Centered Cocircular Central Configurations for Homogeneous and Logarithmic Potentials

Chih-Yao Wu
andersonwu2000@gmail.com

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Abstract

We classify central configurations of positive masses on a circle centered at their center of mass. For every number of bodies and every potential $r^{-\alpha}$ with $\alpha > 0$, the only such configurations are equal-mass regular polygons. The same conclusion holds for the logarithmic potential. The proof first separates the equality of the masses from the angular geometry. Comparisons obtained from the force equations then reduce mass equality to inequalities for positive kernels and finite probability laws. Different exponent ranges share these comparisons, while requiring different estimates to complete the argument.

The proof and its exposition were developed with AI agents. A companion Lean development is separate from the proof below; no formal theorem is used as a premise. This manuscript has not received an independent human review.

1 Introduction

Albouy, Cabral and Santos asked whether a central configuration on a circle centered at its center of mass must be an equal-mass regular polygon, throughout the attractive power-law range [1, Problem 12]. Their question asks for a symmetry conclusion from the force equations and the geometric constraint, without prescribing the masses. We prove this classification for the full range, including its logarithmic endpoint.

Let $m_1, \dots, m_n$ be positive masses at distinct positions $z_1, \dots, z_n \in \mathbb{R}^2$. We consider the attractive force equations

$$\sum_{j \neq i} m_j \frac{z_j - z_i}{|z_j - z_i|^{\alpha+2}} = -\lambda z_i, \qquad 1 \le i \le n. \tag{1.1}$$

For $\alpha > 0$, these are the central-configuration equations for the potential $\sum_{i<j} m_i m_j |z_i - z_j|^{-\alpha}/\alpha$. At $\alpha = 0$ the potential is $-\sum_{i<j} m_i m_j \log|z_i - z_j|$. The factor $1/\alpha$ fixes the force convention and has no effect on the configurations for positive α.

Our assumption is that the positions lie on a circle centered at their center of mass. After translation, it reads

$$|z_i| = R > 0 \quad (1 \le i \le n), \qquad \sum_i m_i z_i = 0. \tag{1.2}$$

The conclusion determines both the masses and the geometry.

Theorem 1.1. *Let $n \ge 2$ and $\alpha \ge 0$. If positive masses at distinct positions satisfy* (1.1) *and* (1.2)*, then all masses are equal and the positions are the vertices of a regular n-gon. Conversely, every equal-mass regular n-gon satisfies these equations for some $\lambda > 0$.*

We use the physical potential exponent α throughout. In the convention where the pair force has distance exponent $2a$, the corresponding parameter is $a = -1 - \alpha/2$. The logarithmic case is an endpoint of the force family, not the constant potential obtained by substituting zero in $r^{-\alpha}$.

Several parts of this picture were known. Cors, Hall and Roberts proved uniqueness for a prescribed cyclic mass order and obtained the equal-mass and logarithmic classifications [2]. Hampton proved local isolation of the regular polygon in the Newtonian problem [3]. Wang obtained a sufficient inequality in the exponent and the number of bodies; in particular it gives the Newtonian classification for $n \leq 6$ [4]. Tang and Zhu excluded several classes of unequal mass words for power-law potentials [5]. The argument here removes the restrictions on the number of bodies, the positive masses and the nonnegative potential exponent.

The first reduction separates the two parts of the conclusion. For a fixed cyclic ordering of the masses, the angular energy is strictly convex after rotations have been removed. Equal masses therefore imply a regular polygon. We include this short convexity proof in Section 2. The main task is to prove equality of the masses from the full force equations.

Section 2 also settles the logarithmic case and configurations with at most four bodies. For positive exponents below two we use consecutive force rows to constrain nearby masses. The last interval below two combines these bounds with the remainders of a trial angular comparison. For exponents at least two we bound the mass ratio from the force equations and compare two bounds for the common radial value. Between two and four the comparison uses a probability law on the masses, with the diagonal term removed explicitly. Above four a quartic estimate controls the trial energy for the full range of larger exponents. The table locates the proof in each range.

Range	Argument proving mass equality
$\alpha = 0$, or $2 \leq n \leq 4$	Section 2
$0 < \alpha \leq 1$	Section 4
$1 < \alpha \leq 9/5$	Section 7; Appendices D–H
$9/5 < \alpha < 2$	Sections 8–9; Appendices B–C
$2 \leq \alpha \leq 4$	Sections 10–12; Appendix A
$\alpha \geq 4$	Sections 13–14

Section 15 combines the ranges and states explicitly how the hypotheses of each comparison follow from the original equations. All decimal constants used in inequalities denote exact rational numbers.

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2 Angular convexity and elementary cases

Write $z_i = R(\cos\theta_i, \sin\theta_i)$ and $d_{ij} = |z_i - z_j|$. We label the points in their cyclic order. Taking the scalar product of (1.1) with z_i and using

$$z_i \cdot (z_j - z_i) = -\tfrac{1}{2} d_{ij}^2$$

gives the common radial value

$$C_i := \sum_{j \neq i} m_j d_{ij}^{-\alpha} = 2\lambda R^2 =: C > 0. \tag{2.1}$$

The tangential projection says that the angular potential is stationary. For $\alpha > 0$ we also use the unscaled energy

$$U_\alpha(m, \theta) = \sum_{i<j} m_i m_j d_{ij}^{-\alpha}, \qquad M = \sum_i m_i,$$

so that $2U_\alpha = MC$ and $\partial U_\alpha / \partial m_i = C$ at a configuration satisfying (2.1).

Lemma 2.1 (Uniqueness for a cyclic mass word)**.** *Fix positive masses in a cyclic order, a radius $R > 0$, and $\alpha \geq 0$. Up to a common rotation, there is at most one angular critical configuration with that order. If one exists, it is the unique minimum of the angular energy on the ordered chamber.*

Proof. Fix $\theta_1 = 0$ and work on the convex chamber $0 < \theta_2 < \cdots < \theta_n < 2\pi$. For $0 < t < 2\pi$, put

$$v_\alpha(t) = \begin{cases} [2R\sin(t/2)]^{-\alpha}/\alpha, & \alpha > 0, \\ -\log[2R\sin(t/2)], & \alpha = 0. \end{cases}$$

Differentiation gives, in both cases,

$$v_\alpha''(t) = \tfrac{1}{4}[2R\sin(t/2)]^{-\alpha}\{1 + (\alpha+1)\cot^2(t/2)\} > 0.$$

The second variation of $\sum_{i<j} m_i m_j v_\alpha(\theta_j - \theta_i)$ is

$$\sum_{i<j} m_i m_j v_\alpha''(\theta_j - \theta_i)(\eta_j - \eta_i)^2.$$

It vanishes exactly for a common rotation. It is therefore positive definite on the pinned chamber. Strict convexity gives the claim by restricting to the segment from a critical point to any other point. □

Corollary 2.2. *Under the hypotheses of Theorem 1.1, equality of the masses implies regularity. Equal-mass regular polygons satisfy the equations with $\lambda > 0$ for every $\alpha \geq 0$.*

Proof. In a regular polygon, the terms at forward and backward cyclic rank have opposite tangential components. An antipodal term, when present, has zero tangential component. Thus the regular polygon is angular critical. Lemma 2.1 proves uniqueness for equal masses. Its center of mass is the circle center, since the sum of the nth roots of unity is zero. The radial rows are identical and positive by cyclic symmetry, so their common value gives $\lambda = C/(2R^2) > 0$. □

Proposition 2.3. *Theorem 1.1 holds at $\alpha = 0$.*

Proof. Equation (2.1) becomes $M - m_i = C$. Hence the masses are equal, and Corollary 2.2 applies. □

Proposition 2.4. *Theorem 1.1 holds for $2 \leq n \leq 4$ and every $\alpha \geq 0$.*

Proof. For two bodies, $m_1 z_1 = -m_2 z_2$ and equal positive radii imply equal masses at antipodal points. For three bodies, the center-of-mass condition gives

$$\sum_{j \neq i} m_j (z_j - z_i) = -M z_i.$$

Subtracting λ/M times this identity from the force equation gives

$$\sum_{j \neq i} m_j (d_{ij}^{-\alpha-2} - \lambda/M)(z_j - z_i) = 0.$$

Three distinct points on a circle are not collinear, so the two chord vectors in this sum are independent. Every coefficient is zero. Since $\alpha + 2 > 0$, the three distances are equal. The only linear dependence between the three radius vectors of an equilateral triangle is their sum, so the center-of-mass equation forces equal masses.

It remains to consider four bodies with $\alpha > 0$. In their cyclic order, each opposite transposition $P = (1\ 3)$ or $(2\ 4)$ reverses that order. Reflection of the original critical configuration, followed by this relabeling and a rotation, gives an angular critical configuration for the mass word Pm, with the same energy as the original one. By Lemma 2.1, this is the minimum for Pm. In particular, at the original angles,

$$U_\alpha(Pm, \theta) \geq U_\alpha(m, \theta).$$

For a transposition of i and j, put $\delta = (m_j - m_i)(e_i - e_j)$. The energy is quadratic in the masses. Its linear change is $C \sum_k \delta_k = 0$, and its quadratic change is $\delta_i \delta_j d_{ij}^{-\alpha}$. Consequently

$$U_\alpha(Pm, \theta) - U_\alpha(m, \theta) = -(m_i - m_j)^2 d_{ij}^{-\alpha}.$$

Both opposite masses must agree; write the cyclic word as (A, B, A, B). A square is angular critical for this word, by cancellation of the two adjacent terms at each vertex. Angular uniqueness identifies the positions with that square. Comparing an A row and a B row now gives

$$(A - B)\{(2R)^{-\alpha} - 2(\sqrt{2}R)^{-\alpha}\} = 0.$$

The bracket is strictly negative. Thus $A = B$, completing the proof. □

3 Two comparisons for the common radial value

Assume $\alpha > 0$. The radial rows give a lower bound for the common value C. A trial configuration in consecutive cells of widths proportional to the masses gives an upper bound. Sections 10–12 compare these bounds after controlling the mass ratio. For $\alpha \geq 4$, Section 14 pairs the same lower bound with a finite-edge trial upper bound. The proofs below two begin with consecutive force rows and Section 9 also uses the remainders in the cell comparison.

Put $P = \alpha + 1$, $\nu = 1/P$, $w_i = m_i/M$, and $u_i = n w_i$. We distinguish the uniform and mass-weighted expectations

$$\mathbb{E}_U f = \frac{1}{n} \sum_i f_i, \qquad \mathbb{E}_w f = \sum_i w_i f_i = \mathbb{E}_U(uf).$$

In particular $\mathbb{E}_U u = 1$ and $\mathbb{E}_w(1/u) = 1$. The common scale factor is $Q_R = (2R)^{-\alpha}$.

3.1 A continuous extension of the regular-polygon row

Lemma 3.1. *The even density*

$$\mu_\alpha(\xi) = \frac{2^{\alpha-1}}{\pi\Gamma(\alpha)}|\Gamma(\alpha/2 + \mathrm{i}\xi)|^2$$

is a probability density, and

$$\int_{\mathbb{R}} e^{z\xi}\mu_\alpha(\xi)\,\mathrm{d}\xi = \cos(z/2)^{-\alpha} \qquad (|\Re z| < \pi), \tag{3.1}$$

with the analytic branch taking the value one at zero.

Proof. Let $s = \alpha/2$. The substitution $x = e^t$ in Euler's beta integral gives

$$\int_{\mathbb{R}} \frac{e^{st}}{(1+e^t)^{2s}} e^{\mathrm{i}\xi t}\,\mathrm{d}t = \frac{\Gamma(s+\mathrm{i}\xi)\Gamma(s-\mathrm{i}\xi)}{\Gamma(2s)}.$$

Fourier inversion, applied to the smooth exponentially decaying function on the left, gives

$$\int_{\mathbb{R}} e^{\mathrm{i}v\xi}|\Gamma(s+\mathrm{i}\xi)|^2\,\mathrm{d}\xi = \frac{2\pi\Gamma(2s)}{[2\cosh(v/2)]^{2s}} \qquad (v \in \mathbb{R}).$$

For completeness, the needed decay also follows directly from the beta integral. Choose an integer $k > s$. Taking absolute values in $B(s+\mathrm{i}\xi, k-s)$ yields

$$\frac{|\Gamma(s+\mathrm{i}\xi)|}{|\Gamma(k+\mathrm{i}\xi)|} \le \frac{\Gamma(s)}{\Gamma(k)}.$$

The recurrence and reflection formulas give

$$|\Gamma(k+\mathrm{i}\xi)|^2 = \frac{\pi\xi}{\sinh(\pi\xi)} \prod_{j=1}^{k-1}(j^2+\xi^2).$$

The quotient at zero is interpreted continuously. Thus μ_α is bounded by a polynomial times $e^{-\pi|\xi|}$. Its transform is holomorphic on $|\Re z| < \pi$, with all derivatives obtained under the integral. The displayed Fourier identity and analytic continuation prove (3.1). Setting $z = 0$ proves normalization. □

For real $N > 1$ define

$$S_\alpha(N) = \int_{\mathbb{R}} \frac{\sinh((N-1)\pi\xi/N)}{\sinh(\pi\xi/N)}\mu_\alpha(\xi)\,\mathrm{d}\xi. \tag{3.2}$$

This is positive. Its integrand at $\xi = 0$ has the value $N-1$. At infinity its growth is strictly smaller than $e^{\pi|\xi|}$, so Lemma 3.1 proves convergence. For integer $n \ge 2$ the quotient expands as $\sum_{k=1}^{n-1} e^{(1-2k/n)\pi\xi}$. Using $\cos((1-2k/n)\pi/2) = \sin(\pi k/n)$ in (3.1) gives

$$S_\alpha(n) = \sum_{k=1}^{n-1} \sin(\pi k/n)^{-\alpha}. \tag{3.3}$$

The antipodal rank, when present, appears once.

We will also need a representation that keeps the subtracted constant visible. Put

$$\nu_\alpha(\xi) = \frac{\sinh(\pi\xi)|\Gamma(\alpha/2+\mathrm{i}\xi)|^2}{\pi\Gamma(\alpha)}, \qquad B_\alpha(N) = 2^{\alpha+1}\int_0^\infty \frac{\nu_\alpha(\xi)}{e^{2\pi\xi/N}-1}\,\mathrm{d}\xi \quad (N > 0).$$

Then

$$S_\alpha(N) = B_\alpha(N) - B_\alpha(1). \tag{3.4}$$

Indeed, with $t = \pi\xi$,

$$2\sinh t\left(\frac{1}{e^{2t/N}-1} - \frac{1}{e^{2t}-1}\right) = \frac{\sinh(t - t/N)}{\sinh(t/N)},$$

and evenness gives the formula. The function ν_α is $O(\xi)$ at zero and has at most polynomial growth at infinity. These bounds, together with the exponential denominator, justify convergence and count differentiation locally on $N > 0$.

Define the two mass-dependent quantities

$$B_{\mathrm{root}} = Q_R S_\alpha(n)\frac{M}{n}(\mathbb{E}_U u^\nu)^P, \tag{3.5}$$

$$V_{\log} = Q_R \exp\{\mathbb{E}_w \log[m_i S_\alpha(1/w_i)]\}. \tag{3.6}$$

Our goal in this section is $B_{\mathrm{root}} \le C \le V_{\log}$.

3.2 The lower comparison

Lemma 3.2 (One chord support)**.** *For any finite probability law let $t, s \in (0, 2\pi)$, $A > 0$, $c = \mathbb{E}A$, $r = \sin(s/2)/\sin(t/2)$ and $b = (s - t)\cot(t/2)/2$. With $P = 1 + \alpha$ put $D(x, c) = x^P - c^P - Pc^{P-1}(x - c)$. Then*

$$\mathbb{E}(A^P r^{-\alpha}) - c^P + \alpha c^P \mathbb{E}b = \mathbb{E}\{rD(A/r, c) + \alpha c^P(1 + b - r)\} \ge 0.$$

Proof. Expand using $\mathbb{E}A = c$. The two remainders are nonnegative by the tangents to the convex power and the concave sine, respectively; the sine remainder vanishes only at $s = t$. □

Proposition 3.3. *Every configuration under consideration satisfies $B_{\mathrm{root}} \le C$.*

Proof. At forward rank k, take $t = 2\pi k/n$, $s = \theta_{i+k} - \theta_i$ in the cyclic lift, $A = m_i^\nu$ under the uniform index i. Each gap occurs k times, so $\mathbb{E}_U s = t$ and $\mathbb{E}_U b = 0$. Multiply the lemma by $Q_R \sin(t/2)^{-\alpha}$ and sum all nonzero ranks. Each term with mass m_i belongs to the radial row at $i + k$, so the left side before subtraction is the average of those rows. The lower bound is $Q_R S_\alpha(n)(\mathbb{E}_U m^\nu)^P = B_{\mathrm{root}}$ by (3.3). □

3.3 An exact cell identity

Partition the angular circle into consecutive cells of widths $2\pi x_i$, where $x_i > 0$ and $\sum_i x_i = 1$, and let ϕ_i be their midpoints. For $\xi \ge 0$ put

$$K_\xi(t) = \cosh(\xi(\pi - t)), \quad 0 \le t \le 2\pi, \qquad a_i = \frac{\sinh(\pi\xi x_i)}{\pi\xi}, \quad b_i = \frac{\sinh(\pi\xi(1 - x_i))}{\sinh(\pi\xi x_i)}.$$

At zero set $a_i = x_i$, $b_i = (1 - x_i)/x_i$. Write $K_{ij} = K_\xi(\phi_j - \phi_i)$ with angular differences reduced modulo 2π; this is symmetric.

Lemma 3.4. *For arbitrary positive trial masses v_i,*

$$\begin{aligned}\Delta_{\mathrm{cell}} &:= \sum_i v_i^2 S_\alpha(1/x_i) - \sum_{i\ne j} v_i v_j \sin((\phi_j - \phi_i)/2)^{-\alpha} \\ &= 2\int_0^\infty \sum_{i<j} K_{ij} a_i a_j \left(\frac{v_i}{a_i} - \frac{v_j}{a_j}\right)^2 \mu_\alpha(\xi)\,\mathrm{d}\xi \ge 0.\end{aligned} \tag{3.7}$$

Equality holds exactly when all cell widths and all trial masses are equal.

Proof. Fix $\xi > 0$. The integral of $K_\xi(t-\phi_i)$ over cell $j \neq i$ is $2\pi a_j K_{ij}$. The integral over the whole circle is $2\sinh(\pi\xi)/\xi$. Subtracting the own-cell integral leaves $2\sinh(\pi\xi(1-x_i))/\xi$. Thus

$$\sum_{j\neq i} K_{ij} a_j = a_i b_i.$$

Expansion of squares now gives

$$\sum_i b_i v_i^2 - \sum_{i\neq j} K_{ij} v_i v_j = \sum_{i<j} K_{ij} a_i a_j (v_i/a_i - v_j/a_j)^2.$$

The same identity holds at zero by continuity. Integrating against the even Gamma law, (3.2) identifies the diagonal terms, while evenness and (3.1) at $z = \pi - t$ give $2\int_0^\infty K_\xi(t)\mu_\alpha(\xi)\,\mathrm{d}\xi = \sin(t/2)^{-\alpha}$. All separations are strictly between zero and 2π, and all widths lie in $(0,1)$, so each term has a subcritical exponential rate. This justifies the finite integrations and proves (3.7).

If equality holds, positivity of the density and continuity force $v_i/a_i(\xi)$ to be independent of i for every $\xi \geq 0$. At zero this says that v_i/x_i is constant. The expansion

$$a_i(\xi) = x_i\left(1 + \frac{\pi^2\xi^2 x_i^2}{6} + O(\xi^4)\right)$$

then forces x_i^2 to be constant. The converse follows directly from the squares. □

3.4 The upper comparison

Proposition 3.5. *Every configuration under consideration satisfies $C \leq V_{\log}$.*

Proof. Choose the cells of Lemma 3.4 with $x_i = w_i$ in the original cyclic order. Put $f_i = Q_R S_\alpha(1/w_i)$ and choose the trial masses by

$$v_i^2 = \frac{M V_{\log} w_i}{f_i}, \qquad \ell_i = \log(v_i/m_i) = \tfrac{1}{2}\{\log V_{\log} - \log(m_i f_i)\}.$$

Their calibration gives $\mathbb{E}_w \ell = 0$ and $\sum_i v_i^2 f_i = M V_{\log}$.

On ordered distinct pairs the probability $\pi_{ij} = m_i m_j d_{ij}^{-\alpha}/(MC)$ has both marginals w by the radial rows. Apply Lemma 3.2 with t, s the forward lifted physical and trial separations, $\eta = \phi - \theta$ and $A = e^{\nu(\ell_i+\ell_j)}$. Differentiating the *unordered* energy gives $D_\theta U_\alpha[\eta] = -(\alpha MC/2)\mathbb{E}_\pi b = 0$: the angular difference and cotangent both reverse sign on reversing an edge. Thus

$$2U_\alpha(v,\phi)/(MC) = \mathbb{E}_\pi(A^P r^{-\alpha}) \geq (\mathbb{E}_\pi A)^P \geq 1,$$

since $\mathbb{E}_\pi \log A = 2\nu \mathbb{E}_w \ell = 0$. Finally, multiplying (3.7) by Q_R yields

$$M V_{\log} = 2U_\alpha(v,\phi) + Q_R \Delta_{\mathrm{cell}} \geq MC.$$

All force equations used here belong to the original configuration; the cell construction supplies a trial energy. □

Combining the comparisons gives a convenient necessary condition. Define

$$F_\alpha(u) = P \log \mathbb{E}_U u^\nu - \mathbb{E}_w \log\left(\frac{u_i S_\alpha(n/u_i)}{S_\alpha(n)}\right). \tag{3.8}$$

Cancellation of Q_R and M/n in (3.5)–(3.6) shows that $F_\alpha = \log(B_{\mathrm{root}}/V_{\log})$. Every physical configuration therefore satisfies $F_\alpha \leq 0$. For $2 \leq \alpha \leq 4$ the mass-ratio bound yields $F_\alpha > 0$ off uniformity in Section 12. For $\alpha \geq 4$, Section 14 instead compares B_{root} with the finite-edge upper U.

4 The interval $0 < \alpha \le 1$

This section proves rigidity in the first positive interval. All infinite sources used below are periodic repetitions of the given finite source. Their period is a geometric length; it will not be identified with the number of bodies before equality has been proved.

4.1 The positive image kernel

For $p > 0$, $L > 0$ and $s > 0$, define

$$w_{p,L}(s) = \frac{2^{p-1}\pi^{p-2}L^{1-p}}{\Gamma(p)} \sinh(Ls/2) \left|\Gamma\left(\frac{p}{2} + \frac{Ls}{2\pi}\mathrm{i}\right)\right|^2, \tag{4.1}$$

$$k(t) = \int_0^\infty w_{p,L}(s)e^{-st}\,\mathrm{d}s, \qquad U(t) = \int_0^\infty s w_{p,L}(s)e^{-st}\,\mathrm{d}s \quad (t > 0). \tag{4.2}$$

The parameters of k and U will be suppressed when fixed. The density is strictly positive. Stirling's formula on vertical lines gives $w_{p,L}(s) = O(s^{p-1})$ at infinity, while $w_{p,L}(s) = O(s)$ at zero. Consequently the integrals and all their derivatives converge locally uniformly for $t > 0$, and $U = -k'$.

Lemma 4.1 (Exact periodization)**.** *For $0 < t < L$,*

$$\sum_{j\in\mathbb{Z}} k(|t + jL|) = \left(\frac{\pi}{L\sin(\pi t/L)}\right)^p =: K_{p,L}(t). \tag{4.3}$$

For all $p > 0$ the self-image sum

$$S = 2\sum_{j=1}^{\infty} k(jL), \tag{4.4}$$

satisfies $0 < S < \infty$. For $0 < p < 2$ one has

$$\lim_{t\downarrow 0}\{t^{-p} - k(t)\} = S. \tag{4.5}$$

At $p = 1$, $w_{1,L}(s) = \tanh(Ls/2)$ and $S = 2\log 2/L$.

Proof. By evenness, (3.1) at $\alpha = p$ gives, for $|y| < \pi$,

$$\int_0^\infty \frac{2^p|\Gamma(p/2 + \mathrm{i}u)|^2}{\pi\Gamma(p)} \cosh(uy)\,\mathrm{d}u = \cos(y/2)^{-p}. \tag{4.6}$$

For $s > 0$, the geometric series is

$$\sum_{j\in\mathbb{Z}} e^{-s|t+jL|} = \frac{e^{-st} + e^{-s(L-t)}}{1 - e^{-Ls}}, \qquad \sinh(Ls/2)\frac{e^{-st} + e^{-s(L-t)}}{1 - e^{-Ls}} = \cosh\bigl(s(L/2 - t)\bigr).$$

Tonelli's theorem, followed by $u = Ls/(2\pi)$ and $y = \pi - 2\pi t/L$ in (4.6), proves (4.3). The $O(s)$ bound at zero implies $k(t) = O(t^{-2})$ for large t, so (4.4) converges. For $0 < p < 2$, splitting off the $j = 0$ term in (4.3) gives

$$t^{-p} - k(t) = t^{-p} - K_{p,L}(t) + \sum_{j=1}^{\infty}\{k(jL + t) + k(jL - t)\}.$$

The first difference tends to zero because it is $O(t^{2-p})$. The other terms converge to S by dominated convergence, using $k(jL - t) \le k((j - 1/2)L)$ for $0 < t < L/2$. The reflection formula at $1/2 + \mathrm{i}u$ gives the stated tanh density. Finally, in this case $1 - w_{1,L}(s) = 2/(e^{Ls} + 1)$, so (4.5) and dominated convergence give $S = \int_0^\infty 2/(e^{Ls} + 1)\,\mathrm{d}s = 2\log 2/L$. □

Lemma 4.2 (One positive integral for the density orders)**.** *Let $0 < p < 2$, $a = (p-1)/2$, $b = (2-p)/2$, and $W_p(u) = \sinh(\pi u)|\Gamma(p/2 + \mathrm{i}u)|^2/\pi$. Let $Y = \sum_{j\geq 1} Y_j$, where the Y_j are independent Gamma variables of shape p and rate $4\pi^2 j^2$. Then, for $u > 0$,*

$$W_p(u) = \frac{2^{p-2}u}{\Gamma(b)}\mathbb{E}\int_0^\infty \frac{r^a}{(r+Y)^{3/2}} e^{-u^2/(4(r+Y))}\,\mathrm{d}r. \tag{4.7}$$

Consequently $W_p(u)/u$ has strictly negative derivative, and

$$\int_0^T uW_p(u)\,\mathrm{d}u < \frac{T^{p+1}}{p+1} \quad (T > 0). \tag{4.8}$$

For $1 < p < 2$, $W_p(u)/u^{p-1}$ belongs to $(0,1)$ and has strictly positive derivative.

Proof. The sum Y is positive and finite almost surely since $\mathbb{E}Y = p/24$. Every negative moment is finite: for $Np > c$, couple the first N summands to obtain $Y \geq G_N/(4\pi^2 N^2)$ with $G_N \sim \mathrm{Gamma}(Np, 1)$, so $\mathbb{E}Y^{-c} < \infty$. The Euler product gives $\mathbb{E}e^{-t^2 Y} = [t/(2\sinh(t/2))]^p$. Put $h_p(t) = (2\sinh(t/2))^{-p}$ and $c_p = \Gamma(1-p)\cos(\pi p/2)$, with $c_1 = \pi/2$. Beta substitution and reflection first give, for $0 < p < 1$,

$$\int_0^\infty h_p(t)\sin(ut)\,\mathrm{d}t = c_p W_p(u).$$

For complex p use $\Gamma(p/2 + \mathrm{i}u)\Gamma(p/2 - \mathrm{i}u)$ in place of the modulus square. Both sides are holomorphic on $0 < \Re p < 2$: the sine supplies a factor t at zero, h_p decays exponentially at infinity, and the pole at $p = 1$ is removable. The identity theorem extends the equality.

The Gamma integral gives

$$h_p(t) = \frac{t}{\Gamma(a+1)}\mathbb{E}\int_0^\infty r^a e^{-(r+Y)t^2}\,\mathrm{d}r.$$

Since $\int_0^\infty h_p(t)|\sin(ut)|\,\mathrm{d}t < \infty$, Tonelli with absolute values justifies Fubini. The Gaussian formula

$$\int_0^\infty te^{-At^2}\sin(ut)\,\mathrm{d}t = \frac{\sqrt{\pi}u}{4A^{3/2}}e^{-u^2/(4A)}$$

and $c_p = \sqrt{\pi}\Gamma(b)/(2^p\Gamma(a+1))$ prove (4.7). Its integral without the exponential is $Y^{-b}B(a+1, b)$; with one additional factor $(r+Y)^{-1}$ it is $Y^{-b-1}B(a+1, b+1)$. The finite negative moments thus justify differentiation and give $(W_p/u)' < 0$ and $W_p = O(u)$.

The same formula with $Y = 0$ is exactly u^{p-1}, because

$$\int_0^\infty r^{a-3/2}e^{-u^2/(4r)}\,\mathrm{d}r = 4^b\Gamma(b)u^{-2b}.$$

For the cumulative comparison put

$$J_T(A) = A^{-3/2}\int_0^T u^2 e^{-u^2/(4A)}\,\mathrm{d}u = 8\int_0^{T/(2\sqrt{A})} z^2 e^{-z^2}\,\mathrm{d}z.$$

It strictly decreases with $A > 0$. Tonelli, $Y > 0$, and the pure formula therefore give (4.8) directly. This argument uses only positive integrands, also when $p < 1$.

If $1 < p < 2$, substitute $r + Y = u^2 B$ in (4.7) and divide by u^{p-1}:

$$\frac{W_p(u)}{u^{p-1}} = \frac{2^{p-2}}{\Gamma(b)}\mathbb{E}\int_{Y/u^2}^\infty (B - Y/u^2)^a B^{-3/2}e^{-1/(4B)}\,\mathrm{d}B.$$

Here $a > 0$. Comparing with $Y = 0$ puts the ratio in $(0,1)$. Every integrand, extended by zero below its lower endpoint, is nondecreasing in u and has positive derivative on $B > Y/u^2$. Fatou applied to nonnegative forward difference quotients gives a strictly positive derivative of the smooth Gamma ratio. □

Lemma 4.3 (Two density orders)**.** *For $0 < p \le 2$, the function $s \mapsto w_{p,L}(s)/s$ is nonincreasing. For $0 < p \le 1$, writing $\rho_p(s) = s^{p-1}/\Gamma(p)$, one has*

$$H(T) := \int_0^T s\{\rho_p(s) - w_{p,L}(s)\}\,\mathrm{d}s \ge 0 \qquad (T > 0). \tag{4.9}$$

The latter inequality is strict when $p < 1$.

Proof. Put $\eta = 2\pi/L$. Direct scaling gives $w_{p,L}(s) = \eta^{p-1}W_p(s/\eta)/\Gamma(p)$. Lemma 4.2 gives both assertions for $p < 2$, and its zero-end bound gives $w(s) \le Cs$. At $p = 2$ use $w_{2,L}(s) = s$ directly. □

Lemma 4.4 (Decreasing tests)**.** *Let $0 < p \le 1$. If $f \ge 0$ is continuously differentiable and nonincreasing on $(0, \infty)$, bounded near zero, and $s(\rho_p - w_{p,L})f$ is integrable, then*

$$\int_0^\infty s\{\rho_p(s) - w_{p,L}(s)\}f(s)\,\mathrm{d}s \ge 0.$$

In particular,

$$U(1) \le p\{k(1) + S\}. \tag{4.10}$$

Proof. On a finite interval $[a, T]$, integration by parts gives $\int_a^T H'f = H(T)f(T) - H(a)f(a) - \int_a^T Hf' \ge -H(a)f(a)$. Since $H(0) = 0$ and f is bounded there, first let $a \downarrow 0$ and then $T \to \infty$. This proves the testing assertion without a boundary assumption on $H(T)f(T)$ at infinity. For $0 < \tau \le 1$ apply it to

$$f_\tau(s) = p\int_\tau^1 e^{-sx}\,\mathrm{d}x + e^{-s}.$$

This is a nonnegative decreasing test, bounded by $p + 1$. The elementary Laplace identity $\int_0^\infty \rho_p(s)e^{-st}\,\mathrm{d}s = t^{-p}$ gives

$$0 \le p\{\tau^{-p} - k(\tau) - (1 - k(1))\} + p - U(1).$$

Let $\tau \downarrow 0$ and use (4.5). □

4.2 Weighted sums along the configuration

Choose a neighboring pair and one of its endpoints at which mass divided by its incident angular gap is maximal among all $2n$ incidences. Divide masses by this endpoint mass, and angular lengths by this gap. Rotate and, if necessary, reverse the circle. The chosen atoms are then $(0, 1)$ and $(1, b)$, with $0 < b \le 1$, and the period is $L \ge 2$. Every normalized gap is at least each of its endpoint masses. Indeed the quotient defining the chosen incidence is maximal. It follows also that the distance between any two atoms along either oriented arc is at least each of their endpoint masses.

Repeat the finite source periodically on the line. Let the atoms strictly to the left of 0 have positions $-t_j$, masses q_j, and those strictly to the right of 1 have positions $1 + v_j$, masses r_j; both lists are in increasing distance order. They include all translated copies of the two chosen atoms, but exclude the atoms at 0 and 1 themselves. They satisfy $t_j \ge 1$, $v_j \ge b$, and every mass is bounded by the two neighboring distances in its list, the initial distance being measured from 0 or 1. All the sums below converge absolutely by periodicity and the $O(t^{-2})$ decay of k and its derivatives.

Define

$$C_1(t) = k(t) - k(t+1), \qquad A(t) = C_1(t) - U(t+1), \qquad B(t) = U(t) - C_1(t),$$
$$P_A(s) = 1 - (1+s)e^{-s}, \qquad P_B(s) = s - 1 + e^{-s}, \qquad r_0(t) = \frac{t(3t+1)}{(t+1)(3t+2)}.$$

Thus A and B are Laplace transforms with respective densities wP_A and wP_B. Both densities are positive, and $P_A < P_B$.

Lemma 4.5 (Kernel comparisons)**.** *For $0 < p \le 2$ and $t > 0$,*

$$0 < A(t) < B(t), \qquad r_0(t)B(t) \le A(t). \tag{4.11}$$

The functions $A - B/4$ on $[1, \infty)$ and $B - A/4$ on $(0, \infty)$ are positive and strictly convex. They have $O(t^{-3})$ decay on each closed positive half-line.

Proof. Differentiation gives $P_A' = se^{-s}$, $P_B' = 1 - e^{-s}$, and

$$\left(\frac{P_A}{P_B}\right)' = \frac{e^{-s}\{s^2 + 2 - e^s - e^{-s}\}}{P_B(s)^2} < 0.$$

The last sign follows from $2\sinh(s/2) > s$. By Lemma 4.3, $w(s)/s$ is also nonincreasing. Therefore, for $u, v > 0$, the product

$$\{P_A(u)P_B(v) - P_A(v)P_B(u)\}\{w(u)v - w(v)u\}$$

is nonnegative. Multiply it by $u^j v^j e^{-t(u+v)}$ and integrate in both variables. Symmetrizing gives, for every integer $j \ge 0$,

$$\frac{\int_0^\infty s^{j+1} P_A(s) e^{-st}\,\mathrm{d}s}{\int_0^\infty s^{j+1} P_B(s) e^{-st}\,\mathrm{d}s} \le \frac{\int_0^\infty s^j w(s) P_A(s) e^{-st}\,\mathrm{d}s}{\int_0^\infty s^j w(s) P_B(s) e^{-st}\,\mathrm{d}s}. \tag{4.12}$$

For $j = 0$, the left ratio is $r_0(t)$, by $\int_0^\infty s^j e^{-ts}\,\mathrm{d}s = j! t^{-j-1}$. This proves (4.11). For positivity and convexity of $A - B/4$, use (4.12) for $j = 0, 1, 2$. The corresponding reference differences $\int s^{j+1}(P_A - P_B/4)e^{-st}\,\mathrm{d}s$ are

j	difference, with $y = t - 1 \ge 0$
0	$(9y^2 + 17y + 6)/(4t^3(t+1)^3)$
1	$(18y^3 + 56y^2 + 51y + 10)/(2t^4(t+1)^4)$
2	$(90y^4 + 390y^3 + 585y^2 + 315y + 18)/(2t^5(t+1)^5)$.

Every numerator is positive. Derivatives of the Laplace transforms insert $-s$, so the case $j = 2$ proves strict convexity. For $B - A/4$, the positivity of every moment follows directly from $P_B > P_A > 0$. Finally $w(s) \le Cs$ and $0 < P_A(s), P_B(s) \le s^2/2$ give an $O(t^{-4})$ bound and hence the stated decay. □

Lemma 4.6 (A packing inequality and its equality)**.** *Let K be a nonnegative continuous convex function on $[d, \infty)$, where $d > 0$, with $K(x) = O(x^{-3})$. Let $t_0 < t_1 < \cdots$ be a sequence of bodies on a ray with $t_0 \ge d$, $s_0 = t_0$, $s_j = t_j - t_{j-1} > 0$, and masses $0 < q_j \le \min(s_j, s_{j+1})$. If the sum converges, then*

$$\sum_j q_j K(t_j) \le d \sum_{j=1}^{\infty} K(jd). \tag{4.13}$$

For a strictly positive Laplace transform of a positive density on $(0, \infty)$, equality with $d = 1$ forces $t_j = j + 1$ and $q_j = s_j = 1$ for every j.

Proof. An equidistant comparison mesh consists of the points $t + js$ with step $s > 0$, each assigned weight s. It is an auxiliary point sequence, not a new physical configuration. Put $H_K(t, s) = s\sum_{j\ge 0} K(t + js)$ and $T_K(t, s) = s\sum_{j\ge 1} K(t + js)$. Convex secants show that $H_K(t, s)$ is nondecreasing in s when $t \ge d$, and $T_K(t, s)$ is nonincreasing whenever $t + s \ge d$. Here is a finite justification that also allows corners of K. A finite convex function is locally

absolutely continuous in the interior of its domain and differentiable almost everywhere. On a compact interval of positive step sizes, each finite sum below is absolutely continuous away from a possible endpoint $t + js = d$. For almost every step size, put $v_j = K(t + js)$ and $z_j = K'(t+js)$ for $j \geq 1$. Set $z_0 = 0$: the value $v_0 = K(t)$ is constant in the step size. Convexity gives $v_{j+1} - v_j \leq sz_{j+1}$ and $sz_{j+1} \leq v_{j+2} - v_{j+1}$. Summing these after multiplication by $j + 1$ yields

$$\sum_{j=0}^{N}(v_j + jsz_j) \geq (N + 1)v_N \geq 0, \quad \sum_{j=1}^{N}(v_j + jsz_j) \leq Nv_{N+1}.$$

The second upper bound tends locally uniformly to zero by cubic decay. Integrate these almost-everywhere inequalities between two interior step sizes and pass to the limit in the finite sums. Cubic decay gives locally uniform convergence. Continuity then includes any endpoint where a sample equals d. This proves both monotonicities, including the boundary function $T_K(0, s)$ for $s \geq d$. The identity $H_K(t, s) = sK(t) + T_K(t, s)$ now gives

$$\min(s, h)K(t) + H_K(t + h, h) \leq H_K(t, s).$$

For $s \leq h$ this is tail monotonicity; for $h \leq s$ it is head monotonicity. Apply it successively with $(t, s, h) = (t_j, s_j, s_{j+1})$. The nonnegative remainders telescope and give $\sum q_j K(t_j) \leq H_K(t_0, s_0) = T_K(0, s_0) \leq T_K(0, d)$. For a positive Laplace density the head and tail monotonicities are strict: this follows by integrating the strict scalar monotonicities of $s/(1 - e^{-us})$ and $s/(e^{us} - 1)$ for $u > 0$. If equality holds, no nonnegative local deficit in the telescoping inequality can be positive. The boundary strictness first gives $s_0 = 1$; the strict one-step inequality gives $s_{j+1} = s_j$, and positivity of $K(t_j)$ gives $q_j = \min(s_j, s_{j+1}) = 1$. This proves the equality claim. □

Lemma 4.7 (The force balances)**.** *With the incident normalization above, put $D = k(1) + S$. Then*

$$\sum_j q_j A(t_j) + \sum_j r_j B(v_j) = U(1) + (1 - b)D, \tag{4.14}$$

$$\sum_j q_j B(t_j) + \sum_j r_j A(v_j) = bU(1) - (1 - b)D. \tag{4.15}$$

For $0 < p \leq 1$, these identities imply

$$Q_p(b) := (3p + 6)b^3 + 2pb^2 + (p - 4)b - 2 > 0 \tag{4.16}$$

and

$$0 \leq F_{p,L}(b) := \sum_{j=1}^{\infty}\{A(j) - B(j)/4\} + b\sum_{j=1}^{\infty}\{B(jb) - A(jb)/4\} - (1 - b/4)U(1) - \tfrac{5}{4}(1 - b)D. \tag{4.17}$$

Proof. At a body position x, sum the image potential over all other atoms and their translates, including the nonzero translates of x. By (4.3) the result is the scaled radial row plus $m_x S$, where m_x is the mass at x. Differentiating the chord potential gives the tangential equation; the derivatives from nonzero translates of x cancel in pairs. Subtracting the radial row at 1 from the row at 0 gives

$$\sum_j q_j C_1(t_j) - \sum_j r_j C_1(v_j) = (1 - b)(k(1) + S). \tag{4.18}$$

The two zero tangential rows at 0 and 1 give, respectively,

$$\sum q_j U(t_j) - \sum r_j U(v_j + 1) = bU(1), \qquad \sum r_j U(v_j) - \sum q_j U(t_j + 1) = U(1). \tag{4.19}$$

Combining these three identities proves the asserted balances. The ratio r_0 is strictly increasing, lies in $(0,1)$, and every $v_j \geq b$. Lemma 4.5 therefore gives

$$r_0(b)\{\sum q_j A(t_j) + \sum r_j B(v_j)\} < \sum q_j B(t_j) + \sum r_j A(v_j).$$

The strict part follows already from $A < B$ on the nonempty left ray. Substitute the balances. Since $U(1), D > 0$, this implies $b > r_0(b)$ and

$$(1-b)(1+r_0(b)) < \frac{U(1)}{D}(b - r_0(b)) \leq p(b - r_0(b)),$$

where the last step is (4.10). Multiplication by the positive denominator $(b+1)(3b+2)$ gives exactly (4.16). Finally subtract one quarter of (4.15) from (4.14) and apply Lemma 4.6 to the left ray with $K = A - B/4$, $d = 1$, and the right ray with $K = B - A/4$, $d = b$. Lemma 4.5 verifies all its hypotheses. The result is (4.17). □

4.3 Transfer to the line and the scalar exclusion

For $s, b > 0$ write

$$\begin{aligned} T_b(s) &= \frac{b}{e^{bs} - 1}, \quad P(s) = P_B(s) - \tfrac{1}{4}P_A(s), \quad c_b = \tfrac{5}{4}(1-b), \\ E_b(s) &= P(s)\{T_b(s) - T_1(s)\} - (1-b)(s/4 + 5/4)e^{-s}, \\ \mathcal{L}_p(b) &= \int_0^\infty s^{p-1} E_b(s)\,\mathrm{d}s. \end{aligned}$$

These are the full infinite sampling expressions, rather than truncated approximations.

