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INFINITELY MANY DECREASES IN THE THRESHOLDS FOR SUMS OF DISTINCT POWERS

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Abstract. Let T_k be the least integer such that every integer at least T_k is a sum of distinct positive kth powers. We prove that $T_k > T_{k+1}$ for infinitely many positive integers k. The principal upper estimate holds on a set of degrees meeting every sufficiently large dyadic interval:

$$\log T_k \le (1+o(1))\, k \exp\left(\frac{2}{5}\frac{\log k}{\log\log k}\right).$$

We obtain it from flat integral recurrences, cyclotomic coefficient estimates, an exact reduction of polynomial degree, and a subset-sum construction. A shifted-prime divisor theorem of Fan and Pollack supplies larger lower bounds along an unbounded sequence of degrees. The two estimates contradict eventual monotonicity of the thresholds.

1. The problem and the two estimates

For $A \subseteq \mathbb{Z}_{\ge 0}$, write

$$\Sigma(A) = \left\{ \sum_{a\in B} a : B \subseteq A,\ B \text{ finite} \right\}.$$

The empty sum is allowed. Let $\mathcal{P}_k = \{n^k : n \in \mathbb{N},\ n \ge 1\}$, and put

$$T_k = \min\{T \in \mathbb{Z}_{\ge 0} : [T,\infty) \cap \mathbb{Z} \subseteq \Sigma(\mathcal{P}_k)\}.$$

The finiteness of T_k for every positive k is classical; see Kim [4]. We have $T_1 = 0$. If $k \ge 2$, the integer two is not representable, so $T_k \ge 3$. For $k \ge 2$, using the largest exceptional integer instead subtracts one from each threshold, and therefore preserves adjacent comparisons. The question considered here is Erdős problem 345; see Erdős and Graham [2, p. 55].

Theorem 1.1. *There are infinitely many positive integers k for which $T_k > T_{k+1}$.*

All logarithms are natural, and $\mathcal{L}(x) = \log x/\log\log x$ for $x > \mathrm{e}$. An unspecified positive absolute constant may be enlarged when it reappears; dependence on a fixed parameter is indicated by a subscript. All asymptotic estimates below are uniform in the degrees and frequencies to which they are applied.

Proposition 1.2. *There is a set $\mathcal{G} \subseteq \mathbb{N}$ meeting every sufficiently large interval $[2^m, 2^{m+1}]$ such that*

$$\log T_k \le (1+o(1))\, k \exp\left(\frac{2}{5}\mathcal{L}(k)\right) \qquad (k \in \mathcal{G}). \tag{1.1}$$

This is proved in Sections 2–7. For comparison, define

$$\omega^*(n) = \#\{p : p \text{ prime},\ p-1 \mid n\}, \qquad D(n) = \prod_{\substack{p \text{ prime}\\ p-1\mid n}} p.$$

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Fan and Pollack [3, Theorem 1] prove unconditionally that

$$\omega^*(n) > \exp((0.6736 \log 2)\mathcal{L}(n)) \tag{1.2}$$

for infinitely many n. Only this infinitely-often conclusion is used.

Lemma 1.3. *There are arbitrarily large j such that*

$$\log T_j \geq j \exp\left(\frac{9}{20}\mathcal{L}(j)\right). \tag{1.3}$$

Proof. Choose n satisfying (1.2), and put $j = n + 1$ and $Q = D(n)$. For every integer b, $b^j \equiv b \pmod{Q}$. Modulo a prime factor p of Q, this follows from $p - 1 \mid n$ if $p \nmid b$; if $p \mid b$, both sides vanish. The factors of Q are distinct.

Set $H = \lfloor \sqrt{Q} \rfloor$ and $S = H(H + 1)/2 < Q$. Subset sums of powers with bases at most H have residues in $\{0, \ldots, S\}$. Choose $M \equiv S + 1 \pmod{Q}$ with

$$(H + 1)^j - Q \leq M < (H + 1)^j.$$

For large Q and $j \geq 3$, this is positive. It cannot be represented: a base larger than H contributes too much, while the smaller bases cannot produce its residue. Hence

$$\log T_j \geq \log M \geq \frac{j}{2} \log Q - O(j).$$

Since $\log Q \geq (\log 2)\omega^*(n)$, $0.6736 \log 2 > 9/20$, and $\mathcal{L}(n + 1) - \mathcal{L}(n) \to 0$, the fixed factors and the error are absorbed by the strict difference of constants. This proves (1.3). □

2. Flat products and integral recurrences

For $P(z) = \sum_h w_h z^h$, put $\|P\|_1 = \sum_h |w_h|$. Let E denote the shift operator on sequences: $(Ex)_n = x_{n+1}$, and write $\Delta = E - 1$. A rational polynomial is called integer-valued if its values at all integers are integers.

We use the following consequence of Belov and Konyagin [1, Corollary 0.2]. For every positive integer J, there are nonincreasing nonnegative integers $a_1, \ldots, a_J$, of sum J, and a real a_0, such that

$$a_0 + \sum_{h=1}^{J} a_h \cos(h\theta) \geq 0 \quad (\theta \in \mathbb{R}), \qquad a_0 \leq 2^{32}(1 + \log J)^5. \tag{2.1}$$

Lemma 2.1. *For every $J \geq 1$, one can choose*

$$P_J(z) = \prod_{h=1}^{J} (1 - z^h)^{a_h} = (1 - z)^J Q_J(z) \in \mathbb{Z}[z]$$

so that

$$\deg P_J \leq J^2, \qquad \log \|P_J\|_1 \leq C(1 + \log J)^6, \tag{2.2}$$

$$Q_J \in \mathbb{Z}_{\geq 0}[z], \qquad M_J := Q_J(1) = \prod_h h^{a_h} \mid J!, \tag{2.3}$$

$$\sum_{\substack{h \leq J \\ v \mid h}} a_h \leq J/v \qquad (v \geq 1). \tag{2.4}$$

The constant and leading coefficients of P_J have absolute value one.

