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Monotone classes of bounded game chromatic number have bounded arboricity

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Lean development: folder `lean` of https://github.com/6/clank-lab/tree/game-chromatic-arboricity-v1/results/game-chromatic-arboricity

Abstract

In the coloring game on a finite graph, Alice and Bob alternately color uncolored vertices with one of k colors so that adjacent vertices get different colors. Alice moves first and wins if every vertex gets colored; Bob wins if some uncolored vertex sees all k colors among its neighbors. The game chromatic number of a graph is the least k for which Alice can force a win. We prove that for every K there is d such that every graph of average degree at least d has a subgraph on which Bob wins the k-color game for every $k \le K$. Consequently, if a class of graphs is closed under taking subgraphs and its game chromatic numbers are bounded, then the edges of each of its graphs can be partitioned into a bounded number of forests, that is, the class has bounded arboricity. This answers Zhu's Question 3 (1999) in the affirmative. The proof combines the theorem of Kühn and Osthus that every graph of large enough average degree contains a subgraph of large average degree with no 4-cycle, with a strategy for Bob that adapts the weighting argument behind Beck's criterion for positional games: Bob uses weights to keep some uncolored vertex that can still be blocked from every color. The rest of the proof is checked in Lean, with the theorem of Kühn and Osthus as an explicit hypothesis.

This work has not yet been reviewed by experts. The proofs, the Lean formalization and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

Keywords: graph coloring game, game chromatic number, arboricity, monotone graph class, 4-cycle-free subgraph, Maker–Breaker game.
MSC 2020: 05C15, 05C57, 91A43.

1 Introduction

1.1 Main results

Graphs are finite and simple. In the *k-color game* on a graph G, Alice and Bob alternately choose an uncolored vertex and give it one of the colors $1, \dots, k$ that no neighbor of the vertex has; Alice moves first. The game ends when the player to move has no legal move, and Alice wins if every vertex is then colored; otherwise Bob wins. We use the rules as stated by Zhu [15, pp. 1–2]; the game goes back to Brams, was published by Gardner and was introduced for general graphs by Bodlaender [2, Section 1]. The *game chromatic number* $\chi_g(G)$ is the least k for which Alice has a winning strategy. A class of graphs is *monotone* if it is closed under taking subgraphs, that is, any

vertex subset with any subset of the edges among it. The *arboricity* of G is the least number of forests into which its edge set can be partitioned. The average degree of a graph with at least one vertex is $2|E(G)|/|V(G)|$.

Theorem 1.1. *For every integer $K \geq 0$ there is an integer $d = d(K)$ such that every graph with at least one vertex and average degree at least d has a subgraph H on which Bob has a winning strategy in the k-color game for every integer k with $0 \leq k \leq K$; equivalently, $\chi_g(H) > K$.*

Whether a win for Alice with k colors implies one with $k+1$ colors is Zhu's Question 1 [15, p. 11], still open [2, Section 8], [7, Question 1.1]; so a win for Bob with K colors alone would leave the smaller palettes to check.

Corollary 1.2. *For every integer $K \geq 0$ there is an integer $d(K)$ with the following property. If a class of graphs is closed under taking subgraphs and each of its members has game chromatic number at most K, then each of its members has arboricity at most $d(K)$.*

This answers in the affirmative Zhu's Question 3, whose hereditary classes are those "closed under taking (not necessarily induced) subgraphs" [15, p. 12]:

Question 1.3 (Zhu [15, Question 3])**.** *"Suppose a hereditary class of graphs has bounded game chromatic number. Is it true that the arboricity of this class of graphs is also bounded?"*

Both results rest on a theorem of Kühn and Osthus [10, Theorem 1], in the form of Theorem 4.2; given it as a hypothesis, both are proved in Lean (Section 6).

1.2 Proof overview

Given a graph of very large average degree, the theorem of Kühn and Osthus gives a subgraph G_0 of large average degree with no 4-cycle, and inside it a nonempty vertex set S each of whose vertices has many neighbors in S (Lemma 3.1). Labeling S at random and averaging (Lemma 3.2) gives disjoint sets $U_1, \dots, U_K \subseteq S$ and a set $A \subseteq S$ outside them, more than $2^K + 1$ times as large as $U = U_1 \cup \cdots \cup U_K$, such that every vertex of A has a neighbor in every U_i. Keeping one such edge from each $a \in A$ to each U_i gives a subgraph H in which every $a \in A$ has exactly one neighbor a_i in each U_i. Two vertices of A share at most one of these neighbors, or there would be a 4-cycle.

