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COMPLETENESS OF TWO FLOOR-DOUBLING SEQUENCES

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Abstract. We prove that, whenever $\alpha, \beta > 0$ and α/β is irrational, every sufficiently large integer is a sum of distinct indexed terms of the two sequences $(\lfloor 2^n\alpha \rfloor)_{n\geq 0}$ and $(\lfloor 2^n\beta \rfloor)_{n\geq 0}$. This answers the doubling case of Erdős Problem 354 affirmatively. The proof associates a symbolic tower system to each sequence and turns infinitely many missing sums into a nonproduct invariant coupling. Binary carries give two complementary estimates: many marked transitions force decay of correlations, whereas a single common marked transition controls one-step displacement by displacement at a prescribed large time. The latter estimate forces a forward relation between the positions of binary ones in the two parameters. Applying that relation in both directions bounds all zero runs, contradicting the correlation estimate. A complete Lean 4 formalization of the exact conjecture accompanies the paper; its final theorem has no admitted proof and uses only the standard axioms `propext`, `Classical.choice`, and `Quot.sound`.

1. Introduction

A sequence of nonnegative integers is called complete if its finite subsequence sums contain every sufficiently large integer. The number of summands is unrestricted, but each index may be used at most once. The question considered here appears in the study of complete sequences initiated by Graham [4] and is recorded by Erdős and Graham [2, p. 58]. It is Problem 354 in Bloom's catalogue [1]. Earlier investigations include Hegyvári [5]; his later discussion [6] also describes the connection with binary bracket sequences.

Theorem 1.1. *Let α, β be positive real numbers such that α/β is irrational. There is an integer N_0 such that every integer $N \geq N_0$ has a representation*

$$N = \sum_{s\in S} \lfloor 2^s\alpha \rfloor + \sum_{t\in T} \left\lfloor 2^t\beta \right\rfloor, \qquad S, T \subset \mathbb{N}_0 \text{ finite.} \tag{1.1}$$

Here and throughout, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The sets S and T are chosen independently. Thus two equal numerical values occurring at different indices remain two available terms. This is precisely the multiset convention in the formalized problem [3]. The theorem concerns the multiplier 2. The separate question about a multiplier in $(1, 2)$ is outside its scope.

The binary recurrence is the starting point. Put

$$h_\gamma(n) = \lfloor 2^n\gamma \rfloor, \qquad d_\gamma(n) = h_\gamma(n+1) - 2h_\gamma(n) \in \{0, 1\}.$$

The digit $d_\gamma(n)$ records whether one spacer is inserted after doubling a word of length $h_\gamma(n)$. The resulting tower systems retain both the subset-sum information and the binary carries generated by large translations.

The main estimate is a comparison of two displacement energies. If a large translation has a suitable marked carry at stage m, then every stage-m observable f, bounded by F, satisfies

$$\|U^{-1}f - f\|_2^2 \leq 16\|U^t f - f\|_2^2 + 36F^2\varepsilon_m + 676F^2(\ell + 1)w_m. \tag{1.2}$$

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Here U is the Koopman operator of the tower, $w_m = (2^m\gamma)^{-1}$, $\varepsilon_m = 1 - h_\gamma(m)w_m$, and 2^ℓ bounds the relevant return query. The constants are absolute. A pair of three-bit carry paths produces the factor 16, and elementary interval counting controls both return regions simultaneously. This estimate is proved in Section 6.

The final argument has a simple form. Failure of disjointness forces every sufficiently late binary one of α to be followed, at a fixed positive offset and within a bounded distance, by a one of β. Reversing the parameters gives a bounded forward return to the ones of α. This bounds the zero runs of α. But bounded zero runs already imply disjointness, by a second use of the carry paths. The contradiction proves completeness.

All arguments needed for Theorem 1.1, including the construction of the symbolic systems and the estimates above, are implemented in the accompanying Lean development. Section 8 gives the exact target, verification information, and a guide to the proof files.

2. Binary heights and the symbolic tower model

We first work with a parameter $\gamma \geq 1$, so that every height is positive. We suppress γ from the notation when it is fixed.

Lemma 2.1. *For $n \geq 0$,*

$$h_{n+1} = 2h_n + d_n, \quad d_n \in \{0,1\}, \tag{2.1}$$

$$h_n - \sum_{j<n} h_j = h_0 + \sum_{j<n} d_j \geq 1. \tag{2.2}$$

Moreover,

$$2^m\gamma = h_m + \sum_{j\geq 0} \frac{d_{m+j}}{2^{j+1}}. \tag{2.3}$$

Every digit sequence d_γ has zeros arbitrarily far to the right. If γ is irrational, it has both ones and adjacent digit transitions arbitrarily far to the right. The digit assertions and (2.3) *remain valid for every real γ, without the normalization $\gamma \geq 1$.*

Proof. Writing $2^n\gamma = h_n + \theta_n$ with $0 \leq \theta_n < 1$ gives $d_n = \lfloor 2\theta_n \rfloor \in \{0,1\}$. Induction gives (2.2), and iteration of (2.1), followed by division by the corresponding power of 2, gives (2.3). If a tail consisted entirely of ones, the sum on the right of (2.3) would be 1, contrary to the definition of h_m. If a tail consisted entirely of zeros, γ would be dyadic rational. Finally, finitely many transitions would make the digit sequence eventually constant. □

Define words by

$$W_0 = 1\,0^{h_0-1}, \qquad W_{n+1} = W_n W_n 0^{d_n}.$$

The word W_n has length h_n. Its one positions are exactly $\{\sum_{j\in A} h_j : A \subset \{0,\dots,n-1\}\}$, by (2.2). The increasing union of these words is the indicator of

$$\mathcal{P}_\gamma = \left\{ \sum_{j\in A} h_\gamma(j) : A \subset \mathbb{N}_0 \text{ finite} \right\}$$

on the nonnegative integers. Set that indicator equal to zero at negative integers, and denote the resulting two-sided binary name by $v_\gamma \in \{0,1\}^{\mathbb{Z}}$.

