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AN EXACT FORMULA FOR FORK-FREE SETS OF INTEGERS AND THE IRRATIONALITY OF THEIR DENSITY

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Abstract. A set of positive integers is fork-free if no element of the set divides two other elements of the set. Let $f(n)$ be the largest size of a fork-free subset of $\{1,\dots,n\}$. Erdős asked how large $f(n)$ can be, and whether $f(n)/n$ tends to an irrational limit. Lebensold proved $0.6725n \le f(n) \le 0.6736n$ for large n, and Davis recently proved that $f(n)/n$ converges to an effectively computable constant. We prove the exact formula

$$f(n) = \sum_{\substack{m\le n\\(m,6)=1}} \big(R(n/m) - \mathbf{1}_{\mathcal{B}}(n/m)\big) \qquad (n \ge 1),$$

where $R(x) = 1 + \lfloor \log_4 x \rfloor + \lfloor \log_3 x \rfloor + \mathbf{1}_{\mathcal{E}}(x)$ is the largest size of a fork-free set of integers $2^a 3^b \le x$ for $x \ge 1$, and $\mathcal{E}, \mathcal{B}$ are explicit unions of intervals. Consequently $f(n)/n \to c$ with

$$180c = 32 + \sum_{k\ge 0} \varphi([4^k]_3) 4^{-k} + \sum_{k\ge 0} \psi([3^k]_4) 3^{-k} = 121.13381\ldots,$$

where $[x]_b \in [1,b)$ is the mantissa of x in base b and φ, ψ are explicit integer step functions. Numerically $c = 0.67296565119948934747\ldots$. We prove that c is irrational, which answers Erdős's question affirmatively. The proof of irrationality combines exact return identities for the rotation by $\log 4/\log 3$, a two-cutoff 2-adic nonvanishing argument, and a three-place case of the p-adic Subspace Theorem. The lower bound in the exact formula rests on a finite computation. The whole argument, including the finite computation (replayed as checked certificates) and a self-contained proof of the required case of the Subspace Theorem, is formalized in a single Lean 4 file that proves the Formal Conjectures statement of Erdős Problem 1062(ii) using only the standard axioms.

1. Introduction

A set A of positive integers is fork-free if there are no distinct $a, b, c \in A$ with $a \mid b$ and $a \mid c$. Equivalently, every element of A has at most one proper multiple in A. Let

$$f(n) = \max\{|A| : A \subseteq \{1,\dots,n\},\ A \text{ fork-free}\}.$$

Every element of $(n/3, n]$ has at most one proper multiple up to n, namely its double, so $f(n) \ge \lceil 2n/3 \rceil$.

Erdős asked how large $f(n)$ can be, and whether $\lim_{n\to\infty} f(n)/n$ exists and is irrational. See Problem B24 in Guy's collection [14] and Problem 1062 in Bloom's list [3]. Lebensold [15] proved that $0.6725n \le f(n) \le 0.6736n$ for all large n. Davis [6] recently showed that $f(n) = cn + o(n)$ for an effectively computable constant c, as a special case of a general theorem on forbidden subgraphs of divisor graphs, and left the irrationality of c open. The irrationality question is also recorded in the Formal Conjectures project [11] as the Lean statement `Erdos1062.erdos_1062.parts.ii`, and it is posed as a formal challenge on the conjectures.io platform [5].

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TABLE 1. The coefficient step functions.

$t \in [1,3)$	$\varphi(t)$	$s \in [1,4)$	$\psi(s)$
$[1, 16/15)$	78	$[1, 4/3)$	30
$[16/15, 10/9)$	30	$[4/3, 8/5)$	35
$[10/9, 4/3)$	-30	$[8/5, 5/3)$	29
$[4/3, 3/2)$	30	$[5/3, 2)$	24
$[3/2, 2)$	0	$[2, 4)$	-60
$[2, 8/3)$	-60		
$[8/3, 3)$	-12		

1.1. **Results.** All intervals below are closed on the left and open on the right. Put

$$\begin{aligned} \mathcal{E} &= \left(\bigcup_{j\ge 0}[2\cdot 4^j, 4^{j+1})\right) \cap \left(\bigcup_{b\ge 0}[2\cdot 3^b, 3^{b+1})\right), \\ \mathcal{B} &= \left(\bigcup_{b\ge 0}[10\cdot 3^b, 12\cdot 3^b)\right) \cap \left(\bigcup_{j\ge 0}\left[4^j, \tfrac{5}{4}4^j\right)\right), \end{aligned} \tag{1}$$

and define, for real x,

$$R(x) = 1 + \lfloor \log_4 x \rfloor + \lfloor \log_3 x \rfloor + \mathbf{1}_{\mathcal{E}}(x) \quad (x \ge 1), \qquad R(x) = 0 \quad (x < 1). \tag{2}$$

The first intervals of $\mathcal{B}$ are $[270, 320)$, $[65610, 78732)$ and $[16777216, 19131876)$.

Theorem 1.1 (Exact formula)**.** *For every positive integer n,*

$$f(n) = U(n) := \sum_{\substack{1\le m\le n\\ (m,6)=1}} \big(R(n/m) - \mathbf{1}_{\mathcal{B}}(n/m)\big). \tag{3}$$

Moreover, a fork-free set of this size can be chosen inside $(n/8, n]$.

Here $R(x)$ is the largest size of a fork-free set of integers of the form $2^a 3^b$ not exceeding x (Theorem 2.2). The terms $\mathbf{1}_{\mathcal{B}}$ are forced losses between adjacent powers-of-five layers (Lemma 3.1). Formula (3) is evaluated quickly by summing the jumps of R. For instance,

$$f(10^6) = 672964, \qquad f(10^{12}) = 672965651200, \qquad f(10^{24}) = 672965651199489347471228.$$

Theorem 1.1 answers the first half of Erdős's question, how large $f(n)$ can be, in the strongest possible form. For the density, write $[x]_b = x/b^{\lfloor \log_b x \rfloor} \in [1, b)$ for the mantissa of $x > 0$ in base b, and define integer step functions φ on $[1, 3)$ and ψ on $[1, 4)$ by Table 1.

Theorem 1.2 (Density)**.** *The limit $c = \lim_{n\to\infty} f(n)/n$ exists, and*

$$c = \frac{11}{18} + \frac{1}{3}\left(\int_{\mathcal{E}} \frac{dt}{t^2} - \int_{\mathcal{B}} \frac{dt}{t^2}\right). \tag{4}$$

Equivalently,

$$180c = 32 + \sum_{k\ge 0} \frac{\varphi([4^k]_3)}{4^k} + \sum_{k\ge 0} \frac{\psi([3^k]_4)}{3^k}. \tag{5}$$

Numerically,

$$c = 0.672965651199489347471227919852209644752814733132999025108932699233\ldots.$$

The displayed digits were obtained twice, independently: from (4) by exact rational integration with a tail bound, and from (5) by exact rational summation with a tail bound. The two enclosures agree.

Theorem 1.3 (Irrationality)**.** *The constant c is irrational.*

Corollary 1.4. *The ratio $f(n)/n$ converges to an irrational number. In particular the Formal Conjectures statement* `Erdos1062.erdos_1062.parts.ii` *holds with answer* `True`*.*

1.2. **Outline of the proof.** Every positive integer is uniquely $m2^a3^b$ with $(m, 6) = 1$. Section 2 solves the extremal problem on the two-prime grid $\{2^a3^b \le x\}$ exactly: the answer is $R(x)$, and the maximal configurations are rigid enough to be described. Summing over m gives an upper bound, which Section 3 sharpens by forced losses between the classes m and $5m$. Section 4 constructs matching sets inside $(n/8, n]$, where the only divisibility ratios between distinct elements are $2, 3, \dots, 7$. Ratios $2, 3, 4, 6$ are handled by explicit parity-restricted grid sets, and ratios $5, 7$ by a cutoff recursion. That the recursion never gets stuck is a finite statement, which we verify by computer. Section 5 derives Theorem 1.2.

The proof of irrationality occupies Sections 6 and 7. The two series in (5) are sampled along the rotation by $\alpha = \log 4/\log 3$. Along a sequence of unimodular brackets $p/q < \alpha < P/Q$, the number $(4^q-1)(3^p-1)(180c-32)$ is an integer N plus a lacunary sum of $\{2,3\}$-units with bounded integer coefficients. Truncating this sum at two carefully chosen cutoffs, and comparing 2-adic valuations, shows that if c were rational, one of the two truncation errors would be a nonzero linear form in at most 405 $\{2,3\}$-integers that is exponentially small compared with its height. A three-place case of the Subspace Theorem of Schmidt and Schlickewei, applied through a coordinate-elimination induction, rules this out.

1.3. **Status of the computations and of the formalization.** The upper bound $f(n) \le U(n)$ and the proof of irrationality are ordinary mathematical proofs, presented below. The irrationality proof uses a known case of the p-adic Subspace Theorem. The lower bound $f(n) \ge U(n)$ depends on a finite computation with two parts: a table of exact states of the cutoff recursion below $2 \cdot 7^7$, and the unsatisfiability of an explicit quantifier-free formula in linear rational arithmetic (Lemma 4.4).

