VERTEX-CRITICAL FOUR-CHROMATIC GRAPHS WITH NO SMALL CRITICAL EDGE SET

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Abstract. We construct, for every positive integer r, a finite four-chromatic graph whose every vertex is critical, while deleting any set of at most r edges leaves its chromatic number unchanged. The construction combines punctured cyclic three-colorings, an integral height character, an Andrásfai winding inequality, and a Chinese remainder construction with an exact classification of its small-deficit characters. Together with the established result of Skottová and Steiner for five or more colors, this gives an affirmative answer to the existence statement in Erdős problem 944. The complete existence statement for all $k \geq 4$ is formalized in one Lean file with 570 proved theorems. This includes the four-color parameter construction and character argument, and the higher-color construction, local structure, and propagation proof. The exact challenge theorem is checked without admitted proof placeholders or additional axioms.

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1. Introduction

An edge set of a graph is critical if deleting it lowers the chromatic number. A graph is vertex-critical if deleting any vertex lowers its chromatic number. We use the terminology (k, r)-graph for a finite graph G satisfying

$$\chi(G) = k, \quad \chi(G - v) = k - 1 \quad (v \in V(G)), \quad \chi(G - E') = k \quad (E' \subseteq E(G),\ |E'| \leq r). \tag{1}$$

The existence statement of Erdős problem 944 asks whether a (k, r)-graph exists for every $k \geq 4$ and $r \geq 1$. It is also the statement selected by the Formal Conjectures challenge [2, 3]. Skottová and Steiner [1, Theorem 1.3] prove the existence of such graphs for every $k \geq 5$ and $r \geq 1$. Their result leaves the four-color case.

Theorem 1.1 (Construction)**.** *For every integer $r \geq 1$, there is a finite $(4, r)$-graph.*

Corollary 1.2. *For every $k \geq 4$ and $r \geq 1$, there is a finite (k, r)-graph.*

The corollary combines Theorem 1.1 with the cited higher-color theorem. We do not claim a bound of practical size for the number of vertices, nor the stronger assertion about the existence of graphs of every sufficiently large order.

1.1. Status of the proof and its formalization.

This is a research manuscript for mathematical review. Sections 2–7 give the four-color proof in ordinary mathematics. The accompanying Lean file proves that theorem for every natural r, including the recursive parameters, punctured colorings, winding argument, word construction, finite rounding, exact CRT character classification, and edge-robustness deduction.

The file also formalizes the higher-color construction of Skottová and Steiner and proves its required existence statement. This includes the exceptional cases $k = 5, 6, 8$, with a paired-ring argument for even k and an overlapping-interval count for $k = 5$. All witnesses are transported to the universe required by the challenge. The theorem `Bounty.target` proves exactly the displayed proposition for every $k \geq 4$ and $r \geq 1$. Section 8 gives the proof correspondence and pinned verification environment. The symbolic proof is independent of the supplementary finite computations.

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1.2. **Notation.** For an integer x and a positive integer n, write $[x]_n$ for the least nonnegative residue of x modulo n. We identify elements of $\mathbb{Z}/n\mathbb{Z}$ with their least representatives when inequalities or residues modulo 3 are used. A direction pair is an unordered pair $\{s, -s\}$ in $\mathbb{Z}/n\mathbb{Z}$. If S is an inverse-closed set of nonzero directions, $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ has the edges $\{v, v+s\}$ for $s \in S$. An edge is bad for a coloring when its endpoints receive the same color. All edge counts below count distinct undirected edges.

2. A cyclic graph construction criterion

Suppose that $n \equiv 1 \pmod 3$ and put

$$N_n = \{1, n-1\} \cup \{s : 0 < s < n,\ s \equiv 2 \pmod 3\}. \tag{2}$$

Fix $r \geq 1$, $t \geq r$, and $q = 3t + 1$, with $q \mid n$ and $q < n$. Let $a_0, \ldots, a_{q-1}$ be units modulo n with distinct direction pairs. Let S be inverse-closed, contain all $\pm a_i$, and contain neither zero nor an element of order two. Let $J = \{s \in S : 0 < s < n/2\}$. Thus J contains exactly one representative of every direction pair in S. For $K \in \mathbb{Z}$, define

$$D_K(s) = \max\{0, n - 3\min([Ks]_n, n - [Ks]_n)\}, \qquad F(K) = \sum_{s \in J} D_K(s). \tag{3}$$

The deficit is unchanged when s or K is negated.

Theorem 2.1 (Construction criterion)**.** *Suppose the following conditions hold.*

(i) $S \subseteq a_i N_n$ *for every* i.
(ii) *For* $i \neq j$ *and* $\ell \in \{4, 7, \ldots, q\}$, $[\ell a_i a_j^{-1}]_n \equiv 2 \pmod 3$.
(iii) *Every* $s \in J$ *is the sum of each of* r *three-step words in* S. *Their direction pairs avoid* $\{s, -s\}$ *and the sets of direction pairs used by different words are disjoint.*
(iv) *If* $F(K) \leq r$, *then* $D_K(a_i) > 0$ *for some* i.

Start with $H = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$. *For each* i, *and every vertex* x *with* $x \equiv i \pmod q$, *add the edges*

$$\{x + ja_i, x + (j + \ell)a_i\}, \qquad \ell \in \{4, 7, \ldots, q\}, \quad 0 \leq j \leq q - \ell. \tag{4}$$

Identify repeated edges. The resulting simple graph G *is a* $(4, r)$*-graph.*

2.1. **Punctured colorings.** Delete a vertex v, let $i = [v]_q$, and color every remaining vertex by

$$c_v(u) = [[a_i^{-1}(u - v)]_n]_3 \qquad (u \neq v). \tag{5}$$

If a normalized difference is 2 modulo 3, its color difference is 2 without a wrap and 1 with a wrap, since $n \equiv 1 \pmod 3$. Differences 1 and -1 are also proper after coordinate zero is deleted. Condition (i) therefore makes every core edge proper.