Lemma 4.8 (The finite-period comparison)**.** *For $0 < p \leq 1$ and $2/3 \leq b \leq 1$,*

$$F_{p,L}(b) \leq \mathcal{L}_p(b)/\Gamma(p). \tag{4.20}$$

Proof. Summing the geometric series inside the Laplace representations gives

$$F_{p,L}(b) = \int_0^\infty w_{p,L}(s) E_b(s)\,\mathrm{d}s - c_b S. \tag{4.21}$$

For example $\sum_{j\geq 1} be^{-jbs} = T_b(s)$; the identity $(P_A + P_B)T_1 = (se^{-s})$ accounts for the remaining terms. All integrations converge: at zero E_b is bounded, and at infinity it is bounded by a polynomial times $e^{-2s/3}$ for $b \in [2/3, 1]$.

We need a decreasing test with exactly the constant term c_b. Put $h(x) = x/(e^x - 1)$ and $\Delta_b(s) = T_b(s) - T_1(s)$. Direct differentiation shows $h' < 0$ and

$$h''(x) = \frac{e^x\{(x-2)e^x + x + 2\}}{(e^x - 1)^3} > 0.$$

The numerator in braces has value and first derivative zero at zero, and second derivative $xe^x > 0$. Also $0 \leq -h'(x) \leq 1$, the upper bound following from $e^x \geq 1 + x$. Thus

$$\Delta_b(s) = \int_b^1 -h'(sx)\,\mathrm{d}x$$

lies in $[0, 1-b]$ and is nonincreasing in s. Differentiating $P(s)/s$ shows that it is nondecreasing and belongs to $[0,1]$: its derivative has the sign of $5/4 - (5/4 + 5s/4 + s^2/4)e^{-s}$, which is nonnegative by $e^s \geq 1 + s + s^2/2$. The identity

$$f_b(s) := \frac{c_b + E_b(s)}{s} = (1-b) - \frac{P(s)}{s}\{(1-b) - \Delta_b(s)\}$$

therefore proves that f_b is nonnegative, nonincreasing and bounded by $1-b$. Apply Lemma 4.4 to $e^{-\tau s}f_b(s)$, $\tau>0$. Its product with the signed first density is integrable, and the result is

$$0 \le c_b\{\tau^{-p} - k(\tau)\} + \int_0^\infty (\rho_p - w_{p,L})E_b(s)e^{-\tau s}\,\mathrm{d}s.$$

Let $\tau \downarrow 0$. Equation (4.5) and dominated convergence yield $0 \le c_b S + \int(\rho_p - w)E_b$. Combine this with (4.21) to obtain (4.20). □

Lemma 4.9 (The scalar sign)**.** *For $0<p\le 1$ and $2/3 \le b < 1$, $\mathcal{L}_p(b) < 0$.*

Proof. We first prove the assertion at $b = 2/3$. Put $x = e^{s/3}$ and

$$B_*(s) = \frac{x^3(2x+1)}{3(x+1)(x^2+x+1)}, \qquad R_*(s) = 4 - \frac{25}{s+5} + e^{-s},$$
$$J(s) = \frac{s+5}{4}\{B_*(s)R_*(s) - 1/3\}.$$

Algebra gives $E_{2/3}(s) = e^{-s}J(s)$. The factor B_* is positive and nondecreasing, as is seen from

$$B_* = \tfrac{1}{3}(2 - 1/(x+1))\,\frac{x}{1+1/x+1/x^2}.$$

The function R_* has value zero at zero and positive derivative for $s>0$: indeed $R_*' = 25/(s+5)^2 - e^{-s} > 0$, since $1 + s/5 < e^{s/2}$. Thus B_*R_* increases strictly from zero. Moreover $B_*(5) > 8/27$ and $R_*(5) > 3/2$, so its value at 5 exceeds $1/3$. There is exactly one $c \in (0,5)$ where $J(c) = 0$, with $J<0$ before it and $J>0$ after it.

Here is a fully rational certificate for the sign at $p=1$. Define

$$f(t) = \frac{3t-1}{4t^2(t+1)^2}, \qquad g(t) = \frac{3t+4}{4t^2(t+1)^2}.$$

Termwise integration of the geometric series, or elementary integration of the Laplace transforms of P_A, P_B, gives

$$\mathcal{L}_1(b) = \sum_{k=1}^{\infty}\{f(k) + bg(kb)\} - 9/4 + 3b/2.$$

For $t>0$,

$$\frac{1}{t^3} - f(t) = \frac{t^2+9t+4}{4t^3(t+1)^2} > 0, \qquad \frac{1}{t^3} - g(t) = \frac{(t+2)^2}{4t^3(t+1)^2} > 0.$$

Therefore the tail after N terms is at most $(1+b^{-2})/(2N^2)$, by comparison with $\int_N^\infty x^{-3}\,\mathrm{d}x$. At $b=2/3$, $N=12$, exact rational addition gives

$$\sum_{k=1}^{12}\{f(k) + \tfrac{2}{3}g(2k/3)\} + \frac{13}{1152} - \frac{5}{4} = -\frac{2804360137027488583}{2150082517456954080000} < 0.$$

Thus $\int_0^\infty e^{-s}J(s)\,\mathrm{d}s < 0$. For $p \le 1$, the weight s^{p-1} is nonincreasing. On both sides of the crossing c this gives

$$s^{p-1}e^{-s}J(s) \le c^{p-1}e^{-s}J(s).$$

The integrals converge, so $\mathcal{L}_p(2/3) < 0$.

Finally $b \mapsto T_b(s) = h(bs)/s$ is convex, by the displayed formula for h''. The coefficient $P(s)$ is positive since $P_B > P_A > 0$. Consequently $b \mapsto s^{p-1}E_b(s)$ is convex. Write $b = \tau(2/3) + (1-\tau)1$ with $\tau = 3(1-b) > 0$. Since $E_1 = 0$, pointwise convexity and integration give $\mathcal{L}_p(b) \le \tau\mathcal{L}_p(2/3) < 0$. □

Theorem 4.10. *Under the hypotheses of Theorem 1.1, if $0 < \alpha \le 1$, the masses are equal and the sites form a regular polygon.*

Proof. Use the attained incident normalization and take $p = \alpha$. The cubic inequality (4.16) implies $b > 2/3$. Indeed

$$Q_1(b) - Q_p(b) = (1-p)(3b^3 + 2b^2 + b) \ge 0, \qquad Q_1(b) = (b - 2/3)(9b^2 + 8b + 7/3) - 4/9.$$

If $b < 1$, Lemmas 4.7, 4.8 and 4.9 give $0 \le F_{p,L}(b) < 0$. Thus $b = 1$. We use the undivided balances to recover equality, without dividing by $1 - b$. The identity

$$\sum_{j=1}^{\infty} \{A(j) + B(j)\} = \sum_{j=1}^{\infty} \{U(j) - U(j+1)\} = U(1)$$

and (4.14) show that the separate nonnegative packing deficits for the left A ray and the right B ray both vanish. These kernels have strictly positive Laplace densities, so the equality part of Lemma 4.6 forces every atom on the repeated left ray to have mass one and consecutive gap one. Every original atom occurs there. Hence all original masses are equal; equivalently the normalized word has all masses and gaps one and its actual period is $L = n$. Corollary 2.2 finishes the proof. □

5 Kernel order and row curvature below two

The next two exponent intervals use the same image kernel. We establish its density order and the curvature of its continued row before using either in a source estimate. Throughout this section $1 \le p \le 2$.

Lemma 5.1 (The image density between two powers)**.** *For $1 < p < 2$ and every period $L > 0$, the density of* (4.1) *satisfies*

$$0 < w_{p,L}(s) < \rho_p(s) := \frac{s^{p-1}}{\Gamma(p)}, \qquad p - 1 < \frac{s w'_{p,L}(s)}{w_{p,L}(s)} < 1. \tag{5.1}$$

In particular $w_{p,L}/\rho_p$ increases and $w_{p,L}/s$ decreases. At $p = 1$ the density is $\tanh(Ls/2)$*; at $p = 2$ it is s.*

Proof. Put $x = Ls/(2\pi)$. The exact scale relation gives $w_{p,L}(s)/\rho_p(s) = W_p(x)/x^{p-1}$. Lemma 4.2 gives its strict increase inside $(0, 1)$, and also $(W_p(x)/x)' < 0$. These two slopes give (5.1). Reflection and recurrence give the literal endpoint densities $\tanh(Ls/2)$ and s. □

Two useful consequences keep the original law intact. First, for $k_j(t) = \int_0^\infty s^j w_{p,L}(s) e^{-ts}\, \mathrm{d}s$, integration by parts gives

$$(j + p)k_j(t) < t k_{j+1}(t) < (j + 2)k_j(t) \quad (j \ge 0,\ t > 0, 1 < p < 2). \tag{5.2}$$

All boundary terms vanish because $w = O(s)$ at zero and the exponential controls infinity. Second, with any common positive integrable multiplier, an increasing test has larger expectation under the normalized w law than under the normalized ρ_p law, and smaller expectation than under the normalized s law. For example the first statement is the covariance identity

$$\mathrm{Cov}(f, r) = \frac{1}{2}\mathbb{E}[(f(S) - f(S'))(r(S) - r(S'))] \ge 0, \qquad r = w/\rho_p,$$

with two independent copies of the reference law. Decreasing tests reverse the inequalities. These comparisons will be used only with the same multiplier on both sides.

5.1 Exact cancellation of the self images

Use the canonical period $L = 1$, and abbreviate $w = w_{p,1}$. Let $\beta_B(u) = (e^u - 1)^{-1}$ and

$$S_{\text{self}} = 2\int_0^\infty w(u)\beta_B(u)\,\mathrm{d}u.$$

For $1 \le p < 2$, Lemmas 4.1 and 5.1, followed by monotone convergence in (4.5), give the exact equality

$$\int_0^\infty \{\rho_p(u) - w(u)\}\,\mathrm{d}u = S_{\text{self}}. \tag{5.3}$$

At $p = 1$ this is $\int_0^\infty 2/(e^u + 1)\,\mathrm{d}u = 2\log 2$. At $p = 2$, in contrast, $w = \rho_2$ and $S_{\text{self}} = 2\int_0^\infty u/(e^u - 1)\,\mathrm{d}u = \pi^2/3$. The endpoint's self term must therefore be retained separately.

Let $S_p(N)$ denote the analytic row from (3.2), and put $F_p(N) = S_p(N)/N^p$. The substitution $u = 2\pi\xi$ in the integral representation gives, for every $N > 1$,

$$F_p(N) = \frac{2N^{-p}}{\pi^p}\int_0^\infty w(u)\{\beta_B(u/N) - \beta_B(u)\}\,\mathrm{d}u. \tag{5.4}$$

In particular $F_p(N) > 0$ and $F_p(1) = 0$.

Lemma 5.2 (Strict row curvature)**.** *For $1 \le p \le 2$, the functions $s \mapsto F_p(e^s)$ and $\phi_p(s) = \log F_p(e^s)$ are strictly increasing and strictly concave on $s > 0$. Both F_p and $\log F_p$ are also strictly concave as functions of $N > 1$.*

Proof. Let $D_x = x\,\mathrm{d}/\,\mathrm{d}x$, and write

$$r_B(x) = \beta_B(x) - 1/x + 1/2 = \tfrac{1}{2}\coth(x/2) - 1/x, \quad H_p = (D_x + p)^2 r_B,$$

$$Q_p = (D_x + p)^2\beta_B, \qquad K_p = Q_p + p^2/2 = (p-1)^2/x + H_p.$$

The functions r_B and r_B' are positive because $\sinh z > z$ and $z\cosh z > \sinh z$. With $z = x/2$, exact differentiation gives

$$2\sinh^3 z\, H_1(x) = \cosh z\{\sinh^2 z + 2z^2 - 3z\tanh z\} > 0.$$

The last sign uses $\sinh^2 z > z^2$ and $\tanh z < z$. The identity $H_p = H_1 + 2(p-1)(D_x + 1)r_B + (p-1)^2 r_B$ therefore proves $H_p > 0$ and $K_p > 0$ for every $p \ge 1$. At zero, $H_p = O(x)$; at infinity, $K_p \to p^2/2$.

Differentiating (5.4) twice, with $N = e^s$, yields

$$-\partial_s^2 F_p(e^s) = \frac{2N^{-p}}{\pi^p}\left\{\frac{p^2}{2}S_{\text{self}} - \int_0^\infty w(u)Q_p(u/N)\,\mathrm{d}u\right\}. \tag{5.5}$$

Local domination follows from $w = O(u)$ and $Q_p(u) = O(1/u)$ at zero, and from exponential decay at infinity. At $p = 1$ the sharper $Q_1(u) = -1/2 + O(u)$ applies. The corresponding pure-density differentiated integral is zero:

$$\int_0^\infty \rho_p(u)Q_p(u/N)\,\mathrm{d}u = 0.$$

After $u = Nx$, it is $N^p/\Gamma(p)$ times the endpoint difference $[x^p(D_x + p)\beta_B(x)]_0^\infty$. At zero this tends to zero for $p > 1$ because its leading term is $(p-1)x^{p-1}$; at $p = 1$ that leading coefficient vanishes and the remaining term is $O(x)$. At infinity it decays exponentially.

For $p < 2$, subtract this zero integral in (5.5) and use (5.3). The result is the positive integral

$$-\partial_s^2 F_p(e^s) = \frac{2N^{-p}}{\pi^p}\int_0^\infty (\rho_p - w)(u)K_p(u/N)\,\mathrm{d}u > 0. \tag{5.6}$$

It is integrable: for $p > 1$ the possible zero-end term is $O(u^{p-2})$, and the total deficit mass is finite at infinity. At $p = 1$ use $K_1 = H_1 = O(u)$ directly. At $p = 2$, (5.5) and the zero pure integral give $-\partial_s^2 F_2(e^s) = 4N^{-2}/3 > 0$. Thus the derivative of the positive function $F_p(e^s)$ strictly decreases. It must stay positive, since a nonpositive value would force the function to become negative later. Logarithmic strict concavity follows from $\phi_p'' = F_{p,ss}/F_p - (F_{p,s}/F_p)^2 < 0$. Finally the second derivatives with respect to N are $(F_{p,ss} - F_{p,s})/N^2$ and $(\phi_p'' - \phi_p')/N^2$, both negative. □

For $1 < p \le 2$, normalization by $C_p = 2\zeta(p)/\pi^p$ gives

$$\mathcal{A}_p(N) = F_p(N)/C_p, \qquad 0 < \mathcal{A}_p(N) < 1.$$

Indeed, in (5.4) substitute $u = Nx$. The ratio $w(Nx)/\rho_p(Nx)$ tends to one and is bounded by one. Dominated convergence against $x^{p-1}/(e^x - 1)$ gives $F_p(N) \to C_p$; the subtracted self term tends to zero. Strict increase then proves the upper bound.

6 Cancellation between the equations at adjacent bodies

Lemma 6.1 (Bounds for $1/(e^x - 1)$)**.** *For every $x > 0$,*

$$\frac{1}{x} - \frac{1}{2} + \frac{5x}{60 + x^2} < \frac{1}{e^x - 1} < \frac{1}{x} - \frac{1}{2} + \frac{x}{12}. \tag{6.1}$$

With $h_0(x) = x/(1 - e^{-x})$, we have $h_0(x) - 1 < x/2 + x^2/12$ and $\coth t - 1/t < t/3$ for $t > 0$.

Proof. The complete coth partial fractions, with $a_j = 1/(\pi^2 j^2)$, $\sum a_j = 1/6$ and $\sum a_j^2 = 1/90$, give

$$\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} = \frac{x}{2} \sum_{j \ge 1} \frac{a_j}{1 + (x^2/4)a_j} > \frac{x}{2} \frac{(\sum a_j)^2}{\sum a_j + (x^2/4) \sum a_j^2} = \frac{5x}{60 + x^2}.$$

This is strict Cauchy–Schwarz for the full positive series, since the a_j are not equal. Dropping its denominator correction gives the strict upper bound $x/12$. Use $h_0(x) = x + x/(e^x - 1)$ for the first consequence; the second uses $1/(e^{2t} - 1) = (\coth t - 1)/2$. □

6.1 Cancellation at a pair of adjacent bodies

Retain the attained unit edge $(0, 1), (1, b)$ and $D = k(1) + S$ from Section 4. Write the Laplace sums of the sequences of bodies on the two rays as

$$X(s) = \sum_j q_j e^{-st_j}, \qquad Y(s) = \sum_j r_j e^{-sv_j}, \qquad A_s = 1 - (1 + s)e^{-s}, \quad B_s = s - 1 + e^{-s}.$$

For this section write $H_B(s) = h_0(s)$. The corrected balances and torque give

$$\int w(A_s X + B_s Y) = U(1) + (1 - b)D,$$
$$\int w(B_s X + A_s Y) = bU(1) - (1 - b)D, \qquad T_X := \int wsX > bU(1).$$

Lemma 4.6 with $K(t) = e^{-st}$ and $d = 1$ gives $X(s) \le 1/(e^s - 1)$. Every distance in Y is at least b. The integral identities are the balances in Lemma 4.7, and the strict torque bound follows from the first row of (4.19). The self-image sum (4.4) converges also at $p = 2$, so these identities include that endpoint.

Lemma 6.2. *Fix $0 < \lambda < 1$ and $0 < b_0 < 1$. Suppose, whenever $b \le b_0$, the right-ray contribution $\int w(A_s - \lambda b B_s)Y$ is positive. Let ν be a probability law such that the normalized density $w(s)e^{-s}/k(1)$ stochastically dominates ν. Define*

$$F_{\lambda,b_0}(s) = [1 + \lambda b_0 - \lambda(1 + b_0)H_B(s) + \lambda b_0 s]_+.$$

If $\mathbb{E}_\nu F_{\lambda,b_0} < (1 - b_0)(1 + \lambda b_0)$, then $b > b_0$.

Proof. Set $c = \lambda b$ and $\theta = 1 - \lambda$. Packing gives

$$\int w(B_s - cA_s)X \ge \theta T_X - \mathcal{E}_{\text{edge}}, \quad \mathcal{E}_{\text{edge}} = \int ws[\theta - (B_s - cA_s)/s]_+/(e^s - 1).$$

After division by $k(1)$ the correction term is the expectation of $[1 + c - (\lambda + c)H_B(s) + cs]_+$. The positive partial fractions give $H_B' > 1/2$; this test decreases in s for $b < 1$; it increases with b, since $1 - H_B(s) + s = 1 - s/(e^s - 1) > 0$. Thus the stipulated expectation bounds it. Subtracting c times the first balance from the second, the coefficient of $U(1)$ after the left-ray estimate is $b - c - \theta b = 0$. Consequently $\mathcal{E}_{\text{edge}} > (1 - b)(1 + c)D$. But $(1 - b)(1 + \lambda b)$ decreases in b, and $D > k(1)$, a contradiction. □

6.2 Four bounds for the positive-part correction

Lemma 6.3. *For Gamma laws of shape a and rate one, the following strict bounds hold:*

λ	b_0	a	$\mathbb{E}F_{\lambda,b_0}$
4/9	2/3	1	$< 35/81$
7/16	5/7	3/2	$< 3/8$
1/3	5/8	2	$< 9/20$
1/4	1/2	2	$< 17/32$

Proof. The lower bound in Lemma 6.1 gives $H_B(s) > 1 + s/2 + 5s^2/(60 + s^2)$. In the four rows respectively, this bounds F by $Q(s)\mathbf{1}_{[0,T]}$:

$Q(s)$	T
$(15 - 2s - s^2)/27$	3
$9/16 - s/16 - s^2/24$	3
$2/3 - s/16 - 5s^2/192$	4
$3/4 - s/16 - s^2/48$	9/2

Indeed each Q is nonnegative on its displayed interval, and its quadratic coefficient is no greater than $5\lambda(1 + b_0)/(60 + T^2)$. The first three vanish at T. In the last row the rational bound in Lemma 6.1 on the raw test at T is $-15/3424 < 0$. Monotonicity therefore excludes the entire tail in every row. The first integral is $(11 + 10e^{-3})/27 < 35/81$. For the second, the incomplete-Gamma recurrence and positivity of the remaining shape-1/2 tail give the upper bound

$$5/16 + (9/8)\sqrt{3/\pi}\, e^{-3} < 3/8.$$

Here $\pi > 3$ and $e^3 > 18$. The last two integrals are

$$37/96 + (191/96)e^{-4} < 9/20, \qquad 1/2 + (215/128)e^{-9/2} < 17/32.$$

Use $e^4 > 32$ and $e^{9/2} > 64$. All exponential bounds follow from positive partial sums: $e^3 > 18$, $e^4 > 32$, and $e^{3/2} > 4$. □

6.3 The whole lower band

Lemma 6.4. *For an original positive centered cocircular source with $n \geq 3$, the attained unit edge satisfies $b > 2/3$ when $1 \leq p \leq 2$, and $b > 5/7$ when $3/2 \leq p \leq 2$.*

Proof. Lemma 4.5 gives $A(t)/B(t) \geq r_0(t)$ for $t \geq b$. The function r_0 increases. The two concave polynomials

$$9(3t+1) - 4(t+1)(3t+2) = -12t^2 + 7t + 1, \quad 16(3t+1) - 7(t+1)(3t+2) = -21t^2 + 13t + 2$$

are positive on $[0, 2/3]$ and $[0, 5/7]$, respectively, by their positive endpoint values. Thus the required right rays are positive. For $1 < p < 2$, Lemma 4.2 makes w/ρ_p increasing. Hence $we^{-s}/k(1)$ dominates Gamma$(p, 1)$. At $p = 1$, $w = \tanh(Ls/2)$ increases; at $p = 2$, $w = s$. Thus the first correction bound uses Gamma$(1, 1)$ throughout $[1, 2]$, and the second uses Gamma$(3/2, 1)$ throughout $[3/2, 2]$. Apply Lemmas 6.2 and 6.3; the two charges are exactly $35/81$ and $3/8$. □

7 The interval $1 \leq \alpha \leq 9/5$

We continue to use the attained incident normalization and the kernels $k, U, w_{p,L}$ of Section 4. This section uses two further bodies of the original configuration. Proposition 2.4 has already settled $n \leq 4$, so it remains to consider $n \geq 5$ and $p = \alpha$ in $[1, 9/5]$.

The force rows give two inequalities constraining the third and fourth masses. We use them to bound an expression that the first two rows require to be nonnegative. For the gap $u \leq 7/8$ we bound the third mass by one root and compare the resulting mesh along a family of kernels. For $u > 7/8$ we use both mass constraints and compare kernels at fixed geometry.

7.1 Force inequalities at the next two bodies

Suppose the configuration is not already equal-mass regular. In cyclic order write four consecutive atoms as

$$(0, 1), \quad (1, b), \quad (1 + u, m), \quad (1 + u + v, r). \tag{7.1}$$

Lemma 6.4 gives $b > 2/3$. The mass caps give $b \leq \min(1, u)$, $m \leq \min(u, v)$ and $r \leq v$. In fact $u < 1$. To see this, if $u \geq 1$, pack the left A ray at unit spacing. On the right, first pack with head $H_B(u, m)$, increase the step from m to u, then use the decreasing boundary tail to bound it by $T_B(0, 1)$. The notation H_K, T_K is that of Lemma 4.6; their strict sampling properties hold for A, B because their Laplace densities are positive. The first force balance (4.14) and $T_A(0, 1) + T_B(0, 1) = U(1)$ now force $b = 1$. The undivided equality argument in Theorem 4.10, which used only $0 < p \leq 2$, makes the whole configuration regular, a contradiction. Thus

$$2/3 < b \leq u < 1, \qquad 0 < m \leq u, \qquad v \geq \max(m, r) > 0. \tag{7.2}$$

For $h > 0$ introduce the positive Peano kernels

$$A_h(t) = k(t) - k(t + h) - hU(t + h), \qquad B_h(t) = hU(t) - k(t) + k(t + h).$$

Their Laplace densities are $w(s)a_0(hs)$ and $w(s)b_0(hs)$, where

$$a_0(x) = 1 - (1 + x)e^{-x}, \qquad b_0(x) = x - 1 + e^{-x}.$$

Both are strictly positive for $x > 0$. Therefore all positive-kernel packing and strict head/tail comparisons in Lemma 4.6 apply at every positive starting point.

Lemma 7.1 (Two necessary force inequalities)**.** *Set* $D_u = k(u) + S$, $M_u = uU(u) + D_u$, *and*

$$q(v, m, r) = T_{B_u}(0, 1) + uD_u - mM_u + H_{A_u}(v, r), \tag{7.3}$$
$$H(v, r; m) = uB_v(u) + T_{B_v}(u, 1) + T_{A_v}(0, r) - vrU(v) + (m - r)(k(v) + S). \tag{7.4}$$

At the actual configuration parameters (7.1), $q(v, m, r) \geq 0$ *and* $H(v, r; m) \geq 0$.

Proof. The second Peano balance across a gap h, with left and right endpoint masses a, c, is

$$\text{left } B_h \text{ ray} + \text{right } A_h \text{ ray} = hcU(h) - (a - c)(k(h) + S).$$

Indeed, with $C_h(t) = k(t) - k(t + h)$, the radial difference is left C_h ray − right C_h ray = $(a-c)(k(h)+S)$ by the calculation in Lemma 4.7. Subtract this from h times the left tangential row, left U ray − right $U(\cdot + h)$ ray = $cU(h)$. The coefficients on the two rays are then B_h and A_h. Apply it across the gap u. The left ray begins with the unit atom at distance one and is bounded by $T_{B_u}(0, 1)$. The right ray begins with mass r at distance v and is bounded by $H_{A_u}(v, r)$. Replacing b by its upper bound u increases the resulting expression by $(u - b)D_u \geq 0$, giving (7.3). Apply the same identity across the gap v. On its left, retain the atom of mass b at distance u, bound b by u, and pack all more remote atoms by $T_{B_v}(u, 1)$. On its right, packing gives $T_{A_v}(0, r)$. These are exactly the four terms in (7.4), so $H \geq 0$. All repeated copies, including self images, occur in these rays; the single term S in each difference is the correction for the omitted physical diagonal. □

For the following source estimates we allow the continuous family

$$w_\theta = (1 - \theta)\rho_p + \theta w_{p,L}, \qquad S_\theta = \theta S, \qquad 0 \leq \theta \leq 1, \tag{7.5}$$

and use it to evaluate k, U, A_h, B_h, q, H at any fixed arguments. The configuration inequalities apply at $\theta = 1$. For $1 \leq p < 2$, $w_\theta(s)/s$ is nonincreasing: the line term is $s^{p-2}/\Gamma(p)$ and the image term has this property by Lemma 4.3. Write

$$\beta_h(s) = \frac{h}{e^{hs} - 1}, \quad T_u(s) = e^{-us}(u + \beta_1(s)).$$

Here $h_0(x) = x/(1 - e^{-x})$ as in Lemma 6.1.

7.2 Signed sampling and positive parts

For later use we record explicitly the properties of $x/(e^x - 1)$ needed for signed sampling. Put $h(x) = x/(e^x - 1)$,

$$C(x) = 1 - h_0'(x) = -h'(x) = \frac{(x - 1)e^x + 1}{(e^x - 1)^2}, \qquad E(x) = 1 - C(x).$$

For $x > 0$, $0 < C(x) < 1$, $C' < 0$, $C'' > 0$, and $\log C$ is concave. The first two signs follow from $e^x > 1 + x$ and the formula for h'' in Lemma 4.8. To check the last two without an uncontrolled series remainder, direct differentiation gives

$$C'' = \frac{e^x V(x)}{(e^x - 1)^4}, \quad V(x) = (x - 3)e^{2x} + 4xe^x + x + 3,$$

and

$$C''C - (C')^2 = -\frac{e^x J(x)}{(e^x - 1)^6}, \quad J(x) = e^{3x} + (-2x^2 + 3x - 5)e^{2x} + (-2x + 7)e^x - x - 3.$$

The values of V, V', V'', V''' at zero vanish, while

$$V^{(4)}(x) = 16(x-1)e^{2x} + (4x+16)e^x > 0,$$

because $(1-x)e^x < 1$. The values of J through its fourth derivative at zero also vanish, and

$$J^{(5)}(x) = 243e^{3x} - (64x^2 + 224x + 240)e^{2x} - (2x+3)e^x > 0.$$

For the last inequality divide by e^{2x}, use $e^{-x} \le 1$ and $e^x \ge 1 + x + x^2/2$; what remains is at least $17x + (115/2)x^2 > 0$. Repeated integration proves $V, J > 0$. It follows that $C(x)/x$ and $E(x)/x$ are decreasing. For E, use concavity and $E(0+) = 1/2$ to get $xE'(x) \le E(x)$. Also, for every $q \ge 1$, $C(qx)/C(x)$ is decreasing: differentiate its logarithm, use the negative nonincreasing logarithmic derivative of C, and multiply its value at qx by $q \ge 1$.

We shall repeatedly use a simple signed comparison. If an integrable function f is nonnegative before a single crossing $s_0 > 0$ and nonpositive after it, $\int f > 0$, and $g > 0$ is decreasing, then

$$\int_0^\infty f(s)g(s)\,\mathrm{d}s \ge g(s_0)\int_0^\infty f(s)\,\mathrm{d}s > 0, \tag{7.6}$$

provided the product is integrable. This is a pointwise comparison on the two sides of the crossing, so it applies to the entire signed tail.

The force-row combination below needs coefficients with $a + z = u$. The bound $a \le 6/25$ ensures the left sampling signs in Lemma 7.2. We use the following choice throughout:

$$\begin{aligned} a(u) &= \frac{6}{25}u(2-u), \quad z(u) = u - a(u), \\ \ell(s) &= 1 - e^{-s} - a(u)s - z(u)se^{-s}, \\ g_u(s) &= z(u) + a(u)e^{-s} - \frac{1-e^{-s}}{s}. \end{aligned}$$

Define the signed left and right kernels

$$\begin{aligned} F_u(t) &= k(t) - k(t+1) - a(u)U(t) - z(u)U(t+1), \\ G_u(t) &= z(u)U(t) + a(u)U(t+1) - k(t) + k(t+1), \\ J_u(t) &= \max(0, G_u(t)). \end{aligned} \tag{7.7}$$

Their densities are respectively $w_\theta(s)\ell(s)$ and $w_\theta(s)sg_u(s)$.

Lemma 7.2 (The signed left ray and positive right part)**.** *For $0 < u \le 1$, the left kernel F_u has nonnegative values on $[1, \infty)$, increasing head sampling and decreasing tail sampling there, and decreasing boundary tail for step sizes at least one. Consequently the sum along its left-hand sequence of bodies is at most $T_{F_u}(0,1)$. The function J_u is nonnegative, decreasing and convex on $(0, \infty)$, with summable samples on every positive arithmetic ray. It therefore satisfies all packing inequalities of Lemma 4.6.*

Proof. Write $a = a(u)$ and $z = z(u)$. Then $0 < a \le 6/25$, $z \ge 0$, $a + z \le 1$. The multiplier ℓ has exactly one crossing from positive to negative. Indeed $e^s\ell(s) = (1-as)e^s - 1 - zs$; its derivative has derivative $(1 - 2a - as)e^s$, which changes sign once. The first derivative starts at $1 - a - z \ge 0$, and the function itself starts at zero and is negative at $s = 1/a$. Thus it first increases and then decreases through a unique positive zero. Two complete reference moments suffice:

$$\begin{aligned} &\int_0^\infty s\ell(s)se^{-s}\,\mathrm{d}s = 7/4 - 6a - 3z/8 \ge 1/40, \\ &\int_0^\infty s\ell(s)C(s)\,\mathrm{d}s = 2(1-2(a+z))\zeta(3) + 6z\zeta(4) - 1 - 2z > 0.0079. \end{aligned}$$

The first is elementary exponential integration. For the second, integrate $-h'$ by parts and expand $1/(e^s-1)$ into its positive geometric series; the boundary terms vanish. It is affine in a, z. Its minimum on $0 \le a \le 6/25$, $0 \le z \le 1-a$ occurs at $(a,z) = (6/25, 19/25)$; the signs of the coefficients follow, for example, from $1.2 < \zeta(3) < 1.203$ and $1.082 < \zeta(4) < 1.083$. At that point the lower bound is $-2(1.203) + (114/25)(1.082) - 63/25 = 0.00792$. All four zeta bounds follow from ten terms and neighboring integral tails. Multiplication by the decreasing ratio $w_\theta(s)/s$ in (7.6) makes both corresponding actual-law moments positive.

For $t \ge 1$, apply (7.6) to the first actual moment and the decreasing multipliers

$$e^{-(t-1)s}/s, \qquad e^{-(t-1)s}E(qs)/s, \qquad e^{-(t-1)s}C(qs)/s.$$

This proves $F_u(t) > 0$ and the required signs of the head and tail derivatives: the head and tail factors $q/(1-e^{-qs})$ and $q/(e^{qs}-1)$ have derivatives $E(qs)$ and $-C(qs)$, respectively. For the boundary derivative at $q \ge 1$, use the second actual moment and the decreasing ratio $C(qs)/C(s)$. The densities have a polynomial bound and all the exponential factors have positive distance; dominated convergence justifies the derivatives. The one-step telescope in Lemma 4.6 uses exactly these sampling monotonicities, so proves the claimed signed left bound even though no positive Laplace density for F_u was assumed.

For the right kernel, g_u is strictly increasing, since

$$s^2 g_u'(s) = 1 - (1 + s + as^2)e^{-s} > 0$$

when $a < 1/2$. It starts at $u - 1 \le 0$ and ends at $z > 0$. Therefore its density has at most one crossing, from negative to positive. Whenever $G_u(t) \ge 0$, multiplication by the increasing positive functions s and s^2 shows that its first and second Laplace moments are positive. Hence $G_u'(t) < 0$ and $G_u''(t) > 0$ on its positive part. For $u < 1$, the small-frequency limit $g_u(0+) = u - 1$ shows $G_u(t) < 0$ eventually, whereas the positive high-frequency tail makes $G_u(t) > 0$ for sufficiently small t. The same one-crossing comparison shows that once G_u has become nonpositive it cannot become positive again. Thus its positive part is decreasing and convex; at its zero the derivative jumps upwards to zero. For $u = 1$ its density is positive, giving the same conclusion directly.

The decay needed for packing concerns J_u, not the signed kernel G_u. If $u < 1$, the crossing just proved makes J_u identically zero beyond a finite point. If $u = 1$, then $g_u(s) = O(s)$ as $s \downarrow 0$. Since $0 < w_\theta \le \rho_p = s^{p-1}/\Gamma(p)$, its density $w_\theta(s)sg_u(s)$ is $O(s^{p+1})$ near zero. The contribution from $s \ge 1$ decays exponentially in t, whereas integration on $(0,1)$ gives $J_u(t) = O(t^{-p-2}) = O(t^{-3})$ for $p \ge 1$. Thus J_u meets the continuous convex version of Lemma 4.6, including convergence of its samples. The signed kernel need not have cubic decay; for the pure power law, $G_u(t) = p(u-1)t^{-p-1} + O(t^{-p-2})$. □

7.3 The comparison functional and the fourth mass

The comparison functional is the upper bound in (7.8). Its terms are a negative baseline, the third mass contribution and the positive part of the ray after it. For $v \ge u$ the inequality $q \ge 0$ bounds the third mass directly. For $v \le u$ we use the root of H constructed below to bound the fourth mass.

Lemma 7.3 (The comparison functional)**.** *Let* $Z = \sum_{j=1}^{\infty} U(j)$ *and*

$$\mathcal{B}_u = u\{k(1) + a(u)U(1) - Z\} - (1-u)S.$$

At the configuration geometry (7.1)*,*

$$0 \le \mathcal{B}_u + mG_u(u) + H_{J_u}(u+v, r). \tag{7.8}$$

Moreover $\mathcal{B}_u < 0$. Thus the only possible case has $G_u(u) > 0$. For an outer geometry $v \geq u$, define

$$F = \int_0^\infty w_\theta(s) b_0(us)(\beta_u(s) - \beta_1(s))\,\mathrm{d}s > 0, \qquad m_R = u - F/M_u.$$

The force equations at the bodies imply $m \leq m_R < u$, and its comparison functional is bounded by

$$0 \leq \mathcal{B}_u + m_R G_u(u) + T_{J_u}(u, u). \tag{7.9}$$

The physical statements use $\theta = 1$.

Proof. Take the radial difference (4.18), subtract a times the tangent row at 0, and add z times the tangent row at 1, in the orientations of (4.19). This gives

$$\text{left } F_u \text{ ray} + \text{right } G_u \text{ ray} = (1-b)(k(1) + S) - abU(1) + zU(1).$$

The signed left sampling bound is $T_{F_u}(0,1) = k(1) - uZ + zU(1)$, by telescoping $k(j) - k(j+1)$. On the right retain the signed first atom $mG_u(u)$ and pack the remaining positive part from the fourth atom. Rearrangement gives

$$0 \leq b\{k(1) + aU(1) + S\} - uZ - S + mG_u(u) + H_{J_u}(u+v, r).$$

Its coefficient of b is positive, so $b \leq u$ proves (7.8). To see $\mathcal{B}_u < 0$, the positive sampling head of U at step one bounds its integral strictly below its head sum: $Z > \int_1^\infty U(t)\,\mathrm{d}t + \frac{1}{2}U(1) = k(1) + \frac{1}{2}U(1)$, by strict trapezoidal convexity. Since $a \leq 6/25 < 1/2$ and $S > 0$, the base is strictly negative. If $G_u(u) \leq 0$, decreasing J_u vanishes throughout $[u, \infty)$, contradicting (7.8).

For an outer source, $v \geq u$ and $r \leq v$ give $H_{A_u}(v, r) \leq H_{A_u}(v, v) = T_{A_u}(0, v) \leq T_{A_u}(0, u)$. Insert this in $q \geq 0$. The identity $T_{A_u}(0,u) + T_{B_u}(0,u) = u^2 U(u)$ then gives $mM_u \leq uM_u - F$, so $m \leq m_R$. The same positive-part packing gives $H_{J_u}(u+v, r) \leq H_{J_u}(u+v, v) = T_{J_u}(u, v) \leq T_{J_u}(u, u)$. Since $G_u(u) > 0$, the bound on m proves (7.9). □

The geometric series turns (7.4) into the exact integral

$$H(v, r; m) = \int_0^\infty w_\theta(s)\{b_0(vs)T_u(s) + a_0(vs)\beta_r(s) - vrse^{-vs} + (m-r)e^{-vs}\}\,\mathrm{d}s + (m-r)S_\theta. \tag{7.10}$$

Lemma 7.4 (The fourth comparison root)**.** *For $0 < m \leq v \leq u < 1$ the equation $H(v, r; m) = 0$ has a unique solution $r = R(v; m)$ in $(0, v)$. On the diagonal, $R_D(v) = R(v; v)$ obeys*

$$0 < R_D'(v) < 2 \qquad (0 < v \leq u). \tag{7.11}$$

At fixed m with $m/v \geq 1/2$, $0 < \partial_v R(v; m) < 2$.

Proof. Differentiating (7.10), with $x = vs$, $y = rs$, gives

$$D := -H_r = \int_0^\infty w_\theta(s)\{e^{-x}(1+x) + a_0(x)C(y)\}\,\mathrm{d}s + S_\theta > 0.$$

Since $u < 1$, the strict step comparison gives $T_u(s) < e^{-us}(u + \beta_u(s)) = \beta_u(s) \leq \beta_v(s)$. At $r = v$, direct geometric-series algebra gives

$$H(v, v; m) = \int_0^\infty w_\theta(s) b_0(vs)\{T_u(s) - \beta_v(s)\}\,\mathrm{d}s + (m-v)(k(v) + S_\theta) < 0.$$

As $r \downarrow 0$, $\beta_r(s) \uparrow 1/s$. Since $a_0(vs) \to 1$ and $w_\theta(s) \sim s^{p-1}/\Gamma(p)$ at infinity by Stirling's formula, the limiting $a_0(vs)$ integrand has a divergent s^{p-2} tail. The other terms stay bounded below.

Thus a unique root exists in $(0, v)$; implicit differentiation is legitimate by positive-distance Laplace domination.

At $m = v$, for every $0 < r \le v$ (not only a root),

$$H_v + H_m = \int_0^\infty w_\theta(s)\{s(1-e^{-x})T_u(s) + e^{-x}[1 - x + xh_0(y)]\}\,\mathrm{d}s + S_\theta > 0.$$

Since $y \le x$, use $T_u < \beta_v$, increasing h_0 and decreasing C to obtain

$$H_v + H_m - 2D < \int_0^\infty w_\theta(s)N(vs)\,\mathrm{d}s - S_\theta, \quad N(x) = e^{-x}[-1 - 2x + xh_0(x)] - 2a_0(x)C(x).$$

We verify its sign on the entire frequency ray. Clearing its positive denominator gives

$$e^{2x}(1-e^{-x})^2 N(x) = (x^2 - 4x + 1)e^x + x^2 + 4x - 2 + e^{-x}.$$

The coefficients of degrees zero and one vanish; degrees two, three and four are $-1, -1, -1/12$, degree five is zero, and for every $j \ge 6$ the coefficient multiplied by $j!$ is $j^2 - 5j + 1 + (-1)^j > 0$. After division by x^4, the negative part is $-x^{-2} - x^{-1} - 1/12$ and the positive part is strictly increasing. Thus N has exactly one crossing, from negative to positive. Expanding the geometric denominators, integrating their exponential terms and collecting the absolutely convergent series gives

$$\int_0^\infty xN(x)\,\mathrm{d}x = 1 + 4\zeta(3) - 6\zeta(4) < -0.68.$$

The last bound follows from $\zeta(3) < 1.203$ and $\zeta(4) > 1.082$; these follow by summing the first ten positive terms and bounding the remaining decreasing series by the two neighboring integral tails. At a negative-to-positive crossing, multiplication by any nonnegative decreasing function can only decrease the integral relative to its value at the crossing. Apply this to $w_\theta(x/v)/(x/v)$. The complete w_θ integral of $N(vs)$ is therefore negative. This proves $0 < (H_v + H_m)/D < 2$, namely (7.11).

