Subexponential bounds for distinct-distance subsets: $f_d(n) = 2^{O_n(\sqrt{d}\log d)}$ for every fixed n

(an affirmative answer to Erdős Problem #1088, subject to expert verification)

Will Howe Grant Huffman Jensen Kohlmeyer
Liam Kruer

The mathematics was developed by AI agents (Claude, Anthropic) in sessions directed by the authors; see the statement on AI assistance and verification in Section 13.

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Abstract

For integers $n \geq 3$ and $d \geq 1$ let $f_d(n)$ be the least m such that every set of m points in $\mathbb{R}^d$ contains n points whose $\binom{n}{2}$ pairwise distances are all distinct. Erdős asked whether $f_d(n) = 2^{o(d)}$ for every fixed n; the classical bound is exponential in d. We prove that $f_d(n) \leq 2^{C_n\sqrt{d}\log(d+2)}$ for every fixed $n \geq 3$, which would answer the question affirmatively. The main new tool is a rank–rectangle theorem (Theorem LR): if every entry of a real matrix of rank r equals one of k prescribed functions, each depending only on the row or only on the column, then there is a rectangle of relative size $2^{-O(k^3\sqrt{r}\log r)}$ in each direction on which a single one of these functions describes all entries. Its proof combines Rothvoss's hyperplane-rounding proof of Lovett's bound with a rescaling at the largest gap and a cleaning step. The geometric part of the argument is a downward induction over "levels" (rainbow sets S together with a ground set W of points extending S), with a polynomial-method kill step at each level and a structural class-count statement proved by repeated use of Theorem LR. *Verification status:* every step was written out and checked by independent AI referee passes. The whole argument has also been formalized in Lean 4 with Mathlib. The formalization proves the statement `Erdos1088.erdos_1088` of the Google DeepMind *formal-conjectures* repository verbatim, using only Lean's standard axioms (Section 12). No human expert has reviewed the informal argument yet. See Section 13.

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1 Introduction

1.1 The problem

A finite set $A \subset \mathbb{R}^d$ is *rainbow* if its $\binom{|A|}{2}$ pairwise distances are pairwise distinct (sets with at most two points are rainbow). A set X is *rainbow-n-free* if it contains no rainbow subset of size n. For $n \geq 3$ and $d \geq 1$ let $f_d(n)$ be the least m such that every set of m points of $\mathbb{R}^d$ contains a rainbow n-subset, and let

$$s_n(j) := \sup\{|X| : X \subset \mathbb{R}^N \text{ finite, rainbow-}n\text{-free, } \dim \operatorname{aff} X \leq j\}$$

(the ambient dimension N is irrelevant). Then $f_d(n) = s_n(d) + 1$, and $s_n(0) = 1$.

Erdős asked whether $f_d(n) = 2^{o(d)}$ for every fixed n; this is Problem #1088 on the Erdős problems website [4].[1] We prove that the answer is yes, with a much stronger quantitative bound.

Theorem 1.1 (Main Theorem)**.** *For every fixed $n \geq 3$ there is a constant C_n such that every set of more than $2^{C_n\sqrt{d}\,\log_2(d+2)}$ points of $\mathbb{R}^d$ contains n points with all $\binom{n}{2}$ pairwise distances distinct. That is, $f_d(n) = 2^{O_n(\sqrt{d}\log d)} = 2^{o(d)}$.*

1.2 History

As recorded on the problem page [4], Erdős and Straus showed $f_d(n) \leq c_n^d$ for a constant c_n; the problem asks whether the exponential dependence on d is necessary. In the other direction, s-distance sets with $s = \binom{n}{2} - 1$ are rainbow-n-free, since a rainbow n-set needs $\binom{n}{2}$ distinct distances. For instance the 0/1-vectors of weight w in $\mathbb{R}^{d+1}$ lie in a d-dimensional affine subspace and determine only the distances $\sqrt{2i}$, $1 \leq i \leq w$; with $w = \binom{n}{2} - 1$ this gives $f_d(n) > \binom{d+1}{\binom{n}{2}-1}$ for $d+1 \geq \binom{n}{2} - 1$. So $f_d(n)$ grows at least polynomially in d, with degree depending on n, and we are not aware of any super-polynomial lower bound.

For $n = 3$ the problem is classical and polynomial: a set is rainbow-3-free exactly when every three of its points span an isosceles triangle, where collinear triples with two equal distances count as (degenerate) isosceles triangles. Blokhuis [2] proved that an isosceles set in $\mathbb{R}^d$ has at most $\binom{d+2}{2}$ points. His argument is purely metric, so it covers degenerate triples, and it gives $f_d(3) \leq \binom{d+2}{2} + 1$. For $n \geq 4$ we know of no bound of the form $2^{o(d)}$ in the literature before the present work.

1.3 The new tool: Theorem LR

The geometric part of our argument repeatedly produces matrices of the form $M[x, w] = |x-w|^2$, of rank at most $D + 2$, in which every entry is "explained" by one of boundedly many simple causes. A typical cause is $|x-w| = |w-t|$ for a fixed point t, which depends only on the column

[1] Within the project the problem is also tracked under the identifier JSP-000905.

w. Another is $|x-w| = |x-t|$, which depends only on the row x. A third is $|x-w| = d$ for a fixed number d. The tool that converts such a cover into geometric structure is the following rank–rectangle theorem (Theorem 4.1).

Theorem LR (informal). *Let $M \in \mathbb{R}^{m\times n}$ have rank at most r, and let $\tau_1, \dots, \tau_k$ be functions on $[m] \times [n]$, each depending only on the row or only on the column. If every entry satisfies $M[x,w] = \tau_i(x,w)$ for some i, then there are $R \subseteq [m]$, $C \subseteq [n]$ and a single i with $M = \tau_i$ on $R \times C$, $|R| \geq 2^{-L}m$ and $|C| \geq 2^{-L}n$, where $L = O(k^3\sqrt{r+2}\,\log(r+2))$.*

For Boolean matrices this is the rectangle form of Lovett's bound [10], in Rothvoss's formulation [12]. Our proof follows Rothvoss's hyperplane-rounding argument and adds two ingredients. The first is a normal form, obtained on a sub-rectangle of constant relative size, which allows us to rescale the matrix at its largest gap *by a row function, a column function or a constant.* This keeps a margin even when the type values accumulate. The second is a cleaning step, the Gavinsky–Lovett lemma (Lemma 3.6), applied to a Hadamard product, whose rank enters only logarithmically.

Remark 1.2 (possible independent interest; to be confirmed by experts). The special case in which all types are column functions concerns *k-listable* matrices, in which every column takes at most k distinct values. Here Theorem LR yields a column-constant submatrix of relative size $2^{-O(k^3\sqrt{d}\log d)}$ in each direction for rank-d matrices (Corollary 4.6). By Singer–Sudan [14, Prop. 2.3], a parallel k-partitioned point–hyperplane configuration in $\mathbb{R}^d$ yields a k-listable matrix of rank at most d. A column-constant ("1-listable") submatrix of it then yields a complete bipartite subgraph of the incidence graph, of proportional size up to a factor depending on k. Singer and Sudan write that "the best bound we know for parallel 3-partitioned configurations is $2^{-\tilde{O}(d)}$" ([14], arXiv version 5, Section 1). As far as we can tell, Corollary 4.6 improves this to $2^{-O(\sqrt{d}\log d)}$ for every fixed k. Our literature search (summarised in Section 11) was limited, and the observation follows from Rothvoss's proof in a few lines, so experts may regard it as known. The claim of improvement must be confirmed by experts before it is relied on. Theorem LR does not bear on the log-rank conjecture, which would require $2^{-\mathrm{polylog}(d)}$.

1.4 Prior in-project work

This manuscript builds on earlier notes of a peer session in the same project [11]. Those notes recorded the problem, its literature and the lower-bound constructions above. They also contained polynomial-method "peeling" lemmas: a point all of whose distances to the rest of a set lie in a short list can be removed using a low-degree polynomial in the lifted coordinates $(|y|^2, y)$. Lemma H below (Lemma 3.8) is a two-sided variant of those peeling lemmas, in which each pair needs to be "killed" from one side only, at the price of squaring the dimension. The present argument does not use the other results of those notes, which include reductions to prime sets and twin analyses.

1.5 Verification status

We state this plainly, because the result would resolve a problem of Erdős and the argument is long.

- The mathematics was developed by AI agents (Claude, Anthropic) in sessions directed by the team.
- Each component was checked by at least one independent AI referee pass. These passes found and repaired one genuine gap, a threshold gap in the top-level argument (fix F1, incorporated in Section 7), together with a number of expository gaps, all addressed here.

- **Machine-checked.** The complete argument has been formalized in Lean 4 (Section 12). The formal proof establishes the statement of Problem #1088 in the formal-conjectures repository verbatim. It depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`, and it has no `sorry`. The formalization proves Theorem LR with a weaker dependence on k than stated here; this suffices for the Main Theorem.
- **No human expert has reviewed the informal argument.** Theorem LR is new and unpublished, and every later step depends on it. It is the component we most want checked by specialists.
- This manuscript consolidates the project notes. A few routine changes made in the consolidation, listed in Section 13, have not themselves been refereed; among them are some explicit constants and the direct single-cap write-up of Theorem R^+.
- The finite, threshold-free implications were also tested by computer in exact arithmetic. Because every threshold in the argument is astronomically large, such tests are sanity checks only and cannot test the asymptotic claims (Section 11).

Until an expert check has been carried out, the Main Theorem should be regarded as a *candidate* solution.

1.6 Organisation

Section 2 gives an overview of the proof, the linear order of dependencies and the threshold tower. Section 3 collects the classical tools: John's theorem, Sheppard's formula, the cleaning lemma (with proof), Lemma H, the Caro–Wei and König theorems, and the orthogonal split. Section 4 proves Theorem LR. Section 5 proves the label lemma and its consequence G1 for pairs with disjoint label sets. Section 6 sets up the level-configurations, the hereditary relative bound HR and the kill step. Section 7 treats the top three levels. Sections 8 and 9 prove the class-count statement at the lower levels: the Core Lemma and the Slicing Lemma. Section 10 closes the induction and proves the Main Theorem. Section 11 contains concluding remarks and the computational sanity checks, and Section 13 the statement on AI assistance and the acknowledgements.

2 Structure of the proof

Throughout, $n \geq 4$ is fixed; the case $n = 3$ is Blokhuis's theorem. X is a finite rainbow-n-free set of affine dimension at most $D \geq 1$, and $h := s_n(\lfloor D/2 \rfloor)$. The proof is a *halving recursion*: we show

$$s_n(D) \;\leq\; 2^{C_n \sqrt{D} \log_2(D+2)} \left(s_n(\lfloor D/2 \rfloor) + 1 \right) \qquad (D \geq 1), \tag{1}$$

and iterating (1) gives $s_n(d) = 2^{O_n(\sqrt{d} \log d)}$ (Lemma 3.12). The quantity h enters only through the *orthogonal split* (Lemma 3.11). If $R, C \subseteq X$ and $\langle x - x', w - w' \rangle = 0$ for all $x, x' \in R$ and $w, w' \in C$, then one of R, C has affine dimension at most $\lfloor D/2 \rfloor$, so $\min(|R|, |C|) \leq h$. Every rectangle produced by Theorem LR in the geometric part ends in such an orthogonal split.

2.1 Levels

A *level-j configuration* (Definition 6.3) is a triple (S, W, σ). Here $S \subseteq X$ is a rainbow set with $|S| = j$, and $W \subseteq X \setminus S$ consists of points x with $S \cup \{x\}$ rainbow. The number σ is a *cap*: every sphere centred at a point of S meets W in fewer than σ points. The statement

$$\textbf{Claim}(j)\textbf{:}\ \text{every level-}j\ \text{configuration with } \sigma \geq \sigma_{\min} \text{ has } |W| \leq \Gamma_j \sigma$$

is proved by downward induction on j, from $j = n-2$ to $j = 0$. Claim(0), applied to $(\emptyset, X, \sigma_{\min})$, gives $|X| \leq \Gamma_0 \sigma_{\min}$. Here $\Gamma_0 = (D+2)^{O_n(1)}$ and $\sigma_{\min} = 2^{O_n(\sqrt{D} \log D)}(h+1)$, which is (1).

2.2 The induction step

Each step from Claim$(j+1)$ to Claim(j) follows one pattern.

(i) *Kill step* (Lemma 6.8, Theorem 7.1). Each $b \in W$ receives a polynomial F_b of bounded degree in the lifted coordinates $(|y|^2, y)$. Its factors are spheres about b with the radii of the *big* classes of b, together with some degenerate loci. Apart from at most $j^2\sigma$ points where a distance from x to S equals one from b to S, each $x \neq b$ with $F_b(x) \neq 0$ extends $S \cup \{b\}$ to a rainbow set and lies in a small class of b in the kill step. These extending points form a level-$(j+1)$ configuration and are few by Claim$(j+1)$. Caro–Wei and Lemma H (a pair needs to be killed from one side only) then bound $|W|$. This works as long as every $b \in W$ has at most $K = O_n(1)$ big classes.

(ii) *Few big classes.* At the top levels $j \geq n-4$ this is not needed. If b has two big classes, then $n-4$ points of S serve as "spread" points: together with b and an auxiliary point a they form an $(n-2)$-set. A label-counting argument (G1) and Theorem LR then show that the auxiliary points a which are not killed are few (Section 7; this is Theorem R^+, written out in full for a single cap). At the lower levels $j = r \leq n-5$ there are not enough spread points. Instead we prove the class-count statement $\mathbf{E1}_r$: every $b \in W$ has fewer than $K_r = 12 \cdot 6^{n-5-r}$ big classes (Theorem 9.2).

(iii) *Proof of E1$_r$.* Suppose b has K_r big classes. We pass to a *rainbow family*: centres $B \ni b$ and pairwise disjoint classes C_k on which every point of B has a constant distance, with all these constant labels distinct. The *Slicing Lemma* (Section 9) adds one centre at a time and loses a factor 6 in the number of classes. Its proof uses König's theorem, the relative bound HR and Theorem LR. After $n-5-r$ slicing rounds there are $n-4-r$ centres and 12 classes. The *Core Lemma* (Section 8) shows that this is impossible, using two auxiliary points, G1, HR and three applications of Theorem LR.

(iv) *Hereditary relative bound HR$_r$* (Lemma 6.6). It is a direct consequence of Claim$(r+1)$: for $c \in W$, $V \subseteq W \setminus \{c\}$ and $\theta \geq \sigma$, at most $\Gamma'_r\theta$ points of V lie in classes of c that meet V in fewer than θ points, after excluding the prescribed radii and bisectors in Lemma 6.6. The explicit constant is $\Gamma'_r = \Gamma_{r+1} + r^2$. This is the only place where the induction hypothesis enters the proof of E1$_r$, and it turns "c has few heavy classes" into a *covering* statement, which is what the cleaning lemma needs.

2.3 Linear order of dependencies

The results are proved in the following order. Each item uses only earlier items, and none uses the statement it is proving (there is no circularity).

1. Classical tools (Section 3). John's theorem and Sheppard's formula (cited); the cleaning lemma GL (proved); the Hadamard-product rank bound HP and the dimension count DIM (proved); Lemma H (proved); Caro–Wei and König (cited); the orthogonal split OR (proved); the halving lemma (proved).

2. Theorem LR (Section 4). Lemma R (margin rounding), Lemmas A and B, Lemma NF (normal form), then Theorem LR, followed by Corollary 4.6.

3. The label lemma and G1 (Section 5), from LR and OR.

4. Configurations; Lemma HR$_r$ and the kill Lemma 6.8, both assuming Claim$(r+1)$ (Section 6).

5. Theorem R^+: Claim(j) for $j = n-2, n-3, \ldots, \max(n-4, 0)$ (Section 7), from G1, LR, GL, HP, OR, Lemma H, Caro–Wei, DIM and the two-distribution Lemma 7.3.

6. For $n \geq 5$: the Core Lemma (Section 8) and the Slicing Lemma (Section 9), each assuming HR_r; then Theorem $\mathrm{E1}_r$.

7. The downward induction $\mathrm{Claim}(r+1) \Rightarrow \mathrm{HR}_r \Rightarrow \mathrm{E1}_r \Rightarrow \mathrm{Claim}(r)$ for $r = n-5, \ldots, 0$ (Section 10).

8. The Main Theorem, from $\mathrm{Claim}(0)$ and the halving lemma (Section 10).

The induction over levels takes place at a fixed dimension D, and the only input from dimensions below D is the finite number $h = s_n(\lfloor D/2 \rfloor)$. The recursion (1) is therefore an ordinary induction on D.

2.4 The threshold tower

All constants depend only on n and D, except $\sigma_{\min}$, which also depends on h. They are defined in the following order, and each definition uses only quantities defined before it (Section 10 gives the details).

(T1) Type counts k', $k_c(r)$, $k_{\mathrm{sl}}(r,p)$; the class counts K_r; their maximum k^*. These depend on n only.

(T2) $L^* := L_{k^*}(D+2)$ from Theorem LR; then $\mu := 2^{L^*}(h+1)$, $e := 2^{L^*+1}(h+1)$, $e_A := 2^{L^*+2}(h+1)$, and

$$\sigma_{\min} := 2^{L^*+2+2n}(h+1).$$

So $\sigma_{\min}$ depends only on n, D and h, and not on any Γ.

(T3) $Q := 2^{44} n^4 (D+2)^{8k'}$ and $\Gamma_{n-2} := 4(n-2)^2$; then Γ_{n-3} and Γ_{n-4} by (3) (for $n = 4$ these are Γ_1, Γ_0, and the tower ends here).

(T4) For $r = n-5, n-6, \ldots, 0$, in this order: $\Gamma'_r := \Gamma_{r+1} + r^2$, then the ratios θ_{rel}, ε, $\hat{M}_{\mathrm{core}}$, θ_{sl}, $\hat{m}_p$ and ratio_r (all functions of n, D and Γ'_r), and then Γ_r by (8).

Every Γ_j, Γ'_r and ratio_r is a pure number, at most $(D+2)^{c_n}$, and does not depend on the cap σ. Every absolute error term occurring in the proofs (μ, e, e_A, $2 \cdot 4^{n-2}\mu + 4$, $4^n \mu$, and 1) is at most $\sigma_{\min} \leq \sigma$. These are the two facts that allow the caps to vary freely inside the induction: $\mathrm{Claim}(j+1)$ is applied with caps $\theta \geq \sigma$, $Q\sigma$ and $\mathrm{ratio}_r \sigma$, all of which are at least $\sigma_{\min}$.

3 Notation, conventions and classical tools

3.1 Notation

For $x, y \in \mathbb{R}^D$ we write $|xy| := |x - y|$. For $c \in \mathbb{R}^D$ and $t > 0$, $\mathrm{S}(c,t) := \{y : |yc| = t\}$ is the sphere with centre c and radius t, and for a set U of positive reals $\mathrm{S}(c,U) := \bigcup_{u \in U} \mathrm{S}(c,u)$. For $p \neq q$, $\mathrm{H}(p,q) := \{y : |yp| = |yq|\}$ is the perpendicular bisector hyperplane. For a finite set A, $\mathcal{D}(A) := \{|xy| : x \neq y \in A\}$ is its set of distances. If A is rainbow then $|\mathcal{D}(A)| = \binom{|A|}{2}$. If c is a point and V a set, a *class of c in V* is a nonempty set $V \cap \mathrm{S}(c,t)$ with $t > 0$. Two spheres about the same centre with different radii are disjoint, so the classes of c in V partition $V \setminus \{c\}$.

