ERDŐS PROBLEM 859: A REFUTATION OF THE LOGARITHMIC DENSITY LAW

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ABSTRACT. For a positive integer t, let d_t be the natural density of positive integers n for which t is a sum of distinct positive divisors of n. Erdős proposed that d_t might be asymptotic to $c_1/(\log t)^{c_2}$ for positive constants c_1, c_2. The accepted Lean submission refutes this proposal. It proves both $\liminf_{t\to\infty}(-\log d_t)/\log\log t = \delta$ and $(\log t)^{\delta} d_t \to 0$, where $\delta = 1 - (1+\log\log 2)/\log 2$. These force a proposed exponent to be δ and its positive leading coefficient to be zero, a contradiction. We give the exact finite prime model, derive the density and change-of-measure reductions, explain the divisor-interval saving and the arithmetic construction, and map the argument to the complete formal source. The exposition follows the authors' supplied manuscript, with verification details tied to the publicly accepted artifact.

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1. The question and the result

All logarithms are natural. Limits in the target parameter t are taken through positive integers. Put

$$A_t = \left\{ n \geq 1 : \exists S \subseteq \{d \geq 1 : d \mid n\},\ \sum_{d\in S} d = t \right\}.$$

The subset S is finite and contains no repeated divisor. If it exists, the natural density is

$$d_t = \lim_{N\to\infty} \frac{\#(A_t \cap \{1,\dots,N\})}{N}.$$

In his 1970 paper, printed page 130, Erdős introduces this set, notes the existence of its density, and proposes the asymptotic

$$d_t \sim \frac{c_1}{(\log t)^{c_2}}, \qquad c_1, c_2 > 0, \tag{1}$$

as equation (33) [1]. This is Erdős Problem 859 [2]. The symbol $\sim$ means that the ratio of the two sides tends to one. A bound up to constants would be a different assertion.

Theorem 1.1 (Accepted result)**.** *The density d_t exists and is positive for every $t \geq 1$. There are no positive constants c_1, c_2 satisfying* (1)*.*

The proof supplies two complementary estimates. Set

$$\delta = 1 - \frac{1+\log\log 2}{\log 2}, \qquad 0 < \delta < 1. \tag{2}$$

Then the submitted development proves

$$\liminf_{t\to\infty} \frac{-\log d_t}{\log\log t} = \delta, \tag{3}$$

$$(\log t)^{\delta} d_t \longrightarrow 0. \tag{4}$$

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Submitted by JenW1N. Expanded proof exposition prepared with Codex assistance from the authors' manuscript and the accepted Lean submission. The long technical estimates are explained with references to their detailed manuscript and formal proofs. This prose is not itself formally verified.

The distinction between a lower limit in (3) and an ordinary limit is essential. Neither a full limit for that expression nor an exact replacement asymptotic for d_t is asserted here.

2. Density exists for an elementary reason

Lemma 2.1 (Periodicity)**.** *For fixed $t \geq 1$, membership in A_t among positive integers depends only on the residue class modulo $t!$.*

Proof. Every divisor appearing in a sum equal to t is at most t. Each positive integer $d \leq t$ divides $t!$. Consequently, if $n \equiv m \pmod{t!}$, then $d \mid n$ if and only if $d \mid m$ for every such d. The same finite subsets of these divisors can be used for n and m, and their sums agree. □

Let $M = t!$ and choose the positive representatives $1, \ldots, M$. Writing $N = qM + r$ with $0 \leq r < M$ gives

$$\#(A_t \cap [1, N]) = q\,\#(A_t \cap [1, M]) + O(M).$$

For fixed t, divide by N and let N tend to infinity. This proves

$$d_t = \frac{\#(A_t \cap [1, t!])}{t!}. \tag{5}$$

Thus the density is an exact finite residue-class average. Moreover, every positive multiple of t belongs to A_t, by using the single divisor t. Hence $1/t \leq d_t \leq 1$ for $t \geq 1$. In particular the logarithm of d_t is defined. The Lean proof uses a periodic real-valued indicator and its average. The choice of counting $0 \leq n < N$ rather than $1 \leq n \leq N$ changes only a bounded number of terms and therefore preserves the density.

3. An exact finite model for prime divisibility

The lower construction uses finite probability spaces. Let P be a finite set of primes. For $T \subseteq P$, define

$$\mu_P(T) = \prod_{p \in T} \frac{1}{p} \prod_{p \in P \setminus T} \left(1 - \frac{1}{p}\right), \qquad n_T = \prod_{p \in T} p. \tag{6}$$

The Chinese remainder theorem shows that $\mu_P(T)$ is exactly the natural density of integers whose prime divisors in P are precisely T. The coordinates are independent, with inclusion probability $1/p$.