For a fixed m, direct differentiation gives

$$H_m = k(v) + S_\theta, \qquad H_v(v, r; m) = H_v(v, r; v) + (v - m)U(v).$$

Here $H_v(v, r; v)$ is a partial derivative evaluated at $m = v$, not the diagonal total derivative $H_v + H_m$. The diagonal estimate above holds at every $r \le v$, not just a root. Also $vU(v) \le 2k(v)$: integrate the derivative of $sw_\theta(s)e^{-vs}$ and use $sw'_\theta \le w_\theta$; the boundary terms vanish at zero and infinity. Consequently, whenever $m/v \ge 1/2$,

$$H_v(v, r; m) < 2D - (k(v) + S_\theta) + (v - m)U(v) \le 2D.$$

Its positivity follows directly from the complete density

$$H_v = \int w_\theta(s)s\{(1-e^{-vs})T_u(s) + e^{-vs}[vh_0(rs) - m]\}\,\mathrm{d}s > 0.$$

Thus $0 < \partial_v R = H_v/D < 2$ on this domain. □

7.4 Comparison with the power kernel

For $u \le 7/8$ we use the auxiliary adjacent root and move it together with the kernel. The near-unit branch instead compares the augmented comparison functional at fixed geometry. Define

$$\begin{aligned} q_A(u, m) = &\int w_\theta(s)[b_0(us)\beta_1(s) + a_0(us)\beta_m(s)]\,\mathrm{d}s \\ &- umU(u) + (u - m)(k(u) + S_\theta). \end{aligned}$$

For a configuration we will use $q_A(u,m) \geq 0$. Its derivative in m is strictly negative, and

$$q_A(u,u) = -\int w_\theta(s) b_0(us)[\beta_u(s) - \beta_1(s)]\,\mathrm{d}s < 0.$$

For sufficiently small $m > 0$, the positive term $(u-m)k(u)$ alone dominates $umU(u)$. Thus there is a unique root $m_A \in (0,u)$. For every configuration geometry $v \geq m$ and $r \leq v$, positive head and boundary-tail sampling give

$$H_{A_u}(v,r) \leq H_{A_u}(v,v) = T_{A_u}(0,v) \leq T_{A_u}(0,m) = H_{A_u}(m,m).$$

Thus $q_A(u,m) \geq q(v,m,r) \geq 0$ directly from the equations at the selected bodies, and strict mass decrease implies $m \leq m_A$. On $G_u(u) > 0$, the identical sampling chain for the clipped kernel gives $H_{J_u}(u+v,r) \leq T_{J_u}(u,m)$. Adding the first atom and increasing head sampling therefore bound the comparison functional by $\mathcal{B}_u + H_{J_u}(u,m_A)$, for inner and outer geometries alike. If $G_u(u) \leq 0$, the original negative baseline and one-crossing suffice.

Fix p,u,L as θ runs from the power law at 0 to the circle law at 1. The root $m_A(\theta)$ moves according to $q_A(u,m_A;\theta) = 0$. Define the density deficit

$$e(s) = \rho_p(s) - w_{p,L}(s), \qquad \partial_\theta w_\theta = -e, \qquad S = \int_0^\infty e(s)\,\mathrm{d}s.$$

The last identity follows from (4.5) by monotone convergence, since $e \geq 0$. The ratio e/w_θ decreases because $w_{p,L}/\rho_p$ increases by Lemma 4.2 for $p > 1$ and by the tanh formula for $p = 1$.

Lemma 7.5 (The needed adjacent-root bounds)**.** *For $1 \leq p \leq 9/5$ and every law in* (7.5)*,*

$$\max(0, 2u-1) < m_A < u.$$

Proof. Only $u > 1/2$ needs proof beyond the root construction above. Put $\delta = 1-u$ and $m = 2u-1$. Convexity of β gives

$$\beta_m(s) - \beta_u(s) > \delta C(us), \qquad \beta_u(s) - \beta_1(s) < \delta C(us).$$

Use the same frequency function N whose complete negative moment was proved in Lemma 7.4. Its exact identity is

$$(1+x)e^{-x} + [a_0(x) - b_0(x)]C(x) = -N(x).$$

Subtract the adjacent diagonal value from its value at m and use the displayed bounds. Including the self term gives

$$q_A(u, 2u-1) > \delta\left\{S_\theta - \int_0^\infty w_\theta(s) N(us)\,\mathrm{d}s\right\} > 0.$$

Strict decrease of q_A proves the lower bound. For the pure law the quantity in braces equals $u^{-p}(Q - 2P_m - 1)$, where $P_m = p(p-1)\zeta(p+1)$ and $Q = p(p+1)\zeta(p+2)$. We use this power-law value in Lemma 7.9. □

Let

$$\mathcal{L}_A(u; w_\theta) = \mathcal{B}_u + m_A \sum_{j \geq 0} [G_u(u + j m_A)]_+.$$

The next result allows its sign to be checked with the power law.

Lemma 7.6 (Adjacent-root comparison)**.** *If $\mathcal{L}_A(u;\rho_p) < 0$, then the comparison functional is strictly negative for every configuration bounded by this adjacent-root mesh.*

Proof. If $G_u(u) \le 0$, the one-crossing property makes every following positive part vanish. Moreover

$$Z - k(1) - aU(1) = \int w_\theta(s) e^{-s} \left[\frac{s}{1 - e^{-s}} - 1 - as \right] \mathrm{d}s > 0,$$

since $s/(1 - e^{-s}) > 1 + s/2$ and $a \le 6/25$. Hence $\mathcal{B}_u < 0$. It remains to consider portions of the law path where $G_u(u) > 0$. Apply the two sampling monotonicities of Lemma 4.6 to $J = [G_u]_+$: since $H_J = mJ(u) + T_J$, they give

$$0 \le \partial_m \left(m \sum_{j \ge 0} [G_u(u + jm)]_+ \right) \le G_u(u).$$

This includes one-sided derivatives at a clipping contact. Set $\delta = 1 - u$, $X = e^{-s}$, $Y = e^{-us}$ and

$$A_0 = 1 - (1 + as)X, \quad \gamma = A_0 - zs, \quad B = \beta_u(s) - \beta_1(s), \quad D = u + \beta_u(s) - m - \beta_m(s).$$

Let $\mathcal{L}_A^{\mathrm{sgn}}$ denote the comparison functional with the full signed right mesh. The geometric sums give its density $u[X + asX - sX/(1 - X)] - Y(m + \beta_m(s))\gamma$. Adding δ for the self term and simplifying gives R_L below. For q_A, combine its diagonal value at $m = u$ with the change from u to m. Thus the fixed-mass derivatives of $\mathcal{L}_A^{\mathrm{sgn}}$ and q_A are $-\int e R_L \, \mathrm{d}s$ and $-\int e R_q \, \mathrm{d}s$, respectively, where

$$R_L = \delta A_0 + \gamma(YD - B), \qquad -R_q = b_0(us)B + a_0(us)D. \tag{7.12}$$

For clarity, write $D_A = -q_{A,m} > 0$ and use a partial θ derivative to mean that the mass is held fixed. Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}\theta} q_A(m_A(\theta), \theta) = -\int e R_q \, \mathrm{d}s - D_A m'_A, \qquad m'_A = -\frac{\int e R_q \, \mathrm{d}s}{D_A}.$$

The fixed-mass comparison functional derivative is at most $-\int e R_L \, \mathrm{d}s$. Adding the chain-rule term $(\partial_m \mathcal{L}_A) m'_A$ therefore gives, almost everywhere,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathcal{L}_A \le -\int e(s)[R_L + \lambda R_q] \, \mathrm{d}s, \quad \lambda = \frac{\partial_m \mathcal{L}_A}{-q_{A,m}}.$$

At each site where $G_u(t) \le 0$, its negative-to-positive density crossing and the decreasing weight e/w_θ give $\int e(s) e^{-ts} s g_u(s) \, \mathrm{d}s \le 0$. The signed site therefore has nonnegative fixed-mass law derivative. Clipping it to zero can only decrease that derivative, which gives the inequality above. Since $u < 1$, both endpoint kernels are negative beyond a common finite distance, and so are their convex combinations. The clipped sum is locally finite along the positive root path. For the signed sum the power-law bound on the tail justifies termwise differentiation.

The kernel elasticity and $w_\theta(s)/s$ decreasing give, with $R = u/(1 + u)$,

$$0 \le \lambda \le \frac{G_u(u)}{uU(u) + k(u) + S_\theta} \le \frac{1 + R^2 - 2(a/u)(1 - R^3)}{3} < \frac{83}{300}.$$

Here is a direct signed-moment proof of the middle inequality. After dropping the positive self term from the denominator, the ratio is the quotient of the integrals with densities $w_\theta e^{-us} s g_u(s)$ and $w_\theta e^{-us}(us + 1)$. On the positive part of g_u, the function $s g_u(s)/(us + 1)$ strictly increases, since both factors are positive increasing. For a positive constant c, the difference $c(us + 1) - s g_u(s)$ therefore has at most one crossing, from positive to negative. Choose c to make its integral against $s e^{-us}$ zero. Using $\int_0^\infty s^j e^{-ts} \, \mathrm{d}s = j! t^{-j-1}$ gives exactly $c = [1 + R^2 - 2(a/u)(1 - R^3)]/3$. Multiplication by the decreasing weight $w_\theta(s)/s$ makes the

integral nonnegative, proving the ratio bound. The resulting rational expression increases with u and equals $83/300$ at $u = 1$. Also, with $x = us$ and $E(x) = 1 - C(x)$, the root bound gives

$$B = \int_u^1 C(ts)\,\mathrm{d}t \le \delta C(x), \qquad D = \int_m^u E(ts)\,\mathrm{d}t < \delta E(x).$$

Here C decreases, E increases, and $u - m < \delta$. We keep these two bounds together. If $\gamma \ge 0$, then

$$R_L + \lambda R_q \ge \delta\{A_0 - \gamma C(x) - \lambda[b_0(x)C(x) + a_0(x)E(x)]\} \ge \delta\{(z - \lambda u)s + \gamma E(x)\} > 0,$$

since $0 < a_0(x) < b_0(x) < x$, $C + E = 1$, and $z/u \ge 13/25 > 83/300 > \lambda$.

If $\gamma < 0$, put $g = -\gamma > 0$. Dropping $gB > 0$ and using $B \le \delta C(x)$, $D < \delta$, gives

$$R_L + \lambda R_q \ge \delta\{A_0 - Yg - \lambda K(x)\}, \qquad K(x) = a_0(x) + b_0(x)C(x).$$

Write $\sigma = a/u \le 12/25$. Since $X \le Y = e^{-x}$,

$$\frac{A_0 - Yg - (1 - Y)/2}{Y} \ge \sinh x - x + \frac{1 - e^{-x}}{2} - \sigma x e^{-x} > 0.$$

The last sign uses $\sinh x > x$ and $e^x > 1 + x$. For the other term put $T = h_0(x) - 1$. The common envelope (6.1) gives $T < x/2 + x^2/12$. Now $K = 1 - (1 + x)e^{-x} + e^{-x}T^2$, and the positive Taylor sum through fourth degree gives

$$\frac{3}{2}(1 - e^{-x}) - K > e^{-x}\left\{\frac{e^x - 1}{2} + x - \frac{x^2}{4} - \frac{x^3}{12} - \frac{x^4}{144}\right\} > e^{-x}\left(\frac{3x}{2} + \frac{x^4}{72}\right) > 0.$$

Consequently $R_L + \lambda R_q > (17/200)\delta(1 - Y) > 0$. The comparison functional therefore decreases on every active portion of the law path. At a boundary of such a portion it is already negative by the first paragraph. This proves the implication from the power-law value. □

7.5 The adjacent mesh away from the unit gap

We prove the required pure-law sign on $2/3 \le u \le 7/8$. The estimate uses one explicit comparison mass on the whole exponent interval. Put

$$\begin{gathered} p_{\max} = 9/5, \quad U_* = 7/8, \quad b_* = 2304/8575, \quad c = 3/20, \\ \bar\tau = 5/7 + b_* u^2 + c(p_{\max} - p), \qquad \bar m = u\bar\tau. \end{gathered} \tag{7.13}$$

The proof has two parts: the root is below $\bar m$, and the complete clipped mesh at $\bar m$ has negative value.

For the power law the adjacent root equation separates as

$$\widetilde q(p, u, \tau) = \mathcal{F}_p(\tau) + \mathcal{P}_p(u) + 1 - (p + 1)\tau = 0, \tag{7.14}$$

where the convergent positive formulas are

$$\mathcal{F}_p(\tau) = \tau p(p + 1)\sum_{j\ge1}\int_0^1 t(j\tau + t)^{-p-2}\,\mathrm{d}t,$$

$$\mathcal{P}_p(u) = u^{-1}p(p + 1)\sum_{j\ge1}\int_0^1 (1 - t)(j/u + t)^{-p-2}\,\mathrm{d}t.$$

They follow by integrating a_0 and b_0 twice. In particular $\widetilde q_\tau < 0$ and $\widetilde q_{\tau\tau} > 0$.

Lemma 7.7 (The cubic root bound)**.** *For $1 \le p \le p_{\max}$ and $0 < u \le U_*$, the power-law adjacent root satisfies $m_A < \bar{m}$.*

Proof. We first give the endpoint at $p = p_{\max}$. Let $\mathcal{A}_p(\tau) = (p+1)\tau - 1 - \mathcal{F}_p(\tau)$. This is increasing and concave, so its inverse is increasing and convex. If τ_0 is its zero and $\tau_U = m_A(p_{\max}, U_*)/U_*$, then

$$m_A(p_{\max}, u)/u \le \tau_0 + (\tau_U - \tau_0)\mathcal{P}_{p_{\max}}(u)/\mathcal{P}_{p_{\max}}(U_*).$$

The three bounds needed here are

$$\tau_0 < 5/7, \qquad \tau_U < 23/25, \qquad \mathcal{P}_{p_{\max}}(u)/\mathcal{P}_{p_{\max}}(U_*) \le (u/U_*)^2. \tag{7.15}$$

For the first, set $c_0 = 5/7$ and $A_1(x) = x^{-p_{\max}} - (x+1)^{-p_{\max}} - p_{\max}(x+1)^{-p_{\max}-1}$. The Euler–Maclaurin upper for this completely monotone function gives

$$\mathcal{F}_{p_{\max}}(c_0) < c_0 A_1(c_0) + \int_{2c_0}^{\infty} A_1(x)\,\mathrm{d}x + (c_0/2)A_1(2c_0) - (c_0^2/12)A_1'(2c_0).$$

The upper is valid termwise for exponentials and hence for their positive Laplace integral. Its exact value is

$$(7/5)^{4/5} + (123/80)(7/10)^{4/5} - (41/28)(7/12)^{9/5} \\ - (5/4 + 19/34 + 141/1156 + 147/9826)(7/17)^{4/5} < .982 < 1.$$

For the inequality use the fifth-power bounds $33/25, 151/200, 3/8, 49/100$ on the indicated upper, upper, lower, lower sides, and bound the last coefficient below by $97/50$. Since $\mathcal{P}_{p_{\max}}(u) \to 0$ as $u \downarrow 0$, this gives $\tau_0 < 5/7$. For the second bound, integrate the Peano summands in (7.14). At $x = j\tau$ and $y = j/u$ they are

$$\tau[x^{-p} - (x+1)^{-p} - p(x+1)^{-p-1}], \qquad u^{-1}[py^{-p-1} - y^{-p} + (y+1)^{-p}].$$

Replacing $j\tau + t$ and $j/u + t$ by $j\tau$ and j/u in the integrands, the integral test bounds the tails after N by $(p/2)(N\tau)^{-p-1}$ and $(p/2)(u/N)^{p+1}$. At $(p, u, \tau) = (9/5, 7/8, 23/25)$ with $N = 8$, the partial sums are below .636 and .919, and the tails below .0034 and .0019. For these bounds round each power to adjacent multiples of 10^{-8} by fifth-power comparison, using the lower ends for subtracted terms. Substituting the four bounds gives

$$\widetilde{q}(p_{\max}, 7/8, 23/25) < -.0157. \tag{7.16}$$

For the last bound in (7.15), integrate the positive Peano formula and differentiate its convergent series:

$$u\mathcal{P}'_{p_{\max}}(u) - 2\mathcal{P}_{p_{\max}}(u) = u^{p_{\max}-1}\sum_{j\ge 1} j^{-p_{\max}} g(u/j),$$
$$g(x) = (p_{\max} - 3)[(1+x)^{-p_{\max}} - 1] + p_{\max}(p_{\max} - 2)x - p_{\max}x(1+x)^{-p_{\max}-1}.$$

Here $g(0) = g'(0) = 0$ and

$$g''(x) = p_{\max}(p_{\max}+1)(1+x)^{-p_{\max}-3}(p_{\max} - 1 - 3x), \qquad g'(1) = \frac{9}{25}(2^{1/5} - 1) > 0.$$

Thus g' first increases and then decreases on $[0, 1]$, and is strictly positive on $(0, 1]$. Hence $g(x) > 0$ for $0 < x \le 1$. As $u/j \le 7/8 < 1$, every summand is positive. Consequently

$u\mathcal{P}'_{p_{\max}} > 2\mathcal{P}_{p_{\max}}$; integrating proves the last bound in (7.15). The endpoint cubic ceiling follows.

We propagate the source inequality through the whole exponent interval. Its complete negative integrand is

$$h_p(x) = x^{-p-2}[2(2p+1)\log x - p(p+1)(\log x)^2 - 2]_+.$$

Put $r = p/(p+1) \in [1/2, 9/14]$, $t = (p+1)\log x$, and $Q_r(t) = -rt^2 + 2(1+r)t - 2$. We prove the three estimates together:

$$\int_0^\infty h_p(x)\,\mathrm{d}x < 1/3, \qquad \sup h_p < 1/4, \qquad \sup x h_p < 2/5. \tag{7.17}$$

The integral is $J(r) = (1-r)\int e^{-t}[Q_r(t)]_+\,\mathrm{d}t$. Writing $w = \sqrt{1+r^{-2}}$, direct integration gives

$$J(r) = 4r(1-r)e^{-1-1/r}(w\cosh w - \sinh w).$$

Its logarithmic derivative is $1/r - 1/(1-r) + 1/r^2 - \sinh w/[r^3(w\cosh w - \sinh w)] < 0$: indeed $w > 5/3$ and $\tanh w > 3/4 > \sqrt{1+r^2}/(1+r)$. At $r = 1/2$, using $2 < \sqrt{5} < 9/4$, $e^{3/4} > 2$ and $e^5 > 100$,

$$J(1/2) = \tfrac{1}{2}[(\sqrt{5}-1)e^{-3+\sqrt{5}} + (\sqrt{5}+1)e^{-3-\sqrt{5}}] < 131/400 < 1/3.$$

The two heights are the maxima of $e^{-(2-r)t}[Q_r(t)]_+$ and $e^{-t}[Q_r(t)]_+$. Each has one maximum. The first occurs before $t = 1 + 1/r$, where its r derivative is $t^2e^{-(2-r)t}(1 + 2r - rt) > 0$; the second occurs before $t = 2$, where its r derivative is $e^{-t}t(2-t) > 0$. Thus both heights increase with r, and it suffices to use $r = 9/14$. For $(L, \eta) = (1/4, 19/14)$ or $(2/5, 1)$, expand $L(1 + \eta t + (\eta t)^2/2 + (\eta t)^3/6) - Q_{9/14}(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$. Its exact certificates are

(L,η)	$c_1 + 2c_2$	$c_0 + c_1 + c_2$	$4c_0(c_2+c_3) - c_1^2$
$(1/4, 19/14)$	$-941/784$	$277/1568$	$39/343$
$(2/5, 1)$	$-6/5$	$5/14$	$99/245$

Since $c_2, c_3 > 0$, on $[0,1]$ drop the cubic and use the decreasing quadratic; on $[1,\infty)$ replace t^3 by t^2 and complete the square. The exponential lower proves the two heights in (7.17). For nonnegative unimodal h, a mesh of step d satisfies $d\sum_j h(x_j) \le \int h + d\sup h$, with the supremum taken on the sampled ray. Compare rectangles to the integral in the direction of the largest sample, leaving only that rectangle. The negative part of the second p derivative of $p(p+1)x^{-p-2}$ is $h_p(x)$. In (7.14), take $d = \tau$ for the first sum and $d = 1/u$ for the second. In the second sum $x \ge d$, so the rectangle term is bounded by $\sup x h_p(x)$. The weights t and $1-t$ each integrate to $1/2$. Thus for $0 < \tau \le 26/25$ and $0 < u \le U_*$,

$$\partial_p^2 \mathcal{F} > -\tfrac{1}{2}(1/3 + \tau/4) \ge -89/300, \qquad \partial_p^2 \mathcal{P} > -\tfrac{1}{2}(1/3 + 2/5) = -11/30. \tag{7.18}$$

Throughout the rectangle, $0 < \bar\tau \le \bar\tau(1, U_*) = 26/25$. Put $I = -\mathcal{F}_\tau = \int \rho_p(s)a_0(s)C(\tau s)\,\mathrm{d}s$. After $t = \tau s$, the multiplier of the Gamma$(p,1)$ density in $\tau^p I$ is $a_0(t/\tau)e^t C(t)$, which increases with t. Indeed a_0 increases, and $e^t C(t) = b_0(t)/(1-e^{-t})^2$ has positive derivative numerator $1 - 2te^{-t} - e^{-2t} > 0$. Covariance with $\log t$ therefore gives

$$I_p \ge -(\log\tau)I.$$

For $\tau \le 1$ this is nonnegative. For $1 \le \tau \le 26/25$, decrease of C and $a_0(s) < (1-e^{-s})^2$ give

$$I(p,\tau) \le I(p,1) < \int \rho_p(s)e^{-s}b_0(s)\,\mathrm{d}s = p - 1 + 2^{-p} < p.$$

As $\log(26/25) < 1/25$, it follows that $I_p > -9/125$ throughout. Since $\bar\tau_p = -c$ and $\mathcal{F}_{\tau\tau} > 0$, exact differentiation gives

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\widetilde{q}(p,u,\bar\tau) = \partial_p^2\mathcal{F} + \partial_p^2\mathcal{P} + 2c(1+I_p) + c^2\mathcal{F}_{\tau\tau} > -\frac{2887}{7500} > -\frac{2}{5}. \tag{7.19}$$

At $p=1$, put $c_1 = 146/175$. The exact series $\mathcal{F}_1(c_1) = \sum_{j\ge1} 1/[j(jc_1+1)^2]$ is less than .435: the first eight terms are at most .29722, .07022, .02717, .01330, .00749, .00463, .00306, .00213; the remaining decreasing tail is less than $\log(1+175/1168) - 175/1343 < .00932$. On $0 < \tau \le 26/25$, the literal derivative is

$$-\widetilde{q}_\tau(1,u,\tau) = 2 + 2\sum_{j\ge1}(j\tau+1)^{-3} > 2 + 2(51/25)^{-3} > 11/5.$$

Also

$$\mathcal{P}_1(u) = u^2\sum_{j\ge1}\frac{1}{j^2(j+u)} \le u^2[1/(1+u) + .2021 - .059u].$$

For this bound use $\zeta(3) < 1.2021$ and retain $j = 2,\dots,9$ in $\sum_{j\ge2} 1/[j^3(j+7/8)] > .0590054$. Since $\bar\tau(1,u) = c_1 + b_*u^2$, these estimates give, for every $0 < u \le 7/8$,

$$\widetilde{q}(1,u,\bar\tau) < .435 + 1 - \frac{292}{175} + .054 + \frac{1}{5}\frac{36}{175} = -\frac{969}{7000}.$$

Here $b_*u^2 \le 36/175$ and $2.4 - 11/5 = 1/5$. The bound .054 applies to $u^2[1/(1+u) - .4427 - .059u]$ on $[0, 7/8]$. It increases up to $1/2$, with value less than .049 there. Above $1/2$, multiplying the desired inequality by $1+u$ and putting $v = u - .6$ gives $.0017856 - .021948v + .4732v^2 + .6433v^3 + .059v^4$, which is at least $.0017856 - .021948v + .40887v^2 > 0$. Put $\theta = (p-1)/(p_{\max}-1)$. The semiconvex chord from (7.19), with the already proved negative $p_{\max}$ endpoint, gives for $p < p_{\max}$

$$\widetilde{q}(p,u,\bar\tau) < (1-\theta)\left[-\frac{969}{7000} + \frac{16}{125}\theta\right] \le -\frac{73}{7000}(1-\theta) < 0.$$

At $p = p_{\max}$ use the strict endpoint itself. Thus the decreasing source has its root below $\bar\tau$ on the entire exponent interval.

□

Lemma 7.8 (The full adjacent value)**.** *For the power law and $1 \le p \le 9/5$, $2/3 \le u \le 7/8$,*

$$u(1+pa-Z) + \bar m\sum_{j\ge0}[G_u(u+j\bar m)]_+ < 0, \qquad Z = p\zeta(p+1).$$

Consequently both the original inner comparison functional and the original outer comparison functional are strictly negative on this range of u.

Proof. We keep the mass (7.13). Appendix D proves the assertion at $p = p_{\max}$, including its entire tail. The $j \ge 6$ extinction in (D.4) applies for smaller p: the ratio $s^{p-p_{\max}}$ decreases and the density of G has one crossing from negative to positive. Moreover $\bar m(p,u) \ge \bar m(p_{\max},u)$. Hence throughout the stated range only the indices $j = 0,\dots,5$ can be positive.

Write

$$K(p,u) = u(1+pa-Z) + \bar m G_u(u), \qquad R(p,u) = \bar m\sum_{j=1}^{5}[G_u(u+j\bar m)]_+.$$

We shall use the complete logarithmic-moment bound

$$Z''(p) < 1/8 \qquad (1 \le p \le 9/5). \tag{7.20}$$

The common Gamma-shape argument in Lemma E.1 proves this bound on the larger interval $[1,2]$, together with every logarithmic-moment bound subsequently used. We retain $p \le 9/5$ here.

We next control the exponent response of the following atoms. At any active site x, normalize $G_u(x)$ by px^{-p-1}, obtaining

$$F = z + a\left(\frac{x}{x+1}\right)^{p+1} - \int_0^1 (1+t/x)^{-p-1}\,\mathrm{d}t.$$

Differentiation gives $F_p \ge (1/2-a)[x/(x+1)]^{p+1}\log(1+1/x) > 0$; use $\log(1+t/x) \ge t\log(1+1/x)$ under the integral. If $\log x \le 1/p$, it follows that $\partial_p G_u(x) > 0$. Otherwise put $h(t) = t^{-p-1}(p\log t - 1) > 0$; the exact derivative is

$$\partial_p G_u(x) = \int_x^{x+1} h(t)\,\mathrm{d}t - zh(x) - ah(x+1).$$

Its logarithmic decay rate is less than $1/2$. The maximum increases with p, and at $p_{\max}$ it occurs at the root $y > 4/5$ of $(14/5)y^2 - y - 1 = 0$, where $y = \log t - 5/9$; its value is $y^{-2}e^{-y-5/9} < (25/16)e^{-61/45} < 1/2$. Thus $h(x)/h(x+1) < 5/3$. Also

$$h'' = p(p+1)(p+2)t^{-p-3}\left[\log t - \frac{1}{p} - \frac{1}{p+1} - \frac{1}{p+2}\right] < 1/50.$$

The positive part is bounded by $20e^{-4y}(y-7/6)$, whose maximum is $5e^{-17/3} < 1/50$. The trapezoidal remainder now gives

$$\partial_p G_u(x) > (211/800)h(x+1) - (111/800)h(x) - 1/600 > -1/600.$$

For $j = 1$ this derivative is positive, because $x_1 \le U_*(2.19 - .15p)$ and $p\log[U_*(2.19-.15p)] \le p_{\max}\log 1.68 < 1$. The following mesh R decreases with its mesh step by the T_J monotonicity in Lemma 4.6, applied to $J = [G_u]_+$. Therefore the contribution from $\bar{m}_p = -cu$ is nonnegative when differentiating R. Only $j = 2,\dots,5$ can spend the preceding derivative bound, so

$$\partial_p R > -4\bar{m}/600 \ge -(13/1875)u, \quad R(p,u) \le R(p_{\max},u) + (13/1875)u(p_{\max}-p). \tag{7.21}$$

These inequalities pass through clipping contacts by continuity and local Lipschitz regularity.

The term K has a stronger curvature estimate. Put

$$s = .52 + .24u, \quad A = a/(1+u), \quad l = -\log u, \quad t = \log(1+u),$$

and let subscripts p denote exponent derivatives at fixed u. Then

$$\begin{aligned} G_{pp}(u) &= u^{-p}[2sl + (ps-1)l^2] + (1+u)^{-p}[(1+pA)t^2 - 2At], \\ K_{pp}/u &= -Z'' + \bar{\tau}G_{pp} - .3G_p. \end{aligned}$$

The shifted block in the last two terms is nonnegative, since $t > 2A$. The unshifted block is

$$u^{-p}\{s(2q-.3) + (ps-1)l(q-.3)\}, \qquad q = \bar{\tau}l.$$

On $[1/2, 3/4]$, $q > .248$, $s \ge .64$, $l < .694$. If $q \ge .3$, its brace is at least $.3s \ge .192$. If $q \le .3$, its brace is greater than $.64(2(.248) - .3) - .26(.694)(.3 - .248) > .116$; multiplication by $u^{-p} \ge 4/3$ makes the block greater than $.154$. Thus $K_{pp} > 0$ by (7.20). On $[3/4, 7/8]$ one instead has the sufficient estimate

$$K_{pp} > -.027u. \tag{7.22}$$

We provide the derivative bounds proving it. Here $.1228 < \bar\tau l < .284$, $.7 \le s \le .73$ and $.126 \le A < .129$. The unshifted part of $.45G_{pp} - \bar\tau G_{ppp}$ is positive, since its brace is at least

$$.7(.9 - 3(.284)) - .3(.288)(.45 - .1228) > 0.$$

For the shifted block $T_2 = (1+u)^{-p}[(1+pA)t^2 - 2At]$, $T_2 > .0878$ and $-T_{2,p} > .0345$ on this rectangle. Both follow by monotonicity in p, u and their corner $(p_{\max}, 3/4)$; the spatial derivative numerators are

$$2t - p_{\max}t^2 + A(4p_{\max}t - 2 - p_{\max}^2 t^2) + (1+u)A'(p_{\max}t^2 - 2t),$$
$$(1+p_{\max}A)t^2(3 - p_{\max}t) + At(3p_{\max}t - 6) + (1+u)A't^2(p_{\max}t - 3),$$

which are positive using $.559 < t < .629$ and $A' < 0$. The corner lower bounds are

$$.365[(431/350)(.559)^2 - (9/35)(.56)] > .0878, \quad .365(.559)^2[(431/350)(.559) - 27/70] > .0345.$$

Thus $.45G_{pp} - \bar\tau G_{ppp} > .069$. It follows, including $\bar\tau_p = -.15$, that $R_0 = \bar\tau G_{pp} - .3G_p$ decreases with p. At $p_{\max}$ it also decreases with u. To verify the latter, differentiating the two explicit power blocks gives

$$G_{ppu} < -2.304, \quad G_{pu} > -3.15, \quad G_{pp} < .81, \quad \bar\tau_u < .471, \quad \bar\tau > .865.$$

For example the unshifted numerator of G_{ppu} is $-1.04 - .48u - (1.744 + 1.248u)l + p_{\max}(.064 - .192u)l^2$; after its factor $u^{-p_{\max}-1}$ it is less than -2.548, while the shifted part is less than .244. The adverse unshifted part of G_{pu} is at most $2.24[1.34 + .144(.288)] < 3.095$, and its shifted part is less than .049. The displayed bounds yield $R_{0,u} < .471(.81) - .865(2.304) + .3(3.15) < 0$. Finally at $(p_{\max}, U_*)$, the direct fifth-power and logarithm bounds give $G_{pp} > .3598$, $G_p < .775$, and $\bar\tau = .92$. Hence $R_0 > .92(.3598) - .3(.775) = .098516$. Together with (7.20) this proves (7.22).

We can now finish using complete endpoint values. On $[2/3, 7/8]$ set

$$E(p,u) = K(p,u) + R(p_{\max}, u) + (13/1875)u(p_{\max} - p).$$

It bounds $K + R$ by (7.21). Appendix D proves $E(p_{\max}, u) < -1/2000$. At $p = 1$,

$$G_1(u) = \frac{37u^2 + 7u - 12}{25u(1+u)^2},$$

which is positive and increasing for $u \ge 1/2$. Use the complete ratio bound (D.5), not individual tangent estimates for individual summands. The ratio $K(1,u)/u$ increases on $[3/4, 7/8]$. With $\bar\tau(1, U_*) = 26/25$ and $\zeta(2) > 1.6449$,

$$E(1,u)/u < 1.23625 - 1.6449 + (26/25)G_1(U_*) + 7/80 + 52/9375 < -1/100.$$

On $[2/3, 3/4]$, use $G_1(3/4) = 12/49$ and $\bar\tau(1, 3/4) = 338/343$ to obtain

$$E(1,u)/u < 1.225 - 1.6449 + (338/343)(12/49) + 7/80 + 52/9375 < -1/100.$$

Convexity in p proves negativity on this last rectangle. On the upper rectangle use (7.22); for $\theta = (p-1)/(4/5)$ the semiconvex chord gives

$$E(p,u)/u < -(1-\theta)/100 - \theta/2000 + .00864\theta(1-\theta) < -1/2000.$$

This completes the pure-law estimate on $[2/3, 7/8]$.

Lemma 7.6 and the root bound transfer the sign to the actual adjacent mesh. The direct force-equation and complete sampling argument preceding Lemma 7.5 bounds both physical geometries by this mesh, without a joint root. By (7.2), every physical source in this branch has $u > 2/3$. Hence the physical necessity is impossible. □

7.6 Feasible comparison points near the unit gap

Let $7/8 \le u < 1$ and $\delta = 1 - u$. For the inner geometry we raise the comparison mass to at least $2u - 1$. For the outer geometry we use the cap m_R from (7.9). Both choices give a point (m, r) with $q_1(m, m, r) \ge 0$ and $H_1(m, r; m) = 0$. We first bound $m - r$ in two ways. The uniform bound permits comparison with the power law at fixed (m, r), and the stronger bound in terms of $u - m$ controls the power-law value. Its negative sign, after allowing for clipping, will contradict the comparison functional in both geometries.

First construct only the fourth root of Lemma 7.4, and set $q_D(m) = q(m, m, R_D(m))$ for $2u - 1 \le m \le u$. For $W(v, r) = H_{A_u}(v, r)$, its positive Laplace integrand gives $W_r > 0$ and $W_v + 2W_r < 0$: with $y = rs$,

$$h_0(y) - 2h_0'(y) = \frac{(y-2) + (y+2)e^{-y}}{(1-e^{-y})^2} > 0.$$

The numerator and its derivative vanish at zero, and its second derivative is $ye^{-y} > 0$. At $m = u$, $R_D(u) < u$ and positive head monotonicity give $q_D(u) < q(u, u, u) < 0$; the last sign follows from $T_{B_u}(0, 1) < T_{B_u}(0, u)$ after the complete u-mesh telescope.

Lemma 7.9 (Fourth-root strip and boundary sign)**.** *For every law* (7.5)*,* $7/8 \le u < 1$ *and* $2u - 1 \le m < u$*, put* $r = R_D(m)$*. Then*

$$0 < m - r < (4/5)\delta, \qquad r > 13/20, \qquad q_D(2u - 1) > 0. \tag{7.23}$$

Proof. The following moments refer to the power law:

$$P_m = p(p-1)\zeta(p+1), \quad Q = p(p+1)\zeta(p+2),$$
$$D = Q - P_m, \quad I = P_m + 1, \quad I_x = 2p + P_m - Q + 2^{-p}.$$

They are complete integral representations. In particular

$$I = \int \rho_p b_0 C, \quad D = p + 1 + \int \rho_p a_0 C, \quad I_x = \int \rho_p b_0(s) e^{-s} C(s)\, \mathrm{d}s.$$

Put $t = p-1$ and $\bar{x} = 1/2-(10641/40000)t$. Convexity gives $2^{-p} \le \bar{x}$, and the Euler–Maclaurin upper at two gives

$$P_m \le \bar{P} = p(p-1) + (p-1)\bar{x}[1 + p/4 + p(p+1)/48].$$

The complete lower $\zeta(19/5) > 1.09751$ gives $Q > Q_- = 1.09751p(p+1)$. These moment bounds can be obtained by the positive-power enclosure procedure of Appendix D. The rational polynomial

$$(9/5)Q_- - (14/5)\bar{P} - 2p - 1 - \bar{x}$$

decreases on $0 \le t \le 4/5$ and is greater than .037 at its right endpoint. To check monotonicity, its derivative is convex and is less than $-.415$ and $-.47$ at the two endpoints, respectively. Consequently

$$(I + I_x)/D < 4/5, \qquad Q - 2P_m - 1 > 6/7. \tag{7.24}$$

For the second inequality the same rational bounds give

$$Q - 2P_m - 1 \ge 1.19502 - .52775875(16/25) > 6/7;$$

the discarded cubic and quartic coefficients are positive.

Temporarily set $m = 2u - 1$. At $r = m$, direct collection of (7.10) gives

$$-H(m, m; m) = \int \rho_p(s) b_0(ms)\{\beta_m(s) - \beta_u(s) + e^{-us}[\beta_u(s) - \beta_1(s)]\}\, \mathrm{d}s.$$

Since C decreases, this is at most $\delta m^{-p}(I+I_x)$. On the other hand $-H_r \geq m^{-p}D$ for $r \leq m$. Integrating this derivative to the root gives $m - R_D(m) < (4/5)\delta$. More generally, if $m < u$ and $V = (1-m)/\delta \leq 2$, the same proof gives the sharper bound $m - R_D(m) \leq \delta[(V-1)I + I_x]/D < (4/5)\delta$.

To test the adjacent equation at $m = 2u-1$, first retain its diagonal head $r = m$. The calculation in Lemma 7.5 gives $q_A > \delta u^{-p}(Q - 2P_m - 1)$. Replacing that head by $r = R_D(m)$ reduces it by at most

$$(4/5)\delta(u/m)^2 m^{-p} B_*, \qquad B_* := \int \rho_p a_0(s) e^{-s}(1 - C(s))\,\mathrm{d}s = p^2 + 3p + 1 + P_m - 2Q.$$

Here convexity in the head mass and $a_0(us) \leq (u/m)^2 a_0(ms)$ give the bound. Also $B_* < 1 - (1 + p/2)2^{-p} < 91/200$, using $1 - C < 1$. Since $u/m \leq 7/6$, the reduction after division by δu^{-p} is less than $(4/5)(7/6)^4(91/200) < 27/40$. Together with (7.24) this leaves more than $9/50$. Thus the joint adjacent equation is positive at $m = 2u-1$.

These root tests persist along the entire law path. For fixed geometry both q and H increase when ρ_p is replaced by $\rho_p - \theta e$, including their self terms. Explicitly, with $R = re^{-ms}/(1-e^{-rs})$, define the fixed-geometry response densities by $\partial_\theta q = -\int eR_q\,\mathrm{d}s$ and $\partial_\theta H = -\int eR_H\,\mathrm{d}s$. Geometric summation gives

$$\begin{aligned} -R_q &= b_0(us)[\beta_u(s) - \beta_1(s)] + a_0(us)\{u + \beta_u(s) - m - R\}, \\ -R_H &= b_0(ms)\{\beta_m(s) - \beta_u(s) + e^{-us}[\beta_u(s) - \beta_1(s)]\} \\ &\quad + a_0(ms)\{m + \beta_m(s) - r - \beta_r(s)\}. \end{aligned} \tag{7.25}$$

Both right sides are positive, by decrease of β and increase of $h + \beta_h(s)$ in h. At fixed m, increase of H and strict decrease in r imply $R_{D,\theta}(m) \geq R_{D,0}(m)$. Since q increases with its head mass, the reduced adjacent value at $m = 2u-1$ remains positive. For clarity, at fixed m the complete response is $\partial_\theta q_D = q_\theta + (q_r/(-H_r))H_\theta > 0$; only the fourth root is differentiated. The lower boundary is therefore positive for every law. The gap bound applies at every m in the strip, and fixed-m fourth-root order only decreases the actual gap. Thus $r > 2u - 1 - (4/5)\delta \geq 13/20$. □

Lemma 7.10 (The inner feasible point)**.** *For a physical inner geometry $m \leq v \leq u$ with $7/8 \leq u < 1$, set $M = \max\{m, 2u-1\}$ and $r = R_D(M)$. Then*

$$2u - 1 \leq M < u, \qquad q_D(M) \geq 0, \qquad H(M, r; M) = 0,$$

and the full comparison functional obeys

$$0 \leq \mathcal{B}_u + H_{J_u}(u, M). \tag{7.26}$$

Proof. If $m \leq 2u-1$, the boundary sign in Lemma 7.9 gives feasibility. Otherwise $m/v \geq m/u > 6/7$. The original $H \geq 0$ bounds the actual fourth mass by $R(v; m)$; increasing that mass preserves $q \geq 0$. Along this fixed-m root, $W_v + R_v W_r < W_v + 2W_r < 0$. Decreasing v to m therefore gives $q_D(m) \geq 0$. Since $q_D(u) < 0$, we have $m < u$. In both cases $M \geq m$. Positive-part packing first bounds the comparison functional by $\mathcal{B}_u + H_{J_u}(u, m)$, and increasing head sampling gives (7.26). □

7.7 A cone from the source deficit

For the power-law value estimate we need $m - r \leq 9(u-m)/10$. The adjacent equation supplies this stronger bound through its deficit. For $r = R_D(m)$ define

$$D_m = \int w_\theta(s)[(1 + ms)e^{-ms} + a_0(ms)C(ms)]\,\mathrm{d}s + S_\theta.$$

The complete outer deficit is $F = \delta I_q$, with I_q below.

Lemma 7.11 (Feasible-point cone)**.** *For every law* (7.5)*, let* $7/8 \le u < 1$*,* $u/2 < m < u$ *and* $r = R_D(m)$*. If* $F \le (u-m)D_m$*, then*

$$0 < m - r < \frac{25}{28}(u-m) < \frac{9}{10}(u-m).$$

In particular this holds whenever $q_D(m) \ge 0$*.*

Proof. First the pure moments of Lemma 7.9 satisfy $I < 3D/4$. Indeed, for $t = 9/5 - p \in [0, 4/5]$, the same $\bar{P}, Q_-$ give

$$\frac{3}{4}D - I > \frac{169847}{50000000} + \frac{33149569}{30000000}t - \frac{8202443}{32000000}t^2 - \frac{6078317}{38400000}t^3 + \frac{24829}{2560000}t^4 > 0.$$

The linear term pays the two negative terms using $t^2 \le 4t/5$ and $t^3 \le 16t/25$; its remaining coefficient is $31944771/40000000$. Write $w = w_\theta$, $e = \rho_p - w_{p,L}$ and retain $S_\theta = \theta \int e$. For $m > 0$ put

$$D_m = \int w[(1+ms)e^{-ms} + a_0(ms)C(ms)]\,\mathrm{d}s + S_\theta, \qquad I_m = \int w b_0(ms)C(ms)\,\mathrm{d}s.$$

The complete SELF identity gives

$$D_m - m^{-p}D = \theta \int e\, a_0(ms)[1 - C(ms)]\,\mathrm{d}s \ge 0, \qquad I_m \le m^{-p}I < 3D_m/4. \tag{AC1}$$

No constant integral has been split off: e is integrable.

