier weighted characterization is credited through the latter's attribution to [15]. The two historical originals [20, 21] remain to be compared in full. Accordingly, the paper asserts and proves the stated uniform refinements while leaving first-discovery priority for them open to that comparison.

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OpenAI's Codex was used for mathematical exploration, proof development, exact arithmetic checks, literature searches, independent-agent review, and drafting this manuscript. Attribution of OpenAI's pre-existing rigidity and transfer results is given separately throughout Part I. This acknowledgment records assistance in preparing the present text.

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