Proposition 3.1 (Finite witnesses and actual density)**.** *For each $t \geq 1$,*

$$\sum_{T \subseteq P} \mu_P(T) \mathbf{1}_{\{n_T \in A_t\}} \leq d_t.$$

Proof. A representation using divisors of n_T remains valid for every multiple of n_T. On the prime-divisibility pattern T, the integer n_T divides the sampled integer. Each successful pattern is therefore contained in A_t. The patterns are disjoint and have the densities (6), proving the claim by finite addition. □

3.1. Exponential tilting.

For $\lambda > 0$, set

$$Z_P(\lambda) = \prod_{p \in P} \left(1 + \frac{\lambda - 1}{p}\right), \qquad \mu_{P,\lambda}(T) = \frac{\lambda^{|T|} \mu_P(T)}{Z_P(\lambda)}. \tag{7}$$

Expanding a finite product proves $\sum_T \lambda^{|T|} \mu_P(T) = Z_P(\lambda)$, so this is a probability law. Its independent inclusion probabilities are $z_p = \lambda/(p + \lambda - 1)$. Under the original law, its likelihood ratio is $L_\lambda(T) = \lambda^{|T|}/Z_P(\lambda)$. For every real q,

$$\mathbb{E}_{P,1} L_\lambda^q = \frac{Z_P(\lambda^q)}{Z_P(\lambda)^q}. \tag{8}$$

This follows by substituting the likelihood and applying the same finite product identity. The denominator is positive.

The critical tilt is $\lambda_* = 1/\log 2$. A logarithmic prime band $u < \log p \le 2u$ has reciprocal-prime mass approaching $\log 2$, and its tilted expected prime count approaches $\lambda \log 2$. Thus $\lambda > \lambda_*$ makes that expected count exceed one. The construction uses this excess to build a rich system of divisor logarithms while keeping the probability cost explicit.

3.2. **Conditioning on an adaptively chosen window.** One later prime window is selected after observing earlier primes. If $A \subseteq P$ is the observed universe and $Q(S) \subseteq P \setminus A$ is chosen from the observation $S \subseteq A$, then for any function G,

$$\mathbb{E}_{P,\lambda} G(T \cap A, T \cap Q(T \cap A)) = \sum_{S \subseteq A} \mu_{A,\lambda}(S) \sum_{U \subseteq Q(S)} \mu_{Q(S),\lambda}(U) G(S,U).$$

For each fixed S, the remaining coordinates retain their product law. Coordinates outside $A \cup Q(S)$ sum to one. Consequently a bound uniform over successful past observations can be applied conditionally, without paying a union bound over all past configurations.

4. The contradiction from the two estimates

This section converts (3)–(4) into the refutation. The proofs of those two inputs are discussed in the next sections and are provided by the linked Lean artifact.

Lemma 4.1 (An asymptotic power determines the logarithmic rate)**.** *Let $a_t > 0$ eventually, and suppose $a_t \sim c(\log t)^{-p}$, where $c > 0$ and $p \in \mathbb{R}$. Then*

$$\frac{-\log a_t}{\log\log t} \longrightarrow p, \qquad (\log t)^p a_t \longrightarrow c.$$

Proof. There is a positive sequence $u_t \to 1$ with $a_t = c(\log t)^{-p} u_t$ for all sufficiently large t. Taking logarithms yields

$$\frac{-\log a_t}{\log\log t} = p - \frac{\log c + \log u_t}{\log\log t}.$$

Since $\log u_t \to 0$ and $\log\log t \to \infty$, the last fraction tends to zero. The second claim follows from $(\log t)^p a_t = c u_t$. □

Proof of Theorem 1.1, using the two estimates. Density existence and positivity were established above. Suppose (1) holds. Applying Lemma 4.1 with $a_t = d_t$ shows that the logarithmic rate has limit c_2. Its lower limit is therefore c_2. Equation (3) forces $c_2 = \delta$. The second part of the lemma gives $(\log t)^\delta d_t \to c_1$. By (4) the same sequence tends to zero, so uniqueness of the limit gives $c_1 = 0$, contradicting $c_1 > 0$. □

This argument excludes every real exponent p with a positive constant coefficient. Only positive exponents are needed for the original question. Values at $t = 1$, where $\log t = 0$, play no role in an eventual asymptotic statement.

5. The upper estimate: reducing sums to divisor intervals

For integers $0 \le y \le t$, write

$$B(y,t) = \{n \ge 1 : \exists d \mid n,\ y < d \le t\}, \qquad D(y,t) = \operatorname{dens} B(y,t).$$

These densities also exist by periodicity. Define the sum of the small divisors by $S_y(n) = \sum_{1 \le d \le y,\ d \mid n} d$.