Fix $k \leq K$ and play the k-color game on H with $U_1, \dots, U_k$; call i the *designated* color of U_i. An uncolored $a \in A$ whose $a_1, \dots, a_k$ carry colors $1, \dots, k$ can never be colored, since colors are never removed, and Bob wins. Alice protects a by making some a_i *dead*: coloring it with another color, or giving color i to a neighbor of it. A move changes the status of at most one vertex of U, but may cost every target adjacent to it. While some vertex of U is *free* (uncolored, with no neighbor of its designated color), Bob gives one its designated color; afterwards he plays any legal move. He maximizes a potential, the sum of $2^{t(a)}$ over the a with no dead a_i, where $t(a)$ counts the a_i carrying their designated color, placed by either player. As for the Erdős–Selfridge theorem [6, p. 299] and Beck's criterion [1], Bob's greedy move gains at least what Alice's next move can lose, up to an error the overlap bound controls. The potential stays above a reserve proportional to the number of free vertices, so once none is free, some uncolored a sees all k colors. Finally, if all subgraphs of G have average degree below d, removing vertices with fewer than d remaining neighbors one at a time orders the vertices so that each has fewer than d earlier neighbors; giving these edges distinct colors splits the edges into d forests.

1.3 Prior work and related questions

Zhu's questions concern hereditary classes because a subgraph can have larger game chromatic number than the graph itself [15, p. 11], as Tuza and Zhu also note in their survey [14, Section 1, p. 306].

Zhu's Question 2 asks the same with the game coloring number in place of the arboricity. In Bodlaender's marking game, where Alice and Bob alternately mark vertices, Alice tries to keep small the largest number of previously marked neighbors of any vertex; the least bound she can guarantee is the *Go number*, and the game coloring number is one more [8, Section 2]. A yes answer to Question 2 would imply one to Question 3 [15, p. 12]. In 2001 Kierstead and Trotter gave "an example which shows that the game chromatic number of a class of graphs may be bounded even when the Go number is not" [8, Section 6], and with a theorem of Dinski and Zhu [3, Theorem 2] their Example 6.1 gives a monotone class with game chromatic number at most 12 and unbounded Go number, a negative answer to Question 2 (Appendix B). Zhu's Question 4 repeats Conjecture 1 of Dinski and Zhu [3], that every monotone class of bounded game chromatic number has bounded acyclic chromatic number, the fewest colors in a proper coloring with no two-colored cycle. It remains open, and would imply Corollary 1.2, since bounded acyclic chromatic number implies bounded arboricity [15, p. 12]; Appendix B discusses Dvořák's reduction of it.

The new step is Theorem 2.1, which adapts the weighting argument behind Beck's criterion for Maker–Breaker games [1], [9, Theorem 4.3]. Unlike Maker, Bob must move legally, and a new estimate bounds the weight Alice's next move can destroy, since two targets share at most one vertex of U.

2 Bob's potential strategy

Theorem 2.1. *Let $k \geq 0$ and $D \geq 0$ be integers, and let G be a graph with a set A of vertices and pairwise disjoint sets $U_1, \dots, U_k$ of vertices, disjoint from A. Write $U = U_1 \cup \cdots \cup U_k$ and assume:*

1. *every neighbor of every vertex of U lies in A;*
2. *every $a \in A$ has exactly one neighbor in U_i, written a_i, for each i;*
3. *two distinct vertices of A have at most one common neighbor in U;*
4. *every vertex of U has at most D neighbors in A, and $2^k|U| + 8D < 8|A|$.*

Then Bob has a winning strategy in the k-color game on G.

The graph G may have further vertices, in neither A nor U, and further edges not meeting U.