For a chord of type i, the normalized forward length is 1 modulo 3. The chord lies inside a q-step gap between consecutive vertices whose residue modulo q is i. Such a gap has no vertex of that residue class in its interior because a_i remains a unit modulo q. The puncture v consequently cannot lie strictly inside the chord's forward arc. If it is an endpoint, the edge has been deleted. Otherwise there is no wrap and the color difference is 1 modulo 3. Chords of any other type are proper by Condition (ii). Thus $G - v$ is three-colorable for every v, and assigning one new color to v shows that $\chi(G) \leq 4$.

2.2. **Height characters.** For different colors $b, c \in \mathbb{Z}/3\mathbb{Z}$, let $\sigma(b, c)$ be the unique element of $\{-1, 1\}$ congruent to $c - b$ modulo 3.

Lemma 2.2 (Height character)**.** *Let* U *be an inverse-closed set of directions generating* $\mathbb{Z}/n\mathbb{Z}$, *and suppose its Cayley graph is properly three-colored. There is a character* $K \in \mathbb{Z}/n\mathbb{Z}$ *and a translation-invariant mean height increment* θ_u *for every oriented generator, satisfying*

$$|\theta_u| \leq 1, \qquad \theta_u \equiv 3 + \frac{6Ku}{n} \pmod 6. \tag{6}$$

The height increment along a fixed word depends only on its starting group vertex. Its mean is the signed sum of the generator means. In particular, an odd word with endpoint displacement s has mean congruent to $3 + 6Ks/n$ modulo 6.

Proof. Choose representatives $u_1, \dots, u_d$ for the generator pairs and pull the coloring back under the map $\mathbb{Z}^d \to \mathbb{Z}/n\mathbb{Z}$ taking e_j to u_j. Give each oriented grid edge the increment σ of its endpoint colors. The sum around a square is even, divisible by 3, and between -4 and 4. It is therefore zero. Backtracks also have zero sum. Commuting adjacent coordinate steps and cancelling backtracks reduces every closed grid walk, so the increments integrate to an integer height h on $\mathbb{Z}^d$.

Write Λ for the kernel of the map. For $\lambda \in \Lambda$, the difference $h(x + \lambda) - h(x)$ is divisible by 3. Its change when x moves by a unit grid step is in $\{-2, 0, 2\}$ and divisible by 3, so that difference is independent of x. Denote it by $\phi(\lambda)$. It is additive in λ and extends to a real linear functional $\theta \cdot x$, since Λ has finite index. The period ne_j gives $|\theta_j| \le 1$. Height parity and color congruence give

$$\phi(\lambda) \equiv \sum_j \lambda_j \pmod 2, \qquad \phi(\lambda) \equiv 0 \pmod 3.$$

Hence $(\theta - 3\mathbf{1})/6$ takes integral values on Λ. Modulo integral functionals it factors through a character of the cyclic quotient, proving (6).

Changing a lift of a starting vertex adds the same period constant to both heights defining a word increment. The increment is therefore well-defined. For a single generator, summing over an orbit gives its period height, so its average increment is θ_j. Translation permutes the starting vertices, and therefore averaging a word adds the signed generator means. The contribution of the constant 3 is 3 modulo 6 for an odd word. □

2.3. **Bad edges and the deficit bound.** Suppose a three-coloring of G has at most r bad edges. A direction pair in S is called bad if at least one of its translated edges is bad. There are at most r such pairs, since different direction pairs have disjoint edge sets. As $q > r$, one of the chosen unit directions has no bad edge. The remaining proper directions still generate $\mathbb{Z}/n\mathbb{Z}$.

If a target direction s is bad, its r words in Condition (iii) avoid its own pair. At most $r - 1$ other bad pairs can spoil these mutually disjoint words, so at least one is wholly proper. If s is proper, use the word $(s, s, -s)$ instead. Apply Lemma 2.2 to the proper directions and let $\delta_s(v)$ be the height increment of a selected three-step word for s, starting at v. It belongs to $\{-3, -1, 1, 3\}$ and has the endpoint color difference as its residue modulo 3. Consequently it is ± 1 at a proper s-edge and ± 3 at a bad one. If β_s is the number of bad s-edges and θ_s is the mean of these word increments, then

$$\theta_s \equiv 3 + \frac{6Ks}{n} \pmod 6, \qquad |\theta_s| \le 1 + \frac{2\beta_s}{n}. \tag{7}$$

There are exactly n edges in this direction pair: the assumption that s is not an involution makes $v \mapsto \{v, v + s\}$ injective.

If $x = [Ks]_n/n$, the least possible absolute value of a real number congruent to $3 + 6x$ modulo 6 is $3 - 6\min(x, 1 - x)$. Equation (7) gives $\beta_s \ge D_K(s)$. Summing over the representatives in J therefore proves

$$F(K) \le \sum_{s \in J} \beta_s \le r. \tag{8}$$

Moreover, $D_K(s) > 0$ forces $|\theta_s| > 1$.

2.4. **The winding inequality.** Let B_t have vertex set $\{0, \dots, 3t + 1\}$ and an edge ij, with $i < j$, exactly when $j - i \equiv 1 \pmod 3$. In particular it contains the consecutive cycle edges, including the closing edge.

Lemma 2.3 (Winding bound)**.** *Suppose a three-coloring of B_t has at most t bad edges. Along the path $0, 1, \ldots, 3t+1$, assign increments ± 1 to proper edges and ± 3 to bad edges, each congruent modulo 3 to its color difference. The sum of the increments lies between $-(3t+1)$ and $3t+1$.*

Proof. For a color pair define its maximal increment m to be 3 if the colors are equal, 1 if the color increases by 1 modulo 3, and -1 otherwise. For a cyclic color word of length $3t+2$, let C be the sum of these maximal increments. We prove by induction that at most t bad B_t edges imply $C \leq 3t$.

For $t = 0$, the graph is a proper edge traversed in both directions, so $C = 0$. If $t > 0$ and there are no bad cycle edges, then $C \leq 3t+2$. Its parity agrees with $3t+2$ and it is divisible by 3, giving $C \leq 3t$.

Otherwise let b be the number of bad cycle edges, so $1 \leq b \leq t$. The $3t+2-b > 2b$ proper edges imply that some bad cycle edge is followed by at least three proper edges before the next bad one. Rotate this four-edge window to the start and delete its three interior vertices. Rotation preserves B_t, since $3t+2 \equiv 2 \pmod 3$ and reversing an allowed cyclic difference again gives a difference 1 modulo 3.