We have formalized the entire argument in Lean 4 [8] with Mathlib [19]. The single Lean file proves the Formal Conjectures statement, for the answer `True`, and depends only on the axioms `propext`, `Quot.sound` and `Classical.choice`. The finite computations enter the formal proof as certificates that are checked by the Lean kernel, so no external solver is trusted. The formalization does not cite the Subspace Theorem. It contains a complete proof of the special case we need (Theorem 7.1), following the classical method of Roth and Schmidt. Section 8 describes the development and its verification.

1.4. **Notation.** For a prime p, $|\cdot|_p$ is the p-adic absolute value with $|p|_p = p^{-1}$, v_p is the p-adic valuation, and $|\cdot| = |\cdot|_\infty$. For $x \in \mathbb{Q}^n \setminus \{0\}$ the multiplicative, projective height is $H(x) = \prod_v \max_i |x_i|_v$, the product over all places of $\mathbb{Q}$. For a primitive integer vector, $H(x) = \max_i |x_i|$. A $\{2,3\}$-unit is a number 2^a3^b with $a, b \in \mathbb{Z}$, and a $\{2,3\}$-integer is an element of $\mathbb{Z}[1/6]$. We write $\{x\}$ for the fractional part and $\|x\|$ for the distance to the nearest integer. Throughout, $\langle 2,3\rangle = \{2^a3^b : a, b \ge 0\}$.

2. The two-prime grid

For real $x \ge 1$ let $\mathcal{S}(x) = \{2^a3^b \le x : a, b \ge 0\}$. We identify 2^a3^b with the lattice point (a, b). Then divisibility is the coordinatewise order, and $\mathcal{S}(x)$ is a finite down-set. Put

$$a^* = \lfloor \log_2 x \rfloor, \qquad b^* = \lfloor \log_3 x \rfloor, \qquad \delta = \mathbf{1}_{[x \ge 2 \cdot 3^{b^*}]}.$$

Define $g(1,1) = 1$ and $g(w,h) = 1 + \min(w,h)$ for all other positive integers w, h.

Lemma 2.1 (Rectangle decomposition)**.** *Let $W \subseteq \mathcal{S}(x)$ be a nonempty fork-free set. Let its maximal elements be $(a_1, b_1), \dots, (a_r, b_r)$ with $a_1 < \cdots < a_r$ and $b_1 > \cdots > b_r$. Put $a_0 = -1$, $b_{r+1} = -1$, $w_i = a_i - a_{i-1}$, $h_i = b_i - b_{i+1}$, and $\mathcal{R}_i = \{(a, b) : a_{i-1} < a \le a_i,\ b_{i+1} < b \le b_i\}$.*

(i) *$W \subseteq \bigcup_i \mathcal{R}_i$, and $W \cap \mathcal{R}_i$ consists of the corner (a_i, b_i) together with an antichain not containing it.*

(ii) *$|W \cap \mathcal{R}_i| \le g(w_i, h_i) \le h_i + w_i/2$.*

(iii) *The deficits $d_i = h_i + w_i/2 - |W \cap \mathcal{R}_i|$ lie in $\frac{1}{2}\mathbb{Z}_{\ge 0}$, and*

$$|W| = b_1 + 1 + \frac{a_r + 1}{2} - \sum_{i=1}^{r} d_i. \tag{6}$$

Moreover $d_i \ge \frac{1}{2}$ whenever $w_i = 1$, with equality only if $h_i \in \{1, 2\}$. The equality $d_i = 0$ holds exactly when $w_i = 2$, $h_i \in \{1, 2\}$ and $|W \cap \mathcal{R}_i| = h_i + 1$.

(iv) *Conversely, if $(a_1, b_1), \dots, (a_r, b_r) \in \mathcal{S}(x)$ form such an antichain and we choose in each $\mathcal{R}_i$ its corner and an antichain avoiding the corner, the union is fork-free.*

Proof. Every element of W lies below some maximal element. A nonmaximal $w \in W$ below two maximal elements would divide two other elements of W, so it lies below exactly one, say (a_i, b_i). Then $w = (a, b)$ is not below (a_{i-1}, b_{i-1}) when $i > 1$, and since $b \le b_i < b_{i-1}$ this forces $a > a_{i-1}$. For $i = 1$, $a > a_0 = -1$ holds automatically. Similarly $b > b_{i+1}$, so $w \in \mathcal{R}_i$. Two comparable nonmaximal elements below (a_i, b_i) would give a fork, so they form an antichain. This proves (i).

An antichain in a $w \times h$ box has distinct first and distinct second coordinates, so it has at most $\min(w, h)$ elements, and none if it must avoid the corner of a 1×1 box. This is the first inequality in (ii). The second inequality, and the statements about the cases of equality in (iii), follow by direct comparison: $g(w, h) = h + w/2$ exactly when $w = 2$ and $h \in \{1, 2\}$, and for $w = 1$ one has $h + \frac{1}{2} - g(1, h) = \frac{1}{2}$ for $h \in \{1, 2\}$ and $h - \frac{3}{2} \ge \frac{3}{2}$ for $h \ge 3$. Identity (6) follows by summing $\sum_i h_i = b_1 + 1$ and $\sum_i w_i = a_r + 1$. For (iv), a point of $\mathcal{R}_i$ has smaller first coordinate and larger second coordinate than every point of $\mathcal{R}_j$ for $j > i$. Thus points in different rectangles are incomparable, and within one rectangle only the corner lies above the other chosen points. □

Theorem 2.2. *For every real $x \ge 1$, the largest fork-free subset of $\mathcal{S}(x)$ has*

$$R(x) = \lfloor a^*/2 \rfloor + b^* + 1 + \mathbf{1}_{[a^* \text{ odd}]}\delta = 1 + \lfloor \log_4 x \rfloor + \lfloor \log_3 x \rfloor + \mathbf{1}_{\mathcal{E}}(x)$$

elements.

Proof. The two expressions agree because $\lfloor \log_4 x \rfloor = \lfloor a^*/2 \rfloor$, a^* is odd exactly when $x \in \bigcup_j [2 \cdot 4^j, 4^{j+1})$, and $\delta = 1$ exactly when $x \in \bigcup_b [2 \cdot 3^b, 3^{b+1})$.

Upper bound. Let W be fork-free. Since $b_1 \le b^*$ and $a_r \le a^*$, (6) gives $|W| \le b^* + 1 + (a^* + 1)/2$. If a^* is even, integrality gives $|W| \le b^* + 1 + a^*/2$. If a^* is odd and $\delta = 1$, the bound is already $\lfloor a^*/2 \rfloor + b^* + 2$.

If a^* is odd and $\delta = 0$, equality would require $b_1 = b^*$, $a_r = a^*$ and all $d_i = 0$. Then $w_1 = 2$, so $a_1 = 1$ and $2 \cdot 3^{b^*} \in \mathcal{S}(x)$, contradicting $\delta = 0$. Hence $|W| \le b^* + (a^* + 1)/2 = \lfloor a^*/2 \rfloor + b^* + 1$.

Constructions. If $a^* = 0$, then $\mathcal{S}(x) = \{1\}$ and $R(x) = 1$. If $a^* \ge 2$ is even, take corners in the columns $a = 0, 2, \dots, a^*$ at heights $\beta(a) = \lfloor \log_3(x/2^a) \rfloor$. Since $3 < 4 < 9$, consecutive heights drop by 1 or 2, and $\beta(a^*) = 0$. The first rectangle has width 1 and height $h_1 \in \{1, 2\}$ and holds h_1 points. Every other rectangle has width 2 and height 1 or 2 and holds $h_i + 1$ points (Lemma 2.1(iv)). The total is $\beta(0) + 1 + a^*/2 = b^* + 1 + a^*/2$.

If a^* is odd, use the columns $1, 3, \dots, a^*$. Now every rectangle has width 2, and the first corner has height $\lfloor \log_3(x/2) \rfloor = b^* - 1 + \delta$, so the total is $b^* + \delta + (a^* + 1)/2$. □

3. The upper bound

Lemma 3.1 (Forced elements)**.** *Let $t \in \mathcal{B}$, say $10 \cdot 3^b \le t < 12 \cdot 3^b$ and $4^j \le t < \frac{5}{4}4^j$. Then every largest fork-free subset of $\mathcal{S}(t)$ contains $2 \cdot 3^b$ and $4 \cdot 3^b$, and every largest fork-free subset of $\mathcal{S}(t/5)$ contains $2 \cdot 3^b$. Consequently every fork-free subset of $\mathcal{S}(t) \cup 5\mathcal{S}(t/5)$ has at most $R(t) + R(t/5) - 1$ elements.*

Proof. For $\mathcal{S}(t)$ we have $a^* = 2j$, $b^* = b+2$ and $\delta = 0$, because $t < \frac{5}{4}4^j < 2 \cdot 4^j$ and $9 \cdot 3^b \le t < 18 \cdot 3^b$. Also $j \ge 2$ since $t \ge 10$. Thus $R(t) = j + b + 3$. If $|W| = R(t)$, then (6) gives

$$\sum_i d_i + (b + 2 - b_1) + \frac{2j - a_r}{2} = \frac{1}{2}.$$

Hence $b_1 = b + 2$. Since $t < 2 \cdot 3^{b+2}$, the first corner is $(0, b+2)$, so $w_1 = 1$ and $d_1 \ge \frac{1}{2}$. Therefore $d_1 = \frac{1}{2}$, $a_r = 2j$, and $d_i = 0$ for $i \ge 2$. In particular $r \ge 2$, and $w_2 = 2$ gives $a_2 = 2$. As $h_1 \in \{1, 2\}$ we have $b_2 \in \{b, b+1\}$. The bound $4 \cdot 3^{b_2} \le t < 12 \cdot 3^b$ gives $b_2 = b$, so $4 \cdot 3^b \in W$. The rectangle $\mathcal{R}_2$ has width 2, height $h_2 \in \{1, 2\}$ and holds $h_2 + 1$ points. If $h_2 = 1$ it is $\{(1, b), (2, b)\}$. If $h_2 = 2$, the only antichain of size two avoiding the corner is $\{(1, b), (2, b-1)\}$. Either way $2 \cdot 3^b \in W$.

