A PERIODIC-SUMMAND CHARACTERIZATION FOR ERDŐS PROBLEM 335

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Abstract. Erdős and Graham asked which pairs of positive-density sets of positive integers have a sumset whose ordinary density equals the sum of their densities. The formalization gives an unrestricted characterization. A numerical boundary case is controlled by literal modular supports. In the strict-gap case, equality forces parallel Bohr intervals with finite backgrounds, two periodic templates with a literal fringe test, or one sparse exceptional section with an exact terminal condition. This overview presents the statement and proof structure. The accompanying mathematical notes give the detailed arguments, and the Lean project contains the complete formal proof.

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1. Introduction

Throughout, A and B are subsets of the positive integers, and both ordinary densities exist and are positive. All sumsets use the actual elements of the sets. For a positive integer q, let $R_q(A)$ be the set of residues represented by A. Put

$$d(A) = \lim_{N\to\infty} \frac{|A \cap [1,N]|}{N}, \qquad s = d(A) + d(B), \tag{1.1}$$

$$R_q(A) = \{a \bmod q : a \in A\}, \qquad \gamma = \inf_{q\geq 1} \frac{|R_q(A) + R_q(B)|}{q}. \tag{1.2}$$

Theorem 1.1 (Classification)**.** *The equality under study is*

$$d(A+B) = d(A) + d(B). \tag{1.3}$$

Equality always holds at the modular boundary:

$$\gamma = s \quad \Longrightarrow \quad d(A+B) = s. \tag{1.4}$$

If $\gamma < s$, equality is impossible. When $s < \gamma$, equality holds precisely when at least one of the three structural alternatives in Section 2 holds. The alternatives retain the literal exceptional elements of A and B.

2. The three structural alternatives

Parallel Bohr intervals with finite backgrounds. Choose a modulus h, the actual supports $S = R_h(A)$ and $T = R_h(B)$, and distinguished residues $i \in S$ and $j \in T$. Every background section is full up to density zero. The distinguished output residue has a unique expression in the quotient, and the support count satisfies

$$|S+T| = |S| + |T| - 1, \qquad \{(r,t) \in S \times T : r + t = i + j\} = \{(i,j)\}. \tag{2.1}$$

After rescaling the distinguished sections, each is a literal subset of a closed-arc pullback under the same irrational circle rotation. Each fills its pullback up to density zero. The two arc lengths are positive and have sum less than one. The modulus $h = 1$ is allowed.

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Two periodic templates with literal fringes. Write $P_q(S)$ for the positive integers whose residue modulo q lies in S. The notation $\sim_d$ means that the symmetric difference has ordinary density zero. There are nonempty masks S and T such that

$$A \sim_d P_q(S), \qquad B \sim_d P_q(T). \tag{2.2}$$

The baseline formed from the actual supports and the two masks is

$$W = (R_q(A) + T) \cup (S + R_q(B)), \qquad |W| = |S| + |T|. \tag{2.3}$$

For every output residue outside W, the sums of the literal fringes $A \setminus P_q(S)$ and $B \setminus P_q(T)$ satisfy the vanishing interval-packing test described in Section 3. The masks can be proper subsets of the actual supports.

One sparse exceptional section. Use the same finite background frame and unique-expression condition as in the first alternative. One exceptional section U is nonempty and has density zero. Its partner V has density x strictly between zero and one. The two roles may be exchanged. Normalize the sparse section at its least element:

$$u_0 = \min U, \qquad D = \{u - u_0 : u \in U\}. \tag{2.4}$$

If U is a singleton, the terminal condition holds. Otherwise $g = \gcd(D) > 0$ and $V + g \sim_d V$. For finite U this is the remaining terminal condition. For infinite U, the literal sums of D/g with each rescaled g-residue section of V must satisfy the escape-packing test outside that section.

3. The literal interval-packing test

The terminal tests operate on runs of consecutive block indices. They keep initial elements, singleton runs and sparse sections. A residue section and the carry in a sum are given by

$$E_{q,r} = \{k \geq 0 : qk + r \in E\}, \qquad \kappa(r,t) = \left\lfloor \frac{r+t}{q} \right\rfloor, \qquad 0 \leq r, t < q. \tag{3.1}$$

Take consecutive source runs $[a,b]$ and $[c,d]$. The represented output indices fill the complete integer interval

$$[a,b] + [c,d] + \kappa = [a + c + \kappa, b + d + \kappa]. \tag{3.2}$$

Clip every interval to the first N indices, including index zero. In the two-template alternative, collect the intervals in each output residue outside W. In the sparse alternative, intersect them with complementary runs of the relevant section of V. Only finitely many clipped intervals contribute.