Proof. Take the coefficients from (2.1). For $0 < \rho < 1$,

$$\log |P_J(\rho e^{i\theta})| = -\sum_{m \geq 1} \frac{\rho^m}{m} \sum_h a_h \cos(hm\theta) \leq a_0 \log \frac{1}{1 - \rho}.$$

Also

$$|1-\mathrm{e}^{\mathrm{i}t}| \le \frac{2}{1+\rho}|1-\rho\mathrm{e}^{\mathrm{i}t}|.$$

Taking $\rho = 1-J^{-1}$, for $J \ge 2$, gives $\log\|P_J\|_\infty \le C(1+\log J)^6$. By Parseval, $\|P_J\|_1 \le \sqrt{\deg P_J+1}\|P_J\|_\infty$, proving (2.2); $J=1$ is immediate.

The product

$$Q_J(z) = \prod_h (1+z+\cdots+z^{h-1})^{a_h}$$

has nonnegative integral coefficients. Let $\lambda_i = \#\{h : a_h \ge i\}$, $1 \le i \le a_1$. Then $\sum_i \lambda_i = J$ and $M_J = \prod_i \lambda_i!$, which divides $J!$ by the multinomial formula. Finally, the block of v indices ending at iv has coefficient sum at least va_{iv}. Adding the disjoint blocks proves (2.4). The remaining assertions follow from the product. □

Lemma 2.2 (Rounding to an exact recurrence)**.** *Let k, X be positive integers and put $K = 1+\log k$. Suppose $1 \le d \le k$, $F \in \mathbb{R}[x]$ has degree at most d, and*

$$\sum_{0\le n<X} \sin^2(\pi F(n)) \le k^2.$$

If $\log X$ grows faster than every fixed power of K, there is an integer sequence m_n, defined on all of $\mathbb{Z}$, with $P_{d+1}(E)m = 0$, and an interval I of L consecutive integers such that

$$\sum_{n\in I} |F(n)-m_n|^2 \le k^2/4, \qquad L \ge X\exp(-CK^6). \tag{2.5}$$

Proof. On the prefix choose m_n nearest to $F(n)$, and put $\epsilon_n = F(n)-m_n$. Since $\sin(\pi t) \ge 2t$ on $0 \le t \le 1/2$, $\sum \epsilon_n^2 \le k^2/4$. The root of P_{d+1} at one has order $d+1$, so $P_{d+1}(E)F = 0$. The residuals $P_{d+1}(E)m$ are integers, and the finite convolution inequality gives

$$\sum_n |P_{d+1}(E)m_n|^2 \le \|P_{d+1}\|_1^2 \sum_n \epsilon_n^2 \le \|P_{d+1}\|_1^2 k^2/4.$$

Write $D_0 = \deg P_{d+1}$. The valid residual positions are $0,\dots,X-D_0-1$. If B_0 is the number of nonzero residuals, then $B_0 \le \|P_{d+1}\|_1^2 k^2/4$. The zero residuals split into at most B_0+1 runs, so one run has length at least

$$\frac{X-D_0-B_0}{B_0+1}.$$

Here $D_0 \le (d+1)^2$ and $B_0+1 \le \exp(CK^6)$. The hypothesis on X therefore gives a run of r zero residuals, beginning at an integer s, with $r \ge X\exp(-CK^6)$, after enlarging C.

Use the D_0 rounded values at $s,\dots,s+D_0-1$ to define an exact recurrence sequence. The leading coefficient of P_{d+1} is a unit, so the r zero residual equations force agreement with the rounded values throughout $s,\dots,s+r+D_0-1$. Its constant coefficient is also a unit, so the extension to all of $\mathbb{Z}$ is integral in both directions. This whole interval lies in the original prefix. Restricting the error estimate to it proves the assertion. □

3. Cyclotomic heights on a long interval

Write

$$P(z) = \prod_{i=1}^{J}(1-z^{h_i}), \qquad 1 \le h_i \le H, \qquad D = \deg P.$$

Every sequence annihilated by $P(E)$ is a sum $\sum_\zeta p_\zeta(n)\zeta^{-n}$, where the roots of unity have orders at most H, and the degree of each polynomial is less than the multiplicity of its root.

Lemma 3.1 (Long-interval separation)**.** *For such a sum x_n, let a_ζ be the leading coefficient of a nonzero component of degree b. On an interval of L consecutive integers,*

$$\sum_{n\in I} |x_n|^2 \geq \frac{L^{2b+1}|a_\zeta|^2}{2(2b+1)\binom{2b}{b}^2}, \tag{3.1}$$

provided $L \geq CD(J+1)^3H^2$.

Proof. Translate the interval to start at zero. Let ℓ_j be the shifted Legendre polynomials orthonormal on $[0,1]$, with leading coefficient $\sqrt{2j+1}\binom{2j}{j}$. Writing $\ell_j(x) = \sqrt{2j+1}\,\mathsf{P}_j(2x-1)$ in terms of the usual Legendre polynomial, one has

$$\|\ell_j\|_\infty \leq \sqrt{2j+1}, \qquad \|\ell_j'\|_\infty \leq j(j+1)\sqrt{2j+1}.$$

For example, the derivative bound follows by summing $\mathsf{P}_j' = \sum_{0\leq r<j,\ j-r\ \text{odd}}(2r+1)\mathsf{P}_r$ and using $|\mathsf{P}_r| \leq 1$ on $[-1,1]$. Products of two basis polynomials of degrees less than J therefore have supremum and total variation $O((J+1)^3)$.

Use the basis $\ell_j(n/L)\zeta^{-n}$, with the appropriate degrees at each root. Its dimension is at most D. At a common frequency, the normalized discrete inner product differs from the continuous one by $O((J+1)^3/L)$. Distinct root frequencies are separated modulo one by at least H^{-2}. Summation by parts and the geometric-series bound therefore give off-diagonal entries $O((J+1)^3H^2/L)$. The row-sum estimate shows that the normalized Gram matrix has all eigenvalues at least $1/2$ under the stated length condition. The coefficient of the top basis vector in the chosen component is

$$\frac{a_\zeta L^b}{\sqrt{2b+1}\binom{2b}{b}}.$$

Its contribution to the Gram lower bound gives (3.1). Translation multiplies the top coefficient by a number of absolute value one and does not change this estimate. □

Lemma 3.2 (Integer recurrence height)**.** *If a nonzero integral sequence x is annihilated by $P(E)$, and b is its largest component degree, then*

$$(2b+1)\log L \leq \log\left(\sum_{n\in I}|x_n|^2\right) + CJ\log(2H(J+1)) \tag{3.2}$$

whenever L satisfies Lemma 3.1.