Let $V(s) = \delta^{-1}\int_u^1 C(ts)\,\mathrm{d}t$ and

$$I_q = \int w b_0(us)V(s)\,\mathrm{d}s, \qquad X_H = \int w b_0(ms)e^{-us}V(s)\,\mathrm{d}s.$$

We claim $0 < X_H < I_q/7$ for $0 < m \le u$. The weight w is nondecreasing (including the constant pure weight at $p = 1$). Since $b_0(ms) \le b_0(us)$, monotone covariance with e^{-us} under the finite measure $b_0(us)V(s)\,\mathrm{d}s$ gives

$$\frac{X_H}{I_q} \le \frac{\int b_0(us)e^{-us}V(s)\,\mathrm{d}s}{\int b_0(us)V(s)\,\mathrm{d}s}.$$

For fixed $t \in [u, 1]$, put $A = t/u \in [1, 8/7]$ and $y = ts$. Convexity of b_0 and decrease of $b_0(y)/y^2$ imply $A^{-2}b_0(y) \le b_0(y/A) \le A^{-1}b_0(y)$. Also integration by parts gives $\int_0^\infty b_0(y)C(y)\,\mathrm{d}y = 1$. Thus the fixed-$t$ ratio is at most $(8/7)\int b_0(y)e^{-7y/8}C(y)\,\mathrm{d}y$. Since $b_0(y)/(1-e^{-y}) = h_0(y) - 1$, the common envelope (6.1) gives $b_0(y)/(1-e^{-y}) < y/2 + y^2/12$. Using $C = e^{-y}b_0(y)/(1-e^{-y})^2$, we obtain

$$\int b_0(y)e^{-7y/8}C(y)\,\mathrm{d}y < \int e^{-15y/8}(y^2/4 + y^3/12 + y^4/144)\,\mathrm{d}y = \frac{281344}{2278125} < \frac{1}{8}.$$

Positive integration in t proves the claim. All these measures are finite by their polynomial behavior at zero and exponential tail.

Set $h = u - m$, $d = m - r$ and $U = h^{-1}\int_m^u C(ts)\,\mathrm{d}t$, $Z = d^{-1}\int_r^m C(ts)\,\mathrm{d}t$. Then $U \le C(ms) \le Z$. Put $D_0(t) = k(t) + tU_{\mathrm{ker}}(t) + S_\theta$, where $U_{\mathrm{ker}} = -k'$, and define

$$A_q = D_0(u) + \int w a_0(us)U\,\mathrm{d}s, \qquad B_q = \int w a_0(us)e^{-ms}(1-Z)\,\mathrm{d}s,$$
$$D_H = D_0(m) + \int w a_0(ms)Z\,\mathrm{d}s, \quad I_H = \int w b_0(ms)U\,\mathrm{d}s.$$

Every quantity is positive. Integrating the complete adjacent and fourth-row derivatives gives the exact identities

$$q = hA_q - dB_q - \delta I_q, \qquad 0 = H = dD_H - hI_H - \delta X_H. \tag{AC2}$$

For the adjacent head the identity used is $\beta_m(s) - re^{-ms}/(1-e^{-rs}) = de^{-ms}(1-Z)$. The diagonal fourth-row value is $-hI_H - \delta X_H$. Pointwise order, including $a_0(us) \geq a_0(ms)$, gives

$$A_q \leq \int w[1 - a_0(us)(1 - C(ms))]\,\mathrm{d}s + S_\theta \leq D_m \leq D_H, \qquad I_H \leq I_m.$$

Each bracket is an integrable exponential expression, not a difference of separately divergent integrals. If $q \geq 0$, the first identity gives $F = \delta I_q \leq hD_m$. Under this deficit bound, the second identity, (AC1), and $X_H < I_q/7$ give $dD_m < hD_m(3/4 + 1/7) = (25/28)hD_m$, as required. □

Lemma 7.12 (The outer feasible point)**.** *At the physical law, the affine outer cap $m_R = u - F/M_u$ satisfies $2u - 1 \leq m_R < u$. With $r = R_D(m_R)$, it obeys $q_D(m_R) \geq 0$ and the cone of Lemma 7.11.*

Proof. The moment bounds above give $P_m < p$. For completeness, after setting $t = (p-1)/(4/5)$ the degree-four Bernstein coefficients of $p - \bar{P}$ are

$$1, \quad 209/240, \quad 1964957/3000000, \quad 542647/1500000, \quad 34351/12500000,$$

all positive. Since $w \leq \rho_p$ and C decreases,

$$0 < F \leq \delta u^{-p}(1 + P_m).$$

The retained self term gives

$$M_u = (p+1)u^{-p} + \int e(s)[1 - (1+us)e^{-us}]\,\mathrm{d}s \geq (p+1)u^{-p}.$$

Thus $0 < F/M_u < \delta$, which proves the claimed mass bounds. For $m = m_R$, decrease of $(1+x)e^{-x}$ on $x \geq 0$ gives $D_m \geq M_u$. Hence $F = (u-m)M_u \leq (u-m)D_m$, so the deficit form of Lemma 7.11 applies before any sign of $q_D(m)$ is needed. It gives $r > r_0 = 2m - u > 0$.

Write $W(v, r) = H_{A_u}(v, r)$. Both $r/(1-e^{-rs})$ and $-x/\sinh x$ strictly increase on the positive half-line. The common-midpoint identity consequently gives

$$\frac{r_0 e^{-ms}}{1 - e^{-r_0 s}} = \frac{r_0 e^{-us/2}}{2\sinh(r_0 s/2)} > \frac{ue^{-us/2}}{2\sinh(us/2)} = \frac{ue^{-us}}{1 - e^{-us}}.$$

Integration against $w(s)a_0(us) > 0$ yields $W(m, r) > W(u, u)$. The complete outer telescope is

$$q_D(m) = (u-m)M_u - F + W(m, r) - W(u, u).$$

Its first two terms cancel at $m = m_R$, proving $q_D(m_R) > 0$. □

7.8 A comparison with fixed multipliers at the actual geometry

Write $\mathcal{L}_\theta(m) = \mathcal{B}_u + m\sum_{j\geq 0} G_u(u + jm)$, with every kernel taken in law (7.5). Fix the canonical constants

$$\begin{gathered} \lambda_* = 89/500, \qquad \mu_* = \lambda_* B_*(9/5)/D(9/5), \\ \Phi_\theta(m, r) = \mathcal{L}_\theta(m) + \lambda_* q_\theta(m, m, r) + \mu_* H_\theta(m, r; m). \end{gathered} \tag{7.27}$$

The complete moments just defined give $.017762 < \mu_* < .017765$. At a feasible point the positive q_1 multiplier makes Φ_1 an upper bound for the signed mesh, and the H_1 term vanishes. Both multipliers and both comparison masses stay fixed under the change of law.

Lemma 7.13 (Fixed-geometry dual comparison)**.** *For every point in the closed rectangle* $2u-1 \le m \le u$, $0 \le m-r \le 4\delta/5$,

$$\Phi_1(m,r) < \Phi_0(m,r) - \frac{\delta}{30}\int_0^\infty (\rho_p - w_{p,L})(1-e^{-s})\,\mathrm{d}s. \tag{7.28}$$

At an actual feasible point, $H_1 = 0$ *and* $q_1 \ge 0$ *give* $\mathcal{L}_1(m) \le \Phi_1(m,r)$.

Proof. Put $\lambda = \lambda_*$ and $\mu = \mu_*$ for this proof only. Both are fixed positive numbers, with $\lambda < 83/300$. Write $g(s) = s(z + ae^{-s}) - 1 + e^{-s}$.

We calculate the full law response. Set

$$X = e^{-s}, \quad Y = e^{-us}, \quad V = (1-m)/\delta \in [1,2],$$
$$q_s = \frac{1-e^{-\delta s}}{\delta(1-X)}, \quad K = e^{-ms}\int_r^m (1 - C(ts))\,\mathrm{d}t.$$

The retained right transform is $R = \beta_m(s) - K$. Define the auxiliary unit-mesh expression by

$$g_b = s(1-a+aX) - 1 + X, \quad B = sX(1-Y)/(1-X),$$
$$l = X(u-Y)(1+as) - B + mYg_b,$$
$$j = (u-m)Y - musY + uB,$$
$$\mathcal{F} = l + \lambda j + \delta - \lambda(u-m).$$

These are frequency functions; the product $musY$ means $m\,u\,s\,Y$. Substitution of the complete geometric sums gives the exact identity

$$R_{\mathcal{S}} + \lambda R_q = \mathcal{F} + \delta^2 sY(V - q_s) + [\beta_m(s) - \beta_1(s)][Yg + \lambda a_0(us)] - \lambda a_0(us)K. \tag{7.29}$$

For example, $R_{\mathcal{S}} = u(1+as)X - us\beta_1(s) + mYg/(1-e^{-ms}) + \delta$; inserting this and the source response in the proof of Lemma 7.9 verifies (7.29) by cancellation. All three source expressions are affine in the law at fixed arguments. The exact self identity $\int e = S$, with $e = \rho_p - w_{p,L}$, gives

$$\Phi_1(m,r) - \Phi_0(m,r) = -\int e(s)[R_{\mathcal{S}} + \lambda R_q + \mu R_H]\,\mathrm{d}s.$$

The $+\delta$ in $R_{\mathcal{S}}$ is its self correction. Neither q_0 nor H_0 is set to zero, and no root velocity occurs.

We pay the whole frequency ray without subdividing it. Keep the actual rectangle $m \ge 2u-1$. The function g_b is positive: it vanishes at zero and $g_b' = 1 - a - (1 - a + as)X > 0$ by $a < 1/2$ and $e^s > 1 + s$. Hence $\mathcal{F}_m = Yg_b + \lambda a_0(us) > 0$; replace m by $2u-1$, not the smaller value $3u-2$. Exact cancellation gives

$$\frac{\mathcal{F}}{\delta} = A_0 - \lambda(1-Y) + Y(g_b - \lambda us)(q_s - 2), \qquad A_0 = 1 - (1+as)X.$$

Put $\beta_s = s/(e^s - 1)$ and $\zeta_s = s/(1-X) = s + \beta_s$. Then $1 \le q_s \le \zeta_s$, $0 < \beta_s < 1$, and

$$\frac{g_b - \lambda us}{1-X} = (1 - a(1-X) - \lambda u)\zeta_s - 1.$$

The coefficient in parentheses is at least $291/500 > 1/2$. Consequently its product with $q_s - 2$ can be negative only when $q_s < 2$ and the other factor is positive; then $2 - q_s \le 1$. Moreover $a \ge 189/800$, $\lambda u \ge 623/4000 > 3/20$, and $1 - a - \lambda u < 5/8$. The base contributes at least $1 - a\beta_s - \lambda \ge 291/500$, so

$$\frac{\mathcal{F}}{\delta(1-X)} \ge \frac{291}{500} - e^{-7s/8}\left[\frac{5}{8}s - \frac{3}{20}\right]_+ > \frac{291}{500} - \frac{5}{21} > \frac{1}{3}. \tag{FR1}$$

Indeed the subtracted function has maximum $(5/7)e^{-121/100} < (5/7)e^{-181/160} < 5/21$; the positive exponential series through degree three gives $e^{181/160} > 3$. Thus

$$\mathcal{F} > \frac{1}{3}\delta(1-X). \tag{7.30}$$

Keep the next TWO terms of (7.29) together. The same positive g_b gives $g = g_b - \delta s > -\delta s$. Also $0 < \beta_m(s) - \beta_1(s) < V\delta/2$, since $0 < C < 1/2$. As $V \geq 1$ and $\lambda a_0(us) \geq 0$, their sum $\mathcal{N}$ obeys

$$\mathcal{N} \geq \delta^2 sY(V/2 - q_s) \geq -\delta^2 sY(q_s - 1/2).$$

The elementary whole-ray inequality

$$e^{-7s/8}\zeta_s(\zeta_s - 1/2) < 1 \tag{FR2}$$

therefore gives a single bound, for either sign of g:

$$\mathcal{N} > -\frac{1}{8}\delta(1-X). \tag{7.31}$$

For completeness, put $x = s/8$. Clearing positive denominators in (FR2) gives

$$P(x) = e^{16x} - 2e^{8x} + 1 - 64x^2e^{9x} + 4xe^{9x} - 4xe^x > 0.$$

Its factorial coefficients $c_n = P^{(n)}(0)$ have $c_0 = c_1 = 0$ and $(c_2, c_3, c_4) = (64, 576, 6784)$. For $n \geq 4$,

$$c_{n+1} - 9c_n = 7\,16^n + 2\,8^n - 128n\,9^{n-1} + 4\,9^n + 32n - 4 > 0.$$

Here $(16/9)^4 > 4(128/63)$ and multiplication by $16/9$ preserves $(16/9)^n > (128/63)n$ for $n \geq 4$. Thus all c_n for $n \geq 2$ are positive, proving (FR2) by its entire exponential series.

The head needs no separate exponential polynomial: $K \leq (m-r)e^{-ms}$ and $a_0(us) < 1 - e^{-us} \leq 1 - X$ give

$$\lambda a_0(us)K < \frac{4}{5}\lambda\delta(1-X).$$

This includes $m = r$, when the left side is zero.

Finally the exact source response (7.25) gives

$$0 < -R_H < \delta(1-X)(V/3 + (m-r)/\delta) \leq (22/15)\delta(1-X).$$

Use $b_0(x)C(x) < (1-e^{-x})/3$, $C(us) \leq C(ms)$ and $a_0(ms) < 1 - e^{-ms} \leq 1 - X$. [1] Since $\mu_* < 3553/200000$, the same complete normalized response is greater than

$$\frac{1}{3} - \frac{1}{8} - \frac{4}{5}\frac{89}{500} - \frac{22}{15}\frac{3553}{200000} = \frac{1}{30} + \frac{9817}{1500000} > \frac{1}{30}.$$

The fixed-geometry affine identity proves (7.28). Its integral is finite since $0 < 1 - e^{-s} < 1$ and $\int e = S$. □

[1] Clearing denominators gives $(e^x - 1)^3 - 3[(x-1)e^x + 1]^2 > 0$. Its first three coefficients vanish; the factorial coefficients are $c_n = 3^n - 3 \cdot 2^{n-2}(n^2 - 5n + 8) + 9 - 6n$ for $n \geq 3$. Here $(c_3, \ldots, c_7) = (6, 18, 30, 30, 42)$ and $c_{n+1} - 2c_n = 3^n - 3 \cdot 2^n(n-2) + 6n - 15 > 0$ for $n \geq 7$, since $(3/2)^n > 3(n-2)$ by induction.

7.9 A fixed-geometry value on the joint cone

We now bound the signed power-law value, then restore its clipped tail. The estimates use the rectangle earned in Lemma 7.9,

$$2u - 1 \le m \le u, \qquad 0 \le d = m - r \le 4\delta/5.$$

In this subsection $q = q(m, m, m-d)$ and $H = H(m, m-d; m)$. The signed mesh has no dependence on d:

$$S(u, m) = u(1 + pa - Z) + m \sum_{j \ge 0} G_u(u + jm).$$

Use the fixed constants in (7.27) and put $\Phi(m, d) = \Phi_0(m, m-d) = S + \lambda_* q + \mu_* H$; thus d is a gap coordinate, not the fourth mass r. The complete enclosures give $\lambda_*/\mu_* > 8/3$ and $\lambda_* - \mu_* > .1573$. No pure-root equation is assumed in the following value estimates.

Lemma 7.14 (Convexity at zero gap)**.** *At fixed p, u, the function $m \mapsto \Phi(m, 0)$ is strictly convex on $2u - 1 \le m \le u$.*

Proof. Let $h(s) = s/(1 - e^{-s})$. Then $h' = 1 - C$ and $h'' = -C' > 0$. At $d = 0$ the exact adjacent second derivative gives

$$q_{mm} = \int \rho_p(s) a_0(us) s h''(ms)\, \mathrm{d}s \ge m^{-p-1} K_p, \qquad K_p = \int \rho_p(s) a_0(s) s h''(s)\, \mathrm{d}s.$$

The function $e^s h''(s)$ increases, as its derivative has positive numerator $e^{2s} + (4 - 4s)e^s - 2s - 5$. Gamma order consequently gives $K_p/p \ge K_1 = 6[\zeta(3) - \zeta(4)] > 7/10$. For the last inequality retain $1/16 + 2/81 + 3/256 + 4/625$ and the integral tail $1/81$ in $\zeta(3) - \zeta(4)$. The diagonal identity

$$H(m, m; m) = \int \rho_p(s) b_0(ms) e^{-us} [u + \beta(1; s)]\, \mathrm{d}s - (Z - 1) m^{1-p}$$

has convex first term, hence $H_{mm} > -p(p-1)(Z-1)m^{-p-1}$. The common upper bound (6.1) gives $(p-1)(Z-1) < 132/125$. Therefore the source curvature is positive, since $(\lambda_*/\mu_*)K_p > (8/3)(7/10)p > (132/125)p$. The second derivative of S has integrand $\rho_p e^{-us} g(s) s h''(ms)$. After moving e^{-ms} into the signed weight its remaining multiplier $s e^{ms} h''(ms)$ is positive increasing. Apply signed comparison to the positive seed $G_u(u + m)$: $u + m \le 2$, $G_{1,7/8}(2) = 37/1920 > 0$, and raising u or p preserves the seed. Thus the second derivative of S is positive too. □

7.10 Bounding the clipping error

For the actual and every interpolated kernel write

$$c = 1/2 - a, \quad b = 1/6 - a/2, \quad A = c(p+1), \quad B = b(p+1)(p+2), \quad C_2 = \frac{A + \sqrt{A^2 - 4\delta B}}{2\delta}.$$

The square root is real. In fact the number $C_1 = A/\delta - (p+2)/2 > 2$ lies strictly between the two roots:

$$\delta C_1^2 - A C_1 + B = (p+2)[3\delta(p+2) - (p+1)]/12 < 0.$$

To explain why these roots control the tail, put

$$h_a(s) = 1 - a - \left[\frac{1 - e^{-s}}{s} - a e^{-s}\right].$$

This is positive increasing, and on the whole positive ray

$$h_a(s) > \frac{cs}{1 + s/2}, \qquad h_a(s) > cs - bs^2.$$

For the first inequality multiplication gives $2s[(1+s/2)h_a - cs] = (1+as)[s-2+(s+2)e^{-s}] > 0$. For the second, use the positive remainder $R_3(s) = 1 - e^{-s} - s + s^2/2$ and $R_3(ts) > t^3 R_3(s)$ for $0 < t < 1$. Then $h_a(s) - cs + bs^2 = \int_0^1 R_3(ts)\,\mathrm{d}t - aR_3(s) > 0$, since $a < 1/4$. Under the law with density $w_\theta(s)se^{-xs}/U(x)$, we have $G_u(x)/U(x) = \mathbb{E}h_a - \delta$. Its likelihood ratio against Gamma$(p+1, x)$ is increasing. Under that Gamma law, Cauchy–Schwarz gives $\mathbb{E}[s/(1+s/2)] \geq (\mathbb{E}s)^2/\mathbb{E}[s + s^2/2]$, so

$$\frac{G_u(x)}{U(x)} > \max\left\{\frac{A}{x + (p+2)/2},\ \frac{A}{x} - \frac{B}{x^2}\right\} - \delta.$$

The first bound gives positivity up to C_1 and the second extends it to C_2. For $x \geq C_2$, factor the negative of the second bound as $\delta(1 - C_2/x)(1 - C_-/x)$, where $C_- > 0$ is the smaller root. Dropping the last factor and using $U(x) \leq px^{-p-1}$ gives

$$[-G_u(x)]_+ \leq p\delta x^{-p-2}(x - C_2)_+.$$

The majorant is unimodal. Comparing samples to the cells toward its maximum bounds the mesh sum by its integral plus one sample. The integral is $\delta C_2^{-p}/(p+1)$. At the maximizing point $x = C_2(p+2)/(p+1)$ the factor $[(p+1)/(p+2)]^{p+1}$ is at most $4/9 < 1/2$, since $v\log(1+1/v)$ increases for $v > 0$. Thus for mesh step at most one the two contributions give the clipping bound

$$\mathcal{C}(p, \delta) = \delta C_2^{-p}\left[\frac{1}{p+1} + \frac{p}{2(p+2)C_2}\right]. \tag{7.32}$$

A uniform upper bound is $\mathcal{C} < \delta T_*(p)$, where

$$\alpha_*(p) = \frac{p+27}{16}, \qquad T_* = \alpha_*^{-p}(p+1)^{-p-1}\left[1 + \frac{p}{2\alpha_*(p+2)}\right]. \tag{7.33}$$

Indeed $C_2 > \alpha_*(p)(p+1)$: substitute this test point into the quadratic. Its value, divided by $p+1$, increases with $\delta \leq 1/8$; at $1/8$ it is

$$[\alpha_*^2/8 - (211/800)\alpha_*](p+1) + (233/4800)(p+2).$$

This is convex in p and negative at both endpoints, with values $-.011875$ and less than $-.01084$. The function T_* is convex. Writing $T_* = fh$ and $L = -f'/f$ gives $L = \log[\alpha_*(p)(p+1)] + 1 + p/(p+27)$, $2 < L < 3$, $L' < .6$, $h \geq 1$, $|h'| < .3$, $|h''| < .6$. Hence $T_*''/f = h(L^2 - L') - 2Lh' + h'' > 3.4 - 1.8 - .6 > 0$. Its endpoints satisfy $T_*(1) = 23/147$ and $T_*(9/5) < .0224$, the latter by a fifth-power comparison.

7.11 One value variable and two endpoints

Appendix G proves, throughout $2u - 1 \leq m \leq u$, $0 \leq d \leq 9(u-m)/10$,

$$\Phi(m, d) - \Phi(m, 0) \leq dm^{-p}[C_*(p) - \kappa_p(u-m)/m], \qquad \kappa_p > 0.$$

With $C_+ = [C_*]_+$, an upper on this cone is therefore

$$\Psi(m) = \Phi(m, 0) + \frac{9}{10}(u-m)m^{-p}C_+.$$

It is convex by Lemma 7.14 and $[(u-m)m^{-p}]'' = pm^{-p-2}[(p+1)u - (p-1)m] > 0$. The clipping error bound is independent of m, so only two values remain.

At $m = u$, Lemma E.3 and (E.3) give $[\Psi(u) + \mathcal{C}]/\delta < -.0097 - .54(p_{\max} - p) - .0043\delta < 0$. At $m = 1 - 2\delta$, retain the diagonal curvature of Lemma F.1. Also $m^{-p}C_+$ is convex in δ, and $\mathcal{C}/\delta^2$ increases: with $v = \delta C_2$,

$$v' = \frac{A'v - (B + \delta B')}{2v - A} < 0, \quad A'v < .125, \quad B + \delta B' \geq .28,$$

and (7.32) gives $\mathcal{C}/\delta^2 = \delta^{p-1}v^{-p}[1/(p+1) + p\delta/(2(p+2)v)]$. The curvature lemma gives convexity of $\Phi(1-2\delta, 0)/\delta$. The clipping quotient satisfies $\mathcal{C}/\delta \le (\delta/\delta_0)\mathcal{C}(p, \delta_0)/\delta_0$ for $\delta_0 = 1/8$. Combining this linear bound with the convex diagonal terms gives the chord from the unit limit to the upper bound at δ_0. The unit bound (E.3) with $k = 1$ and $c \le 4/5$ already pays $(4/5)C_+$, choosing $c = 4/5$ or 0 according to its sign. The extra tenth costs at most $.0069(p_{\max} - p)$, since $C_*(p_{\max}) = 0$ and $C_*' > -.069$. At the far endpoint, Lemma H.1 already pays a coefficient $1/10$ of $(3/4)^{-p}C_+$. The required coefficient is $(9/10)\delta_0 = 9/80$, leaving $1/80$. Thus the upper is

$$\mathcal{F}_0(p) + \frac{1}{80}(3/4)^{-p}C_+.$$

Lemma H.1 gives $\mathcal{F}_0 < -.00052 - .009(p_{\max} - p)$, while the additional term is less than $.00154(p_{\max} - p)$. Both endpoints of the chord are negative. Consequently $\Psi(m) + \mathcal{C} < 0$ on the full m interval.

Every feasible point supplied by Lemmas 7.10 and 7.12 lies in the rectangle and cone. At that point, $H_1 = 0$, $q_1 \ge 0$ and $\lambda_* > 0$, so

$$\mathcal{L}_1(m) + \mathcal{C} \le \Phi_1(m, r) + \mathcal{C} < \Phi_0(m, r) + \mathcal{C} \le \Psi(m) + \mathcal{C} < 0.$$

The strict inequality is the fixed-geometry comparison of Lemma 7.13. The clipping error bound controls the negative atoms discarded from $\mathcal{L}_1$, proving $\mathcal{B}_u + H_{J_u}(u, m) < 0$.

7.12 Returning to the two physical geometries

For the inner geometry, use $m = M$ from Lemma 7.10. The negative value just proved contradicts (7.26). For the outer geometry, use $m = m_R$ from Lemma 7.12. On the active region $G = G_u(u) > 0$, $J_u(u) = G$ and the complete clipped tail decreases with its step. Since $m_R < u$,

$$\mathcal{B}_u + m_R G + T_{J_u}(u, u) \le \mathcal{B}_u + m_R G + T_{J_u}(u, m_R) = \mathcal{B}_u + H_{J_u}(u, m_R) < 0,$$

contradicting (7.9). The value of the comparison functional supplies strictness even if the tail comparison is an equality. Both geometries therefore use the same feasible-point estimate; the seam $v = u$ is included in either case.

Theorem 7.15. *Every positive centered cocircular central configuration with $1 \le \alpha \le 9/5$ has equal masses and regular-polygon geometry.*

Proof. The small counts are covered by Proposition 2.4. For the remaining counts, an unequal-mass configuration in the attained incident normalization satisfies $2/3 < u < 1$ by (7.2). The negative baseline in (7.8) requires $G_u(u) > 0$. For $u \le 7/8$, the direct adjacent source bound and Lemma 7.8 contradict the comparison functional. For $u \ge 7/8$, the near-unit inner cap or affine outer cap applies according as $v \le u$ or $v \ge u$; the common feasible-point estimate then gives the contradiction. Thus unequal masses are impossible. Strict angular convexity then makes the equal-mass configuration regular. □

8 The mass-ratio bound for $9/5 \le p < 2$

To rule out a small mass ratio, we bound one conserved pressure from below at the attained edge and the maximum mass, then from above at the minimum mass.

Theorem 8.1. *For $9/5 \le p < 2$, every centered cocircular central configuration with positive masses satisfies $\min_i m_i / \max_i m_i > 1/13$. For $n \ge 3$, the two mass ratios across every adjacent edge are strictly less than $p + 1$.*

The assertion for two bodies follows from centering. In the rest of this section $n \ge 3$.

8.1 Normalization and conserved pressure

Among all endpoints of all cyclic gaps, choose a maximum of endpoint mass divided by gap length. Divide masses by the attaining mass and arc lengths by the attaining gap. Orient this edge as $(0,1),(1,b)$. Then $0<b\le 1$, and every gap is at least both endpoint masses. Write a_m and Q_m for the minimum and maximum mass in these units, $q=a_m/Q_m$, and L for the circumference. In particular $a_m\le 1\le Q_m$. The circle kernel in these units is $K_{p,L}$ from (4.3); denote its common radial row by C.

For a site of mass r, let $X_r(s)$ and $Y_r(s)$ be its complete left and right exponential image sums. These include the site's own images at nonzero multiples of L. The original rows say

$$\int_0^\infty w(X_r+Y_r)\,\mathrm{d}s = C+rS, \qquad \int_0^\infty wsX_r\,\mathrm{d}s=\int_0^\infty wsY_r\,\mathrm{d}s, \tag{8.1}$$

where $w=w_{p,L}$ and $S=2\sum_{j\ge1}k(jL)>0$. The packing lemma bounds each full ray by $\beta_r(s)=r/(e^{rs}-1)$. If a ray begins at a neighbor of mass u at distance d, its full sum is at most $e^{-ds}u/(1-e^{-us})$. There is one cut-independent pressure

$$\Pi=\int_0^\infty ws\{r(X_r+Y_r)/2+X_rY_r\}\,\mathrm{d}s=\int_0^\infty ws(r+Y_r)X_r\,\mathrm{d}s. \tag{8.2}$$

To check independence, sum $m_im_jU(t_j-t_i)$ over pairs on opposite sides of a cut in the periodic lift. Moving the cut past one label changes the sum by its mass times its zero torque row. The sum converges: $U(t)=O(t^{-3})$ and the number of crossing pairs at a fixed separation grows at most linearly. Factoring the exponentials gives (8.2); the two forms agree by (8.1).

We also record the adjacent ratio bound directly. On an empty angular interval $(0,\delta)$ between masses r,u, let $P(t)$ be the potential of all other labels, with kernel $f(t)=\sin(t/2)^{-p}$. It is strictly convex on the interval. The radial and torque equations give

$$P(\delta)-P(0)=(u-r)f(\delta),\quad P'(0)=uf'(\delta),\quad P'(\delta)=-rf'(\delta).$$

Strict increase of P' first implies $f'(\delta)<0$ and $\delta<\pi$. The two strict secant bounds then imply

$$\max\{r/u,u/r\}<1+\frac{p\delta}{2}\cot(\delta/2)<p+1. \tag{8.3}$$

The final sign follows from $\sin x-x\cos x>0$ for $0<x<\pi/2$.

8.2 A lower bound at the attained edge

Let X be the full left ray at zero and Y the full ray beyond the atom at one, measured from that atom. Put $U_1=U(1)$ and $T_X=\int wsX$. The complete torque gives $T_X=bU_1+\int wse^{-s}Y>bU_1$. Lemma 6.4 applies throughout $9/5\le p<2$ and gives $b>5/7$.

At the same unit edge, the pressure and torque give $\Pi>T_X+b\int wse^{-s}X$. The upper elasticity for U in (5.2) gives $U(t+1)/U(t)\ge(t/(t+1))^3\ge 1/8$ for $t\ge1$. Consequently

$$\widehat{\Pi}:=\frac{\Pi}{Q_mk(Q_m)}>pb(1+b/8)Q_m^{p-1}>\frac{2745}{1960}>\frac{7}{5}. \tag{8.4}$$

We used $U_1\ge pk(1)$ and $Q_mk(Q_m)\le k(1)Q_m^{1-p}$, both from the same elasticity.

8.3 A second lower bound and the pressure coordinate

At the maximum mass let $X_Q,Y_Q\le\beta_{Q_m}$ be the two full rays. Their product satisfies $X_QY_Q\ge\beta_{Q_m}(X_Q+Y_Q)-\beta_{Q_m}^2$. Since $s(Q_m/2+\beta_{Q_m})>1$, the pressure gives

$$\Pi>C+Q_mS-J_Q,\qquad J_Q=\int_0^\infty ws\beta_{Q_m}(s)^2\,\mathrm{d}s.$$

After scaling by Q_m, the ratio $J_Q/(Q_m k(Q_m))$ is the expectation of the decreasing function $s/[4\sinh^2(s/2)]$ under the normalized actual density times e^{-s}. The order w/ρ_p bounds it by the Gamma$(p,1)$ expectation, namely $d(p) = p(\zeta(p) - \zeta(p+1))$. Set

$$x_c = \frac{C}{Q_m k(Q_m)}, \quad z_c = \frac{S}{k(Q_m)}, \quad R_c = x_c + z_c.$$

The row at the mass b contains the unit atom and gives $C > K_{p,L}(1) > k(1) \geq Q_m k(Q_m)$, so $x_c > 1$. Packing both maximum rays and using the lower elasticity gives $R_c \leq 2\zeta(p)$. For completeness, the convex midpoint tail gives

$$\zeta(9/5) < 1 + \frac{23}{80} + \frac{5}{36} + \frac{37}{80} = \frac{17}{9}.$$

The three bounds follow by raising to the fifth power from $512 \cdot 23^5 > 80^5$, $5^5 > 3 \cdot 2^{10}$, and $7^4 \cdot 37^5 > 16 \cdot 100^5$; the last term includes the integral from $7/2$ to infinity. Each term of $p\sum_{j\geq 2}(j-1)j^{-p-1}$ decreases with $p \geq 9/5$, since $p\log 2 > 1$. At $p = 9/5$, $(t-1)t^{-14/5}$ has second derivative $(14/25)(9t-19)t^{-24/5}$ and is convex for $t > 19/9$. The complete midpoint tail after $j = 3$ therefore gives

$$d(p) < \frac{9}{10}2^{-9/5} + \frac{6}{5}3^{-9/5} + \frac{55}{28}(2/7)^{4/5} < \frac{9}{10}\frac{23}{80} + \frac{6}{5}\frac{5}{36} + \frac{55}{28}\frac{147}{400} = \frac{5507}{4800} < \frac{23}{20}.$$

The tail is the integral from $7/2$ to infinity. The three power bounds follow by fifth powers. Consequently

$$1 < R_c < 4, \qquad d(p) < 23/20, \qquad \widehat{\Pi} > \max\{7/5, R_c - d(p)\}. \tag{8.5}$$

8.4 The minimum and its opposite neighbor

Assume $q \leq 1/13$. At a site of minimum mass a_m, remove the identical self-image ray $a_m/(e^{Ls}-1)$ from both sides. The remaining radial values add to C, and their forces are equal. Choose a side of radial value at most $C/2$, and define $t_* > 0$ by $k(t_*) = C/(2a_m)$. Every atom on that side has distance $t > t_*$: its mass is at least a_m, and its individual radial contribution is strictly less than $C/2$ because other positive atoms remain. The variance identity for the positive Laplace law shows that $U(t)/k(t)$ strictly decreases. Thus the whole force on that side is less than $a_m U(t_*)$.

Let the first atom on the opposite side have mass $r = \kappa_m a_m$ and gap g_m. Equation (8.3) gives $1 \leq \kappa_m < p+1 \leq 3$ and packing gives $g_m \geq r$. Its force is below the common force, so $rU(g_m) < a_m U(t_*)$. The upper elasticity $tk_2(t) < 3U(t)$ then implies

$$g_m > t_* \kappa_m^{1/3}. \tag{8.6}$$

Also $t_* < Q_m$, since $x_c > 1 > 2q$. Integrating the lower elasticity of k between these two scales gives

$$y := a_m/t_* < (q^{p-1}x_c/2)^{1/p}. \tag{8.7}$$

In (8.2), keep this opposite neighbor fixed and use its packing bound. The contribution of an atom at distance $t > t_*$ on the chosen side is bounded by its mass times the Laplace mean of

$$f(s) = a_m s + \frac{rs}{1-e^{-rs}}e^{-g_m s}.$$

The upper bound in (6.1) gives $v/(1-e^{-v}) < 1 + v/2 + v^2/12$. Using the full moments in (5.2), we obtain

$$\begin{aligned}\frac{\int we^{-ts}f(s)\,\mathrm{d}s}{k(t)} &< \frac{2a_m}{t} + \frac{k(t+g_m)}{k(t)}\left(1 + \frac{r}{t+g_m} + \frac{r^2}{2(t+g_m)^2}\right) \\ &< 2yz + W_\eta(dz), \qquad z = t_*/t \in (0,1),\ d = g_m/t_*,\ \eta = r/g_m \leq 1,\end{aligned}$$

where

$$W_\eta(v) = (1+v)^{-p}\left(1 + \frac{\eta v}{1+v} + \frac{\eta^2 v^2}{2(1+v)^2}\right).$$

This function is convex on $v \geq 0$. To verify it put $T = 1+v$; its second derivative, multiplied by T^{p+4}, is

$$P(T) = p(p+1)(1+\eta+\eta^2/2)T^2 - (p+1)(p+2)(\eta+\eta^2)T + (p+2)(p+3)\eta^2/2.$$

On $T \geq 1$, $P'(T) \geq (p+1)(3p-4) > 0$, and $P(1) \geq p^2 - p - 1 \geq 11/25 > 0$. Convexity bounds the mean by $\max\{1, 2y + W_\eta(d)\}$. Writing $u = \kappa_m^{1/3} \in [1, 3^{1/3})$, inequality (8.6) and $\eta d = \kappa_m y$ give

$$2y + W_\eta(d) < B_{p,u}(y) := 2y + (1+u)^{-p}\left(1 + \frac{u^3 y}{1+u} + \frac{u^6 y^2}{2(1+u)^2}\right).$$

Summing all chosen-side atoms bounds their pressure contribution strictly by

$$\frac{C}{2}\max\{1, B_{p,u}(y)\}.$$

The chosen-side self images contribute less than

$$\frac{a_m S}{2}\left(1 + \frac{a_m}{L} + \frac{a_m^2}{2L^2}\right) < a_m S.$$

This follows by applying the same moment bounds to each image at jL and using $L \geq 2a_m$. We have therefore proved

$$\Pi < \frac{C}{2}\max\{1, B_{p,u}(y)\} + a_m S. \tag{8.8}$$

8.5 A single convex upper envelope

Use (8.7) in (8.8), replace x_c by R_c, and absorb the self term using $q < 1/2$. For fixed q, R_c, the quantity $(q^{p-1}R_c/2)^{1/p}$ decreases with p, its logarithmic derivative being $-\log(R_c/(2q))/p^2 < 0$. The function $B_{p,u}$ increases with y and decreases with p. Thus

$$\widehat{\Pi} < \overline{F}(R_c), \qquad \overline{F}(R) = \max_{1 \leq u \leq 3^{1/3}} \frac{R}{2}\max\{1, B_{9/5,u}(\overline{y}(R))\}, \quad \overline{y}(R) = \left(13^{-4/5}R/2\right)^{5/9}.$$

Each fixed-u branch is a positive linear combination of $R, R^{14/9}, R^{19/9}$, or the affine function $R/2$. Hence the compact maximum $\overline{F}$ is increasing and convex.

Let $R_0 = 7/5 + 23/20 = 51/20$, the intersection of the relaxed lower bounds. Since $R_0 < 13/5$,

$$\overline{y}(R_0)^9 < (13/10)^5/13^4 = 13/100000 < (3/8)^9.$$

For $1 \leq u \leq 3^{1/3} < 29/20$, the two increasing rational functions

$$\frac{u^2(3+u/5)}{1+u} < \frac{17}{6}, \qquad \frac{u^5(3+11u/10)}{(1+u)^2} < 5$$

are bounded by their values at $29/20$. Their monotonicity follows by differentiation; in the second case its logarithmic derivative is $5/u + 11/(30+11u) - 2/(1+u) > 0$. The derivative in u of the nonconstant part of $B_{9/5,u}(y)$, multiplied by $(1+u)^{14/5}$, is therefore at most

$$-\frac{9}{5} + \frac{17}{6}y + 5y^2 \leq -\frac{11}{320} < 0 \qquad (y \leq 3/8).$$

Thus the nonconstant branch at R_0 is largest at $u = 1$. Using $2^{-9/5} < 23/80$ and retaining the outer affine branch gives

$$\overline{F}(R_0) < \frac{51}{40}\left\{\frac{3}{4} + \frac{23}{80}\left(1 + \frac{3}{16} + \frac{9}{512}\right)\right\} = \frac{2290461}{1638400} < \frac{7}{5}.$$

The quantity in braces exceeds one, so it also covers $R_0/2$.

At $R = 4$ the ninth-power comparison $32/13^4 < (19/40)^9$ gives $\overline{y}(4) < 19/40$. Uniformly for $1 \le u \le 3^{1/3} < 3/2$,

$$(1+u)^{-9/5} < 3/10, \quad u^3(1+u)^{-14/5} < 27/100, \quad \tfrac{1}{2}u^6(1+u)^{-19/5} < 3/16.$$

For the latter two, write the power as a rational power times $(1+u)^{1/5} < 5/4$, and use the increasing rational functions at $u = 3/2$. Both branches of the envelope are bounded by

$$\overline{F}(4) < 2\left\{\frac{19}{20} + \frac{3}{10} + \frac{27}{100}\frac{19}{40} + \frac{3}{16}\left(\frac{19}{40}\right)^2\right\} = \frac{181831}{64000} < \frac{57}{20} = 4 - \frac{23}{20}.$$

Monotonicity below R_0 and convexity of $\overline{F}(R) - (R - 23/20)$ on $[R_0, 4]$ imply

$$\overline{F}(R) < \max\{7/5, R - 23/20\} \qquad (1 < R < 4).$$

This contradicts (8.5) and completes the proof of Theorem 8.1.

9 The interval $9/5 \le \alpha < 2$

Set $p = \alpha \in [9/5, 2)$ and divide by the largest mass. Theorem 8.1 gives $1/13 < m_i \le 1$. The radial rows give a lower bound for the normalized row E_r, and trial cells give an upper bound for QE_r^a. Combining them yields (9.13), which the entropy inequality will contradict for unequal masses. Here Q is a mean under the physical pair law defined below.

9.1 The physical pair law and centered mass coordinates

Use half-angle distance $\vartheta_{ij} \in (0, \pi)$ and the angular kernel $\sin(\vartheta_{ij})^{-p}$, with its diagonal defined as zero. Write

$$C_{\mathrm{ang}} = (2R)^p C_{\mathrm{rad}}, \qquad \pi_{ij} = \frac{m_i m_j \sin(\vartheta_{ij})^{-p}}{M C_{\mathrm{ang}}} \quad (i \ne j), \qquad \pi_{ii} = 0.$$

The original common radial row gives

$$\pi_{ij} = \pi_{ji} \ge 0, \qquad \sum_j \pi_{ij} = w_i := m_i/M, \qquad \sum_i \pi_{ij} = w_j. \tag{9.1}$$

Thus $\sum_{i,j} \pi_{ij} = 1$. These formulas refer to the given angles, not to independent draws from the mass marginal.

For positive masses define the following dimensionless quantities:

$$g = \exp\Big(\sum_i w_i \log m_i\Big), \quad X_i = \log(g/m_i), \quad K_c = \sum_i w_i e^{cX_i}, \tag{9.2}$$

$$a = 1/(p+1), \quad \beta = p/(p+1), \quad b = (p-1)/p, \quad c = (p-1)/(p+1),$$

$$t = K_1, \quad K = K_\beta, \quad L = K_b,$$

$$\tau = g^{1/(p(p+1))}, \quad \kappa = t/g^a, \quad z_{ij} = \exp[-c(X_i + X_j)/2],$$

$$Q = \sum_{i,j} \pi_{ij} z_{ij}, \quad A = \frac{1}{n}\sum_i m_i^a, \quad U = \frac{1}{n}\sum_i m_i^{1/p},$$

$$W_f = g^a/t^\beta.$$

Take $\lambda = 28/27$, the coefficient in the rank-gap comparison (9.12). Direct substitution gives

$$\begin{aligned} \sum_i w_i X_i = 0, \quad t = ng/M, \\ A = g^a K/t, \quad U = g^{1/p} L/t, \\ \kappa A = K, \quad \kappa U = \tau L, \quad \kappa(\lambda + U^a) W_f = \lambda t^a + \tau L^a. \end{aligned} \tag{9.3}$$

Jensen's inequality gives $K_c \geq 1$ for every real c and $Q \geq 1$. If the masses are not all equal, strict convexity of the exponential gives $t > 1$.