For $u = t/5$ we have $2 \cdot 3^b \le u < \frac{12}{5}3^b$ and $2^{2j-3} < 4^j/5 \le u < 2^{2j-2}$, so $a^* = 2j - 3$, $b^* = b$, $\delta = 1$ and $R(u) = j + b$. For a largest W', equality holds in $|W'| \le b^* + 1 + (a^* + 1)/2$, so every rectangle has width 2 and the first corner is $(1, b)$, that is, $2 \cdot 3^b \in W'$. If a fork-free subset of $\mathcal{S}(t) \cup 5\mathcal{S}(t/5)$ attained $R(t) + R(t/5)$, it would contain $2 \cdot 3^b$, $4 \cdot 3^b$ and $5 \cdot 2 \cdot 3^b$, and $2 \cdot 3^b$ would divide the other two. □

Lemma 3.2. *$\mathcal{B} \cap 5\mathcal{B} = \varnothing$.*

Proof. $\mathcal{B} \subseteq G := \bigcup_j [4^j, \frac{5}{4}4^j)$, and $5G = \bigcup_j [5 \cdot 4^j, \frac{25}{4}4^j)$. The interval $[5 \cdot 4^j, \frac{25}{4}4^j)$ starts exactly where $[4^{j+1}, 5 \cdot 4^j)$ ends and lies below 4^{j+2}, so $G \cap 5G = \varnothing$. □

Proposition 3.3. *For every positive integer n, $f(n) \le U(n)$.*

Proof. Let $A \subseteq \{1, \dots, n\}$ be fork-free. For $(m, 6) = 1$ the class $\{ms : s \in \mathcal{S}(n/m)\}$ meets A in a set of the form mW_m with $W_m \subseteq \mathcal{S}(n/m)$ fork-free, so $|A| \le \sum_m R(n/m)$ by Theorem 2.2. Group the classes into chains $m = m_0 5^k$ with $(m_0, 30) = 1$. If $t = n/m \in \mathcal{B}$, then $t \ge 10$ and, after division by m, the part of A in the classes m and $5m$ is a fork-free subset of $\mathcal{S}(t) \cup 5\mathcal{S}(t/5)$. Lemma 3.1 forces a loss of one element there. By Lemma 3.2, n/m and $n/(5m)$ are never both in $\mathcal{B}$, so the pairs of classes used for different losses are disjoint and the losses add up. □

4. The construction

4.1. **Parity-restricted grid sets.** Say that $2^a 3^b$ has parity a mod 2. For $p \in \{0, 1\}$ and real $z \ge 0$ let

$$\begin{aligned} W_p(z) = {} & \{s \in \langle 2, 3\rangle : z/4 < s \le z,\ s \text{ of parity } p\} \\ & \cup \{s \in \langle 2, 3\rangle : z/6 < s \le z/2,\ s \text{ of parity } 1 - p\}. \end{aligned} \tag{7}$$

Lemma 4.1. *The set $W_p(z)$ is fork-free and contained in $(z/6, z]$. Its size is*

$$\begin{aligned} h_0(z) &= \lfloor \log_4 z \rfloor + \lfloor \log_3 z \rfloor + 1 && (z \ge 1), \\ h_1(z) &= \lfloor \log_4(z/2) \rfloor + \lfloor \log_3(z/2) \rfloor + 2 && (z \ge 2), \end{aligned}$$

with $h_0(z) = 0$ for $z < 1$ and $h_1(z) = 0$ for $z < 2$. Moreover $R(z) = \max(h_0(z), h_1(z))$.

Proof. An element s of parity p exceeds $z/4$. Among its proper multiples in $\langle 2, 3\rangle$ up to z, only $2s$ and $3s$ are possible, and $2s$ has parity $1 - p$ but exceeds $z/2$. Its only possible selected proper multiple is $3s$. An element s of parity $1 - p$ lies in $(z/6, z/2]$. Its double lies in $(z/3, z]$, has parity p

and is selected. The multiples $3s, 4s$ exceed $z/2$ and have parity $1-p$, while $6s, 8s, 9s, 12s$ exceed z. Thus every element has at most one selected proper multiple.

For the count with $p=0$, for each b with $3^b \le z$, exactly one power $4^i \ge 1$ satisfies $4^i 3^b \in (z/4, z]$. For each i with $4^{i+1} \le z$, exactly one power of 3 satisfies $2 \cdot 4^i \cdot 3^b \in (z/6, z/2]$. For $p=1$, for each b with $3^b \le z/2$, exactly one i satisfies $2 \cdot 4^i 3^b \in (z/4, z]$. For each i with $4^i \le z/2$, exactly one b satisfies $4^i 3^b \in (z/6, z/2]$. Finally, comparing with (2), $h_0 = R - \mathbf{1}_{\mathcal{E}}$ and, for $z \ge 1$, $h_1 = R - \mathbf{1}_{[a^* \text{ even}]}\mathbf{1}_{[\delta=0]}$, so the maximum is R. □

Each h_p is a nondecreasing step function, so for every integer k the level set $\{z \ge 0 : h_p(z) = k\}$ is a possibly empty interval $[\lambda_p(k), \mu_p(k))$.

4.2. **The cutoff recursion.** For $\epsilon \in \{0,1\}$ and $u > 0$ define the target

$$k_\epsilon(u) = R(u) - \epsilon \mathbf{1}_{\mathcal{B}}(5u). \tag{8}$$

We define states $(p_\epsilon(u), z_\epsilon(u)) \in \{0,1\} \times \mathbb{Q}_{\ge 0}$ for rational $u > 0$ by recursion. For $u < 1$ the state is $(0,0)$. For $u \ge 1$, let the two children be $(p_5, z_5) = (p_1(u/5), z_1(u/5))$ and $(p_7, z_7) = (p_\epsilon(u/7), z_\epsilon(u/7))$. For a trial parity p whose level set for $k_\epsilon(u)$ is nonempty, put

$$\zeta_p = \max\left\{\frac{3}{4}u, \lambda_p(k_\epsilon(u)), \gamma(p,p_5)z_5, \gamma(p,p_7)z_7\right\}, \qquad \gamma(p,p') = \begin{cases} 4, & p = p', \\ 6, & p \ne p'. \end{cases} \tag{9}$$

The parity p is available if that level set is nonempty and

$$\zeta_p \le u, \qquad \zeta_p < \mu_p(k_\epsilon(u)). \tag{10}$$

Try first $p = a^*(u) \bmod 2$, where $a^*(u) = \lfloor \log_2 u \rfloor$, and then the other parity. If some parity is available, the state is the first available parity p together with $z_\epsilon(u) = \zeta_p$, and we say that the recursion succeeds at (u, ϵ).

Lemma 4.2. *Suppose the recursion succeeds at every rational $u \ge 1$ for both values of ϵ. Then $f(n) \ge U(n)$ for every n, with an extremal set inside $(n/8, n]$.*

Proof. Write each integer uniquely as ghs with $(g, 210) = 1$, h a product of powers of 5 and 7, and $s \in \langle 2, 3\rangle$. For each pair (g,h) with $gh \le n$, let $u = n/(gh)$ and $\epsilon = \mathbf{1}_{[5|h]}$, and select the fiber $ghW_{p_\epsilon(u)}(z_\epsilon(u))$. Let A_n be the union of the fibers.

Since $\frac{3}{4}u \le z_\epsilon(u) \le u$, the selected elements of the fiber lie in $(u/8, u]$ after division by gh, so $A_n \subseteq (n/8, n]$. If $x \mid y$ are distinct elements of A_n, then $y/x \in \{2, \dots, 7\}$. Ratios $2, 3, 4, 6$ keep (g,h) fixed, and each fiber is fork-free by Lemma 4.1. A ratio $d \in \{5,7\}$ would mean that the same s lies in the fiber (g,h) and in its child fiber (g, dh), whose argument is u/d and whose type is 1 for $d=5$ and ϵ for $d=7$. These are exactly the child states used in (9).

Let p be the parent's parity and p_d the child's. An element of the child of parity $1-p_d$ is at most $z_d/2 \le u/(2d) < u/8$, too small for the parent. An element s of the child of parity p_d satisfies $s \le z_d$. If $p_d = p$, then s would have the parent's principal parity, so $s > \zeta_p/4 \ge z_d$. If $p_d \ne p$, then $s > \zeta_p/6 \ge z_d$. So no such s exists. Hence every divisibility relation in A_n lies inside a fiber, and A_n is fork-free.