Proof. Translate the interval to start at zero. The recurrence first gives a generating-function denominator equal to the reciprocal of P. Since $z^DP(1/z) = (-1)^JP(z)$, a change of sign in the numerator writes the generating function as A/P, with $A \in \mathbb{Z}[z]$ of degree less than D. Cancel the monic common cyclotomic factors. Gauss's lemma gives A_0/C with relatively prime integral polynomials. Choose a primitive root ζ at which C has multiplicity $b+1$. The absolute value of the top component coefficient is

$$\frac{|A_0(\zeta)|}{|C_{\mathrm{reg}}(\zeta)|\,b!}, \qquad C_{\mathrm{reg}}(z) = C(z)/(z-\zeta)^{b+1}.$$

The numerator is a nonzero algebraic integer, and its geometric mean over primitive conjugates is at least one.

If $P = GC$, then $P_{\mathrm{reg}} = G_{\mathrm{reg}}C_{\mathrm{reg}}$. Regularized values of integral polynomials are algebraic integers: successive division by the monic factor $z-\zeta$ preserves algebraic integrality. Thus the geometric mean of $|G_{\mathrm{reg}}(\zeta)|$ is at least one. Each vanishing binomial in P contributes a derivative of absolute value at most H, and each nonvanishing one contributes at most two. The geometric mean of $|C_{\mathrm{reg}}(\zeta)|$ is consequently at most $(2H)^J$. Some conjugate top coefficient has absolute value at least $((2H)^JJ!)^{-1}$. Insert it into (3.1), use $\binom{2b}{b} \leq 4^b$, and take logarithms. □

Lemma 3.3 (Refined cyclotomic height)**.** *Let $K = 1 + \log k$, $Z = \log X$, with Z growing faster than every fixed power of K. Suppose $2 \le q \mid k!(k+1)!$, $\gcd(u, q) = 1$, and $F \in \mathbb{Q}[x]$ has degree at most d, $1 \le d \le k$, with*

$$qF(n) \in \mathbb{Z}, \qquad qF(n) \equiv un^k \pmod q \quad (n \in \mathbb{Z}).$$

If its sine energy on $0 \le n < X$ is at most k^2, there is a nonzero rational sequence ϵ, annihilated by $P_{d+1}(E)$, such that $q\epsilon$ is integral and congruent to $un^k \pmod q$ globally. If h_t denotes its multiplicity at primitive tth roots, then

$$h_t \le \frac{CdK}{tZ} \quad (t \ge 2), \tag{3.3}$$

$$h_1 \le \frac{C(\log q + dK)}{Z}. \tag{3.4}$$

In particular a multiplicity vanishes if its upper bound is less than one.

Proof. Apply Lemma 2.2 and put $\epsilon = F - m$. This gives all the integrality, congruence, and energy assertions, with energy at most $k^2/4$ on an interval of length $L \ge X \exp(-CK^6)$. The sequence is nonzero by its congruence at $n = 1$. Its least annihilator is an integral product of cyclotomic factors dividing P_{d+1}, by cancellation in the generating function of $q\epsilon$.

Put $J = d + 1$. At a nontrivial root the components of ϵ are the negatives of those of the integer sequence m, since F contributes only at one. Fix primitive order $t \ge 2$ with $h_t = b + 1 > 0$. We refine the denominator estimate in the preceding proof. There are at most J/t vanishing factors, by (2.4), and their derivatives have absolute value at most J.

For a nonvanishing factor $1 - \zeta^h$, put $l = t/\gcd(t, h) > 1$. Its geometric mean over primitive tth roots is $\Phi_l(1)^{1/\varphi(l)}$. This is one unless $l = p^a$ is a prime power, when its logarithm is $\log p/\varphi(p^a)$. Such factors have $t/p^a \mid h$; by (2.4) their number with multiplicity is at most Jp^a/t. The total logarithmic contribution of nonvanishing factors is thus at most

$$\sum_{p^a \mid t} \frac{Jp^a}{t} \frac{\log p}{\varphi(p^a)} \le \frac{2J}{t} \sum_{p^a \mid t} \log p = \frac{2J}{t} \log t.$$

Cancellation decreases the geometric-mean denominator bound, as above, and $b! \le J^{J/t}$. Some conjugate top coefficient therefore satisfies

$$|a_\zeta| \ge \exp(-C(J/t)\log(2J)).$$

The interval-length condition in Lemma 3.1 is polynomial in k, and hence holds here. It follows that

$$(2h_t - 1)\log L \le C(J/t)K + C\log k.$$

If $h_t > 0$, then $t \le J$, so $J/t \ge 1$. Now $2h_t - 1 \ge h_t$, $J \le 2d$, and $\log L \sim Z$. This proves (3.3) without an additive constant.

Apply Lemma 3.2 to $q\epsilon$ for the remaining bound. Its energy is at most $q^2k^2/4$, and its largest component degree is at least $h_1 - 1$. This gives (3.4); in particular the term involving $\log q$ is not discarded. □

4. Exact arithmetic of polynomial reductions

For $0 \le d < k$, define

$$A_j(k) = \Delta^j(x^k)\big|_{x=0} = j!\,S(k, j), \qquad G_d(k) = \gcd\{A_j(k) : d < j \le k\},$$

where $S(k, j)$ is a Stirling number of the second kind. In particular $G_0(k) = 1$, since $A_1(k) = 1$. The Newton expansion shows that

$$q \mid G_d(k) \iff \begin{array}{c}\text{there exists an integer-valued } P \in \mathbb{Q}[x],\ \deg P \le d,\\ n^k \equiv P(n) \pmod q \quad \text{for every } n \in \mathbb{Z}.\end{array} \tag{4.1}$$

Indeed, the forward differences above degree d must vanish modulo q; conversely, the truncation $\sum_{j=0}^{d} A_j(k)\binom{x}{j}$ supplies P. This formulation applies on all integers, including negative ones.