The *lifted coordinates* of $y \in \mathbb{R}^D$ are $(|y|^2, y_1, \ldots, y_D)$. Every sphere equation $|yc|^2 - t^2 = |y|^2 - 2\langle y, c\rangle + |c|^2 - t^2$ and every bisector equation $|yp|^2 - |yq|^2 = -2\langle y, p-q\rangle + |p|^2 - |q|^2$ is an affine function of the lifted coordinates.

Since an affine isometry changes neither distances nor affine dimension, we may and do assume that a set of affine dimension at most D lies in $\mathbb{R}^D$.

Convention 3.1 (rectangles and rounding)**.** A *rectangle* of a matrix with row set A and column set B is a product $A' \times B'$ with $A' \subseteq A$ and $B' \subseteq B$. Every rectangle produced below is nonempty, and its size bounds are real inequalities: a conclusion $|A'| \geq 2^{-L}|A|$ with $2^{-L}|A| < 1$ just says that $A' \neq \emptyset$. In the cleaning lemma, "half" means at least $|A|/2$ (hence at least $\lceil |A|/2 \rceil$). Whenever a bound on a set Z is deduced from a bound on a half $Z'' \subseteq Z$, we use $|Z| \leq 2|Z''|$.

3.2 Two classical facts used in the rounding lemma

Theorem 3.2 (John [8]; see also [1])**.** *If $K \subset \mathbb{R}^s$ is a compact convex body that is symmetric about 0 and has nonempty interior, then there is an ellipsoid E centred at 0 with $E \subseteq K \subseteq \sqrt{s}\, E$.*

Theorem 3.3 (Sheppard's formula [13]; see also [6])**.** *Let $u, v \in \mathbb{R}^s$ be unit vectors with $\langle u, v \rangle = a$ and let $g \sim N(0, I_s)$. Then $\Pr[\langle g, u \rangle \geq 0$ and $\langle g, v \rangle \geq 0] = \frac{1}{2} - \frac{\arccos a}{2\pi} = \frac{1}{4} + \frac{\arcsin a}{2\pi}$.*

Consequently this probability is at least $\frac{1}{4}$ if $a \geq 0$. If $a \leq -b$ with $0 \leq b \leq 1$, it is at most $\frac{1}{4} - \frac{\arcsin b}{2\pi} \leq \frac{1}{4}(1 - \frac{2b}{\pi})$, because $\arcsin b \geq b$.

3.3 Rank tools

Lemma 3.4 (Hadamard products, HP)**.** *For matrices A, B of the same shape,* $\operatorname{rank}(A \circ B) \leq \operatorname{rank} A \cdot \operatorname{rank} B$*, where $\circ$ is the entrywise product.*

Proof. Write $A = \sum_{i \leq \operatorname{rank} A} a_i b_i^T$ and $B = \sum_{j \leq \operatorname{rank} B} c_j d_j^T$. Then $A \circ B = \sum_{i,j} (a_i \circ c_j)(b_i \circ d_j)^T$. □

Lemma 3.5 (distance matrices)**.** *Let $x_1, \ldots, x_m, w_1, \ldots, w_{m'} \in \mathbb{R}^D$ and let f, g be real functions on the index sets. The matrix $[\, |x_i w_j|^2 - f(i) - g(j) \,]_{i,j}$ has rank at most $D + 2$.*

Proof. It equals $(|x_i|^2 - f(i))\mathbf{1}^T + \mathbf{1}(|w_j|^2 - g(j))^T - 2[\langle x_i, w_j \rangle]$, a sum of matrices of rank at most 1, 1 and D. □

In particular, each of the following matrices has rank at most $D+2$: $[|xa|^2 - v^2]$ for a constant v; $[|xa|^2 - v(a)^2]$ for a column function v; and $[|xa|^2 - |xs|^2]$ for a fixed point s (take $f(x) = |xs|^2$). By Lemma 3.4, a Hadamard product of k such matrices has rank at most $(D+2)^k$.

Lemma 3.6 (cleaning lemma, GL; Gavinsky–Lovett [5], cf. [7, Lemma 4.7])**.** *Let B be a matrix over a field, with row set A ($|A| = m$) and column set B_0 ($|B_0| = n$), and with $\operatorname{rank} B \leq R$ for some $R \geq 1$. Suppose B has at most $mn/(16R)$ nonzero entries. Then there are $A'' \subseteq A$ and $B'' \subseteq B_0$ with $|A''| \geq 7m/8$, $|B''| \geq n/2$ and $B = 0$ on $A'' \times B''$.*

Proof. Call a column *light* if it has at most $m/(8R)$ nonzero entries. If c columns are heavy then $cm/(8R) < mn/(16R)$ whenever $c \geq 1$, so $c < n/2$, and the set B'' of light columns has $|B''| \geq n/2$. The columns in B'' span a space of dimension at most R. Choose a basis of it consisting of at most R columns from B''. The union of their supports has at most $R \cdot m/(8R) = m/8$ rows; let A'' be the remaining rows. Every basis column vanishes on A'', and every column of B'' is a linear combination of basis columns, so $B = 0$ on $A'' \times B''$. □

We use Lemma 3.6 in two forms. In the *average form*, the total number of nonzero entries is at most $mn/(16R)$. In the *per-column form*, every column has at most $m/(16R)$ nonzero entries, which implies the average form.

Lemma 3.7 (dimension count, DIM)**.** *The polynomials of degree at most e in the $D+1$ lifted coordinates form a vector space of dimension $\binom{D+1+e}{e} \leq (D+2)^e$. A product of e affine functions of the lifted coordinates lies in this space.*

Proof. The count of monomials of degree at most e in $D+1$ variables is $\binom{D+1+e}{e} = \prod_{i=1}^{e} \frac{D+1+i}{i}$, and each factor is at most $D+2$. □

Lemma 3.8 (Lemma H, symmetric form)**.** *Let I be a finite set and, for each $x \in I$, let F_x be a real function on I. Suppose all F_x lie in a vector space of functions of dimension at most m, that $F_x(x) \neq 0$ for all x, and that for every pair $x \neq y$ in I we have $F_x(y) = 0$ or $F_y(x) = 0$. Then $|I| \leq m^2$.*

Proof. Let $A_{y,x} := F_y(x)$. Its rows are restrictions of functions from an m-dimensional space, so $\operatorname{rank} A \leq m$. The matrix $A \circ A^T$ has entries $F_y(x)F_x(y)$, which vanish for $x \neq y$ and are nonzero for $x = y$. It is therefore a nonsingular diagonal matrix of size $|I|$. By Lemma 3.4, $|I| = \operatorname{rank}(A \circ A^T) \leq m^2$. □

Lemma H is the symmetric special case of a "bi-peeling" lemma from earlier in the project, which extends the one-sided peeling lemmas of [11]. The squaring of the dimension is harmless for us.

3.4 Graph theory

Theorem 3.9 (Caro–Wei [3, 16])**.** *A finite graph with N vertices and average degree $\bar{d}$ has an independent set of size at least $N/(\bar{d}+1)$.*

Theorem 3.10 (König [9])**.** *In a finite bipartite graph, the maximum size of a matching equals the minimum size of a vertex cover.*

3.5 The orthogonal split

Lemma 3.11 (orthogonal split, OR)**.** *Let $X \subset \mathbb{R}^D$ be rainbow-n-free, $h = s_n(\lfloor D/2 \rfloor)$, and let $R, C \subseteq X$ be nonempty. Suppose one of the following holds:*

(a) every $w \in C$ is equidistant from all points of R;

(b) there is a point $p \notin C$ with $R \subseteq \mathrm{H}(p, w)$ for every $w \in C$.

Then $\langle x - x', w - w' \rangle = 0$ for all $x, x' \in R$ and $w, w' \in C$, and $\min(|R|, |C|) \leq h$.

Proof. (a) $|xw|^2 - |x'w|^2 = |x|^2 - |x'|^2 - 2\langle x - x', w \rangle$ does not depend on $w \in C$. Subtracting the values at w and w' gives $\langle x - x', w - w' \rangle = 0$. (b) $x \in \mathrm{H}(p, w)$ means $2\langle x, w - p \rangle = |w|^2 - |p|^2$. Hence $\langle x - x', w - p \rangle = 0$ for $x, x' \in R$ and $w \in C$, and subtracting the identities for w and w' gives the claim. Now let $U, V \subseteq \mathbb{R}^D$ be the direction spaces of $\operatorname{aff} R$ and $\operatorname{aff} C$. They are orthogonal, so $\dim U + \dim V \leq D$ and $\min(\dim U, \dim V) \leq \lfloor D/2 \rfloor$. The corresponding set (R or C) is a rainbow-n-free set of affine dimension at most $\lfloor D/2 \rfloor$, so its size is at most h. □

3.6 The halving recursion

Lemma 3.12 (halving)**.** *Let $s : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 1}$ satisfy $s(0) = 1$ and $s(D) \leq A(D)\,(s(\lfloor D/2 \rfloor) + 1)$ for all $D \geq 1$, where $A(D) = 2^{C\sqrt{D}\log_2(D+2)}$ with $C \geq 1$. Then $s(D) \leq 2^{6C\sqrt{D}\log_2(D+2)}$ for all $D \geq 1$.*

Proof. Put $u(D) := \log_2(s(D) + 1)$. Then $u(0) = 1$, and for $D \geq 1$,

$$\begin{aligned} u(D) &\leq \log_2\bigl(2A(D)(s(\lfloor D/2 \rfloor) + 1)\bigr) \\ &= 1 + C\sqrt{D}\log_2(D+2) + u(\lfloor D/2 \rfloor). \end{aligned}$$

Let $D_i := \lfloor D/2^i \rfloor$ and $m := \lfloor \log_2 D \rfloor$, so that $D_m \geq 1$ and $D_{m+1} = 0$. Iterating, and using $\sqrt{D_i} \leq 2^{-i/2}\sqrt{D}$,

$$u(D) \leq (m+1) + 1 + C\log_2(D+2)\sqrt{D}\sum_{i \geq 0} 2^{-i/2} \leq \log_2 D + 2 + 3.5\,C\sqrt{D}\log_2(D+2).$$

Since $\log_2 D + 2 \leq 2\sqrt{D}\log_2(D+2)$ for $D \geq 1$ and $C \geq 1$, we get $u(D) \leq 6C\sqrt{D}\log_2(D+2)$. □

4 Theorem LR: low-rank matrices covered by row and column types

4.1 Statement

Let $M \in \mathbb{R}^{R_0 \times C_0}$ be a matrix with finite row set R_0 and column set C_0. A *type* is a function τ on $R_0 \times C_0$ which is either a *row function*, $\tau(x,w) = a(x)$, or a *column function*, $\tau(x,w) = b(w)$. A constant function is both. A set $\mathcal{T}$ of types *covers* M on a rectangle $R \times C$ if for every $(x,w) \in R \times C$ there is $\tau \in \mathcal{T}$ with $M[x,w] = \tau(x,w)$. For $k, r \geq 1$ put

$$L_k(r) := 2^{14}\, k^3 \sqrt{r+2}\, \log_2(r+2).$$

This is increasing in k and in r.

Theorem 4.1 (Theorem LR). *Let $M \in \mathbb{R}^{m \times n}$ have rank at most r, where $m, n, r \geq 1$. Let $\mathcal{T}$ be a set of at most k types covering M on $[m] \times [n]$. Then there are nonempty $R \subseteq [m]$ and $C \subseteq [n]$ and a single $\tau^* \in \mathcal{T}$ such that*

$$|R| \geq 2^{-L_k(r)} m, \qquad |C| \geq 2^{-L_k(r)} n, \qquad M[x,w] = \tau^*(x,w) \ \text{ for all } (x,w) \in R \times C.$$

Since L_k is increasing in k, the theorem also holds with any $k' \geq k$ in place of k. When it is applied to a family of fewer than k types, the family may be padded by repeating a type.

4.2 Margin rounding

Lemma 4.2 (margin rounding; quantitative form of Rothvoss [12, Sec. 3]). *Let $N \in \mathbb{R}^{m \times n}$ have rank at most s, where $s \geq 1$, and suppose $\frac{1}{2} \leq |N_{xw}| \leq K$ for all entries. Let $Q^+ := \{(x,w) : N_{xw} > 0\}$ and $Q^- := \{(x,w) : N_{xw} < 0\}$, and suppose $|Q^+| \geq \delta mn$ with $0 < \delta \leq \frac{1}{2}$. Put $T := \lceil \pi K \sqrt{s}\, \ln(2/\delta^2) \rceil$. Then there are $X' \subseteq [m]$ and $Y' \subseteq [n]$ with*

$$|X'| \geq \tfrac{\delta}{2} 4^{-T} m, \qquad |Y'| \geq \tfrac{\delta}{2} 4^{-T} n, \qquad |(X' \times Y') \cap Q^-| \leq \delta |X'||Y'|.$$

Proof. Let $s_0 := \operatorname{rank} N$. Then $1 \leq s_0 \leq s$, since N has nonzero entries. Write $N_{xw} = \langle u_x, v_w \rangle$ with $u_x, v_w \in \mathbb{R}^{s_0}$, where the u_x span $\mathbb{R}^{s_0}$. The body $K_0 := \operatorname{conv}\{\pm u_x\}$ is symmetric with nonempty interior. By Theorem 3.2 there is an ellipsoid E with $E \subseteq K_0 \subseteq \sqrt{s_0} E$. Choose a linear map A with $A(E) = s_0^{-1/4} B$, where B is the unit ball, and replace u_x by Au_x and v_w by $A^{-T} v_w$. This preserves all inner products, and now $s_0^{-1/4} B \subseteq K_0 \subseteq s_0^{1/4} B$. Hence $|u_x| \leq s_0^{1/4}$. Moreover $v_w \neq 0$, because $N_{xw} \neq 0$. The point $z := s_0^{-1/4} v_w / |v_w|$ lies in K_0, so

$$s_0^{-1/4} |v_w| = \langle z, v_w \rangle \leq \max_x |\langle u_x, v_w \rangle| \leq K,$$

that is, $|v_w| \leq K s_0^{1/4}$. For the unit vectors $\hat{u}_x, \hat{v}_w$ we get $|\langle \hat{u}_x, \hat{v}_w \rangle| \geq \frac{1/2}{K\sqrt{s_0}} \geq b := \frac{1}{2K\sqrt{s}}$, with the sign of N_{xw}.

Let $g_1, \ldots, g_T$ be independent standard Gaussian vectors in $\mathbb{R}^{s_0}$. Put $X' := \{x : \langle \hat{u}_x, g_t \rangle \geq 0 \ \forall t\}$ and $Y' := \{w : \langle \hat{v}_w, g_t \rangle \geq 0 \ \forall t\}$, and $\mathcal{R} := X' \times Y'$. By Theorem 3.3 and independence, $\Pr[(x,w) \in \mathcal{R}] \geq 4^{-T}$ for $(x,w) \in Q^+$, and $\Pr[(x,w) \in \mathcal{R}] \leq 4^{-T}(1 - 2b/\pi)^T \leq 4^{-T} e^{-2bT/\pi}$ for $(x,w) \in Q^-$. Since $2bT/\pi = T/(\pi K \sqrt{s}) \geq \ln(2/\delta^2)$, we get

$$\mathbb{E}|\mathcal{R} \cap Q^-| \leq \tfrac{\delta^2}{2} mn 4^{-T} \leq \tfrac{\delta}{2}\, \mathbb{E}|\mathcal{R} \cap Q^+|, \qquad \mathbb{E}|\mathcal{R} \cap Q^+| \geq \delta mn 4^{-T}.$$

Hence $\mathbb{E}\big[|\mathcal{R} \cap Q^+| - \delta^{-1} |\mathcal{R} \cap Q^-|\big] \geq \frac{1}{2} \mathbb{E}|\mathcal{R} \cap Q^+| \geq \frac{\delta}{2} mn 4^{-T}$. Fix an outcome attaining at least this value. For it, $|\mathcal{R} \cap Q^-| \leq \delta |\mathcal{R} \cap Q^+| \leq \delta |\mathcal{R}|$ and $|\mathcal{R}| \geq \frac{\delta}{2} mn 4^{-T}$. Since $|X'| \leq m$ and $|Y'| \leq n$, both size bounds follow. $\square$

4.3 Refinements and the normal form

A *refinement of cost* (c_1, c_2) of a rectangle $R_0 \times C_0$ is a rectangle $R \times C \subseteq R_0 \times C_0$ with $|R| \geq |R_0|/c_1$ and $|C| \geq |C_0|/c_2$. Every property stated "everywhere on" a rectangle is inherited by its refinements. A *basic quantity* is a positive row function, a positive column function or a positive constant. A function Δ on a rectangle is *c-comparable* to a basic quantity Q if $Q \leq |\Delta| \leq cQ$ everywhere on it.

Lemma 4.3 (Lemma A)**.** *Let a be a row function and b a column function on $R_0 \times C_0$. There is a refinement of cost $(4, 2)$ on which either (i) a is constant, or (ii) for some θ we have $a(x) > \theta \geq b(w)$ everywhere, or $a(x) < \theta \leq b(w)$ everywhere.*

Proof. Let θ be a median of a on R_0, so that at least half of the rows have $a \leq \theta$ and at least half have $a \geq \theta$. If $|\{a = \theta\}| \geq |R_0|/4$, take $\{a = \theta\} \times C_0$; this is case (i). Otherwise $\{a > \theta\}$ and $\{a < \theta\}$ each contain at least $|R_0|/4$ rows. One of $\{b \leq \theta\}$ and $\{b \geq \theta\}$ contains at least $|C_0|/2$ columns. Take $\{a > \theta\} \times \{b \leq \theta\}$ or $\{a < \theta\} \times \{b \geq \theta\}$ accordingly; this is case (ii). □

Lemma 4.4 (Lemma B)**.** *Let $\Delta > 0$ be a row function and $E \geq 0$ a column function on $R_0 \times C_0$. There is a refinement of cost $(4, 4)$ on which (a) $\Delta \geq 2E$ everywhere, or (b) $E \geq 2\Delta$ everywhere, or (c) for some $s > 0$ both Δ and E take values in $(s/4, 4s)$.*

Proof. Let s be a median of E on C_0. If $s = 0$, then $\{E = 0\}$ contains at least half of the columns and $\Delta > 0 = 2E$ there: case (a). Let $s > 0$. If $|\{\Delta \geq 2s\}| \geq |R_0|/4$, take $\{\Delta \geq 2s\} \times \{E \leq s\}$: case (a). Else, if $|\{\Delta \leq s/2\}| \geq |R_0|/4$, take $\{\Delta \leq s/2\} \times \{E \geq s\}$: case (b). Otherwise $R_1 := \{s/2 < \Delta < 2s\}$ contains more than $|R_0|/2$ rows. If $|\{E \leq s/4\}| \geq |C_0|/4$, take $R_1 \times \{E \leq s/4\}$: case (a). Else, if $|\{E \geq 4s\}| \geq |C_0|/4$, take $R_1 \times \{E \geq 4s\}$: case (b). Otherwise $\{s/4 < E < 4s\}$ contains more than $|C_0|/2$ columns, and $R_1 \times \{s/4 < E < 4s\}$ gives case (c). □

Lemma 4.5 (normal form, Lemma NF)**.** *Let a set $\mathcal{T}$ of k types cover M on $R_0 \times C_0$. There is a refinement $R \times C$ of cost at most $(2^{2k^2+3}, 2^{2k^2+3})$, and a subset $\mathcal{T}' \subseteq \mathcal{T}$ with $k' := |\mathcal{T}'| \geq 1$ that still covers M on $R \times C$, such that on $R \times C$:*

(NF1) the types of $\mathcal{T}'$ are strictly ordered, $\tau_{(1)}(x, w) < \cdots < \tau_{(k')}(x, w)$, in the same order at every (x, w);

(NF2) if $k' \geq 2$, every gap $g_j := \tau_{(j+1)} - \tau_{(j)}$ is 16-comparable to a basic quantity Q_j;

(NF3) if $k' \geq 2$, there is j^* with $Q_{j^*} \geq Q_j/16$ for all j.