Put $e(x) = e^x - 1 - x$, and define the pair expression

$$\begin{aligned} \mathcal{E}_{p,\lambda}(x,y) &= p\tau\{e(bx) + e(by)\} + 2(p+1)(\lambda+\tau)e[-c(x+y)/2] \\ &\quad + (p+1)\lambda\{e(\beta x) + e(\beta y)\} - \lambda\{e(x) + e(y)\}, \\ B &= \frac{a}{2} \sum_{i,j} \pi_{ij} \mathcal{E}_{p,\lambda}(X_i, X_j). \end{aligned} \tag{9.4}$$

Every occurrence of τ in this expression is the same whole-word quantity defined above.

Lemma 9.1 (The whole entropy identity and strict remainder)**.** *For $p > 1$,*

$$B = \beta\tau(L-1) + (\lambda+\tau)(Q-1) - \lambda a(t-1) + \lambda(K-1). \tag{9.5}$$

If the mass word is nonconstant, then

$$B < \kappa\{Q(\lambda A + U) - (\lambda + U^a) W_f\}. \tag{9.6}$$

The same strict upper bound holds with any $0 < W \leq W_f$ in place of W_f on the right.

Proof. For every real d,

$$\sum_i w_i e(dX_i) = K_d - 1,$$

because the centered linear term vanishes. The two marginals (9.1) then give

$$\sum_{i,j} \pi_{ij}\{e(dX_i) + e(dX_j)\} = 2(K_d - 1).$$

Likewise the mean of $e[-c(X_i + X_j)/2]$ is $Q - 1$. Substitution in (9.4) proves (9.5). The identities (9.3) show that the right side of (9.6) is $Q(\lambda K + \tau L) - \lambda t^a - \tau L^a$. Subtracting B gives exactly

$$\begin{aligned} &(Q-1)\{\lambda(K-1) + \tau(L-1)\} \\ &\quad + \lambda\{1 + a(t-1) - t^a\} + \tau\{1 + a(L-1) - L^a\}. \end{aligned}$$

All three terms are nonnegative. The middle one is strictly positive because $0 < a < 1$ and $t > 1$, by the strict concave tangent for x^a. Finally $\kappa(\lambda + U^a) > 0$, so replacing W_f by a smaller positive W only increases the right side. □

9.2 Calibration of the original cell comparison

We use the complete cell comparison in Proposition 3.5. Set $B_m = M/n$, $\nu = p+1$, $S_n = S_p(n)$, and

$$E_r = C_{\mathrm{ang}}/S_n, \qquad N_0 = M/g = n/t, \qquad W^\nu = gS_p(N_0)/S_n.$$

Use $\Psi(N) = \log A_p(N)$, with A_p defined after Lemma 5.2, and put

$$\Delta = \Psi(n) - \Psi(N_0), \qquad \overline{\Psi} = \sum_i w_i \Psi(M/m_i), \qquad F_c = e^{\Psi(N_0) - \overline{\Psi}}.$$

Since $t \geq 1$, row monotonicity gives $\Delta \geq 0$. In particular

$$W = W_f e^{-a\Delta} \leq W_f. \tag{9.7}$$

Choose the consecutive cells of widths $2\pi w_i$ as in Proposition 3.5, with midpoints ϕ_i, and define

$$V_i = m_i S_p(M/m_i), \quad V = \exp \sum_i w_i \log V_i, \quad h_i = \tfrac{1}{2} \log(V/V_i), \quad \widetilde{m}_i = m_i e^{h_i}.$$

For $i < j$ let $\theta_{ij} = \theta_j - \theta_i \in (0, 2\pi)$ in ordered lifts, and use the corresponding lifted trial difference ϕ_{ij}. Put

$$r_{ij} = \frac{\sin(\phi_{ij}/2)^{-p}}{\sin(\theta_{ij}/2)^{-p}}.$$

The drift $b = (\phi_{ij} - \theta_{ij}) \cot(\theta_{ij}/2)/2$ has $\mathbb{E}_\pi b = 0$ by the same original torque calculation. Write $z_i = e^{-cX_i/2}$, so $z_{ij} = z_i z_j$, and introduce the tilted probability law $\widehat{\pi}_{ij} = \pi_{ij} z_i z_j / Q$. The row calibration is

$$b_{ij} = \exp\{a[\Psi(N_0) - \tfrac{1}{2}\Psi(M/m_i) - \tfrac{1}{2}\Psi(M/m_j)]\}, \qquad q_b = \mathbb{E}_{\widehat{\pi}} b.$$

Put $s_0 = \log N_0$ and $\varphi(s) = \log F_p(e^s)$. Since $\Psi(e^s) = \varphi(s) - \log C_p$ and $s_0 + X_i = \log(M/m_i) > 0$, the increasing concave function φ gives

$$\log b_{ij} \geq -\tfrac{a}{2} \varphi'(s_0)(X_i + X_j).$$

Under π, $Y = X_i + X_j$ has mean zero. Pairing independent copies shows $\mathbb{E}_\pi[Y e^{-cY/2}] \leq 0$, since the exponential decreases. Jensen's inequality therefore gives

$$\mathbb{E}_{\widehat{\pi}} \log b \geq 0, \qquad q_b \geq 1. \tag{9.8}$$

For $A_{ij} = e^{a(h_i + h_j)}$, the definitions give

$$A_{ij} = F_c^{-a} z_i z_j b_{ij}, \qquad \mathbb{E}_\pi A = F_c^{-a} Q q_b, \qquad V F_c = W^\nu S_n.$$

The diagonal side of the cell comparison in Lemma 3.4 equals MV, by the choice of $\widetilde{m}_i$. Dividing that comparison by M and applying Lemma 3.2 with $\mathbb{E}_\pi b = 0$ gives

$$V \geq C_{\mathrm{ang}} \mathbb{E}_\pi(A_{ij}^\nu r_{ij}) \geq C_{\mathrm{ang}} (\mathbb{E}_\pi A)^\nu.$$

Multiply by F_c/S_n and use $q_b \geq 1$ from (9.8) to get

$$W^\nu \geq E_r Q^\nu q_b^\nu \geq E_r Q^\nu. \tag{9.9}$$

9.3 The two complete rank profiles

Let $\omega_k = \sin(\pi k/n)^{-p}/S_n$ for $1 \leq k < n$. For positive x, y define

$$\begin{aligned} G_p(x,y) &= ((x+y)/2)^a - (x^a + y^a)/2, \\ D_p(x,y) &= (x^{1/p} + y^{1/p})/2 - ((x^{1/p} y + y^{1/p} x)/2)^a. \end{aligned} \tag{9.10}$$

Write $G_k = n^{-1}\sum_i G_p(m_i, m_{i+k})$, and similarly D_k, and put $G = \sum_k \omega_k G_k$, $D = \sum_k \omega_k D_k$. The strengthened rank comparisons give

$$E_r^a \geq A + G, \qquad U^a E_r^a \geq U - D. \tag{9.11}$$

Write $f_{ij} = \sin(\vartheta_{ij})^{-p}$. Before splitting the sums by rank, use $f_{ij} = f_{ji}$ to symmetrize:

$$nC_{\mathrm{ang}} = \sum_{i\neq j} \frac{m_i + m_j}{2} f_{ij},$$
$$nUC_{\mathrm{ang}} = \sum_{i\neq j} \frac{m_i^{1/p} m_j + m_j^{1/p} m_i}{2} f_{ij}.$$

The second identity comes from multiplying row i by $m_i^{1/p}$. At rank k apply Lemma 3.2 as in Proposition 3.3, using the ath power of the symmetrized coefficient as amplitude. The two mean amplitudes are $A + G_k$ and $U - D_k$. Dividing by nS_n and summing over ranks gives $E_r \geq \sum_k \omega_k (A + G_k)^{p+1}$ and $UE_r \geq \sum_k \omega_k (U - D_k)^{p+1}$. Jensen's inequality and the ath root give (9.11).

Appendix C proves

$$D_p(x, y) \leq \lambda G_p(x, y), \quad \lambda = 28/27, \quad 9/5 \leq p \leq 2, \quad 1/16 \leq x, y \leq 1. \tag{9.12}$$

The source floor $1/13$ is within this domain. Multiplying the first inequality in (9.11) by λ and adding the second yields

$$(\lambda + U^a) E_r^a \geq \lambda A + U + \lambda G - D \geq \lambda A + U.$$

The calibrated upper bound (9.9) and (9.7) now imply the necessary inequality

$$Q(\lambda A + U) \leq (\lambda + U^a) W \leq (\lambda + U^a) W_f. \tag{9.13}$$

9.4 Completion on the full source-supported mass interval

Put $\sigma = -\log g$. The source floor gives $0 \leq \sigma < \log 13$ and $X_i \in [-\sigma, \log 13 - \sigma]$. Lemma B.1 proves $\mathcal{E}_{p,\lambda}(X_i, X_j) \geq 0$ for every pair, with the same whole-word value of τ. Thus $B \geq 0$ under the original zero-diagonal pair law. For unequal masses, Lemma 9.1 and (9.13) give

$$0 \leq B < \kappa\{Q(\lambda A + U) - (\lambda + U^a) W_f\} \leq 0,$$

a contradiction. Hence the masses are equal for $9/5 \leq p < 2$ and all $n \geq 3$. The scalar entropy and ratio lemmas include $p = 2$; the physical floor is used only for $p < 2$. The physical endpoint $p = 2$ is covered by the closed interval $[2, 4]$ in Sections 10–12.

10 The source mass bound between two and four

We derive the mass-ratio bound for Section 12 by the pressure comparison of Section 8, with estimates for the kernel at $a = \alpha \in [2, 4]$. The pressure argument uses $n \geq 4$.

10.1 The normalized source and its kernel

Choose an attained maximum of endpoint mass divided by incident arc length, and divide the masses and arc lengths by the attaining mass and length. Orient the attaining edge as $(0, 1), (1, b)$, $0 < b \leq 1$. Every consecutive gap dominates both endpoint masses. Let L be the

resulting circumference, $Q \geq 1$ the maximum mass, v the minimum mass, and $q = v/Q$. In these units write C for the common radial value.

Use the image density and kernel in (4.1)–(4.2), now at exponent a, and write

$$k_j(t) = \int_0^\infty w(s) s^j e^{-ts}\,\mathrm{d}s, \quad k = k_0, \quad U = k_1, \quad S = 2\sum_{l\geq 1} k(lL).$$

Lemma 4.1 gives the complete circle kernel, including this positive self-image sum. We will need the following shape information.

Lemma 10.1. *Let $\rho_a(s) = s^{a-1}/\Gamma(a)$. For $a \geq 2$, w/ρ_a is nonincreasing and w/s is nondecreasing. For $a > 2$ both orders are strict. Moreover*

$$(j+2)k_j(t) \leq t k_{j+1}(t) \leq (j+a)k_j(t), \qquad t > 0. \tag{10.1}$$

At $a = 2$, $w(s) = s$ and $k(t) = t^{-2}$; both inequalities are equalities.

Proof. Let $d = a/2$, $X = Ls/(2\pi)$, and $\psi = \Gamma'/\Gamma$. For $X > 0$ the reflection formula gives

$$\Im\psi''(1/2 + \mathrm{i}X) = \pi^3 \operatorname{sech}^2(\pi X)\tanh(\pi X) > 0.$$

This positivity extends to $d > 1/2$. One way to see the extension explicitly is to use the sine-integral formula

$$\Im\psi''(d + \mathrm{i}X) = \int_0^\infty \frac{t^2 e^{-dt}\sin(Xt)}{1 - e^{-t}}\,\mathrm{d}t.$$

Increasing d from $1/2$ multiplies the sine-transform integrand by $e^{-(d-1/2)t}$. Equivalently it convolves the odd boundary function with the Poisson kernel $P_\eta(x) = \eta/[\pi(\eta^2 + x^2)]$. For $X, y > 0$, $P_\eta(X - y) - P_\eta(X + y) > 0$. The boundary function is positive on the positive half-line and exponentially integrable, so the convolution is positive. The integral formula and the exponential multiplier justify this identity by Fourier inversion and Fubini.

Define

$$H_d(X) = \partial_X \log \frac{\sinh(\pi X)|\Gamma(d + \mathrm{i}X)|^2}{\pi X^{2d-1}}.$$

Then $\partial_d^2 H_d = -2\Im\psi''(d + \mathrm{i}X) < 0$. Reflection gives $H_1 = 0$ and $H_{1/2} = \partial_X \log\tanh(\pi X) > 0$. Concavity in d implies $H_d < 0$ for $d > 1$. This proves the first order. For the second, the convergent trigamma series gives $\Im\psi'(d + \mathrm{i}X) < 0$, so $\partial_d\partial_X \log(w/s) = -2\Im\psi'(d + \mathrm{i}X) > 0$. At $d = 1$ reflection gives $w = s$, proving the second order and the endpoint formula.

The two orders say $w \leq sw' \leq (a-1)w$, strictly for $a > 2$. Integration by parts of $s^{j+1}w(s)e^{-ts}$ proves (10.1). At zero $w = O(s)$; at infinity the differential bounds imply polynomial growth of w, w'. Thus the boundary terms vanish. The same bounds give $k_j(t) = O(t^{-j-2})$ at infinity. □

A sequence of bodies on a ray starting at a boundary mass r obeys the packing estimate

$$\sum_j m_j H(t_j) \leq r\sum_{l\geq 1} H(lr) \tag{10.2}$$

for a positive decreasing convex kernel with quadratic decay. The proof of Lemma 4.6 applies verbatim: its only decay requirement is that the finite boundary term $NH(Nr)$ tends to zero. In particular, with

$$\beta_r(s) = \frac{r}{e^{rs} - 1}, \qquad G_r(s) = \frac{r}{1 - e^{-rs}},$$

the exponential ray sums satisfy $X_r, Y_r \leq \beta_r$. If the first atom of a ray has mass u and distance g, its full sum is at most $e^{-gs}G_u(s)$.

Lemma 10.2. *Both mass ratios across every actual adjacent pair are less than $a+1$.*

Proof. Place the adjacent endpoints at angles zero and $\delta \in (0, 2\pi)$, with masses r, u. Remove these two atoms and call the remaining angular potential $P(t)$. Each term is a translate of $f(t) = \sin(t/2)^{-a}$ on the empty interval, and

$$f''(t)/f(t) = \frac{a}{4}\{1 + (a+1)\cot^2(t/2)\} > 0.$$

When there are no remaining atoms the potential is zero and is still convex. The radial and tangential equations give

$$P(\delta) - P(0) = (u-r)f(\delta), \quad P'(0) = uf'(\delta), \quad P'(\delta) = -rf'(\delta).$$

Monotonicity of P' implies $f'(\delta) \le 0$, so $\delta \le \pi$. The two secant inequalities give

$$u\delta f'(\delta) \le (u-r)f(\delta) \le -r\delta f'(\delta).$$

Rearranging both inequalities and dividing by positive quantities yields

$$\max\{r/u, u/r\} \le 1 - \delta f'(\delta)/f(\delta) = 1 + a(\delta/2)\cot(\delta/2) < a+1.$$

For the last strict inequality, $\sin x - x\cos x > 0$ on $0 < x \le \pi/2$: its derivative is $x\sin x > 0$ and its value at zero is zero. The argument includes a gap of π and also applies to $n = 2, 3$. □

Let X_r, Y_r be the full exponential ray sums on either side of a mass r. The original equations are

$$\int w(X_r + Y_r) = C + rS, \qquad \int wsX_r = \int wsY_r. \tag{10.3}$$

All integrals without bounds in this section run over $(0, \infty)$. There is one conserved cut pressure,

$$\Pi = \int ws\{r(X_r + Y_r)/2 + X_rY_r\} = \int ws(r + Y_r)X_r. \tag{10.4}$$

To verify conservation, sum $m_im_jU(t_j - t_i)$ over pairs on opposite sides of a cut in the periodic lift. Moving the cut past one body changes the sum by that body's mass times its zero tangential row. Absolute convergence follows from $U(t) = O(t^{-3})$ and the linear bound on the number of crossing pairs in each separation interval. Tonelli and the exponential factorization give the last expression in (10.4); the two expressions agree by (10.3).

10.2 The attained edge has a substantial second mass

Lemma 10.3. *At the normalized unit edge, $b > (12-a)/16$.*

Proof. Set $\sigma = (a+4)/2$, $u = 1/\sigma \in [1/4, 1/3]$, and $b_0 = 1 - \sigma/8 = (12-a)/16$. Suppose $b \le b_0$, and put $c = b/\sigma$, $\theta = 1 - 1/\sigma$. For the pure-power kernel, write

$$A_a(t) = t^{-a} - (t+1)^{-a} - a(t+1)^{-a-1},$$
$$B_a(t) = at^{-a-1} - t^{-a} + (t+1)^{-a}.$$

We claim $\sigma A_a(t) > tB_a(t)$ for $0 < t \le 5/8$, $2 \le a \le 4$. Put $z = t/(1+t)$ and $P(a) = \sigma(1 + a/(1+t)) + t$. After multiplication by t^a, the claim is $F(a) = \sigma - a + t - z^aP(a) > 0$. Let $l = \log(1 + 1/t) > 2/(2t+1)$. The identity

$$2(1+t)\{P - (2t+1)P'\} = a^2 + 3a - 1 - (3a+5)t > 0$$

gives $l > 2P'/P$, while $P'' = 1/(1+t) > 0$. Thus $(z^a P)'' > 0$ and F is concave in a. At the two endpoints its sign reduces, respectively, to

$$1 + 4t - 3t^2 > 0, \qquad t(1 + 5t + 10t^2 - 10t^3) > 0.$$

Both hold on the stated interval, proving the claim.

The ratio A_s/B_s decreases strictly in s: its derivative has numerator $-1+(s^2+2)e^{-s}-e^{-2s} < 0$, equivalent to $2\sinh(s/2) > s$. Since w/ρ_a is nonincreasing, normalized covariance gives $A(t)/B(t) \geq A_a(t)/B_a(t)$. Exponential tilting shows the latter increases with t. The scalar claim gives $A_a(b)/B_a(b) > c$. Each distance t in the right ray is at least b, so $A(t)/B(t) \geq A_a(t)/B_a(t) > c$. Summing over these atoms gives $\int w(A_s - cB_s)Y > 0$.

Use Lemma 6.2 with $\lambda = u = 1/\sigma$. The increasing likelihood w/s makes $w(s)e^{-s}/k(1)$ dominate Gamma$(2,1)$. At $b_0 = 1 - 1/(8u)$ the clipped test is

$$[7/8 + u - (2u - 1/8)H_B(s) + (u - 1/8)s]_+.$$

It is convex in $u \in [1/4, 1/3]$. The last two bounds of Lemma 6.3 therefore give the affine bound $\mathcal{E}_{\text{edge}}/k(1) < 31/40 - 39u/40$. The charge is $(1-b_0)(1+ub_0) = 7/(64u) + 1/8$. Its difference from the correction bound, multiplied by $320u$, is $35 - 208u + 312u^2 = 312(u-1/3)^2 + 1/3 > 0$. The cancellation lemma proves $b > b_0$ with the complete SELF retained. □

10.3 The available period

At a maximum mass Q, centering gives

$$QR \leq (M - Q)R.$$

Equality in this triangle inequality would place every other atom at the same antipodal point. Distinctness and $n \geq 3$ exclude equality. Packing also gives $L \geq M$, so

$$\ell := L/Q \geq M/Q > 2. \tag{10.5}$$

This is the only lower bound on total mass needed in the pressure argument.

10.4 Two lower bounds for the same pressure

At the unit edge, (10.4) and the tangential force equation give

$$\Pi > T_X + b \int wse^{-s}X.$$

Indeed the opposite ray is $e^{-s}(b + Y)$, and the omitted product is positive. Integrating the upper elasticity bound for U gives $U(t+1)/U(t) \geq (t/(t+1))^{a+1} \geq 2^{-a-1}$ for $t \geq 1$. Thus

$$\Pi > b(1 + b2^{-a-1})U(1).$$

At a maximum mass Q, define

$$x = \frac{C}{Qk(Q)}, \qquad z = \frac{S}{k(Q)}, \qquad R_* = x + z, \qquad \widehat{\Pi} = \frac{\Pi}{Qk(Q)}.$$

Here R_* is a scalar pressure coordinate. The row at the other endpoint of the unit edge contains its unit neighbor, so $C > K_L(1) > k(1)$, where K_L is the complete periodic kernel. Equation (10.1) gives $Qk(Q) \leq k(1)/Q$, hence $x > 1$. Packing both maximum rays gives

$$C + QS \leq 2Q \sum_{l \geq 1} k(lQ) \leq 2Qk(Q)\zeta(2).$$

Consequently $1 < R_* < 33/10$, using $2\zeta(2) = \pi^2/3 < 33/10$. With $b_0 = (12-a)/16$, Lemma 10.3 and $U(1) \geq 2k(1)$ give

$$\widehat{\Pi} > A(a) := 2b_0 + b_0^2 2^{-a}. \tag{10.6}$$

For the second lower bound, put $J_Q = \int ws\beta_Q(s)^2$. Since $(\beta_Q - X_Q)(\beta_Q - Y_Q) \geq 0$ and $s(Q/2 + \beta_Q) > 1$, the symmetric pressure formula yields

$$\Pi > C + QS - J_Q.$$

We next prove the full bound

$$\frac{J_Q}{Qk(Q)} < \frac{11-a}{10}. \tag{10.7}$$

Let $\ell = L/Q > 2$ as in (10.5). Scaling the density shows that the correction term is the normalized $w_{a,\ell}(s)e^{-s}$ expectation of $j(s) = s/[4\sinh^2(s/2)]$, a decreasing function. Put $\beta = (a-2)/2 \in [0,1]$. The concavity in d proved in Lemma 10.1 implies that the density elasticity at a is at least the chord between its values at two and four. Since

$$w_{2,\ell}(s) = s, \qquad w_{4,\ell}(s) = s(s^2 + 4\pi^2/\ell^2)/6,$$

the ratio $w_{a,\ell}(s)/[s(s^2 + 4\pi^2/\ell^2)^\beta]$ increases in s. Covariance therefore bounds the correction term above by the expectation of j under the latter reference density times e^{-s}. Increasing its constant $4\pi^2/\ell^2$ tilts the reference density toward smaller s. Since $\ell > 2$ and $\pi^2 < 10$, one may replace that constant by $C_0 = 10$. Thus, for $X \sim \mathrm{Gamma}(2,1)$,

$$\frac{J_Q}{Qk(Q)} \leq \frac{\mathbb{E}[j(X)(1 + X^2/C_0)^\beta]}{\mathbb{E}(1 + X^2/C_0)^\beta}.$$

Let $p_\beta(s)$ be the normalized density proportional to $se^{-s}(s^2+10)^\beta$, and set

$$h_\beta(s) = -1 - \left(\frac{1}{18} + \frac{\beta}{10}\right)s + \left(\frac{1}{80} + \frac{\beta}{30}\right)s^2, \qquad \mathcal{L}_\beta h = h' + h\left(\frac{1}{s} - 1 + \frac{2\beta s}{s^2+10}\right).$$

The boundary values of $p_\beta h_\beta$ vanish at zero and infinity, so $\mathbb{E}_{p_\beta}\mathcal{L}_\beta h_\beta = 0$. Indeed $p_\beta(s) = O(s)$ at zero, while $j(s)$ and $\mathcal{L}_\beta h_\beta(s)$ are $O(1/s)$; their expectations are therefore integrable there. At infinity the density has exponential decay and both terms have at most polynomial growth. Thus this integration by parts uses convergent integrals and vanishing boundary terms at both ends. The positive series for $\cosh s$ gives $j(s) < J(s) := 360/[s(s^4 + 30s^2 + 360)]$ for $s > 0$. Put $D = s^4 + 30s^2 + 360$ and $R_\beta = 9/10 - \beta/5 - J - \mathcal{L}_\beta h_\beta$. Direct expansion gives

$$R_\beta = \frac{P(s)}{720D} + \frac{\beta s^2\{(1-\beta)(12s^2 - 81s + 160) + \beta(12s^2 - 105s + 232)\}}{360(s^2+10)},$$

where

$$\begin{aligned} P(s) = {} & s^4(9s^2 - 67s + 125) + s^2(153s^2 - 1290s + 2800) \\ & + 680s^2 - 2520s + 2880. \end{aligned}$$

All five displayed quadratics are positive: their values of $4ac - b^2$ are, respectively, $1119, 111, 11, 49500, 1483200$, and their leading coefficients are positive. Thus $R_\beta > 0$ for every $s > 0$ and $0 \leq \beta \leq 1$. Integrating against p_β proves $\mathbb{E}_{p_\beta} j < 9/10 - \beta/5 = (11-a)/10$, including both endpoints. This proves (10.7).

Finally A in (10.6) is convex in a. Its value at four is $65/64$ and its derivative there is $-33/256 - (\log 2)/64 < -107/768$. Since $a - 4 \leq 0$, its tangent gives

$$A(a) \geq \overline{A}(a) := \frac{1208 - 107a}{768}.$$

The same conserved pressure therefore satisfies

$$\widehat{\Pi} > \max\{\overline{A}(a), R_* - J(a)\}, \qquad J(a) = \frac{11-a}{10}. \tag{10.8}$$

10.5 An upper bound at a minimum mass

Assume for contradiction that $q \le 2/(7a)$. At a minimum mass v, remove the identical self-image ray from both sides. The remaining radial values sum to C, and the two remaining torques agree. Choose the side with radial value at most $C/2$, and define $t_* > 0$ by $k(t_*) = C/(2v)$. Existence follows from the positive Laplace kernel's endpoint limits.

Every nonself atom on this side has distance strictly greater than t_*. Its mass is at least v and other positive nonself terms remain in the row. Also U/k decreases strictly: its derivative is minus the variance of the frequency under the normalized Laplace density. Thus the torque of this side, and hence that of the opposite side, is strictly less than $vU(t_*)$.

Let r, g be the first mass and gap on the opposite side. Then $v \le r < (a+1)v$, $g \ge r$, and $rU(g) < vU(t_*)$. Integrating the upper elasticity bound for U gives

$$\kappa := r/v \in [1, a+1), \qquad g > t_* \kappa^{1/(a+1)}. \tag{10.9}$$

Indeed, for $g \ge t_*$ that elasticity bounds $U(g)$ below by $U(t_*)(t_*/g)^{a+1}$; for $g < t_*$, the force inequality is already impossible. Since $x > 1 > 2q$, one has $t_* < Q$. The quadratic kernel bound then gives

$$y := v/t_* \le \sqrt{qx/2}. \tag{10.10}$$

At $a = 2$ this is an equality; the following estimates only require the displayed weak bound.

We retain this particular neighboring body throughout the comparison. For each $z_0 \ge 1$, the function

$$f(s) = vs + H_B(rs)e^{-gs}$$

crosses z_0 at most once, from below to above. To verify this, on $0 < s < z_0/v$ take the logarithm of the equivalent comparison between $H_B(rs)e^{-gs}$ and $z_0 - vs$:

$$l(s) = -gs + \log H_B(rs) - \log(z_0 - vs).$$

It has $l(0) = -\log z_0 \le 0$ and

$$l''(s) = \frac{v^2}{(z_0 - vs)^2} - \left\{ \frac{1}{s^2} - \frac{r^2}{4\sinh^2(rs/2)} \right\}.$$

The expression in braces decreases. Equivalently, with $u = rs/2$, one needs $\sinh^3 u > u^3 \cosh u$; differentiation and $\sinh^2 u - u^2 > u^4/3$ prove this inequality. Hence l'' increases strictly. If $l'(0) > 0$, then $g \ge r$ implies $r < 2v/z_0$, so $l''(0) = v^2/z_0^2 - r^2/12 > 0$. Otherwise l first decreases and then increases. In either case it can cross zero at most once, upward, and $f > z_0$ automatically beyond z_0/v.

Let $Z = \max\{1, \mathbb{E}_{\mathrm{Gamma}(a,t_*)} f\}$. The likelihood ratio $(w/\rho_a)e^{-(t-t_*)s}$ is decreasing for every actual $t > t_*$. The one-crossing property at level Z therefore implies

$$\int w e^{-ts} f(s)\, \mathrm{d}s \le Z k(t).$$

For clarity, subtract Z inside both expectations and split at its crossing. A decreasing likelihood gives a nonpositive weighted integral whenever the unweighted integral is nonpositive. This proves the claim also when $Z = 1$. Summing over the chosen nonself ray bounds its pressure contribution by $CZ/2$.

The chosen self ray contributes at most

$$\int w(s) H_B(vs) \frac{v}{e^{Ls} - 1}\, \mathrm{d}s.$$

Here we used the complete opposite packing bound, including its self images. Since $H_B(t) < 1 + t/2 + t^2/12$, likelihood comparison at each positive image jL, followed by $L \geq 2v$, bounds this contribution by $d(a)vS$, where

$$d(a) = \frac{1 + a/4 + a(a+1)/48}{2} \leq 29/24.$$

Consequently $\Pi \leq CZ/2 + d(a)vS$; the strictness of this bound will not be needed.

The Gamma expectation defining Z is bounded using (10.9). Set $h = \kappa y$ and $A = 1 + \kappa^{1/(a+1)}$. Expanding the exponential denominator gives

$$\mathbb{E}_{\mathrm{Gamma}(a,t_*)} f < ay + ah \sum_{j \geq 0} (A + jh)^{-a-1}.$$

For the terms $j \geq 1$, midpoint convexity bounds the complete sum by the integral over $[1/2, \infty)$, so

$$\sum_{j \geq 0} (A + jh)^{-a-1} \leq A^{-a-1} + (A + h/2)^{-a}/(ah).$$

Now $A \geq 2$, $h \geq y$, and

$$\frac{a(a+1)}{[1 + (a+1)^{1/(a+1)}]^{a+1}} < 5/12 \qquad (a \geq 2). \tag{10.11}$$

Here is a verification on the whole range. With $\eta = a + 1 \geq 3$ and $b = \eta^{1/\eta}$, the logarithmic derivative of the left side is

$$\frac{1}{\eta - 1} + \frac{1}{\eta} - \log(1 + b) + \frac{b}{1 + b} \frac{\log \eta - 1}{\eta}.$$

By AM–GM, $\log(1 + b) \geq \log 2 + (\log b)/2$. Since $\log \eta - 1 > 0$ and $b/(1 + b) < 1$, the derivative is strictly less than $1/(\eta - 1) + \log \eta/(2\eta) - \log 2$. This expression decreases for $\eta \geq 3$ and is negative at 3. The last inequality follows from $e < 11/4$ and $(11/4)^3 < 64/3$. Thus the coefficient decreases. At $a = 2$, $\sqrt[3]{3} > 10/7$ gives $(1 + \sqrt[3]{3})^3 > 706/49 > 72/5$, proving (10.11). In the midpoint bound the term ah/A^{a+1} increases with κ, since it equals $ay[\kappa^{1/(a+1)}/(1 + \kappa^{1/(a+1)})]^{a+1}$. Apply (10.11) at $\kappa = a + 1$ to obtain

$$Z \leq \max\{1, B_a(y)\}, \qquad B_a(y) = (a + 5/12)y + (2 + y/2)^{-a}.$$

This function increases in y. Combining (10.10), $q \leq 2/(7a)$, and $d(a)q \leq 29/168 < 1/2$ yields

$$\widehat{\Pi} \leq F(a, R_*), \qquad F(a, R) = \frac{R}{2} \max\left\{1, B_a\left(\sqrt{R/(7a)}\right)\right\}. \tag{10.12}$$

The factor $R_* = x + z$ retains the full self term together with the radial value.

10.6 The scalar pressure comparison

Lemma 10.4. *For $2 \leq a \leq 4$ and $1 \leq R \leq 33/10$,*

$$F(a, R) < \max\{\overline{A}(a), R - J(a)\}.$$

Proof. The two affine lower bounds meet at

$$R_0(a) = \overline{A}(a) + J(a) = \frac{10264 - 919a}{3840}.$$

First consider the slightly larger curve $R^*(a) = (67 - 6a)/25$, for which $R^* - R_0 = (136 - 13a)/19200 > 0$. On this curve put

$$y^2 = A/a - b, \quad A = 67/175, \quad b = 6/175, \qquad L = (a + 5/12)y, \quad E = (2 + y/2)^{-a}.$$

We show $L + E < 7/6$ throughout $[2, 4]$ by a differential comparison. Since $y^2 \geq 43/700 > b$ and $\log(1 + y/4) \geq y/(4 + y)$,

$$-E'/E = \log 2 + \log(1 + y/4) - \frac{y^2 + b}{2y(4 + y)} > \log 2 > 2/3.$$

For the other term,

$$1/32 < -L'' < 1/16. \tag{10.13}$$

Here

$$-L'' = \frac{67(308a - 335)}{16\sqrt{175}\, a^{5/2}(67 - 6a)^{3/2}}.$$

Its logarithmic derivative has the sign of $11088a^2 - 77988a + 112225$, which decreases from 601 to -22319 on $[2, 4]$. Thus $-L''$ has one interior maximum and no interior minimum. At the endpoints,

$$-L''(2) = \frac{18827}{17600\sqrt{770}} > \frac{18827}{492800} > 1/32,$$

$$-L''(4) = \frac{60099}{110080\sqrt{301}} > \frac{60099}{1915392} > 1/32,$$

using $\sqrt{770} < 28$ and $\sqrt{301} < 87/5$. For the upper bound rewrite

$$-L'' = \frac{A^2[77/67 - 5/(4a)]}{4\{a(A - ba)\}^{3/2}}.$$

Use $a(A - ba) \geq 22/35$, the bracket at most $897/1072$, and $\sqrt{22/35} > 79/100$. The rational inequality $4A^2(897/1072) = 60099/122500 < 60830/122500 = (22/35)(79/100)$ proves the upper half of (10.13).

At $a = 3$, $y = \sqrt{21}/15$, $L = 41\sqrt{21}/180 < 21/20$, and $L' = 781/(37800y) < 7/100$. With $h = a - 3 \in [-1, 1]$, integration of (10.13) gives

$$L \leq L(3) + L'(3)h - h^2/64, \qquad L' \leq L'(3) - 3h/64 + |h|/64.$$

Since $1 + 2h/3 > 0$, these imply

$$L' + 2L/3 < 77/100 + (19/1200)|h| - h^2/96 \leq 46561/60000 < 7/9.$$

For $H = 7/6 - L$, therefore, $H' + 2H/3 > 0$, while $E' + 2E/3 < 0$. At $a = 2$, $39/100 < y < 793/2000$ and

$$L + E < (29/12)(793/2000) + (200/439)^2 < 22997/24000 + 5/24 = 7/6 - 1/8000.$$

Multiplication of $H - E$ by $e^{2(a-2)/3}$ proves $L + E < 7/6$ on the whole curve.

The function $B_a(y)$ increases in y, so this also holds at $R_0 < R^*$. Hence

$$F(a, R_0) < 7R_0/12, \qquad \overline{A} - 7R_0/12 = (5\overline{A} - 7J)/12 \geq 329/23040 > 1/80.$$

This proves the desired strict bound at the switch, including the clipped branch.

For the upper boundary $R_{max} = 33/10$, write $B_a(y) = L_R + E_R$ with $y = \sqrt{R/(7a)}$. Direct differentiation gives

$$L_R'' = \frac{(5-4a)\sqrt{R/7}}{16a^{5/2}} < 0, \qquad (\log E_R)'' = \frac{y(2+y)}{2a(4+y)^2} > 0.$$

Thus the tangent to L_R at four and the chord of E_R between two and four give an affine upper bound. At four, $1/3 < y < 11/32$ yields

$$L_4 < 583/384, \qquad L_4' > 43/288, \qquad E_4 < 1/20.$$

At two, $y > 2/5$ gives $E_2 < 25/121$. At the two endpoints the affine upper is therefore below the affine target $2(R_{max} - J)/R_{max}$, because

$$1405/1152 + 25/121 < 16/11, \qquad 583/384 + 1/20 = 3011/1920 < 52/33.$$

The target is greater than one, so clipping causes no change.

Finally $B_y > 0$ and $B_{yy} > 0$ imply that the unclipped function $RB_a(\sqrt{R/(7a)})/2$ increases and is convex in R; explicitly its second derivative is $y(3B_y + yB_{yy})/(8R) > 0$. Taking the maximum with $R/2$ preserves both properties. For $R \le R_0$, increase and the switch bound give $F < \overline{A}$. For $R \ge R_0$, convexity of $F - (R - J)$ and its negative values at both endpoints give $F < R - J$. This proves the entire rectangle. □

Theorem 10.5. *Under the original physical hypotheses, for $n \ge 2$ and $2 \le \alpha \le 4$,*

$$\frac{\min_i m_i}{\max_i m_i} > \frac{2}{7\alpha}.$$

Proof. For $n = 2, 3$, every pair is adjacent. Lemma 10.2 gives $q > 1/(a+1) > 2/(7a)$, since $5a - 2 > 0$. For $n \ge 4$, if the ratio were at most $2/(7a)$, the same pressure would satisfy the strict lower bound (10.8) and the upper bound (10.12), contrary to Lemma 10.4. All source estimates used above are valid at $a = 2$: the kernel orders there are weak, but the strict correction and scalar bounds remain. The physical equations were applied at their given exponent throughout. □

11 A positive representation of the analytic row

The integral formula of (3.4) has a positive representation that supplies the comparison on the closed interval $[2, 4]$. We prove it for every positive exponent, since the same argument handles the endpoints without a separate limiting construction. The probability space in this section is auxiliary and independent of the finite mass laws $\mathbb{E}_U$ and $\mathbb{E}_w$.

Let $X_{\alpha,j}$, $j \ge 1$, be independent Gamma variables of shape α and rate $4\pi^2 j^2$, and set

$$X_\alpha = \sum_{j \ge 1} X_{\alpha,j}.$$

Independently, let $\Pr(J = j) = j^{-2}/\zeta(2)$ and let E_0 be exponential of mean one. Put $\chi = E_0/(4\pi^2 J^2)$. Since $\mathbb{E}X_\alpha = \alpha/24$, the positive sum defining X_α is finite almost surely. The product for sinh gives

$$\mathbb{E}e^{-tX_\alpha} = \varphi(t)^\alpha, \qquad \varphi(t) = \frac{\sqrt{t}}{2\sinh(\sqrt{t}/2)}, \tag{11.1}$$

$$\mathbb{E}e^{-zt\chi} = \frac{6}{\sqrt{zt}}\coth(\sqrt{zt}/2) - \frac{12}{zt}. \tag{11.2}$$

For example, the second formula follows by summing $[\zeta(2)j^2(1 + zt/(4\pi^2 j^2))]^{-1}$ using the logarithmic derivative of the same product. Both right sides take the value one at zero.

Theorem 11.1. *For $\alpha > 0$, $N > 0$, and $\beta = (\alpha - 2)/2$,*

$$B_\alpha(N) = \frac{4^\beta N^2}{3\Gamma(\beta+1)} \mathbb{E}(X_\alpha + N^2\chi)^\beta. \tag{11.3}$$

All positive and negative real moments appearing here are finite. In particular, $B_2(N) = N^2/3$ and $S_2(N) = (N^2-1)/3$.

Proof. For $0 < \alpha < 1$, put $\vartheta = \pi\alpha/2$ and $f_\alpha(v) = (2\sinh(v/2))^{-\alpha}$. The sine identity proved in Lemma 4.2, together with Gamma reflection, gives

$$\nu_\alpha(\xi) = \frac{2\sin\vartheta}{\pi} \int_0^\infty f_\alpha(v)\sin(\xi v)\,\mathrm{d}v.$$

The interchange with the integral representation is justified by

$$\int_0^\infty \int_0^\infty \frac{f_\alpha(v)|\sin(\xi v)|}{e^{2\pi\xi/N}-1}\,\mathrm{d}\xi\,\mathrm{d}v \le \frac{N^2}{24}\int_0^\infty v f_\alpha(v)\,\mathrm{d}v < \infty.$$

Expanding the exponential denominator as a geometric series and integrating each sine term gives

$$\int_0^\infty \frac{\sin(\xi v)}{e^{2\pi\xi/N}-1}\,\mathrm{d}\xi = \frac{N}{4}\coth(Nv/2) - \frac{1}{2v}.$$

Thus

$$B_\alpha(N) = \frac{2^\alpha \sin\vartheta}{\pi}\int_0^\infty \{N\coth(Nv/2) - 2/v\} f_\alpha(v)\,\mathrm{d}v. \tag{11.4}$$

The expression in braces is kept together; it is $N^2 v/6 + O(v^3)$ at zero.

Here $\beta < 0$. The negative-moment identity and Tonelli's theorem give

$$\mathbb{E}(X_\alpha + N^2\chi)^\beta = \frac{1}{\Gamma(-\beta)}\int_0^\infty t^{-\beta-1}\varphi(t)^\alpha \mathbb{E}e^{-N^2 t\chi}\,\mathrm{d}t.$$

With $t = v^2$, the measure $t^{-\beta-1}\varphi(t)^\alpha\,\mathrm{d}t$ becomes $2v f_\alpha(v)\,\mathrm{d}v$. In (11.2) the factor $2v$ then gives $2vN^2\mathbb{E}e^{-N^2v^2\chi} = 12\{N\coth(Nv/2) - 2/v\}$. The reflection formula $\Gamma(\beta+1)\Gamma(-\beta) = \pi/\sin\vartheta$ proves (11.3) on this interval.

For complex $\Re\alpha > 0$ and $z > 0$, put $L(\alpha, z; t) = \varphi(t)^\alpha \mathbb{E}e^{-zt\chi}$, using the real logarithm of $\varphi(t) > 0$. For an integer $k \ge 1$ define, on $0 < \Re\alpha < 2(k+1)$,

$$M_k(\alpha, z) = \frac{(-1)^k}{\Gamma(k-\beta)}\int_0^\infty t^{k-\beta-1}\partial_t^k L(\alpha, z; t)\,\mathrm{d}t, \qquad \beta = (\alpha-2)/2.$$

The explicit Laplace transforms are analytic in t near zero, so $\partial_t^k L$ is bounded there. Their differentiated formulas give $|\partial_t^k L| \le C(1+t)^b e^{-\epsilon\sqrt{t}}$ at infinity, locally uniformly in α. Since $\Re\beta < k$, these bounds give holomorphy of M_k on its strip by dominated parameter integration. No subtractions or removable poles occur.

For real $\alpha > 0$ put $Y = X_\alpha + z\chi > 0$. Boundedness of $y^k e^{-ty}$, locally uniformly in $t > 0$, justifies $(-1)^k\partial_t^k L = \mathbb{E}[Y^k e^{-tY}]$. For any real $h < k$, Tonelli and the ordinary Gamma integral therefore give

$$\frac{(-1)^k}{\Gamma(k-h)}\int_0^\infty t^{k-h-1}\partial_t^k L\,\mathrm{d}t = \mathbb{E}Y^h.$$

The integral is finite by the same bounds. Choosing $k > h$ proves every asserted real moment; taking $h = \beta$ identifies M_k, including integer β, without a separate seam argument.