Lemma 4.1 (A product bound for the reduction modulus)**.** *For $1 \le d < k$, put*

$$F_{k,j} = \gcd_{a \in \mathbb{Z}_{\ge 0}} (a^k - a^j) \quad (1 \le j \le d).$$

Then

$$G_d(k) \mid \prod_{j=1}^{d} F_{k,j}, \qquad F_{k,j} \mid 2(k-j)D(k-j). \tag{4.2}$$

Proof. Start with $q_d = G_d(k)$, and recursively put $q_{j-1} = \gcd(q_j, A_j(k))$. Then $q_j \mid G_j(k)$, and the canonical truncation

$$P_j(x) = \sum_{\ell=0}^{j} A_\ell(k)\binom{x}{\ell}$$

is congruent to $x^k \pmod{q_j}$ on the integers. For every nonnegative integer a, the polynomial $P_j(ax) - a^k P_j(x)$ is integer-valued and divisible by q_j at every integer. Its jth forward difference is $A_j(k)(a^j - a^k)$. Thus

$$q_j/q_{j-1} \mid F_{k,j}.$$

Since $q_0 = 1$, multiplication gives the first assertion.

Write $h = k - j > 0$. A prime p divides $F_{k,j}$ exactly when $p - 1 \mid h$. Necessity follows by testing units modulo p; sufficiency also includes nonunits, because $j \ge 1$. Testing $a = p$ bounds $v_p(F_{k,j})$ by j. For odd p, testing $a = 1 + p$ bounds it by $1 + v_p(h)$. For $p = 2$, testing $a = 3$ bounds it by $2 + v_2(h)$. These lifting identities show that $F_{k,j} \mid 2hD(h)$. Taking logarithms gives

$$\log G_d(k) \le d \log(2k) + \sum_{j=1}^{d} \log D(k-j). \tag{4.3}$$

□

We next control a prime-power modulus from the cyclotomic multiplicities of an annihilator. Here v_p denotes the p-adic valuation.

Lemma 4.2 (Dilation of an integral recurrence)**.** *Let p be prime, $e \ge 1$, and*

$$B(z) = (z-1)^{m_0} \prod_{a=1}^{A} \Phi_{p^a}(z)^{m_a}, \qquad 0 \le m_a \le k \quad (0 \le a \le A).$$

If $B(E)n^k \equiv 0 \pmod{p^e}$ for all $n \in \mathbb{Z}$, then

$$e \le C\left\{ \frac{m_0}{p-1} + \sum_{a=1}^{A} (a+1)m_a + (A+1)(1 + \log_p(k+1)) \right\}. \tag{4.4}$$

Proof. Put $D_B = \deg B$. The rational B-recurrence sequences on $\mathbb{Z}$ have an integral lattice identified with $\mathbb{Z}^{D_B}$ by their first D_B entries, since the leading and constant coefficients of B are units. For a positive integer g prime to p, the dilation $S_g(x)_n = x_{gn}$ preserves this space and its integral lattice: on the complex basis $n^j\zeta^n$, it permutes primitive roots of a fixed p-power order and multiplies the degree-j vector by g^j.

Extend $0^k, 1^k, \ldots, (D_B-1)^k$ by the recurrence to an integral sequence y. The assumed congruence makes $y_n \equiv n^k \pmod{p^e}$ for all integers n. The cases $D_B = 0, 1$ are impossible, because the values at zero and one are respectively zero and one. Thus the initial vector contains a unit, and

$$S_g y \equiv g^k y \pmod{p^e}.$$

If χ_g is the characteristic polynomial of this integral matrix, Cayley–Hamilton gives

$$p^e \mid \chi_g(g^k). \tag{4.5}$$

For odd p, choose g primitive modulo p^2, hence modulo every p^a. The primitive p^ath roots form one cycle of length $\varphi(p^a)$. Their degree-j characteristic factor is $T^{\varphi(p^a)} - g^{j\varphi(p^a)}$, for $0 \le j < m_a$. At $T = g^k$, its valuation is

$$a + v_p(k-j).$$

The root-one factor $T - g^j$ has valuation zero unless $p-1 \mid k-j$; in that case its valuation is $1 + v_p(k-j)$. None of these factors vanishes, since $j < m_a \le k$.

For any interval of $m \le k$ consecutive positive integers ending at k, counting multiples of p^ν gives

$$\sum_{j=0}^{m-1} v_p(k-j) \le \frac{m}{p-1} + \log_p(k+1). \tag{4.6}$$

For the root-one contribution, restrict to the single residue class $k - j \equiv 0 \pmod{p-1}$, and divide its terms by $p-1$. The resulting consecutive integers number at most $m_0/(p-1)+1$. Applying (4.6) gives a contribution at most $Cm_0/(p-1) + C(1+\log_p(k+1))$. The other cycles contribute at most $\sum_a (a+1)m_a + A\log_p(k+1)$. Equation (4.5) proves the claim for odd p.

For $p = 2$, take $g = 5$. There are at most two cycles at each primitive 2^ath order, of length 2^{a-2} for $a \ge 3$; the orders two and four are handled directly. The valuation of each factor is at most $C(a+1+v_2(k-j))$. The root-one contribution is $2 + v_2(k-j)$. Summing with (4.6) proves the same bound. □

Lemma 4.3 (One contraction step)**.** *Under the hypotheses of Lemma 3.3, if the sine energy is at most k^2, there is an integer $M \ge 1$ satisfying*

$$M \le \frac{CK^2(\log q + dK)}{Z} + CK^4 \tag{4.7}$$

such that, whenever $M < k$, one has $q \mid G_M(k)$. All constants are uniform for $1 \le d \le k$.

Proof. Choose

$$T = C_0 \frac{\log q + dK}{Z}$$

large enough that Lemma 3.3 gives $h_1 \le T$ and $h_{p^a} \le T/p^a$. These multiplicities are less than k for large k, since $\log q = O(kK)$ and Z dominates powers of K.