Proposition 5.1 (Exact density bound)**.** *For $t > 0$ and integers $0 \le y \le t$,*

$$d_t \le D(y,t) + \frac{y}{t}. \tag{9}$$

Proof. If $n \in A_t$ but $n \notin B(y,t)$, all the divisors in a representation of t are at most y. Thus $S_y(n) \geq t$. For every positive integer n this gives

$$t\mathbf{1}_{A_t}(n) \leq t\mathbf{1}_{B(y,t)}(n) + S_y(n).$$

For each fixed positive d, multiples of d have density $1/d$. Therefore the finite sum defining S_y has limiting mean

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S_y(n) = \sum_{d=1}^{y} d \cdot \frac{1}{d} = y.$$

Average the pointwise inequality and divide by t to obtain (9). □

5.1. **A finite dyadic reduction.** For a positive integer J, put $y_j = \lfloor t/2^{j+1} \rfloor$ for $0 \leq j < J$. The interval above $\lfloor t/2^J \rfloor$ is covered by these dyadic pieces. Integer rounding adds at most the single divisor $2y_j + 1$ at each step, whose multiples have density $1/(2y_j+1)$. The union bound and Proposition 5.1 yield

$$d_t \leq \sum_{j=0}^{J-1} \left(D(y_j, 2y_j) + \frac{1}{2y_j+1} \right) + 2^{-J}. \tag{10}$$

In particular, a uniform bound $D(y_j, 2y_j) \leq M(t)$ gives

$$d_t \leq JM(t) + \frac{J}{2\lfloor t/2^J \rfloor + 1} + 2^{-J}. \tag{11}$$

The submitted proof supplies a nonnegative envelope $M(t)$, valid uniformly for $\sqrt{t} \leq y \leq t$, such that

$$(\log\log t)(\log t)^{\delta} M(t) \longrightarrow 0. \tag{12}$$

Its divisor-interval analysis develops Ford-type bounds, including control of nonsquarefree contributions by powerful-number sums. Ford's classical divisor-distribution work concerns $D(y, 2y)$ [5]. The reduction above is what connects this counting problem to sums of distinct divisors.

For the limiting argument take $J = \lceil \log_2 \log t \rceil$. Then $J = O(\log\log t)$, $2^{-J} \leq 1/\log t$, and eventually $\lfloor t/2^J \rfloor \geq \sqrt{t}$. After multiplying (11) by $(\log t)^{\delta}$, its three terms tend to zero. The first does so by (12), the second because $\sqrt{t}$ dominates logarithmic powers, and the third because $\delta < 1$. This proves (4) from the stated envelope. The Lean source provides the full envelope proof and exact rounded-endpoint estimates.

5.2. **Logarithmic support and its weighted size.** For a finite prime set S and thickness $h \geq 0$, put

$$\mathcal{L}_h(S) = \bigcup_{U \subseteq S} \left[\sum_{p\in U} \log p,\ \sum_{p\in U} \log p + h \right], \qquad L_h(S) = |\mathcal{L}_h(S)|,$$

where $|\cdot|$ denotes Lebesgue measure. This is a union of intervals around logarithms of actual divisors of the squarefree product n_S. For every $I \subseteq S$,

$$L_h(S) \leq 2^{|S|-|I|} \left(h + \sum_{p\in I} \log p \right), \qquad L_h(S \cup \{p\}) \leq 2L_h(S). \tag{13}$$

Indeed, fix the selected subset outside I. The remaining intervals are contained in one translate of $[0, h + \sum_{p\in I} \log p]$. There are at most $2^{|S|-|I|}$ translations. Separating subsets according to the presence of one additional prime gives the second inequality.

Set $w_p = 1/(p-1)$, $w(S) = \prod_{p\in S} w_p$, and $Z_h(P) = \sum_{S\subseteq P} w(S)L_h(S)$. The source proves constants $C_s, C_c > 0$ and a threshold, uniform in finite prime sets $P \subseteq [2, N]$ and $h \geq 0$, such that

$$Z_h(P) \leq 2h + C_s(h + C_c)\frac{(\log N)^{2-\delta}}{(\log\log N)^{5/4}}. \tag{14}$$

Three ingredients explain the saving. First, the disjoint intervals of lengths $1/(p-1)$ in cumulative prime-weight coordinates convert a sum over distinct primes to an integral over disjoint cells. Permutation symmetry supplies the factor $1/k!$ for a k-element prime set. Tuples with repeated primes are not assigned an arithmetic meaning. Second, an ordered-coordinate barrier bounds the measure of configurations whose prefix subset sums have large support. Third, a bin-occupancy estimate controls exceptionally crowded cells. With the occupancy cutoff of order $v^{1/4}$ and the square-root factor in Stirling's bound, the calculation leaves a factor $v^{-5/4}$, where v is comparable to $\log\log N$. The manuscript gives these steps as Lemmas 3.3–3.6 and Proposition 3.7 [3], and the formal source proves their finite counterparts.