It is useful to retain every tower label, rather than just the binary name. For $m, q \geq 0$ define the full return position

$$G_{\gamma,m}(q) = qh_\gamma(m) + \sum_{j\geq 0} d_\gamma(m+j) \left\lfloor \frac{q}{2^{j+1}} \right\rfloor. \tag{2.4}$$

The sum is finite for fixed q. Directly from this formula,

$$qh_m \le G_m(q) \le q(h_m+1), \qquad G_m(q)+h_m \le G_m(q+1), \tag{2.5}$$

$$G_m(2^K) = h_{m+K}, \qquad G_m(2^K q + r) = G_{m+K}(q) + G_m(r) \quad (0 \le r < 2^K). \tag{2.6}$$

Alternatively, $G_m(q) = \sum_{j\ge 0} q_j h_{m+j}$, where q_j is the jth binary bit of q, counted from the least significant bit. This also proves (2.6).

For $t \ge 0$, the stage-m label $\lambda_m(t)$ is i if $t = G_m(q) + i$ for some $q \ge 0$ and $0 \le i < h_m$. All remaining positions, including negative t, have label h_m, called the outside label. These rules define a name

$$x_\gamma \in X_\gamma := \left(\prod_{m\ge 0} \{0, \dots, h_\gamma(m)\} \right)^{\mathbb{Z}}.$$

The space is compact and metrizable. Let $\sigma^k x(t) = x(t+k)$. A tower-name limit is any subsequential weak limit of

$$\frac{1}{N+1} \sum_{t=0}^{N} \delta_{\sigma^t x_\gamma}$$

as $N \to \infty$. Such limits exist and are invariant under σ. Write $E_{m,i} = \{x : x(0,m) = i\}$ for an ordinary level, and $B_m = \bigcup_{i<h_m} E_{m,i}$ for the tower body.

Proposition 2.2. *Every tower-name limit μ has the following properties.*

(i) *Each ordinary level has mass*

$$\mu(E_{m,i}) = w_m := \frac{1}{2^m \gamma}.$$

Consequently, $\mu(B_m^c) = \varepsilon_m := 1 - h_m w_m$, with $0 \le \varepsilon_m \le 2^{-m}$.

(ii) *The stage-$(m+K)$ tower contains 2^K ordinary copies of the stage-m tower, starting at $G_m(r)$, $0 \le r < 2^K$. Each fine level has mass $w_{m+K} = 2^{-K} w_m$. The stage partitions refine one another almost everywhere.*

(iii) *Functions of a single current tower label, that is, $f(x) = a(x(0,m))$, are dense in $L^2(\mu)$ as m and a vary. Such a function at a fixed stage can be expressed at every later stage without increasing its bound.*

(iv) *The system (X_γ, μ, σ) is ergodic. Thus the fixed vectors of $Uf = f \circ \sigma$ are precisely the constant functions.*

Proof. Formula (2.6) proves the copy and refinement identities on the original name, away from its initial endpoint. Their cylinder formulations therefore hold almost everywhere in every empirical limit. Shifting within a copy identifies all ordinary level masses.

One can calculate those masses directly. Formula (2.3) and dominated convergence give $G_m(q)/q \to 2^m\gamma$ as $q \to \infty$. Each fixed ordinary label occurs once in every complete return block. Hence its frequency is $(2^m\gamma)^{-1}$. The assertions about the body and fine levels follow.

To prove density, fix a cylinder depending on finitely many stages and on times t with $|t| \le R$. For all sufficiently large m, the current stage-m level determines every specified coordinate whenever that level is at least R from both ends of the tower. The exceptional mass is at most $\varepsilon_m + 2Rw_m$. Thus cylinder indicators can be approximated in L^2 by functions of the current stage-m label. Cylinders generate the Borel sigma-algebra, which proves density. Refinement gives the assertion about a fixed observable and its bound.

For ergodicity, let A be an invariant measurable set. Invariance implies that $\mu(A \cap E_{m,i})/w_m$ is the same number p_m on every ordinary level. Since the outside mass tends to zero, $p_m \to \mu(A)$. The conditional expectation of $\mathbf{1}_A$ onto the stage-m partition therefore converges in L^2 to the constant

$\mu(A)$. Density and refinement imply that the same conditional expectations converge to $\mathbf{1}_A$. Hence $\mathbf{1}_A = \mu(A)$ almost everywhere, so $\mu(A) \in \{0, 1\}$. □

All Hilbert spaces below are real. The operator U is an invertible linear isometry, and $\langle f, g\rangle = \int fg\, d\mu$. We write $L^2_0(\mu)$ for the functions of integral zero. The coordinate map $x \mapsto \mathbf{1}_{\{x(t,0)=0\}}$, $t \in \mathbb{Z}$, is a continuous shift factor sending x_γ to v_γ. Every empirical limit of v_γ is the image of a tower-name limit: take a further convergent subsequence of the corresponding labeled empirical measures. In particular its one cylinder has mass $w_0 = 1/\gamma > 0$.

3. Missing sums, couplings, and rigidity

A binary-name limit for γ is a subsequential empirical limit of the shifts of v_γ. Given binary-name limits ν_α and ν_β, an anti-joining is a probability measure η on $\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ with those marginals, invariant under $\sigma \times \sigma^{-1}$. We write $\mathcal{D}(\alpha, \beta)$ if every such anti-joining is the product of its marginals. Swapping coordinates and using inverse invariance shows that $\mathcal{D}(\alpha, \beta)$ is symmetric.

Lemma 3.1. *If $\alpha, \beta \geq 1$ and $\mathcal{D}(\alpha, \beta)$ holds, then $\mathcal{P}_\alpha + \mathcal{P}_\beta$ contains all sufficiently large integers.*

Proof. If this failed, choose missing integers $N_r \to \infty$ and consider

$$\eta_r = \frac{1}{N_r + 1} \sum_{t=0}^{N_r} \delta_{(\sigma^t v_\alpha, \sigma^{N_r - t} v_\beta)}.$$

Compactness gives a convergent subsequence. The two endpoint terms are the only discrepancy between η_r and its image under $\sigma \times \sigma^{-1}$, so the limit η is an anti-joining. Its marginals are binary-name limits. Every η_r gives mass zero to the clopen cylinder $\{(x, y) : x(0) = y(0) = 1\}$, since a point in that cylinder would represent N_r. Its limiting mass is also zero. The product of the marginals gives it mass $(1/\alpha)(1/\beta) > 0$, a contradiction. □

We will rule out anti-joinings through their Hilbert-space operators. An anti-joining defines a contraction $J_0 : L^2(\nu_\beta) \to L^2(\nu_\alpha)$ by

$$\langle g, J_0 f\rangle = \int g(x) f(y)\, d\eta(x, y).$$

It preserves constants and integrals and satisfies $U_\alpha^k J_0 = J_0 U_\beta^{-k}$. If it annihilates $L^2_0(\nu_\beta)$, then η is product, first on rectangles and hence on the whole product sigma-algebra.