The integral formula is holomorphic for $\Re\alpha > 0$: the ordinary-beta bound in Lemma 3.1 bounds its Gamma product on compact complex α sets by $C(1+|\xi|)^b e^{-\pi|\xi|}$; the exponential

denominator pays the tail, and $\sinh(\pi\xi)$ cancels its origin singularity. Thus $B_\alpha(N)$ and $4^\beta N^2 M_k(\alpha, N^2)/(3\Gamma(\beta+1))$ are holomorphic on the same strip and agree on $0<\alpha<1$ by the ordinary entrance above. The identity theorem gives equality throughout the strip. Their union over $k \geq 1$ covers every real $\alpha > 0$; each carrier is identified with the same row, with no compatibility calculation needed. At $\alpha = 2$, $Y^0 = 1$ gives $B_2(N) = N^2/3$. For the consumer $2 \leq \alpha \leq 4$, one choice $k = 2$ suffices throughout, including both endpoints. □

For $\alpha \geq 2$ define

$$L_\alpha = \frac{4^\beta \mathbb{E} X_\alpha^\beta}{3\Gamma(\beta+1)}, \qquad f(z) = \frac{\mathbb{E}(X_\alpha + z\chi)^\beta}{\mathbb{E} X_\alpha^\beta}, \qquad \tau = f(1). \tag{11.5}$$

Then $f(0) = 1$ and

$$B_\alpha(N) = L_\alpha N^2 f(N^2), \qquad S_\alpha(N) = L_\alpha\{N^2 f(N^2) - \tau\}.$$

On a compact interval $0 < a \leq z \leq b$, the bound $\chi/(X_\alpha + z\chi) \leq 1/a$ dominates every fixed derivative by a constant times $(X_\alpha + b\chi)^\beta$. At zero, dominated convergence gives continuity. These facts justify the differentiations of f below without requiring derivatives at zero.

12 The mass comparison between two and four

Assume $2 \leq \alpha \leq 4$ and $n \geq 2$, and put $\beta = (\alpha-2)/2 \in [0,1]$. The source estimates give $\max m_i / \min m_i < 7\alpha/2$. We shall prove that this bound is incompatible with $F_\alpha \leq 0$ unless the masses are equal. The analytic comparison below holds for every real anchor greater than one.

12.1 The mean and the exact self fraction

For a real anchor $N > 1$ set $x = N^2$, and tilt the joint law of Theorem 11.1 by $(X_\alpha + x\chi)^\beta$. Let

$$m = \frac{\mathbb{E}[x\chi(X_\alpha + x\chi)^{\beta-1}]}{\mathbb{E}(X_\alpha + x\chi)^\beta}, \qquad \delta = \frac{B_\alpha(1)}{B_\alpha(N)}.$$

Both lie strictly between zero and one. Write $r_\alpha(N) = N(\log S_\alpha)'(N)$. Differentiation gives

$$N(\log B_\alpha)'(N) = 2 + 2\beta m, \qquad r_\alpha(N) = \frac{2 + 2\beta m}{1 - \delta}. \tag{12.1}$$

We estimate the two terms together, rather than requiring $m > 1/2$.

Put $Z = 60\chi$. Formula (11.2) gives, with $a = \sqrt{15t}$,

$$\Psi(t) = \mathbb{E} e^{-tZ} = \frac{3}{a}(\coth a - 1/a), \qquad 2t\Psi'(t) = 3 - 3\Psi(t) - 5t\Psi(t)^2.$$

The bound $Z \leq 15E_0/\pi^2$ gives a moment-generating function near zero, so comparison of Taylor coefficients is justified. For $\mu_k = \mathbb{E} Z^k$ it gives

$$\mu_0 = 1, \qquad \mu_k = \frac{5k}{2k+3} \sum_{j=0}^{k-1} \binom{k-1}{j} \mu_j \mu_{k-1-j}.$$

In particular

$$(\mu_1, \ldots, \mu_6) = \left(1, \frac{20}{7}, \frac{90}{7}, \frac{6000}{77}, \frac{4146000}{7007}, \frac{5400000}{1001}\right).$$

For $p(z) = 1 - z/6 + z^2/108$, direct multiplication of these six moments gives

$$A = \mathbb{E}[Zp(Z)] = 9/14, \qquad B = \mathbb{E}[Zp(Z)^2] = 140267/309582,$$
$$C = \mathbb{E}[Z^2p(Z)^2] = 51410/63063, \quad 10A^2 - 9B = 944/17199 > 0, \quad 9B - 5C = 1/594 > 0.$$

Let $b = 5(1+\beta)/x$. Conditioning the numerator defining m on Z, convexity of the power $\beta - 1$ and concavity of the power β give

$$m \geq \mathbb{E}\frac{Z}{(b+Z)^{1-\beta}(b+1)^{\beta}} \geq \mathbb{E}\frac{Z}{b+\beta+(1-\beta)Z}.$$

Here $\mathbb{E}X_\alpha = (1+\beta)/12$ and $\mathbb{E}Z = 1$; the second step is weighted arithmetic–geometric mean in the denominator. Cauchy–Schwarz applied to $\sqrt{Z/d(Z)}$ and $p(Z)\sqrt{Zd(Z)}$, where $d(Z) = b+\beta+(1-\beta)Z$, now gives

$$m \geq \frac{A^2}{(b+\beta)B + (1-\beta)C} > \frac{9x}{x(18-8\beta)+50(1+\beta)}. \tag{12.2}$$

All denominators are positive, including at $\beta = 0, 1$.

For the self fraction we retain $W = X_\alpha + \chi$ in its numerator. Independence and the Gamma moments give

$$\mathbb{E}W = \frac{h}{60}, \qquad \operatorname{Var} W = \frac{35\beta+48}{25200}, \qquad h = 5\beta + 6.$$

Indeed $\operatorname{Var} X_\alpha = (\beta+1)/720$ follows by adding the independent variances and using $\zeta(4) = \pi^4/90$; $\operatorname{Var}\chi = 13/25200$ follows from the second moment above. These operations on the infinite sum are justified by convergence in L^2 of its partial sums. Put $v = \operatorname{Var} W/(\mathbb{E}W)^2 \leq 2/h$. Hölder interpolation gives

$$\mathbb{E}W \leq (\mathbb{E}W^\beta)^{1/(2-\beta)}(\mathbb{E}W^2)^{(1-\beta)/(2-\beta)}, \qquad \mathbb{E}W^\beta \geq \frac{(\mathbb{E}W)^\beta}{(1+v)^{1-\beta}}.$$

The second inequality also holds at zero and one by direct substitution. Jensen in the denominator of δ then gives

$$\begin{aligned}\delta &\geq \frac{1}{x(1+v)^{1-\beta}[1+(x-1)/h]^\beta} \\ &\geq \frac{1}{x[1+(1-\beta)v+\beta(x-1)/h]} \geq \frac{5\beta+6}{x[8+\beta(x+2)]}.\end{aligned} \tag{12.3}$$

Lemma 12.1. *For $2 \leq \alpha \leq 4$ and every real $N > 1$, one has $r_\alpha(N) > 2 + \beta$.*

Proof. By (12.1), it suffices to prove $\beta(2m-1)+(2+\beta)\delta > 0$. Write $D = x(18-8\beta)+50(1+\beta)$ and $E = 8+\beta(x+2)$. The bounds (12.2)–(12.3) bound this expression below by $P/(xED)$, where

$$P = \beta xE(18x - D) + (2+\beta)(5\beta+6)D.$$

Its full expansion in powers of β and $1-\beta$ is

$$\begin{aligned}P/2 = {}& 12(9x+25)(1-\beta)^3 + 20(11x+80)\beta(1-\beta)^2 \\ &+ (7x^2 - 153x + 2825)\beta^2(1-\beta) \\ &+ (x+10)(4x^2 - 50x + 165)\beta^3.\end{aligned}$$

Every coefficient is positive for $x > 0$: the only nonimmediate ones are

$$7x^2 - 153x + 2825 = \frac{(14x-153)^2 + 55691}{28}, \quad 4x^2 - 50x + 165 = \frac{(4x-25)^2+35}{4}.$$

At least one basis term is positive for every $\beta \in [0,1]$. Thus $P > 0$. □

12.2 A tangent at every real anchor above one

Lemma 12.2. *For $n > 1$, $nt > 1$ and $t \neq 1$,*

$$T_\beta(t) = \{r_\alpha(n) - (2+\beta)\}(t-1) - \log \frac{S_\alpha(nt)}{S_\alpha(n)t^2[(1+t^2)/2]^\beta} > 0.$$

At $t = 1$ the expression is zero.

Proof. Use the joint law tilted at the anchor n, and put $Y = n^2\chi/(X_\alpha + n^2\chi)$ and $m = \mathbb{E}_n Y$. Concavity of s^β gives

$$\frac{B_\alpha(nt)}{t^2 B_\alpha(n)} = \mathbb{E}_n[1 + Y(t^2-1)]^\beta \le (1 - m + mt^2)^\beta.$$

Define

$$B_m(t) = (2m-1)(t-1) - \log \frac{1-m+mt^2}{(1+t^2)/2}, \qquad H(t) = t - \frac{3}{2} + \frac{1}{2t^2} = \frac{(t-1)^2(2t+1)}{2t^2}.$$

The raw-row contribution to T_β is therefore at least $\beta B_m(t)$. We claim

$$B_m(t) \ge -(1-2m)_+ H(t), \qquad 0 \le m \le 1. \tag{12.4}$$

For $m \ge 1/2$, apply $\log a \le a - 1$ to its logarithm; the resulting lower bound is $(2m-1)t(t-1)^2/(1+t^2) \ge 0$. For $m \le 1/2$, differentiate with respect to m. Direct simplification gives

$$2H(t) - \partial_m B_m(t) = \frac{(t-1)^2(1-m)(t+1)^2}{t^2(1-m+mt^2)} \ge 0.$$

Integrating from m to $1/2$, where $B_{1/2} = 0$, proves (12.4).

It remains to estimate the self correction quantitatively. Write $B_\alpha(N) = cN^2 f(N^2)$ with $c > 0$, $f > 0$, $f' \ge 0$ and $f'' \le 0$; these signs follow from the joint representation. Put $d = B_\alpha(1) > 0$ and

$$Q(N) = \log(1 - d/B_\alpha(N)), \qquad N > 1.$$

For $z = N^2$, differentiation gives

$$N^3 Q'(N) = \frac{2d}{c} \frac{(f + zf')/f^2}{1 - d/(czf)}.$$

The numerator is positive and nonincreasing, since its z-derivative is $zf''/f^2 - 2z(f')^2/f^3 \le 0$. The denominator is positive and strictly increasing. Hence $N^3 Q'(N)$ strictly decreases. It follows that $Q(N) + n^3 Q'(n)/(2N^2)$ increases up to n and decreases after n. Its strict maximum at n yields

$$Q(n) + nQ'(n)(t-1) - Q(nt) > nQ'(n)H(t), \qquad t \neq 1.$$

At the anchor, (12.1) gives $q := nQ'(n) = \delta(2 + 2\beta m)/(1-\delta) > 0$. If $m < 1/2$, Lemma 12.1 gives

$$q - \beta(1-2m) = \frac{\beta(2m-1) + (2+\beta)\delta}{1-\delta} > 0.$$

If $m \ge 1/2$, $q > 0$ already suffices. Adding the self remainder to (12.4) proves $T_\beta(t) > 0$ because $H(t) > 0$ off one. This argument includes $\beta = 0$: the raw part vanishes and the self remainder is strict. □

12.3 Application to the finite mass distribution

For $0 \le \beta \le 1$ put

$$j_\beta(u) = (1+2\beta)u\log u - \beta u\log((1+u^2)/2) - (1+\beta)(u-1).$$

The finite-law inequality proved in Appendix A states that

$$P\log \mathbb{E}_U u^{1/P} + \mathbb{E}_U j_\beta(u) \ge 0, \qquad P = 3+2\beta, \tag{12.5}$$

whenever $\mathbb{E}_U u = 1$ and the support ratio is at most $7(1+\beta)$. For the original masses, put $t_i = 1/u_i$. Then $nt_i = M/m_i > 1$ and $\mathbb{E}_w t_i = 1$, since $\mathbb{E}_w h(u) = \mathbb{E}_U[uh(u)]$. Averaging T_β cancels the term proportional to $t_i - 1$. Using $\log[(1+u^{-2})/2] = \log[(1+u^2)/2] - 2\log u$ in the remaining logarithm gives

$$F_\alpha = \mathbb{E}_w T_\beta(1/u) + P\log \mathbb{E}_U u^{1/P} + \mathbb{E}_U j_\beta(u).$$

If the masses are not all equal, Lemma 12.2 makes the first term strictly positive. The support condition is exactly the bound on the maximum-to-minimum mass ratio. Equation (12.5) makes the remaining sum nonnegative. Hence $F_\alpha > 0$, contradicting the necessary condition at the end of Section 3.

13 A uniform mass-ratio bound above four

Theorem 13.1. *For $\alpha \ge 4$ and $n \ge 5$, the original force and centering equations imply* $\min_i m_i / \max_i m_i > 1/5$.

The minimum-mass site will bound the conserved current from above, and the maximum-mass site will bound it from below. We use both bounds for the mass ratio and then for the count. Denote the common radial row by $C_0 > 0$ and the largest mass by m_*. Replace masses by m_i/m_* and positions by ℓz_i, where $\ell^\alpha = C_0/m_*$. The new common row is $C_0/(m_*\ell^\alpha) = 1$, and the largest mass is one. Both sides of the force equations scale by $1/(m_*\ell^{\alpha+1})$ when the multiplier is replaced by $\lambda/(m_*\ell^{\alpha+2})$. Centering and distinctness are preserved. We use these normalized variables throughout this section. Write $x = (2R)^{-2}$, $r = 4/\alpha$, $q = \min_i m_i$, $U = M - 1$, $T = M - q$. Centering at a unit mass gives $U \ge 1$. Fix a maximum-mass site o. Its contribution to any other radial row gives $d_{oj} > 1$. On the other atoms write $y = d_{oj}^{-2}$ and $d\nu = y^{\alpha/2-2}d\mu$, where μ is the mass measure. Thus $x \le y < 1$, $\nu \le \mu$, and $\int y^2 d\nu = 1$. As $W = \int (y^{\alpha/2})^{1-r} d\mu$, Hölder gives $W \le U^r$. At any site $\sum_{j\ne i} m_j d_{ij}^2 = M/(2x)$ by centering. Jensen at the minimum therefore yields $2x \le M/T^{1+2/\alpha}$, and

$$k = Wx^2 \le [M/(2T)]^2 (U/T)^r. \tag{A}$$

13.1 One two-sided current primitive

Route each pair along its shorter arc of length l with weight $m_i m_j K(l)$, $d(l) = \sin(\sqrt{x}\,l)/\sqrt{x}$, $K(l) = d(l)^{-\alpha-1}\cos(\sqrt{x}\,l)$ up to $L = \pi R$, zero thereafter. An antipodal pair has zero current. The crossing sum J is constant since the jump at a label is its zero tangential force; each incident directional sum is at most J. From every label the cumulative mass $F(t)$ on either side up to arclength t, excluding the marked atom, satisfies $F(t) \le t$: each successive mass is at most one and below its preceding chord to power α, hence below its preceding gap. Restarting at the first atom of an arc gives $\mu(I) \le 1 + |I|$. In particular, if $f \ge 0$ has interval superlevel sets on one semicircle, the layer-cake identity gives

$$\int f\,d\mu = \int_0^{\sup f} \mu\{f > s\}\,ds \le \sup f + \int f(l)\,dl.$$

This is the one-hump occupation bound used below; its integral is with respect to actual arclength, not the mass measure.

At the minimum take the side with nonantipodal radial sum at most $1/2$, put $A_{\min} = (2q)^{-1/\alpha}$ and $z = x/A_{\min}^2$. To bound z, keep both adjacent terms in each unit radial row and use the mass floor q. If d_i are the adjacent chords, summing over the sites gives $n^{-1}\sum_i d_i^{-\alpha} \le 1/(2q)$. Jensen's inequality for the inverse power and then for the chord sine gives

$$(2q)^{1/\alpha} \le n^{-1}\sum_i d_i \le 2R\sin(\pi/n), \qquad z \le \sin^2(\pi/n) \le \sin^2(\pi/5).$$

On the chosen side each incident chord has $d^{-1} \le A_{\min}$. Multiplying the radial terms by $qd^{-1}\sqrt{1-xd^2}$ bounds twice the incident current by $qA_{\min}\sqrt{1-z}$. For nonincident crossings let F_+, F_- be the cumulative masses on the two semicircles measured from the minimum site. A crossing pair follows the short path through this site if its length $u+v$ is at most L. The total nonincident current is $\iint F_+(u)F_-(v)K''(u+v)\,du\,dv$ in the distributional sense: integrating twice from any atom pair gives its exact kernel. In the interior $K'' = \cos(\sqrt{x}\,l)d^{-\alpha-3}[(\alpha+1)(\alpha+2) - \alpha^2 xd^2] \ge 0$; the zero extension adds the positive mass $x^{1+\alpha/2}\delta_L$ since $K'(L-) = -x^{1+\alpha/2}$. On $u \le v$ charge only $F_-(v) \le v$ and use symmetry on $v \le u$. The resulting inner primitive for $u < L/2$ is $\int_u^\infty vK''(u+v)\,dv = K(2u) - uK'(2u)$, zero for $u > L/2$. The primitive includes the boundary mass of K'' at L. Integrating from a surviving atom at distance t to $L/2$ yields

$$\frac{3}{4\alpha}[d(2t)^{-\alpha} - d(L)^{-\alpha}] + \frac{t}{2}K(2t)$$

since $K = -(d^{-\alpha})'/\alpha$. Both triangles thus give twice the full nonincident cost at most $\sum_i m_i d(t_i)^{-\alpha}P_\alpha(\theta_i)$, $\theta_i = \sqrt{x}\,t_i$, where $P_\alpha = 0$ for $\theta \ge \pi/4$ and

$$P_\alpha(\theta) = (2\cos\theta)^{-\alpha}\left[\frac{3(1-\sin^\alpha(2\theta))}{2\alpha} + \theta\cot(2\theta)\right].$$

Here t_i is measured from the marked minimum, and the weights $m_i d(t_i)^{-\alpha}$ sum to its unit radial row.

Put $w = \tan^2\theta$, $Y = w/(1+w)$ and $b = \sin(2\theta)$. Since $(1-b^\alpha)/\alpha = \int_b^1 s^{\alpha-1}ds$ and $2\cos\theta > 1$, $P_\alpha \le P_4$. Write $\theta\cot(2\theta) = (1-w)\mathcal{A}(w)/2$, where $\mathcal{A}(w) = \int_0^1 (1+wt^2)^{-1}dt$. Then $\mathcal{A}(w) = 1 - w\int_0^1 t^2/(1+wt^2)\,dt \le 1 - w/(3+9w/5)$ by Jensen under $3t^2dt$. Substitution gives

$$16(1/16 - 5Y^2/32 - P_\alpha) \ge \frac{(3+w)(5-16w)^2 + w^2(82+94w+295w^2+405w^3+80w^4)}{120(1+w)^2(5+3w)} \ge 0.$$

Thus twice the nonincident current is at most $1/16$. For the count bound we use $m_i \ge q$ in the unit radial row, so $d(t_i)^\alpha \ge q$ and $Y = xd(t_i)^2 \ge z2^{-2/\alpha} \ge z/\sqrt{2}$. The stronger bound is $e(z) = 1/16 - 5z^2/64$. It also bounds the zero contributions since $z \le \sin^2(\pi/5)$ gives $e(z) > 0$.

13.2 One barycenter at the maximum

Use the whole measure $d\sigma = (y-x)d\nu$ there, $D = \int d\sigma > 0$, $\beta = D^{-1}\int y\,d\sigma$, $v = x/\beta < 1$. Antipodes have zero weight and contribute no current. Then $H = 1 - k = \int(y+x)d\sigma = D\beta(1+v)$. The two incident directions give $2J \ge \int y^2/\sqrt{y-x}\,d\sigma$. The integrand is convex in $y > x$: at $w = y - x$ its second derivative times $4w^{5/2}$ is $3w^2 - 2xw + 3x^2 > 0$. Jensen's inequality therefore gives

$$(2J)^2 \ge \frac{H^2\beta}{(1-v)(1+v)^2}. \tag{J}$$

The comparison atom β denotes a mean, not a physical solution.

13.3 The strong one-fifth ratio

If $q \le 1/5$, $T \ge U + 4/5$ and (A) gives $k \le [(U+1)/(2U+8/5)]^2 \le 25/81$. At the minimum $qA_{\min} \le (5/2)^{1/4}/5 < 63/250$, so $2J < 629/2000 =: B_0$. Set $B = 3\beta$. The definition $\int y\, d\sigma = D\beta$ gives the exact signed identity $2D/3 = \int (y-x)(1-y/B)d\nu$. Restricting this integral to $x \le y \le B$ can only increase it, since its omitted integrand is negative. On that interval $\nu \le \mu$. On each true semicircle the positive part under μ has one hump of height at most $(B-x)^2/(4B)$. With $w = \sqrt{y}$ the arclength element is $|dw|/(w^2\sqrt{1-x/w^2})$; its weighted integral is at most $\int_{\sqrt{x}}^{\sqrt{B}} (1 - w^2/B)dw$. Thus the preceding arc occupation gives

$$D \le \beta(3-v)^2/4 + \sqrt{\beta}[2\sqrt{3} - \sqrt{v}(3 - v/3)].$$

The first term after multiplication by $\beta(1+v)$ is at most $3\beta^2$, since $(1+v)(3-v) \le 4$ and $3 - v \le 3$. For the second, $(1+v)(3-v/3) \ge 2\sqrt{3v}$ gives

$$(1+v)\{2\sqrt{3} - \sqrt{v}(3-v/3)\} \le 2\sqrt{3} < 4.$$

Thus $H = \beta(1+v)D \le 3\beta^2 + 4\beta^{3/2}$. But $H \ge h = 56/81$ and (J) gives $H^2\beta \le (32/27)B_0^2 < h^2/4$, since $(1-v)(1+v)^2 \le 32/27$. Hence $H < 3/16 + 1/2 = 11/16 < h$, impossible. This proves Theorem 13.1.

13.4 The physical count for every exponent above four

Lemma 13.2. *If $n \ge 5$ and $\alpha \ge 4$, then $M/\max_i m_i > 4$.*

Proof. Suppose now $M \le 4$. Strict centering at the maximum gives $1 < U \le 3$ (equality at the lower end would force coincident antipodes). Let $d = \sin^2(\pi/5) = (5-\sqrt{5})/8$, $q_* = 3/4$. We have $q \le U/(n-1) \le q_*$ and $T \ge 1 + 3U/4$. Hence $k \le \bar{k} = 64/169$ by (A). The adjacent-chord bound above gives $0 \le z \le d$. Since $W \le U^r \le 3^r$, also $k \le (3/(2q))^r z^2$. At the maximum center of mass gives $\int d\sigma/y \le \int (1 - x/y)d\mu = (U-1)/2 \le 1$. By Cauchy $D \le \beta \int d\sigma/y \le \beta$; hence $\beta^2(1+v) \ge H$. Squaring (J) now gives $(2J)^4 \ge H^5/[(1-v)^2(1+v)^5]$. The denominator peaks at $3/7$ by differentiation. Thus $2J \ge c(1-k)^p$, with $p = 5/4$, $c = \sqrt{7/4}(7/10)^{5/4}$; this power is a consequence of (J), not a switch of measure. Conversely the same two-sided primitive gives $2J \le q(2q)^{-r/4}\sqrt{1-z} + e(z)$. Set $k_* = 2^r z^2$. The two radius bounds give the correlated estimate

$$k - k_* \le \bar{k}[1 - (q/q_*)^r] \le (\bar{k}/q_*)(q_* - q);$$

indeed $k_* \ge (q/q_*)^r k$. Thus the lower bound loses at most $cp(\bar{k}/q_*)(q_* - q)$ relative to $c(1-k_*)^p$. On the right the incident term gains at least $(3/4)(2/3)^{1/4}\sqrt{1-d}(q_* - q)$ when q increases to q_* at fixed r, z, while $e(z)$ is independent of q. Using $c < 8471/10000$, $(2/3)^{1/4} > 903/1000$, and $\sqrt{1-d} > 809/1000$ leaves a positive net slope $26843567/2028000000$. Thus it suffices at fixed r, z to disprove the bound at $q_* = 3/4, k_* = 2^r z^2$. This is a scalar comparison, not a variation of the configuration.

The remaining difference

$$V(r,z) = c(1 - 2^r z^2)^p - q_*(2/3)^{r/4}\sqrt{1-z} - e(z)$$

is separately concave on $0 \le r \le 1$, $0 \le z \le d_0 = 691/2000 > d$. In r the first term's second derivative has the sign of $p2^r z^2 - 1 < 0$ when $z > 0$, and the negative exponential is concave. In z the first second derivative is $-2Bpc(1 - Bz^2)^{p-2}[1 - (2p-1)Bz^2]$ for $B = 2^r$. Its magnitude exceeds $2p(847/1000)(5/8)$ since $2(2p-1)d_0^2 < 3/8$. The two positive contributions sum to at most $3/8 + 5/32$, since $(1-d_0)^3 > 1/4$. Thus only the four corners remain. At $z = 0$ use $c > 847/1000$, $(2/3)^{1/4} < 904/1000$. At $z = d_0$ also use $\sqrt{1-d_0} < 8091/10000$, $(1-d_0^2)^{1/4} > 968/1000$ and $(1-2d_0^2)^{1/4} > 934/1000$. All follow by positive powers; substitution at all four corners gives $V > 1/2500$. This contradiction proves the strict count bound. □

14 One quartic anchor for all exponents at least four

Let $a = \alpha \ge 4$, $b = \pi/n$, $F(y) = -\log\sin(\pi y)$, $h(u) = u\log u - u + 1$, $b_0 = u - 1 - \log u$, $H = \mathbb{E}_U h(u)$. For $n \ge 5$ the original current bounds (Theorem 13.1, Lemma 13.2) give $u_i = nw_i \ge 1/5$ and $w_i \le 1/4$. The purpose of this section is to contradict the two physical comparisons for every nonuniform mass word on this support. We first calibrate the trial configuration at exponent four. We then bound its change with $a-4$ by a single affine function. Two scalar inequalities, one for the intercept and one for the slope, will pay this bound for the whole ray $a \ge 4$.

14.1 The quartic anchor

Prescribe consecutive cells of widths w_i with midpoints ϕ_i; d_{ij} is their forward fractional separation, $k_{ij}^{\phi} = Q_R \sin(\pi d_{ij})^{-a}$. For $v_i > 0$ put $Z = \sum_{i\ne j} k_{ij}^{\phi} v_i v_j$, $P_{ij} = k_{ij}^{\phi} v_i v_j / Z$ and $U = (Z/M)\exp[2\mathbb{E}_w \log(m_i/v_i)]$. By the radial rows the directed $\pi_{ij} = m_i m_j |z_i - z_j|^{-a}/(MC)$ has both marginals w. For the physical and trial forward angles t, s write $r = \sin(s/2)/\sin(t/2)$, $B = (s-t)\cot(t/2)/2$. The unordered torque derivative is $-aMC\mathbb{E}_\pi B/2 = 0$. Indeed, the displacement from the physical angles to the trial angles is a permitted angular variation, and every physical angular derivative vanishes. The two marginals of π then give the entropy identity

$$D(\pi\|P) = 2\mathbb{E}_w \log(m_i/v_i) + \log[Z/(MC)] + a\mathbb{E}_\pi \log r = \log(U/C) + a\mathbb{E}_\pi \log r.$$

Since $\mathbb{E}_\pi B = 0$, this proves $\log(U/C) = D(\pi\|P) + a\mathbb{E}_\pi(B - \log r) \ge 0$ by the log-sum inequality and the tangent inequality for log-sine. Proposition 3.3 gives the other bound $B_{\mathrm{root}} \le C$.

Use (3.7) only at four; its real diagonal is

$$S_4(N) = (N^2-1)(N^2+11)/45 \qquad (N > 1).$$

To verify this at noninteger N, let K be the ratio in (3.2), $z = |\xi|$. The Gamma recurrence and reflection give $\mathrm{d}\mu_2 = 2z\,\mathrm{d}\xi/\sinh(\pi z)$ and $\mathrm{d}\mu_4 = 2(1+z^2)\,\mathrm{d}\mu_2/3$. Expanding $\coth(\pi z/N) - \coth(\pi z)$ in exponentials gives $\int K\,\mathrm{d}\mu_2 = (N^2-1)/3$ and $\int z^2 K\,\mathrm{d}\mu_2 = (N^4-1)/30$ by the ordinary even sums $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. This gives the displayed polynomial.

Choose the gauge

$$\begin{aligned} r_4(y) &= (1-y^2)(1+11y^2)[\sin(\pi y)/y]^4/45, \quad c(y) = r_4(y)^{-1/2}, \\ v_i &= \sqrt{w_i}\, c(w_i)\sin(\pi w_i)^{a/2}, \quad \rho_{ij} = \frac{\sqrt{\sin(\pi w_i)\sin(\pi w_j)}}{\sin(\pi d_{ij})}, \\ Z_s &= \sum_{i\ne j} \sqrt{w_i w_j}\, c(w_i)c(w_j)\rho_{ij}^s, \\ p(s) &= \frac{R_n(s)}{R_n(4)}, \quad R_n(s) = \sum_{r=1}^{n-1} \left(\frac{\sin b}{\sin(rb)}\right)^s. \end{aligned}$$

Both arcs contain the half endpoint cells, hence $0 < \rho_{ij} \le 1$ by monotonicity on the first half and log-sine concavity. At $a = 4$ the gauge satisfies $v_i^2 S_4(1/w_i) = w_i$. The diagonal in the cell square therefore sums to one and the off-diagonal sum is Z_4, giving $\Delta_4 = 1 - Z_4 \ge 0$. At uniform masses $Z_s = p(s)$. Dropping the positive power mean in (3.5) to the geometric mean at this very a, cancellation of Q_R, M, v yields

$$\begin{aligned} \log(B_{\mathrm{root}}/U) &\ge L_4 + (a-4)D + \log p(a) - \log Z_a, \qquad (14.1) \\ L_4 &= \log S_4(n) + \mathbb{E}_U \log u - \mathbb{E}_w \log[uS_4(1/w)], \qquad D = F(1/n) - \mathbb{E}_w F(w). \end{aligned}$$

For example $Z = Q_R Z_a$ and $2\log v_i = \log w_i - \log r_4(w_i) + a\log\sin(\pi w_i)$, which verifies the normalization directly.

14.2 Two mass tangents

Put $P(y) = \log[(1-y^2)(1+11y^2)]$ and $J = 2yP' + y^2P''$. For $0 < y \le 1/4$,

$$\frac{J}{y} = \frac{22y(3+11y^2)}{(1+11y^2)^2} - \frac{2y(3-y^2)}{(1-y^2)^2} \le \frac{22y(3+11y^2)}{(1+11y^2)^2} - 6y < 6.2.$$

For the last inequality subtract the middle expression from 6.2 and multiply by $5(1+11y^2)^2$. The result is $K = 3630y^5 + 3751y^4 - 550y^3 + 682y^2 - 300y + 31$. Since $K'' = 44(1650y^3 + 1023y^2 - 75y + 31) > 1200$ by completing the quadratic square, its tangent at $2/11$ gives $K \ge 685/1331 - (420/121)^2/2400 = 14923/29282 > 0$.

The second derivative of $uP(u/n)$ is $J(u/n)/u < 6.2/n$. Subtract its unit tangent in L_4; the affine term cancels in the uniform mean. The remaining terms of L_4 equal $3h - b_0$ modulo affine functions. Hence $L_4 \ge \mathbb{E}_U[3h - b_0 - 3.1(u-1)^2/n]$.

For the principal sine, $-y(yF)'' = C_0 = 2z\cot z - z^2\csc^2 z$, $z = \pi y$. The cot partial fractions give nonnegative even powers in $1 - z\cot z$; as $1 - C_0 = (1 - z\cot z) + z(1 - z\cot z)'$, $(1-C_0)/z^2$ increases. The endpoint $\pi/4$ gives $C_0 \ge 1 - 43y^2/4$ since $64(1 - \pi/2 + \pi^2/8) < 43$. Integrate the negative perspective from its unit tangent: $D \ge \mathbb{E}_U[h - 43(u-1)^2(u+2)/(24n^2)]$.

14.3 The two intervening arcs

At a nonneighbor half-rank $2 \le r \le \lfloor n/2 \rfloor$, $j = i + r$, put $T = (u_i + u_j)/2$. Let I, I' be the sums of the two strictly intervening sets of labels, of sizes $k = r - 1, k' = n - r - 1$, both positive. Then $2T + I + I' = n$. Write $g = -\log \rho_{ij}$ and $g_r = \log[\sin(rb)/\sin b]$. Endpoint log-sine concavity and two integrations give

$$2g \ge 2\log\frac{\sin b(T+I)}{\sin bT} = \int_0^I \int_0^{I'} b^2 \csc^2[b(T+s+s')]\,\mathrm{d}s'\,\mathrm{d}s > 0.$$

The middle sines in the mixed difference agree, as do the outer two. Scale the integration to $[0,1]^2$ and take log Jensen with the fixed probability of density $kk'b^2\csc^2[b(1+kz+k'z')]/(2g_r)$ there, applied to the ratio of the two squared cosecants. The resulting lower for $\log(g/g_r)$ is $\log(I/k) + \log(I'/k')$ minus twice the integral of the change in $\log\sin$ from $b(1+kz+k'z')$ to $b(T + Iz + I'z')$. Over all cyclic translates of the pair, $\mathbb{E}_U T = 1$, $\mathbb{E}_U I = k$ and $\mathbb{E}_U I' = k'$ because each label occurs once in each translated slot. Hence the mean of $T + Iz + I'z'$ is $1 + kz + k'z'$ for each z, z'. Concavity of log-sine makes the negative integral nonnegative in this mean. By log concavity each intervening average has mean logarithm at least $\mathbb{E}_U \log u$ (use all cyclic translates). Thus $\mathbb{E}_U \log(g/g_r) \ge 2\mathbb{E}_U \log u$.

14.4 One supporting line for the entire ray

Write $x = 1/n$, $t = a - 4$, $R = \mathbb{E}_U b_0$, $V = \mathbb{E}_U(u-1)^2$ and

$$A = 7R + 11.5x^2V, \quad B = R + 2x^2V, \quad q_r = e^{-g_r}, \quad p_\infty = 2/R_n(4).$$

For $G(y) = \log[\sin(\pi y)/y]$, the even partial fractions of $-G''$ increase from $\pi^2/3$ to $2\pi^2 - 16 < 4$ on the cap. Thus $(\log r_4)'' \le -2 + 22 - 4\pi^2/3 < 7$. The mean logarithm of the actual prefactor $\sqrt{u_iu_j}c(w_i)c(w_j)$ relative to $c(x)^2$ is at least $-R - 3.5x^2V$ by its unit tangent. The numerator log-sine in g has nonpositive mean displacement by concavity; its endpoint log-sine equals $\log u + G(xu)$ up to constants. Hence $\mathbb{E}_U(g - g_r) \le R + 2x^2V$. To see how these three estimates enter the derivative, fix a nonneighbor rank and let the average below run over its n cyclic translates. Put

$$a_i = \frac{\sqrt{u_iu_{i+r}}c(w_i)c(w_{i+r})}{c(x)^2}, \qquad s = 4 + t.$$

The estimates just proved give

$$\begin{aligned}\mathbb{E}_U \log a_i &\geq -R - 3.5x^2V,\\ \mathbb{E}_U \log(g_i/g_r) &\geq -2R,\\ \mathbb{E}_U(g_i - g_r) &\leq R + 2x^2V.\end{aligned}$$

Consequently their sum, with the last line multiplied by $-s$, is at least $-A - Bt$. Exponential Jensen now gives the explicit bound

$$\mathbb{E}_U[a_i g_i e^{-sg_i}] \geq g_r e^{-sg_r} e^{-A-Bt}.$$

For each directed rank, $c(x)^2 = 1/R_n(4)$ and $w_i = xu_i$. Thus the contributions to $(Z_s)'$ and $p'(s)$ are, respectively,

$$-\frac{\mathbb{E}_U[a_i g_i e^{-sg_i}]}{R_n(4)} \qquad \text{and} \qquad -\frac{g_r e^{-sg_r}}{R_n(4)}.$$

Their difference is at most $g_r e^{-sg_r}(1 - e^{-A-Bt})/R_n(4)$. Since $A, B \geq 0$, $1 - e^{-A-Bt} \leq A + Bt$. The nearest regular rank has $g_r = 0$; its actual derivative is nonpositive and can be discarded in this upper bound. Swapping the pair endpoints gives equal contributions at ranks r and $n - r$. Pairing these ranks, with half weight at an even antipode, yields

$$(Z - p)'(4 + t) \leq p_\infty \sum{}^* q_r^4 g_r e^{-g_r t}(A + Bt)$$

where the star means half weight at the even antipodal rank and unit weight at other nonneighbor half-ranks. This does not substitute a surrogate inside ge^{-sg}. For $g = g_r$, $q = e^{-g}$ let $K(t) = \int_0^t ge^{-gs}(A + Bs)\,\mathrm{d}s$. As $g \geq \log(2\cos(\pi/5)) > 1/4$ and $B \leq 4A/23$, $K'' = -ge^{-gt}[g(A + Bt) - B] \leq 0$. Its tangent at one bounds the whole ray. The slope is at most $(A + B)/e$. On $[0, 1]$ the weighted mean of s against the decreasing e^{-gs} is at most $1/2$ by reflection. The intercept is thus at most $(A + B/2)(1 - q - qg) \leq (A + B/2)(1 - q)^2/(1 + q)$, using $g \geq 2(1 - q)/(1 + q)$.

The ordinary sine moments, the second by cotangent multiplication and the fourth as above, give for $y = x^2$, $c_0 = \pi^2$,

$$2 + 2\sum{}^* q_r^2 = c_0(1 - y)\operatorname{sinc}^2 b/3, \quad 2 + 2\sum{}^* q_r^4 = c_0^2(1 - y)(1 + 11y)\operatorname{sinc}^4 b/45.$$

Here $\operatorname{sinc} b = \sin(b)/b \leq e^{-b^2/6}$. For an elementary verification, $\sin b - b\cos b \geq 0$ follows by differentiation from zero. The derivative of $(1 - b^2/3)\sin b - b\cos b$ is $b(\sin b - b\cos b)/3 \geq 0$. Hence $e^{b^2/6}\operatorname{sinc} b$ is nonincreasing from its value one at zero, proving the bound. The first sum is $< .65$. For the second the logarithmic factor $(1 + 10y - 11y^2)e^{-2c_0 y/3}$ increases to $y = 1/25$: the derivative sign factor $10 - 22y - (2c_0/3)(1 + 10y - 11y^2)$ decreases but exceeds .024 there. The resulting upper for $2 + 2\sum{}^* q_r^4$ is < 2.29998, using $3.14159 < \pi < 3.14160$ and the first six exponential terms at $2(3.14159)^2/75$. Hence $\sum{}^* q_r^4 < .15$. Since $q^2(1 - q)^2/(1 + q) \leq 1/20$ (for $q \leq 1/4$ the numerator is at most $(3/16)^2$; otherwise use $1/16$), the integrated bound is

$$Z_a - p(a) \leq -\Delta_4 + p_\infty\{\rho(A + B/2) + t\lambda(A + B)\}, \qquad \rho = .0325, \quad \lambda = .0552 > .15/e.$$

14.5 Bounding the error using the mass distribution

We can now state the comparison delivered by these estimates.

Proposition 14.1. *For a nonuniform positive mass word with $n \geq 5$, $u_i = nw_i \geq 1/5$ and $w_i \leq 1/4$, the preceding trial gauge satisfies, for every $a \geq 4$,*

$$\log(B_{\mathrm{root}}/U) > H/50 + \Delta_4/p(a) > 0.$$

At the uniform word $B_{\mathrm{root}} = U$.

Proof. Use $\log(Z_a/p(a)) \le (Z_a - p(a))/p(a)$ in (14.1), with $p_\infty/p(a) \le 1$ only on the nonnegative error term. By the two mass tangents above it suffices to prove $W = W_0 + tW_1 > 0$ off $u = 1$, where $k = 43/24$ and

$$W_0 = 2.98h - (1 + 7.5\rho)b_0 - (3.1x + 12.5\rho x^2)(u-1)^2,$$
$$W_1 = h - 8\lambda b_0 - [13.5\lambda + k(u+2)]x^2(u-1)^2.$$

This pays the literal intercept and the sine slope simultaneously without a separate uniform dropout maximum. The function u^2W'' is concave: its respective parts are $2.98u - (1+7.5\rho) - 2(3.1x + 12.5\rho x^2)u^2$ and $u - 8\lambda - x^2u^2(27\lambda + 6ku)$. Apply this observation separately to W_0 and W_1. For each, the strict superlevel set $\{u : u^2W_i''(u) > 0\}$ is an interval containing one, because u^2W_i'' is concave and positive at one. On that interval, $W_i(1) = W_i'(1) = 0$ and strict convexity give $W_i(u) > 0$ for $u \ne 1$. On either remaining portion of the support $[1/5, 1/(4x)]$, $W_i'' \le 0$, so W_i is bounded below by the chord between its endpoint values. The value adjoining the central interval is positive by the preceding argument. It therefore remains only to check the two original support endpoints. This also covers the case where one or both outer portions are empty. At $u = 1/5$ both costs are worst at $x = 1/5$. On the other end write $s = 1/(4x) \ge 5/4$; each $W_i(s)$ is $E(s) = Ph(s) - Qb_0(s) - (ls + m)(1 - 1/s)^2$, with rows

$$(P, Q, l, m) = (2.98, 1 + 7.5\rho, 3.1/4, 12.5\rho/16), \quad (1, 8\lambda, k/16, (2k + 13.5\lambda)/16).$$

Put $J = s^4E'' = Ps^3 - Qs^2 - 2(l - 2m)s - 6m$. Direct substitution gives the following strict lower bounds, the first two at $x = 1/5$ and the last five at $s = 5/4$:

	$W_i''(1)$	$W_i(1/5)$	J	J'	J''	E	E'
0	.46	.0107	1.91	9.41	19.86	.0130	.1307
1	.068	.00064	.712	4.44	6.61	.00064	.0252

Here $\log 5 < 1.6095$, $\log(5/4) > .22314$ follow from the positive $2\operatorname{arctanh} z$ series at $z = 2/3, 1/9$, with 24 and four terms and tail at most $2z^{2j+1}/[(2j+1)(1-z^2)]$. Machin's alternating formula gives the stated π bounds. Since $J''' = 6P > 0$, both cap endpoint functions are positive on their whole ray. The left endpoint and unit curvature therefore complete the claim for every label. Since $h''(u) = 1/u > 0$ and $h(1) = h'(1) = 0$, a nonuniform word has $H > 0$. Averaging $W_0 + (a-4)W_1$ thus gives

$$\log(B_{\mathrm{root}}/U) > H/50 + \Delta_4/p(a) > 0.$$

At the uniform word $B_{\mathrm{root}} = U$. □

15 Completion of the classification

We now collect the interval arguments. Their hypotheses all refer to the original configuration in Theorem 1.1; the auxiliary probability laws and comparison meshes have been eliminated before each argument returns to that configuration.