The remaining induced graph, in its cyclic order, is B_{t-1}. At least the initial bad edge disappears. If $s \in \{-3, -1, 1, 3\}$ is the sum of the following three proper increments, the four removed increments sum to $3+s$. The maximal increment on the new bridge is, respectively, $3, -1, 1, 3$. Thus the old cycle sum is at most the new one plus 3. Induction proves $C \leq 3t$.

The maximal path increments dominate the assigned increments. The closing increment is at least -1, so a path sum exceeding $3t+1$ would give $C > 3t$. This proves the upper bound. Reverse the path and negate its increments to obtain the lower bound. □

Proof of Theorem 2.1. The punctured colorings prove the vertex-deletion assertions and $\chi(G) \leq 4$. If a three-coloring had at most r bad edges, (8) and Condition (iv) would supply an a_i with $|\theta_{a_i}| > 1$. Reverse the orientation if necessary. The sum of its word increments around its unit Hamiltonian cycle then exceeds n. Partition the cycle into its n/q designated gaps of length $q = 3t+1$. Some gap has sum exceeding q. Its vertices span B_t by (4) and the unit core edges, contradicting Lemma 2.3, since the entire graph has at most $r \leq t$ bad edges. Reversal preserves the same gap graphs.

Thus every three-coloring of G has more than r bad edges. In particular G is not three-colorable, so $\chi(G) = 4$. A three-coloring after deleting at most r edges would have at most r bad edges in G, which is impossible. Finally, $\chi(G - v)$ cannot be at most 2, since adding v in a new color would three-color G. Therefore (1) holds with $k = 4$. □

3. The integer parameter construction

Fix $r \geq 1$. Choose $t = r$ unless $5 \mid 3r+1$, in which case choose $t = r+1$. Then $q = 3t+1$ is coprime to 15. Put

$$D = 30q, \qquad M = 150q = 5D, \qquad a_0 = 1. \tag{9}$$

For $1 \leq j < q$, recursively choose integers κ_j, d_j, a_j satisfying

$$\kappa_j \equiv 0 \pmod 3, \qquad \kappa_j \equiv 1 \pmod{M-5}, \tag{10}$$

$$d_j = D(\kappa_j - 1) + 1, \qquad a_j = 5d_j, \tag{11}$$

$$\kappa_j \geq 6, \qquad a_j > 10qa_{j-1}, \tag{12}$$

$$d_j \equiv \kappa_i^{-1} \pmod{d_i} \qquad (1 \leq i < j). \tag{13}$$

Lemma 3.1. *These choices exist. The d_j are pairwise coprime, and for every $j \geq 1$,*

$$d_j \equiv 1 \pmod{15}, \quad \kappa_j \equiv 6 \pmod{15}, \quad \gcd(\kappa_j, a_j) = 1, \quad a_j \equiv 2 \pmod 3, \quad a_j \equiv 5 \pmod q. \tag{14}$$

Proof. Every d_i is 1 modulo D. Since $\kappa_i \equiv 1 \pmod{5(D-1)}$, it is also 1 modulo $D-1$, and $d_i \equiv 1 \pmod{M-5}$ by (11). Thus d_i is coprime to both D and $M-5 = 5(D-1)$. A common divisor of κ_i and d_i divides $D-1$. As $\kappa_i \equiv 1 \pmod{D-1}$, this proves $\gcd(\kappa_i, d_i) = 1$.

For a new index j, condition (13) is equivalent to

$$\kappa_j \equiv D^{-1}(\kappa_i^{-1} + D - 1) \pmod{d_i} \qquad (i < j). \tag{15}$$

The moduli 3, $M-5$, and the earlier d_i are pairwise coprime. Indeed, $M-5 \equiv 1 \pmod 3$, the d_i are coprime to $3(M-5)$ by the observations above, and their pairwise coprimality follows inductively from (13). The Chinese remainder theorem supplies an arithmetic progression of choices for κ_j. Take a sufficiently large member to satisfy (12).

Finally D is divisible by 15, and κ_j is 0 modulo 3 and 1 modulo 5, proving the first two congruences in (14). The coprimality with a_j follows from the coprimality with d_j and $\kappa_j \equiv 1 \pmod 5$. The two congruences for a_j follow from $a_j = 5d_j$ and $D = 30q$. □

Set

$$P = 5\prod_{j=1}^{q-1} d_j, \qquad L = 3P. \tag{16}$$

Choose a positive integer n with

$$n \equiv 1 \pmod 3, \qquad n \equiv 0 \pmod q, \tag{17}$$

$$n \equiv -\kappa_j^{-1} \pmod{a_j} \qquad (1 \le j < q), \tag{18}$$

$$n > 1000(r+1)qL^2. \tag{19}$$

Such integers form a nonempty progression of step qL. To see this, separate the common factor 5 of the a_j. Their prescribed residues are all -1 modulo 5, because $\kappa_j \equiv 1 \pmod 5$. The remaining moduli d_j are pairwise coprime and coprime to $3q$, while q is coprime to 15. The Chinese remainder theorem therefore applies to $3, q, 5, d_1, \dots, d_{q-1}$, and an arbitrarily large member may be chosen. For a deterministic choice, take the least member satisfying (19).

Each a_j is a unit modulo n, and its least positive inverse for $j \ge 1$ is

$$[a_j^{-1}]_n = \frac{1 + \kappa_j n}{a_j}. \tag{20}$$

The numerator is divisible by a_j by (18). Since $0 < \kappa_j < a_j$ and n is larger than all the parameters a_j, the displayed quotient is strictly between 0 and n.

Lemma 3.2 (Short compatible ratios)**.** *For all distinct indices i, j,*

$$0 < [a_i a_j^{-1}]_n < \frac{n}{q}, \qquad [a_i a_j^{-1}]_n \equiv 2 \pmod 3. \tag{21}$$

Consequently the a_i have distinct direction pairs.