These operators can be tested on the labeled systems of Section 2. Indeed, the factor maps induce isometric embeddings $I_\gamma : L^2(\nu_\gamma) \to L^2(\mu_\gamma)$. The operator $J = I_\alpha J_0 I_\beta^*$ again preserves constants and integrals and intertwines U_α^k with U_β^{-k}. Moreover $I_\alpha^* J I_\beta = J_0$. It therefore suffices to prove that every such operator between tower-name limits annihilates the space of mean-zero functions.

Lemma 3.2 (Zero blocks)**.** *Let μ be a tower-name limit for $\gamma \geq 1$. If $d_\gamma(n + j) = 0$ for $j + 1 < L$, and f is an observable from a stage at most n with $|f| \leq F$, then*

$$\|U^{h_\gamma(n)} f - f\|_2^2 \leq \frac{4}{2^L} \|f\|_2^2 + 4F^2 \varepsilon_n. \tag{3.1}$$

Consequently:

(i) *if $n_r, L_r \to \infty$ and $d_\gamma(n_r + j) = 0$ for $j + 1 < L_r$, then $U^{\pm h_\gamma(n_r)} \to I$ strongly;*
(ii) *if $n_r \to \infty$ and $d_\gamma(n_r) = 0$, then for every $f \in L^2(\mu)$ and every $\epsilon > 0$, eventually*

$$\|U^{\pm h_\gamma(n_r)} f - f\|_2^2 \leq \|f\|_2^2 + \epsilon. \tag{3.2}$$

For $f \neq 0$, this implies eventually $\langle f, U^{\pm h_\gamma(n_r)} f\rangle \geq \|f\|_2^2/8$.

Proof. Among the 2^L stage-n copies in stage $n+L$, every copy before the last is followed immediately by the next one. Indeed $G_n(r) = rh_n$ for $r < 2^L$ under the stated zero assumptions; the unrestricted last digit can only place a spacer after the last copy. Translation by h_n preserves f on these first $2^L - 1$ copies. The integral of f^2 over their complement is at most $2^{-L}\|f\|_2^2 + F^2\varepsilon_n$. Since $\|U^{h_n} f\|_2 = \|f\|_2$, the two functions have equal square integrals on the complement of their equality set. The inequality $|a-b|^2 \le 2a^2 + 2b^2$ now proves (3.1).

Refine a fixed earlier observable to stage n_r and pass to the limit. Density gives (i) for every L^2 function. Taking $L = 2$ proves (ii), again by density. The negative-time displacement has the same norm as the positive-time displacement. Finally use $\|U^t f - f\|_2^2 = 2\|f\|_2^2 - 2\langle f, U^t f\rangle$ and choose $\epsilon > 0$ sufficiently small. □

Lemma 3.3 (Two disjointness criteria)**.** *For $\alpha, \beta \ge 1$, either of the following conditions implies $\mathcal{D}(\alpha, \beta)$.*

(i) *For every tower-name limit of α, $\langle g, U_\alpha^{h_\beta(n)} f\rangle \to 0$ whenever f has mean zero and $g \in L^2$.*
(ii) *There are $n_r, L_r \to \infty$ with $d_\beta(n_r + j) = 0$ for $j + 1 < L_r$, and, in every tower-name limit of α, the only mean-zero vector satisfying $U_\alpha^{h_\beta(n_r)} f \to f$ is zero.*

Proof. Use the lifted operator J above. For (i), choose an unbounded sequence n_r with $d_\beta(n_r) = 0$, possible by Lemma 2.1. If $g \in L_0^2(\mu_\alpha)$, put $v = J^* g$. The constant and integral identities imply that v and Jv have mean zero. Intertwining gives

$$\langle v, U_\beta^{-h_\beta(n_r)} v\rangle = \langle g, U_\alpha^{h_\beta(n_r)} JJ^* g\rangle \longrightarrow 0.$$

Lemma 3.2(ii) forces $v = 0$. Thus J^* vanishes on the mean-zero space, and so does J.

For (ii), Lemma 3.2(i) gives $U_\beta^{-h_\beta(n_r)} f \to f$. Hence $U_\alpha^{h_\beta(n_r)} Jf \to Jf$. If f has mean zero, the assumed exclusion gives $Jf = 0$. □

4. Carries and the two return regions

Fix a tower-name limit for $\alpha \ge 1$ and write $U = U_\alpha$, $h = h_\alpha(m)$, $G = G_{\alpha,m}$. For an integer $q \ge 0$, let $q_j \in \{0, 1\}$ be its jth bit. For a uniformly chosen K-bit word x, viewed also as an integer $0 \le x < 2^K$, let $c_j(x, q)$ be the carry into position j when adding x and q. Thus $c_0 = 0$ and $c_{j+1} = \mathbf{1}_{\{x_j + q_j + c_j \ge 2\}}$. Define

$$C_q^K(x) = \sum_{j<K} d_\alpha(m + j) c_{j+1}(x, q). \tag{4.1}$$

The dependence on α, m is suppressed. The elementary addition identity is

$$G(x + q) = G(x) + G(q) + C_q^K(x) \qquad (x + q < 2^K). \tag{4.2}$$

It follows either by subtracting the floor sums in (2.4), or by the usual binary carrying algorithm.

Lemma 4.1 (Mean carry cost)**.** *If $q + 1 \le 2^\ell$ and $\Delta = G(q+1) - G(q) - h$, then*

$$0 \le \Delta \le \ell + 1, \qquad \mathbb{E}(C_q^K + \Delta + C_{q+1}^K) \le 3(\ell + 1) \tag{4.3}$$

for every K.