Proof of Theorem 1.1. The logarithmic endpoint is Proposition 2.3. Proposition 2.4 handles every exponent when $2 \le n \le 4$. We may therefore assume $\alpha > 0$ and $n \ge 5$.

For $0 < \alpha \le 1$, Theorem 4.10 proves mass equality. For $1 < \alpha \le 9/5$, the force inequalities at two additional bodies and the resulting bounds on the comparison functional give Theorem 7.15. For $9/5 < \alpha < 2$, Theorem 8.1 supplies the mass bound used in Section 9. The last comparison of that section is a strict contradiction for unequal masses.

For $2 \le \alpha \le 4$, Theorem 10.5 gives $\max m_i / \min m_i < 7\alpha/2$. Write $\beta = (\alpha - 2)/2$; this is exactly the support bound $7(1 + \beta)$ required in Section 12. The strict tangent inequality there,

together with the finite-law inequality in Appendix A, gives $F_\alpha > 0$ for unequal masses. This contradicts the physical inequality $F_\alpha \le 0$ from Section 3. The argument includes $\alpha = 2$.

For $\alpha \ge 4$, Theorem 13.1 gives $m_i > \max m/5 \ge M/(5n)$; Lemma 13.2 gives $M/\max m > 4$, so all widths $x_i = m_i/M$ meet the cap in Proposition 14.1. Its finite-edge upper uses the actual angular equilibrium, and Proposition 3.3 uses the actual radial rows. For nonuniform mass the strict comparison contradicts $B_{\text{root}} \le C \le U$. Thus masses are equal.

These intervals cover all exponents. Corollary 2.2 now supplies regularity in every branch. Conversely, an equal-mass regular polygon has center of mass zero. Its opposite tangential ranks cancel, the antipodal tangent is zero when present, and every radial row has the same positive value C. Thus it satisfies the force equations with $\lambda = C/(2R^2) > 0$. □

A A finite-law entropy inequality

Theorem A.1. *Let $3 \le P \le 5$, $\beta = (P-3)/2$, and let u be a positive random variable on a finite probability space. Suppose $\mathbb{E}u = 1$ and*

$$\frac{\max u}{\min u} \le \frac{7(P-1)}{2}.$$

Then

$$P \log \mathbb{E}u^{1/P} + \mathbb{E}j_\beta(u) \ge 0,$$

with equality exactly when $u = 1$ almost surely. Here

$$j_\beta(u) = (P-2)u\log u - \beta u \log((1+u^2)/2) - (1+\beta)(u-1).$$

A.1 An affine change of probability

Write $a = P - 1 \in [2,4]$, $k = 2/7$, and $R = a/k$. Since $\mathbb{E}u = 1$, the support-ratio hypothesis implies $1/R \le u \le R$. Choose

$$c = \frac{1}{a-k}, \qquad d = 1 - c, \qquad \rho(u) = cu + d.$$

This choice makes $\rho(1/R) = 1 - 1/a$ and $\rho(R) = 9/2$. Thus $\rho > 0$ on the interval and $\mathbb{E}\rho = 1$. Define

$$h(u) = j_\beta(u) + \rho(u)\log u - P\rho(u)\log\rho(u) - (1 - Pc)(u-1).$$

Then $h(1) = h'(1) = 0$. If the original probabilities are p_i and $A = \mathbb{E}u^{1/P}$, cancellation gives

$$P\log A + \mathbb{E}j_\beta(u) = \mathbb{E}h(u) + P\sum_i p_i\rho(u_i)\log\frac{\rho(u_i)A}{u_i^{1/P}}. \tag{A.1}$$

The last sum is relative entropy between the probability vectors $p_i\rho(u_i)$ and $p_i u_i^{1/P}/A$, so is nonnegative. We prove $h(u) > 0$ on $[1/R, R] \setminus \{1\}$ by locating its two changes of curvature and checking the endpoints.

A.2 Two changes of curvature

Differentiation gives

$$u^2\rho(u)h''(u) = N(u) = du - d^2 - kc^2u^2 + (a-2)\frac{u\rho(u)(1-u^2)}{(1+u^2)^2}.$$

Put $D = (7a-2)^2$, $t = a-2 \in [0,2]$, and $W(u) = D(1+u^2)^2 N(u)$. We have $W(1) = 28(7t+3) > 0$ and $W(0) = -(7t+5)^2 < 0$. We show that W has exactly one zero on each side of one.

First write $W(1+v) = \sum_{i=0}^{6} p_i v^i$. Direct expansion gives $p_0 = W(1)$, $p_6 = -14$, and

$$\begin{aligned} p_1 - 112 &= 2(2-t)(49t^2 + 168t + 46) \geq 0, \\ p_2 + 160 &= 3(2-t)(49t^2 + 168t + 88) \geq 0, \\ p_3 - 228 &= t(752 + 77t - 49t^2) \geq 0, \\ p_4 &= 147t^2 + 441t + 37 > 0. \end{aligned}$$

Consequently, for $v > 0$,

$$\begin{aligned} -v^6 \frac{d}{dv} \frac{W(1+v)}{v^5} &= 5p_0 + 4p_1 v + 3p_2 v^2 + 2p_3 v^3 + p_4 v^4 + 14v^6 \\ &\geq 5p_0 + v(448 - 480v + 456v^2) + p_4 v^4 + 14v^6 > 0. \end{aligned}$$

The quadratic is positive since its discriminant is negative. The quotient decreases from $+\infty$ to $-\infty$, so it has exactly one zero, giving the required zero $\eta > 1$.

For the interval below one, put $V(v) = (1+v)^6 W((1+v)^{-1}) = \sum_{i=0}^{6} q_i v^i$. Here $q_0 = W(1) > 0$, and expansion gives

$$\begin{aligned} q_1 &= 98t^3 + 140t^2 + 596t + 208 > 0, \\ q_2 &= 343t^3 + 490t^2 + 784t + 148 \geq 148, \\ q_3 &= 441t^3 + 483t^2 + 344t - 36 \geq -36, \\ q_5 &= (7t+5)(7t^2 - 23t - 18) < 0, \\ q_6 &= -(7t+5)^2 \leq -25. \end{aligned}$$

The sign of q_5 follows from $7t^2 \leq 14t$. It follows that

$$\begin{aligned} -v^5 \frac{d}{dv} \frac{V(v)}{v^4} &= 4q_0 + 3q_1 v + 2q_2 v^2 + q_3 v^3 - q_5 v^5 - 2q_6 v^6 \\ &\geq 4q_0 + 3q_1 v + v^2(296 - 36v + 50v^4) > 0, \end{aligned}$$

where $36v \leq 27 + 9v^4$ by weighted arithmetic–geometric mean. Again the quotient decreases from $+\infty$ to $-\infty$. Since $u = (1+v)^{-1}$ reverses order, this gives exactly one zero $\xi \in (0,1)$ of W below one. Thus h'' is negative, positive, negative on the intervals separated by ξ and η.

A.3 The lower endpoint

Put $K_0(a) = ah(k/a)$. Substitution gives

$$\begin{aligned} K_0(a) = (k+1)(a-1)\log(k/a) - \frac{k(a-2)}{2} \log \frac{1 + k^2/a^2}{2} \\ - (a^2-1)\log(1 - 1/a) + \frac{a(a-k)}{2} - (k+1). \end{aligned}$$

Write $B = -k(a-2)\log(1 + k^2/a^2)/2$. Differentiating three times gives

$$\begin{aligned} K_0^{(3)} &= \frac{9(a+2)}{7a^3} + \frac{2(2a-1)}{a^3(a-1)^2} + B^{(3)}, \\ a^3 B^{(3)} &= k^3 \frac{3a^5 - 24a^4 - 6a^3k^2 - 12a^2k^2 - ak^4 - 4k^4}{(a^2+k^2)^3}. \end{aligned}$$

For $t = a - 2 \in [0, 2]$,

$$\frac{9(a+2)}{7} + \frac{2(2a-1)}{(a-1)^2} - \frac{17}{2} = \frac{(t-1)^2(18t+25) + 12t + 12}{14(t+1)^2} > 0.$$

Discard the positive term in $a^3 B^{(3)}$ and use $a \geq 2$ to get

$$a^3 B^{(3)} \geq -k^3(6 + 3k^2/2 + 3k^4/32) > -7/27 > -1/2.$$

Hence $K_0^{(3)} > 8/a^3$. The initial values satisfy

$$\begin{aligned} K_0(2) &= (3 + 21\log 2 - 9\log 7)/7 > 1/200, \\ K_0'(2) &= -\log 7 - (2/7)\log 5 - 2/7 + 4\log 2 > 2/25, \\ K_0''(2) &= -299/175 + 2\log 2 > -1/3. \end{aligned}$$

The rational logarithmic bounds used here are proved below. Twice integrating gives

$$K_0'(a) > 2/25 + (2/3)(a-2) + 4/a - 2 > 0,$$

since multiplication of the last expression by a gives $(2/3)(a - 61/25)^2 + 58/1875$. Thus $h(1/R) > 1/(200a) > 0$.

A.4 The upper endpoint and logarithmic bounds

At $R = 7a/2$,

$$\begin{aligned} h(R) = \frac{7a+9}{2}\log\frac{7a}{2} + \frac{7a(a-2)}{4}\left\{\log 2 - \log\left(1 + \frac{4}{49a^2}\right)\right\} \\ - \frac{7a^2}{4} + \frac{a}{2} - \frac{9(a+1)}{2}\log\frac{9}{2} + \frac{9}{2}. \end{aligned}$$

Concavity gives $\log(a/2) \geq (a-2)\log 2/2$ on $[2, 4]$. Use $\log(1+x) \leq x$, $1/a \leq 1/2$, and $\log 7 > 97/50$, $\log 2 > 69/100$, $\log(9/2) < 151/100$ to obtain

$$h(R) > \frac{133}{200}t^2 - \frac{543}{2800}t + \frac{17}{40} \geq \frac{13}{350} > 0, \qquad t = a - 2.$$

Here are rational checks for all logarithmic bounds used above. For $0 < z < 1$ set $S_m(z) = 2\sum_{j=0}^{m-1} z^{2j+1}/(2j+1)$. Integrating the geometric series and bounding its positive tail gives

$$S_m(z) < \log\frac{1+z}{1-z} < S_m(z) + \frac{2z^{2m+1}}{(2m+1)(1-z^2)}.$$

At $z = 1/3$, $m = 3, 5$ give $842/1215 < \log 2 < 69315/100000$. For $0 < t < 1$ the logarithmic series also gives

$$-t - \frac{t^2}{2(1-t)} < \log(1-t) < -t - t^2/2 - t^3/3.$$

Apply this at $t = 1/8$ to $\log 7 = 3\log 2 + \log(7/8)$ to obtain $97/50 < \log 7 < 973/500$. Using $\log 5 = 2\log 2 + \log(5/4)$, the S_m bound at $z = 1/9, m = 1$ gives $\log 5 < 161/100$. Finally $\sum_{j=0}^{5}(151/100)^j/j! > 9/2$ gives the upper bound for $\log(9/2)$. These estimates imply the three initial inequalities for K_0 as well.

A.5 Conclusion

Strict convexity and $h(1) = h'(1) = 0$ give $h > 0$ on $[\xi, \eta] \setminus \{1\}$. On each remaining part of $[1/R, R]$, concavity bounds h below by the chord between positive endpoint values. Empty outer parts require no argument. Thus $h > 0$ on the whole required interval except at one. Equation (A.1) proves the theorem and its equality statement.

B Weighted entropy on the whole mass support

The coefficient $\lambda = 28/27$ matches the two complete rank gaps in Section 9. Write $e(t) = e^t - 1 - t$ in (9.4). Each pair uses a common τ.

Lemma B.1 (Whole-support weighted entropy)**.** *Let $p \geq 9/5$, $\lambda = 28/27$, and $L = \log 13$. If $0 \leq \sigma \leq L$, $X, Y \in [-\sigma, L - \sigma]$, and $\tau = e^{-\sigma/[p(p+1)]}$ in (9.4), then $\mathcal{E}_{p,\lambda}(X, Y) \geq 0$, with equality exactly at $X = Y = 0$.*

Proof. For fixed σ, X, Y, every positive coefficient and nonnegative scale in (9.4) increases with $p > 1$, while its negative terms are fixed: $\partial_k e(kt) = t(e^{kt} - 1) \geq 0$ for either sign of t. It suffices to use $p_0 = 9/5$. Put

$$\nu = 14/5, \quad b = 4/9, \quad \beta = 9/14, \quad s = 1/7, \quad d = 25/126,$$

and $r = 1/18$, $A = 13^{-d}$, $\eta(t) = Ae^{dt}$. Then $b + d = \beta$, $d - s = r$, $p_0 b = 2\nu s$, and

$$3/5 < A < 5/8, \qquad 5/2 < L < 13/5. \tag{B.1}$$

The first lower bound is $3^{126}13^{25} < 5^{126}$. Positive exponential series and geometric tails give $e < 11/4$, $\sqrt{e} < 5/3$, $e^2 > 109/15$, $e^{3/5} > 9/5$, hence the bounds for L. Finally $1 + 125/252 + (125/252)^2/2 > 8/5$ gives $13^d > 8/5$. Define

$$F_\tau(t) = p_0 \tau e(bt) + \nu\lambda e(\beta t) - \lambda e(t), \qquad G(t) = F_{\eta(t)}(t) + 2\nu(\lambda + \eta(t))e(-st).$$

The integral remainder gives $e(-av) \geq a^2 e(-v)$ for $0 < a < 1$, and makes $e(-v)/v^2$ decrease. Thus, for $0 < v \leq L$ and $\tau \geq A$,

$$F_\tau(-v) > \left(\frac{16}{75} + \lambda\frac{11}{70}\right) e(-v) > \frac{93}{1000} v^2. \tag{B.2}$$

Indeed $(t - 12/13)/t^2$ decreases for $t > 24/13$, so

$$\frac{e(-L)}{L^2} = \frac{L - 12/13}{L^2} > \frac{545}{2197}, \qquad \left(\frac{16}{75} + \lambda\frac{11}{70}\right)\frac{545}{2197} = \frac{27686}{296595} > \frac{93}{1000}.$$

The cancellation $p_0 b = 2\nu s$ gives the exact finite expression

$$G(t) = (p_0 A + \nu\lambda)e^{\beta t} + 2\nu\lambda e^{-st} + 2\nu A e^{rt} - \lambda e^t - \frac{37}{5}(Ae^{dt} + \lambda).$$

In particular $G(0) = G'(0) = 0$ and $G''(0) = 148A/315 + 38/135 > 2662/4725 > 14/25$. For every $t \geq 0$, the positive fourth-derivative terms have exponents at most $\beta < 1$. Moreover

$$p_0 A + \nu\lambda < 81/20, \quad \beta < 2/3, \quad 2\nu\lambda s^4 < 1/200, \quad 2\nu A r^4 < 1/200, \quad \lambda > 1.$$

Dropping the other negative term therefore gives $G^{(4)}(t) < e^{\beta t}(4/5 + 1/100 - 1) < 0$. Taylor's integral formula and $G(0) = G'(0) = 0$ give $G(t) = t^2 \int_0^1 (1 - u)G''(ut)\, \mathrm{d}u$. Thus the quotient

$H(t) = G(t)/t^2$ extends to $H(0) = G''(0)/2$. Differentiating this fixed-interval integral twice is legitimate because G is a finite sum of exponentials. It gives

$$H''(t) = \int_0^1 u^2(1-u)G^{(4)}(ut)\,\mathrm{d}u < 0.$$

At L, $\eta(L) = 1$ and

$$5G(L) = 9\,13^{4/9} + 14\lambda\,13^{9/14} + 28(\lambda+1)13^{-1/7} - 37 - 102\lambda > \frac{4643}{14040}.$$

The lower bounds $13^{4/9} > 25/8$, $13^{9/14} > 26/5$, $13^{-1/7} > 9/13$ are respectively $25^9 < 8^9 13^4$, $26^{14} < 5^{14}13^9$, $9^7 < 13^6$. Thus $H(0) > 7/25$ and $H(L) > 4643/474552 > 9/1000$. Concavity proves the single chord

$$G(t) > t^2\left\{\frac{7}{25}(1-t/L) + \frac{9}{1000}t/L\right\} \quad (0 < t \le L). \tag{B.3}$$

The support implies $\tau \ge A$ and $\tau \ge \eta(t)$ for every positive coordinate t. Two nonpositive coordinates are covered by (B.2) and the nonnegative coupling. For two nonnegative coordinates, the convex increasing function $e(-st)$ vanishes at zero and is superadditive on the positive ray. Hence $\mathcal{E}_{p_0,\lambda}(X,Y) \ge G(X) + G(Y)$. For opposite signs write $X = x > 0$, $Y = -v < 0$. The common support interval has width L, so $x + v \le L$. Lower the same τ everywhere to $\eta(x)$. Its nonnegative coefficients and

$$e[-s(x-v)] - e(-sx) = e^{-sx}(e^{sv} - 1) - sv \ge -sv(1 - e^{-sx}) \ge -s^2xv$$

bound the coupling loss by $(44/189)xv < (7/30)xv$. Set $q = v/x$, so $x/L \le 1/(1+q)$. Equations (B.2)–(B.3) give

$$\frac{\mathcal{E}_{p_0,\lambda}(x,-v)}{x^2} > \frac{279q^3 - 421q^2 + 140q + 27}{3000(1+q)} > 0.$$

For $0 \le q \le 1$ its numerator is $279q(q - 421/558)^2 + 27 - (21001/1116)q \ge 9131/1116$; for $q = 1 + z \ge 1$ it is $25 + 135z + 416z^2 + 279z^3$. Zero coordinates and closed endpoints follow from the same strict one-coordinate bounds. At $X = Y = 0$ all terms vanish. Monotonicity in p completes the proof. □

C A uniform ratio of the two rank gaps

We prove (9.12) on $9/5 \le p \le 2$ and $1/16 \le x, y \le 1$. Order a distinct pair as $y < x$ and put $t = (y/x)^{1/p}$. The one-variable comparison is at $p_0 = 9/5$ and uses $v_0 = -(p_0/2)\log t \in [0, \log 4]$. The gaps have different homogeneities, so we restore their mass factors after comparing them at this ratio.

C.1 A positive differential equation for the concavity gap

Put $a_0 = 5/14$, $\beta_0 = 9/14$, $b_0 = 4/9$, $c_0 = 5/9$, and

$$G(v) = \cosh(v)^{a_0} - \cosh(a_0 v), \qquad D(v) = \cosh(c_0 v) - \cosh(b_0 v)^{a_0}.$$

We will work on $0 \le v \le \ell_0 = \log 4$, contained in $v^2 < 2$. Set

$$g_2 = 45/392, \quad b_* = 1/10, \quad k_* = 99/784, \quad q_* = b_* + k_* = 887/3920, \quad C(v) = g_2 v^2 \frac{1 + b_* v^2}{1 + q_* v^2}.$$

We claim $G(v) \geq C(v)$, strictly for $v > 0$. Differentiating G gives the exact first-order equation

$$G' - a_0 \tanh(v) G = a_0 \frac{\sinh(\beta_0 v)}{\cosh v}.$$

The residual for the comparison C is $E(v) = a_0 \sinh(\beta_0 v) + a_0 C(v) \sinh v - C'(v) \cosh v$. Multiplying by the positive denominator gives the entire function

$$\begin{aligned} F(v) := \frac{(1+q_* v^2)^2 E(v)}{g_2} = \frac{2}{\beta_0}(1+q_* v^2)^2 \sinh(\beta_0 v) \\ + a_0 v^2 (1+b_* v^2)(1+q_* v^2) \sinh v \\ - 2v(1 + 2b_* v^2 + b_* q_* v^4) \cosh v. \end{aligned}$$

Here is a coefficient check that includes its whole tail. Write $F(v) = \sum_{j \geq 0} A_j v^{2j+1}/(2j+1)!$ and $B_0 = \beta_0^2$. The first two coefficients vanish. The next three are

$$\begin{aligned} A_2 &= 2B_0^2 + 80 q_* B_0 + 240 q_*^2 + (132 q_* - 158)/7 > 3/2, \\ A_3 &= 2B_0^3 + 168 q_* B_0^2 + 1680 q_*^2 B_0 - 53 - 24 q_* > -17, \\ A_4 &= 2B_0^4 + 288 q_* B_0^3 + 6048 q_*^2 B_0^2 + 216 q_* - 3006/35 > 10. \end{aligned}$$

These are rational comparisons with $B_0 = 81/196$. For $j \geq 5$, set $n_j = 2j+1$. The contribution from the first line of F is positive. The other two lines contribute

$$\begin{aligned} n_j \{ 2a_0 j - 2 + (2j)(2j-1)[a_0(b_* + q_*)(2j-2) - 4b_*] \\ + b_* q_* (2j)(2j-1)(2j-2)(2j-3)[a_0(2j-4) - 2] \} > 0. \end{aligned}$$

Every bracket is positive already at $j = 5$ and increases thereafter. Consequently, for $0 < v^2 \leq 2$,

$$F(v) > v^5 \left(\frac{3/2}{120} - \frac{17 v^2}{5040} \right) \geq \frac{29}{5040} v^5 > 0.$$

The integrating factor gives the desired comparison explicitly:

$$G(v) - C(v) = \cosh(v)^{a_0} \int_0^v \frac{E(t)}{\cosh(t)^{a_0+1}} \, \mathrm{d}t \geq 0. \tag{C.1}$$

Cross-multiplication also gives $C(v) \geq g_2 v^2/(1 + k_* v^2)$, since the numerator of the difference is $b_* k_* v^4$ after canceling the positive leading factor.

C.2 A bound for the second gap with its full series

Put

$$d = a_0 b_0 = 10/63, \quad K = (c_0^2 - d^2)/2 = 125/882, \quad q_D = (c_0^2 + d^2)/12 = 1325/47628,$$

$$J_0 = g_2 b_0^2 = 10/441, \quad e_0 = k_* b_0^2 = 11/441, \quad d_2 = K - J_0 = 5/42,$$

$$\eta_* = (K q_D + J_0 e_0)/d_2 = 37877/1000188, \quad q_2 = 1/90, \quad h_* = \eta_* - q_2 > 0.$$

In the positive series of $\cosh(c_0 v) - \cosh(dv)$, the ratio following the quadratic coefficient is q_D; all subsequent ratios are less than q_2. For the first of these later ratios the assertion is

$$\frac{c_0^4 + c_0^2 d^2 + d^4}{30(c_0^2 + d^2)} < 1/90.$$

For all later terms the geometric-sum identity $(1-r^{j+1})/(1-r^j) \le (j+1)/j$ gives the bound $c_0^2/[2j(2j+1)] \le c_0^2/42 < 1/90$ for $j \ge 3$. Thus the complete series satisfies, with $w = v^2$,

$$\cosh(c_0 v) - \cosh(dv) \le Kw\left(1 + \frac{q_D w}{1 - q_2 w}\right).$$

Subtract (C.1) at $b_0 v$, using its weaker rational lower, to obtain

$$\frac{D(v)}{v^2} \le K\left(1 + \frac{q_D w}{1 - q_2 w}\right) - \frac{J_0}{1 + e_0 w} \le d_2 \frac{1 + h_* w}{1 - q_2 w}. \tag{C.2}$$

The second inequality follows by expanding the first difference as $d_2 + w[Kq_D/(1-q_2w) + J_0 e_0/(1+e_0 w)]$ and increasing the latter denominator's reciprocal to $1/(1-q_2 w)$.

The ratio of these two explicit bounds is

$$R_*(v) = \frac{28}{27} e^{-25v/126} \frac{(1 + h_* v^2)(1 + q_* v^2)}{(1 - q_2 v^2)(1 + b_* v^2)}.$$

It is log convex on $[0, \ell_0]$. The second derivatives of $\log(1 + h_* v^2)$ and $-\log(1 - q_2 v^2)$ are positive there. For the remaining logarithmic difference the numerator has sign

$$1 - (q_* + b_*)v^2 - 3q_* b_* v^4 > 1 - .654 - .2724 > 0.$$

All decimals here denote the displayed rational numbers. At zero $R_*(0) = 28/27$, and at the other endpoint $R_*(\ell_0) < 1019/1000$. Here is an exact bound including all tails. Write

$$L_- := 1684/1215 < \ell_0 < L_+ := 1733/1250.$$

The first three terms of $\ell_0 = 4\sum_{j \ge 0} 3^{-(2j+1)}/(2j+1)$ give L_-; bounding $1/(2j+1) \le 1/7$ in the remaining tail gives $\ell_0 < 23581/17010 < L_+$. Monotonicity of the numerator factors and separate lower bounds for the positive denominator factors give

$$(1 + h_* \ell_0^2)(1 + q_* \ell_0^2) < (1 + h_* L_+^2)(1 + q_* L_+^2) < 1509/1000,$$
$$(1 - q_2 \ell_0^2)(1 + b_* \ell_0^2) > (1 - q_2 L_+^2)(1 + b_* L_-^2) > 5833/5000.$$

These are rational comparisons with the exact coefficients above. Set $x = (25/126)L_- = 4210/15309$. The positive exponential series gives

$$e^x > \sum_{j=0}^{4} \frac{x^j}{j!} = \frac{216938038367782253}{164781536053084083} > \frac{2500}{1899}.$$

Consequently

$$R_*(\ell_0) < \frac{28}{27}\frac{1899}{2500}\frac{1509/1000}{5833/5000} = \frac{742931}{729125} < \frac{1019}{1000},$$

with final strict margin $379/5833000$. Convexity bounds R_* by its larger endpoint value, hence

$$R_*(v) \le \lambda = 28/27 \qquad (0 \le v \le \ell_0). \tag{C.3}$$

C.3 Restoring the physical exponent and the common mass scale

For fixed $0 < t < 1$ define

$$d_p(t) = \frac{1+t}{2} - \left(\frac{t + t^p}{2}\right)^{1/(p+1)}, \qquad g_p(t) = \left(\frac{1 + t^p}{2}\right)^{1/(p+1)} - \frac{1 + t^{p/(p+1)}}{2}.$$

Put $z = -\log t/2$ and $v_0 = p_0 z$. At $p = p_0$,

$$g_{p_0}(t) = e^{-a_0 v_0} G(v_0), \qquad d_{p_0}(t) = e^{-z} D(v_0).$$

The preceding estimates give the positive lower and upper

$$g_{\mathrm{low}} = e^{-a_0 v_0} C(v_0), \qquad d_{\mathrm{up}} = e^{-z} d_2 v_0^2 \frac{1 + h_* v_0^2}{1 - q_2 v_0^2},$$

with $d_{\mathrm{up}} \le \lambda g_{\mathrm{low}}$.

For fixed t, d_p decreases and g_p increases with p. For the first statement differentiation gives

$$-\partial_p d_p(t) = e^{-z} \cosh((p-1)z)^{1/(p+1)} \frac{(p+1)z \tanh((p-1)z) - \log\cosh((p-1)z)}{(p+1)^2} > 0.$$

For the second, express g_p as the difference between the power mean of $1, t^{p/(p+1)}$ of order $p+1$ and their arithmetic mean. Increasing the order increases this difference. Decreasing the smaller argument also increases it, since its derivative is $[(y/M_{p+1}(1,y))^p - 1]/2 < 0$ for $y < 1$. Consequently, for $p_0 \le p \le 2$ and $v_0 \le \ell_0$,

$$d_p(t) \le d_{\mathrm{up}} \le \lambda g_{\mathrm{low}}, \qquad g_p(t) \ge g_{\mathrm{low}} > 0. \tag{C.4}$$

Finally let $1/16 \le y < x \le 1$, put $u = x^{1/p}$ and $t = (y/x)^{1/p}$. Then $0 < u \le 1$ and $v_0 = p_0 |\log(y/x)|/(2p) \le \ell_0$. Exactly

$$D_p(x,y) = u d_p(t), \qquad G_p(x,y) = u^{p/(p+1)} g_p(t).$$

Equation (C.4) gives

$$D_p(x,y) \le \lambda u g_p(t) \le \lambda u^{p/(p+1)} g_p(t) = \lambda G_p(x,y).$$

This proves (9.12). Equal pairs have both gaps zero before division, and symmetry covers the other ordering.

D A power-kernel endpoint estimate

This appendix proves the finite endpoint estimate used for the adjacent mesh. All decimal constants in this appendix are rational numbers. Set

$$P = 9/5, \quad U_* = 7/8, \quad b_* = 2304/8575,$$
$$\tau = 5/7 + b_* u^2, \quad m = u\tau, \quad a = 6u(2-u)/25, \quad z = u - a.$$

In this appendix only, $k(x) = x^{-P}$ and

$$G(x) = Pzx^{-P-1} + Pa(x+1)^{-P-1} - x^{-P} + (x+1)^{-P}, \qquad Z = P\zeta(P+1).$$

We prove

$$u(1 + Pa - Z) + m \sum_{j \ge 0} [G(u + jm)]_+ < 0 \qquad (2/3 \le u \le 7/8). \tag{D.1}$$

The bound (7.15) in Lemma 7.7 supplies this comparison mass at P. The calculation below controls its complete value and the following tail used in Lemma 7.8.

D.1 Power and sum enclosures

The following elementary rules suffice to reproduce the numerical enclosures below. For positive rational x, l, h, the inequalities $l < x^{-9/5} < h$ are equivalent to $(lx^2)^5 < x < (hx^2)^5$. Thus every power enclosure used here reduces to integer arithmetic. For the single zeta value we use

$$2.2446558 < Z < 2.24465706. \tag{D.2}$$

One reproducible enclosure is Euler–Maclaurin at $N = 32$, with the terms through B_6:

$$\zeta(s) = \sum_{j=1}^{31} j^{-s} + \frac{32^{1-s}}{s-1} + \frac{1}{2}32^{-s} + \frac{s}{12}32^{-s-1} - \frac{(s)_3}{720}32^{-s-3} + \frac{(s)_5}{30240}32^{-s-5} + \mathcal{E}.$$

Here $(s)_j = s(s+1)\cdots(s+j-1)$. A bound requiring no remainder-sign assertion is

$$|\mathcal{E}| \le \frac{2(s)_7}{1209600}\, 32^{-s-7}.$$

Indeed, first include the next Euler–Maclaurin term $-(s)_7 32^{-s-7}/1209600$. The absolute value of the remaining periodic-Bernoulli integral is at most

$$\frac{|B_8|}{8!}(s)_8 \int_{32}^{\infty} x^{-s-8}\,\mathrm{d}x = \frac{(s)_7}{1209600}\, 32^{-s-7},$$

using $|B_8(\{x\})| \le |B_8| = 1/30$. Adding the absolute value of the omitted term gives the displayed two-term bound. At $s = 14/5$ it is less than $4.1 \cdot 10^{-16}$. Enclose each positive fifth root by rational bisection to width 10^{-12}, and round each multiplication outwards. The total interval width contributed by the 31 finite terms is less than 10^{-9}; substitution gives (D.2) with a margin greater than that width. This describes finite rational operations, including the error of the infinite tail.

D.2 One complete following-tail integral

Put $\delta = 1 - u$, $x_1 = u + m$ and $t = u + 3m/2$. The primitive

$$P_u(x) = zx^{-P} + a(x+1)^{-P} + \frac{(x+1)^{1-P} - x^{1-P}}{P-1}$$

satisfies $P_u' = -G$ and $P_u(\infty) = 0$. For $2/3 \le u \le 7/8$ and every $x \ge 3/2$, we claim

$$P_u(x) > -\frac{10}{9}\delta^{P+1}. \tag{D.3}$$

Here is a root-free proof. Write $c = 7/30$ and $b = 1 - 2a$. The signed measure

$$(1 - a - s)\,\mathrm{d}s - (1/2 - a)\delta_c, \qquad 0 \le s \le 1,$$

has zero total mass. Its twice integrated distribution function is

$$I(s) = (1-a)s^2/2 - s^3/6 - (1/2 - a)(s-c)_+ \ge 0.$$

Indeed, on $[c, 1]$ its only interior minimum is at $s = b$, and $I(b) = (b/2)(c - b/2 + b^2/6) > 0$ because $13/25 \le b \le 43/75$ and $b/2 - b^2/6 \le 3913/16875 < 7/30$. On $[0, c]$ positivity is immediate. Twice integrating by parts against a decreasing convex function therefore gives

$$\int_0^1 (1 - a - s)(x+s)^{-P-1}\,\mathrm{d}s \ge (1/2 - a)(x+c)^{-P-1}.$$

Consequently, with $B = P(1/2-a)$, $P_u(x) \geq -\delta x^{-P} + B(x+c)^{-P-1}$. Set $y = (x+c)^{-1} \leq 15/26$. Convexity and the endpoint inequality $(52/45)^9 < (13/10)^5$ imply $(1-cy)^{-P} \leq 1 + (13/25)y$. Also

$$B - 13\delta/25 = 117/250 - 13\delta/25 + 54\delta^2/125 \geq 257/750 \qquad (1/8 \leq \delta \leq 1/3).$$

Minimizing $-\delta y^P + (257/750)y^{P+1}$ now proves (D.3): its minimum is $-P^P\delta^{P+1}/[(P+1)^{P+1}(257/750)^P]$, whose coefficient is less than $10/9$ by the rational ninth/fifth-power comparison.

The convexity of $J = [G]_+$ in Lemma 7.2 gives midpoint sampling, with every term of the infinite tail retained:

$$m\sum_{j\geq 2} J(u+jm) \leq \int_t^\infty J(x)\,\mathrm{d}x \leq P_u(t) + (10/9)\delta^{14/5} =: T(u).$$

Here $t \geq 38597/25725 > 3/2$. The last bound holds whether t is before or after the unique zero of G: before it subtract the minimum of P_u, and after it the integral is zero while the same primitive floor makes $T(u) > 0$. No comparison root is introduced.

D.3 Complete endpoint and downstream interfaces

Write $K = u(1+Pa-Z) + mG(u)$ and $S_1 = mG(x_1)$. The following closed-interval inequalities are proved below:

$$\begin{aligned} G(u+6m) &< 0 && (1/2 \leq u \leq 7/8), \\ K+T,\ K+S_1+T &< -1/2000 && (2/3 \leq u \leq 7/8), \\ T,\ S_1+T &< (7/80)u && (2/3 \leq u \leq 7/8). \end{aligned} \tag{D.4}$$

The summand $G(u)$ is positive since $Pz - u = u(54u-8)/125 > 0$. The complete remainder is bounded by T on $[2/3, 7/8]$. Taking the maximum of the two displayed signed bounds at a possible first-atom clipping contact proves (D.1), with margin $-1/2000$. It proves at the same time the interface needed for exponent transport:

$$R(P,u) := m\sum_{j\geq 1}[G(u+jm)]_+ < (7/80)u \qquad (2/3 \leq u \leq 7/8). \tag{D.5}$$

The separate $j \geq 6$ extinction in (D.4) retains exactly the finite count required by (7.21).

D.4 Rational whole-interval verification

For reproducibility we give the polynomial rule for every inequality in (D.4). For a positive monotone argument $x = x(u)$ on $[l,h]$, put $M = \max\{x(l), x(h)\}$, $d = \min\{x(l), x(h)\}$, $w = 1 - x/M$, $w_* = 1 - d/M$. For $\beta \in \{9/5, 1/5\}$ define

$$C_4(w) = \sum_{j=0}^{4} \frac{(\beta)_j}{j!} w^j, \qquad r = w_* \max\{1, (\beta+5)/6\} < 1.$$

Choose consecutive multiples L, H of 10^{-18} with $L^5 M^{5\beta} \leq 1 < H^5 M^{5\beta}$ by integer bisection. Then

$$LC_4(w) \leq x^{-\beta} \leq H\left(C_4(w) + \frac{(\beta)_5 w^5}{5!(1-r)}\right).$$

This follows from the positive binomial series and the geometric bound on all coefficient ratios after its fifth term. Use these polynomials on the appropriate side of each signed term, with

Z replaced by its lower bound in (D.2). For the primitive use $P_u(t) = [z - t/(P-1)]t^{-P} + [a + (t+1)/(P-1)](t+1)^{-P}$; write $\delta^{14/5} = \delta^3\delta^{-1/5}$. All coefficients have fixed sign on their stated interval. Multiply the negative value inequalities by $D = u(u+1)x_1(x_1+1)$, and the following-ratio inequalities by $D_1 = x_1(x_1+1)$. For extinction at $x = u + jm$ multiply $-G(x)$ by $x(x+1)$. The targets below are bounded from below by rational polynomials. The degrees and Bernstein coefficient bounds refer to those lower polynomials.

positive expression	$[l, h]$	degree	all Bernstein coefficients exceed
$-x(x+1)G(x),\ x = u + 6m$	$[1/2, 7/8]$	21	$1/200$
$-D(K + T + 1/2000)$	$[2/3, 7/8]$	26	$9/100$
$-D(K + S_1 + T + 1/2000)$	$[2/3, 7/8]$	26	$3/2000$
$D_1(7u/80 - T)$	$[2/3, 7/8]$	24	$1/8$
$D_1(7u/80 - S_1 - T)$	$[2/3, 7/8]$	24	$19/10000$

These are integer-arithmetic tests, not sampling. Explicitly, if the lower polynomial after $u = l + (h-l)v$ is $\sum_{j=0}^{n} q_j v^j$, its Bernstein coefficients are $\sum_{j=0}^{k} q_j \binom{k}{j}/\binom{n}{j}$, $0 \le k \le n$. The displayed bounds follow by this formula, the preceding degree-four binomial rule, and the fixed rational definitions of a, m, z. Thus all endpoints and every clipping contact are included.

E Moment estimates for the joint comparison

We collect the estimates used on $1 \le p \le P = 9/5$, $7/8 \le u < 1$. Every zeta expression below is a convergent positive moment; exponent derivatives are differentiated complete sums. Use the notation

$$Z = p\zeta(p+1), \quad P_m = (p-1)Z, \quad Q = p(p+1)\zeta(p+2), \quad D = Q - P_m, \quad I = P_m + 1,$$

$$I_x = 2p + P_m - Q + 2^{-p}, \quad B_* = p^2 + 3p + 1 + P_m - 2Q, \quad M = 1.24Q - P_m - p - 1 - .24p(p+2).$$

Primes in this appendix denote p derivatives, except where a different variable is explicitly indicated.

The estimates are organized by their subsequent use. Lemma E.1 controls the displayed moments and their exponent derivatives. Lemma E.2 controls the value at the unit gap, and Lemma E.3 controls the upper edge of the mass rectangle. Appendix F treats its lower diagonal; Appendix G bounds the increment in $d = m - r$, and Appendix H bounds the far corner. The subsection on the two endpoints combines these estimates on the rectangle obtained from the force equations at the bodies.

Lemma E.1 (Logarithmic moments)**.** *Putting $t = p - 1$, the following bounds hold on the whole interval:*

$$\begin{aligned} &.074 < Z'' < .125, && .7 < Z' \le .708 + .125t, && Z \le 1.645 + .75t, \\ &1.4 < P_m'' < 1.716, && 1.6449 + 1.4t < P_m' < 1.645 + 1.458t + .125t^2, \\ &1.6 < Q'' < 1.848, && 3.20 + 1.6t < Q' < 3.211 + 1.848t. \end{aligned}$$

If $M_2 = Q - p(p+1)$ and $M_3 = (p)_3[\zeta(p+3) - 1]$, then

$$M_2 > .4, \quad M_2' > .02, \quad -.4 < M_2'' < -.152, \qquad M_3 < .85, \quad M_3' < 1, \quad M_3'' < 0.$$

Proof. We use one shape-parameter argument and ordinary zeta values only. If X_p has the Gamma$(p, 1)$ law, then

$$Z(p) = \mathbb{E}h_0(X_p), \qquad h_0(x) = \frac{x}{1 - e^{-x}}, \quad h_0(0) = 1.$$

Indeed, positive geometric expansion gives $\int_0^\infty x^p/(e^x-1)\,\mathrm{d}x = \Gamma(p+1)\zeta(p+1)$. For independent Y_j of Gamma$(r_j,1)$ laws, Gamma additivity gives

$$\Delta_{r_1}\cdots\Delta_{r_k}\mathbb{E}f(X_p) = \mathbb{E}[\Delta_{Y_1}\cdots\Delta_{Y_k}f(X_p)].$$

The inner difference is the integral of $f^{(k)}$ over $\prod_j[0,Y_j]$. Thus its sign is preserved; if $|f^{(k)}|\le B$, its expectation has magnitude at most $B\prod_j r_j$. Division and passage to zero give the corresponding derivative bounds. Here all expectations are finite and smooth: h_0 grows linearly, its derivatives are bounded, and the zeta series converges locally normally.

Writing $h_0(x) = x + x/(e^x-1)$, the derivative signs of $x/(e^x-1)$ in the signed sampling argument give $0 < h_0' < 1$, $h_0'' > 0$ and $h_0''' < 0$. For a uniform bound put $\ell(y) = e^y/(1+e^y)^2$, $g(y) = 1/(1+e^y)$. The substitution $r = e^y$ gives

$$(\ell * g)(x) = \int_0^\infty \frac{r\ \mathrm{d}r}{(1+r)^2(r+e^x)} = C(x).$$

Since $-g' = \ell$ and $C = 1-h_0'$, this implies $h_0'' = \ell*\ell$. The logistic substitution gives $\|\ell\|_2^2 = 1/6$ and $\|\ell'\|_2^2 = 1/30$. Cauchy–Schwarz therefore gives $|h_0'''| \le 1/(6\sqrt{5}) < 3/40$. The shape argument proves, for every $p > 0$,

$$0 \le Z' \le 1, \qquad Z'' \ge 0, \qquad -3/40 < Z''' \le 0. \tag{E.1}$$

In particular Z'' decreases.

Write $\mathcal{D}_hF(x) = [F(x+2h)-2F(x+h)+F(x)]/h^2$ and $\mathcal{C}_hF(x) = [F(x+h)-F(x-h)]/(2h)$. The following ordinary-value bounds suffice:

$\mathcal{D}_{1/4}Z(1/2)$	$< .123$	$\mathcal{D}_{1/4}Z(2)$	$> .076$
$\mathcal{D}_{1/4}Z(3)$	$> .051$	$\mathcal{D}_{1/4}Q(1/2)$	> 1.635
$\mathcal{D}_{1/20}Q(9/5)$	< 1.8461	$Z(9/5)$	< 2.245
$\mathcal{C}_{1/10}Z(1)$	$\in (.7073, .7074)$	$\mathcal{C}_{1/10}Q(1)$	$\in (3.2102, 3.2103)$
$Z(1)$	$\in (1.6449, 1.645)$		

All value enclosures here and below use the complete Euler–Maclaurin rule of Appendix D, at $N = 32$, with error $2(s)_7 32^{-s-7}/1209600$ for $\zeta(s)$. For rational $s = a/b$, enclose j^{-s} between $q/10^{18}$ and $(q+1)/10^{18}$ by $q^b j^a \le 10^{18b} < (q+1)^b j^a$. There are 28 distinct ordinary Z values, with root degrees at most 20; only rational interval operations are used after those enclosures. These are finite endpoint checks, not sampled derivative signs.