Fix $p^e \| q$. Factor the integral annihilator of ϵ as BC, where B contains precisely its root-one and p-power cyclotomic factors. Then $C(1)$ is a unit modulo p: $\Phi_t(1)$ is one unless t is a prime power, and the prime-power factors in C correspond to primes other than p. Put $c_0 = C(1)$ and write $C(1+\Delta) = c_0 + \Delta R(\Delta)$. On polynomial sequences of degree at most k modulo p^e, $\Delta^{k+1} = 0$. An explicit inverse is therefore

$$c_0^{-1} \sum_{j=0}^{k} (-c_0^{-1}\Delta R(\Delta))^j.$$

Because $B(E)C(E)(q\epsilon) = 0$ and $q\epsilon \equiv un^k \pmod{p^e}$, the unit u may be cancelled. Apply the displayed inverse, which commutes with $B(E)$, to obtain

$$B(E)n^k \equiv 0 \pmod{p^e}.$$

Apply Lemma 4.2. The largest a is at most $\log_p(d+1)$, and $\sum_{a\ge1}(a+1)p^{-a} = O((p-1)^{-1})$. Hence

$$e \le CT/(p-1) + CK^2. \tag{4.8}$$

If $p > T$, all nontrivial p-power roots are absent. Then $B = (z-1)^{h_1}$; necessarily $h_1 \ge 1$, and

$$p^e \mid G_{h_1-1}(k).$$

Set $M = \lceil C_1(T+K^2)K^2 \rceil$, with C_1 fixed and sufficiently large. For $p \le T$, the elementary bound $v_p(M!) \ge M/(2p)$, valid here since $M \ge 2p$, dominates (4.8). Thus the part of q supported on primes at most T divides $M!$. Now assume $M < k$, as in the conclusion. The remaining prime powers divide $G_M(k)$, because $M \ge h_1$ and $G_{h_1-1}(k) \mid G_M(k)$. Each $A_j(k) = j!S(k,j)$, $j > M$, is divisible by $M!$. Thus the small-prime part also divides $G_M(k)$. The two parts are coprime, so $q \mid G_M(k)$. The chosen value of M satisfies (4.7). □

5. Good degrees and the exact-center estimate

Fix throughout

$$c = \frac{2}{5}, \qquad a = \frac{1}{10}, \qquad \beta = \frac{1}{100}. \tag{5.1}$$

Lemma 5.1 (Simultaneous degree selection)**.** *For every sufficiently large dyadic Y, all but $o(Y)$ integers $k \in [Y, 2Y]$ satisfy*

$$\log D(k-j) \le \exp(a\mathcal{L}(Y)) \quad \textit{for } 1 \le j \le \lceil (2Y)^{\beta} \rceil. \tag{5.2}$$

Proof. Let τ be the divisor function and d_z the ordered z-fold divisor function. For every integer $m \ge 1$,

$$\tau(n)^m \le d_{2^m}(n).$$

For a prime power exponent b, this follows by induction from

$$\frac{\binom{b+2z-1}{b}}{\binom{b+z-1}{b}} = \prod_{i=0}^{b-1} \frac{2z+i}{z+i} \ge b+1 \qquad (z \ge 1);$$

each factor increases with z, and the product at $z=1$ is $b+1$. Multiplicativity completes the proof. Counting ordered factorizations and summing the first $z-1$ factors harmonically gives, with no constant depending on z,

$$\sum_{n \le U} d_z(n) \le U(1+\log U)^{z-1}. \tag{5.3}$$

Since a prime predecessor divisor of n is at most $n+1$, $\log D(n) \le \tau(n)\log(n+1)$.

Take

$$m = \left\lfloor \frac{\log\log Y - 3\log\log\log Y}{\log 2} \right\rfloor, \qquad B_Y = \frac{\exp(a\mathcal{L}(Y))}{\log(2Y+1)}.$$

By (5.3) and Markov's inequality, the number of $n \le 2Y$ with $\log D(n) > \exp(a\mathcal{L}(Y))$ is at most

$$2Y(1+\log(2Y))^{2^m-1} B_Y^{-m} = Y^{1-a/\log 2+o(1)}.$$

Indeed $2^m \le \log Y/(\log\log Y)^3$, so the positive logarithmic error is $o(\log Y)$, while $m\log B_Y = (a/\log 2 + o(1))\log Y$. The union over at most $O(Y^{\beta})$ shifts has size $Y^{1-a/\log 2+\beta+o(1)} = o(Y)$, because $\beta < a/\log 2$. All shifted arguments are positive for large Y. □

Let $\mathcal{G}$ be the union of the sets selected in Lemma 5.1, and for $k \in \mathcal{G}$ put

$$K = 1+\log k, \qquad N = \left\lceil \exp(\exp(c\mathcal{L}(k))) \right\rceil, \qquad X = \lfloor N\exp(-BK^6) \rfloor, \tag{5.4}$$

where $B > 0$ is any fixed constant. Then

$$Z := \log X \sim \log N \sim \exp(c\mathcal{L}(k)), \qquad K^A = o(Z) \ (A \text{ fixed}), \qquad \log N = o(k). \tag{5.5}$$

In particular N and X exceed every fixed power of k.

We will also use the following fixed-parameter consequence of Shiu's short-interval estimate [7], in the form stated by Nair and Tenenbaum [6, Introduction]:

$$\sum_{j=1}^{d} \tau(k-j) \le C_\beta d\log k \quad (k^\beta \le d \le k/3). \tag{5.6}$$

For clarity, its parameters are fixed: apply the theorem to the multiplicative function $\tau = \tau_2$, modulus one, and interval $(k-d-1, k-1]$, with exponent $\beta/2$. This interval has length d and lies at height comparable to k. The hypotheses $\tau(p^\nu) = \nu + 1 \le 2^\nu$ and $\tau(n) \ll_\varepsilon n^\varepsilon$ hold. The resulting bound is $O_\beta(d(\log k)^{-1} \exp(\sum_{p \le k} 2/p)) = O_\beta(d \log k)$. Neither the function nor the interval exponent varies with k.

Proposition 5.2 (Exact-center energy)**.** *For all sufficiently large $k \in \mathcal{G}$, and every reduced fraction u/q with $2 \le q \mid k!(k+1)!$,*

$$\sum_{0 \le n < X} \sin^2(\pi u n^k/q) > k^2. \tag{5.7}$$

Here X may be chosen as in (5.4) for any fixed $B > 0$.