Proof. For $1 \le j \le K$, counting residues modulo 2^j gives $\mathbb{E}c_j = (q \bmod 2^j)/2^j$. The first ℓ terms of their sum are at most ℓ and the remaining geometric tail is at most 1. The weights in (4.1) lie in $\{0, 1\}$, so $\mathbb{E}C_q^K \le \ell + 1$, and likewise for $q + 1$. Nonnegativity of Δ follows from (2.5). For its upper bound apply (4.2) with starting position q, increment 1, and $K = \ell + 1$. Since $G(1) = h$, the gap is a carry cost and is at most K. □

With an infinite sequence of independent fair bits, addition of a fixed integer has only finitely many nonzero carries almost surely. Above its highest nonzero bit, a carry survives only through a run of ones, whose probability decreases geometrically. Thus $C_q^K \to C_q < \infty$ almost surely. The contractions

$$P_{m,q}^K f = \mathbb{E} U^{-C_q^K} f, \qquad P_{m,q} f = \mathbb{E} U^{-C_q} f \tag{4.4}$$

satisfy $P_{m,q}^K f \to P_{m,q} f$ in norm for each f.

Fix a time $t \geq 0$, and choose the unique q with $G(q) \leq t < G(q+1)$. Set $r = t - G(q)$, so $0 \leq r < h + \Delta$. Among the ordinary stage-m levels there are two regions:

$$I_0 = \bigcup_{\substack{0 \leq i < h \\ i+r < h}} E_{m,i}, \qquad I_1 = \bigcup_{\substack{0 \leq i < h \\ i+r \geq h}} E_{m,i}.$$

The split uses the same r for every fine copy.

Proposition 4.2 (Adaptive correlation)**.** *If f, g are stage-m observables with $|f| \leq F$, $|g| \leq H$, and $q + 1 \leq 2^\ell$, then*

$$\langle g, U^t f \rangle = \langle \mathbf{1}_{I_0} g, U^r P_{m,q} f \rangle + \langle \mathbf{1}_{I_1} g, U^{r-h-\Delta} P_{m,q+1} f \rangle + e, \tag{4.5}$$

$$|e| \leq FH\varepsilon_m + 6FH(\ell+1)w_m. \tag{4.6}$$

In particular,

$$|\langle g, U^t f \rangle| \leq \|g\|_2 \big(\|P_{m,q} f\|_2 + \|P_{m,q+1} f\|_2 \big) + FH\varepsilon_m + 6FH(\ell+1)w_m. \tag{4.7}$$

Proof. Work first in stage $m + K$, whose ordinary stage-m copies are indexed by $x < 2^K$. If $x + q + 1 < 2^K$, identity (4.2) gives the proposed destination coordinates

$$\Phi_{c_0,c_1}(i) = \begin{cases} i + r - c_0, & i + r < h, \\ i + r - h - \Delta - c_1, & i + r \geq h, \end{cases} \qquad (c_0, c_1) = (C_q^K(x), C_{q+1}^K(x)). \tag{4.8}$$

Whenever the indicated coordinate lies in $[0, h)$, the proposed translation and the actual translation end at the same fine level. The numbers of exceptions in the two regions are at most c_0 and $\Delta + c_1$, respectively. In the first region only the lower endpoint can fail. In the second, the lower endpoint and the possible excess past h together account for at most $\Delta + c_1$ levels.

Each exceptional product changes the correlation by at most $2FH$. Summing over the equally weighted copies gives an error at most $2FHw_m \mathbb{E}(C_q^K + \Delta + C_{q+1}^K)$. At most $q + 1$ top copies are excluded; they contribute at most $2FH(q+1)hw_{m+K}$. The outside body contributes at most $FH\varepsilon_m$. Integrating the proposed shifts over all copies gives the finite version of (4.5), with P^K replacing P. Use (4.3), let $K \to \infty$, and use $w_{m+K} \to 0$ and the strong convergence in (4.4). Cauchy–Schwarz and $\|\mathbf{1}_{I_j} g\|_2 \leq \|g\|_2$ give (4.7). □

5. Many marked transitions and bounded zero runs

We begin with the local carry calculation that will also drive Section 6. Suppose $q_k = a$, $q_{k+1} = 1 - a$, $q_{k+2} = e$, and $d_\alpha(m+k+1) = 1$. Conditional on any incoming carry, the two random three-bit words

$$(a, 0, e), \qquad (a, 1, e) \tag{5.1}$$

produce the same outgoing carry e and costs differing by exactly one. To check this, note the three elementary rules for the carry map $b(q, x, c) = \mathbf{1}_{\{q+x+c \geq 2\}}$:

$$b(a, a, c) = a, \qquad b(a, 1-a, c) = c, \qquad b(1-a, x, a) = x.$$

The first bit resets the carry to a, the middle bit selects 0 or 1, and the last bit resets it to e. Both words have conditional probability $1/8$. The incoming prefix contributes the same cost, and the common outgoing carry ensures that every fixed suffix also contributes the same cost.

If the first digit equal to one after position $m+k$ occurs at $m+k+1+s$, the same construction can be delayed. After the branching bit, choose random bits complementary to the query bits until that first one has been counted, and then reset the carry using a random bit equal to its query bit. The intervening zero weights contribute nothing. The resulting pair has length $s+3$, has common outgoing carry, and still has costs differing by one.

Lemma 5.1 (Smoothing)**.** *For an invertible linear isometry U of a real Hilbert space, set $A = (I+U^{-1})/2$. If f is orthogonal to* $\mathrm{Fix}(U)$*, then $A^n f \to 0$ in norm.*

Proof. The parallelogram identity gives $\|Af\|^2 + \frac{1}{4}\|f - U^{-1}f\|^2 = \|f\|^2$. The norms $\|A^n g\|$ decrease. Summing the identity and using commutation shows

$$(n+1)\|A^n(g - U^{-1}g)\|^2 \le 4\|g\|^2.$$

The closure of the range of $I - U^{-1}$ is $\mathrm{Fix}(U)^\perp$: the orthogonal complement of that range is the kernel of $I - U$. Contraction and density now prove the assertion. □

Lemma 5.2 (Amplification of carry pairs)**.** *Suppose a query word contains N disjoint blocks, each of length at most $M \ge 1$, such that, for every incoming carry, two equally likely random words have a common outgoing carry and costs differing by one. Put $\delta = 2^{1-M}$. If*

$$\|A^j f\| \le \epsilon + Cs^j \quad (j \ge 0), \qquad 0 \le s < 1, \quad C \ge 0,$$

then its carry average satisfies

$$\|\mathbb{E}U^{-\mathrm{cost}} f\| \le \epsilon + C(1 - \delta + \delta s)^N. \tag{5.2}$$

The bound is uniform in the incoming carry and in any unmarked prefix, intervening blocks, or suffix.