Proof. When $j = 0$, the representative is a_i. Equation (14) and the size of n give the result. Suppose $j > 0$ and set $\ell = [a_i \kappa_j]_{a_j}$. Equation (20) gives the integer representative

$$R_{ij} = \frac{a_i + \ell n}{a_j}. \tag{22}$$

We check that it is less than n/q, which also confirms that it is the least positive representative. If $i = 0$, then $\ell = \kappa_j$, and $\ell/a_j < 1/(4q)$. If $i > j \ge 1$, condition (13) gives $\ell = 5$, again with $\ell/a_j < 1/(4q)$. If $0 < i < j$, use

$$\frac{a_i \kappa_j}{a_j} = (\kappa_i - 1) + \frac{5}{M} + \frac{a_i}{a_j}\left(1 - \frac{5}{M}\right). \tag{23}$$

The fractional part is positive and less than $1/(30q)+1/(10q) < 1/(4q)$. The integer part is $\kappa_i - 1$. Thus $\ell \equiv 2 \pmod 3$ in both cases $i, j > 0$, $i \neq j$. For $i = 0$ we instead have $\ell \equiv 0 \pmod 3$. Using $a_j \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 3$ in (22) now gives $R_{ij} \equiv 2 \pmod 3$ in every case.

Since $a_i \leq P = L/3$, the bound on n implies $n > 4q \max_i a_i$. Equation (22) consequently gives $0 < R_{ij}/n < 1/(2q)$ when $j > 0$. This proves (21). Equality $a_i = a_j$ modulo n would make their ratio 1, while $a_i = -a_j$ would make it $n - 1$. Neither has residue 2 modulo 3. □

4. The core and its periodic residue classes

Define

$$S = \left(\bigcap_{i=0}^{q-1} a_i N_n \right) \setminus \{n/2\}, \tag{24}$$

where the last removal is made only when n is even. The set N_n is inverse-closed because the negative of a residue 2 modulo 3 has the same residue, since $n \equiv 1 \pmod 3$. Thus S is inverse-closed and has no involution. Lemma 3.2 puts all $\pm a_i$ in S. It proves Conditions (i) and (ii) as well. For $\ell \leq q$ with $\ell \equiv 1 \pmod 3$, the product of ℓ with an off-diagonal ratio is less than n and is 2 modulo 3.

For $1 \leq j < q$ and an integer s, write $e_j(s) = [\kappa_j s]_{a_j}$. For $0 < s < n$, equation (20) yields the exact identity

$$[a_j^{-1} s]_n = \frac{s + e_j(s) n}{a_j}. \tag{25}$$

The right side is a positive integer less than n, because $0 \leq e_j(s) < a_j$ and $0 < s < n$.

Apart from the residues $\pm a_i$, membership in the intersection in (24) is equivalent to

$$s \equiv 2 \pmod 3, \qquad e_j(s) \equiv 2 \pmod 3 \qquad (1 \leq j < q). \tag{26}$$

Indeed, a normalized coordinate equals 1 or $n-1$ only at the two corresponding orientations. Away from those exceptions, every normalized coordinate must be 2 modulo 3. Equation (25), with $a_j \equiv 2 \pmod 3$, gives (26). Conversely those congruences make every coordinate 2 modulo 3.

Call (26) the generic condition. It has period L, independently of n. Let $\mathcal{R} \subseteq \mathbb{Z}/L\mathbb{Z}$ be its residue classes. Every positive representative below n of a class in $\mathcal{R}$ belongs to S, except the possible involution. The Chinese remainder theorem describes $\mathcal{R}$ as five products. Put $m_j = (d_j - 1)/3$. For $h \in \{0, 1, 2, 3, 4\}$, impose

$$s \equiv 2 \pmod 3, \qquad s \equiv 2 + 3h \pmod 5, \qquad e_j(s) = 2 + 3h + 15 y_j, \tag{27}$$

where the coordinates are independent and range over

$$0 \leq y_j \leq m_j \quad (h = 0), \qquad 0 \leq y_j \leq m_j - 1 \quad (h > 0). \tag{28}$$

To check this, use $\kappa_j \equiv 1 \pmod 5$, so that $e_j(s)$ and s have the same residue modulo 5. The residue 2 modulo 3 fixes the least representative modulo 15 as $2 + 3h$. Since $d_j \equiv 1 \pmod{15}$, listing these representatives in $[0, 5d_j)$ gives exactly (28). The e_j coordinates modulo d_j, together with the residues modulo 3 and 5, are independent CRT coordinates. Every block is nonempty, so in particular $\mathcal{R}$ contains more than one class.

5. Exact character rigidity

Let $B = [1/3, 2/3] \subseteq \mathbb{R}/\mathbb{Z}$. The following lemma is the arithmetic classification on which the construction rests.

Lemma 5.1 (Rigidity of the periodic part)**.** *Suppose $\alpha \in L^{-1}\mathbb{Z}$, $\beta \in \mathbb{R}$, and $K = \alpha n + \beta \in \mathbb{Z}$, where n satisfies* (17)–(18)*. If*

$$\alpha\rho + \beta x \pmod 1 \ \in B \qquad (\rho \in \mathcal{R},\ 0 \leq x \leq 1), \tag{29}$$

then, for some $i \in \{0, \dots, q-1\}$,

$$Ka_i \equiv \pm\frac{n-1}{3} \pmod{n}. \tag{30}$$

Proof. A continuous linear path contained in an arc of length $1/3$ has $|\beta| \le 1/3$. Taking the two endpoints in (29) shows that all $\alpha\rho$ have real representatives in a common interval $I \subseteq B$ of width at most $1/3 - |\beta|$.

Decompose the character in the independent CRT coordinates:

$$\alpha s \equiv \frac{u(s \bmod 3)}{3} + \frac{v(s \bmod 5)}{5} + \sum_{j=1}^{q-1} \frac{c_j e_j(s)}{d_j} \pmod 1, \tag{31}$$

with integer coefficients u, v, c_j, taken modulo $3, 5, d_j$, respectively. In the $h = 0$ product in (28), increasing y_j by one changes the phase by $15c_j/d_j$ modulo one. Since all real representatives lie in I, their actual difference lies in $[-1/3, 1/3]$. There is at most one representative of a given phase difference in that interval, so the actual difference is constant throughout the product. Denote it by δ_j. The phase on the product is therefore affine in its coordinates, and its width is

$$\sum_j m_j|\delta_j| \le \frac{1}{3}. \tag{32}$$

Each $\delta_j = z_j/d_j$ for an integer z_j. Since $m_j = (d_j - 1)/3$, a nonzero z_j must equal 1 or -1. Moreover at most one z_j can be nonzero, since two would contribute at least $2/3 - 1/(3d_i) - 1/(3d_j) > 1/3$. Here all $d_j > 8$.