5.3. **Stopping primes and the squarefree estimate.** Let $R(y)$ be the probability that the product of primes selected with probabilities $1/p$, among primes at most $2y$, has a divisor in $(y, 2y]$. Select the largest prime in a witnessing divisor. Its smaller prime factors provide a point in logarithmic support. A reciprocal-prime interval estimate controls the possible stopping primes, and a support-weighted fourth moment controls configurations with an unfavorable denominator. The resulting formal estimate has an absolute constant C with

$$R(y) \leq \frac{C}{(\log(2y))^2} Z_{\log 2}(\{p : p \leq 2y\}).$$

Combining this with (14), and absorbing rounding in the prime cutoff, gives absolute constants C_1, C_2 such that

$$R(y) \leq \frac{C_1}{(\log(2y))^2} + \frac{C_2}{(\log(2y))^{\delta}(\log\log(2y))^{5/4}} \tag{15}$$

for all sufficiently large y. This is a probability for the finite prime law, whereas $D(y, 2y)$ also includes integers with prime powers.

5.4. **Restoring prime powers.** Call $b \geq 1$ powerful if every prime dividing b occurs to exponent at least two, including $b = 1$. The useful summability bounds are

$$\sum_{b \text{ powerful}} \frac{\tau(b)}{b} < \infty, \qquad \sum_{\substack{b \text{ powerful} \\ b>L}} \frac{1}{b} \ll L^{-1/4}. \tag{16}$$

To see their scale, write a powerful integer uniquely as a^2c^3 with c squarefree. Then $\sum_{b \text{ powerful}} b^{-3/4}$ converges by comparison with $\sum_{a,c\geq 1} a^{-3/2}c^{-9/4}$. The elementary divisor bound $\tau(b) \ll b^{1/4}$ proves the first claim. For the second use $b^{-1} \leq L^{-1/4}b^{-3/4}$ when $b > L$.

Cap each prime valuation at a finite integer $c_p \geq 2$. The one-prime weights are

$$\nu_{p,c_p}(e) = \begin{cases} (1-1/p)p^{-e}, & 0 \leq e < c_p, \\ p^{-c_p}, & e = c_p. \end{cases}$$

The final atom includes all valuations at least c_p. The Chinese remainder theorem identifies these weights with actual residue-class densities. Large enough caps determine all divisors in the interval under consideration exactly.

Fixing the powerful part b leaves primes of exponent zero or one. Their conditional inclusion probability is $1/(p+1)$, at most $1/p$. The total mass of the fixed powerful part is at most $1/b$.

Every divisor has the form $fa(U)$, with $f \mid b$ and U a subset of the remaining primes whose product is $a(U)$. Monotonicity of the divisor-interval event and a union bound therefore give

$$D(y,2y) \le \sum_{\substack{b \text{ powerful} \\ b \le \sqrt{y}}} \frac{1}{b} \sum_{f \mid b} R(y/f) + \sum_{\substack{b \text{ powerful} \\ b > \sqrt{y}}} \frac{1}{b}.$$

The formal argument first obtains this domination in a finite capped model. The convergent majorants in (16) then bound it uniformly. If $f \mid b \le \sqrt{y}$, then $\sqrt{y} \le y/f \le y$, so (15) is uniform over all these shifts. It follows that suitable positive constants satisfy

$$D(y,2y) \le F(y) := \frac{A_1}{(\log y)^2} + \frac{A_2}{(\log y)^{\delta}(\log\log y)^{5/4}} + \frac{A_3}{y^{1/8}} \tag{17}$$

for every sufficiently large integer y. The last exponent is $(\sqrt{y})^{-1/4} = y^{-1/8}$. Each term in F is eventually decreasing, and

$$(\log y)^{\delta} \log\log y \, F(y) = A_1 \frac{\log\log y}{(\log y)^{2-\delta}} + A_2 (\log\log y)^{-1/4} + A_3 \frac{(\log y)^{\delta} \log\log y}{y^{1/8}} \longrightarrow 0.$$

Thus $M(t) = F(\sqrt{t})$ is an envelope in (12). The exponent $5/4$ is useful because the final dyadic union costs one factor of $\log\log t$, leaving a factor tending to zero.