Proof. Process the $\binom{k}{2}$ pairs $\{\tau, \tau'\}$ one after the other, refining each time. If one of the two types has already been deleted, skip the pair. A constant is treated as a row function when it is paired with a row function, and as a column function when it is paired with a column function (editorial item (i) of the referee report on Theorem LR).

- *Two row functions* (or a row function and a constant, or two constants). Split the rows into three parts according to the sign of $\tau - \tau'$, and keep a largest part: cost $(3, 1)$. If the part with $\tau = \tau'$ is kept, delete τ'; coverage is preserved. Otherwise $|\tau - \tau'|$ is a positive row function, hence 1-comparable to a basic quantity, and has constant sign. Two column functions are treated symmetrically, at cost $(1, 3)$.

- *A row function a and a column function b* (neither constant). Apply Lemma 4.3, at cost $(4, 2)$. In case (i) a is now constant, and from now on it is treated as a column function. Split the columns by the sign of $a - b$ (cost $(1, 3)$), deleting b if the part with $a = b$ is kept. In case (ii), say $a > \theta \geq b$ (the other case is symmetric), we have $a - b = \Delta + E$ with $\Delta := a - \theta > 0$ a row function and $E := \theta - b \geq 0$ a column function. Lemma 4.4, at cost

$(4,4)$, gives one of three cases: (a) $\Delta \le a-b \le \frac{3}{2}\Delta$; (b) $E \le a-b \le \frac{3}{2}E$, where $E \ge 2\Delta > 0$; or (c) $s/2 < a-b < 8s = 16\cdot(s/2)$. In each case $a-b>0$ is 16-comparable to a basic quantity.

Each pair costs at most $(16,8)$, so all pairs together cost at most $16^{\binom{k}{2}}$ per side. On the resulting rectangle, every difference of two surviving types has a constant strict sign and is 16-comparable to a basic quantity. Since the values at a fixed (x,w) are real numbers, these signs define one strict total order, and every entry has exactly one type in $\mathcal{T}'$. This gives (NF1) and (NF2).

For (NF3), let J_{row}, J_{col}, J_{const} be the sets of j for which Q_j is a row function, a column function or a constant. Split the rows according to which $j \in J_{\mathrm{row}}$ maximises $Q_j(x)$, keeping a largest part (cost at most k). On the result some j_r satisfies $Q_{j_r} = R^* := \max_{J_{\mathrm{row}}} Q_j$. Do the same for the columns, which gives j_c with $Q_{j_c} = C^* := \max_{J_{\mathrm{col}}} Q_j$. Let j_s attain $s^* := \max_{J_{\mathrm{const}}} Q_j$. Apply Lemma 4.4 to (R^*, C^*). In case (a) ($R^* \ge 2C^*$), split the rows into $\{R^* \ge s^*\}$ and $\{R^* < s^*\}$ (cost 2). On the first part $j^* = j_r$ works, and on the second $j^* = j_s$ works. Case (b) is symmetric, splitting the columns. In case (c) ($R^*, C^* \in (s/4, 4s)$): if $s^* \ge s$, then $j^* = j_s$ works with factor 4; if $s^* < s$, then $R^* > s/4 > s^*/4$ and $C^* < 4s < 16R^*$, so $j^* = j_r$ works with factor 16. If some $J_\bullet$ is empty, the corresponding comparison is simply omitted. The cost of this step is at most $(8k, 8k)$. The total cost is at most $16^{\binom{k}{2}} \cdot 8k \le 2^{2k^2+3}$ per side. □

4.4 Proof of Theorem LR

We use induction on k. For $k=1$ take $R=[m]$ and $C=[n]$. Let $k \ge 2$, and apply Lemma 4.5 to $[m]\times[n]$, obtaining $R_0 \times C_0$ and $\mathcal{T}'$. If $k'=1$ we are done. Otherwise, by (NF2) and (NF3), $g_{j^*} \ge Q_{j^*}$ and $g_j \le 16Q_j \le 256Q_{j^*}$ for every j. So the spread $\mathcal{S} := \tau_{(k')} - \tau_{(1)} = \sum_j g_j$ satisfies $\mathcal{S} \le 256(k'-1)Q_{j^*}$. Put

$$z := \tfrac{1}{2}\big(\tau_{(j^*)} + \tau_{(j^*+1)}\big), \qquad \sigma := Q_{j^*}, \qquad N := (M-z)/\sigma \quad \text{(entrywise, on } R_0\times C_0\text{)}.$$

Each τ is a row or a column function, so $z = \varphi(x) + \vartheta(w)$ for functions φ, ϑ. Since σ is a basic quantity, $1/\sigma = \alpha(x)\beta(w)$ with $\alpha, \beta > 0$ (one of them $\equiv 1$). Hence $N = \operatorname{diag}(\alpha)\,(M - \varphi\mathbf{1}^T - \mathbf{1}\vartheta^T)\operatorname{diag}(\beta)$ has rank at most $r+2$. By (NF1) each entry has a unique type $\tau_{(i)}$. If $i \le j^*$, then $-256(k'-1) \le -\mathcal{S}/\sigma \le N \le -g_{j^*}/(2\sigma) \le -\frac{1}{2}$. If $i > j^*$, then $\frac{1}{2} \le N \le 256(k'-1)$. So $\frac{1}{2} \le |N| \le K := 256(k-1)$, and the sign of N records on which side of the gap j^* the type of the entry lies.

Replacing N by $-N$ if necessary, we may assume that $Q^+ = \{N > 0\}$ contains at least half of the entries of $R_0 \times C_0$. Let $\mathcal{T}_{\mathrm{maj}}$ be the types on the positive side and $\mathcal{T}_{\mathrm{min}}$ those on the negative side. Both are nonempty, because $1 \le j^* < k'$, so $|\mathcal{T}_{\mathrm{maj}}| \le k'-1 \le k-1$. If $Q^- = \emptyset$, every entry of $R_0 \times C_0$ has a type in $\mathcal{T}_{\mathrm{maj}}$, and we apply the induction hypothesis to $\mathcal{T}_{\mathrm{maj}}$ on $R_0 \times C_0$ directly. Otherwise put $\delta := 1/(16(r+1)^{k-1})$ and apply Lemma 4.2 (with $s = r+2$). This gives $X' \times Y' \subseteq R_0 \times C_0$ with $|X'| \ge \frac{\delta}{2}4^{-T}|R_0|$ and $|Y'| \ge \frac{\delta}{2}4^{-T}|C_0|$, containing at most $\delta|X'||Y'|$ entries of Q^-, where $T = \lceil \pi K\sqrt{r+2}\ln(2/\delta^2)\rceil$.

The entries of Q^- are *exactly* the support of

$$H := \bigcirc_{\tau\in\mathcal{T}_{\mathrm{maj}}} (M-\tau).$$

Indeed, an entry whose type lies in $\mathcal{T}_{\mathrm{maj}}$ makes one factor vanish. By (NF1), an entry whose type lies in $\mathcal{T}_{\mathrm{min}}$ differs from every majority-type value at that position. Each matrix $M-\tau$ has the form $M - a\mathbf{1}^T$ or $M - \mathbf{1}b^T$, of rank at most $r+1$. By Lemma 3.4, $\operatorname{rank} H \le (r+1)^{k-1}$. So H restricted to $X' \times Y'$ has at most $|X'||Y'|/(16(r+1)^{k-1})$ nonzero entries, and Lemma 3.6 gives $X'' \times Y''$ with $|X''| \ge |X'|/2$, $|Y''| \ge |Y'|/2$ and $H = 0$ there. Every entry of $X'' \times Y''$ therefore has its type in $\mathcal{T}_{\mathrm{maj}}$. The submatrix has rank at most r, and we apply the induction hypothesis to it with the at most $k-1$ types $\mathcal{T}_{\mathrm{maj}}$.

Loss. We bound the loss of one round, measured in bits per side, i.e. $\log_2$ of the ratio of side lengths. Lemma 4.5 costs at most $2k^2+3$, Lemma 3.6 costs 1, and Lemma 4.2 costs $\log_2(2/\delta)+2T$. Here $\log_2(2/\delta) = 5+(k-1)\log_2(r+1)$. Since $r \geq 1$ we have $\ln(r+2) \geq 1$, so $\ln(2/\delta^2) = \ln 512 + 2(k-1)\ln(r+1) \leq 6k\ln(r+2)$. Hence $T \leq 1+256\pi(k-1)\sqrt{r+2}\cdot 6k\ln(r+2) \leq 1+3345\,k^2\sqrt{r+2}\log_2(r+2)$. Using $\sqrt{r+2}\log_2(r+2) \geq 1$, one round costs at most

$$2k^2+11+k\log_2(r+2)+6690\,k^2\sqrt{r+2}\log_2(r+2) \leq 2^{13}k^2\sqrt{r+2}\log_2(r+2)$$

bits per side. The number of types drops by at least one per round, so the total loss is at most $\sum_{j=2}^{k} 2^{13}j^2\sqrt{r+2}\log_2(r+2) \leq L_k(r)$. □

4.5 The column-type special case

Corollary 4.6 (RQ-list)**.** *Let $M \in \mathbb{R}^{m\times n}$ have rank at most r, where $m,n,r \geq 1$, and suppose every column of M takes at most k distinct values. Then there are $R \subseteq [m]$ and $C \subseteq [n]$ with $|R| \geq 2^{-L_k(r)}m$ and $|C| \geq 2^{-L_k(r)}n$ such that every column of M is constant on $R\times C$.*

Proof. List the values of column w as $v_1(w) < \cdots < v_{k_w}(w)$, and pad the list to length k by repeating $v_{k_w}(w)$. The k column functions v_i cover M. Theorem 4.1 gives a rectangle on which $M[x,w] = v_i(w)$ for a single i. □

See Remark 1.2 for the relation of Corollary 4.6 to [14]. In the geometric applications, Theorem 4.1 is always used together with the following observation.

Lemma 4.7 (types give orthogonal splits)**.** *Let $X \subset \mathbb{R}^D$ be rainbow-n-free, and let $R, C \subseteq X$ be disjoint and nonempty. Suppose that on $R\times C$ the matrix $[|xa|^2]_{x\in R, a\in C}$ equals a single type τ of one of the following forms: a constant v^2; a column function $v(a)^2$; or a row function $|xs|^2$ with $s \in X\setminus C$ fixed. Then $\min(|R|,|C|) \leq h$.*

Proof. In the first two cases every $a \in C$ is equidistant from R, so Lemma 3.11(a) applies. In the third, $|xa| = |xs|$ means $x \in \mathrm{H}(s,a)$ for all $x \in R$ and $a \in C$, so Lemma 3.11(b) applies with $p = s$. □

5 The label lemma and pairs with disjoint labels (G1)

Throughout this section $n \geq 4$, $X \subset \mathbb{R}^D$ is finite and rainbow-n-free, and $h = s_n(\lfloor D/2\rfloor)$. Let $T \subseteq X$ be a rainbow set with $|T| = n-2$. For $x \notin T$ the *label set* of x is $L_x := \{|xt| : t \in T\}$. A point x is *T-generic* if $x \notin T$ and $T\cup\{x\}$ is rainbow; then $|L_x| = n-2$. Let Y_T be the set of T-generic points of X. Two points x, w are *label-disjoint* if $L_x \cap L_w = \emptyset$. Put

$$k_G := \binom{n-2}{2} + 2(n-2), \qquad \mu_G := 2^{L_{k_G}(D+2)}(h+1).$$

Lemma 5.1 (label lemma)**.** *If $x, w \in Y_T$ are label-disjoint, then $x \neq w$ and $|xw| \in \mathcal{D}(T)\cup L_x \cup L_w$.*

Proof. $L_x \neq \emptyset$, so $x \neq w$. The n-set $T\cup\{x,w\}$ is not rainbow. Its distances other than $|xw|$ are $\mathcal{D}(T)\cup L_x\cup L_w$, and these $\binom{n}{2}-1$ numbers are pairwise distinct: $\mathcal{D}(T)\cup L_x$ and $\mathcal{D}(T)\cup L_w$ are rainbow, and $L_x\cap L_w = \emptyset$. Hence $|xw|$ equals one of them. □

Theorem 5.2 (G1, bipartite form)**.** *Let $P, Q \subseteq Y_T$ be such that every $x \in P$ and $w \in Q$ are label-disjoint. Then $\min(|P|,|Q|) \leq 2^{L_{k_G}(D+2)}h \leq \mu_G$.*

Proof. We may assume $P, Q \neq \emptyset$. Then $P \cap Q = \emptyset$, by Lemma 5.1. By Lemma 5.1, for $x \in P$ and $w \in Q$ the entry $|xw|^2$ of the matrix $[|xw|^2]_{x\in P, w\in Q}$ equals one of the following k_G types: the constants d^2 with $d \in \mathcal{D}(T)$; the column functions $w \mapsto |wt|^2$ with $t \in T$; and the row functions $x \mapsto |xt|^2$ with $t \in T$. The matrix has rank at most $D+2$ (Lemma 3.5). Theorem 4.1 gives $R \subseteq P$ and $C \subseteq Q$ with $|R| \geq 2^{-L}|P|$ and $|C| \geq 2^{-L}|Q|$, where $L = L_{k_G}(D+2)$, and a single type on $R \times C$. The points of T are not in $Q \subseteq Y_T$, so Lemma 4.7 applies and gives $\min(|R|, |C|) \leq h$. Hence $\min(|P|, |Q|) \leq 2^L h$. □

Corollary 5.3 (G1, counting forms)**.** *Let $\mu \geq \mu_G$.*

(a) For every $Y \subseteq Y_T$, the number of ordered pairs $(x, w) \in Y^2$ that are label-disjoint is at most $4^{n-2}\mu|Y|$.

(b) For disjoint $P, Q \subseteq Y_T$, the number of label-disjoint pairs in $P \times Q$ is at most $4^{n-2}\mu(|P| + |Q|)$.

(c) (degree form) For $Y \subseteq Y_T$ and a real v put $\deg_Y(v) := |\{x \in Y : v \in L_x\}|$. If $\deg_Y(v) \leq |Y|/(4(n-2))$ for every v, then $|Y| \leq \frac{4}{3}(4^{n-2}\mu + 1) \leq 2 \cdot 4^{n-2}\mu + 4$.

Proof. (a) Let N_{ld} be the number of label-disjoint ordered pairs. Colour every element of $\bigcup_{x\in Y} L_x$ red or blue, independently and uniformly. Let P be the set of $x \in Y$ all of whose labels are red, and Q the set of $w \in Y$ all of whose labels are blue. Every pair in $P \times Q$ is label-disjoint, so by Theorem 5.2, $|P||Q| \leq \mu \max(|P|, |Q|) \leq \mu|Y|$. A label-disjoint pair (x, w) involves $2(n-2)$ distinct labels, so it lies in $P \times Q$ with probability $4^{-(n-2)}$. Taking expectations, $4^{-(n-2)}N_{\mathrm{ld}} \leq \mathbb{E}|P||Q| \leq \mu|Y|$. (b) Apply (a) to $Y = P \cup Q$. (c) The ordered pairs (x, w) with $x \neq w$ that share a label number at most $\sum_v \deg_Y(v)^2 \leq \max_v \deg_Y(v) \cdot \sum_v \deg_Y(v) \leq \frac{|Y|}{4(n-2)}(n-2)|Y| = |Y|^2/4$. So $N_{\mathrm{ld}} \geq |Y|(|Y|-1) - |Y|^2/4$, and (a) gives $\frac{3}{4}|Y|^2 - |Y| \leq 4^{n-2}\mu|Y|$. □

In the sequel G1 is applied with $\mu := 2^{L^*}(h+1)$, where $L^* = L_{k^*}(D+2)$ and $k^* \geq k_G$ (Section 10). Then $\mu \geq \mu_G$, since L_k is increasing in k.

6 Configurations, the relative bound HR and the kill step

6.1 Standing assumptions and absolute parameters

From now on $n \geq 4$ is fixed, $D \geq 1$, $X \subset \mathbb{R}^D$ is a finite rainbow-n-free set, and $h := s_n(\lfloor D/2 \rfloor)$. All parameters below are explicit. Their roles are explained where they are used, but they are defined here, so that every later definition refers only to earlier ones.

Definition 6.1 (type counts)**.** Put $t := \binom{n-2}{2}$ and $k' := t + 2n$. For $0 \leq r \leq n-5$ put $p_r := n - 4 - r$ and

$$w_1(r) := (n-2) + t + p_r, \quad w_2(r) := t + p_r, \quad k_3(w) := w + 1 + r, \quad k_4(w) := w + 2r + \binom{r}{2},$$
$$k_5 := \binom{n-4}{2} + 2(n-4), \qquad k_c(r) := k_3(w_1(r)) + k_4(w_1(r)).$$

For $0 \leq r \leq n-5$ and $1 \leq p \leq n-4-r$ put $K^{(r,p)} := 12 \cdot 6^{n-4-r-p}$, and let $K_r := K^{(r,1)} = 12 \cdot 6^{n-5-r}$. For $0 \leq r \leq n-6$ and $1 \leq p \leq n-5-r$ put

$$k_{\mathrm{sl}}(r, p) := 3K^{(r,p+1)} - 1 + \binom{r+p}{2} + 2p + 2r.$$

Let k^* be the maximum of k', of all $k_c(r)$ and of all $k_{\mathrm{sl}}(r, p)$. For $n = 4$ only k' is defined, and $k^* := k'$.

One checks directly that $k_G \leq k' \leq k^*$, and that k_5, $k_3(w_2)$, $k_4(w_2)$, $k_3(w_1)$, $k_4(w_1)$ are all at most $k_c(r) \leq k^*$. For instance $k_3(w_1) = \binom{n-4}{2} + 4n - 12 \geq k_5$, using $r + p_r = n - 4$. Also $k^* \leq 6^{n-4} + 2n^2$.

Definition 6.2 (absolute parameters)**.**

$$L^* := L_{k^*}(D+2), \quad \mu := 2^{L^*}(h+1), \quad e := 2^{L^*+1}(h+1),$$
$$e_A := 2^{L^*+2}(h+1), \quad \sigma_{\min} := 2^{L^*+2+2n}(h+1).$$

These depend only on n, D and h. We shall use the following inequalities, all immediate from the definitions (recall $h \geq 1$, so $\mu \geq 2$):

$$1 \leq \mu \leq e \leq e_A \leq \sigma_{\min}, \qquad 4^n\mu \leq \sigma_{\min}, \qquad 2 \cdot 4^{n-2}\mu + 4 \leq \sigma_{\min}, \qquad e > 2^{L^*+1}h. \tag{2}$$

Since $L^* \geq L_k(D+2)$ for every $k \leq k^*$, any rectangle given by Theorem 4.1 with at most k^* types for a matrix of rank at most $D+2$ has both sides of relative size at least 2^{-L^*}.