The fiber (g,h) has $h_p(\zeta_p) = k_\epsilon(u)$ elements. Writing $m = gh$, the total is

$$\sum_{\substack{m \le n \\ (m,6)=1}} \left(R(n/m) - \mathbf{1}_{[5|m]}\mathbf{1}_{\mathcal{B}}(5n/m)\right).$$

In the second sum put $m = 5m'$. Since every element of $\mathcal{B}$ exceeds 5, the range $m' \le n/5$ may be extended to $m' \le n$, giving exactly $U(n)$. □

Thus each forced loss of Section 3 is assigned to the upper of the two powers-of-five layers involved.

4.3. **The finite verification.**

Proposition 4.3. *The recursion succeeds at every rational $u \geq 1$ for $\epsilon = 0$ and $\epsilon = 1$.*

The proof has two computer-verified ingredients.

The base range. Let $T_0 = 2 \cdot 7^7 = 1{,}647{,}086$. On $[1, T_0)$ the recursion was evaluated exactly. There are 109 pieces for $\epsilon = 0$ and 108 for $\epsilon = 1$. Each piece is a half-open interval with rational endpoints on which the state is a fixed parity and a cutoff $z = au + b$ with $a, b \in \mathbb{Q}$. The table is verified by refining it at every breakpoint into 306 and 305 intervals. On each interval one checks, with exact rational arithmetic, the target size, the four lower bounds in (9), the upper bounds in (10), which term of the maximum is active, and the priority rule. For pieces using the second parity, one also checks that the preferred parity is unavailable. The check covers all u in the interval.

A finite closure. Normalize the root argument u to 1. The powers of 4 and 3 below u are encoded by the two parameters

$$X = 4^{\lfloor \log_4 u \rfloor}/u \in (1/4, 1], \qquad Y = 3^{\lfloor \log_3 u \rfloor}/u \in (1/3, 1],$$

which the finite statement allows to vary independently. Its nodes are $\nu = (i, j)$ with $i, j \geq 0$ and $i + j \leq D$. The node ν has normalized argument $v_\nu = 5^{-i}7^{-j}$, type $\epsilon_\nu = 1$ if $i > 0$ and $\epsilon_\nu = \epsilon$ if $i = 0$, and children $(i + 1, j)$ and $(i, j + 1)$. At a node, let L_4 be the largest element of $X4^{\mathbb{Z}}$ not exceeding v_ν, and L_3 the largest element of $Y3^{\mathbb{Z}}$ not exceeding v_ν. Put $P = \mathbf{1}_{[v_\nu \geq 2L_4]}$ and $\Delta = \mathbf{1}_{[v_\nu \geq 2L_3]}$, and let L_4', L_3' be the analogous quantities at $5v_\nu$. The obstruction bit

$$Q = \epsilon_\nu \mathbf{1}_{\left[5v_\nu < \frac{5}{4}L_4',\ \frac{10}{9}L_3' \leq 5v_\nu < \frac{4}{3}L_3'\right]}$$

encodes $\epsilon_\nu \mathbf{1}_{\mathcal{B}}(5u_\nu)$, where $u_\nu = uv_\nu$. The errors $e_0 = Q - P\Delta$ and $e_1 = Q - (1 - P)(1 - \Delta)$ measure how many levels each parity must drop from its value at v_ν. The most recent jumps of h_0 at or below v_ν are $(a, b) = (L_4, L_3)$. Those of h_1 are $a = 2L_4$ or $L_4/2$ according as $P = 1$ or 0, and $b = 2L_3$ or $\frac{2}{3}L_3$ according as $\Delta = 1$ or 0. The window of parity p, which is the normalized set of admissible cutoffs, is

- empty if $e_p = -1$;
- $[\max(a, b), v_\nu]$ if $e_p = 0$;
- $[\max(a/4, b), \max(a, b))$ if $e_p = 1$ and $a > b$, and $[\max(a, b/3), \max(a, b))$ if $e_p = 1$ and $a \leq b$.

Each window endpoint is a rational-linear expression in (X, Y) on each of finitely many explicitly described pieces.

Lemma 4.4 (Closure)**.** *For $D = 5$ and $\epsilon \in \{0, 1\}$, the following system in the unknowns X, Y and, for every node ν, a parity π_ν and a cutoff $\zeta_\nu \in \mathbb{Q}$, is unsatisfiable.*

- *Each leaf ($i + j = D$) has ζ_ν in the window of π_ν and $\zeta_\nu \geq \frac{3}{4}v_\nu$.*
- *Each node of depth $1, \ldots, D - 1$ has the state produced by (9), (10) and the priority rule from its children's states, and some parity is available there.*
- *At the root, neither parity is available.*

The system has two real parameters, 21 nodes and Boolean choices, and all its coefficients are rational. No logarithms or floating-point numbers occur. For $D = 7$ (36 nodes) the corresponding system was proved unsatisfiable independently by Z3 [7] and by cvc5 [2]. For $D = 3, 4$ it is satisfiable. For $D = 5$ it is unsatisfiable, and the refutation is replayed in the formal proof as a certificate checked by the Lean kernel (Section 8).

Proof of Proposition 4.3. Use strong induction on $\lfloor u \rfloor$ for both values of ϵ. Below T_0 we use the table. Let $u \geq T_0$. Every proper descendant has smaller integer part than u, so its state exists by induction. Every node of depth at most 7 below u has argument $u_\nu = u5^{-i}7^{-j} \geq T_0/7^7 = 2$, and $5u_\nu \geq 10$. For such arguments the normalized powers, windows and obstruction bits above coincide

with the actual ones. Here $h_p(z) = k_{\epsilon_\nu}(u_\nu)$ and $z \le u_\nu$ exactly when z/u lies in the corresponding window.

Assume that neither parity is available at (u, ϵ). Take X, Y as above, and for every node of depth at most 5 let π_ν be the actual parity and ζ_ν the actual cutoff divided by u, wherever the state is used. The leaves and internal nodes satisfy their constraints because the recursion succeeds at the smaller arguments $u_\nu \le u/5$, and the root fails by assumption. This contradicts Lemma 4.4. □

With Proposition 3.3 and Lemma 4.2, this proves Theorem 1.1. Independently of the formal proof, an exhaustive search confirms (3) for $n \le 40$.

5. The density and the coefficient series

Proof of (4). The function R is nondecreasing and right-continuous. It jumps only at points of $\langle 2, 3\rangle$, and each jump $\Delta R(s)$ is 0 or 1, because adding one point to a grid raises the largest fork-free size by at most one. Let $N_6(y) = \#\{m \le y : (m, 6) = 1,\ m \ge 1\} = y/3 + O(1)$. Then

$$\sum_{\substack{m \le n \\ (m,6)=1}} R(n/m) = \sum_{s \in \langle 2,3\rangle} \Delta R(s) N_6(n/s).$$

After division by n, the terms are dominated by $2/s$, and $\sum_{s\in\langle 2,3\rangle} 1/s = 3$. Dominated convergence therefore gives the limit $\frac{1}{3}\sum_s \Delta R(s)/s = \frac{1}{3}\int_1^\infty R(t)t^{-2}\,dt$. For the $\mathcal{B}$-term, each interval $[\alpha, \beta)$ of $\mathcal{B}$ contributes $N_6(n/\alpha) - N_6(n/\beta)$. The intervals beyond any cutoff M contribute at most $n/M + 1$ in total. Hence $n^{-1}\#\{m : (m, 6) = 1,\ n/m \in \mathcal{B},\ m \ge 1\} \to \frac{1}{3}\int_{\mathcal{B}} t^{-2}\,dt$. Finally, by (2) and $\int_1^\infty \lfloor \log_b t \rfloor\, t^{-2}\,dt = \sum_{k\ge 1} b^{-k} = 1/(b-1)$,

$$\frac{1}{3}\int_1^\infty \frac{R(t)}{t^2}\,dt = \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{2} + \int_{\mathcal{E}} \frac{dt}{t^2}\right) = \frac{11}{18} + \frac{1}{3}\int_{\mathcal{E}} \frac{dt}{t^2}.$$

□

Proof of (5). Write $\mathcal{E} = E_4 \cap E_3$ with $E_4 = \bigcup_j [2\cdot 4^j, 4^{j+1})$ and $E_3 = \bigcup_b [2\cdot 3^b, 3^{b+1})$, and $\mathcal{B} = B_3 \cap B_4$ with $B_3 = \bigcup_b [10\cdot 3^b, 12\cdot 3^b)$ and $B_4 = \bigcup_j [4^j, \frac{5}{4}4^j)$. For such an intersection, $\int t^{-2}\,dt$ is the sum over the jump points x of its indicator of $\pm 1/x$, positive where the set is entered and negative where it is left. At an endpoint of one factor at which the other factor is constant, the indicator jumps exactly when the point lies in the other factor.