Proof. A *position* is a partial proper coloring reached by legal moves. In a position c, call a vertex $w \in U_i$ *taken* if $c(w) = i$, *free* if w is uncolored and no neighbor of w has color i, and *dead* otherwise. Colors never change, so a taken vertex stays taken and a dead vertex stays dead. Call $a \in A$ *alive* if none of $a_1, \dots, a_k$ is dead, and let $t(a)$ be the number of taken vertices among them. Put

$$\Phi(c) = \sum_{a \text{ alive}} 2^{t(a)}, \qquad \Phi_y(c) = \sum_{a \text{ alive},\ a \sim y} 2^{t(a)} \quad (y \in U),$$

and let $F(c)$ be the number of free vertices.

(a) *An alive vertex is uncolored.* If $a \in A$ has color j, then a_j is not taken (the coloring is proper) and not free (its neighbor a has color j), so a_j is dead.

(b) *One move changes the status of at most one vertex of U.* Let a move give the vertex v the color j. If $v \in U_i$, then by (1) no vertex of U is adjacent to v, so only v can change status: if it was free, it becomes taken if $j = i$ and dead otherwise, and if it was dead, it stays dead. If $v \in A$, a free vertex $w \in U_i$ stops being free only if $i = j$ and $w \sim v$, that is, by (2), only if $w = v_j$. If v lies in neither A nor U, then by (1) nothing changes.

(c) *The potential.* If a move makes a free vertex y taken, every alive $a \sim y$ stays alive with $t(a)$ one larger, so Φ grows by $\Phi_y(c)$. If a move makes a free vertex y dead, exactly the alive $a \sim y$ stop being alive, so Φ falls by $\Phi_y(c)$. Otherwise Φ is unchanged.

(d) *Bob's strategy.* When some vertex is free, Bob takes a free vertex x with $\Phi_x(c)$ largest: if $x \in U_i$ he colors it i, which is legal because x is free. Otherwise he makes any legal move.

(e) *The invariants.* For a position c with Bob to move let $I_B(c)$ say $8\Phi(c) > 2^k F(c)$; for a position with Alice to move let $I_A(c)$ say $8\Phi(c) > 2^k F(c)$ and $8(\Phi(c) - \Phi_y(c)) > 2^k(F(c) - 1)$ for every free y. At the start every vertex of U is free, every $a \in A$ is alive with $t(a) = 0$, $\Phi = |A|$, $F = |U|$ and $\Phi_y \le D$, so I_A holds by (4). If $I_A(c)$ holds and Alice moves to c', then by (b) and (c) either a free y became dead, with $\Phi(c') = \Phi(c) - \Phi_y(c)$ and $F(c') = F(c) - 1$, which is the second part of $I_A(c)$; or a free y became taken, and Φ grew while F fell; or nothing changed. So $I_B(c')$ holds.

Now let $I_B(c)$ hold, let some vertex be free, and let Bob take x as in (d), reaching c'. Then $\Phi(c') = \Phi(c) + \Phi_x(c)$ and $F(c') = F(c) - 1$, which gives the first part of $I_A(c')$. Let y be free in c'; then $y \neq x$, y was free in c, and $\Phi_y(c) \le \Phi_x(c)$. The alive vertices of c and c' are the same, and t changed only on those adjacent to x. By (3) at most one alive a is adjacent to both x and y; if there is one, x and y are two of $a_1, \dots, a_k$, both free in c, so $t(a) \le k - 2$ in c and the term of a in Φ_y grew by $2^{t(a)} \le 2^k/4$. Hence $8(\Phi_y(c') - \Phi_y(c)) \le 2 \cdot 2^k$ and

$$8(\Phi(c') - \Phi_y(c')) \ge 8\Phi(c) - 2 \cdot 2^k > 2^k F(c) - 2 \cdot 2^k = 2^k(F(c') - 1).$$

(f) *Bob wins.* Let Bob follow (d), and let p be the first position of the play with no free vertex; there is one, because the last position has none (a free vertex can be colored). By (e) and induction along the play, I_A or I_B holds at every position up to p, according to who moves; the step for Bob is used only at positions with a free vertex. So $\Phi(p) > 0$ and some a is alive at p. As no vertex is free, $a_1, \dots, a_k$ are all taken, so a sees all k colors, and by (a) a is uncolored. It can never be colored, and Bob wins. □

For $k \ge 2$ this adapts the weighting argument behind Beck's criterion [1], [9, Theorem 4.3]: U plays the role of the board and the sets $\{a_1, \dots, a_k\}$ the winning sets of a k-uniform family in which two sets share at most one element. New are the legality of Bob's moves, tracked by the statuses free, taken and dead, and the overlap estimate in (e). The term $8D$ pays for Alice's first move.