6.2 Configurations

Definition 6.3 (level-j configuration)**.** Let $0 \leq j \leq n-2$. A *level-j configuration* is a triple (S, W, σ) consisting of a rainbow set $S \subseteq X$ with $|S| = j$, a set $W \subseteq X \setminus S$ such that $S \cup \{x\}$ is rainbow for every $x \in W$, and a real $\sigma > 0$ (the *cap*) such that

$$|W \cap \mathrm{S}(s,t)| < \sigma \qquad \text{for all } s \in S,\ t > 0.$$

For $j = 0$ the cap condition is void, and $(\emptyset, W, \sigma)$ is a configuration for every $W \subseteq X$ and every $\sigma > 0$. For $b \in W$ put

$$R_b := \mathcal{D}(S \cup \{b\}) = \mathcal{D}(S) \cup \{|bs| : s \in S\},$$
$$\mathrm{Big}_b(\tau) := \{\lambda > 0 : \lambda \notin R_b,\ |W \cap \mathrm{S}(b,\lambda)| \geq \tau\} \quad (\tau > 0).$$

If (S, W, σ) is a level-j configuration, $W' \subseteq W$ and $\sigma' \geq \sigma$, then (S, W', σ') is again a level-j configuration.

Definition 6.4 (Claim(j))**.** For $0 \leq j \leq n-2$ and a real $\Gamma_j \geq 1$, **Claim**(j) (with constant Γ_j) is the statement: *every level-j configuration (S, W, σ) with $\sigma \geq \sigma_{\min}$ satisfies $|W| \leq \Gamma_j \sigma$.*

Claim(j) quantifies over *all* configurations and *all* caps $\sigma \geq \sigma_{\min}$, and Γ_j does not depend on σ. This uniformity is what allows the lemmas below to apply Claim$(j+1)$ with caps other than σ.

Lemma 6.5 (extension)**.** *Let (S, W, σ) be a level-j configuration, and let $b \in W$ and $x \in W \setminus \{b\}$. Suppose that $|xb| \notin R_b$, that $x \notin \bigcup_{s \in S} \mathrm{H}(b,s)$, and that $|xs| \neq |bs'|$ for all $s, s' \in S$. Then $S \cup \{b, x\}$ is rainbow. Moreover, at most $j^2\sigma$ points $x \in W$ satisfy $|xs| = |bs'|$ for some $s, s' \in S$; there are no such points if $j = 0$.*

Proof. The distances of $S \cup \{b, x\}$ are those of S, the numbers $|bs|$ and $|xs|$ ($s \in S$), and $|bx|$. The sets $\mathcal{D}(S) \cup \{|bs|\}$ and $\mathcal{D}(S) \cup \{|xs|\}$ are rainbow because $b, x \in W$. By hypothesis $|bs| \neq |xs'|$, $|bx| \notin \mathcal{D}(S) \cup \{|bs|\}$ and $|bx| \neq |xs|$, and $b, x \notin S$ are distinct. For the second statement: such points lie on one of the j^2 spheres $\mathrm{S}(s, |bs'|)$, each of which meets W in fewer than σ points. □

6.3 The hereditary relative bound

Lemma 6.6 (HR$_r$)**.** *Let $0 \leq r \leq n-3$ and assume Claim$(r+1)$ with constant Γ_{r+1}. Put $\Gamma'_r := \Gamma_{r+1} + r^2$. Let (S, W, σ) be a level-r configuration with $\sigma \geq \sigma_{\min}$, let $c \in W$, $V \subseteq W \setminus \{c\}$ and $\theta \geq \sigma$. Then*

$$\left|\left\{x \in V :\ |xc| \notin R_c,\ x \notin \textstyle\bigcup_{s \in S} \mathrm{H}(c,s),\ |V \cap \mathrm{S}(c, |xc|)| < \theta\right\}\right| \ \leq\ \Gamma'_r\,\theta.$$

Proof. Let V_0 be the set on the left. Let V_2 be the set of $x \in V_0$ with $|xs| = |cs'|$ for some $s, s' \in S$. By Lemma 6.5, $|V_2| \le r^2\sigma \le r^2\theta$. Put $V_1 := V_0 \setminus V_2$. For $x \in V_1$ the set $S \cup \{c, x\}$ is rainbow, by Lemma 6.5. We claim that $(S \cup \{c\}, V_1, \theta)$ is a level-$(r+1)$ configuration. Indeed, $S \cup \{c\}$ is rainbow because $c \in W$, and $V_1 \subseteq X \setminus (S \cup \{c\})$. A sphere about $s \in S$ meets $V_1 \subseteq W$ in fewer than $\sigma \le \theta$ points. If a sphere $\mathrm{S}(c,t)$ meets V_1 in a point x, then $t = |xc|$ and $|V_1 \cap \mathrm{S}(c,t)| \le |V \cap \mathrm{S}(c,|xc|)| < \theta$. Since $\theta \ge \sigma_{\min}$, Claim$(r+1)$ gives $|V_1| \le \Gamma_{r+1}\theta$. □

Remark 6.7 (the bisector exception). Points of V on the bisectors $\mathrm{H}(c,s)$ are not controlled by Lemma 6.6, and neither are points on spheres about c with radius in R_c. In every application such points are handled by Theorem 4.1. A point $x \in \mathrm{H}(c,s)$ satisfies $|xc|^2 = |xs|^2$, which is a *row type* of the matrix $[|xc|^2]$ (rows x, columns c). A single-type rectangle of this type has $R \subseteq \mathrm{H}(s,c)$ for all its columns c, so Lemma 4.7 applies. The radii in R_c are column functions ($|cs|$) or constants ($d \in \mathcal{D}(S)$).

6.4 The kill step

Lemma 6.8 (kill step). *Let $0 \le r \le n-3$ and assume Claim$(r+1)$ with constant Γ_{r+1}. Let (S, W, σ) be a level-r configuration with $\sigma \ge \sigma_{\min}$, let $\tau \ge \sigma$, and let $K \ge 0$ be an integer such that $|\mathrm{Big}_b(\tau)| \le K$ for every $b \in W$. Then*

$$|W| \le P(K,r)\,\bigl(2\Gamma_{r+1}\tau + 2r^2\sigma + 1\bigr), \qquad P(K,r) := \binom{D+1+e_{K,r}}{e_{K,r}}^2, \quad e_{K,r} := K + 2r + \tbinom{r}{2}.$$

Proof. For $b \in W$ define

$$\Delta_b(y) := \prod_{s\in S}\bigl(|yb|^2 - |ys|^2\bigr)\prod_{s\in S}\bigl(|yb|^2 - |bs|^2\bigr)\prod_{d\in\mathcal{D}(S)}\bigl(|yb|^2 - d^2\bigr),$$
$$F_b(y) := \Delta_b(y)\prod_{\lambda\in\mathrm{Big}_b(\tau)}\bigl(|yb|^2 - \lambda^2\bigr).$$

F_b is a product of at most $e_{K,r}$ affine functions of the lifted coordinates, so by Lemma 3.7 all F_b lie in one space of dimension $\binom{D+1+e_{K,r}}{e_{K,r}}$. At $y = b$ the factors equal $-|bs|^2$, $-|bs|^2$, $-d^2$ and $-\lambda^2$, all nonzero (recall $b \notin S$), so $F_b(b) \ne 0$.

Claim: for every $b \in W$, at most $\Gamma_{r+1}\tau + r^2\sigma$ points $x \in W \setminus \{b\}$ have $F_b(x) \ne 0$. Let $F_b(x) \ne 0$. Then $|xb| \notin \mathrm{Big}_b(\tau) \cup R_b$ and $x \notin \bigcup_s \mathrm{H}(b,s)$. By Lemma 6.5, either x is one of at most $r^2\sigma$ points with $|xs| = |bs'|$, or $S \cup \{b,x\}$ is rainbow and $x \in W' := \{x \in W \setminus \{b\} : S \cup \{b,x\}$ rainbow, $|xb| \notin \mathrm{Big}_b(\tau)\}$. The triple $(S \cup \{b\}, W', \tau)$ is a level-$(r+1)$ configuration. Indeed, $S \cup \{b\}$ is rainbow. Spheres about $s \in S$ meet W' in fewer than $\sigma \le \tau$ points. If $x \in W' \cap \mathrm{S}(b,t)$, then $t = |xb| \notin R_b$, because $S \cup \{b,x\}$ is rainbow, and $t \notin \mathrm{Big}_b(\tau)$; hence $|W' \cap \mathrm{S}(b,t)| \le |W \cap \mathrm{S}(b,t)| < \tau$. Since $\tau \ge \sigma_{\min}$, Claim$(r+1)$ gives $|W'| \le \Gamma_{r+1}\tau$.

Let G be the graph on W in which $\{x,b\}$ is an edge if $F_b(x) \ne 0$ and $F_x(b) \ne 0$. By the claim, G has at most $|W|(\Gamma_{r+1}\tau + r^2\sigma)$ edges, so its average degree is at most $2(\Gamma_{r+1}\tau + r^2\sigma)$. By Theorem 3.9, G has an independent set I with $|I| \ge |W|/(2\Gamma_{r+1}\tau + 2r^2\sigma + 1)$. For distinct $x, y \in I$ we have $F_x(y) = 0$ or $F_y(x) = 0$, so Lemma 3.8 gives $|I| \le \binom{D+1+e_{K,r}}{e_{K,r}}^2 = P(K,r)$. □

Remark 6.9 (on the constants of Lemma 6.8). The square in $P(K,r)$ comes from Lemma H, where a pair is killed from one side only. The term $+1$ comes from Caro–Wei. When the lemma is applied with $\tau = \rho\sigma$ for a number $\rho \ge 1$, we use $1 \le \sigma$ (by (2)) to write the bound as $|W| \le P(K,r)(2\rho\Gamma_{r+1} + 2r^2 + 1)\,\sigma$. This is a pure multiple of the cap.

7 The top levels: Theorem R$^+$ with a single cap

In this section we prove Claim(j) for $j = n-2, n-3, n-4$ (for $n = 4$: $j = 2, 1, 0$). At these levels a point with two big classes is handled directly, using $n-4$ points of S as "spread" points; no class-count statement is needed.

Theorem 7.1 (Theorem R$^+$)**.** *Put* $Q := 2^{44}n^4(D+2)^{8k'}$, $\Gamma_{n-2} := 4(n-2)^2$, *and for* $j = n-3, n-4$

$$\Gamma_j := P_j^+\left(2Q\Gamma_{j+1} + 2(j^2+2n) + 1\right), \qquad P_j^+ := \binom{D+1+e_j}{e_j}^2, \quad e_j := 2 + 2j + \tbinom{j}{2}. \tag{3}$$

Then Claim(j) holds with constant Γ_j *for* $j = n-2, n-3, n-4$.

Remark 7.2 (provenance and referee fixes). This is Theorem R$^+$ of the project notes. There it was formulated with a sequence of thresholds $\tau_0 < \tau_1 < \cdots$, one for each point of S. An AI referee pass (report "REFEREE_R") found it correct subject to three fixes: **F1**, a genuine threshold gap in Step A, repaired by the branch at $\theta = Q^{1/2}\sigma$ in Step 4 below; **F2**, the auxiliary points a must be taken in $W \setminus \{b\}$; **F3**, an undefined threshold τ_{-1}, which is moot here because only the points of S' are used and $S' = \emptyset$ is allowed. A later referee pass ("referee15", item X1) asked that the single-cap version be written out in full rather than obtained by a translation remark. This section does so. In particular the two-class count ("Lemma S2") and the "four-case analysis" of the notes appear below as Step 3 and Lemma 7.3. The "level switch" for $n \geq 6$ that was sketched in the notes after that referee pass is not needed: the configuration formalism of Section 6 replaces it. For $n = 4$ the Main Theorem follows from Theorem 7.1 alone. An earlier, longer proof of the case $n = 4$ in the project notes, together with the editorial items of its referee report that concern only that proof, is superseded and not used here.

7.1 The base level

Proof of Claim($n-2$). Let (S, W, σ) be a level-$(n-2)$ configuration with $\sigma \geq \sigma_{\min}$. Then S is a rainbow $(n-2)$-set and every $x \in W$ is S-generic, i.e. $W \subseteq Y_S$ in the notation of Section 5. For every real v, $\deg_W(v) \leq \sum_{s\in S} |W \cap \mathrm{S}(s, v)| < (n-2)\sigma$. Suppose $|W| \geq 4(n-2)^2\sigma$. Then $\deg_W(v) < |W|/(4(n-2))$ for all v, and Corollary 5.3(c) gives $|W| \leq 2 \cdot 4^{n-2}\mu + 4 \leq \sigma_{\min} \leq \sigma < 4(n-2)^2\sigma$, a contradiction. Hence $|W| < 4(n-2)^2\sigma = \Gamma_{n-2}\sigma$. □

7.2 Two distributions

Lemma 7.3 (two-distribution lemma)**.** *Let* $p = (p_t)$ *and* $q = (q_t)$ *be finitely supported probability distributions on* $\mathbb{R}$, *let* $\rho \neq \rho'$ *be reals and* $0 \leq \eta \leq 1/16$. *Suppose*

$$(1 - p_{\rho'})(1 - q_\rho) - \sum_{t \notin \{\rho, \rho'\}} p_t q_t \;\leq\; \eta.$$

Then there is a real $t^* \notin \{\rho, \rho'\}$ *with* $p_{\rho'} + p_{t^*} \geq 1 - \eta^{1/4}$ *or* $q_\rho + q_{t^*} \geq 1 - \eta^{1/4}$.

Proof. Let $\alpha := 1 - p_{\rho'}$ and $\beta := 1 - q_\rho$. If $\alpha \leq \eta^{1/4}$ or $\beta \leq \eta^{1/4}$, the claim holds with any $t^* \notin \{\rho, \rho'\}$. Otherwise $\alpha\beta > \sqrt{\eta}$. For $t \notin \{\rho, \rho'\}$ put $p'_t := p_t/\alpha$ and $q'_t := q_t/\beta$. Then $p'_t, q'_t \leq 1$, $\sum_t p'_t \leq 1$ and $\sum_t q'_t \leq 1$ (sums over $t \notin \{\rho, \rho'\}$), and the hypothesis gives $\sum_t p'_t q'_t \geq 1 - \eta/(\alpha\beta) \geq 1 - \epsilon$ with $\epsilon := \sqrt{\eta} < 1$. Let $t^* \notin \{\rho, \rho'\}$ maximise q'_t. Then $1 - \epsilon \leq \sum_t p'_t q'_t \leq q'_{t^*}$, and $1 - \epsilon \leq p'_{t^*} q'_{t^*} + \sum_{t \neq t^*} q'_t \leq p'_{t^*} + (1 - q'_{t^*}) \leq p'_{t^*} + \epsilon$. So $p'_{t^*} \geq 1 - 2\epsilon$, and $p_{\rho'} + p_{t^*} = (1-\alpha) + \alpha p'_{t^*} \geq 1 - 2\sqrt{\eta} \geq 1 - \eta^{1/4}$, where the last step uses $\eta \leq 1/16$. □

7.3 A column with two big classes (Step A)

For a level-j configuration (S,W,σ) and $b\in W$ let Δ_b be the polynomial of Lemma 6.8: $\Delta_b(y)=\prod_s(|yb|^2-|ys|^2)\prod_s(|yb|^2-|bs|^2)\prod_{d\in\mathcal{D}(S)}(|yb|^2-d^2)$. If $\Delta_b(a)\neq 0$ then $|ab|\notin R_b$ and $a\notin\bigcup_s \mathrm{H}(b,s)$.

Lemma 7.4 (Step A). *Let $\max(n-4,0)\le j\le n-3$, and let (S,W,σ) be a level-j configuration with $\sigma\ge\sigma_{\min}$. Let $b\in W$ and let $\rho_1\neq\rho_2$ be elements of $\mathrm{Big}_b(Q\sigma)$. Put $C_\nu:=W\cap\mathrm{S}(b,\rho_\nu)$ for $\nu=1,2$, so that $|C_\nu|\ge Q\sigma$. Then the set*

$$A:=\{a\in W\setminus\{b\}:\ |ab|\notin\{\rho_1,\rho_2\},\ \Delta_b(a)\neq 0\}$$

has $|A|\le e_A+(j^2+2(n-4))\sigma$.

Proof. Fix $S'\subseteq S$ with $|S'|=n-4$; this is possible since $j\ge n-4$. Let A_0 be the set of $a\in A$ such that $|as|\neq|bs'|$ for all $s,s'\in S$, and $a\notin\mathrm{S}(s,\rho_1)\cup\mathrm{S}(s,\rho_2)$ for all $s\in S'$. By Lemma 6.5 and the cap condition, $|A\setminus A_0|\le j^2\sigma+2(n-4)\sigma$. It remains to show $|A_0|\le e_A$.

Step 1: the $(n-2)$-set T. For $a\in A_0$ let $T=T(a):=\{b,a\}\cup S'$, so $|T|=n-2$. Since $|ab|\notin R_b$, $a\notin\bigcup_s\mathrm{H}(b,s)$ and $|as|\neq|bs'|$, Lemma 6.5 shows that $S\cup\{b,a\}$ is rainbow, hence so is T. Moreover $\rho_\nu\notin\mathcal{D}(T)$, where $\mathcal{D}(T)=\mathcal{D}(S')\cup\{|bs|,|as|:s\in S'\}\cup\{|ab|\}$. Indeed, $\rho_\nu\notin R_b\supseteq\mathcal{D}(S')\cup\{|bs|\}$ by the definition of Big_b; $\rho_\nu\neq|ab|$ since $a\in A$; and $\rho_\nu\neq|as|$ for $s\in S'$ by the definition of A_0.

Step 2: non-generic points. Fix $a\in A_0$ and $\nu\in\{1,2\}$, and let $x\in C_\nu$. Then $x\notin T$: $x\neq b$ since $|xb|=\rho_\nu>0$, $x\neq a$ since $|ab|\neq\rho_\nu$, and $x\notin S$ since $W\cap S=\emptyset$. The label set of x is $L_x=\{\rho_\nu,|xa|\}\cup\{|xs|:s\in S'\}$. If x is not T-generic, then x lies in

$$\mathrm{S}\big(a,\mathcal{D}(T)\cup\{\rho_\nu\}\big)\ \cup\ \bigcup_{s\in S'}\mathrm{H}(a,s)\ \cup\ \mathrm{Sm}_\nu(a),$$

where

$$\mathrm{Sm}_\nu(a):=\bigcup_{s\in S'}\mathrm{S}\big(s,\{\rho_\nu,|ab|\}\cup\{|bs'|,|as'|:s'\in S'\}\big).$$

Indeed, the possible coincidences among $\mathcal{D}(T)\cup L_x$ are the following. $\mathcal{D}(T)$ itself is rainbow. Equalities among the $|xs|$, or between some $|xs|$ and $\mathcal{D}(S')$, are impossible because $S\cup\{x\}$ is rainbow. Equalities $|xs|\in\{\rho_\nu,|ab|,|bs'|,|as'|\}$ put x in $\mathrm{Sm}_\nu(a)$. An equality $|xs|=|xa|$ puts x in $\mathrm{H}(a,s)$. An equality $|xa|\in\mathcal{D}(T)\cup\{\rho_\nu\}$ puts x on a sphere about a. Finally $\rho_\nu\notin\mathcal{D}(T)$ by Step 1. The set $\mathrm{Sm}_\nu(a)$ is a union of $(n-4)(2n-6)$ spheres about points of S, so $|C_\nu\cap\mathrm{Sm}_\nu(a)|\le 2n^2\sigma$.