6. The lower construction: why the exponent cannot increase

The upper estimate alone would allow a faster logarithmic decay. The complementary construction prevents any larger exponent from holding uniformly at large targets. For every $\varepsilon > 0$ there are arbitrarily large integers t with

$$d_t \ge \exp\left(-(\delta+\varepsilon)\log\log t\right) = (\log t)^{-(\delta+\varepsilon)}. \tag{18}$$

Proposition 6.1. *Equations* (4) *and* (18) *imply* (3).

Proof. Since the positive sequence $(\log t)^{\delta} d_t$ tends to zero, it is eventually at most one. Thus $-\log d_t / \log\log t \ge \delta$ once $\log\log t > 0$. This proves the lower bound on the lower limit. For each $\varepsilon > 0$, (18) gives arbitrarily large t for which the same quotient is at most $\delta + \varepsilon$. Its lower limit is consequently at most $\delta + \varepsilon$ for every $\varepsilon > 0$, proving equality. □

6.1. Proper divisor cubes and logarithmic coverage.

Fix a tilt $\lambda_* < \lambda < 2$ and write

$$J(\lambda) = \lambda \log 2 - 1 - \log(\lambda \log 2).$$

The lower construction has a scale $k \to \infty$. A block event in disjoint logarithmic prime bands produces a core of k selected primes at probability at least $\exp(-(J(\lambda)+o(1))k)$ under the tilted law. The prefix counts obey a ballot-type condition. Carefully placed single-prime suffixes control the endpoint size of the product. Section 4.2 of the manuscript gives explicit choices of the block lengths and the other parameters [3]. The accepted source verifies their eventual inequalities.

For a positive integer H, a divisor cube of b is called H-proper when

$$(c_d)_{d \mid b} \longmapsto \sum_{d \mid b} c_d d \quad \text{is injective on} \quad \{-H, \dots, H\}^{\{d : d \mid b\}}.$$

This condition means that the many bounded signed combinations are actually distinct integers. It is stronger than the divisors themselves being distinct. The construction removes improper cores at a probability cost negligible compared with the block event. A first failed extension by a new prime would give a nontrivial equation $U + pV = 0$ between old cube differences. Properness of the old cube forces $V \ne 0$. Then $U \ne 0$, and the bad new prime must divide one of a controlled finite collection of nonzero integers. The very large prime windows make the total forbidden mass small. Properness passes to divisor subcubes by extending coefficients by zero.

In the same base window, a supercritical tilted selection has a positive amount of logarithmic support with probability bounded below. The source uses second-moment estimates for smoothed subset sums to establish this finite statement. The event of containing a suitable proper core and the event of having large support are increasing in the selected prime set. Positive correlation for a finite product law can therefore combine them.

6.2. **Prime pairs fill most of a logarithmic interval.** A pair of fresh primes p_i, q_i supplies a logarithmic displacement $g_i = \log p_i - \log q_i$. After adding the common baseline $\sum_i \log q_i$, choosing a subset of these displacements is the same as choosing one prime from each pair. The resulting points are logarithms of actual divisors.

The first stage reduces the uncovered fraction of an interval of length L to at most 2^{-10}. The second stage uses narrower windows. Suppose A covers all but measure E of an interval I, and a random shift g is supported on $[-w, w]$ with density at most $2/w$. On the interval I' obtained by trimming w from each endpoint,

$$\mathbb{E}|I' \setminus (A \cup (A+g))| \le 2E^2/w.$$

Indeed, an uncovered point must lie outside A, and its shift-preimage must also lie outside A. Since the shift-preimage belongs to I, Fubini and the density bound give the product of the two missing measures. Hence at least half the shifts leave missing measure at most $4E^2/w$.

For widths $w_j = L/2^{j+4}$, the normalized errors obey

$$z_{j+1} \le 2^{j+6} z_j^2, \qquad z_0 \le 2^{-10}, \qquad z_j \le 2^{-(3\cdot 2^j + j + 7)}.$$

The last inequality follows by induction. Reciprocal-prime estimates couple the continuous comparison shifts to the actual prime-pair law with a bounded displacement error, which is absorbed by thickening the support intervals. Duplicate-prime samples are removed before interpreting pairs arithmetically. The selected windows and the number of stages make the combined extra cost $\exp(-o(k))$.

The output is extremely small missing measure. If the complement of $\bigcup_{d \mid n_0} [\log d, \log d + h]$ has measure less than ω in the range being used, every contained interval $[x - \omega, x]$ meets that union. An actual divisor then satisfies $x - (h + \omega) \le \log d \le x$. This gives the logarithmic mesh needed by the arithmetic argument.