Corollary 2.2. *Let $K \ge 2$ and let G consist of pairwise disjoint vertex sets $A, U_1, \dots, U_K$ and edges joining each $a \in A$ to exactly one vertex a_i of each U_i, with no other edges (further isolated vertices are allowed). If G has no 4-cycle and $(2^K + 8)|U| < 8|A|$, where $U = U_1 \cup \cdots \cup U_K$, then $\chi_g(G) > K$.*

Proof. Fix $k \le K$ and apply Theorem 2.1 with the sets $U_1, \dots, U_k$; the vertices of the other sets lie in neither A nor the union of the sets used. Conditions (1) and (2) are immediate, and (3) holds because two vertices of A with two common neighbors span a 4-cycle. For (4), let $w \in U_i$ and pick $j \neq i$. Two distinct neighbors a, a' of w have $a_j \neq a'_j$, since otherwise w and a_j would be two common neighbors. So w has at most $|U_j| \le |U|$ neighbors, and (4) holds with $D = |U|$ because $2^k|U_1 \cup \cdots \cup U_k| + 8|U| \le (2^K + 8)|U|$. □

3 Extraction

Lemma 3.1. *A graph with at least one vertex and average degree at least $t > 0$ has a nonempty vertex set S in which every vertex has at least $t/2$ neighbors in S.*

Proof. Write $e(S)$ for the number of edges inside S, and take S nonempty of least size with $2e(S) \geq t|S|$; the whole vertex set qualifies. If some $v \in S$ had fewer than $t/2$ neighbors in S, then $S \setminus \{v\}$ would be nonempty, since $e(S) > 0$, and $2e(S \setminus \{v\}) > 2e(S) - t \geq t|S \setminus \{v\}|$, against the choice of S. □

Lemma 3.2. *Let $K, M \geq 0$ be integers and let S be a nonempty set of vertices of a graph such that each vertex in S has at least $\delta = 2(M+1)K(K+1)$ neighbors in S. Then there are disjoint sets $U_1, \dots, U_K \subseteq S$ and a set $A \subseteq S \setminus (U_1 \cup \cdots \cup U_K)$ such that every $a \in A$ has a neighbor in each U_i and $M|U_1 \cup \cdots \cup U_K| < |A|$.*

Proof. For $K = 0$ take $A = S$. Let $K \geq 1$, $N = 2(M+1)K$ and $n = |S|$, and consider all N^n labelings $\ell\colon S \to \{0, \dots, N-1\}$. For a labeling, let $U_i(\ell) = \ell^{-1}(i-1)$ for $i \leq K$, let $U(\ell)$ be their union, and let $A(\ell)$ be the set of $x \in S$ with $\ell(x) \geq K$ that have a neighbor in every $U_i(\ell)$. Each x lies in $U(\ell)$ for KN^{n-1} labelings, so $\sum_\ell |U(\ell)| = nKN^{n-1}$. For fixed x and i, fix δ neighbors of x in S; the labelings in which none of them has label $i-1$ number $(N-1)^\delta N^{n-\delta}$, and they include those in which no neighbor of x in S has label $i-1$. So x lies in $A(\ell)$ for at least $N^n - KN^{n-1} - K(N-1)^\delta N^{n-\delta}$ labelings. Since $N \geq 2$ and, by the binomial theorem, $(N/(N-1))^N \geq 1 + N/(N-1) > 2$, we have $2(N-1)^N \leq N^N$; as $\delta = N(K+1)$, this gives $2^{K+1}(N-1)^\delta \leq N^\delta$, and since $2^{K+1} > 2K$ it follows that $K(N-1)^\delta N^{n-\delta} < N^n/2$. Also $(M+1)KN^{n-1} = N^n/2$. Therefore

$$\sum_\ell \bigl(|A(\ell)| - M|U(\ell)|\bigr) \geq n\bigl(N^n - (M+1)KN^{n-1} - K(N-1)^\delta N^{n-\delta}\bigr) > 0,$$

and some labeling has $|A(\ell)| > M|U(\ell)|$. □

4 The cited theorem

Theorem 4.1 (Kühn and Osthus [10, Theorem 1])**.** *"For every k there exists $d = d(k)$ such that every graph of average degree at least d contains a subgraph of average degree at least k whose girth is at least six."*

We use it in this form, the hypothesis `C4FreeSubgraphTheorem` of the Lean proof (Section 6).