Step 3: counting cross pairs ("Lemma S2"). Fix $a\in A_0$ and put $P:=C_1\cap Y_T$ and $Q':=C_2\cap Y_T$. These are disjoint, since $\rho_1\neq\rho_2$. Let E be the set of pairs $(x,w)\in P\times Q'$ with $|xa|=|wa|$, or $|xa|=\rho_2$, or $|wa|=\rho_1$. We claim

$$|(P\times Q')\setminus E|\le\big(4^{n-2}\mu+n(n-4)\sigma\big)(|P|+|Q'|).\tag{4}$$

By Corollary 5.3(b), at most $4^{n-2}\mu(|P|+|Q'|)$ pairs are label-disjoint. Let $(x,w)\notin E$ share a label, where $L_x=\{\rho_1,|xa|\}\cup\{|xs|\}_{s\in S'}$ and $L_w=\{\rho_2,|wa|\}\cup\{|ws|\}_{s\in S'}$. Since $\rho_1\neq\rho_2$ and $(x,w)\notin E$, the shared value involves some $s\in S'$, so one of the following holds. Either $|xs|\in\{|ws'|,\rho_2,|wa|\}$: then for fixed w the point x lies on one of $(n-4)(n-2)$ spheres about points of S', which gives at most $(n-4)(n-2)\sigma|Q'|$ pairs. Or $|ws|\in\{\rho_1,|xa|\}$: this gives at most $2(n-4)\sigma|P|$ pairs. Since $(n-4)(n-2)\le n(n-4)$ and $2(n-4)\le n(n-4)$, (4) follows.

Step 4: one of the two classes is covered. Fix $a\in A_0$ and put $\theta:=Q^{1/2}\sigma$. We claim that there are $\nu\in\{1,2\}$ and a real $t^*=t^*(a)$ such that, with $V(a):=\mathcal{D}(T)\cup\{\rho_1,\rho_2,t^*\}$,

$$\Big|C_\nu\setminus\Big(\mathrm{S}(a,V(a))\cup\bigcup_{s\in S'}\mathrm{H}(a,s)\Big)\Big|\le\frac{|C_\nu|}{16(D+2)^{k'}}.\tag{5}$$

We use $|C_\nu|\ge Q\sigma$ and the bounds $Q^{-1/2}=2^{-22}n^{-2}(D+2)^{-4k'}\le 2^{-5}(D+2)^{-k'}$ and $2n^2/Q\le 2^{-5}(D+2)^{-k'}$.

- If $|P| < \theta$, Step 2 shows that the left side of (5) for $\nu = 1$ is at most $|P| + |C_1 \cap \mathrm{Sm}_1(a)| < (Q^{1/2} + 2n^2)\sigma \le (Q^{-1/2} + 2n^2/Q)|C_1| \le |C_1|/(16(D+2)^{k'})$. The case $|Q'| < \theta$ is symmetric, with $\nu = 2$.

- Otherwise $|P|, |Q'| \ge \theta$. Put $\eta := (4^{n-2}\mu + n(n-4)\sigma)(1/|P| + 1/|Q'|)$. Since $4^n\mu \le \sigma_{\min} \le \sigma$, we have $4^{n-2}\mu \le \sigma/16$, and therefore

$$\eta \le \frac{2(\sigma/16 + n^2\sigma)}{Q^{1/2}\sigma} \le 3n^2 Q^{-1/2} \le 2^{-20}(D+2)^{-4k'} \le \tfrac{1}{16}.$$

  Let $p_t := |P \cap \mathrm{S}(a,t)|/|P|$ and $q_t := |Q' \cap \mathrm{S}(a,t)|/|Q'|$. For independent uniform $x \in P$ and $w \in Q'$,

$$\Pr[(x,w) \notin E] = \Pr[|xa| \ne \rho_2,\ |wa| \ne \rho_1,\ |xa| \ne |wa|] = (1 - p_{\rho_2})(1 - q_{\rho_1}) - \sum_{t \notin \{\rho_1, \rho_2\}} p_t q_t,$$

  and by (4) this is at most η. Lemma 7.3 (with $\rho = \rho_1$, $\rho' = \rho_2$) gives t^* with $p_{\rho_2} + p_{t^*} \ge 1 - \eta^{1/4}$ or $q_{\rho_1} + q_{t^*} \ge 1 - \eta^{1/4}$. We can take $t^* > 0$, since the two distributions are supported on positive distances. In the first case all but $\eta^{1/4}|P|$ points of P lie on $\mathrm{S}(a, \{\rho_2, t^*\})$. Together with Step 2, the left side of (5) for $\nu = 1$ is at most $\eta^{1/4}|C_1| + 2n^2\sigma \le (2^{-5}(D+2)^{-k'} + 2n^2/Q)|C_1|$, which is at most $|C_1|/(16(D+2)^{k'})$. The second case is symmetric, with $\nu = 2$.

Step 5: GL, LR and OR. Let A_ν be the set of $a \in A_0$ for which (5) holds with this ν. Then $A_0 = A_1 \cup A_2$; say $|A_1| \ge |A_0|/2$ (the other case is symmetric). Note $A_1 \cap C_1 = \emptyset$. List $V(a)$ as $v_1(a), \dots, v_{t+3}(a)$, where $t = \binom{n-2}{2}$ and entries are repeated if $|V(a)| < t + 3$. On rows C_1 and columns A_1 consider

$$\mathcal{B}[x,a] := \prod_{i=1}^{t+3} \big(|xa|^2 - v_i(a)^2\big) \prod_{s \in S'} \big(|xa|^2 - |xs|^2\big).$$

It has $t + n - 1 \le k'$ factors, so $\operatorname{rank} \mathcal{B} \le (D+2)^{k'}$ by Lemmas 3.5 and 3.4. By (5), each column has at most $|C_1|/(16(D+2)^{k'})$ nonzero entries. Lemma 3.6 gives $C'' \times A''$ with $|C''| \ge |C_1|/2$, $|A''| \ge |A_1|/2$ and $\mathcal{B} = 0$ on it. So on $C'' \times A''$ the matrix $[|xa|^2]$, of rank at most $D + 2$, is covered by at most $k' \le k^*$ types: the column functions $v_i(a)^2$ (some of them constants) and the row functions $|xs|^2$, $s \in S'$. Theorem 4.1 gives $R \times C' \subseteq C'' \times A''$ with $|R| \ge 2^{-L^*}|C''|$, $|C'| \ge 2^{-L^*}|A''|$ and a single type. Since $S' \cap C' = \emptyset$, Lemma 4.7 gives $\min(|R|, |C'|) \le h$. But $|R| \ge 2^{-L^*-1}Q\sigma \ge 2^{-L^*-1}\sigma_{\min} > h$, so $|C'| \le h$. Hence $|A''| \le 2^{L^*}h$, $|A_1| \le 2^{L^*+1}h$ and $|A_0| \le 2^{L^*+2}h < e_A$. □

7.4 Proof of Theorem 7.1

Claim$(n-2)$ was proved above. Let $j \in \{n-3, n-4\}$ and assume Claim$(j+1)$ with constant Γ_{j+1}. Let (S, W, σ) be a level-j configuration with $\sigma \ge \sigma_{\min}$, and put $\tau := Q\sigma \ge \sigma_{\min}$. For $b \in W$ define

$$F_b := \Delta_b \prod_{\lambda \in \mathrm{Big}_b(\tau)} \big(|yb|^2 - \lambda^2\big) \quad \text{if } |\mathrm{Big}_b(\tau)| \le 1 \text{ (case B)},$$

$$F_b := \Delta_b \big(|yb|^2 - \rho_1^2\big)\big(|yb|^2 - \rho_2^2\big) \quad \text{otherwise (case A)},$$

where in case A, $\rho_1 \ne \rho_2$ are two elements of $\mathrm{Big}_b(\tau)$. In both cases F_b is a product of at most $e_j = 2 + 2j + \binom{j}{2}$ affine functions of the lifted coordinates, and $F_b(b) \ne 0$.

Claim: for each $b \in W$, at most $\max(\Gamma_{j+1}Q\sigma, e_A) + (j^2 + 2n)\sigma$ points $x \in W \setminus \{b\}$ have $F_b(x) \ne 0$. In case B, the proof of the claim in Lemma 6.8 (with $K = 1$ and $\tau = Q\sigma$) gives at

most $\Gamma_{j+1}Q\sigma + j^2\sigma$ such points. In case A, $F_b(x) \neq 0$ implies $|xb| \notin \{\rho_1, \rho_2\}$ and $\Delta_b(x) \neq 0$, i.e. $x \in A$, and Lemma 7.4 gives at most $e_A + (j^2 + 2(n-4))\sigma$ points.

As in the proof of Lemma 6.8, Caro–Wei and Lemma H give $|W| \le P_j^+\big(2\max(\Gamma_{j+1}Q\sigma, e_A) + 2(j^2+2n)\sigma + 1\big)$. Since $e_A \le \sigma \le Q\Gamma_{j+1}\sigma$ and $1 \le \sigma$ by (2), we get $|W| \le P_j^+(2Q\Gamma_{j+1} + 2(j^2 + 2n) + 1)\sigma = \Gamma_j\sigma$. □

Remark 7.5. For $n = 4$ the levels $j = 2, 1, 0$ are all covered by Theorem 7.1. In that case $S' = \emptyset$, the sets $\mathrm{Sm}_\nu(a)$ are empty, and the cost $n(n-4)\sigma$ in (4) vanishes. Claim(0) then follows directly, and Sections 8 and 9 are not needed.

8 Rainbow families and the Core Lemma

Throughout Sections 8 and 9, $n \ge 5$, $0 \le r \le n-5$, and (S, W, σ) is a fixed level-r configuration with $\sigma \ge \sigma_{\min}$. We assume that the conclusion of Lemma 6.6 holds for (S, W, σ) with some constant $\Gamma' \ge 1$; we refer to this as $\mathbf{HR}_r$. For $c \in W$, $R_c = \mathcal{D}(S) \cup \{|cs| : s \in S\}$ has at most $\binom{r}{2} + r$ elements. Recall from (2) that $\mu \le e \le \sigma$.

8.1 Rainbow families

Definition 8.1 (rainbow family at level r)**.** A *rainbow family* $\mathcal{F} = (B, (C_k, \beta(k))_{k\in I})$ consists of a set $B = \{b_1, \dots, b_p\} \subseteq X$ with $p \ge 1$, a finite index set I, and for each $k \in I$ a nonempty set $C_k \subseteq W$ and a p-tuple $\beta(k) = (\beta_1(k), \dots, \beta_p(k))$ of positive reals, such that:

(F1) $S \cap B = \emptyset$ and $S \cup B$ is rainbow;

(F2) $|xb_i| = \beta_i(k)$ for all $k \in I$, $x \in C_k$ and $i \le p$;

(F3) the $p|I|$ numbers $\beta_i(k)$ $(i \le p,\ k \in I)$ are pairwise distinct, and none of them lies in $\mathcal{D}(S \cup B)$;

(F4) for all $k \in I$ and $x \in C_k$: $x \notin S \cup B$ and $S \cup B \cup \{x\}$ is rainbow;

(F5) $|xs| \notin \beta(I) := \{\beta_i(k') : i \le p,\ k' \in I\}$ for all $x \in \bigcup_k C_k$ and $s \in S$.

We write $M(\mathcal{F}) := \min_k |C_k|$, and also use $\beta(k)$ for the *set* of its p values.

The classes C_k are pairwise disjoint, since they lie on spheres about b_1 with distinct radii $\beta_1(k)$, and they are disjoint from $S \cup B$. Passing to a subset $I' \subseteq I$, and to nonempty subsets $C'_k \subseteq C_k$, preserves (F1)–(F5).

8.2 Constants

Let $p := n - 4 - r$ and let $t, w_1, w_2, k_3, k_4, k_5, k_c$ be as in Definition 6.1. Put

$$\theta_{\mathrm{rel}}(w) := \frac{1}{32\,\Gamma'(D+2)^{k_4(w)}}, \qquad \varepsilon := \frac{\theta_{\mathrm{rel}}(w_2)}{96\,(D+2)^{k_3(w_2)}},$$
$$\hat{M}_{\mathrm{core}} := 2^{20}4^n n^2 \Gamma'^2 (D+2)^{2k_c}, \qquad M_{\mathrm{core}} := \hat{M}_{\mathrm{core}}\,\sigma.$$

Theorem 8.2 (Core Lemma)**.** *Assume HR_r with constant Γ'. There is no rainbow family at level r with $p = |B| = n - 4 - r$, $|I| \ge 12$ and $M(\mathcal{F}) \ge M_{\mathrm{core}}$.*

The rest of this section proves Theorem 8.2. Suppose $\mathcal{F}$ is such a family; discarding indices we may assume $|I| = 12$. Put $M := M(\mathcal{F}) \ge M_{\mathrm{core}}$ and

$$\delta_1 := \frac{2 \cdot 4^{n-2}\mu}{\varepsilon M} + \frac{3n^2\sigma}{M}.$$

Lemma 8.3 (parameter inequalities)**.** *(P1)* $\delta_1 \le \theta_{\mathrm{rel}}(w_1)/(16(D+2)^{k_3(w_1)}) = 1/(2^9\Gamma'(D+2)^{k_c})$*;*

(P2) $2n^2\sigma \le \varepsilon M$*;*

(P3) $6\varepsilon = \theta_{\mathrm{rel}}(w_2)/(16(D+2)^{k_3(w_2)})$*;*

(P4) $M \ge 32(D+2)^{k_4(w_1)}\Gamma'(r+1)\sigma$*;*

(P5) $M \ge 80n^2(D+2)^{k_5}\sigma$*;*

(P6) $M \ge \sigma \ge e > 2^{L^*+1}h$.

Proof. (P3) is the definition of ε, and (P6) follows from (2). We have $1/\varepsilon = 96 \cdot 32\,\Gamma'(D+2)^{k_3(w_2)+k_4(w_2)} \le 3 \cdot 2^{10}\Gamma'(D+2)^{k_c}$. For (P1), the first term of δ_1 is at most $6 \cdot 2^{10}\,4^{n-2}\Gamma'(D+2)^{k_c}\sigma/M$, using $\mu \le \sigma$. This is at most $1/(2^{10}\Gamma'(D+2)^{k_c})$ because $M \ge M_{\mathrm{core}} \ge 6 \cdot 2^{20}4^{n-2}\Gamma'^2(D+2)^{2k_c}\sigma$. The second term is at most $1/(2^{10}\Gamma'(D+2)^{k_c})$ because $M \ge 3 \cdot 2^{10}n^2\Gamma'(D+2)^{k_c}\sigma$. (P2) holds because $M \ge 6 \cdot 2^{10}n^2\Gamma'(D+2)^{k_c}\sigma \ge 2n^2\sigma/\varepsilon$. (P4) and (P5) hold because $k_4(w_1), k_5 \le k_c$, $r+1 \le n$ and $\Gamma' \ge 1$. □

8.3 Notation for two auxiliary points

Fix $l \ne m$ in I. For $a_1 \in C_l$ and $a_2 \in C_m$ put $T = T(a_1,a_2) := S \cup B \cup \{a_1,a_2\}$. These are $n-2$ distinct points, since $r+p = n-4$, $a_1 \ne a_2$, and $a_1, a_2 \notin S \cup B$ by (F4). For $x \notin T$ let $L_x := \{|xy| : y \in T\}$. For $x \in C_k$ we have $L_x = \{|xs|\}_{s\in S} \cup \beta(k) \cup \{|xa_1|, |xa_2|\}$ by (F2). Let Y_T be the set of T-generic points, and put

$$\mathrm{Hyp}(a_1,a_2) := \bigcup_{s\in S,\, i=1,2} \mathrm{H}(s,a_i), \qquad \mathrm{Sm}(a_1,a_2) := \bigcup_{s\in S} \mathrm{S}\big(s, \{|a_is'| : i=1,2,\ s' \in S\} \cup \{|a_1a_2|\}\big).$$

Both are empty if $r = 0$. Sm is a union of $r(2r+1)$ spheres about points of S, so it meets W in at most $r(2r+1)\sigma \le 2n^2\sigma$ points. By (F2),

$$\mathcal{D}(T) = \mathcal{D}(S\cup B) \cup \{|a_is| : i=1,2,\ s\in S\} \cup \beta(l) \cup \beta(m) \cup \{|a_1a_2|\}, \qquad |\mathcal{D}(T)| \le t. \tag{6}$$

Definition 8.4 (validity)**.** A pair (k,k') is *valid for* (a_1,a_2) if $k \ne k'$ are in $I \setminus \{l,m\}$, $T(a_1,a_2)$ is rainbow, and $(\beta(k) \cup \beta(k')) \cap \mathcal{D}(T) = \emptyset$.

Validity is used only in Steps 0–2, to obtain the inclusions there. The properties Good_k and Cov_k defined below are properties of *arbitrary* pairs $(a_1,a_2) \in C_l \times C_m$. Steps 3–5 count them over all of $C_l \times C_m$, and validity enters Step 5 only through the set V_{lm} of pairs on which it holds (referee item F).

8.4 Step 0: non-generic loci

Lemma 8.5 (Step 0)**.** *Let T be rainbow, $k \in I \setminus \{l,m\}$ and $\beta(k) \cap \mathcal{D}(T) = \emptyset$. Every $x \in C_k \setminus Y_T$ lies in*

$$\mathrm{S}\big(a_1, \mathcal{D}(T)\cup\beta(k)\big) \cup \mathrm{S}\big(a_2, \mathcal{D}(T)\cup\beta(k)\big) \cup \mathrm{H}(a_1,a_2) \cup \mathrm{Hyp}(a_1,a_2) \cup \mathrm{Sm}(a_1,a_2).$$

Proof. First, $x \notin T$, because C_k is disjoint from $S \cup B$, C_l and C_m. We list the possible coincidences among $\mathcal{D}(T) \cup L_x$. $\mathcal{D}(T)$ is rainbow. Equalities among the $|xs|$ and $\beta(k)$, or between these and $\mathcal{D}(S\cup B)$, are excluded by (F4) and (F3). The case $|xs| \in \beta(l) \cup \beta(m)$ is excluded by (F5). Equalities $|xs| \in \{|a_is'|, |a_1a_2|\}$ give Sm. An equality $|xs| = |xa_i|$ gives Hyp. An equality $|xa_i| \in \mathcal{D}(T) \cup \beta(k)$ gives a sphere about a_i, and $|xa_1| = |xa_2|$ gives $\mathrm{H}(a_1,a_2)$. Finally $\beta(k) \cap \mathcal{D}(T) = \emptyset$ by hypothesis. □

8.5 Step 1: a trichotomy

For $k \in I \setminus \{l, m\}$ put

$$\Lambda_k(a_1, a_2) := \mathrm{S}\big(a_1, \mathcal{D}(T) \cup \beta(k)\big) \cup \mathrm{S}\big(a_2, \mathcal{D}(T) \cup \beta(k)\big) \cup \mathrm{H}(a_1, a_2) \cup \mathrm{Hyp}(a_1, a_2).$$

Definition 8.6 (Good)**.** $\mathrm{Good}_k(a_1, a_2)$ holds if there are a set Ω of at most w_1 positive reals and a set $\mathrm{Ex} \subseteq C_k$ with $|\mathrm{Ex}| \le \delta_1 |C_k|$ such that $C_k \subseteq \mathrm{S}(a_1, \Omega) \cup \mathrm{S}(a_2, \Omega) \cup \mathrm{H}(a_1, a_2) \cup \mathrm{Hyp}(a_1, a_2) \cup \mathrm{Ex}$.