No active coordinate. If all z_j vanish, then $c_j = 0$ modulo d_j, and the five block phases are $2u/3 + vz/5$, with z running through $\mathbb{Z}/5\mathbb{Z}$. For $v \neq 0$ modulo 5 these are five equally spaced points, which cannot lie in I. Thus $v = 0$, and $u = 1$ or 2. The character α is consequently $1/3$ or $2/3$ modulo one. The integrality of $\alpha n + \beta$, together with $n \equiv 1 \pmod 3$ and (29), forces $\beta = -1/3$ or $1/3$, respectively. These give (30) with $a_0 = 1$.

One active coordinate. Suppose only the coordinate j is active. Negating the character and β preserves the hypothesis, so assume $\delta_j = 1/d$, where $d = d_j$ and $a = 5d$. We may take $c_j = (1 - d)/15$, because $d \equiv 1 \pmod{15}$ and $15c_j \equiv 1 \pmod d$. Write $m = (d - 1)/3$. The first phase of block h is

$$P_h \equiv \frac{2u}{3} + \frac{(3v-1)(2+3h)}{15} + \frac{2+3h}{15d} \pmod 1. \tag{33}$$

Each block is a progression with step $1/d$. Its width is m/d when $h = 0$ and $(m-1)/d$ otherwise. Since all blocks lie in I, their first phases have representatives in an interval of width at most

$$\frac{1}{3} - \frac{m-1}{d} = \frac{4}{3d}.$$

Successive first phases differ modulo one by $(3v-1)/5 + 1/(5d)$. Unless $v = 2$ modulo 5, its distance from an integer is at least $1/5 - 1/(5d) > 4/(3d)$, because $d > 8$. Hence $v = 2$.

The first phases then advance by $1/(5d)$. The $h = 0$ block contains the real range of all five blocks because the other starts are shifted forward by at most $4/(5d)$ and their widths are smaller by $1/d$. Its initial phase is $2u/3 + 2/3 + 2/(15d)$ modulo one. Containment in B forces $u = 1$ modulo 3: $u = 0$ starts above $2/3$, and $u = 2$ below $1/3$. The full range of phases is therefore

$$\left[\frac{1}{3} + \frac{2}{3a}, \frac{2}{3} - \frac{1}{a}\right]. \tag{34}$$

Its translates by βx stay in B only if

$$-\frac{2}{3a} \le \beta \le \frac{1}{a}. \tag{35}$$

Using $\kappa_j \equiv 6 \pmod{15}$, formula (31) now simplifies to

$$\alpha \equiv \frac{1}{3} + \frac{\kappa_j}{3a} \pmod 1. \tag{36}$$

Indeed, $2/5 + (1-d)\kappa_j/(15d)$ is congruent to $\kappa_j/(15d)$. The number

$$\left(\frac{1}{3} + \frac{\kappa_j}{3a}\right) n + \frac{1}{3a} = \frac{(a+\kappa_j)n + 1}{3a} \tag{37}$$

is integral, since its numerator is divisible by a by (18) and by 3 by (14) and (17). The interval (35) has length $5/(3a) < 1$ and contains $1/(3a)$. Thus integrality leaves precisely $\beta = 1/(3a)$. Multiplying (37) by $3a$ shows that $3Ka \equiv 1 \pmod n$. As $n \equiv 1 \pmod 3$, this gives $Ka \equiv (2n+1)/3 \equiv -(n-1)/3 \pmod n$. Negation covers the other sign. □

6. From finite deficits to exact rigidity

We now prove Condition (iv) with the finite size bound (19). No compactness or limiting argument is needed.

Proposition 6.1. *For the parameters above, $F(K) \le r$ implies (30) for some i. In particular $D_K(a_i) = 1 > 0$.*

Proof. Each individual deficit is at most r. By symmetry in s and $-s$, every $s \in S$ satisfies

$$\frac{[Ks]_n}{n} \in B_\varepsilon = \left[\frac{1}{3} - \varepsilon, \frac{2}{3} + \varepsilon\right], \qquad \varepsilon = \frac{r}{3n}. \tag{38}$$

The width $w = 1/3 + 2\varepsilon$ is less than $1/2$. Choose a residue class in $\mathcal{R}$ not containing the possible involution $n/2$. Its positive representatives below n form an uninterrupted progression of step L, with at least $n/L - 1$ elements. The phases of consecutive representatives have differences in $[-w, w]$ and are all congruent to KL/n modulo one. Since $2w < 1$, these real differences are equal. Write their common value as $\delta = KL/n - p$, where $p \in \mathbb{Z}$. Telescoping gives $|\delta|(n/L - 2) \le w$. Set $\alpha = p/L$ and $\beta = n\delta/L$. Then $K = \alpha n + \beta$, and (19) gives

$$|\beta| \le \frac{w}{1 - 2L/n} < \frac{1}{2}. \tag{39}$$

For an arbitrary $\rho \in \mathcal{R}$, its first and last positive representatives below n lie within L of 0 and n, respectively. Neither is the involution, since $n > 2L$. Their phases in (38) show that $\alpha\rho$ and $\alpha\rho + \beta$ have circular distance from B at most

$$\varepsilon + |\beta|\frac{L}{n} < \frac{r/3 + L/2}{n} < \frac{1}{L}. \tag{40}$$

Both phases belong to the grid $L^{-1}\mathbb{Z}/\mathbb{Z}$, because $\alpha = p/L$ and $\beta = K - pn/L$. The endpoints of B belong to the same grid since $3 \mid L$. A grid point outside B is at distance at least $1/L$ from B, so both phases actually belong to B.