The endpoints of E_4 and E_3 coincide only at $x = 2$, where both begin and $\mathcal{E}$ jumps once. The endpoints of B_3 and B_4 never coincide. Hence

$$\int_{\mathcal{E}} \frac{dt}{t^2} = \sum_{k\ge 0}\left(\frac{\mathbf{1}_{E_3}(2\cdot 4^k)}{2\cdot 4^k} - \frac{\mathbf{1}_{E_3}(4^k)}{4^k}\right) + \sum_{k\ge 0}\left(\frac{\mathbf{1}_{E_4}(2\cdot 3^k)}{2\cdot 3^k} - \frac{\mathbf{1}_{E_4}(3^k)}{3^k}\right) - \frac{1}{2},$$

$$\int_{\mathcal{B}} \frac{dt}{t^2} = \sum_{k\ge 0}\left(\frac{\mathbf{1}_{B_3}(4^k)}{4^k} - \frac{\mathbf{1}_{B_3}(\frac{5}{4}4^k)}{\frac{5}{4}4^k}\right) + \sum_{k\ge 0}\left(\frac{\mathbf{1}_{B_4}(10\cdot 3^k)}{10\cdot 3^k} - \frac{\mathbf{1}_{B_4}(12\cdot 3^k)}{12\cdot 3^k}\right).$$

Here the terms $\mathbf{1}_{E_3}(1)$ and $\mathbf{1}_{E_4}(1)$ vanish. Let $t = [4^k]_3$ and $s = [3^k]_4$. Elementary manipulation of mantissas gives

- $\mathbf{1}_{E_3}(2\cdot 4^k) = \mathbf{1}_{[t<3/2]}$ and $\mathbf{1}_{E_3}(4^k) = \mathbf{1}_{[t\ge 2]}$;
- $\mathbf{1}_{B_3}(4^k) = \mathbf{1}_{[10/9\le t<4/3]}$;
- $\mathbf{1}_{B_3}(\frac{5}{4}4^k) = \mathbf{1}_{[t<16/15 \text{ or } t\ge 8/3]}$ for $k \ge 1$, while for $k = 0$ the right side is 1 but $5/4 \notin B_3$;
- $\mathbf{1}_{E_4}(2\cdot 3^k) = \mathbf{1}_{[s<2]}$ and $\mathbf{1}_{E_4}(3^k) = \mathbf{1}_{[s\ge 2]}$;
- $\mathbf{1}_{B_4}(10\cdot 3^k) = \mathbf{1}_{[8/5\le s<2]}$ and $\mathbf{1}_{B_4}(12\cdot 3^k) = \mathbf{1}_{[4/3\le s<5/3]}$.

Multiplying by 60 therefore gives $60(I-J)=\Sigma_4+\Sigma_3-30-48$, where I,J are the two integrals and

$$\varphi(t)=301_{[t<3/2]}-601_{[t\ge2]}-601_{[10/9\le t<4/3]}+481_{[t<16/15\text{ or }t\ge8/3]},$$
$$\psi(s)=301_{[s<2]}-601_{[s\ge2]}-61_{[8/5\le s<2]}+51_{[4/3\le s<5/3]},$$

which are the functions of Table 1. Here $\Sigma_4=\sum_k\varphi([4^k]_3)4^{-k}$ and $\Sigma_3=\sum_k\psi([3^k]_4)3^{-k}$. Since $180\cdot11/18=110$, we obtain $180c=110+60(I-J)=32+\Sigma_4+\Sigma_3$. □

Remark 5.1. The value $c=0.67296\ldots$ lies in Lebensold's interval $[0.6725,0.6736]$ and above the lower bound $c\ge0.6729$ reported by Davis.

6. Returns of the rotation and a small nonzero linear form

6.1. **Rotation coordinates and jumps.** Let $\alpha=\log4/\log3=\log_3 4$, which is irrational because no positive power of 4 is a power of 3. For $k\in\mathbb{Z}$ we have $[4^k]_3=3^{\{k\alpha\}}$ and $[3^k]_4=4^{\{k/\alpha\}}$. Let $\widetilde{\varphi}(x)=\varphi(3^{\{x\}})$ and $\widetilde{\psi}(y)=\psi(4^{\{y\}})$ be the corresponding 1-periodic, right-continuous step functions. Their jumps on the circle, at the points θ listed through 3^θ and 4^θ respectively, are

3^θ	1	16/15	10/9	4/3	3/2	2	8/3
jump of $\widetilde{\varphi}$	+90	−48	−60	+60	−30	−60	+48

and

4^θ	1	4/3	8/5	5/3	2
jump of $\widetilde{\psi}$	+90	+5	−6	−5	−84.

Jump points related by the rotation are grouped into orbits. For $\widetilde{\varphi}$ we use the representatives $\theta_1,\ldots,\theta_4=\log_3(1,2,\frac{9}{5},\frac{10}{9})$, with the logarithm applied coordinatewise. Then

$$\log_3\tfrac43=\theta_1+\alpha-1,\qquad \log_3\tfrac32=\theta_2-\alpha+1,$$
$$\log_3\tfrac83=\theta_2+\alpha-1,\qquad \log_3\tfrac{16}{15}=\theta_3+2\alpha-3.$$

For $\widetilde{\psi}$ we use $\theta'_1,\ldots,\theta'_4=\log_4(1,2,\frac85,\frac54)$, with $\log_4\frac43=\theta'_1+1-1/\alpha$ and $\log_4\frac53=\theta'_4+1-1/\alpha$.

Weighting each jump by the power of 4 (respectively 3) corresponding to its orbit shift gives the orbit coefficients

$$(C_1,C_2,C_3,C_4)=(105,-168,-3,-60),\qquad (D_1,D_2,D_3,D_4)=(105,-84,-6,-15).$$

For example, $C_1=90+60/4$, $C_2=-30\cdot4-60+48/4$, $D_1=90+5\cdot3$ and $D_4=-5\cdot3$. Each list represents four distinct orbits for its own rotation. No two elements of either list of mantissas have ratio 4^a3^b with $a,b\in\mathbb{Z}$. The two lists share the representatives 1 and 2.

6.2. **Brackets.**

Lemma 6.1. *There are infinitely many quadruples of positive integers (p,q,P,Q) with*

$$qP-pQ=1,\qquad q\le Q,\qquad \delta:=q\alpha-p>0,\qquad 0<\delta':=P-Q\alpha<\delta.$$

Along them $Q\to\infty$ and $\delta\le1/Q\to0$. Put $L=P\log3$. Then $p\log3,q\log4<L$ and $Q\log4>L-\log3$. Moreover $\|m\alpha\|\ge\delta$ for $1\le m<Q$, and every block of $q+Q$ consecutive integers k contains one with $\{k\alpha-\theta\}\in[1-\delta,1)$, for every $\theta\in\mathbb{R}$.

Proof. Run the mediant (Stern–Brocot) process for α starting from $1/1<\alpha<2/1$. Each bracket $p/q<\alpha<P/Q$ satisfies $qP-pQ=1$, and the identity $q\delta'+Q\delta=qP-pQ=1$ holds. When the right endpoint is replaced by the mediant, Q becomes $q+Q\ge q$ and δ' decreases by δ. Thus at the end of every maximal run of right replacements we have $q\le Q$ and $0<\delta'<\delta$. Since α is irrational, there are infinitely many such runs, and $Q\delta\le1$. The inequalities involving L follow from $p/q<P/Q$, $Q\alpha<P<Q\alpha+1$ and $q\le Q$.

Given $1 \le m < Q$ and $n \in \mathbb{Z}$, write $(m,n) = x(q,p) + y(Q,P)$ with $x, y \in \mathbb{Z}$, which is possible because $qP - pQ = 1$. Then $m\alpha - n = x\delta - y\delta'$. Since $0 < m < Q$ and $q \le Q$, either $y = 0 < x$, or x and y are nonzero with opposite signs. In both cases $|x\delta - y\delta'| \ge \delta$. The last assertion is the three-distance theorem: the $q+Q$ points $\{k\alpha\}$ for k in a block of consecutive integers cut the circle into gaps of lengths δ and δ' only [25]. In this case the cyclic successor on the indices $0, \dots, q+Q-1$ adds q modulo $q+Q$, stepping by δ if the index increases by q and by δ' if it decreases by Q. The successor is one cycle because $\gcd(q,Q) = 1$, and the step lengths sum to $Q\delta + q\delta' = 1$. Thus every half-open arc of length δ contains a point. □

6.3. **The return identity.** For a bounded integer sequence (c_k), a base $\beta \in \{3,4\}$ and $m \ge 0$,

$$(\beta^m - 1)\sum_{k\ge 0} \frac{c_k}{\beta^k} = \sum_{k<m} c_k \beta^{m-k} + \sum_{k\ge 0} \frac{c_{k+m} - c_k}{\beta^k}. \tag{11}$$

Let $\varphi_k = \varphi([4^k]_3)$ and $\psi_k = \psi([3^k]_4)$. Since $q\alpha = p + \delta$ and $p/\alpha = q - \delta/\alpha$, we have $\varphi_{k+q} = \widetilde{\varphi}(k\alpha + \delta)$ and $\psi_{k+p} = \widetilde{\psi}(k/\alpha - \delta/\alpha)$. So the differences in (11) are small forward and backward rotations, respectively. Define the visit indicators

$$H_i(k) = \mathbf{1}_{[\{k\alpha - \theta_i\} \ge 1-\delta]}, \qquad K_i(k) = \mathbf{1}_{[\{k/\alpha - \theta_i'\} < \delta/\alpha]} \qquad (k \in \mathbb{Z}).$$