Proof. Read the blocks in order and use induction on the number of selected blocks remaining. At a marked block of length $m \le M$, the selected pair has total mass $p = 2^{1-m} \ge \delta$. Their common outgoing carry lets their two contributions be combined into the suffix average applied to Af, followed by an isometry. The other words are bounded by the suffix estimate for f. The geometric coefficient is therefore multiplied by at most $(1-p) + ps \le 1 - \delta + \delta s$. An unmarked block only averages isometric translates and cannot increase the bound. This proves (5.2). □

For a vector with $A^j f \to 0$, and each $\epsilon > 0$, a bound of the form in Lemma 5.2 exists with $s = 1/2$: choose a stage beyond which the norm is at most ϵ and absorb the finitely many preceding values into C. It follows that arbitrarily many such blocks force the carry average to tend to zero. The conclusion passes to $P_{m,q}$ by first including all selected blocks in a finite word and then letting its remaining length tend to infinity.

Definition 5.3. The digits of α have bounded zero runs if there is an integer $H > 0$ such that every interval $[a, a+H) \cap \mathbb{N}_0$ contains a position j with $d_\alpha(j) = 1$.

Under this hypothesis, each transition $q_k \ne q_{k+1}$ starts a delayed pair of length at most $H+2$. Transitions separated by at least $H+2$ therefore yield disjoint marked blocks, with the uniform parameter $\delta = 2^{-H-1}$.

Lemma 5.4 (Queries at the other height)**.** *Let $\alpha, \beta \ge 1$, $n \to \infty$, and put*

$$m = \lfloor n/2 \rfloor, \quad L = n - m, \quad t = h_\beta(n), \quad G_{\alpha,m}(q) \le t < G_{\alpha,m}(q+1).$$

Then

$$\frac{q}{2^L} \longrightarrow \frac{\beta}{\alpha}, \qquad \frac{q+1}{2^L} \longrightarrow \frac{\beta}{\alpha}. \tag{5.3}$$

There is a constant integer C such that $q+1 \le 2^{n+C}$ for all n, and

$$\varepsilon_m \longrightarrow 0, \qquad (n+C+1)w_m \longrightarrow 0. \tag{5.4}$$

Proof. The return bounds give

$$\frac{t}{h_\alpha(m)+1} - 1 < q \le \frac{t}{h_\alpha(m)}.$$

Divide by 2^L, use $t/2^n \to \beta$ and $h_\alpha(m)/2^m \to \alpha$, and squeeze. The successor query has the same limit because $L \to \infty$. Since $h_\alpha(m) \ge 1$, one has $q+1 \le t+1 \le 2^n\beta + 1$, which gives the stated bound after choosing C. Finally, exponential decay in $m = \lfloor n/2 \rfloor$ dominates n. □

Lemma 5.5 (Stability of query bits)**.** *Suppose $q_r/2^{L_r} \to c$, where c is irrational and $L_r \to \infty$. For every fixed $b \ge 0$, eventually*

$$(q_r)_{L_r-b-1} = d_c(b). \tag{5.5}$$

Consequently, the queries have arbitrarily many pairwise separated adjacent bit transitions.

Proof. Neither $2^b c$ nor $2^{b+1}c$ is an integer. Their floor functions are therefore locally constant, so $d_{q_r/2^{L_r}}(b) = d_c(b)$ eventually. The former is exactly the bit on the left of (5.5). By Lemma 2.1, the digits of c have infinitely many transitions. Choose any prescribed number with the required separation, and apply stability to this finite collection. The order of the chosen positions reverses when converted into least-significant-bit coordinates, but their separations are unchanged. □

Proposition 5.6 (The bounded-zero-run case)**.** *If $\alpha, \beta \ge 1$, α/β is irrational, and d_α has bounded zero runs, then $\mathcal{D}(\alpha, \beta)$.*

Proof. Work in an arbitrary tower-name limit for α. For $f \in L^2_0$, ergodicity and Lemma 5.1 give $A^j f \to 0$. Apply Lemmas 5.4 and 5.5 to q and $q+1$. The preceding delayed-pair construction and Lemma 5.2 imply

$$\|P_{m,q} f\|_2 \longrightarrow 0, \qquad \|P_{m,q+1} f\|_2 \longrightarrow 0.$$

For fixed bounded tower observables f, g, with f of mean zero, refine them to stage m without changing their bounds. Proposition 4.2, with $\ell = n + C$, and (5.4) now give $\langle g, U^{h_\beta(n)} f\rangle \to 0$. Centered tower observables are dense in L^2_0 and arbitrary tower observables are dense in L^2. The isometry bounds extend this convergence to all $f \in L^2_0$ and $g \in L^2$. Lemma 3.3(i) proves disjointness. □

6. One common mark controls displacement energy

The argument in this section uses a single marked three-bit block shared by the queries q and $q+1$. The two return regions in (4.5) can have quite different sizes. To handle them together, we compare squared displacements before estimating either region separately.

6.1. **A finite interval estimate.** Fix integers $h > 0$, $\Delta \ge 0$, and $0 \le r < h + \Delta$. For nonnegative integers c_0, c_1, use the split coordinate Φ_{c_0,c_1} in (4.8).