Their representatives in B differ by at most $1/3$. By (39) that difference must equal β, since any nonzero integer shift of β has magnitude greater than $1/2$. The interpolating segment stays in B, proving the hypothesis of Lemma 5.1. This yields (30). If $m = (n-1)/3$, then $\min(m, n-m) = m$ and $n - 3m = 1$, so the resulting deficit is exactly one. □

7. Disjoint three-step words

We finish the construction by proving Condition (iii). Fix $z \in J$, so $0 < z < n/2$. As $S \subseteq N_n$, either $z = 1$ or $z \equiv 2 \pmod 3$. If z is nongeneric, it is one of the positive orientations a_j, because every negative orientation exceeds $n/2$ by the parameter bounds. Set

$$W = z + cn, \qquad c = \begin{cases} 2, & z = 1, \\ 1, & z \ne 1. \end{cases} \tag{41}$$

Thus $n < W \le 2n+1$ and $W \equiv 0 \pmod 3$.

For each $j \ge 1$, prescribe the sum of three e_j coordinates as

$$M_j = \begin{cases} a_j + \kappa_j - 2, & z = 1, \\ 2a_j - 1, & z = a_j, \\ a_j + e_j(z) - 1, & \text{otherwise.} \end{cases} \tag{42}$$

In the last case $e_j(z) \equiv 2 \pmod 3$: for a generic z this is the definition, and for another orientation it follows from compatibility and (25). In all cases

$$M_j \equiv 0 \pmod 3, \quad a_j < M_j < 2a_j, \quad M_j \equiv z - c \pmod 5, \quad M_j \equiv \kappa_j W \pmod{a_j}. \tag{43}$$

These follow directly from $a_j \equiv 2 \pmod 3$, $\kappa_j \equiv 6 \pmod{15}$, and $\kappa_j n \equiv -1 \pmod{a_j}$.

For an explicit choice, put

$$E_j = 2 + 15 \left\lfloor \frac{M_j/3 - 2}{15} \right\rfloor, \qquad E_j^1 = E_j^2 = E_j, \qquad E_j^3 = M_j - 2E_j. \tag{44}$$

Writing $b_j = M_j/3 - E_j$, we have $0 \le b_j \le 14$ and $E_j^3 = M_j/3 + 2b_j$. Since $a_j > 100$ and $a_j < M_j < 2a_j$, we obtain $E_j > a_j/3 - 14 > 0$ and $0 < E_j^3 < 2a_j/3 + 28 < a_j$. Also $E_j < 2a_j/3 < a_j$. All three coordinates are 2 modulo 3. Their residues modulo 5 are $2, 2, z - c - 4$, independently of j.

For each $\ell = 1, 2, 3$, the Chinese remainder theorem therefore gives a generic class ρ_ℓ modulo L with these e_j coordinates and residue 2 modulo 3. Equation (43) implies

$$\rho_1 + \rho_2 + \rho_3 \equiv W \pmod L. \tag{45}$$

Choose s_1, s_2 in the first two classes, respectively, in the real interval

$$\left[\frac{W}{3} - \frac{n}{24}, \frac{W}{3} + \frac{n}{24} \right], \tag{46}$$

and set $s_3 = W - s_1 - s_2$. Then s_3 belongs to the third class and lies within $n/12$ of $W/3$. All three integers lie strictly between 0 and n. Each of the first two choices has at least $n/(12L) - 1$ possibilities.

Lemma 7.1 (Uniform avoidance)**.** *For every set $F \subseteq \mathbb{Z}/n\mathbb{Z}$ with $|F| \le 6r + 2$, there is a three-step word summing to z, using directions in S and avoiding F.*

Proof. Add the possible involution to F, obtaining a forbidden set of size at most $6r + 3$. Choose s_1 outside it. Once s_1 is fixed, at most $2(6r + 3)$ values of s_2 are excluded by requiring both s_2 and $W - s_1 - s_2$ to avoid it. The bound on n implies $n/(12L) - 1 > 2(6r + 3)$, so both choices can be made. The three summands are generic and avoid the involution, hence lie in S. Their sum is $W \equiv z \pmod n$, as required. □

Initially forbid z and $-z$. After choosing a word, also forbid both signs of each of its three directions. Before the rth choice there are at most $2 + 6(r - 1)$ forbidden residues, so Lemma 7.1 applies at every stage. The resulting r words avoid the target pair and have mutually disjoint sets of direction pairs. This proves Condition (iii).

Proof of Theorem 1.1. Lemma 3.1 and the subsequent CRT choice supply the integer parameters for every $r \ge 1$. Define S by (24) and add the chords (4). Lemma 3.2 and the discussion in Section 4 prove Conditions (i) and (ii). Proposition 6.1 proves Condition (iv), and the preceding word construction proves Condition (iii). Theorem 2.1 gives a finite $(4, r)$-graph. □

8. Lean verification and computational checks

The original formal development was supplied as a single file, `Solution.lean`, with the two imports requested by the challenge:

```
import FormalConjectures.ErdosProblems.«944»
import TaskSupport
```

The code uses the namespace `Bounty.Erdos944Proof`. It is checked with Lean 4.33.1 and with warnings treated as errors. The 570 proved theorems in the original manuscript's count have no admitted proof placeholders. Their transitive proof dependencies use only the standard axioms `propext`, `Classical.choice`, and `Quot.sound`. None uses `sorryAx`. Reading the challenge theorem's type with the supplied elaborator does not use that theorem as a proof.

8.1. **The complete target.**

The original file contains 9,548 lines, including every supporting proof and the following final theorem in the namespace `Bounty`:

```
theorem target :  fcTypeOfName% "Erdos944.erdos_944" := by
  constructor
  · intro _ k hk r _hr
    exact Erdos944Proof.erdos944_all hk r
  · intro _
    trivial
```

The audit driver separately checks that its type is

$$\text{True} \longleftrightarrow \forall k \geq 4\ \forall r \geq 1\ \exists V\ \exists G, \quad G \text{ has the Erdős 944 property for } k, r.$$

It also applies the theorem directly to this explicit proposition. The three existence lemmas are `erdos944_four`, `erdos944_five`, and `erdos944_ge_six`. They have no additional mathematical hypotheses, and also cover $r = 0$ and arbitrary target universes. The imported conjecture theorem is used only to obtain the requested type. Its admitted proof does not occur in the proof dependencies.