Thus $H_i(k) = 1$ when the step from $k\alpha$ to $k\alpha + \delta$ crosses θ_i, and $K_i(k) = 1$ when the step from k/α to $k/\alpha - \delta/\alpha$ crosses θ_i', with the endpoints as specified by the indicators. Counting crossings of jump points, and shifting indices within orbits, gives

$$\begin{aligned}
\varphi_{k+q} - \varphi_k &= 90H_1(k) + 60H_1(k-1) - 30H_2(k+1) - 60H_2(k) \\
&\quad + 48H_2(k-1) - 48H_3(k-2) - 60H_4(k), \\
\psi_{k+p} - \psi_k &= -90K_1(k) - 5K_1(k+1) + 84K_2(k) + 6K_3(k) + 5K_4(k+1).
\end{aligned}$$

Always $H_1(0) = 0$ and $K_1(0) = 1$. For every other fixed (i,k), the number $k\alpha - \theta_i$ (respectively $k/\alpha - \theta_i'$) is not an integer, so the corresponding $H_i(k)$ and $K_i(k)$ are zero for all sufficiently late brackets. Inserting these expansions into (11), and regrouping the finitely many boundary terms, proves the following.

Proposition 6.2 (Return identity)**.** *Let*

$$A_q = \sum_{k<q} \varphi_k 4^{q-k}, \qquad B_p = \sum_{k<p} \psi_k 3^{p-k}, \qquad N = (3^p - 1)A_q + (4^q - 1)(B_p - 90) \in \mathbb{Z}.$$

For all sufficiently late brackets,

$$\begin{aligned}
&(4^q - 1)(3^p - 1)(180c - 32) - N \\
&\quad = (3^p - 1)\sum_{i=1}^{4} C_i \sum_{k\ge 1} \frac{H_i(k)}{4^k} - (4^q - 1)\sum_{i=1}^{4} D_i \sum_{k\ge 1} \frac{K_i(k)}{3^k}.
\end{aligned} \tag{12}$$

The constant -90 comes from the backward visit $K_1(0) = 1$ at index zero, whose step crosses the jump $+90$ of $\widetilde{\psi}$ at 0. The forward convention excludes the corresponding forward visit, since $H_1(0) = 0$.

6.4. **Events and two cutoffs.** Expanding the right side of (12) writes it as a sum of events (d,u), each consisting of an integer coefficient d and a $\{2,3\}$-unit $u = e^{-\tau}$ with time τ. A visit $H_i(k) = 1$ produces the events $(C_i, 3^p/4^k)$ and $(-C_i, 4^{-k})$, with times $k\log 4 - p\log 3$ and $k\log 4$. A visit $K_i(k) = 1$ produces $(-D_i, 4^q/3^k)$ and $(D_i, 3^{-k})$, with times $k\log 3 - q\log 4$ and $k\log 3$. There are sixteen event families, given by four unit types times four orbits. For a real T, let $S(T)$ be the sum of du over the events with time $\tau < T$.

Lemma 6.3. *For all sufficiently late brackets:*

(i) *Within each family, distinct event times differ by at least* $L/2$.
(ii) *Distinct visits* $H_i(k) = H_j(k') = 1$, *with* $(i,k) \neq (j,k')$, *satisfy* $|k-k'| \geq 2$.
(iii) *There are cutoffs* $4L < T_1 < 5L$ *and* $10L < T_2 < 11L$ *such that no event time lies in* $[T_j, T_j + L/100]$.
(iv) *For* $T \in \{T_1, T_2\}$, *the right side of* (12) *differs from* $S(T)$ *by at most* $e^{-T-L/200}$.

Proof. For (i), two visits $k < k'$ to the same arc of length δ satisfy $\|(k'-k)\alpha\| < \delta$, so $k'-k \geq Q$ by Lemma 6.1, and $Q \log 4 \geq L/2$. The backward families are similar. The inequality $\|(k'-k)/\alpha\| < \delta/\alpha$ gives a positive integer n for sufficiently small δ with $|(k'-k) - n\alpha| < \delta$. Hence $n \geq Q$ and $k'-k > n\alpha - 1$, so $(k'-k)\log 3 > L - 2\log 3 \geq L/2$.

For (ii), otherwise $\pm\alpha$ or 0 would be within δ of $\theta_i - \theta_j$ modulo 1, for one of finitely many pairs. For $i \neq j$ this fails once δ is small, because the representatives lie in distinct orbits. For $i = j$ and $|k-k'| = 1$, the irrationality of α gives the same exclusion.

For (iii), by (i) each family has at most three events in a window of length $1.01L$, so at most 48 events are relevant. Among the 49 disjoint candidate intervals

$$[4L + \ell L/50, 4L + \ell L/50 + L/100], \qquad 1 \leq \ell \leq 49,$$

at least one is free of them. The choice of T_2 is the same. For (iv), the omitted events have times at least $T + L/100$, spaced by $L/2$ within each family, and all coefficients satisfy $|d| \leq 168$. So the omitted part is at most

$$16 \cdot 168\, e^{-T-L/100}(1 - e^{-L/2})^{-1} \leq e^{-T-L/200}$$

for large L. □

Proposition 6.4 (Two truncations differ)**.** *For all sufficiently late brackets,* $S(T_1) \neq S(T_2)$.

Proof. Write $S = D + G$, where D collects the events with denominator 4^k and G those with denominator 3^k. Every $G(T)$ is 2-adically integral. In $D(T)$, the two events of a visit $H_i(k) = 1$ combine to a term with numerator $C_i(3^p - 1)$ if both are retained, and $C_i 3^p$ if only the earlier one, $3^p/4^k$, is retained. Note that $(v_2(C_1), v_2(C_2), v_2(C_3), v_2(C_4)) = (0, 3, 0, 2)$.

At $T_1 < 5L$, every retained dyadic event has $k\log 4 < 5L + p\log 3 \leq 6L$, so either $D(T_1) = 0$ or $v_2(D(T_1))\log 2 > -6L$. At T_2, Lemma 6.1 gives a visit k with $T_2 - 2L < k\log 4 < T_2$. Its events are retained. Indeed, the open interval for k has length $2L/\log 4 > 2Q \geq q + Q$. Let k^* be the largest retained dyadic index. Then $k^* \log 4 > 8L$, and by Lemma 6.3(ii) all other retained indices are at most $k^* - 2$.

If the unshifted event at k^* is retained, all retained indices have both events retained. Then $D(T_2) = (3^p - 1)\sum C_i 4^{-k}$, where the term with $k = k^*$ has strictly smallest valuation. Using $v_2(3^p - 1)\log 2 \leq p\log 3 \leq L$ gives $v_2(D(T_2))\log 2 \leq L + 3\log 2 - k^*\log 4 < -6L$. Otherwise the k^* term has numerator $C_i 3^p$ and again has strictly smallest valuation, so $v_2(D(T_2))\log 2 \leq 3\log 2 - k^*\log 4 < -6L$. In both cases $D(T_2) - D(T_1)$ has valuation less than $-6L/\log 2 < 0$. It cannot be cancelled by the 2-adically integral number $G(T_2) - G(T_1)$. □

Corollary 6.5. *Suppose* $c \in \mathbb{Q}$, *and put* $z = 180c - 32$. *For all sufficiently late brackets there are* $T \in \{T_1, T_2\}$, *an integer* $m \leq 404$, $\{2,3\}$*-units* $u_1, \ldots, u_m \geq e^{-T}$ *and coefficients* $d_1, \ldots, d_m$ *in the finite set* $\{\pm z, \pm 3, \pm 6, \pm 15, \pm 60, \pm 84, \pm 105, \pm 168\}$ *such that*

$$E := \sum_{i=1}^{m} d_i u_i - N \quad \textit{satisfies} \quad 0 < |E| \leq e^{-T-L/200} \quad \textit{and} \quad H(N : u_1 : \cdots : u_m) \leq e^{27L}.$$

Proof. Let $E(T) = (4^q - 1)(3^p - 1)z - N - S(T)$. By Proposition 6.4, $E(T_1) - E(T_2) = S(T_2) - S(T_1) \neq 0$, so some $E(T)$ is nonzero, and by (12) and Lemma 6.3(iv) it is at most $e^{-T-L/200}$ in absolute value.