6.3. **Small Fourier phases and a fresh prime window.** For a nonempty system of distinct positive integer weights $w_1, \dots, w_N$, write $W = \sum_i w_i$ and

$$F_w = \left\{ \theta \in \mathbb{R}/\mathbb{Z} : \sum_i \|w_i \theta\|^2 \le Y^{-2} \right\},$$

where $\|x\|$ is the distance to the nearest integer. Properness controls the size of this small-phase set. Let C be an H-proper signed cube and $M = (2H+1)^N$ its cardinality. The polynomial

$$\Phi(\theta) = \frac{1}{M} \sum_{x \in C} e^{2\pi i x \theta}$$

satisfies $\int_{\mathbb{R}/\mathbb{Z}} |\Phi|^2 = 1/M$ by orthogonality of distinct integer characters. For uniform signed coefficients, cancellation of cross terms and $\cos u \ge 1 - u^2/2$ give

$$\operatorname{Re} \Phi(\theta) \ge 1 - 2\pi^2 H^2 \sum_i \|w_i \theta\|^2.$$

Assume $Y > 0$, $Y \ge 32H$, and $YW \ge 2$. Points within $1/(YW)$ of F_w have $|\Phi| \ge 3/4$. Their Haar measure is at most $16/(9M)$. A finite grid-packing argument consequently gives, for every positive

integer q,

$$\#\{0 \le a < q : \operatorname{dist}(a/q, F_w) \le 1/q\} \le \frac{6(q + YW)}{M}. \tag{19}$$

These estimates connect the arithmetic properness of the core to control of rational phases.

The construction next observes the previous prime configuration and chooses a fresh high-prime window. Bounds such as (19), applied to several core prefixes, control rational approximants at all required denominator scales. Selected high primes multiply an old phase by several fresh residues. The only phases surviving all these tests lie near zero. The conditional identity in Section 3 justifies applying the uniform estimate to this adaptively chosen window. Lemma 4.7 of the manuscript and the source's high-prime declarations provide the detailed probability and scale estimates. This stage also costs $\exp(-o(k))$.

Combining localization with the logarithmic mesh makes the following arithmetic statement hold for a range of reserve divisors $r \mid n_0$. With $D_r = \{d : d \mid n_0,\ d < r\}$,

$$\sum_{d \in D_r} \|ad/r\|^2 > Y^{-2} \qquad (1 \le a < r). \tag{20}$$

The mesh propagates a putatively small phase from smaller to larger divisors. At the endpoint r, the quantity $ar/r = a$ is an integer, so a nonzero residue cannot survive this propagation.

6.4. **Fourier inversion gives bounded coefficients.** Take a fair Boolean sum of the coins $d \in D_r$ modulo r. At frequency a its normalized Fourier transform has absolute value

$$\prod_{d \in D_r} |\cos(\pi ad/r)| \le \exp\left(-2 \sum_{d \in D_r} \|ad/r\|^2\right).$$

Here $|\cos(\pi x)| \le e^{-2\|x\|^2}$ follows from $|\sin(\pi x)| \ge 2\|x\|$ and $1 - u \le e^{-u}$. Under (20), every nontrivial coefficient is at most e^{-2/Y^2} in absolute value.

Take $\ell_r = \lceil Y^2 \log(2r) \rceil$ independent copies. The sum of the absolute values of their nontrivial Fourier coefficients is at most $(r-1)e^{-2\ell_r/Y^2} \le (r-1)/(2r)^2 < 1$. Fourier inversion therefore assigns positive probability to every residue modulo r. Each residue has a representative $\sum_{d \in D_r} c_d d$ with $0 \le c_d \le \ell_r$.

Choose an integer Z with $\ell_r \le Z$ and $4\ell_r \sum_{d \in D_r} d \le Zr$. If $Zr/4 \le t \le 3Zr/4$, the representative of t is at most $Zr/4 \le t$. Adding a nonnegative multiple of the reserve divisor r gives

$$t = br + \sum_{d \in D_r} c_d d, \qquad 0 \le b \le Z, \quad 0 \le c_d \le Z. \tag{21}$$

6.5. **A practical factor makes the summands distinct.** A positive integer b_0 is practical if every integer between zero and b_0 is a sum of distinct divisors of b_0. The source supplies such a divisor from small prime coordinates, independently of the rough-prime construction. It has $b_0 \ge Z$ and $\gcd(b_0, n_0) = 1$, and its probability cost is $\exp(-o(k))$.