Proof. Suppose the energy is at most k^2. Begin with $F(x) = ux^k/q$, of degree $d = k$. Lemma 4.3 and $\log q = O(kK)$ give a new degree

$$M \le CkK^3/Z + CK^4 = o(k).$$

In particular $M < k/3$, and $q \mid G_M(k)$. By (4.1), replace F by

$$\frac{u}{q} \sum_{j=0}^{M} A_j(k) \binom{x}{j}.$$

Its difference from un^k/q is integral at every integer, so the sine energy has not changed. The modulus q and prefix X also remain unchanged. Every subsequent invocation of the rounding lemma starts again on this same prefix, so its interval-length loss is not accumulated across steps.

Whenever $k^\beta \le d \le k/3$ and $q \mid G_d(k)$, (4.3), $\log D(v) \le \tau(v) \log(v+1)$, and (5.6) give

$$\log q \le C_\beta d K^2.$$

A further contraction therefore gives

$$M \le C_\beta d K^4/Z + CK^4 < d/2$$

for large k, uniformly in this range. After finitely many steps the degree is at most k^β.

Choose the dyadic Y for which (5.2) holds for k. Put $E_Y = \exp(a\mathcal{L}(Y))$. For every $1 \le d \le k^\beta$, the same bound (5.2) gives

$$\log q \le d \log(2k) + dE_Y.$$

The contraction estimate now reads

$$M \le \rho_k d + C_0 K^4, \qquad \rho_k = \frac{C_2 K^2 (K + E_Y)}{Z} \longrightarrow 0. \tag{5.8}$$

The constants are fixed. Since $Y \le k \le 2Y$, $\mathcal{L}(Y) = \mathcal{L}(k) + O(1/\log\log k)$. Thus $E_Y = \exp(a\mathcal{L}(k) + o(1))$, whereas $Z \sim \exp(c\mathcal{L}(k))$; the inequality $a < c$ gives $\rho_k \to 0$, uniformly over the selected degrees. Take k large enough that $\rho_k < 1/4$. While $d > 2C_0K^4$, this gives $M < 3d/4 < k$, so the same exact replacement remains valid. Eventually $q \mid G_d(k)$ with $1 \le d \le 2C_0K^4$. A reduction to degree zero would already give $q \mid G_0(k) = 1$, which is impossible. At the terminal positive degree,

$$\log q \le CK^4(K + E_Y) = o(Z). \tag{5.9}$$

Thus $q = X^{o(1)}$.

Return to the original phase, not to an individual cyclotomic component. Each full block of q consecutive inputs contains an $n \equiv 1 \pmod q$. Since $\gcd(u, q) = 1$,

$$\sin^2(\pi u/q) \ge 4\|u/q\|_{\mathbb{R}/\mathbb{Z}}^2 \ge 4/q^2.$$

Consequently the original energy is at least

$$\frac{4\lfloor X/q\rfloor}{q^2} \geq \frac{2X}{q^3} = X^{1-o(1)} > k^2,$$

by (5.5) and (5.9). This is the desired contradiction. □

6. From rational centers to collision estimates

Retain $k \in \mathcal{G}$ and N from (5.4). Put

$$S = 512\left\lfloor \frac{\lfloor N/k\rfloor}{512}\right\rfloor, \quad \mathcal{V} = \{N-S+1,\dots,N\}, \quad R = \{1,\dots,N-S\}\setminus\{2,\dots,257\}.$$

Thus $\mathcal{V}$ is the reserved set of bases and R is the available initial population. Write

$$M_R = |R|, \qquad t = \lfloor N/8192\rfloor, \qquad \ell = 128.$$

For $b \in \mathcal{V}$, let $q_b = b^k$, and define

$$D_b(\alpha) = \sum_{n\in R} \sin^2(\pi\alpha n^k),$$
$$F_b(h) = \frac{1}{M_R}\sum_{n\in R}\cos^2(\pi h n^k/q_b) = 1 - \frac{D_b(h/q_b)}{M_R}.$$

We will prove

$$\sup_{b\in\mathcal{V}} \sum_{h=1}^{q_b-1} F_b(h)^t = o(1). \tag{6.1}$$

Notice that $M_R \sim N$ and $t/M_R \geq 1/16384$ for large k. Also $S \leq N/k$, so $b/N \geq 1-1/k$ for $b \in \mathcal{V}$. Since $(1-1/k)^k \to \mathrm{e}^{-1} > 1/4$, we have $b^k \geq N^k/4$ uniformly over the reserved bases.

Lemma 6.1 (A common rational approximation)**.** *Let $A > 0$ be fixed. There are a constant C and an integer $B_k \mid k!(k+1)!$, independent of α, such that*

$$D_b(\alpha) \leq Ak\log N \quad\Longrightarrow\quad \left|\alpha - \frac{u}{B_k}\right| \leq N^{-k}\exp(CkK^6)$$

for some integer u. The assertion is uniform in $b \in \mathcal{V}$.

Proof. The population is independent of b. Add back the at most 256 excluded terms and use the prefix $0 \leq n < N-S$; the term at zero vanishes. Its energy E_0 is at most $Ak\log N + 256$.

Take $P = P_{k+1} = (1-z)^{k+1}Q$ from Lemma 2.1, and $M = Q(1) \mid (k+1)!$. Round $f(n) = \alpha n^k$ to integers m_n. As in Lemma 2.2, the number of nonzero integer residuals $P(E)m_n$ is at most $\|P\|_1^2 E_0/4$. We have $\log(1+E_0) = O(K)$, including when $E_0 = 0$. The zero-run count in Lemma 2.2 therefore provides $L \geq N\exp(-C_1K^6)$ consecutive valid residual positions, say $s,\dots,s+L-1$.

Put $D_0 = \deg P$. As $\deg Q = D_0 - k - 1$, the integers $b_n = Q(E)m_n$ are defined for every $s \leq n \leq s+L+k$. On the L residual positions, $\Delta^{k+1}b_n = 0$. Successive finite differences consequently show that all these values agree with one rational polynomial of degree at most k. Its Newton coefficients about the integer s are integral, so its coefficient of n^k is $u/k!$ for some $u \in \mathbb{Z}$. The polynomial $Q(E)f$ has coefficient $M\alpha$ at n^k. Since Q has nonnegative coefficients of sum M, the finite convolution inequality bounds the squared error of their difference, on the first L positions of this interval, by $M^2E_0/4$.