Lemma 6.1 (Two translated intervals)**.** *If $\psi : \mathbb{Z} \to \mathbb{R}$ satisfies $|\psi| \le M$, then*

$$\left| \sum_{i=0}^{h-1} \psi(\Phi_{c_0,c_1}(i)) - \sum_{i=0}^{h-1} \psi(i) \right| \le 2M(c_0 + \Delta + c_1). \tag{6.1}$$

Proof. For any finite integer interval, translating it by a changes the sum of a function bounded by M by at most $2M|a|$. This follows by canceling the intersection of the two intervals, since there are at most $|a|$ terms at each end. If $r \le h$, the two regions with $c_0 = c_1 = \Delta = 0$ map onto $[r, h)$ and $[0, r)$, which partition $[0, h)$. Translate their images by $-c_0$ and $-\Delta - c_1$. If $r > h$, the first region is empty. The second image is the interval of length h starting at $r - h - \Delta - c_1$. Its initial point lies between $-\Delta - c_1$ and 0. The same cancellation estimate proves (6.1). □

Let $a : \mathbb{Z} \to \mathbb{R}$ be zero outside $[0, h)$ and satisfy $|a| \le F$. For two costs C_0, C_1 on uniformly random K-bit words, put

$$E_K = \mathbb{E} \sum_{i=0}^{h-1} \big(a(i) - a(\Phi_{C_0,C_1}(i))\big)^2, \tag{6.2}$$

$$D(a) = \sum_{i=0}^{h-1} (a(i) - a(i-1))^2, \qquad \mathbb{E}(C_0 + \Delta + C_1) \le B. \tag{6.3}$$

Lemma 6.2 (A paired word detects one-step energy)**.** *Suppose $K = k+3+s$. For each fixed prefix of length k and suffix of length s, replacing a middle word $(b, 0, e)$ by $(b, 1, e)$ increases both C_0 and C_1 by exactly one. Then*

$$D(a) \le 16 E_K + 64 F^2 B. \tag{6.4}$$

One also has, without the paired-word assumption,

$$\mathbb{E} \sum_{i=0}^{h-1} a(\Phi_{C_0,C_1}(i))^2 \le \sum_{i=0}^{h-1} a(i)^2 + 2F^2 B. \tag{6.5}$$

Proof. The coordinate identity $\Phi_{c_0+1,c_1+1}(i) = \Phi_{c_0,c_1}(i) - 1$ holds in both regions, since the region is selected by $i + r < h$ independently of the costs. Use

$$\tfrac{1}{2}(x-y)^2 \le (z-x)^2 + (z-y)^2$$

with $z = a(i)$ and with x, y the two paired destinations. Each middle word has probability $1/8$. Averaging over all prefixes and suffixes gives

$$\frac{1}{16} \mathbb{E}_{\mathrm{ctx}} \sum_{i=0}^{h-1} \psi(\Phi_{C_0,C_1}(i)) \le E_K, \qquad \psi(u) = (a(u) - a(u-1))^2,$$

where $\mathbb{E}_{\mathrm{ctx}}$ fixes the middle word to $(b, 0, e)$. Here $0 \le \psi \le 4F^2$. The nonnegative cost in (6.3), conditioned on an event of probability $1/8$, has expectation at most $8B$. Lemma 6.1 therefore yields

$$D(a) \le \mathbb{E}_{\mathrm{ctx}} \sum_{i<h} \psi(\Phi_{C_0,C_1}(i)) + 8F^2(8B) \le 16 E_K + 64 F^2 B.$$

Apply the same interval estimate to $\psi = a^2$, with bound F^2, and take the unconditional average to obtain (6.5). □

6.2. **Comparison with the actual tower.** Return to a tower-name limit for $\alpha \ge 1$, with $h = h_\alpha(m)$, $w = w_m$, and $\varepsilon = \varepsilon_m$. Let f be a stage-m observable, with value a_i on $E_{m,i}$ and value a_{out} outside the body, all bounded in absolute value by F. Use the array $a(i) = a_i$ for $0 \le i < h$ and $a(i) = 0$ otherwise. Set

$$C_0 = C_q^K, \quad C_1 = C_{q+1}^K, \quad Q_K = w \mathbb{E} \sum_{i<h} a(i) a(\Phi_{C_0,C_1}(i)), \quad T_K = (q+1) h w_{m+K}.$$

Lemma 6.3 (Array and tower energies)**.** *If $G(q) \le t < G(q+1)$ and $q + 1 \le 2^\ell$, then*

$$|\langle f, U^t f \rangle - Q_K| \le F^2 \varepsilon + 12 F^2 (\ell + 1) w + 2 F^2 T_K, \tag{6.6}$$

$$w E_K \le \|U^t f - f\|_2^2 + 2F^2 \varepsilon + 30 F^2 (\ell+1) w + 4 F^2 T_K, \tag{6.7}$$

$$\|U^{-1} f - f\|_2^2 \le w D(a) + 4 F^2 (\varepsilon + w). \tag{6.8}$$

Proof. The finite-copy argument in Proposition 4.2 has error at most $F^2\varepsilon + 6F^2(\ell+1)w + 2F^2T_K$. Replace its proposed shifted values by the zero-extended array. On each copy, equality holds whenever the proposed destination lies in $[0,h)$. The number of remaining levels is at most $C_0 + \Delta + C_1$, and each product difference is at most $2F^2$. Lemma 4.1 adds at most $6F^2(\ell+1)w$. This proves (6.6).

Write $A_0 = \sum_{i<h} a(i)^2$. Expanding (6.2) gives

$$wE_K = wA_0 + w\mathbb{E}\sum_{i<h} a(\Phi_{C_0,C_1}(i))^2 - 2Q_K.$$

By (6.5) and (4.3), the second term is at most $wA_0 + 6F^2(\ell+1)w$. Also $wA_0 \le \|f\|_2^2$, since the latter includes the nonnegative outside contribution. Substitute (6.6) and use $2\|f\|_2^2 - 2\langle f, U^t f\rangle = \|U^t f - f\|_2^2$. This gives (6.7).