Lemma 6.2 (Distinct-divisor lifting)**.** *Suppose b_0 is practical, $\gcd(b_0, n_0) = 1$, and $0 \le c_d \le b_0$ for every $d \mid n_0$. Then $\sum_{d \mid n_0} dc_d$ is a sum of distinct divisors of $b_0 n_0$.*

Proof. For each d, choose a subset E_d of the divisors of b_0 with $c_d = \sum_{e \in E_d} e$. The desired sum is $\sum_{d \mid n_0} \sum_{e \in E_d} de$. Every product de divides $b_0 n_0$. Coprimality makes its decomposition into a divisor of n_0 and a divisor of b_0 unique. Equality $de = d'e'$ forces $d = d'$ and $e = e'$. Thus no product is repeated. □

One elementary way to grow the seed starts from $210 = 2 \cdot 3 \cdot 5 \cdot 7$. If b is practical and a fresh prime $p \le b+1$ is adjoined, then bp is practical. For $0 \le m \le bp$, write $m = qp + r$ with $q \le b$ and $r < p \le b+1$. Represent q and r by divisors of b. The two groups, one multiplied by p, are disjoint. Selected prime windows grow this seed past Z using only $O(\log\log Z)$ stages. The manuscript and Lean source supply the quantitative success estimates. Lemma 6.2 converts (21) into the required representations by distinct divisors.

6.6. **Harmonic averaging preserves the rate.** For a nonempty finite target set $\mathcal{H}$ and positive weights w_t, let

$$F(T) = \frac{\sum_{t\in\mathcal{H}} w_t \mathbf{1}_{\{n_T \in A_t\}}}{\sum_{t\in\mathcal{H}} w_t}, \qquad \overline{d} = \frac{\sum_{t\in\mathcal{H}} w_t d_t}{\sum_{t\in\mathcal{H}} w_t}.$$

Thus $0 \le F \le 1$, and Proposition 3.1 gives $\mathbb{E}_{P,1} F \le \overline{d}$. For $q > 1$, Hölder's inequality, $F^{q/(q-1)} \le F$, and (8) yield the single change-of-measure bound

$$\mathbb{E}_{P,\lambda} F = \mathbb{E}_{P,1}(L_\lambda F) \le \overline{d}^{\,1-1/q} \left(\frac{Z_P(\lambda^q)}{Z_P(\lambda)^q} \right)^{1/q}. \tag{22}$$

All stages of the construction belong to this one observable.

In the actual construction $w_t = 1/t$ and the target sets escape to infinity. Logarithmically separated reserve divisors produce disjoint target intervals. Every interval $[N/4, 3N/4]$ with integer $N \ge 4$ has reciprocal weight at least $1/4$. Comparing the total weight supplied by these intervals to the harmonic weight of the whole target horizon gives $F(T) \ge \exp(-o(k))$ on successful configurations. Together with the core probability and the later subexponential costs, this proves

$$\mathbb{E}_{P_k,\lambda} F_k \ge \exp\left(-(J(\lambda) + o(1))k\right). \tag{23}$$

The finite universe is chosen so that its prime cutoff has logarithmic-logarithmic size $k\log 2 + o(k)$. Mertens's estimate and (8) then give, for fixed λ, q,

$$\log \mathbb{E}_{P_k,1} L_\lambda^q = \big(\lambda^q - 1 - q(\lambda - 1) + o(1)\big) k \log 2. \tag{24}$$

To see the coefficient, expand $\log(1 + (u-1)/p) = (u-1)/p + O_u(p^{-2})$ for fixed $u > 0$. The reciprocal-square errors have bounded sum. The first-order terms in $\log Z_{P_k}(\lambda^q) - q \log Z_{P_k}(\lambda)$ give the displayed coefficient.

Rearrange (22), substitute (23)–(24), and take logarithms:

$$-\log \overline{d}_k \le \frac{qJ(\lambda) + (\lambda^q - 1 - q(\lambda-1))\log 2}{q-1} k + o(k).$$

A positive weighted average has an entry at least as large as itself, so some t_k in the horizon satisfies $d_{t_k} \ge \overline{d}_k$. The horizon has $\log\log t_k \ge k \log 2$ and its lower endpoint tends to infinity. This gives arbitrarily large targets with the rate bound used below.