Lemma 8.7 (Step 1)**.** *Let (k, k') be valid for (a_1, a_2). Then $\mathrm{Good}_k(a_1, a_2)$ holds, or*

$$\big|C_{k'} \setminus \big(\Lambda_{k'}(a_1, a_2) \cup \mathrm{S}(a_1, \beta(k)) \cup \mathrm{S}(a_2, \beta(k))\big)\big| < 2\varepsilon |C_{k'}|. \tag{ii}_{k,k'}$$

Proof. Put $P := C_k \cap Y_T$, $Q := C_{k'} \cap Y_T$ and $Q' := \{w \in Q : L_w \cap \beta(k) = \emptyset\}$.

Case $|Q'| \ge \varepsilon |C_{k'}|$. P and Q' are disjoint subsets of Y_T. By Corollary 5.3(b), at most $4^{n-2}\mu(|C_k| + |C_{k'}|)$ pairs $(w, x) \in Q' \times P$ are label-disjoint. So some $w_0 \in Q'$ is label-disjoint from at most $\mathrm{ex} := 4^{n-2}\mu(|C_k| + |C_{k'}|)/(\varepsilon |C_{k'}|)$ points of P. Let $x \in P$ be any other point, so that $L_x \cap L_{w_0} \ne \emptyset$. Now $\beta(k) \cap L_{w_0} = \emptyset$, because $\beta(k) \cap \beta(k') = \emptyset$ (F3), $|w_0 s| \notin \beta(k)$ (F5), and $w_0 \in Q'$. So the common value is $|xs|$ for some s, or $|xa_i|$ for some i, and it lies in L_{w_0}. If it is $|xs|$, then $|xs| \notin \beta(k')$ by (F5), so $|xs| \in \{|w_0 s'|\}_{s'} \cup \{|w_0 a_1|, |w_0 a_2|\}$, and x lies on one of $r(r+2)$ spheres about points of S. If it is $|xa_i|$, then $x \in \mathrm{S}(a_i, L_{w_0})$. Combining this with Lemma 8.5, C_k is covered as in Definition 8.6 with $\Omega := L_{w_0} \cup \mathcal{D}(T) \cup \beta(k)$, where $|\Omega| \le (n-2) + t + p = w_1$, and with Ex consisting of the at most ex label-disjoint partners, the fewer than $r(r+2)\sigma$ points on the spheres just described, and the fewer than $r(2r+1)\sigma$ points of $C_k \cap \mathrm{Sm}$. Since $|C_k|, |C_{k'}| \ge M$,

$$|\mathrm{Ex}| \le \Big(\frac{4^{n-2}\mu}{\varepsilon}\Big(\frac{1}{|C_{k'}|} + \frac{1}{|C_k|}\Big) + \frac{3n^2\sigma}{M}\Big)|C_k| \le \delta_1 |C_k|,$$

so $\mathrm{Good}_k(a_1, a_2)$ holds.

Case $|Q'| < \varepsilon |C_{k'}|$. By Lemma 8.5 applied to k' (valid since $\beta(k') \cap \mathcal{D}(T) = \emptyset$), $C_{k'} \setminus Q \subseteq \Lambda_{k'} \cup \mathrm{Sm}$. If $w \in Q \setminus Q'$, then $L_w \cap \beta(k) \ne \emptyset$. Since $|ws| \notin \beta(k)$ (F5) and $\beta(k') \cap \beta(k) = \emptyset$ (F3), this means $|wa_i| \in \beta(k)$ for some i, i.e. $w \in \mathrm{S}(a_1, \beta(k)) \cup \mathrm{S}(a_2, \beta(k))$. Hence the set in $(\mathrm{ii})_{k,k'}$ is contained in $Q' \cup (C_{k'} \cap \mathrm{Sm})$, which has fewer than $\varepsilon |C_{k'}| + 2n^2\sigma \le 2\varepsilon |C_{k'}|$ points, by (P2). □

8.6 Step 2: summing over k

Lemma 8.8 (Step 2)**.** *Fix a_1, a_2 and $k' \in I \setminus \{l, m\}$. Let $J \subseteq I \setminus \{l, m, k'\}$ with $|J| \ge 3$ be such that, for every $k \in J$, (k, k') is valid and $(\mathrm{ii})_{k,k'}$ holds. Then*

$$\mathrm{Cov}_{k'}(a_1, a_2): \qquad |C_{k'} \setminus \Lambda_{k'}(a_1, a_2)| < 6\varepsilon |C_{k'}|.$$

Proof. Let $c := |C_{k'} \setminus \Lambda_{k'}|$ and assume $c \ge 2\varepsilon |C_{k'}|$. For each $k \in J$, by $(\mathrm{ii})_{k,k'}$, more than $c - 2\varepsilon |C_{k'}|$ points of $C_{k'} \setminus \Lambda_{k'}$ lie in $\mathrm{S}(a_1, \beta(k)) \cup \mathrm{S}(a_2, \beta(k))$. By (F3) the radius sets $\beta(k)$, $k \in J$, are pairwise disjoint, so the sets $\mathrm{S}(a_1, \beta(k))$, $k \in J$, are pairwise disjoint, and likewise for a_2. Summing over J gives $|J|(c - 2\varepsilon |C_{k'}|) < 2c$, i.e. $c(|J| - 2) < 2|J|\varepsilon |C_{k'}|$, so $c < 6\varepsilon |C_{k'}|$ as $|J| \ge 3$. □

Note that $\mathrm{Cov}_{k'}(a_1, a_2)$ has the shape of $\mathrm{Good}_{k'}(a_1, a_2)$ with $\Omega := \mathcal{D}(T) \cup \beta(k')$, so that $|\Omega| \le t + p = w_2$, and $\mathrm{Ex} := C_{k'} \setminus \Lambda_{k'}$, so that $|\mathrm{Ex}| < 6\varepsilon |C_{k'}|$. Here Ω depends on (a_1, a_2) through $\mathcal{D}(T)$.

8.7 Step 3: the two-centre lemma

Definition 8.9 (w-few)**.** Let $j \in I \setminus \{l,m\}$, $w \in \{w_1,w_2\}$ and $a_2 \in C_m$. Put $\theta_w := \max(\theta_{\mathrm{rel}}(w)|C_j|,\sigma)$ and $\theta'_w := \theta_w + r\sigma$. We say that a_2 *is w-few on C_j* if there is a set Ξ of at most $w + r + \binom{r}{2}$ positive reals with

$$\Big|C_j \setminus \Big(\mathrm{S}(a_2,\Xi) \cup \bigcup_{s\in S} \mathrm{H}(a_2,s)\Big)\Big| \le \Gamma'\theta'_w.$$

This property depends only on (a_2, j, w).

Lemma 8.10 (Step 3, two-centre lemma)**.** *Let $j \in I \setminus \{l,m\}$, $a_2 \in C_m$, $w \in \{w_1,w_2\}$ and $0 < \delta \le \theta_{\mathrm{rel}}(w)/(16(D+2)^{k_3(w)})$. Let $Z \subseteq C_l$, and suppose that for each $a_1 \in Z$ there are a set $\Omega(a_1)$ of at most w positive reals and a set $\mathrm{Ex}(a_1) \subseteq C_j$ with $|\mathrm{Ex}(a_1)| \le \delta|C_j|$ such that*

$$C_j \subseteq \mathrm{S}(a_1,\Omega(a_1)) \cup \mathrm{S}(a_2,\Omega(a_1)) \cup \mathrm{H}(a_1,a_2) \cup \mathrm{Hyp}(a_1,a_2) \cup \mathrm{Ex}(a_1).$$

If a_2 is not w-few on C_j, then $|Z| \le (w+1)e$.

Proof. For a real $\lambda > 0$ let $R_\lambda := (C_j \cap \mathrm{S}(a_2,\lambda)) \setminus \bigcup_{s\in S} \mathrm{S}(s,\lambda)$ and $Z_\lambda := \{a_1 \in Z : \lambda \notin \Omega(a_1)\}$. Let $a_1 \in Z_\lambda$ and $x \in R_\lambda \setminus \mathrm{Ex}(a_1)$. Then $x \notin \mathrm{S}(a_2,\Omega(a_1))$, since $|xa_2| = \lambda \notin \Omega(a_1)$. Also $x \notin \mathrm{H}(s,a_2)$, since otherwise $|xs| = \lambda$. If $x \in \mathrm{H}(a_1,a_2)$ then $|xa_1| = \lambda$, and if $x \in \mathrm{H}(s,a_1)$ then $|xa_1| = |xs|$. Hence

$$|xa_1|^2 \in \{v^2 : v \in \Omega(a_1)\} \cup \{\lambda^2\} \cup \{|xs|^2 : s \in S\} \qquad (x \in R_\lambda \setminus \mathrm{Ex}(a_1)). \tag{7}$$

Claim: if $|R_\lambda| \ge \theta_w$, then $|Z_\lambda| \le e$. Take rows R_λ and columns Z_λ; they are disjoint, as $C_j \cap C_l = \emptyset$. List $\Omega(a_1)$ as $v_1(a_1),\dots,v_w(a_1)$, repeating entries if necessary, and put

$$\mathcal{B}[x,a_1] := \prod_{i\le w} \big(|xa_1|^2 - v_i(a_1)^2\big) \cdot \big(|xa_1|^2 - \lambda^2\big) \cdot \prod_{s\in S} \big(|xa_1|^2 - |xs|^2\big).$$

This is a product of $k_3(w) = w + 1 + r$ factors, so $\operatorname{rank}\mathcal{B} \le (D+2)^{k_3(w)}$. By (7), column a_1 has nonzero entries only in $\mathrm{Ex}(a_1)$, hence at most $\delta|C_j| \le \theta_{\mathrm{rel}}(w)|C_j|/(16(D+2)^{k_3(w)}) \le |R_\lambda|/(16(D+2)^{k_3(w)})$ of them. Lemma 3.6 gives $R'' \times Z''$ with $|R''| \ge |R_\lambda|/2$, $|Z''| \ge |Z_\lambda|/2$ and $\mathcal{B} = 0$ there. On $R'' \times Z''$ the matrix $[|xa_1|^2]$ is covered by $k_3(w) \le k^*$ types: the column functions $v_i(a_1)^2$, the constant λ^2, and the row functions $|xs|^2$. Theorem 4.1 and Lemma 4.7 (note $S \cap Z'' = \emptyset$) give $R''' \times Z'''$ with $\min(|R'''|,|Z'''|) \le h$, where $|R'''| \ge 2^{-L^*}|R_\lambda|/2 \ge 2^{-L^*-1}\sigma > h$ by (P6). So $|Z'''| \le h$, hence $|Z''| \le 2^{L^*}h$ and $|Z_\lambda| \le 2^{L^*+1}h \le e$.

Suppose first that there are $w+1$ distinct radii $\lambda_1,\dots,\lambda_{w+1}$ with $|R_{\lambda_i}| \ge \theta_w$. Since $|\Omega(a_1)| \le w$, every $a_1 \in Z$ lies in some Z_{λ_i}, so $|Z| \le (w+1)e$. Otherwise let Ξ_h be the set of the at most w radii with $|R_\lambda| \ge \theta_w$, and put $\Xi := \Xi_h \cup R_{a_2}$, so that $|\Xi| \le w + r + \binom{r}{2}$. Let $x \in C_j$ with $x \notin \bigcup_s \mathrm{H}(a_2,s)$ and $\lambda := |xa_2| \notin \Xi$. Then $|R_\lambda| < \theta_w$, so $|C_j \cap \mathrm{S}(a_2,\lambda)| < \theta_w + r\sigma = \theta'_w$, because the spheres $\mathrm{S}(s,\lambda)$, $s \in S$, meet W in fewer than $r\sigma$ points in total. Apply HR_r with $c := a_2 \in W$, $V := C_j \subseteq W \setminus \{a_2\}$ and threshold $\theta'_w \ge \sigma$. Since $\lambda \notin R_{a_2}$, every such x is counted there, so there are at most $\Gamma'\theta'_w$ of them. Hence a_2 is w-few on C_j. □

So Lemma 8.10 says: *if a_2 is not w-few on C_j, then $|Z| \le (w+1)e$* (referee item X4).

8.8 Step 4: few w-few centres

Lemma 8.11 (Step 4)**.** *For $j \in I \setminus \{l,m\}$ and $w \in \{w_1,w_2\}$, the set F of points $a_2 \in C_m$ that are w-few on C_j has $|F| \le e$.*

Proof. Take rows C_j and columns F (disjoint). For $a_2 \in F$ let $\Xi(a_2)$ be as in Definition 8.9, padded to exactly $w + r + \binom{r}{2}$ entries, and put $\mathcal{B}[x, a_2] := \prod_{v\in\Xi(a_2)}(|xa_2|^2 - v^2)\prod_{s\in S}(|xa_2|^2 - |xs|^2)$. This is a product of $k_4(w)$ factors, so $\operatorname{rank}\mathcal{B} \le (D+2)^{k_4(w)}$. Column a_2 has at most $\Gamma'\theta'_w$ nonzero entries, and

$$\Gamma'\theta'_w \le \Gamma'\theta_{\mathrm{rel}}(w)|C_j| + \Gamma'(r+1)\sigma \le \frac{|C_j|}{32(D+2)^{k_4(w)}} + \frac{|C_j|}{32(D+2)^{k_4(w)}},$$

by the definition of θ_{rel} and (P4) (as $k_4(w) \le k_4(w_1)$). Lemma 3.6 gives a zero rectangle $C'' \times F''$ with halves. There $[|xa_2|^2]$ is covered by $k_4(w) \le k^*$ types: the column functions $a_2 \mapsto v(a_2)^2$ given by the (padded) entries v of $\Xi(a_2)$, and the row functions $|xs|^2$. Theorem 4.1 and Lemma 4.7 give a rectangle with $\min \le h$, whose row side is at least $2^{-L^*}|C_j|/2 > h$ by (P6). So $|F''| \le 2^{L^*}h$ and $|F| \le e$. □

Corollary 8.12. *Let $j \in I \setminus \{l, m\}$, $w \in \{w_1, w_2\}$ and δ be as in Lemma 8.10, and suppose that for every $a_2 \in C_m$ the set $Z(a_2) \subseteq C_l$ satisfies the hypothesis of Lemma 8.10. Then $\sum_{a_2\in C_m}|Z(a_2)| \le e|C_l| + (w+1)e|C_m|$.*

Proof. The at most e points a_2 that are w-few on C_j (Lemma 8.11) contribute at most $|C_l|$ each. Every other a_2 contributes at most $(w+1)e$, by Lemma 8.10. □

We apply Corollary 8.12 in two ways, for each $j \in I \setminus \{l, m\}$:

(a) $Z(a_2) := \{a_1 \in C_l : \mathrm{Good}_j(a_1, a_2)\}$, with $w = w_1$ and $\delta = \delta_1$, which is admissible by (P1). This gives $|\{(a_1, a_2) : \mathrm{Good}_j(a_1, a_2)\}| \le e|C_l| + (w_1+1)e|C_m|$.

(b) $Z(a_2) := \{a_1 \in C_l : \mathrm{Cov}_j(a_1, a_2)\}$, with $w = w_2$ and $\delta = 6\varepsilon$, which is admissible by (P3); see the remark after Lemma 8.8. This gives $|\{(a_1, a_2) : \mathrm{Cov}_j(a_1, a_2)\}| \le e|C_l| + (w_2+1)e|C_m|$.

8.9 Step 5: counting, and the contradiction

Let Π be the set of pairs $(a_1, a_2) \in C_l \times C_m$ with $|a_1 s| = |a_2 s'|$ for some $s, s' \in S$. For fixed a_2, at most $r^2\sigma$ points a_1 qualify (cap), so $|\Pi| \le r^2\sigma|C_m|$; and $\Pi = \emptyset$ if $r = 0$. Put

$$\mathrm{Ty}(a_1, a_2) := \mathcal{D}(S \cup B) \cup \beta(l) \cup \beta(m) \cup \{|a_1 s| : s \in S\} \cup \{|a_2 s| : s \in S\},$$

a list of at most $\binom{r+p}{2} + 2p + 2r = k_5$ numbers, and $V_{lm} := \{(a_1, a_2) \notin \Pi : |a_1 a_2| \notin \mathrm{Ty}(a_1, a_2)\}$.

Lemma 8.13. *$T(a_1, a_2)$ is rainbow if and only if $(a_1, a_2) \notin \Pi$ and $|a_1 a_2| \notin \mathrm{Ty}(a_1, a_2)$.*

Proof. The distances of T are $\mathcal{D}(S \cup B)$, $|a_1 s|$, $\beta(l)$, $|a_2 s|$, $\beta(m)$ and $|a_1 a_2|$. The sets $S \cup B \cup \{a_i\}$ are rainbow (F4), $\beta(l) \cap \beta(m) = \emptyset$ (F3), and $|a_1 s| \notin \beta(m)$ and $|a_2 s| \notin \beta(l)$ (F5). So the only possible coincidences are $|a_1 s| = |a_2 s'|$ (that is, Π) and $|a_1 a_2| \in \mathrm{Ty}$. □

Let $(a_1, a_2) \in V_{lm}$. Then T is rainbow. By (F3) at most one index $k^\bullet \in I$ has $|a_1 a_2| \in \beta(k^\bullet)$, and $k^\bullet \notin \{l, m\}$ since $\beta(l) \cup \beta(m) \subseteq \mathrm{Ty}$. Put $I_0 := I \setminus \{l, m, k^\bullet\}$, so $|I_0| \ge 9$. For $k \in I_0$ we have $\beta(k) \cap \mathcal{D}(T) = \emptyset$ by (6), (F3), (F5) and $k \ne k^\bullet$. Hence every (k, k') with $k \ne k'$ in I_0 is valid. Let $g := |\{k \in I_0 : \mathrm{Good}_k(a_1, a_2)\}|$. If $g \le |I_0| - 4$, then for each $k' \in I_0$ the set $J := \{k \in I_0 \setminus \{k'\} : \text{not } \mathrm{Good}_k(a_1, a_2)\}$ has $|J| \ge 3$, and $(\mathrm{ii})_{k,k'}$ holds for $k \in J$ by Lemma 8.7; so $\mathrm{Cov}_{k'}(a_1, a_2)$ holds by Lemma 8.8, for every $k' \in I_0$. In either case

$$|\{k \in I \setminus \{l, m\} : \mathrm{Good}_k(a_1, a_2)\}| + |\{k \in I \setminus \{l, m\} : \mathrm{Cov}_k(a_1, a_2)\}| \ge 6 \qquad ((a_1, a_2) \in V_{lm}).$$

Summing over $(a_1, a_2) \in C_l \times C_m$ and applying (a) and (b) to each of the 10 indices $j \in I \setminus \{l, m\}$,

$$6|V_{lm}| \le 10\big[2e|C_l| + (w_1 + w_2 + 2)e|C_m|\big] \le 10(w_1 + w_2 + 2)\, e\,(|C_l| + |C_m|).$$

So the density of V_{lm} in $C_l \times C_m$ is at most $\frac{10}{6}(w_1+w_2+2)e(\frac{1}{|C_m|}+\frac{1}{|C_l|}) \le 4(w_1+w_2+2)e/M$ (referee item B: the factor 10/6 is the ratio of the number of indices summed, 10, to the guaranteed count, 6). The density of Π is at most $r^2\sigma/|C_l| \le r^2\sigma/M$. Since $w_1+w_2+2 = n^2-2n-2-2r < n^2$ and $e \le \sigma$, the set $V_{lm}\cup\Pi$ has density at most $5n^2\sigma/M \le 1/(16(D+2)^{k_5})$, by (P5).