For a one-step backward translation, the array agrees exactly with the observable on every ordinary level except level zero. That level and the outside body have mass at most $w+\varepsilon$. The squared displacement there is at most $4F^2$. This gives (6.8). □

Theorem 6.4 (The common-mark estimate)**.** *Suppose $G(q) \le t < G(q+1)$, $q+1 \le 2^\ell$, and for some $k \ge 0$,*

$$q_k \ne q_{k+1}, \qquad d_\alpha(m+k+1) = 1, \qquad q_{k+j} = (q+1)_{k+j} \quad (0 \le j < 3). \tag{6.9}$$

Then every stage-m observable bounded by F satisfies

$$\|U^{-1}f - f\|_2^2 \le 16\|U^t f - f\|_2^2 + 36F^2\varepsilon_m + 676F^2(\ell+1)w_m. \tag{6.10}$$

Proof. Take $K = k+3+s$ and let $s \to \infty$. The three-bit words in (5.1) work for both queries, because they share those three bits. The two costs both increase by one. Thus Lemma 6.2 applies with $B = 3(\ell+1)$. Combining (6.4), (6.7), and (6.8) gives

$$\begin{aligned}\|U^{-1}f - f\|_2^2 \le{}& 16\|U^t f - f\|_2^2 + 36F^2\varepsilon_m \\ &+ (480+192)F^2(\ell+1)w_m + 4F^2 w_m + 64F^2T_K.\end{aligned}$$

Since $\ell + 1 \ge 1$, the two width terms are at most $676F^2(\ell+1)w_m$. Finally $T_K \to 0$. □

The bound does not require either return region to occupy a specified fraction of the tower. Its proof uses the same pair in both regions, together with interval cancellation. This uniformity is what permits its application at arbitrary heights of the other parameter.

6.3. **Rigidity at marked cross-heights.**

Proposition 6.5. *Let $\alpha, \beta \ge 1$ with $c = \beta/\alpha$ irrational. Fix $b > 0$ such that $d_c(b) \ne d_c(b+1)$. Suppose $a_r \to \infty$, $d_\alpha(a_r) = 1$, and $n_r = a_r + b + 1$. In every tower-name limit of α,*

$$U^{h_\beta(n_r)} f \longrightarrow f \quad \Longrightarrow \quad U^{-1}f = f.$$

In particular a mean-zero vector with this rigidity property is zero.

Proof. Set $m_r = \lfloor n_r/2 \rfloor$, $L_r = n_r - m_r$, and let q_r be the return query for $t_r = h_\beta(n_r)$ at stage m_r. By Lemmas 5.4 and 5.5, eventually the bits of both q_r and $q_r + 1$ at positions $k_r = L_r - b - 2$, $k_r + 1$, $k_r + 2$ are respectively $d_c(b+1)$, $d_c(b)$, $d_c(b-1)$. The first two differ, the three bits agree for the two queries, and

$$m_r + k_r + 1 = n_r - b - 1 = a_r.$$

The common-mark assumptions therefore hold.

For a fixed bounded tower observable g, refine it to stage m_r. Theorem 6.4, with $\ell = n_r + C$, gives

$$\|U^{-1}g - g\|_2^2 \le 16\|U^{t_r}g - g\|_2^2 + R_r(g), \qquad R_r(g) \longrightarrow 0, \tag{6.11}$$

by (5.4). The bound on the refined observable is independent of r.

Now let $U^{t_r} f \to f$ and choose such a g with $\|f-g\|_2 < \eta$. Isometry gives $\limsup_r \|U^{t_r} g - g\|_2 \le 2\eta$. Equation (6.11) implies $\|U^{-1} g - g\|_2 \le 8\eta$. Another triangle inequality yields $\|U^{-1} f - f\|_2 \le 10\eta$. Density lets $\eta \downarrow 0$. Ergodicity then gives the mean-zero conclusion. □

7. Forward transport and the proof of completeness

Put $A_\gamma = \{n \ge 0 : d_\gamma(n) = 1\}$. For sets $A, B \subset \mathbb{N}_0$ and an integer $s > 0$, write $A \rightsquigarrow_s B$ if there are integers $M > 0$ and N such that

$$a \in A,\ a \ge N \quad \Longrightarrow \quad B \cap [a+s, a+s+M) \neq \varnothing. \tag{7.1}$$

The constants are uniform over all sufficiently late $a \in A$.

Proposition 7.1 (Non-disjointness forces transport)**.** *If $\alpha, \beta \ge 1$, α/β is irrational, and $\mathcal{D}(\alpha, \beta)$ fails, then $A_\alpha \rightsquigarrow_s A_\beta$ for some $s > 0$.*

Proof. Choose $b > 0$ with $d_{\beta/\alpha}(b) \neq d_{\beta/\alpha}(b+1)$, possible by Lemma 2.1. We claim $s = b+1$ works. If (7.1) fails for this s, then for each r there is $a_r \ge r$ with $d_\alpha(a_r) = 1$ and

$$d_\beta(a_r + b + 1 + j) = 0 \qquad (0 \le j < r+2).$$

Set $n_r = a_r + b + 1$ and $L_r = r + 2$. Lemma 3.2 gives strong rigidity for the β system at $\pm h_\beta(n_r)$. Proposition 6.5 excludes nonzero mean-zero rigid vectors in the α system at the positive times. Lemma 3.3(ii) implies $\mathcal{D}(\alpha, \beta)$, contrary to the hypothesis. □

Lemma 7.2 (Two transports bound all gaps)**.** *Let $A, B \subset \mathbb{N}_0$, with A unbounded. If $A \rightsquigarrow_s B$ and $B \rightsquigarrow_t A$, where $s, t > 0$, then A meets every interval of some fixed positive integer length in $\mathbb{N}_0$.*

Proof. If M and M' are the two interval lengths, composing the relations shows that every sufficiently late $a \in A$ has another point of A in $[a+s+t, a+s+t+M+M')$. Start from one such $a_0 \in A$ and iterate. The resulting sequence increases strictly and has successive gaps bounded by $s + t + M + M'$. It tends to infinity and therefore meets every sufficiently late interval of that length. Increasing the length to include the finite initial segment proves the result. □

Proof of Theorem 1.1. First suppose $\alpha, \beta \ge 1$. If $\mathcal{D}(\alpha, \beta)$ failed, at least one of α, β would be irrational, since their ratio is irrational. Exchange the parameters if necessary so that α is irrational. Then A_α is unbounded. Proposition 7.1 yields $A_\alpha \rightsquigarrow_s A_\beta$ for some $s > 0$. Symmetry of $\mathcal{D}$ and irrationality of β/α give $A_\beta \rightsquigarrow_t A_\alpha$ for some $t > 0$. Lemma 7.2 says that the digits of α have bounded zero runs. Proposition 5.6 now gives $\mathcal{D}(\alpha, \beta)$, a contradiction. Thus $\mathcal{D}(\alpha, \beta)$ always holds, and Lemma 3.1 proves completeness in the normalized case.