Theorem 4.2. *For every integer $t \geq 0$ there is an integer d such that every graph with at least one vertex and average degree at least d has a subgraph with at least one vertex, no 4-cycle and average degree at least t.*

Derivation from Theorem 4.1. For $t = 0$ take $d = 0$: a single vertex is such a subgraph. For $t \geq 1$ take $d = \lceil d(t) \rceil$ from Theorem 4.1 with $k = t$. Since $d \geq d(t)$, a graph of average degree at least d has a subgraph of girth at least six and average degree at least $t \geq 1$. That subgraph has an edge, hence a vertex, and its girth excludes 4-cycles. □

Appendix C gives an explicit d.

5 Proofs of the main results

Proof of Theorem 1.1. The conclusion for K contains the conclusion for every smaller K, so we may assume $K \geq 2$. Let $M = 2^K + 1$, $\delta = 2(M+1)K(K+1)$, $t = 2\delta$, and let d be given by Theorem 4.2 for this t. Let G have at least one vertex and average degree at least d. Theorem 4.2 gives a subgraph G_0 with no 4-cycle and average degree at least t; Lemma 3.1 gives a set S in which every vertex has at least δ neighbors in S; and Lemma 3.2 gives A and $U_1, \ldots, U_K$ with $M|U| < |A|$. Choose for each $a \in A$ one neighbor $a_i \in U_i$ for each i, and let H be the subgraph of G_0 on the vertex set of G_0 whose edges are the pairs aa_i. Then H has no 4-cycle, and $(2^K + 8)|U| \leq 8M|U| < 8|A|$, also when U is empty, since then A is not. By Corollary 2.2, $\chi_g(H) > K$. □

Proof of Corollary 1.2. Let $d = d(K)$ be as in Theorem 1.1, let $\mathcal{C}$ be a monotone class whose members have game chromatic number at most K, and let $G \in \mathcal{C}$. Every subgraph of G lies in $\mathcal{C}$, so by Theorem 1.1 every subgraph of G with at least one vertex has average degree less than d. Hence every nonempty vertex set S contains a vertex with fewer than d neighbors in S, since otherwise the subgraph induced by S would have average degree at least d. Removing such vertices one at a time orders the vertices as $v_1, \ldots, v_n$ so that each v_m has fewer than d neighbors among $v_1, \ldots, v_{m-1}$. Color the edges from each v_m to earlier vertices with distinct colors from $1, \ldots, d$. In each color class every vertex has at most one edge to an earlier vertex, so the class has no cycle, as the latest vertex of a cycle would have two. So G has arboricity at most d. □

6 Formal verification

Theorem 1.1 and Corollary 1.2 are proved in Lean 4 [11] with the Mathlib library [12], from definitions of the game given verbatim in Appendix A. A position gives each vertex one of the k colors, indexed by `Fin k`, that is $\{0, \ldots, k-1\}$, or `none`, meaning uncolored. A move gives one uncolored vertex a color that none of its neighbors has. The inductive predicate `AliceWins G k b c` says that Alice can force a win from the position `c`, with Alice to move when `b` is `true`: every vertex is colored; or it is Alice's turn and some move of hers leads to such a position; or it is Bob's turn, he has a move, and all his moves lead to such positions. `AliceWinsGame G k` is this predicate at the empty position with Alice to move. On a finite graph, `¬ AliceWinsGame G k` says exactly that Bob has a winning strategy in the k-color game (Appendix A). The cited theorem is the proposition

```
def C4FreeSubgraphTheorem : Prop :=
  ∀ t : ℕ, ∃ d : ℕ, ∀ (V : Type) [Finite V] [Nonempty V] (G : SimpleGraph V),
    d * Nat.card V ≤ 2 * G.edgeSet.ncard →
    ∃ H : G.Subgraph, H.verts.Nonempty ∧ C4Free H.coe ∧
      t * H.verts.ncard ≤ 2 * H.coe.edgeSet.ncard
```