The theorem `exists_cyclic_integer_data` constructs the recursive CRT family, a sufficiently large order, and all unit directions. The full three-step word construction and its finite-set avoidance property are proved. The character argument is carried out with integers: finite rounding expresses KL as an integer multiple of n plus a bounded remainder, and the five CRT blocks force the frequency into the canonical cases. The Lean proof establishes constant coordinate steps, their exact divisibility, and coefficient sparsity, then determines the active frequency from adjacent block endpoints modulo fifteen. Integrality determines the remainder and gives a unit with deficit exactly one. Thus the arithmetic conditions of the graph construction criterion are all discharged.

For $k \geq 5$, the theorem `higher_punctured` proves the vertex-deleted colorings for every positive m and natural q. The row formula handles both parities of k. Equal-colored vertices have restricted index differences, which avoid all three distance families in both cyclic directions. These definitions and colorings follow Skottová and Steiner [1, Section 4 and Appendix A]. The proof also bounds the affected vertices by twice the number of monochromatic edges. Dividing a window into $2r + 1$ disjoint blocks then produces an unaffected block of length $3n_{k,m}$. Local propagation from such a block is proved below and then converted into the required graph property.

The theorem `higher_clean_block_periodic` proves periodicity inside every unaffected block of three periods, for all $k \geq 5$, including $k = 6$ and $k = 8$. Odd k uses a partition into cliques. Even k uses ten-vertex subgraphs of independence number two and an interval-counting argument. This formalizes the local-periodicity claim of Skottová and Steiner [1, Claim 4.4]. The periodic expected coloring retains the block's exact color counts and short-distance constraints. A color class larger than $2r$ in an interval of length $2m$ has a single parity: it contains an unaffected vertex, which excludes the opposite parity. Finally, for odd $k \geq 5$, the unaffected prefix leaves only the expected colors at i and $i - 1$ as options at later indices $i < (q/2)n_{k,m}$. The even-color restriction and extension through the larger window are also proved.

The theorem `higher_expected_local_structure_all` establishes the local color structure for every $k \geq 6$: on each parity, changes of expected color occur at exactly one residue modulo $2m$. For odd $k \geq 7$, each expected color class is one even-step block repeated periodically. For even $k \geq 10$, it is two such blocks. The exceptional cases $k = 6, 8$ use finite ring configurations. These force two local color options and periodicity of the change positions. A duplicate color in a ring would remain fixed along an entire parity, contradicting an edge of length $2m$. Ring injectivity and an exact count of changes then give one change on each parity per period. These arguments formalize Skottová and Steiner [1, Claims 4.6–4.7 and Appendix B], using finite certificates checked by Lean's kernel.

8.2. **Completing propagation.** Write $L = n_{k,m}$ for the period of the expected coloring and take $q = 24r + 12$, so the graph has order $qL + 1$. We use $m = 18r + 3$ for $k \geq 6$ and $m = 18r + 20$ for $k = 5$. A coloring with at most r bad edges has at most $2r$ affected vertices. A color occurring more than $2r$ times in any interval of length $2m$ therefore occurs at only one parity in that interval.

For even k, put $n = k - 1$ and $s = \lfloor n/2 \rfloor$. Two adjacent rings have positions $2ma + b$, where $0 \leq a < n$ and $b \in \{0, 1\}$. Two distinct positions are adjacent whenever their row distance is neither s nor $s + 1$. Any three positions contain an edge: three distinct rows cannot have all three pairwise distances in $\{s, s + 1\}$ when $s \geq 2$, and two positions in one row are adjacent. Thus every color occurs at most twice and, by counting $2n$ positions in n colors, exactly twice. For a later vertex, the distance conditions exclude every row except $0, s, s + 1$. The positions in rows $s, s + 1$ form a clique, so the two occurrences of its color cannot both lie there. One occurs in row zero, giving the two expected-color options at indices i and $i - 1$. This proves the restriction for every even $k \geq 6$ at once.

For $k \geq 6$, suppose the first deviation from the expected coloring occurs at i. The two options force its color to equal the preceding color. The even-run structure supplies a nearby interval of m consecutive positions with only this color and one other color available. The repeated color occurs at both parities in a containing $2m$-interval, so it occurs at most $2r$ times there. The other color must then occur at least $m - 2r$ times in the m-interval. It exceeds $2r$, and the single-parity property bounds its count by $\lceil m/2 \rceil$. These inequalities contradict $m \geq 4r + 3$.

For $k = 5$, write e_j for the expected color and c_j for the actual color. The period is $4m$. Every expected color occurs exactly m times per period. Opposite parities use disjoint colors, and the two colors on the parity of a are e_a and e_{a+2m}. At a first deviation i, set

$$\gamma = e_{i-1} = c_{i-1} = c_i, \qquad \delta = e_{i-2} = c_{i-2}, \qquad \varepsilon = e_{i-2+2m}.$$

Let $I = [i - 2m + 1, i]$ and $J = [i - 2, i + 2m - 3]$, and let A, B be the numbers of expected colors γ, ε, respectively, in J. The actual color γ occurs at both parities in each interval, so its count in each is at most $2r$. On I the expected γ-count is no larger than the actual count. The two intervals cover $4m - 3$ positions. The missing three positions in a $4m$-window give

$$A \geq m - 2r - 3. \tag{47}$$

In J, at most $2r$ occurrences of δ can be at the parity opposite $i - 2$. Otherwise the single-parity rule contradicts $c_{i-2} = \delta$. At a position with expected color γ, the two available colors force the actual color to be γ, δ, or the preceding expected color ε. Shifting a color count by one changes it by at most one. Consequently

$$A \leq 4r + B + 1. \tag{48}$$

Let C_γ, C_ε denote the actual counts in J. At each position, the number of occurrences of γ, ε among the two expected options is at most one plus the indicator that the actual color lies in $\{\gamma, \varepsilon\}$. Summing, and allowing one boundary error for each expected color, gives

$$2A + 2B - 2 \leq 2m + C_\gamma + C_\varepsilon.$$

Together with (47)–(48) and $C_\gamma \le 2r$, this implies $C_\varepsilon \ge 2m - 18r - 16 > m$. But every actual color in a $2m$-interval occurs at most m times: a count at most $2r$ is already bounded by m, and a larger count has a single parity. This contradiction excludes the first deviation for $k = 5$.