6.7. **The rate optimization in the source.** The formal development uses a tilt λ and a Hölder parameter $q > 1$. Define

$$\lambda_* = \frac{1}{\log 2}, \qquad J(\lambda) = \lambda \log 2 - 1 - \log(\lambda \log 2), \qquad R(\lambda, q) = \frac{qJ(\lambda)/\log 2 + \lambda^q - 1 - q(\lambda - 1)}{q-1}.$$

Its completed arithmetic construction and transfer estimates give, for each fixed $\lambda_* < \lambda < 2$, $q > 1$, and $\varepsilon > 0$, arbitrarily large targets satisfying

$$\frac{-\log d_t}{\log\log t} \le R(\lambda, q) + \varepsilon.$$

To see the critical value of this rate, note that $J(\lambda_*) = 0$. Differentiation in q at $q = 1$ gives

$$\lim_{q\downarrow 1} R(\lambda_*, q) = \lambda_* \log \lambda_* - \lambda_* + 1 = 1 - \frac{1 + \log\log 2}{\log 2} = \delta.$$

For fixed $q > 1$, continuity in λ permits a choice strictly above λ_* with rate as close to this value as desired. First choose q near one, then choose such a λ, and finally let the finite construction scale grow. This order keeps the parameters fixed during the construction's limiting estimates. It supplies (18) and therefore the required lower limit.

The detailed block estimates, support comparisons, and multiscale phase bounds occupy a substantial part of the 36,197-line accepted submission. Sections 3–4 of the supplied manuscript give a longer prose treatment [3]. The present account proves the elementary reductions and develops the main construction and transfer mechanisms, while identifying the technical estimates used at each stage.

7. Formal source, scope, and attribution

The complete proof is available as the accepted Lean source. It proves the negation of the native declaration `Erdos859.erdos_859`. The statement quantifies over a density function as well as over the positive constants. The submission constructs the density independently and uses uniqueness of limits to identify any claimed density function with it. The refutation therefore concerns the asymptotic itself.

7.1. **Guide to the submitted file.**

The following line numbers refer to the immutable submitted `Main.lean`. Within the namespace `Erdos859Verification`, the principal declarations are:

Lines	Declaration or interface
24–44	`divisorSumSet_iff_of_modEq_factorial`: periodicity.
205–213	`divisorDensity`, `divisorSumSet_hasDensity`: the canonical density.
433–478	`bounded_coefficients_mem_divisorSumSet`: practical-factor lifting.
1590–1636	Exact normalizer and tilted prime law.
1719–1735	`primeTiltLikelihood_moment`: exact likelihood moment.
388–408	`divisorDensity_le_intervalDensity_add`: the small-divisor split.
694–709	`not_equivalent_of_liminf_and_scaled_limit`: analytic contradiction.
822–833	`counterexample_of_estimates`: transfer to the native target.
3364–3418	Finite dyadic bound and endpoint corrections.
29056–29111	Uniform divisor-interval envelope transferred to density decay.
29179–29190	`divisorDensity_critical_scaled_tendsto_zero`.
30083–30181	Target-average observable and single Hölder transfer.
35342–35355	`corePastHigh_event_probability_lower`: combined good event.
35890–35912	`coreArithmeticCoverage_exponential_eventually`.
29936–30016	Frequent lower estimates and weighted extraction.
34970–34980	`divisorDensity_liminf_eq_of_global_harmonic_lower`.
35944–35968	Completed lower expectation and `divisorDensity_log_rate_liminf`.
35981–35984	Final target, supplied with both proved estimates.

The submission was accepted on 18 September 2026 and completed production verification that day. Approval was recorded on 21 September. The public verification report [4] records successful Lean kernel, permitted-axiom, and statement-comparison checks. Its environment used Lean 4.33.1. This exposition does not report a new independent kernel replay.

The supplied manuscript also describes a separate companion build. The verification record and line references here refer specifically to the accepted production submission, whose SHA-256 digest is

`c8011826d105daecca4de0f0a58cc4acd887eafd1a78a9e8c9095c399ca97aea`.

Supporting prime estimates include an explicitly credited adaptation from the PNT+ project, with the cited upstream revision and Apache-2.0 license recorded in the submitted file [6]. The divisor-interval analysis uses established mathematics associated with Ford's work. The public

submitter credit is JenW1N. This TeX edition follows the expanded Conjectures.io exposition, which accompanies Justin Sun Prize submission JSP-000712.

References

[1] P. Erdős, *Some extremal problems in combinatorial number theory* (1970), pp. 123–133. The question is on p. 130, equation (33).

[2] Erdős Problems, *Problem 859*, `https://www.erdosproblems.com/859`.

[3] L. Kruer and J. Kohlmeyer, *Divisor sums and a counterexample to Erdős problem 859*, author-supplied working manuscript, 18 September 2026, 27 pp., especially Sections 2–5.

[4] JenW1N, Erdős 859 counterexample submission, Conjectures, accepted 18 September 2026. Public result and review and archived accepted proof body.

[5] K. Ford, *Integers with a divisor in* $(y, 2y]$, version dated 13 March 2013.

[6] PNT+ contributors, PrimeNumberTheoremAnd, Mertens development. The version used is `a5154676af9aa3095150ee410cdda80555aa0642`.