Packing size. For a finite interval family $\mathcal{I}_N$, let $P(\mathcal{I}_N)$ be the greatest total number of integer points in a pairwise disjoint subfamily. The required test is $P(\mathcal{I}_N) = o(N)$, separately in every relevant residue. The elementary comparison is

$$P(\mathcal{I}_N) \leq \left| \bigcup \mathcal{I}_N \right| \leq 3P(\mathcal{I}_N). \tag{3.3}$$

The comparison shows that the test is equivalent to zero density of the designated escaping indices. To prove the upper bound, select a longest remaining interval and remove every interval meeting it. Each removed interval lies in the selected interval enlarged by at most its length on each side. The selected intervals are disjoint, which gives the factor three.

Why literal data matter. A sparse modification of an input can create a whole new residue class in its sumset. The characterization therefore retains the actual supports and tests the exceptional sums. In the infinite sparse case, invariance under each fixed translate is supplemented by a test on the full union of escaping sums.

The numerical case. The parameter γ is the upper Buck density of $A + B$. If $\gamma = s$ and the lower ordinary density were smaller than s, the integer Kneser theorem [2] would give an ultimately periodic output with ordinary density equal to its Buck density. This contradicts the supposed deficit and proves the boundary case.

4. Proof structure in the strict-gap case

Let $b^*(E)$ denote the upper Buck density of E, and set

$$\Delta = \gamma - s, \quad \delta_A = b^*(A) - d(A), \quad \delta_B = b^*(B) - d(B). \tag{4.1}$$

Local structure and the choice of scales. A zero input gap yields a periodic-in-density summand through the compact inverse argument. When both input gaps exceed Δ, the matched local inverse arguments give the parallel-Bohr alternative. In the remaining case, bounded-index local inverse theory makes one actual input periodic in each limiting window.

The global step starts with joint Lipschitz limits of cumulative residue counts. A common subsequence and a countable diagonal construction provide Fourier models at every fixed ambient enlargement. The window scale dominates all approximation errors:

$$H_k \to \infty, \qquad H_k/N_k \to 0, \qquad \mathrm{error}_k\,/H_k \to 0. \tag{4.2}$$

From local labels to one periodic summand. At almost every positive macroscopic time, derivatives of the cumulative limits give a finitely labeled profile. The labels record the periodic masks and saturation of the fixed-translate baselines. Literal translations between windows give the cross-time inequality

$$c_{r+t}(x+y) \geq \max\{a_r(x), b_t(y)\} \quad \text{if } a_r(x) > 0,\ b_t(y) > 0. \tag{4.3}$$

Two times with the same label have that label at their sum, almost everywhere. A bounded positive-measure piece of a label has a self-convolution positive on an interval. Repeated addition gives a tail. Disjoint labels therefore cannot both have positive measure, so a single label holds almost everywhere on the positive axis.

The normalized prefix-profile sequence has vanishing successive increments. Its cluster points lie in finitely many closed slices, and the label rules separate those slices. The sequence therefore selects one slice. One input coordinate converges to a fixed periodic mask, giving an ordinary periodic-in-density summand.

Exact reduction and sufficiency. Once one summand is periodic in density, use its mask and the other input's actual residue support to form the fixed-translate baseline. If that baseline fills the output density, both inputs have periodic templates. If the baseline has smaller density, the reduction gives the finite background frame with one null section. Finite sparse sets use gcd-shift invariance and finite unions. Infinite sparse sets use the escape test to give $d((D+V)\setminus V) = 0$. The Bohr alternative uses literal arc containment, equidistribution and the Kneser lower bound. Recombining the background and active sections proves the converse in every alternative.

The mathematical source package provides the reading order and the detailed supporting arguments. The classification proof links its local inverse arguments and exact reductions. The expanded gluing proof gives the common scale selection, comparison of windows and passage to one ordinary periodic summand.

5. Formal verification and provenance

The complete formal result is the closed theorem

`Erdos335.Proposed.full_periodic_classification`.

Its type is `PeriodicRefinedSolution`. The theorem

`Erdos335.Proposed.full_periodic_classification_original_positive_counts`

writes the same classification using the original counts on $\{1, \ldots, N\}$.

Check	Recorded result
New supporting modules	45 compiled
Exact declaration and axiom audit	2,602 targets passed
Allowed proof axioms	`propext`, `Classical.choice`, `Quot.sound`
Fresh full-closure kernel replay	Passed in 1,460.406 seconds
Project object files bound	890
Source, object and runtime hashes	Unchanged across replay

The runtime is Lean 4.34.0, with Mathlib revision `5ed2965256430c3649e86755f9576b54eca72435`. The checked proof commit is `63e4377a3f62cdf4a989b2559d8f9c7341c78789`. The checker ran `leanchecker --fresh -v Challenge`. The verification record includes the proof revisions, logs and reproduction instructions. The original overview accompanies Justin Sun Prize submission JSP-000278.

The prize submission credits OpenAI Codex assistance in the Lean formalization. The original overview reports that its literature search found no earlier matching full characterization in the public record it checked. Publication priority and Justin Sun Prize acceptance are submitted for independent review.

References

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