Now consider $\mathcal{B}_0[a_1,a_2] := \prod_{v\in \mathrm{Ty}(a_1,a_2)}(|a_1a_2|^2 - v^2)$ on $C_l \times C_m$, with Ty padded to k_5 entries. It has rank at most $(D+2)^{k_5}$, and it vanishes outside $V_{lm}\cup\Pi$. Lemma 3.6 (average form) gives $C_l'' \times C_m''$ with halves and $\mathcal{B}_0 = 0$. There $[|a_1a_2|^2]$ is covered by $k_5 \le k^*$ types: the constants in $\mathcal{D}(S\cup B)\cup\beta(l)\cup\beta(m)$, the row functions $|a_1s|^2$ and the column functions $|a_2s|^2$. By Theorem 4.1 there is a single-type rectangle $R\times C$ with $|R| \ge 2^{-L^*}|C_l|/2 > h$ and $|C| \ge 2^{-L^*}|C_m|/2 > h$, using (P6). Lemma 4.7 (rows $a_1 \in R$, columns $a_2 \in C$, $S\cap C = \emptyset$) gives $\min(|R|,|C|) \le h$, a contradiction. This proves Theorem 8.2. □

Remark 8.14. For $n = 5$ and $r = 0$ (so $p = 1$) the sets Hyp, Sm and Π are empty, $R_{a_2} = \emptyset$, and only column functions and constants occur. In this case the proof is the argument for $f_d(5)$ written in the project's round 13 and refereed separately ("referee14"), up to the values of the constants. For general r the proof adds row types $|xs|^2$, coming from the bisectors $\mathrm{H}(s,a_i)$, and the small classes of points of S, whose costs are absorbed in δ_1, in ε and in Step 5.

9 The Slicing Lemma and the class-count theorem $\mathbf{E1}_r$

The setting is that of Section 8: $n \ge 5$, $0 \le r \le n-5$, (S,W,σ) is a level-r configuration with $\sigma \ge \sigma_{\min}$, and HR_r holds for it with constant $\Gamma' \ge 1$.

9.1 The Slicing Lemma

Lemma 9.1 (Slicing Lemma). *Let $0 \le r \le n-6$, let $K \ge 1$, and let $\mathcal{F} = (B,(C_k,\beta(k))_{k\in I})$ be a rainbow family at level r with $1 \le p = |B| \le n-5-r$ and $|I| \ge 6K$. Put*

$$k_{\mathrm{sl}} := 3K-1+\tbinom{r+p}{2}+2p+2r, \qquad \theta_{\mathrm{sl}} := \frac{1}{16\,\Gamma'(D+2)^{k_{\mathrm{sl}}}}, \qquad M_{\mathrm{sl}} := \frac{2n(n+K)\sigma}{\theta_{\mathrm{sl}}},$$

and assume $k_{\mathrm{sl}} \le k^*$. If $M(\mathcal{F}) \ge M_{\mathrm{sl}}$, then there are a point $c \in \bigcup_k C_k$ and a rainbow family $\mathcal{F}' = (B\cup\{c\},(C_k',\beta'(k))_{k\in I'})$ at level r with $I' \subseteq I$, $|I'| = K$, $C_k' \subseteq C_k$ and $M(\mathcal{F}') \ge \theta_{\mathrm{sl}}M(\mathcal{F})/2$.

Proof. Discard indices so that $|I| = 6K$, and put $M := M(\mathcal{F})$. Fix $\pi \in I$. For $c \in C_\pi$ the set $S\cup B\cup\{c\}$ is rainbow by (F4), and since $|cb_i| = \beta_i(\pi)$,

$$\Phi(c) := \mathcal{D}(S\cup B\cup\{c\}) = \mathcal{D}(S\cup B)\cup\beta(\pi)\cup\{|cs| : s\in S\}, \qquad |\Phi(c)| \le \tbinom{r+p}{2}+p+r.$$

Note that $R_c = \mathcal{D}(S)\cup\{|cs|\} \subseteq \Phi(c)$. Let G_c be the bipartite graph with left vertex set $I\setminus\{\pi\}$ and right vertex set $\mathbb{R}_{>0}$, in which $k \sim \lambda$ if $\lambda \notin \Phi(c)\cup\beta(k)$ and $|C_k \cap \mathrm{S}(c,\lambda)| \ge \theta_{\mathrm{sl}}|C_k|$. Each k has at most $1/\theta_{\mathrm{sl}}$ neighbours, so only finitely many right vertices are non-isolated, and we regard G_c as a finite graph.

(a) Few bad centres. Let Z_{bad} be the set of $c \in C_\pi$ for which G_c has no matching of size $3K$. We show $|Z_{\mathrm{bad}}| \le 2e$. For $c \in Z_{\mathrm{bad}}$, Theorem 3.10 gives a vertex cover $\mathcal{A}(c)\cup\Lambda_0(c)$, with $\mathcal{A}(c) \subseteq I\setminus\{\pi\}$, $\Lambda_0(c) \subseteq \mathbb{R}_{>0}$ and $|\mathcal{A}(c)|+|\Lambda_0(c)| \le 3K-1$. Call c *k-few* for every $k \in I\setminus(\{\pi\}\cup\mathcal{A}(c))$; then every edge at k ends in $\Lambda_0(c)$.

Let c be k-few, and let $x \in C_k$ satisfy $x \notin \bigcup_s \mathrm{H}(c,s)$ and $\lambda := |xc| \notin \Lambda_0(c)\cup\Phi(c)\cup\beta(k)$. Then (k,λ) is not an edge, since $\lambda \notin \Lambda_0(c)$. As $\lambda \notin \Phi(c)\cup\beta(k)$, this means $|C_k\cap\mathrm{S}(c,\lambda)| < \theta_{\mathrm{sl}}|C_k|$. Also $\lambda \notin R_c$. Now apply HR_r with $c \in W$, $V := C_k \subseteq W\setminus\{c\}$ and threshold $\theta := \theta_{\mathrm{sl}}|C_k|$; this is admissible since $\theta \ge \theta_{\mathrm{sl}}M_{\mathrm{sl}} = 2n(n+K)\sigma \ge \sigma$. So the number of such x is at most $\Gamma'\theta_{\mathrm{sl}}|C_k| = |C_k|/(16(D+2)^{k_{\mathrm{sl}}})$. Hence the matrix

$$\mathcal{B}_k[x,c] := \prod_{v\in\Lambda_0(c)\cup\Phi(c)\cup\beta(k)}(|xc|^2-v^2)\prod_{s\in S}(|xc|^2-|xs|^2)$$

(with $\Lambda_0(c)$ padded to $3K-1$ entries, so that there are at most k_{sl} factors, and $\operatorname{rank}\mathcal{B}_k \le (D+2)^{k_{\mathrm{sl}}}$) has at most $|C_k|/(16(D+2)^{k_{\mathrm{sl}}})$ nonzero entries in every column c that is k-few.

Each $c \in Z_{\mathrm{bad}}$ is k-few for at least $(6K-1)-(3K-1)=3K$ indices k. So some k has $Z_k := \{c \in Z_{\mathrm{bad}} : c \text{ is } k\text{-few}\}$ with $|Z_k| \ge 3K|Z_{\mathrm{bad}}|/(6K-1) \ge |Z_{\mathrm{bad}}|/2$. Take rows C_k and columns $Z_k \subseteq C_\pi$ (disjoint). Lemma 3.6 gives a zero rectangle $C_k'' \times Z_k''$ of $\mathcal{B}_k$ with halves. There $[|xc|^2]$ is covered by at most $k_{\mathrm{sl}} \le k^*$ types: column functions (the entries of $\Lambda_0(c)$, and $|cs|^2$), constants (the elements of $\mathcal{D}(S\cup B)$, $\beta(\pi)$ and $\beta(k)$), and row functions $|xs|^2$. Theorem 4.1 and Lemma 4.7 give a single-type rectangle $R \times C$ with $\min(|R|,|C|) \le h$ and $|R| \ge 2^{-L^*}|C_k|/2 > h$ (as $|C_k| \ge M \ge \sigma$). So $|C| \le h$, hence $|Z_k''| \le 2^{L^*}h$, $|Z_k| \le e$ and $|Z_{\mathrm{bad}}| \le 2e$.

(b) Choice of c. Since $|C_\pi| \ge M \ge M_{\mathrm{sl}} > 2\sigma \ge 2e$, there is $c \in C_\pi \setminus Z_{\mathrm{bad}}$. Fix such a c and a matching $\{(k,\lambda_k) : k \in I_1\}$ of G_c with $|I_1| = 3K$; the λ_k are distinct.

(c) Label conflicts. Consider the digraph on I_1 with an arc $k \to k'$ whenever $k' \ne k$ and $\lambda_k \in \beta(k')$. By (F3) each λ_k equals at most one of the pairwise distinct numbers $\beta_i(k')$, so every out-degree is at most 1. The underlying graph therefore has at most $3K$ edges, hence average degree at most 2, and Theorem 3.9 gives an independent set of size at least $3K/3 = K$. Let I' consist of K of its elements. For distinct $k,k' \in I'$ we then have $\lambda_k \notin \beta(k')$.

(d) The new family. Put $B' := B \cup \{c\}$, $\beta'(k) := (\beta(k), \lambda_k)$ and

$$C_k' := \big(C_k \cap \mathrm{S}(c,\lambda_k)\big) \setminus \bigcup_{s\in S} \mathrm{S}\big(s, \{|cs'| : s' \in S\} \cup \{\lambda_{k'} : k' \in I'\}\big) \qquad (k \in I').$$

Then $|C_k'| \ge \theta_{\mathrm{sl}}|C_k| - r(r+K)\sigma \ge \theta_{\mathrm{sl}}M - n(n+K)\sigma \ge \theta_{\mathrm{sl}}M/2$, because $\theta_{\mathrm{sl}}M \ge \theta_{\mathrm{sl}}M_{\mathrm{sl}} = 2n(n+K)\sigma$. We verify (F1)–(F5) for $\mathcal{F}'$.

(F1) $c \notin S \cup B$, and $S \cup B \cup \{c\}$ is rainbow, by (F4) for $c \in C_\pi$.

(F2) $|xb_i| = \beta_i(k)$ and $|xc| = \lambda_k$ on C_k'.

(F3) The old values are pairwise distinct (F3), and the λ_k are pairwise distinct (matching). For $k \in I'$, $\lambda_k \notin \beta(k)$ (edge rule) and $\lambda_k \notin \beta(k')$ for $k' \in I' \setminus \{k\}$ (independence). None of the values lies in $\mathcal{D}(S\cup B\cup\{c\}) = \Phi(c)$: $\lambda_k \notin \Phi(c)$ by the edge rule; $\beta_i(k) \notin \mathcal{D}(S\cup B)$ by (F3); $\beta_i(k) \ne \beta_{i'}(\pi)$ by (F3), as $k \ne \pi$; and $\beta_i(k) \ne |cs|$ by (F5) for c.

(F4) Let $x \in C_k'$. Then $x \notin S \cup B \cup \{c\}$, because $x \in C_k$, which is disjoint from $C_\pi \ni c$. The distances of $S\cup B\cup\{c,x\}$ are those of $S\cup B\cup\{x\}$ (rainbow), those of $S\cup B\cup\{c\}$ (rainbow), and $|xc| = \lambda_k$. The cross coincidences are excluded as follows. $|xs| = |cs'|$ is excluded by the removal in the definition of C_k'. $|xs| \in \beta(\pi)$ is excluded by (F5). $\beta_i(k) = |cs|$ is excluded by (F5) for c. $\beta_i(k) = \beta_{i'}(\pi)$ is excluded by (F3). $\lambda_k \in \Phi(c)$ and $\lambda_k \in \beta(k)$ are excluded by the edge rule. Finally $\lambda_k = |xs|$ is excluded by the removal.

(F5) For $x \in C_k'$ and $s \in S$: $|xs| \notin \beta(I')$ by (F5) for $\mathcal{F}$, and $|xs| \ne \lambda_{k'}$ by the removal.

So $\mathcal{F}'$ is a rainbow family with $p+1$ centres, $|I'| = K$ and $M(\mathcal{F}') \ge \theta_{\mathrm{sl}}M/2$. □

9.2 The class-count theorem

Theorem 9.2 (E1$_r$). *Let $0 \le r \le n-5$, and write $K^{(p)} := K^{(r,p)}$ for $1 \le p \le n-4-r$ (Definition 6.1), so that $K^{(1)} = K_r$ and $K^{(n-4-r)} = 12$. For $1 \le p \le n-5-r$ let $k_{\mathrm{sl}}(p) := k_{\mathrm{sl}}(r,p)$, $\theta_{\mathrm{sl}}(p) := 1/(16\Gamma'(D+2)^{k_{\mathrm{sl}}(p)})$ and $\hat{M}_{\mathrm{sl}}(p) := 2n(n+K^{(p+1)})/\theta_{\mathrm{sl}}(p)$. Define $\hat{m}_{n-4-r} := \hat{M}_{\mathrm{core}}$ (Section 8) and, for $p = n-5-r,\dots,1$,*

$$\hat{m}_p := \max\big(\hat{M}_{\mathrm{sl}}(p),\, 2\hat{m}_{p+1}/\theta_{\mathrm{sl}}(p)\big), \qquad \text{and} \qquad \mathrm{ratio}_r := 2\hat{m}_1 + 2n(n+K_r).$$

These numbers depend only on n, D and Γ'. If HR_r holds with constant Γ', then $|\mathrm{Big}_b(\mathrm{ratio}_r\,\sigma)| < K_r$ for every $b \in W$.

Proof. Put $\tau := \mathrm{ratio}_r\,\sigma$, and suppose some $b \in W$ has K_r radii ρ_k ($k \in I$, $|I| = K_r$) in $\mathrm{Big}_b(\tau)$. Define $B := \{b\}$, $\beta(k) := (\rho_k)$ and

$$C_k := \big(W \cap \mathrm{S}(b, \rho_k)\big) \setminus \bigcup_{s \in S} \mathrm{S}\big(s, \{|bs'| : s' \in S\} \cup \{\rho_{k'} : k' \in I\}\big).$$

Then $|C_k| \geq \tau - r(r + K_r)\sigma \geq \tau/2 \geq \hat{m}_1\sigma$, because $\tau \geq 2n(n + K_r)\sigma$. We check that this is a rainbow family. (F1): $b \in W \subseteq X \setminus S$ and $S \cup \{b\}$ is rainbow. (F2) is clear. (F3): the ρ_k are distinct and do not lie in $R_b = \mathcal{D}(S \cup \{b\})$, by the definition of Big_b. (F4): for $x \in C_k$, $x \notin S \cup \{b\}$ and $S \cup \{x\}$ is rainbow (as $x \in W$). Moreover $|xb| = \rho_k \notin R_b$; $|xs| \neq |bs'|$ and $|xs| \neq \rho_k = |xb|$ by the removal; so $S \cup \{b, x\}$ is rainbow by Lemma 6.5. (F5) holds by the removal.

If $r = n-5$ then $p = 1 = n-4-r$, $K_r = 12$ and $\hat{m}_1 = \hat{M}_{\mathrm{core}}$, which contradicts Theorem 8.2. Otherwise, for $p = 1, \dots, n-5-r$ in turn, we have a family with p centres, $K^{(p)} = 6K^{(p+1)}$ classes and $M \geq \hat{m}_p\sigma \geq \hat{M}_{\mathrm{sl}}(p)\sigma$. Lemma 9.1, with $K := K^{(p+1)}$ and $k_{\mathrm{sl}} = k_{\mathrm{sl}}(r, p) \leq k^*$, gives a family with $p+1$ centres, $K^{(p+1)}$ classes and $M \geq \theta_{\mathrm{sl}}(p)\hat{m}_p\sigma/2 \geq \hat{m}_{p+1}\sigma$. At $p = n-4-r$ we obtain 12 classes and $M \geq \hat{M}_{\mathrm{core}}\sigma = M_{\mathrm{core}}$, which contradicts Theorem 8.2. □

Remark 9.3 (exceptional choices in a slicing round). For the reader's orientation, these are the five ways in which a slicing round can go wrong, and how each is handled. (i) $c \in Z_{\mathrm{bad}}$: at most $2e$ points, which are avoided. (ii) Small classes of points of S through c or with radius $\lambda_{k'}$: removed, at a cost of at most $r(r + K)\sigma$ points per class. (iii) Label conflicts $\lambda_k \in \beta(k')$: handled by the independent set, at a loss of a factor 3. (iv) Radii in $\Phi(c) \cup \beta(k)$: these are never edges, and they enter the cover as constants or column functions. (v) Bisectors $\mathrm{H}(c, s)$: these are row types. Only the covering form of the cleaning lemma is used, never a density-to-structure statement.

10 The threshold tower and the proof of the Main Theorem

10.1 The tower

Let $n \geq 4$ and $D \geq 1$. The constants are defined in the following order.

(1) The type counts and class counts of Definition 6.1; these depend only on n.

(2) The absolute parameters $L^*, \mu, e, e_A, \sigma_{\min}$ of Definition 6.2; these depend only on n, D and $h = s_n(\lfloor D/2 \rfloor)$.

(3) Q and $\Gamma_{n-2}, \Gamma_{n-3}, \Gamma_{n-4}$, by Theorem 7.1 and (3); these depend only on n and D.

(4) For $n \geq 5$ and $r = n-5, n-6, \dots, 0$, in this order:

$$\begin{aligned} \Gamma'_r &:= \Gamma_{r+1} + r^2, \qquad \mathrm{ratio}_r := \text{the number of Theorem 9.2 with } \Gamma' = \Gamma'_r, \\ \Gamma_r &:= P(K_r - 1, r)\big(2\,\mathrm{ratio}_r\,\Gamma_{r+1} + 2r^2 + 1\big), \end{aligned} \tag{8}$$

with $P(K, r)$ as in Lemma 6.8. Here ratio_r depends only on n, D and Γ'_r, and Γ_r only on n, D and Γ_{r+1}.

No step refers to a later one. In particular $\sigma_{\min}$ involves no Γ, and no Γ involves $\sigma_{\min}$ or h (referee item X5). The quantities Q and $L_{k'}(D+2) \leq L^*$ enter only through Γ_{n-3}, Γ_{n-4} and $\sigma_{\min}$ respectively.