Take $H = \lfloor L/(2k)\rfloor \geq L/(4k)$. The kth difference with step H of this error polynomial is the constant

$$k!H^kM\left(\alpha - \frac{u}{k!M}\right).$$

There are at least $L/2$ valid starting points. The difference operator has coefficient norm 2^k, so

$$\frac{L}{2}(k!H^kM)^2\left|\alpha-\frac{u}{k!M}\right|^2\le 4^kM^2E_0/4.$$

Using $k!\ge(k/\mathrm{e})^k$, we obtain

$$\left|\alpha-\frac{u}{k!M}\right|\le(8\mathrm{e})^kL^{-k}\sqrt{\frac{E_0}{2L}}\le N^{-k}\exp(CkK^6).$$

Take $B_k=k!M$. All filters and this denominator depend only on k. □

Lemma 6.2 (Small symmetric numerators)**.** *Put $K_*=10^{10}(1+\log N/\log k)$. Uniformly for $b\in\mathcal{V}$,*

$$\sum_{\substack{1\le h<q_b\\ \min(h,q_b-h)\le(b/(K_*k^2))^k}}F_b(h)^t=o(1).$$

Proof. By symmetry take $a=\min(h,q_b-h)$ and put $r=ba^{-1/k}$. Then $K_*k^2\le r\le b\le N$, and the sine phase is $(n/r)^k$.

If $r>N/2$, use the interval

$$[r(1/8)^{1/k},\,r(1/4)^{1/k}].$$

It has $\gg N/k$ integer points, all in R for large k: its upper endpoint is at most $N4^{-1/k}<N-S$, and its lower endpoint exceeds 257. On it the sine square is at least $1/8$. Thus $D_b(h/q_b)\gg N/k$.

If $r\le N/2$, set $Y_r=\lfloor r/(1000k)\rfloor$. For $1\le j\le Y_r$, use the disjoint intervals

$$I_j=[r(j+1/8)^{1/k},\,r(j+1/4)^{1/k}].$$

The mean value theorem gives

$$|I_j|\ge\frac{r}{16k(j+1)}>2.$$

Their integer points lie in R. Indeed their lower endpoints exceed 257, while their upper endpoints are at most $r(Y_r+1)^{1/k}\le 3N/4$ eventually, because $\log N=o(k)$. Each interval contains at least half its length in integers, with sine square at least $1/8$. Therefore

$$D_b(h/q_b)\ge\frac{r}{256k}\sum_{j=1}^{Y_r}\frac{1}{j+1}\ge\frac{r\log(Y_r+1)}{512k}\ge\frac{K_*k\log k}{1024}.$$

The first case satisfies the same final bound, since $N/k\gg K_*k\log k$.

Now $F_b(h)^t\le\exp(-D_b(h/q_b)/16384)$. There are fewer than N^k frequencies, and $K_*>2^{24}\log N/\log k$ by a fixed factor. Hence their total contribution is bounded by

$$N^k\exp\left(-\frac{K_*k\log k}{2^{24}}\right)=o(1).$$

□

Proposition 6.3. *The uniform collision estimate (6.1) holds.*

Proof. Choose a fixed A sufficiently large. Frequencies with $D_b(h/q_b)>Ak\log N$ have total contribution at most

$$N^k\exp(-Ak\log N/16384)=o(1).$$

Every remaining frequency is in an arc

$$\left|\frac{h}{q_b}-\frac{u}{B_k}\right|\le\delta,\qquad\delta=N^{-k}\exp(CkK^6),$$

by Lemma 6.1. We have $\log B_k=O(kK)$, and $\delta B_k=o(1)$. Thus at most B_k+2 centers are relevant in or adjacent to $[0,1]$. Each arc contains at most $2q_b\delta+2$ points of the actual grid. The total

number of grid points in these arcs is therefore at most $\exp(C_3 k K^6)$. Consequently those among them with $D_b \geq A_3 k K^6$ also contribute $o(1)$, for a fixed $A_3 > 2 \cdot 16384 C_3$.

It remains to consider $D_b < A_3 k K^6$. Choose the constant B in (5.4) larger than $C + 2$. On the prefix $n < X$, replacing the frequency by its center changes the square root of the sine energy by at most

$$\pi \delta X^k \sqrt{X} \leq \pi \exp(-(B - C) k K^6 + \tfrac{1}{2} \log X) = o(1).$$

If the center is not an integer, its reduced denominator is a divisor of B_k, and Proposition 5.2 gives center energy greater than k^2. Removing the at most 256 auxiliary terms leaves

$$D_b(h/q_b) \geq k^2/2 - 256$$

for large k, contradicting $D_b < A_3 k K^6$.

If the center is an integer, then $\min(h, q_b - h) \leq q_b \delta$. Since $\log N$ dominates K^6 and $\log(K_* k^2) = O(K)$, we have $\delta \leq (K_* k^2)^{-k}$ eventually. These frequencies lie in the range of Lemma 6.2, whose total contribution is $o(1)$. All cases have now been covered. □

7. A represented interval and the infinite tail

The last step converts (6.1) into ordinary subset sums, with no repetition of bases.

Lemma 7.1. *For all sufficiently large $k \in \mathcal{G}$, there are $\ell = 128$ pairwise disjoint sets of bases, all at most N, whose power subset-sum sets $A_1, \ldots, A_\ell$ satisfy*

$$|A_i| \geq s N^k/16, \qquad A_i \subseteq [0, 2 s N^k], \qquad s = S/512. \tag{7.1}$$

No A_i is contained in an arithmetic progression of integer difference greater than one.

Proof. Choose ℓ disjoint random t-subsets C_i of R, uniformly. This is possible since $\ell t < M_R$, and each marginal is a uniform t-subset. Independence between the C_i is not required.