where `C4Free H` says that there are no a, b, c, d with $a \neq c$, $b \neq d$ and $a \sim b \sim c \sim d \sim a$; this is Theorem 4.2, with "average degree at least d" written as $d|V| \leq 2|E|$. The two main results, in `Main.lean`, are

```
theorem GCG.exists_subgraph_bob_wins (hC4 : C4FreeSubgraphTheorem) (K : ℕ) :
    ∃ d : ℕ, ∀ (V : Type) [Finite V] [Nonempty V] (G : SimpleGraph V),
      d * Nat.card V ≤ 2 * G.edgeSet.ncard →
      ∃ H : G.Subgraph, ∀ k ≤ K, ¬ AliceWinsGame H.coe k
```

```
theorem GCG.bounded_arboricity (hC4 : C4FreeSubgraphTheorem) (K : ℕ) :
    ∃ d : ℕ, ∀ C : (V : Type) → SimpleGraph V → Prop,
      (∀ V G, C V G → Finite V) →
      (∀ V G, C V G → ∀ H : G.Subgraph, C H.verts H.coe) →
      (∀ V G, C V G → ∃ k ≤ K, AliceWinsGame G k) →
      ∀ V G, C V G → ArboricityLE G d
```

The first is Theorem 1.1, where `H.coe` is the graph formed by a Mathlib subgraph `H`. In the second, a class is a predicate `C` whose three hypotheses say that its members are finite, closed under subgraphs, and of game chromatic number at most K; `ArboricityLE G d` gives arboricity at most d (Appendix A). The build fails unless the axioms of the two main theorems and of Theorem 2.1 are exactly Lean's `propext`, `Classical.choice` and `Quot.sound`. Theorem 2.1 and Lemma 3.2 are formalized as written. Lemma 3.1 is checked for integer t, and Corollary 2.2 under the stronger hypothesis $(2^K + 1)|U| < |A|$; these are the cases Section 5 uses, and Appendix A notes the differences in detail and gives the commands that reproduce the check.

Appendix A The Lean definitions and correspondence

A position is a function `c : V → Option (Fin k)`: `Fin k` is the set $\{0, \dots, k-1\}$ indexing the k colors, and the value `none` means uncolored. The game is defined in `GCG/Game.lean`:

```
def Move (G : SimpleGraph V) {k : ℕ} (c c' : V → Option (Fin k)) : Prop :=
  ∃ v i, c v = none ∧ (∀ w, G.Adj v w → c w ≠ some i) ∧ c' v = some i ∧
    ∀ u, u ≠ v → c' u = c u

inductive AliceWins (G : SimpleGraph V) (k : ℕ) :
    Bool → (V → Option (Fin k)) → Prop
  | full (b : Bool) (c : V → Option (Fin k)) :
      (∀ v, c v ≠ none) → AliceWins G k b c
  | alice (c c' : V → Option (Fin k)) :
      Move G c c' → AliceWins G k false c' → AliceWins G k true c
  | bob (c : V → Option (Fin k)) :
      (∃ c', Move G c c') → (∀ c', Move G c c' → AliceWins G k true c') →
      AliceWins G k false c

def AliceWinsGame (G : SimpleGraph V) (k : ℕ) : Prop :=
  AliceWins G k true (fun _ => none)
```

Why ¬ `AliceWinsGame G k` means that Bob wins. Each move colors a vertex, so on a finite graph every play ends after at most $|V|$ moves. If `AliceWins` holds at a position, the constructors used in its derivation give Alice a strategy that wins from there: they never let Bob reach a position where it fails, and a play can end only at a full coloring. Suppose it fails at a position. If the player to move has no legal move, play ends there, and since `full` does not apply, some vertex is uncolored: Bob has won. If it is Alice's turn, then by `alice` it fails after each of her moves. If it is Bob's turn and he has a move, then by `bob` it fails after some move of his, and he plays that move. Following this, Bob keeps `AliceWins` false until play ends, so he wins. Zhu [15, pp. 1–2] awards Bob the game as soon as an uncolored vertex has all k colors among its neighbors. Such a vertex stays uncolorable, so under the rules of Section 1 play ends with it uncolored and Bob wins as well;

conversely, a play that ends with an uncolored vertex reached such a vertex. The two conventions give the same winner.