Expanding the head gives the units $4^q3^p, 4^q, 3^p, 1$ with coefficients $z, -z, -z, z$. The retained events contribute their units with the negatives of their coefficients. Each family has at most $2\cdot 11+3$ retained events because the times lie in $[-L, 11L)$ and are $L/2$-separated. Thus $m \le 4+16\cdot 25 = 404$. All denominators divide $4^{\lfloor 12L/\log 4\rfloor}3^{\lfloor 12L/\log 3\rfloor} \le e^{24L}$, and all coordinates, including N, are at most $366e^{2L}$ in absolute value. Hence $H(N : u_1 : \cdots : u_m) \le 366e^{26L} \le e^{27L}$ for large L. □

7. A three-place Subspace Theorem and the proof of irrationality

7.1. **The covering theorem.**

Theorem 7.1. *Let $d \ge 2$, let $A \in \mathrm{GL}_d(\mathbb{Q})$, and let $c_{v,i}$ ($v \in \{\infty, 2, 3\}$, $1 \le i \le d$) be real numbers with $c_{2,i}, c_{3,i} \le 0$ and $\sum_{v,i} c_{v,i} < 0$. There is a finite set of proper linear subspaces of $\mathbb{Q}^d$ such that, for every real $q \ge 0$, every $x \in \mathbb{Z}^d$ with*

$$|(Ax)_i| \le e^{c_{\infty,i}q}, \qquad |x_i|_2 \le e^{c_{2,i}q}, \qquad |x_i|_3 \le e^{c_{3,i}q} \qquad (1 \le i \le d) \tag{13}$$

lies in one of them.

Remark 7.2. Theorem 7.1 is a special case of the p-adic Subspace Theorem of Schlickewei [21], which extends Schmidt's theorem [22, 23]. See also [24, 4, 10, 9]. Indeed, let $M = \max(0, \max_i c_{\infty,i})$ and $\rho = -\sum c_{v,i}$. If $M = 0$ the solutions are bounded, hence finite in number. Otherwise, for $C \ge 1$ sufficiently large we have $\max_i |x_i| \le Ce^{Mq}$, so the solutions of (13) with $\max_i |x_i| \ge C^2$ satisfy

$$\prod_{v\in\{\infty,2,3\}} \prod_{i=1}^{d} |L_{v,i}(x)|_v \le e^{-\rho q} \le \left(\max_i |x_i|\right)^{-\rho/(2M)},$$

with $L_{\infty,i}(x) = (Ax)_i$ and $L_{2,i}(x) = L_{3,i}(x) = x_i$. This is the inequality of the Subspace Theorem.

For a rational linear form $\ell(x) = \sum_i a_i x_i$ with $a_k \ne 0$, put

$$\Phi_{\ell,k}(x) = |\ell(x)| \prod_{i\ne k} |x_i| \prod_i |x_i|_2 |x_i|_3.$$

Theorem 7.3 (One-form covering)**.** *Let $n \ge 2$, ℓ and k be as above, and $\varepsilon > 0$. There are finitely many nonzero $a^{(1)}, \dots, a^{(s)} \in \mathbb{Q}^n$ such that every nonzero $x \in \mathbb{Z}[1/6]^n$ with $\Phi_{\ell,k}(x) < H(x)^{-\varepsilon}$ satisfies $a^{(j)} \cdot x = 0$ for some j.*

Proof. Multiplying x by 2^a3^b changes neither $\Phi_{\ell,k}$ nor H, so we may assume $x \in \mathbb{Z}^n$. Dividing x by its gcd g multiplies $\Phi_{\ell,k}$ by $g_0^{-n} \le 1$, where g_0 is the part of g prime to 6, and preserves H. So we may assume x primitive. Vectors with a zero coordinate, or with $\ell(x) = 0$, lie in coordinate hyperplanes or in $\ker \ell$, and primitive vectors of bounded height are finitely many.

For the rest put $q = \log H(x)$. The logarithms of the $3n$ local factors, divided by q, lie in a bounded range and sum to less than $-\varepsilon$. For the infinite place, the nonzero rational form value is bounded below by the reciprocal of a fixed common denominator of its coefficients and bounded above by a constant times $H(x)$. At 2 and 3, a nonzero integer coordinate has p-adic absolute value between $H(x)^{-1}$ and 1. Rounding the normalized logarithms up to a grid of mesh $\varepsilon/(6n)$ gives one of finitely many profiles c, each with $\sum c \le -\varepsilon/2$ and $c_{2,i}, c_{3,i} \le 0$, such that x satisfies (13) for the matrix A obtained from the identity by replacing row k with $(a_1, \dots, a_n)$. Apply Theorem 7.1 to each profile, and replace each proper subspace by a hyperplane containing it. □

7.2. **Rigidity of small linear forms.**

Proposition 7.4. *Let $m \ge 1$, $d \in \mathbb{Q}^m$ and $\varepsilon > 0$. There is $C > 0$ such that for all $N \in \mathbb{Z}$, all positive $\{2,3\}$-units $u_1, \dots, u_m$, and all $0 < t \le \min(1, u_1, \dots, u_m)$,*

$$E = \sum_{i=1}^{m} d_i u_i - N \ne 0 \quad \Longrightarrow \quad |E| \ge CtH(N : u_1 : \cdots : u_m)^{-\varepsilon}.$$

The same holds for pure forms $E = \sum d_i u_i$, with $H(u_1 : \cdots : u_m)$.

Proof. Use induction on m, first for pure forms and then for mixed forms. For a pure form with $m = 1$, $|E| = |d_1|u_1 \geq |d_1|t$, and for a mixed form with no units, $|E| = |N| \geq 1$. For a pure form with $m \geq 2$ and $E \neq 0$, pick k with $d_k \neq 0$. By the product formula $|u|\,|u|_2|u|_3 = 1$ for units, so $\Phi(u) = |E|/u_k$. If $\Phi(u) \geq H(u)^{-\varepsilon}$, then $|E| \geq u_k H(u)^{-\varepsilon} \geq tH(u)^{-\varepsilon}$. Otherwise, by Theorem 7.3, u lies on one of finitely many hyperplanes $a \cdot u = 0$. Solving for a coordinate with $a_{i_0} \neq 0$ writes E as a pure form in the remaining $m-1$ units with one of finitely many coefficient vectors. The value E is unchanged and the height of the shorter vector is at most $H(u)$, so the induction hypothesis applies.

For a mixed form, take N as the distinguished coordinate. If $N = 0$, we are in the pure case. Otherwise $\Phi(N, u) = |E|\,|N|_2|N|_3 \leq |E|$, which gives the bound off the exceptional hyperplanes. On a hyperplane $a_0 N + \sum a_i u_i = 0$, if $a_0 \neq 0$, eliminate N and obtain a pure form in the same units. If $a_0 = 0$, eliminate a unit and obtain a mixed form with $m - 1$ units. Finitely many coefficient vectors occur, and C is the minimum of the resulting positive constants. □

7.3. **Proof of Theorem 1.3.**

Proof. Suppose $c \in \mathbb{Q}$. By Corollary 6.5 and the pigeonhole principle, there are an infinite sequence of brackets with $L \to \infty$, a fixed $m \leq 404$, and a fixed coefficient vector d, for which $0 < |E| \leq e^{-T-L/200}$ and $H(N : u) \leq e^{27L}$. Apply Proposition 7.4 with $\varepsilon = \varepsilon_0 = 1/10800$ and $t = e^{-T}$. This is allowed because every unit is at least e^{-T}. Since $27\varepsilon_0 = 1/400$,

$$Ce^{-T-L/400} \leq Ce^{-T}H(N : u)^{-\varepsilon_0} \leq |E| \leq e^{-T-L/200},$$

so $C \leq e^{-L/400}$ for all L in an unbounded set. This is impossible. Hence c is irrational. □

7.4. **On the proof of Theorem 7.1 in the formalization.** The formalization proves Theorem 7.1 from scratch, along the classical route of Roth and Schmidt as organized by Evertse and Ferretti [9] and in the exposition of Goel, Lunia and Ray [12]. Restricting to rational coefficients and to coordinate forms at 2 and 3 avoids number fields and adelic geometry. The main steps are the following.

(1) *Lattice model.* Let $u_i = \lceil -c_{2,i}q/\log 2 \rceil$ and $v_i = \lceil -c_{3,i}q/\log 3 \rceil$. The solutions of (13) are the points of the lattice $\Lambda_q = \operatorname{diag}(2^{u_i}3^{v_i})\mathbb{Z}^d$ in the box $\|\operatorname{diag}(e^{-c_{\infty,i}q})Ax\|_\infty \leq 1$. The finite places thus enter only through divisibility.

(2) *Successive minima.* For every full-rank lattice $\Gamma \subset \mathbb{R}^n$ the Euclidean successive minima satisfy $\operatorname{covol}(\Gamma)/n! \leq \prod_i \lambda_i \leq (2/\sqrt{3})^{n(n-1)/2} \operatorname{covol}(\Gamma)$. The upper bound comes from a non-algorithmic reduced basis in the spirit of Lenstra–Lenstra–Lovász [16].

(3) *A gap and Davenport's lemma.* For large q with a nonzero solution, the covolume forces an index k with $\lambda_{k+1}/\lambda_k > e^{\gamma q}$, where $\gamma = \rho/(2d(d-1))$. A rounding lemma of Davenport type produces an integral basis $h_1, \ldots, h_d$ of Λ_q with good local bounds such that all solutions lie in $\operatorname{span}(h_1, \ldots, h_k)$.

(4) *Compound reduction.* Passing to the $(d-k)$th compound matrix, the $\binom{d}{k} - 1$ vectors complementary to the prefix satisfy a three-place system of the same shape, with a profile of negative sum. So it suffices to show that the hyperplanes they span take only finitely many values. A height-free recovery map returns from compound hyperplanes to the prefix subspaces.