For a fixed $C = C_i$, choose each subset $U \subseteq C$ with probability 2^{-t}; equivalently, include its bases independently with probability $1/2$. Let

$$\mu_b(x) = 2^{-t} \# \left\{ U \subseteq C : \sum_{n \in U} n^k \equiv x \pmod{q_b} \right\}.$$

This is a distribution on residues, not the uniform distribution on distinct represented values. Its Fourier transform is

$$\widehat{\mu}_b(h) = \prod_{n \in C} \frac{1 + \exp(2\pi \mathrm{i} h n^k/q_b)}{2}.$$

Finite Fourier orthogonality therefore gives the collision excess

$$q_b \sum_x \mu_b(x)^2 - 1 = \sum_{h=1}^{q_b - 1} \prod_{n \in C} \cos^2(\pi h n^k/q_b).$$

For $x_n = \cos^2(\pi h n^k/q_b)$, the expectation of the product over a uniform t-subset equals $e_t((x_n)_{n \in R})/\binom{M_R}{t}$, where e_t is the elementary symmetric polynomial of degree t. Maclaurin's inequality bounds this by $(M_R^{-1} \sum_{n \in R} x_n)^t = F_b(h)^t$.

Write $\eta_k = \sup_{b \in \mathcal{V}} \sum_{h=1}^{q_b - 1} F_b(h)^t = o(1)$. If the support has fewer than $q_b/4$ elements, Cauchy–Schwarz gives $q_b \sum_x \mu_b(x)^2 > 4$, so the collision excess is greater than three. For each fixed group, the expected number of such bad reserved bases is at most $S\eta_k/3$. Markov's inequality bounds the probability of more than $S/2$ bad bases by $2\eta_k/3$. A union bound over the fixed ℓ groups proves that, with probability $1 - o(1)$, every group has at least $S/2$ good bases.

We may simultaneously require

$$\sum_{n \in C_i} n^k \leq 4 t N^k/k \qquad (1 \leq i \leq \ell). \tag{7.2}$$

Indeed the expectation is at most $2tN^k/k$, because $\sum_{n\le N} n^k \le N^{k+1}/(k+1) + N^k$ and $M_R \sim N$. Sampling without replacement has variance at most tN^{2k}. Chebyshev's inequality makes the failure probability at most $k^2/(4t) = o(1)$, uniformly in the fixed number of groups. Fix a realization with both properties.

Allocate s good reserved bases to each group, with no base used twice. Before any individual choice, fewer than $\ell s = S/4$ bases have been used, whereas the group in question has at least $S/2$ good choices. The greedy allocation is therefore possible; the completed allocation uses exactly $S/4$ bases. If a finite integer set A occupies r residue classes modulo a positive integer q, then

$$|A + \{0, q\}| \ge |A| + r.$$

For each occupied residue, its largest element supplies a new element after addition of q. Applying this with $q = b^k$ shows that each allocated base increases the subset-sum cardinality by at least $q_b/4 \ge N^k/16$. The original support is retained after earlier additions, so the support condition continues to hold. After s additions the cardinality is at least $sN^k/16$.

Finally add the distinct auxiliary pairs

$$(2, 3), (4, 5), \dots, (256, 257)$$

to the 128 groups, one pair per group. They are disjoint from R and from the reserved bases. Each resulting subset-sum set contains zero and two coprime kth powers, so it cannot lie in a proper integer arithmetic progression. The initial span in (7.2) is at most $sN^k/3$ eventually, because $4t/(ks) \to 1/4 < 1/3$. The reserved bases add at most sN^k, and the two auxiliary powers add at most $2 \cdot 257^k = o(sN^k)$. This proves (7.1). □

We use Lev's consecutive-interval theorem in the following sufficient form [5, Corollary 1]. If $A_1, \dots, A_\ell$ are subsets of $[0, l] \cap \mathbb{Z}$, each of size at least $n \ge 3$, none contained in an integer arithmetic progression of difference greater than one, and

$$\ell \ge 2 \left\lceil \frac{l-1}{n-2} \right\rceil,$$

then their sumset contains $\ell(n-1)+1$ consecutive integers. Here the length of a block means its largest element minus its smallest, as in Lev's statement.

Proof of Proposition 1.2. Apply this result to Lemma 7.1, with

$$l = \lceil 2sN^k \rceil, \qquad n = \lfloor sN^k/16 \rfloor, \qquad \ell = 128.$$

The ratio $(l-1)/(n-2)$ tends to 32, so the strict inequality has a fixed margin. The sumset contains an interval $[a, a+L] \cap \mathbb{Z}$ with

$$L \ge 128(n-1) \sim 8sN^k > (N+1)^k.$$

Also $0 \le a \le 256sN^k \le 256N^{k+1}$. Every integer in this interval is a sum of distinct kth powers with bases at most N, because the groups of bases are disjoint.

Adjoin the unused powers with bases $N+1, N+2, \dots$. If the currently represented interval is $[a, a+L]$ and a new power b^k satisfies $b^k \le L+1$, its translate joins it without a gap, producing $[a, a+L+b^k]$. Initially $L \ge (N+1)^k$. If $L \ge b^k$, then the new length is at least $2b^k > (b+1)^k$, since $b \ge N$ and $N/k \to \infty$. This preserves the induction invariant. At each stage the interval consists of subset sums of the finitely many bases used up to that stage, all distinct. Its length tends to infinity. Every fixed integer at least a consequently belongs to one finite-stage interval, and is represented by finitely many distinct positive kth powers. We have proved

$$T_k \le 256N^{k+1}.$$

Taking logarithms and using (5.5) gives (1.1). □

Proof of Theorem 1.1. If there were only finitely many strict adjacent decreases, then $T_{k+1} \geq T_k$ for all sufficiently large k. Choose arbitrarily large j from Lemma 1.3. Let Y be a dyadic integer with $j \leq Y < 2j$, and choose $k \in \mathcal{G} \cap [Y, 2Y]$. Then $j \leq k < 4j$, and $\mathcal{L}(k) = \mathcal{L}(j) + o(1)$. Proposition 1.2 gives

$$\log T_k \leq (4 + o(1)) j \exp\left(\frac{2}{5}\mathcal{L}(j)\right),$$

whereas

$$\log T_j \geq j \exp\left(\frac{9}{20}\mathcal{L}(j)\right).$$

The latter is larger for all sufficiently large j, contradicting $T_j \leq T_k$. Thus strict adjacent decreases occur infinitely often. □

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