Thus a clean prefix determines the actual coloring throughout the required larger window. Every interval of length $(6r + 3)L$ contains a clean $3L$-block. Reflecting that block and applying prefix agreement proves $c(v + L) = c(v)$ for every vertex v. Since $\gcd(L, qL + 1) = 1$, such a coloring is constant, contradicting a proper edge in the clean block. Every $(k - 1)$-coloring therefore has more than r bad edges. Combining this with the punctured colorings proves the higher-color existence theorems and completes `Bounty.target`.

The main correspondences are listed below. Names are relative to `Bounty.Erdos944Proof`, except the final target.

Mathematical component	Lean theorem
Punctured colorings of the augmented graph	`augmentedGraph_punctured`
Filled Hamiltonian gaps	`augmentedGraph_filled_gaps`
Height character for any unit	`exists_group_flux_character`
Deficit estimate with retained lifts	`exists_character_with_lifts`
Winding induction	`andrasfai_cycle_flux_le`
Two-sided gap bound	`path_gain_bounds_of_badEdges_bound`
Flux bound from all gaps	`flux_bounds_of_andrasfai_gaps`
The graph construction criterion	`erdos944_four_of_cyclic_data`
Short compatibility of constructed units	`cyclic_data_shortCompatible`
Disjoint words from avoidance	`hasSignedWords_of_avoidance`
Transport to arbitrary universes	`erdos944_universe_lift`
Arbitrarily large finite CRT solutions	`exists_large_crt`
Choice of $q = 3t + 1$ coprime to 15	`exists_gap_parameter`
Canonical deficit equal to one	`canonical_deficit_one`
Recursive integer data for every r	`exists_cyclic_integer_data`
Exact normalized-coordinate formula	`normalized_coordinate_mul`
Unique five-block CRT grid	`existsUnique_generic_grid_class`
Generic residues give signed connections	`cyclic_data_generic_signed_mem`
Avoiding a finite set in a progression	`exists_progression_point_avoiding`
Splitting a coordinate sum into three	`split_three_residue_coordinates`
Uniform three-step word avoidance	`cyclic_data_word_avoidance`
Finite rounding to an exact character	`cyclic_data_rounded_character`
Affine phases on a CRT block	`middleThirdCharacter_grid_affine`
Sparsity of the CRT coefficients	`middleThirdCharacter_sparse_coefficients`
Exact periodic character rigidity	`cyclic_data_periodic_rigidity`
Four-color existence for every r	`erdos944_four`
Higher-color punctured colorings	`higher_punctured`
An unaffected consecutive block	`higher_exists_unaffected_block`
Local periodicity for every $k \ge 5$	`higher_clean_block_periodic`
Periodic expected-color counts	`higherExpectedColor_window_count`
Parity of large short color classes	`higher_short_color_class_single_parity`
Two color options for odd k	`higher_odd_clean_prefix_two_options`
Local color structure for every $k \ge 6$	`higher_expected_local_structure_all`
Injective rings for $k = 6, 8$	`higher_small_ring_color_injective`
Reduction to local propagation	`higher_erdos944_of_local_propagation`
Two color options for even k	`higher_even_clean_prefix_two_options`
First-deviation count for $k = 5$	`five_first_deviation_impossible`
Existence for $k = 5$	`erdos944_five`
Existence for every $k \ge 6$	`erdos944_ge_six`
All-color existence	`erdos944_all`

Mathematical component	Lean theorem
Exact challenge proposition	`Bounty.target`

8.3. **Pinned environment.** The original checked environment used the following revisions. Its `validation.json` recorded the exact source hash and audit results.

- Lean compiler: `819816b2e0a3bf405af45ae5c7af2491d8f5bee6`.
- Formal Conjectures audited revision: `8432eac998110a563e03df65a28c117e97c8c142`.
- mathlib revision [4]: `0df444a360eaa60ab8c11dca51a86af692955474`.

The challenge source uses `google.answer=always_true` to select the displayed proposition. The original audit covered all 570 theorem dependency closures. The manuscript and original archive used the same checked snapshot.

The later pinned reproducible prize package links the historical and accepted sources and checks the explicit finite target. Its recorded clean target build and axiom audit covers 2,339 declarations using cached dependencies. The separate Conjectures verification record applies to the accepted challenge source. The source history distinguishes these versions from one another and from the original manuscript snapshot.

8.4. **Finite checks.** The exact-arithmetic program checks the integer parameters, residue and ratio identities, and selected word certificates. Recorded instances follow. The graphs are not materialized at these enormous orders.

r	q	Digits of n	Digits of L	Words checked
1	4	78	38	16
2	7	714	355	38
3	13	50,220	25,108	75

Character enumerations were also recorded for moduli 465 and 28,365. These finite checks supplement the symbolic Lean proofs of general rigidity and graph existence. The complete target is proved symbolically in Lean.

Acknowledgment of AI assistance

OpenAI Codex assisted with mathematical exploration, computational checks, Lean proof development, and manuscript drafting. Liam Kruer and Jensen Kohlmeyer are the listed authors. Section 8 records what Lean has verified, the source correspondence, and the pinned environment.

References

[1] E. Skottová and R. Steiner, *Critical edge sets in vertex-critical graphs*, preprint, arXiv:2508.08703 (2025), `https://arxiv.org/abs/2508.08703`.

[2] The Formal Conjectures Authors, *Formal Conjectures*, `FormalConjectures/ErdosProblems/944.lean`, `https://github.com/google-deepmind/formal-conjectures`.

[3] Conjectures.io, *Erdős 944 formalized challenge*, `https://conjectures.io/problems/erdos944-erdos-944/challenge?mode=formalized`, accessed September 10, 2026.

[4] The mathlib Community, *mathlib4*, `https://github.com/leanprover-community/mathlib4`.