The decreasing Z'' and the first three bounds imply $.074 < Z'' < .125$ on $[1,2]$ and $Z''(3) > 1/20$. Since $Q(p) = pZ(p+1)$,

$$Q'''(p) = 3Z''(p+1) + pZ'''(p+1) > .15 - .075p \ge 0 \qquad (0 \le p \le 2).$$

Thus Q'' increases. Its backward difference at one and forward difference at P give $1.6 < Q'' < 1.848$ on $[1,P]$. For the centered slopes, the errors are at most $1/8000$ for $Z'(1)$ and $1/1600$ for $Q'(1)$; use $|Z'''| < 3/40$ and $0 < Q''' \le 3/8$ on $[9/10, 11/10]$. Consequently $.7069 < Z'(1) < .708$ and $3.20 < Q'(1) < 3.211$. Integration gives the asserted Z' and Q' bounds. Convexity and the two endpoint values give $Z \le 1.645 + .75t$. Now $P_m' = Z + tZ'$ and $P_m'' = 2Z' + tZ''$ give all the stated P_m inequalities, including $P_m'' < 1.416 + .375t \le 1.716$.

For M_2, the Q'' bounds give $-.4 < M_2'' < -.152$. The same ordinary-value rule gives

$$M_2(1) > .4, \qquad 20\{M_2(9/5) - M_2(7/4)\} - .01 > .0216.$$

The second quantity is a lower bound for $M_2'(P)$, since $M_2'' > -.4$ and the interval length is $1/20$. As M_2' decreases, this proves $M_2' > .02$ and then $M_2 > .4$.

For M_3 put $R(x) = Z(x) - x$. Its value decreases, its derivative increases, and $R'' = Z''$ decreases, by (E.1). Exactly,

$$M_3''(p) = 2R(p+2) + (4p+2)R'(p+2) + p(p+1)Z''(p+2).$$

For $a \le p \le b = a + 1/5$, set $x = a + 2$, $y = b + 2$. Its upper bound is

$$2R(x) + (4a+2)\frac{R(y+.1) - R(y)}{.1} + b(b+1)\mathcal{D}_{.1}Z(x-.2).$$

The negative first-derivative term uses the right endpoint and the smallest positive coefficient. The other terms use the left endpoint. The following rational upper bounds, from the same ordinary values, make this expression strictly negative on all four full rectangles:

a	$R(x)$	$[R(y+.1) - R(y)]/.1$	$\mathcal{D}_{.1}Z(x-.2)$	combined upper
1	.247	−.1107	.0597	−787/62500
6/5	.224	−.1008	.0552	−812/15625
7/5	.202	−.0916	.0509	−2513/31250
8/5	.182	−.0832	.0468	−1547/15625

Hence $M_3'' < 0$ throughout $[1, P]$. Finally the ordinary values give

$$M_3(3/2) > M_3(7/5), \qquad 3R(3) < .741,$$

$$M_3(7/5) + 4\{M_3(7/5) - M_3(13/10)\} < .8492.$$

Concavity and the first inequality give $M_3'(7/5) > 0$, so the left portion is bounded by $M_3(7/5)$. On its right, the tangent slope is at most the backward secant from 13/10; the last inequality pays that portion up to 9/5. Thus $M_3 < .85$. Also $M_3' \le M_3'(1) = 3R(3) + 2R'(3) \le 3R(3) < 1$. □

The values at P used in the finite comparison are

$$\begin{gathered} 1.79572464 < P_m < 1.795725648, \qquad 5.531452416 < Q < 5.531453928, \\ 3.735726 < D < 3.73573, \qquad .372816 < B_* < .372821, \\ I > 2.79572464, \qquad I_x > .151440712, \qquad M > .62167534784. \end{gathered} \tag{E.2}$$

They follow by rational fifth-root enclosures in the convergent Euler–Maclaurin sums described in Appendix D. Their widths also prove the bounds on μ_* in (7.27).

Lemma E.2 (Unit coefficients)**.** *Set $C_*(p) = \mu_* D - \lambda_* B_*$ and, for $0 \le k \le 1$, $0 \le c \le 4/5$, define*

$$g_*(p,k,c) = P_m - .24Q - \lambda_* I - \mu_* I_x + k[-M + \lambda_* D - \mu_* I] + cC_*(p).$$

Then

$$g_*(p,k,c) + T_*(p) < -.0097 - .54(P - p). \tag{E.3}$$

Moreover $C_*(P) = 0$, $C_*' > -.069$ and $C_*'' > -.095$.

Proof. The formulas result by taking the first coefficient in δ of Φ, at $m = 1 - (1+k)\delta$, $d = c\delta$. For C_* the exact derivatives and Lemma E.1 give the asserted lower bounds; in particular

$$C_*'' = (2\lambda_* + \mu_*)Q'' - (\lambda_* + \mu_*)P_m'' - 2\lambda_*.$$

The second derivatives of the base and k coefficient in g_* are

$$\begin{aligned} &(1 - \lambda_* - \mu_*)P_m'' + (\mu_* - .24)Q'' - \mu_*(\log 2)^2 2^{-p}, \\ &(1 - \lambda_* - \mu_*)P_m'' + (\lambda_* - 1.24)Q'' + .48. \end{aligned}$$

They exceed .710 and $-.357$, respectively, while the last contribution exceeds $-.076$. Thus $g_*'' > .27$. At $p = 1$, direct substitution gives base $< -.75496$, the k coefficient $< .149226$, and $C_* < .008606$. With $T_*(1) = 23/147$, this gives $g_* + T_* < -.4423$. At P, $C_* = 0$ and (E.2) gives the k coefficient $< -.0063$ and base $< -.03215$. Hence $g_* + T_* < -.00975$ there. Convexity of $g_* + T_*$ proves the slightly weaker strict chord (E.3). □

Lemma E.3 (The upper zero-gap edge)**.** *For the fixed coefficients $\lambda = \lambda_*$ and $\mu = \mu_*$ of (7.27), put $\Psi = S + \lambda q + \mu H$. Its upper zero-gap value satisfies*

$$\Psi(u, 0) \le \delta g(p, 0, 0) - .0043\delta^2,$$

Here $\Psi(u, 0)$ denotes evaluation at $m = r = u$, and $g(p, 0, 0) = P_m - .24Q - \lambda I - \mu I_x$ is the first coefficient in $\delta = 1 - u$ at this edge.

Proof. At $m = r = u$, the three complete densities are

$$S: \ -\delta(1 + as)e^{-s} + g(s)(\beta_u - \beta_1), \quad q: \ -b_0(us)(\beta_u - \beta_1), \quad H: \ -e^{-us}b_0(us)(\beta_u - \beta_1).$$

Let $a_* = 6/25$, $g_0 = s(1 - a_* + a_*e^{-s}) - 1 + e^{-s}$, $R_0 = g_0 - \lambda b_0(s) - \mu e^{-s}b_0(s)$ and $J(t) = \int \rho_p(s)C(ts)R_0(s)\,\mathrm{d}s$. Since $a = a_*(1 - \delta^2)$ and $b_0(us) \ge u^2 b_0(s)$, the bracket multiplying $\beta_u - \beta_1$ is at most $u^2 R_0 + \delta K(s, u)$, where

$$K(s, u) = us[1 - 2a_*(1 - e^{-s})] - (1 + u)(1 - e^{-s}).$$

Scaling frequency gives, for $u \le t \le 1$,

$$\int \rho_p C(ts)K(s, u)\,\mathrm{d}s \le t^{-p}\{(p - 1 - u)Z - .48Q + .48p + 1 + u\} < -.0583t^{-p}.$$

For the final inequality use $Z > .6398 + .96395p - .049p^2$ and $Q > 1.0968p(p+1)$. The resulting upper polynomial is $-.049p^3 + .529361p^2 - 1.21407025p + .675375$. It is convex and its two endpoint values are $-.05833425$ and $-.08058981$, both below $-.0583$.

We also need $J(t) \le t^{-p-2}J(1)$. First $R_0 > .16b_0 > 0$, since $g_0 > .52b_0$. Set $F = 2R_0 - sR_0'$. After multiplication by e^{2s} its factorial coefficients at degrees three and four are $.28 - \lambda - 4\mu$ and $2.16 - 6\lambda - 10\mu$. The degree-five coefficient is $9.8 - 23\lambda - 18\mu > 0$; for $j \ge 6$ the coefficient is

$$2^{j-1}[(.76 - \lambda)j - 4(1 - \lambda)] + 2(1 - \lambda + \mu) + (1.24 - \lambda)j + (.24 - \mu)j(j - 1) > 0.$$

If the cubic coefficient is nonnegative, the quartic one is positive. Thus F is positive or has one crossing from negative to positive. Its unit integral is positive. Indeed complete moment integration, then $\lambda \le .28$, $\mu \le .08$, gives

$$\int \rho_p C(s)F(s)\,\mathrm{d}s > .16[8P_m + (p - 3)Q - (p)_3\zeta(p + 3) + 4p + 9 - (p + 2)2^{-p}/2] > 0.$$

For the last sign use $P_m > p(p-1) + .28717(p-1)(1+p/4)$, $Q < 1.2021p(p+1)$, $\zeta(p+3) < 1.083$ and $2^{-p} \le 1/2$. The resulting polynomial in $t = p - 1$ is $.9436 - 2.0997t + 3.27844t^2 + .1191t^3 > 3/5$. Since C is log concave, $C(ts)/C(s)$ increases with s when $t \le 1$. The signed comparison therefore gives $\int \rho_p C(ts)F > 0$. Integration by parts gives

$$tJ'(t) + (p + 2)J(t) = \int \rho_p(s)C(ts)[2R_0(s) - sR_0'(s)]\,\mathrm{d}s > 0.$$

Thus $t^{p+2}J(t)$ is increasing, proving the claimed bound.

Finally

$$\frac{u^2}{\delta}\int_u^1 t^{-p-2}\,\mathrm{d}t \le \frac{u^{1-p} + u}{2} \le 1.$$

The first inequality is the trapezoidal bound for t^p; the second follows from convexity and its endpoint values on $[7/8, 1]$. Inserting the two integral estimates into the exact density gives

$$\Psi(u,0)/\delta \le -(1+ap) + J(1) - .0583\delta = g(p,0,0) + .24p\delta^2 - .0583\delta \le g(p,0,0) - .0043\delta.$$

This proves the stated finite-edge estimate. □

F The lower diagonal of the joint comparison

This appendix treats $u = 1-\delta$, $m = 1-2\delta$, $d = 0$, with $0 \le \delta \le 1/8$. All derivatives in this appendix are with respect to δ, unless an exponent derivative is specified. The positive normalization is $\mathcal{N} = p(p+1)m^{-p-2}$.

Lemma F.1 (Diagonal curvature)**.** *For $1 \le p \le 9/5$ and the fixed coefficients* (7.27)*,*

$$\Phi'''(\delta)/\mathcal{N} > .08.$$

In particular $\Phi(\delta)/\delta$ is strictly convex, with its continuous value at zero given by $g_*(p,1,0)$.

Proof. We split the derivative as

$$\Phi''' = S''' + (\lambda_* - \mu_*)q''' + \mu_*(H''' + q''').$$

We will prove

$$q'''/\mathcal{N} > 8.03 + 5(p-1), \tag{F.1}$$

$$H''' + q''' > 0, \tag{F.2}$$

$$S'''/\mathcal{N} > .811153846 - 1.587692308t + .28t^2 - 1.89 + 1.52t - .1, \quad t = p-1. \tag{F.3}$$

For the adjacent row write

$$q = q_{\text{known}} + q_{\text{right}}, \quad q_{\text{known}} = T_{B_u}(0,1) + (\delta - pm)u^{-p}, \quad q_{\text{right}} = m^{1-p}F(U),$$

where $U = u/m$, $\ell = 2U - 1 = 1/m$ and $F(U) = \sum_{j\ge1} A_U(j)$. In the known part, $B_u(j) = upj^{-p-1} - j^{-p} + (j+u)^{-p}$. Its only contribution after three derivatives is the positive sum $(p)_3 \sum_{j\ge1}(j+u)^{-p-3}$. For the finite term, write $(\delta - pm)u^{-p} = (p+1)u^{-p} - (2p+1)u^{1-p}$. Thus, with $A = (p+1)(p+2)$ and $B = (2p+1)(p-1)$,

$$q'''_{\text{known}}/\mathcal{N} > (m/u)^{p+2}(A - Bu)/u.$$

The logarithmic δ derivative of this lower bound is $-(p+2-m)/(mu) + B/(A-Bu) < 0$, since $A > 2B$ on $[1, 9/5]$. At $\delta = 1/8$ use $(6/7)^{p+2} \ge (6/7)^4(9-p)/7$, from $e^s \ge 1+s$ and $\log(7/6) > 1/7$. This gives

$$q'''_{\text{known}}/\mathcal{N} > \frac{10368(9-p)}{117649}(-3p^2/4 + 31p/8 + 23/8) > 4.23.$$

The polynomial on the right increases on $[1, 9/5]$: its derivative decreases and is positive at $9/5$. The normalized right derivative is the pole-free expression

$$\widehat{R} = 8(p-1)F + 12\ell F' + 6\ell^2 F''/p + \ell^3 F'''/[p(p+1)],$$

where these primes denote U derivatives. This increases with U. To verify the whole-series sign, in each summand put $z = \ell/(j+U) \le 8/13$ and $k = \ell/U \ge 1$. After positive factors are removed, the derivative is affine in k with positive coefficient. At $k = 1$ its numerator is

$$N(p,z) = 4(p+1)(2p+3) - 12(p+2)^2 z + 3(p+3)(2p+5)z^2 - (p+3)(p+4)z^3.$$

It decreases with z and increases with p at $z = 8/13$. In particular $N(1, 8/13) = 1512/2197 > 0$ and $N_p(1, 8/13) = 14580/2197 > 0$; the second derivatives preserve these signs on the indicated rectangle. At $U = 1$ the complete expression is

$$8(p-1)(p+1-Z) + 6(2p+1)(p+1)[\zeta(p+2)-1] \\ - (6p+8)(p+2)[\zeta(p+3)-1] + (p+2)(p+3)[\zeta(p+4)-1].$$

Here $p + 1 - Z \geq .31534 + .3t$. If $b = 9/5 - p$, complete monotonicity of zeta and its endpoint moments give

$$\zeta(p+2) - 1 > .09751 + .0835b + .0405b^2 + (.094/6)b^3 + (.126/24)b^4.$$

For these five coefficients, sum $n^{-19/5}(\log n)^j$ from $n = 2$ through 128, with $j = 0, 1, 2, 3, 4$, and divide by $j!$. Each sum is a lower bound because the omitted terms are positive. Enclose the power by fifth-root bisection to width 10^{-18} as in Appendix D. For the logarithm write $n = 2^\ell v$, $1 \leq v < 2$, and keep the first 12 terms of $2\sum_{k\geq 0}[(v-1)/(v+1)]^{2k+1}/(2k+1)$ and the same series at $v = 2$ for $\log 2$. These rational lower sums give the five displayed bounds. Also $\zeta(p+3) - 1 < .0824 - .049t$ and $\zeta(p+4) - 1 \geq 2^{-p-4} \geq .03125 - .0216875t$. Substitution gives the lower polynomial

$$L_r = 4.1286624 + 3.5105408t + 1.3516743t^2 + .9608485t^3 + .00452t^4 - .1691t^5 + .063t^6 > 3.8 + 5t.$$

For the last inequality discard the positive fourth and sixth terms, use $-.1691t^5 \geq -.108224t^3$, and put $v = t - .4$. The difference is at least $.003714576 + .00114v + 2.0337739v^2 > 0$. This proves (F.1).

For (F.2), the finite terms of $H_{\text{left}} + q_{\text{known}}$ cancel to $-mu^{-p} + u(3u-1)^{-p}$. The first term has third derivative $p(p+1)u^{-p-3}[p + 2 - 2(p-1)u] > 0$. For the second, write it as $[(3u-1)^{1-p} + (3u-1)^{-p}]/3$; both third derivatives are nonnegative. The paired left tail has third derivative $(p)_3$ times

$$\sum_{j\geq 1} x^{-p-3}[(p+3)r - 6 + 27(1+r)^{-p-3}], \qquad x = u + j, \quad r = m/x \leq 1/2.$$

The bracket decreases with r, and at $r = 1/2$ decreases with $p + 3 \in [4, 24/5]$. It is positive at $24/5$, since $(3/2)^{24/5} < 15/2$. The remaining right comparison is $\widehat{R} > 8(p-1)(Z-1)$. The right side is at most $5.16t + 6t^2$, while $L_r - 5.16t - 6t^2$ decreases and is greater than .28 at $t = .8$. This establishes the sign, including $p = 1$ without division by $p - 1$.

We next estimate the third derivative of S, retaining its first two sites separately. For every site x, the convergent central expansion is

$$G_u(x) = \sum_{n\geq 0} \frac{[2n/(2n+1) - \delta](p)_{2n+1}}{2^{2n}(2n)!}(x + 1/2)^{-p-2n-1} \\ + b\sum_{n\geq 0} \frac{(p)_{2n+2}}{2^{2n+1}(2n+1)!}(x+1/2)^{-p-2n-2}, \quad b = .52 - \delta + .48\delta^2.$$

For $j \geq 2$, put $y_j = u + jm + 1/2$ and $X = 1/y_j \leq 8/23$. The derivatives of my_j^{-q} are

$$D_1 = 2y_j^{-q}[q - 1 - qX],$$
$$D_2 = (4q/m)y_j^{-q}(1-X)[q - 1 - (q+1)X],$$
$$D_3 = (8q(q+1)/m^2)y_j^{-q}(1-X)^2[q - 1 - (q+2)X].$$

The ratio D_3/D_2 increases with q and decreases with X. For every odd correction $n \geq 1$, its minimum is $450/77 > 5.84$, whereas its coefficient is at least $13/24$; therefore the derivative of

that complete correction is positive. For the first even correction the ratio is at least $15/2$ and $b \geq 161/400$, giving the same sign; all higher corrections are stronger. It suffices to retain

$$G_{\text{base}}(y) = -p\delta y^{-p-1} + [bp(p+1)/2]y^{-p-2}.$$

To record the complete tail calculation, put $h = \delta/m$, $\ell = 1 + 2h$, $Y = 7/2 + 2h$, $B = .52 + 1.08h + .56h^2$, $e = 1 + 1.04h$. The normalized third derivative of each base site is $\sum_{k=0}^{4} C_k (j + 3/2 + 2h)^{-p-1-k}$, where

$$\begin{aligned}
C_0 &= -8hp(p+2) - 12p, \\
C_1 &= [24h(p+2)(p+1) + 24(p+1)]\ell + (p+1)[4B(p+2)(p+3) - 6e(p+2) + 2.88], \\
C_2 &= -[24h(p+2)^2 + 12(p+2)]\ell^2 + [-12B(p+2)^2(p+3) + 12e(p+2)^2 - 2.88(p+2)]\ell, \\
C_3 &= 8h(p+2)(p+3)\ell^3 + [12B(p+2)(p+3)^2 - 6e(p+2)(p+3)]\ell^2, \\
C_4 &= -4B(p+2)(p+3)(p+4)\ell^3.
\end{aligned}$$

The signs alternate. Apply the Euler upper through B_2 to the negative coefficients and the lower through B_4 to the positive ones. After dividing by Y^{-p}, the lower is

$$L = \sum_{k=0}^{4} C_k \left[\frac{1}{(p+k)Y^k} + \frac{1}{2Y^{k+1}} + \frac{p+1+k}{12Y^{k+2}} \right] - \sum_{k=1,3} \frac{C_k (p+k+1)_3}{720 Y^{k+4}}.$$

To verify $L > -(6.898 - .303p)$, expand $Y^7 (p)_5 (L + 6.898 - .303p)$ as $\sum q_{ij} v^i w^j$ with $p = 1 + 4v/5$, $h = w/6$. Its bidegree is $(11, 7)$. The minimum of the Bernstein coefficients

$$\sum_{i=0}^{r} \sum_{j=0}^{s} q_{ij} \frac{\binom{r}{i}\binom{s}{j}}{\binom{11}{i}\binom{7}{j}}, \qquad 0 \leq r \leq 11, \quad 0 \leq s \leq 7,$$

is $415341507/640 > 0$. The complete $j \geq 2$ tail therefore exceeds

$$-(6.898 - .303p)(7/2)^{-p} > -[1.89 - 1.52t].$$

The final comparison is the convex chord of the positive expression, whose endpoint uppers are 1.89 and .674.

For $j = 0$, put $\alpha_0 = .48p(p-1)(2-p)/(p+1) < .075$. The unshifted part, divided by $p(p+1)$, is

$$u^{-p-3}[-\alpha_0 u^2 + (p-1)(4p/5 - 2)u + (p+2)(1 - 13p/25)].$$

The known term and the shifted part together are positive. After multiplication by v^{p+4}, $v = u + 1 \geq 15/8$, their numerator is

$$\frac{1.44}{p+1} v^{p+4} + \alpha_0 v^3 + (p-1)(2 + 66p/25)v^2 - (p+2)(3 + 108p/25)v + (54/25)(p+2)(p+3).$$

It increases with v. At $15/8$ it is greater than $1.43 - 4.4625t + 4.34125t^2 > 0$. The remaining unshifted bracket is bounded using $B_0 = .28p^2 - 2.84p + 4 - \alpha_0$, $[B_0]_+ \leq 1.8(P - p)$, and $1 \geq m^{p+2} u^{-p-3} > 8/13$. This gives the lower $.811153846 - 1.587692308t + .28t^2$.

For $j = 1$, put $x = 3u - 1$, $y = 3u$. Its derivative divided by $p(p+1)$ is

$$x^{-p-4}[-\alpha_0 x^3 + Ax^2 + B_1 x - C_1] + y^{-p-3}[\alpha_0 y^2 + D_1 y - E_1],$$

with

$$A = (p-1)(96p/25 - 18), \quad B_1 = (p+2)(39p/25 + 9),$$
$$C_1 = (9/5)(p+2)(p+3).$$

$$D_1 = (18/5)(p-1)(p+5), \qquad E_1 = 27(p+2)(1+4p/25).$$

The first numerator divided by x is concave in x and its minimum exceeds ten. Use $(y/x)^{p+3} \ge (3/2)^{p+3} > (2479/1000)(p+1)$. The remaining numerator is bounded below by

$$\begin{aligned} T_j &= k(p+1)[(87/13)p^2 - (37581/1300)p + 10557/260] \\ &\quad - k(2028/1600)p(p-1)(2-p) + (513/100)p^2 + (54/25)p - 405/4, \qquad k = 2479/1000. \end{aligned}$$

Its positive normalization is at most $(8/7)3^{-p-3}$. To prove a lower $-.1$, set $W = T_j + (189/80)3^p$. Differentiation gives $W'' > 8$ on $[1, 6/5]$, $W'(6/5) < -10$, $W(6/5) > 1.5$. On $[6/5, P]$, $W'' > 30$, $W(7/5) > .57$ and $|W'(7/5)| < .8$, whose quadratic lower is positive. These tests use $4.65 < 3^{7/5} < 4.656$ and $1.098 < \log 3 < 1.099$. Hence $W > 0$, proving the claimed $-.1$. This finishes (F.3).

Finally $\lambda_* - \mu_* > .1573$ and (F.1)–(F.3) give

$$\Phi'''/\mathcal{N} > .08427 + .7188t + .28t^2 > .08.$$

Since $\Phi(0) = 0$, write $\Phi(\delta)/\delta = \int_0^1 \Phi'(t\delta)\,\mathrm{d}t$. Its second derivative is the positive integral $\int_0^1 t^2\Phi'''(t\delta)\,\mathrm{d}t$. This proves the last assertion. □

G A coupled increment on the joint cone

This appendix keeps both source residuals coupled. Put $c = 9/10$, $\bar\mu = 3553/200000$, $A = 47/500$, $\lambda = 89/500$ and $\mu = \mu_* \le \bar\mu$. All moments here are for the pure law. Throughout this appendix, $7/8 \le u < 1$.

Lemma G.1 (Whole-cone increment)**.** *For $1 \le p \le P = 9/5$, $7/8 \le u < 1$, $2u - 1 \le m \le u$ and $0 \le d \le 9(u-m)/10$,*

$$\begin{aligned} \Phi(m,d) - \Phi(m,0) &\le d m^{-p}[C_*(p) - \kappa_p(u-m)/m], \\ \kappa_p &= \lambda p(p+1)(13/6)^{-p-2}\frac{288479}{1588261500} > 0. \end{aligned}$$

Proof. Write $g = (u-m)/m \in [0, 1/6]$, $\eta = d/m \le cg$ and $Q_B = -C'$. The rescaled difference is

$$\begin{aligned} m^p\frac{\Phi(m,d) - \Phi(m,0)}{d} &= \int \rho_p(x) R_{g,\eta}(x)\,\mathrm{d}x, \\ R_{g,\eta} &= \mu[(1+x)e^{-x} + a_0(x)\bar C_\eta(x)] - \lambda e^{-x} a_0((1+g)x)[1 - \bar C_\eta(x)], \\ \bar C_\eta(x) &= \int_0^1 C((1-\eta t)x)\,\mathrm{d}t. \end{aligned}$$

At $d = 0$ use the continuous interpretation. The right side increases in η, so replace it by cg. Its full derivative in g is

$$[\mu a_0(x) + \lambda e^{-x} a_0((1+g)x)]cx\int_0^1 tQ_B((1-cgt)x)\,\mathrm{d}t - \lambda(1+g)x^2 e^{-(2+g)x}[1 - \bar C_{cg}(x)]. \quad \text{(CI1)}$$

To bound the integral of (CI1), use

$$C(x) \le e^{-x/3}/2, \qquad Q_B(x) \le \widehat q(x) := e^{-x}[(1+x)/6 + Ax^2]. \quad \text{(G.1)}$$

For the first, the numerator of $3Q_B - C$ is $(2x-5)e^{2x} + (4x+4)e^x + 1$; its coefficients through degree three vanish and the factorial coefficients for $n \ge 4$ are $2^n(n-5) + 4n + 4 > 0$. Integrate

$C' = -Q_B$ from $C(0) = 1/2$. For the second, multiply $\widehat{q} - Q_B$ by the positive factor $e^x(e^x-1)^3$. The numerator is

$$(Ax^2 - 5x/6 + 13/6)e^{3x} + (-3Ax^2 - 3x/2 - 5/2)e^{2x} \\ + (3Ax^2 + x/2 + 1/2)e^x - Ax^2 - x/6 - 1/6.$$

Coefficients through degree four vanish. For $n \geq 5$ the factorial coefficient is $3^n a_n - 2^n b_n + c_n$, where

$$a_n = An(n-1)/9 - 5n/18 + 13/6, \qquad b_n = 3An(n-1)/4 + 3n/4 + 5/2, \\ c_n = 3An(n-1) + n/2 + 1/2.$$

For $n = 5, \ldots, 17$ these are respectively

$$82/25, 538/25, 451/5, 7664/25, 116376/125, 67854/25, 8197, \\ 687196/25, 13046998/125, 53925106/125, 1838499, 38969256/5, 4041974276/125.$$

For $n = j + 18$, the identities $a_n - n^2/1000 = 17j^2/1800 + 233j/4500 + 29/750 > 0$ and $n^2/8 - b_n = 109j^2/2000 + 513j/400 + 2927/1000 > 0$, together with $(3/2)^{18} > 125$, pay the entire remaining tail.

This integrand need not be negative pointwise; its positive tail is kept. Set $U = 1 + g$, $b = 2 + g$, $r(t) = 1 - cgt$, $r_0 = 1 - cg$, $P_r(x) = (1 + rx)/6 + Ar^2x^2$ and $M_2(b) = p(p+1)b^{-p-2}$. Define

$$V(p,g) = c\int_0^1 t \int \rho_p(x) x e^{-r(t)x} P_{r(t)}(x)[\bar{\mu} a_0(x) + \lambda e^{-x} a_0(Ux)]\,\mathrm{d}x\,\mathrm{d}t, \\ F(p,g) = V(p,g)/[\lambda U M_2(b)].$$

Equation (G.1) bounds the integral of (CI1) by

$$\lambda U M_2(b)\{F(p,g) - 1 + \tfrac{1}{2}[b/(b + r_0/3)]^{p+2}\}. \tag{G.2}$$

All differentiated integrals have a uniform positive exponential decay on the closed parameter box and polynomial control at zero.

The function F increases in g. Before integrating, the $a_0(Ux)$ bracket increases. The only possibly adverse logarithmic derivative is from $\widehat{q}(rx)$:

$$\frac{\widehat{q}'(y)}{\widehat{q}(y)} = \frac{y(8/375 - Ay)}{(1+y)/6 + Ay^2}.$$

It is nonpositive for $y \geq 32/141$, and otherwise is at most $(16/125)y$. Thus its g derivative is bounded below by $-(9/10)(16/125)(32/141)^2/(17/20) > -1/100$. The normalization contributes $(p+2)/b - 1/U \geq 3/(13/6) - 1 = 5/13$. Every positive normalized integrand consequently increases in g.

At fixed g, put $P_3(p) = (p+2)(p+3)(p+4)$. The function $F(p,g)/P_3(p)$ is log-convex in p. Indeed $a_0(vx) = v^2x^2\int_0^1 te^{-vtx}\,\mathrm{d}t$. After positive integration in x, the three terms of P_r give factors $\Gamma(p+3+j)/\Gamma(p+5)$, $j = 0, 1, 2$. These are respectively $\int_0^1 y^{p+2}(1-y)\,\mathrm{d}y$, $\int_0^1 y^{p+3}\,\mathrm{d}y$, and 1. Here the measure, rather than the moment, is independent of p. More explicitly, split the positive bracket into the two choices $(e, v, \omega) = (0, 1, \bar{\mu})$ and $(1, U, \lambda)$, and write $P_r(x) = \sum_{j=0}^2 c_j(r)x^j$. The positive x integral has rate $q = r + e + vs$ with $0 \leq s \leq 1$. After normalization its exponent-dependent factor is

$$\frac{\Gamma(p+3+j)}{\Gamma(p+5)} b^{p+2} q^{-p-3-j}.$$

Use the preceding beta integrals for the Gamma ratio (and an atom at $y = 1$ for $j = 2$). This factor becomes $(by/q)^p$ times a nonnegative weight independent of p. Summing and integrating these weights defines one positive measure, so Hölder proves log-convexity. The rates satisfy $q \geq 17/20$; the auxiliary beta weights are integrable at zero. These observations also justify Tonelli and finiteness of all the moments on $[1, 2]$.

Only two integer moments are needed:

$$F(1, 1/6) < 5/16, \qquad F(2, 1/6) < 79/100. \tag{CI2}$$

Using the positive integral for a_0 above, first integrate in x and then the resulting rational functions in $r = 1 - 3t/20$:

$$\begin{aligned} F(1, 1/6) &= R_1 + \frac{41743}{225} \log \frac{190}{181} - \frac{1337737921}{8010000} \log \frac{40}{37} - \frac{148312879}{8010000} \log \frac{20}{17}, \\ R_1 &= \frac{13514936015768257772038125 1}{18418881944938832443965000}, \\ F(2, 1/6) &= \frac{28154758322130217276902457123422743}{35698839250311983480758423155840000}. \end{aligned}$$

To check (CI2) using rational arithmetic alone, for $0 < t < 1$ use

$$2(t + t^3/3) < \log \frac{1 + t}{1 - t} < 2(t + t^3/3) + \frac{2t^5}{5(1 - t^2)}.$$

Use the upper at $t = 9/371$ and the lowers at $3/77, 3/37$. Substitution gives the first inequality in (CI2) by rational arithmetic; the second follows directly from its displayed rational value.

The identity $P_3(p) - 60p = (p - 1)(p - 2)(p + 12)$ gives $P_3(p) \leq 60p$ on $[1, 2]$. Log-convexity first gives

$$\frac{F(p, 1/6)}{P_3(p)} \leq \left(\frac{F(1, 1/6)}{60}\right)^{2-p} \left(\frac{F(2, 1/6)}{120}\right)^{p-1}.$$

Substituting (CI2) and multiplying by $60p$ therefore yields

$$F(p, g) \leq F(p, 1/6) < B(p) := \frac{5p}{16}(158/125)^{p-1}.$$

The adverse factor in (G.2) is at least $N(p) = 1 - \frac{1}{2}(130/147)^{p+2}$. Here B is convex and N concave. At $p = 1$, $N - B > 3/16$. At $P = 9/5$, the exact fifth-power comparisons

$$(9/16)^5(158/125)^4 < (17/25)^5, \qquad (130/147)^{19} < (63/100)^5$$

give $B(P) < 17/25$ and $N(P) > 137/200$. The concave difference therefore exceeds $1/200$ throughout $[1, P]$. This is larger than $288479/1588261500$. Since $U \geq 1$ and $b \leq 13/6$, (G.2) is at most $-\kappa_p$ with the unchanged displayed constant. Integrate from $g = 0$, where $\int \rho_p R_{0,0} = C_*(p)$, and use the earlier η order. This proves the whole-cone estimate. □

H The zero-gap finite far envelope

Continue with the coefficients (7.27). Put $m_0 = 3/4$, $u_0 = 7/8$, $\delta_0 = 1/8$, and define

$$\mathcal{F}_0(p) = N + \lambda_* \bar{Q} + \mu_* \bar{H} + T_*/8 + (1/10) m_0^{-p} [C_*]_+. \tag{H.1}$$

Its contributions from S, q, H are explicitly

$$\begin{aligned} N &= u_0(1+pa-Z) + m_0 G_u(u_0) + I_G(x) + (m_0/2)G_u(x) - (m_0^2/12)G_u'(x), \\ &\quad a = 189/800, \quad z = 511/800, \quad x = 13/8, \quad y = x+1 = 21/8, \\ I_G(x) &= zx^{-p} + ay^{-p} - \int_x^y t^{-p}\,\mathrm{d}t, \\ \bar{Q} &= s[p+1-Z+13M_2/72 - M_3/72 + 1063/1296000] \\ &\quad + Z - 1 - p/8 + (1/8 - 3p/4)r + M_2/128 + 1/2048, \\ &\quad s = (4/3)^{p-1}, \quad r = (8/7)^p, \\ \bar{H} &= -(9/128)[I(13/16)^{-p-2} + I_x(15/16)^{-p-2}]. \end{aligned}$$

The integral form of I_G also applies at $p = 1$, where its last term is $-\log(y/x)$.

Lemma H.1 (Complete far value)**.** *At $u = u_0$ and $p \in [1, 9/5]$, the function* (H.1) *bounds*

$$\Phi(m_0, 0) + \mathcal{C}(p, \delta_0) + \tfrac{1}{10} m_0^{-p} [C_*]_+ \le \mathcal{F}_0(p).$$

It satisfies $\mathcal{F}_0(p) < -.00052 - .009(P - p)$ for every $1 \le p \le 9/5$.

Proof. We first justify the three envelopes rather than treating them as formal expressions. For N the signed Euler upper is valid because

$$E_2(z) = 1/z + 1/2 + z/12 - 1/(1 - e^{-z})$$

is positive increasing. Its derivative sign reduces to $(z^2 - 12)(\cosh z - 1) + 6z^2 > 0$, whose coefficients through degree four vanish and whose later even coefficients are positive. The Laplace density of $G_u(x)$ has one crossing, from negative to positive, by Lemma 7.2. Its integral is positive: trapezoidal convexity on $[x, y]$ gives

$$G_u(x) \ge p x^{-p-1} \left[z - 1/2 + (a - 1/2)(x/y)^{p+1} \right] > 0,$$

since $x/y = 13/21$ and $p \ge 1$. The difference between the three Euler terms in N and the exact following sum is the integral of this density times $m_0 E_2(m_0 s)$. The positive increasing multiplier preserves positivity across the crossing. For $\bar{Q}$, use $F(U) = \sum_{j \ge 1} A_U(j)$. Its first two jets at one are

$$F(1) = p + 1 - Z, \quad F'(1) = M_2, \quad F''(1) = M_2 - M_3.$$

On $1 \le U \le 7/6$, $F'''(U) < 1.063$. Indeed the full differentiated series is bounded above by

$$(p)_3 \sum_{n \ge 2} n^{-p-3} [7(p+3)/(6n) - 4/3].$$

All $n \ge 5$ terms are negative; the other three positive parts increase with p and at P are less than 1.008347, .0524, .001646. For the adjacent left stream, the entire third derivative is bounded by

$$(p)_3 (15/8)^{-p-3} + p(p+1)(15/8)^{-p-2} < 3/2.$$

Taylor's integral formula at $\delta_0 = 1/8$, $U - 1 = 1/6$ now gives exactly $\bar{Q}$. Finally the diagonal identity for H in Lemma 7.9, the elasticity bound for b_0, and Jensen's inequality on the intervals $[m_0, u_0]$, $[u_0, 1]$ give the negative upper $\bar{H}$. Their interval midpoints are 13/16 and 15/16, and $m_0^2 \delta_0 = 9/128$.

The complete exponent curvature bounds are

$$N'' > .049, \qquad \bar{Q}'' > -.25, \qquad \bar{H}'' > -.51. \tag{H.2}$$

For N, separate the five printed terms. The first two together have second derivative greater than $.75(.358) - .875(.125) = .159125$. Here $G_{pp}(u_0) > .358$: the fourth derivative is positive and at P, $G_{pp}(u_0) > .3598$, $G_{ppp}(u_0) < .002$. To check these claims, for an affine power use

$$\partial_p^k[(Ap+B)v^{-p-r}] = v^{-p-r}[(-\log v)^k(Ap+B) + kA(-\log v)^{k-1}].$$

At $u_0 < 1$ this gives a positive fourth derivative for $(pz/u_0 - 1)u_0^{-p}$ because $4z/u_0 > |\log u_0|$. At $v = u_0 + 1$ the other block $(1 + pa/v)v^{-p}$ also has positive fourth derivative because $\log v(1 + a/v) > 4a/v$. The endpoint bounds then give $G_{pp}(u_0) > .3598 - .8(.002) > .358$.

For $G_{pp}(x)$, put $\ell = \log x$, $L_y = \log y$, $A_x = z\ell^2/x$ and $A_y = aL_y^2/y$. Its derivative is

$$x^{-p}[\ell^3 + (3 - p\ell)A_x] - y^{-p}[L_y^3 + (pL_y - 3)A_y].$$

On each of the four intervals $[l, h] = [1 + j/5, 1 + (j+1)/5]$, the second bracket increases, the first decreases, and the second is positive. The difference after division by x^{-p} is at most $\ell^3 + (3 - l\ell)A_x - (x/y)^h[L_y^3 + (lL_y - 3)A_y] < -.007$. Thus $G_{pp}(x) \geq G_{pp}(x)|_{p=P} > -.029$.

For $I_G^{(3)} > 0$, the log-concave function $v^{-p}\log^3 v$ on $[x, y]$ has endpoint ratio between three and five. The integral is at least the logarithmic mean of the two endpoint values, which exceeds .4 times their sum on this ratio interval. This pays the two endpoint terms, since $.4 - z + 3(.4 - a) > 0$. At $p = 1$ the second derivative exceeds

$$.393(.485)^2 + .09(.965)^2 - [(.966)^3 - (.485)^3]/3 > -.087.$$

For the four terms of $(-G_x)_{pp}$, write $D(v) = v^{-p-2}[(\log v)^2 p(p+1) - 2\log v(2p+1) + 2]$. The terms and their lower bounds are

$zD(x)$	$-.191614$
$\partial_p^2(-px^{-p-1})$	$.1399$
$\partial_p^2(py^{-p-1})$	$-.146113$
$aD(y)$	$-.03318.$

The two affine terms attain their minima at P and 1, respectively, by the third-derivative formula. For the two quadratic terms, evaluate the displayed products by interval substitution on intervals of length $1/20$ starting at one. The same power rule checks the four intervals for $G_{ppp}(x)$ and the endpoint derivatives above. For reproduction, enclose $\log v$ by the first 24 terms of the odd logarithm series of Appendix A and its geometric tail, using inversion for $v < 1$. For v^{-p-r}, use the exponential series through degree 32 at the appropriate endpoints of the exponent interval. For a negative exponent enclose the reciprocal of the positive exponential. At $t \geq 0$ the omitted tail is at most $t^{33}/[33!(1 - t/34)]$. Rounding interval operations outwards to multiples of 10^{-18} gives the stated strict bounds. Thus $(-G_x)_{pp} > -.25$, and $N'' > .159125 - .087 - .375(.029) - .046875(.25) > .049$.

For $\bar{Q}$, keep its common Z before differentiating. Lemma E.1 gives the following lower bounds for its second-derivative terms, in order:

$(1 - s)Z''$	$-.03235$
$2\log(4/3)s(1 - Z')$	$.1355117568$
$s\log^2(4/3)(p + 1 - Z)$	$.029240995$
$[(1/8 - 3p/4)r]''$	$-.2836510392$
$(13s/72 + 1/128)M_2''$	$-.0940383333$
$(13s/36)\log(4/3)M_2'$	$.0020727778$
$(13s/72)\log^2(4/3)M_2$	$.0059488722$
$-(sM_3)''/72$	$-.01130301696$

The omitted constant-times-s derivative is positive. Their sum exceeds $-.248568$, proving the stated bound. For $\bar{H}$, differentiation of the two products uses $P_m'' < 1.716$, $P_m' < 2.892$, $I < 2.8$, $I_x'' < .357$, $I_x' < .247$, $I_x < .5$, the powers $< 2.21, 1.28$ and logarithms $< .208, .065$. These give $\bar{H}'' > -.51$. The bound $I_x < .5$ follows directly from $e^s C(s) \le 1/2 + s$ and $b_0(s) \le s^2/2$: $I_x \le .25(p)_2 2^{-p-2} + .5(p)_3 2^{-p-3} < .435456$.

The remaining positive part in (H.1) has second derivative at least $-.1(1.679)[.095 + 2(.288)(.069)] > -.023$. This is a distributional lower curvature: the first derivative jumps upward at an activation of $[C_*]_+$. Since T_* is convex,

$$\mathcal{F}_0'' > .049 - .178(.25) - .017765(.51) - .023 > -.03.$$

At P, the printed finite expressions and the full moment enclosures give $N < -.042376$, $\bar{Q} < .26363$, $\bar{H} < -.4438$. Since $C_*(P) = 0$ and $T_*(P)/8 < .0028$,

$$\mathcal{F}_0(P) < -.042376 + .178(.26363) - .017762(.4438) + .0028 < -.00052.$$

At one, direct powers and logarithms give $N < -.115$, $\bar{Q} < .232$, $\bar{H} < -.13$, $C_*(1) < .008606$, $T_*(1)/8 = 23/1176$. Consequently $\mathcal{F}_0(1) < -.054$. For $\theta = (p-1)/(4/5)$ the semiconvex chord is

$$\mathcal{F}_0(p) < -.054(1-\theta) - .00052\theta + .0096\theta(1-\theta) \le -.00052 - .009(P-p).$$

This proves the full finite value, including every activation contact. □

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