For arbitrary positive α, β, choose $k \ge 0$ such that $\alpha' = 2^k \alpha$ and $\beta' = 2^k \beta$ are at least 1. Their ratio is unchanged. The normalized result gives the representation in Theorem 1.1 for α', β'. Since $h_{\alpha'}(n) = h_\alpha(n+k)$ and likewise for β, these representations use only terms of the original two sequences. This proves the theorem. □

8. Lean formalization and reproducibility

The accompanying source proves the exact declaration `Erdos354.erdos_354.parts.i` from the pinned Formal Conjectures repository [3]. The challenge supplies its statement through the macro `fcTypeOfName%`. The final declaration is named `Bounty.target`. Its expanded proposition is

$$\text{True} \longleftrightarrow \big[\forall \alpha, \beta > 0,\ \alpha/\beta \notin \mathbb{Q} \Longrightarrow \text{Complete}(\alpha, \beta)\big],$$

where $\text{Complete}(\alpha, \beta)$ denotes the conclusion of Theorem 1.1. The library's completeness predicate uses finite sets of indices. It is applied to the interleaving of the α and β floor sequences with multiplier 2, exactly as in the theorem.

TABLE 1. Guide to the main blocks of the formal proof.

Mathematical component	Lean source blocks
Heights, digits, finite sums	`BinaryFloors`, `FiniteSums`, `FullReturnPositions`
Symbolic model and ergodicity	`TowerLabels`, `TowerMeasures`, `TowerGeneration`, `TowerErgodicity`
Missing-sum obstruction and operators	`SubsetSumCoding`, `TowerJoiningOperators`, `IntertwinerCriteria`
Zero blocks and correlations	`TowerRigidityIntegrals`, `TowerPartialRigidity`, `InfiniteAdaptiveCorrelation`
Bounded zero runs	`CarryAmplification`, `BoundedCarryDecay`, `BoundedZeroDisjointness`
The common-mark energy estimate	`SplitWindowEnergy`, `AdaptiveArrayCorrelations`, `MarkedTowerEnergy`
Rigidity exclusion and transport	`CrossHeightRigidity`, `TransportDisjointness`, `DigitTransport`
Assembly of the exact target	`ConcreteReduction`, `FullTarget`

The proof is written in Lean 4 [7] using Mathlib [8]. Its single-file submission contains 601 theorem declarations, assembled from 130 local source files. The development constructs the compact symbolic spaces, empirical limits, factor and coupling operators, finite carry averages, and their limits. It proves the tower approximation, ergodicity, zero-block rigidity, carry amplification, and displacement estimates before assembling the final digit argument.

In particular, the formal proof does not assume a disjointness theorem for a preexisting family of transformations. The systems used by the estimates are the actual empirical limits of the labeled subset-sum names. The main proof blocks are listed in Table 1. The same names appear as source-block comments in the standalone file.

The last step of the proof, after all auxiliary declarations have been supplied, is:

```
theorem target :
    fcTypeOfName% "Erdos354.erdos_354.parts.i" := by
  exact Erdos354Formal.full_target_of_symbolic_digit_criteria
    (fun _ _ ha hb hc hz =>
      Erdos354Formal.symbolicallyDisjoint_of_boundedZeroRuns
        ha hb hc hz)
    (fun _ _ ha hb hc _ hn =>
      Erdos354Formal.forwardTransport_of_not_symbolicallyDisjoint
        ha hb hc hn)
```

The two arguments are the fully proved concrete digit criteria of Propositions 5.6 and 7.1. They are not additional hypotheses of the final theorem.

8.1. **Verification record.** The proof package records successful local checks with Lean 4.33.1 and the following repository revisions:

Repository	Commit
Formal Conjectures	`8432eac998110a563e03df65a28c117e97c8c142`
Mathlib	`0df444a360eaa60ab8c11dca51a86af692955474`
Challenge validator	`922a912f3777ac11451e2d145356e78de4dbedb2`

The standalone file compiles with `autoImplicit=false` and `warningAsError=true`. The exact challenge header, submission body, and footer also compile together. The pinned validator's static submission checks pass, and its target inspector reports that the intended and proved propositions coincide. The transitive axiom audit of the completed declaration reports:

```
'Bounty.target' depends on axioms:
```

```
[propext, Classical.choice, Quot.sound]
```

In particular, `sorryAx` is absent. The upstream conjecture file contains an admitted declaration. It is imported to recover the target type, and that admitted proof is not a dependency of `Bounty.target`. No extra axioms are introduced by the submitted proof.

The canonical target hash recorded by the inspector is

```
de4262d428cc1a0deb7df106c6951e4d
b5d808b9a6d6a4e21c29f05775c50488
```

where the two printed lines are concatenated. The complete compiler and inspector output, source hashes, and pinned task files are included in the original companion package.

8.2. **Files and verification command.** In the original package, `Submit.lean` contains the submission body for the website. The site supplies its imports and outer `Bounty` namespace. `Challenge.lean` contains the complete standalone source for local use, including those commands. `Audit.lean`, `verify.py`, and `verification.json` provide the audit and its recorded results. With the pinned repositories and their built dependencies available, the original verification is reproduced by:

```
python verify.py --lean PATH_TO_LEAN \
  --formal-conjectures PATH_TO_FORMAL_CONJECTURES \
  --mathlib PATH_TO_MATHLIB --validator PATH_TO_VALIDATOR
```

Here the backslashes denote shell line continuation. The command may also be entered on one line. The record establishes local kernel verification and compliance with the pinned static submission checks. A website submission or an official server verdict is not part of that record.

A later public version in the submitted prize package includes a direct base-two Lean endpoint and a separate build and axiom record. See the pinned reproduction guide for that Mathlib-only package and its attribution for the separate variable-base results.

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