Theorem 10.1. *For every $0 \leq j \leq n-2$, Claim(j) holds with the constant Γ_j defined above.*

Proof. For $j \geq n-4$ this is Theorem 7.1. Let $n \geq 5$ and $0 \leq r \leq n-5$, and assume Claim$(r+1)$ with constant Γ_{r+1}. Let (S, W, σ) be a level-r configuration with $\sigma \geq \sigma_{\min}$. By Lemma 6.6, HR_r holds for (S, W, σ) with constant Γ'_r. By Theorem 9.2, $|\mathrm{Big}_b(\mathrm{ratio}_r\sigma)| \leq K_r - 1$ for every $b \in W$. Lemma 6.8, with $K = K_r - 1$ and $\tau = \mathrm{ratio}_r\sigma \geq \sigma$, together with $1 \leq \sigma$ (Remark 6.9), gives $|W| \leq P(K_r - 1, r)(2\mathrm{ratio}_r\Gamma_{r+1} + 2r^2 + 1)\sigma = \Gamma_r\sigma$. Downward induction on r completes the proof. □

The induction is legitimate because each Claim(j) is a statement about all configurations and all caps $\sigma \geq \sigma_{\min}$, with a constant independent of σ. The proofs apply Claim$(j+1)$ with caps $\theta \geq \sigma$ (Lemma 6.6, and through it Lemma 8.10 and Lemma 9.1(a)), $Q\sigma$ (Theorem 7.1) and $\mathrm{ratio}_r\sigma$ (Lemma 6.8). Each of these caps is at least $\sigma_{\min}$.

10.2 Size of the constants

Lemma 10.2 (growth)**.** *Put $a := D + 2$. There is c_n, depending only on n, with $\Gamma_0 \leq a^{c_n}$ for all $D \geq 1$. More precisely, $g_r := \log_a \Gamma_r$ satisfies $g_r \leq (n-2-r)\, g_{r+1} + b_n$ for $0 \leq r \leq n-5$, where b_n depends only on n. Hence $g_0 \leq \frac{(n-2)!}{2}\,(g_{n-4} + b_n)$, and $g_{n-4} = O(n^2)$.*

Proof. Call a quantity *n-polynomial* if it is at most a^c with c depending only on n. Since $a \geq 3$, every number depending only on n is n-polynomial, and sums, products and maxima of boundedly many (in terms of n) n-polynomial quantities are n-polynomial. Q is n-polynomial, and so are $P_j^+ \leq a^{2e_j}$ and $P(K, r) \leq a^{2e_{K,r}}$ (Lemma 3.7). Hence $\Gamma_{n-2}, \Gamma_{n-3}, \Gamma_{n-4}$ are n-polynomial. Since $e_j = O(n^2)$, $k' = O(n^2)$ and $\log_a(2^{44}n^4) = O(\log n)$, two applications of (3) give $g_{n-4} = O(n^2)$.

For the recursion, fix $r \leq n-5$ and write $\Gamma' = \Gamma'_r$. First, $\Gamma_{r+1} \geq r^2$: this is clear from (3) and (8). Hence $\Gamma' \leq 2\Gamma_{r+1}$. Next, $2/\theta_{\mathrm{sl}}(p) = 32\Gamma' a^{k_{\mathrm{sl}}(p)}$ and $\hat{M}_{\mathrm{sl}}(p) = 32\Gamma' a^{k_{\mathrm{sl}}(p)} n(n + K^{(p+1)})$. Using $\max(x, y) \leq xy$ for $x, y \geq 1$, we get $\hat{m}_p \leq 32\Gamma' a^{k_{\mathrm{sl}}(p)} n(n + K^{(p+1)})\, \hat{m}_{p+1}$, and hence

$$\hat{m}_1 \leq \Big[\prod_{p=1}^{n-5-r} 32\Gamma' a^{k_{\mathrm{sl}}(p)} n(n + K^{(p+1)})\Big]\, 2^{20} 4^n n^2 \Gamma'^2 a^{2k_c(r)} \leq A_n\, a^{B_n}\, \Gamma'^{\,n-3-r},$$

with A_n, B_n depending only on n. Also $\hat{m}_1 \geq 2n(n + K_r)$, so $\mathrm{ratio}_r \leq 3\hat{m}_1$. Finally $2r^2 + 1 \leq \mathrm{ratio}_r\Gamma_{r+1}$, so (8) gives $\Gamma_r \leq 3P(K_r - 1, r)\,\mathrm{ratio}_r\,\Gamma_{r+1} \leq 9a^{2e_{K_r-1,r}} A_n a^{B_n} 2^n \Gamma_{r+1}^{n-2-r}$. Taking $\log_a$ proves the recursion, and iterating it from $r = n-5$ down to $r = 0$ gives the bound on g_0, since $\prod_{r=0}^{n-5}(n-2-r) = (n-2)!/2$. □

By Definition 6.1, $k^* \leq 6^{n-4} + 2n^2$. So $L^* = L_{k^*}(D+2) = O(6^{3n}\sqrt{D}\log D)$ and

$$\sigma_{\min} = 2^{L^*+2+2n}(h+1) = 2^{O(6^{3n}\sqrt{D}\log D)}\,(h+1).$$

We have made no attempt to optimise the constants: the exponent c_n grows like $(n-2)!$ and the constant in L^* like 6^{3n}. Only the shape $2^{O_n(\sqrt{d}\log d)}$ is claimed.

10.3 Proof of the Main Theorem

Theorem 10.3 (Main Theorem)**.** *For every $n \geq 3$ there is C_n such that $f_d(n) \leq 2^{C_n\sqrt{d}\log_2(d+2)}$ for all $d \geq 1$.*

Proof. For $n = 3$ this follows from Blokhuis's theorem: rainbow-3-free sets are exactly the isosceles sets, including degenerate triples, and these have at most $\binom{d+2}{2}$ points [2]. Hence $f_d(3) \leq \binom{d+2}{2} + 1$.

Let $n \geq 4$ and $D \geq 1$, and let X be a finite rainbow-n-free set of affine dimension at most D. We may assume $X \subset \mathbb{R}^D$. The triple $(\emptyset, X, \sigma_{\min})$ is a level-0 configuration with $\sigma_{\min} \geq \sigma_{\min}$, so Theorem 10.1 (Claim(0)) gives

$$|X| \leq \Gamma_0 \, \sigma_{\min} = \Gamma_0 \, 2^{L^*+2+2n} \, (h+1).$$

By induction on D, $h = s_n(\lfloor D/2 \rfloor)$ is finite (with $s_n(0) = 1$). So $s_n(D)$ is finite and $s_n(D) \leq A(D)\,(s_n(\lfloor D/2 \rfloor) + 1)$ with $A(D) := \Gamma_0 \, 2^{L^*+2+2n}$. By Lemma 10.2 and the definition of L_k, $\log_2 A(D) \leq c_n \log_2(D+2) + 2^{14} k^{*3} \sqrt{D+4} \log_2(D+4) + 2 + 2n$. For $D \geq 1$ we have $\sqrt{D+4} \leq \sqrt{5}\sqrt{D}$, $\log_2(D+4) \leq 2\log_2(D+2)$ and $\sqrt{D}\log_2(D+2) \geq 1$, so $A(D) \leq 2^{C\sqrt{D}\log_2(D+2)}$ for a constant $C \geq 1$ depending only on n. Lemma 3.12 gives $s_n(d) \leq 2^{6C\sqrt{d}\log_2(d+2)}$, and $f_d(n) = s_n(d) + 1 \leq 2^{(6C+1)\sqrt{d}\log_2(d+2)}$. □

Remark 10.4. The proof uses Theorem 4.1 in five places: in G1 (through μ); in Step A of Theorem 7.1; in Steps 3–5 of the Core Lemma; and in Lemma 9.1(a). If Theorem 4.1 were replaced by the trivial bound $2^{-O(r)}$ for single-type rectangles, the same argument would give only $s_n(D) \leq 2^{O(D)} s_n(\lfloor D/2 \rfloor)$, that is, $f_d(n) = 2^{O_n(d)}$, the classical order of magnitude.

11 Concluding remarks

11.1 What the argument rests on

Apart from classical results (John, Sheppard, Caro–Wei, König, Blokhuis), every step is proved in this manuscript. The load-bearing new ingredient is Theorem 4.1. The geometric induction (Sections 6–10) uses it only as a black box, and all other inputs are elementary. In decreasing order of what we regard as the risk of an error, the parts that most deserve an expert's attention are the following.

1. *Theorem LR*, in particular the normal form (Lemma 4.5) and the rescaling at the largest gap by a row function, a column function or a constant. Its constants-only and column-only special cases are close to known arguments [10, 12]. The mixed row/column case rests on Lemmas 4.3, 4.4 and 4.5.

2. *Step A* of Theorem 7.1 (Lemma 7.4). This is the only place where a density statement (the cross-pair count (4)) is converted into a covering statement, at the cost of a fourth root (Lemma 7.3). An earlier version contained a genuine threshold gap here (fix F1).

3. The bookkeeping of the Core and Slicing Lemmas, in particular the completeness of the coincidence lists in Lemmas 8.5, 8.7 and 8.13, and the verification of (F1)–(F5) in Lemma 9.1(d).

If Theorem 4.1 failed, the argument would give only the classical bound $2^{O(d)}$.

11.2 Computational sanity checks

Several finite, threshold-free parts of the argument were tested by computer, in the project and by the referees. We list these tests for completeness. They are *sanity checks only*. Every threshold in the argument ($\sigma_{\min} \geq 2^{L^*}(h+1)$, the cleaning densities, the sides required in the orthogonal splits) is astronomically larger than any set that can be handled numerically, so no threshold-dependent conclusion was, or could be, tested. Only inclusions and implications were tested.

- *Rounding and cleaning* (referee pass on Theorem LR). John positions were computed numerically (as minimum-volume enclosing ellipsoids) on k-listable instances, giving $\max |u_x|/s^{1/4} \leq 1.0002$ (the solver tolerance) and $\max |v_w|/(Ks^{1/4}) \leq 1$. Sheppard's formula was compared with Monte Carlo estimates and agreed to within 10^{-3}. The cleaning lemma was tested on adversarial sparse low-rank matrices, and a zero block was always found.

- *Normal form* (same referee pass). An independent implementation of Lemmas A, B, NF and one round of the LR induction was run on 1600 adversarial block-structured instances: accumulating labels $\kappa \pm 2^{-j}$ with j up to 40, ties between row and column functions on blocks, magnitudes $2^{\pm 40}$, and up to 5 types of mixed kinds. None of (NF1)–(NF3), the bounds on $|N|$, the identification of the minority set with the support of H, or the rank bound for H was ever violated.

- *Genericity in Step A* (referee pass on Theorem R^+). In 70,213 random grid configurations ($n = 4, 5, 6$), 20,361 of them non-generic, no non-generic point fell outside the listed loci.

- *The case $n = 5$* (referee pass on the $n = 5$ argument). About 150,000 exact-arithmetic checks of the threshold-free inclusions of Steps 0–3 and 5 were run on rainbow-5-free families (Johnson-type schemes, products, and greedy grid sets), with no violation.

- *General n, author's checks.* 348,884 exact-integer checks for $n = 6, 7$ and all relevant (r, p) were run, covering the non-generic loci of Step 0, the inclusions of Step 1, the characterisation in Lemma 8.13, the equivalence behind Lemma 6.6, and the family conditions in Lemma 9.1(d). There was no violation. These checks sampled *arbitrary* integer points, not points of rainbow-n-free sets (referee item X6). So they do not test the one implication that uses rainbow-n-freeness, namely the label lemma behind G1.

- *General n, independent checks* (referee pass on the general-n argument). About 340,000 exact checks were run on genuinely rainbow-n-free sets for $n = 6, 7$. The sets were orthogonal unions of integer points on small spheres, arranged so that cross distances collide, together with few-distance lattices. The checks included the full pipeline of Lemmas 6.6 and 6.8, the initial family of Theorem 9.2, slicing rounds with actual maximum matchings and König covers, and Steps 0–2 and 5 of the Core Lemma. They also included the label lemma (Lemma 5.1) on label-disjoint generic pairs. There was no violation. On these sets families die after one or two slicing rounds, and at most 7 (for $n = 6$) or 12 (for $n = 7$) classes were available. So the contradiction in the Core Lemma itself was never exercised; only its steps were.

11.3 Literature

In the referee pass on Theorem LR, the following works were searched for prior statements of Theorem 4.1 or Corollary 4.6: [7], which proves bounds for nonnegative integer matrices with bounded average and remarks that binarity is not crucial; Sudakov–Tomon [15]; Rothvoss [12]; Lovett [10]; and works citing [14]. No statement of a $2^{-\tilde{O}(\sqrt{d})}$ bound for k-listable real matrices was found. The search was not exhaustive. As stressed in Remark 1.2, the possible improvement over the bound stated in [14] for parallel partitions should be confirmed by experts. Corollary 4.6 follows from Rothvoss's argument in a few lines, and it may be regarded as folklore.

11.4 Further remarks

- The class counts $K_r = 12 \cdot 6^{n-5-r}$, the type counts and all constants are far from optimal. For $n = 5$ a referee observed that the count in Step 5 already works with 7 classes instead of 12, at the cost of larger constants.

- The super-polynomial factor 2^{L^*} in the thresholds cannot be removed from the class-count statement $\mathrm{E1}_r$ alone. If Y is rainbow-$(n-1)$-free and o is any point, then $Y \cup \{o\}$ is rainbow-n-free. A referee observed that, for $n = 5$, taking Y to be a Johnson-type 5-distance set and o a suitable weight vector gives a point with many more than 12 classes, each of size polynomial in D. So E1 with a threshold polynomial in D would be false. With the thresholds used here, E1 is consistent with every example known to us.

- No super-polynomial lower bound for $f_d(n)$ is known to us, so the gap between $d^{\Omega_n(1)}$ and $2^{O_n(\sqrt{d}\log d)}$ remains wide.

12 Formal verification in Lean

The complete argument of this paper has been formalized in Lean 4 (toolchain `v4.33.1`) on Mathlib revision `0df444a3`, the same pins as the Google DeepMind *formal-conjectures* repository at commit `c77fdd446d23539302aa73bcfae4a9fa962b8382`. The formalization is about 9,800 lines.

The target statement. The theorem proved is `Erdos1088.erdos_1088`:

$$\forall n \geq 3, \qquad \big(d \mapsto \log_2 f(d,n)\big) = o(d) \quad (d \to \infty),$$

where `f d n` is defined as $\inf\{m \geq n$: every m-point subset of $\mathbb{R}^d$ contains an n-point subset with $\binom{n}{2}$ distinct distances$\}$. The statement and all definitions it uses (`f`, `cardSet`, `HasSubsetWithDistinctDistances`, `PairwiseDistancesDistinct`, `distinctDistances`, `distanceSet` and the notation $\mathbb{R}^d$) are copied character for character from formal-conjectures. A script checks this against the upstream files. Only module headers and metadata attributes are dropped.

Lean's `sInf` of an empty set of naturals is 0, so a proof could in principle exploit an empty `cardSet`. The formal proof does not do this: it exhibits an explicit element of `cardSet d n` and proves $f(d,n) \leq n \cdot 2^{C_n\sqrt{d}\log_2(d+2)}$, which is the Main Theorem.

Trust base. `#print axioms Erdos1088.erdos_1088` reports only `propext`, `Classical.choice` and `Quot.sound`, the standard axioms of Lean and Mathlib. The development contains no `sorry`, no `axiom` declarations and no `native_decide`.

Differences from the informal proof. The formalization follows Sections 4–10 with the following changes, none of which affects the Main Theorem.

- The normalization step (NF3) of Lemma 4.5 is replaced by a tournament argument. As a result, the formal version of Theorem 4.1 has loss $2^{O(k)}\sqrt{r}\log r$ instead of $O(k^3\sqrt{r}\log r)$; for fixed k the shape $2^{-O_k(\sqrt{r}\log r)}$ is the same, and this is all the later sections use.
- John's theorem in Lemma 4.2 is replaced by a factorization derived from the Kiefer–Wolfowitz optimal-design theorem, proved from the matrix determinant lemma and the spectral theorem. Sheppard's formula is proved from the rotation invariance of the standard Gaussian.
- Several constants are larger. These include those in the Core Lemma and in Theorem 7.1, the number of big radii at which Step A applies (3 instead of 2), and k^*. The case $n = 3$ is derived from G1 rather than from Blokhuis's theorem.
- Lean made explicit some hypotheses that the text leaves implicit, for example $\sigma > 0$ and $\Gamma \geq 0$ in the kill step, and $B \subseteq X$ in the Core Lemma. It found no gap in the argument.

Availability. The Lean source, build instructions and audit scripts are available at https://github.com/H0W4RD-dev/erdos-1088-lean.

13 AI assistance and verification; credits

AI assistance and verification

The mathematics in this manuscript was developed by AI agents (Claude, Anthropic) in research sessions directed by the team. The agents proposed the strategy, proved the lemmas, and wrote the project notes from which this manuscript was assembled. The manuscript itself was also prepared by an AI agent.

The arguments were checked by *independent AI referee passes*: separate agent sessions that had not written the material under review, re-derived each step by hand, and ran their own computer tests. In summary:

- one pass on Theorem 4.1 (with Lemma 4.2, the cleaning lemma, Lemmas A, B and NF), on Corollary 4.6 and its relation to [14], and on an earlier proof of the case $n = 4$. Verdict: correct, with editorial remarks only;
- one pass on the kill step, the two-class count and Theorem R^+. Verdict: correct after fixes F1–F3, of which F1 repaired a genuine gap;
- one pass on the case $n = 5$ of the class-count argument. Verdict: correct with expository gaps; items A–G of that report are addressed in Sections 6–8;
- one pass on the general-n argument (configurations, HR, kill step, Core Lemma, Slicing Lemma, threshold tower). Verdict: correct with expository gaps; items X1–X6 of that report are addressed in Sections 3, 6–10 and 11.

Each referee found the material under review correct, conditional on its inputs. All of them identified Theorem 4.1, which is new and unpublished, as the weakest link.

No human expert has reviewed any part of this argument. AI referee passes can miss errors, and several passes sharing similar blind spots is a real possibility. The Lean formalization of Section 12 was also written by AI agents. It removes the dependence on the referees for the *formal* statement: Lean's kernel checks every step. Two things remain for human judgement: whether the formal statement faithfully captures Erdős's question, and the sharper form of Theorem 4.1 stated here, whose NF3 step is not formalized.

We also note what has *not* been refereed in the present form. While consolidating the project notes into this manuscript we made some routine changes, and these have not been checked by any referee pass. They are: the explicit choices $Q = 2^{44}n^4(D+2)^{8k'}$, $\sigma_{\min} = 2^{L^*+2+2n}(h+1)$ and $e_A = 2^{L^*+2}(h+1)$; the explicit constant 2^{14} in $L_k(r)$ and the explicit cost bounds in Lemma 4.5; the column-basis proof of the cleaning lemma; the self-contained statement and proof of Lemma 7.3; the direct (rather than translated) single-cap proof of Theorem 7.1; and the explicit growth recursion in Lemma 10.2. We believe these changes are correct and harmless, but they are part of what an expert should check.

Authors and credits

The authors are Will Howe (H0W4RD-dev), Grant Huffman (Ghuffy1), Jensen Kohlmeyer (jenw1n) and Liam Kruer (lkruer). Will Howe directed the AI research sessions that produced the proof and the Lean formalization. We acknowledge the prior in-project work of a peer session [11], on which Section 1 and Lemma 3.8 build.

Contact

Will Howe, `howew@purdue.edu`; Grant Huffman, `huffma97@purdue.edu`; Jensen Kohlmeyer, `jenwin@purdue.edu`; Liam Kruer, `kruerl@purdue.edu`.

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