(5) *Height dichotomy.* A hyperplane spanned by solutions at parameter q either belongs to a finite exceptional family, indexed by support patterns, or has a normal vector of logarithmic height at least $\rho q/4 - O(1)$. The proof uses only the elementary inequality $|z|\,|z|_2|z|_3 \geq 1$ for nonzero integers z.

(6) *Roth's lemma.* For $\beta = a/b \in \mathbb{Q}$ in lowest terms let $h(\beta) = \log \max(|a|, |b|)$, and let $\|P\|_1$ be the sum of the absolute values of the coefficients of P. If $0 \neq P \in \mathbb{Z}[t_1, \dots, t_m]$ has $\deg_{t_i} P \leq r_i$, $r_{i+1} \leq \theta r_i$ with $\theta = \eta^{2^{m-1}}$ and $0 < \eta \leq \frac{1}{2}$, and $\log \|P\|_1 + 4mr_1 \leq \theta r_i h(\beta_i)$ for all i, then the index of P at $\beta \in \mathbb{Q}^m$ with weights r is at most $2m\eta$. The proof uses generalized Wronskians, as in [20, 12], and is extended to products of rational hyperplanes.

(7) *Auxiliary polynomial.* Siegel's lemma, in the form available in Mathlib, together with an exact second-moment count of monomials, produces a nonzero integer multihomogeneous polynomial that is balanced with respect to the forms at all three places and has controlled coefficients.

(8) *Contradiction.* Suppose solutions spanned hyperplanes of heights growing linearly in q along an unbounded set of parameters. Choose m well-separated parameters and degrees $r_h \approx D/q_h$. Roth's lemma and a grid nonvanishing lemma yield an integer point at which a low-order derivative of the auxiliary polynomial is a nonzero integer. Its three local sizes multiply to less than 1, which is impossible. With the dichotomy, only finitely many hyperplanes occur for each profile, and finitely many profiles suffice.

(9) *Assembly.* The recovery map gives finitely many prefix subspaces, and bounded parameters give finitely many integer points. This proves Theorem 7.1.

The argument is ineffective. Thus Theorem 1.3 provides no irrationality measure for c.

8. The Lean formalization

8.1. **The statement.** The file `FormalConjectures/ErdosProblems/1062.lean` in Formal Conjectures [11] defines fork-freeness and the extremal function by

```
def ForkFree (A : Set ℕ) :  Prop :=
  ∀a ∈ A, ({b | b ∈ A \ {a} ∧ a | b} :  Set ℕ).Subsingleton

noncomputable def f (n :  ℕ) :  ℕ :=
  open scoped Classical in
  Nat.findGreatest (fun k => ∃ A ⊆ Set.Icc 1 n,
    ForkFree A ∧ A.ncard = k) n
```

and states the problem as

```
theorem erdos_1062.parts.ii :
  (∃ l, Tendsto (fun n => (f n :  ℝ) / n) atTop (𝓝 l)
    ∧ Irrational l) ↔ answer(sorry)
```

The conjectures.io task [5] asks for a proof of this statement with the answer `True`, as the declaration `Bounty.target`. The original file ends with

```
theorem target : fcTypeOfName% "Erdos1062.erdos_1062.parts.ii" :=
  target_of_integer_oneForm_cover
    (integer_oneForm_cover_of_three_place
      rational_three_place_subspace_cover)
```

The development first shows that `Nat.findGreatest` computes the maximum size, and it never uses the imported open statement as a proof.

8.2. **Organization.** The file has 74,210 lines and 7,702 declarations. It is assembled from 368 components, each labelled in the source with the SHA-256 hash of the module it came from. Table 2 shows how the components correspond to the sections of this paper.

8.3. **Certifying the finite computations.** The base-range table is stored as 217 affine pieces and checked on the 611 refined intervals. Each check is a closed proposition about rational numbers, settled by kernel evaluation (`decide +kernel`).

For the closure, the failure system in Lemma 4.4 is stated in Lean as an explicit proposition. A lemma derives it from the corresponding depth-seven hypothesis, and that hypothesis

TABLE 2. Components of the formal proof.

Content	Representative components	Lines
Grid theorem, upper bound, construction, recursion, bridge to the closure (§§ 2–4)	`GridBridge`, `GlobalUpper`, `CutoffRecursion`, `ClosureInduction`, `GlobalExact`	7,816
Base-range table and its checks (§ 4)	`BaseTableData`, `PolicyBaseIntervalsCompact`	9,976
Closure certificate and its checkers (Lemma 4.4)	`PolicyBitsetRUP`, `PolicyLinearCertificate`, `ClosureInputReflection`	32,649
Density, series, returns, nonvanishing, rigidity (§§ 5–7)	`CoefficientSeries`, `ReturnDifferenceIdentity`, `ReturnTruncationNonzero`, `OneFormRigidity`	8,469
Roth's lemma (§ 7.4, step 6)	`GeneralizedWronskian`, `IntegerRothTheorem`	3,568
Theorem 7.1 (§ 7.4)	`LatticeSuccessiveMinima`, `BalancedAuxiliaryPolynomial`, `RationalThreePlaceSubspaceTheorem`	11,732

is proved, in turn, to follow from a failure of the actual recursion. Thus the correctness of the encoding itself is proved. The relevant declarations are `ClosureRawHypothesis5` and `closureRawHypothesis5_of_raw`.

The formula is a typed expression DAG over the rationals. A refutation found with Z3 was decomposed into four kinds of steps:

- Boolean circuit clauses defining the DAG's gates;
- instances of 190 rewrite schemas, each proved once in Lean;
- arithmetic lemmas, each justified by a sparse Farkas certificate with rational coefficients;
- a propositional refutation of the resulting clause set in reverse unit propagation (RUP) form [13, 26], produced with the SAT solver Glucose [1].

The arithmetic lemmas were reduced by a deletion-based minimal unsatisfiable subset search, and solver seeds were chosen to minimize the kernel cost of the whole certificate. Every checker is an ordinary Lean function with a soundness theorem, and every certificate check is closed by kernel evaluation. The depth-five closure has 21 nodes and 59 assertions, against 36 and 102 at depth seven. The smaller depth roughly halves the certificate, which was necessary to meet the time limit.

8.4. **Constraints of the challenge and verification.** For the pinned conjectures.io task, the submission is the body of a single file placed between a trusted header, which imports the Formal Conjectures statement and a small support library, and a trusted footer. The body may not contain imports, `set_option`, attributes, instances, custom syntax, `native_decide` or axioms. It must fit in 10 MiB, and building, exporting and replaying it must finish within the task's verification timeout of 3,600 seconds. The kernel may use only `propext`, `Quot.sound` and `Classical.choice`.

Because declaration attributes and instance commands are unavailable, the development passes the relevant lemmas and instances explicitly. Every proof except the final one is declared with `def` in place of `theorem`. Lean elaborates theorem bodies as parallel tasks. In our measurements each pending task reserved about 1 GiB of address space, which for thousands of declarations would exceed the checker's 64 GiB address-space limit. Definitions are elaborated sequentially.

The submitted body has 4,965,602 bytes and 752,155 tokens as counted by the challenge's static scanner, whose limit is 10^6. Its SHA-256 hash begins `2ebc87b0`. We ran the verification pipeline locally with the toolchain pinned by the challenge (Lean 4.33.1). The build (`lake build`) took 1,647 s of wall time (1,872 s of CPU time, 11.8 GB resident memory). Exporting the challenge and the solution with `lean4export` [18] took 7 s and 51 s. Replaying the exported solution in a fresh kernel with the comparator [17] took 1,362 s. The total is 3,067 s. The comparator confirmed

that the target statement matches the challenge, and that the proof uses only the three permitted axioms. The file contains no `sorry`.

8.5. **Trusted base and public reproduction.** A reader who accepts the formal proof must trust the Lean kernel, the Formal Conjectures statement, the challenge's header and footer, and the exporter and comparator used for the challenge replay. The SMT and SAT solvers, the Python scripts that produced the tables and certificates, and all intermediate data are untrusted in this verification: their outputs are checked in the kernel.

The later public prize submission links the explicit finite-formula and limit wrapper, with stated endpoint `Erdos1062Full.full_scope`, and the corresponding clean build and target axiom record. The wrapper exposes the exact arithmetic identity for every natural n, including the empty sum for $n = 0$, together with convergence and irrationality.

9. Concluding remarks

(1) Theorem 1.1 shows that the extremal structure is governed by the two-prime grid and one family of forced conflicts between adjacent powers-of-five layers. The finite closure behind the lower bound is small at depth five, with 21 nodes and two real parameters. A short conceptual proof of Lemma 4.4 would make the theorem computer-free.
(2) The irrationality proof never uses irrationality of the two series in (5) separately, and it needs no assumption on the continued fraction of $\log 4/\log 3$. It uses the Subspace Theorem only for linear forms in $\{2, 3\}$-integers, and no finiteness theorem for S-unit equations.
(3) Davis's theorem [6] shows that the extremal densities for other finite families of forbidden divisibility configurations are computable as well. The method of this paper, namely exact grid analysis followed by returns of a rotation and the Subspace Theorem, may decide irrationality in some of those cases.

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