Arboricity. `ArboricityLE G d` says that some map from all unordered pairs of vertices to d colors makes every color class of edges of G a forest. Restricting the map to the edges partitions them into at most d forests, so G has arboricity at most d; conversely, for $d > 0$ a partition into d forests extends to such a map. For $d = 0$ the predicate fails on every graph with a vertex, which does not affect an existential upper bound.

Correspondence. Both main theorems take the cited theorem as the hypothesis `hC4`, and their proofs establish its hypotheses (finiteness, a vertex and the density bound) for each graph to which they apply it. `C4Free` is in `GCG/Game.lean` and `ArboricityLE` in `GCG/Forest.lean`. Theorem 2.1 is `GCG.bob_wins_of_structure` in `GCG/Potential.lean`, in which conditions (1)–(4) appear as hypotheses on a set `A`, a family of classes and a map assigning to each $a \in A$ and i its neighbor a_i. Lemma 3.1 is `GCG.exists_minDegree_subset` in `GCG/Extract.lean`, for a positive integer threshold t, the case used in Section 5; the written proof covers real $t > 0$. Lemma 3.2 is `GCG.extract` in the same file. `GCG.bob_wins_pickGraph` in `Main.lean` is Corollary 2.2 with the hypothesis $(2^K + 8)|U| < 8|A|$ replaced by the stronger $(2^K + 1)|U| < |A|$, which is what Lemma 3.2 provides in Section 5; the graph whose edges are the pairs aa_i is `GCG.pickGraph`. The printed Corollary 2.2 follows from the checked Theorem 2.1 by the written proof. The degeneracy step in the proof of Corollary 1.2 is `GCG.arboricityLE_of_degenerate` in `GCG/Forest.lean`. The axiom checks are in `Axioms.lean`.

The formal proof of Theorem 1.1 applies the cited theorem with $t = 4(2^{K+2}+2)(K+2)(K+3)$, which is the value of t in Section 5 for $K + 2$ in place of K; this covers $K < 2$ without a separate case. Its subgraph H keeps all the vertices of G_0, as in the written proof.

To reproduce the check, run `lake exe cache get` and `lake build` in the `lean` folder of the published development (linked below the title), which pins Lean v4.34.0 and a Mathlib commit; with the Mathlib cache the build takes about a minute.

Appendix B Zhu's Questions 2 and 4

Kierstead and Trotter [8, Example 6.1] subdivide every edge of $K_{n,n}$ once; their proof shows that these graphs G_n have acyclic chromatic number at most 3 and game chromatic number at most 4, and that their Go number tends to infinity. By a theorem of Dinski and Zhu [3, Theorem 2], every subgraph of a graph with acyclic chromatic number at most k has game chromatic number at most $k(k + 1)$. So the subgraphs of the graphs G_n form a monotone class with game chromatic number at most 12 and unbounded game coloring number, which answers Question 2 in the negative.

Dvořák [4, Section 3.1.5] showed that the conjecture of Dinski and Zhu follows from Conjecture 3.1 of [4] (Conjecture 1 of [5], derived there from its Corollary 4): "There exists a function f such that for each graph G, if $\chi(G) \geq f(c)$ then the game chromatic number of $\mathrm{sd}_1(G)$ is at least c", where $\chi(G)$ is the chromatic number, c ranges over the positive integers and $\mathrm{sd}_1(G)$ is obtained by subdividing every edge of G once. Dvořák calls it "wide open". The present argument does not reach these graphs. In a subgraph of $\mathrm{sd}_1(G)$, every edge meets a subdivision vertex and these have degree at most 2, so there are at most twice as many edges as subdivision vertices; a subgraph with an edge also has a vertex of G, so its average degree is less than 4, and Theorem 1.1 says nothing about it.

Appendix C An explicit bound

Montgomery, Pokrovskiy and Sudakov [13, Theorem 1.1] showed that there is $c > 0$ such that for every integer $t \geq 1$, every graph of average degree at least $2^{ct^2 \log t}$ has a subgraph of average degree at least t and girth at least 6; so $d = \lceil 2^{ct^2 \log t} \rceil$ serves in Theorem 4.2 for $t \geq 1$. Section 5 applies it with t of order $K^2 2^K$, so $d(K)$ is at most doubly exponential in K; we made no attempt to optimize it.

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