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Cofinite derivative zeros at the order-one threshold

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Abstract

We give a complete probabilistic construction of a transcendental entire function for which every nonempty open set contains a zero of every sufficiently high derivative. The construction has $\log M(r) = O(r \log r \log\log r)$, hence order exactly one and infinite exponential type. No transcendental entire function of finite exponential type can have this zero property. We also establish a local limiting zero-counting measure for the constructed derivatives. The original existence assertion already has prior solution claims, including Hou's 2026 preprint. This draft does not claim first discovery of that assertion. The order-one construction and growth obstruction are the proposed strengthening; their novelty remains subject to literature review. All mathematical arguments are supplied below.

1 Problem, scope, and attribution

The prize catalogue lists JSP-000752 as Open on the source snapshot read on October 4, 2026 [1]. It asks for an entire function with dense derivative zero sets along every infinite subsequence of derivative orders. The natural, nontrivial interpretation requires a *transcendental* entire function: a nonzero polynomial would satisfy the literal version because all sufficiently high derivatives vanish identically. This is Erdős Problem 906 in Bloom's database [3].

The prize rules permit a complete mathematical solution to be supplied before Lean formalization [2]. Its catalogue flag "Eligible to claim" has a different meaning: an award claim requires both an accepted mathematical solution and verified formalization. This manuscript supplies mathematical proof evidence for review; no award entitlement or maintainer acceptance is asserted.

The bounded random-series and zero-free-disk framework has a clear precedent in Hou [4], whose stated construction has order two. Earlier probabilistic proposals are linked in the discussion of Problem 906 [3]. The existence answer itself is therefore *not claimed as a new breakthrough*. The construction below replaces the Fock weights with slowly increasing derivative weights and proves a stronger growth conclusion. It uses an inverse-amplitude moment and Jensen's inequality; no Laguerre asymptotics or external random-zero concentration theorem is needed. The order-one strengthening was not located in the cited prior sources; this is not a claim of verified first-discovery priority.

Write $B(c,R) = \{z : |z-c| < R\}$ and $M_f(r) = \max_{|z|=r} |f(z)|$. The zero property we will prove is

$$\forall U \subset \mathbb{C} \text{ nonempty and open}, \quad \exists N_U \quad \forall n \geq N_U, \quad \exists z \in U : \ f^{(n)}(z) = 0. \tag{C}$$

All derivative orders are nonnegative integers.

Lemma 1 (The required quantifier)**.** *An entire function has property (C) if and only if the union of the zero sets of $f^{(n_k)}$ is dense for every strictly increasing sequence (n_k) of derivative orders.*

Proof. If (C) holds, every such sequence eventually exceeds N_U, so the union meets each nonempty open U. Conversely, if (C) fails for one U, infinitely many derivative orders have no zero in U. Enumerating those orders gives a strictly increasing sequence whose union of zero sets misses U. □

2 A sharp order threshold

Theorem 2. *There is a transcendental entire function f with property (C) such that*

$$\log M_f(r) = O(r \log r \log\log r) \qquad (r \longrightarrow \infty). \tag{1}$$

It has order exactly one and infinite exponential type. Moreover, no transcendental entire function satisfying (C) has finite exponential type. Thus one is the smallest possible order, and this order is attained.

Finite exponential type here means that $|g(z)| \le Ce^{T|z|}$ for some finite $C, T > 0$ and all z. The usual order is $\rho(g) = \limsup_{r\to\infty} \log\log M_g(r)/\log r$.

Proposition 3 (Finite exponential type is impossible)**.** *A transcendental entire function of finite exponential type cannot satisfy (C).*

Proof. Suppose $|g(z)| \le Ce^{T|z|}$ with $T > 0$. Fix $R > 0$, and let

$$m_n(R) = \max_{|z|\le R} |g^{(n)}(z)|.$$

If (C) holds, for all $n \ge N_R$ choose $a_n \in B(0, R)$ with $g^{(n)}(a_n) = 0$. The line segment from a_n to any z in the closed disk lies in that disk. The fundamental theorem of calculus along this segment gives

$$|g^{(n)}(z)| \le |z - a_n| m_{n+1}(R) \le 2R m_{n+1}(R).$$

Consequently

$$m_n(R) \ge m_{N_R}(R)(2R)^{-(n-N_R)} \qquad (n \ge N_R). \tag{2}$$

The factor $m_{N_R}(R)$ is strictly positive, since a derivative of a transcendental entire function is not identically zero.

Cauchy's estimate on a circle of radius t about z, $|z| \le R$, gives

$$m_n(R) \le Ce^{TR} n! \frac{e^{Tt}}{t^n}.$$

Take $t = n/T$ and use $n! \le n^n$. For $n \ge 1$ this yields

$$m_n(R) \le Ce^{TR}(eT)^n. \tag{3}$$

Choosing $0 < R < (2eT)^{-1}$ contradicts (2)–(3), because $(2R)^{-1} > eT$. □

3 The weighted random entire function

Set $A = e^e$ and, for $t \ge 0$, define

$$w(t) = \log(t + A) \log\log(t + A), \qquad W_n = w(n). \tag{4}$$

Define $b_0 = 1$ and $b_n = \prod_{j=1}^{n} W_j$ for $n \geq 1$. Let $(\xi_j)_{j\geq 0}$ be independent random variables, each uniformly distributed with respect to area on the closed unit disk. Our entire function is

$$f(z) = \sum_{j=0}^{\infty} \xi_j \frac{b_j}{j!} z^j. \tag{5}$$

These are genuine random complex coefficients; the probabilistic argument extracts one deterministic sample at the end.

We will use only the following elementary facts about w:

$$W_n \geq 1, \quad w \text{ is increasing}, \quad \frac{W_{2n}}{W_n} \longrightarrow 1, \quad \frac{W_n}{\log n} \longrightarrow \infty, \quad \frac{W_n}{\sqrt{n}} \longrightarrow 0. \tag{6}$$

Indeed, for $t \geq 0$, $w'(t) = (\log\log(t + A) + 1)/(t + A) > 0$, and the remaining limits follow from $w(t) \sim \log t \log\log t$.

Lemma 4 (Normal convergence and deterministic growth)**.** *Every sample in* (5) *is entire. For all sufficiently large r, uniformly over all samples,*

$$M_f(r) \leq 2\exp\bigl(rw(\lceil r^2 \rceil)\bigr). \tag{7}$$

Almost surely all Taylor coefficients are nonzero, so f is transcendental almost surely.

Proof. Monotonicity gives $b_j \leq W_j^j$. The elementary factorial bound $j! \geq (j/e)^j$ gives

$$\left(\frac{b_j}{j!}\right)^{1/j} \leq \frac{eW_j}{j} \longrightarrow 0.$$

Thus the series converges normally on compact sets, uniformly in the bounded coefficients. The same is true after any fixed number of termwise differentiations.

For the growth bound put $J = \lceil r^2 \rceil$. For $0 \leq j \leq J$ we have $b_j \leq W_J^j$, so the sum of these absolute terms is at most $\exp(rW_J)$. For $j > J$, $r \leq \sqrt{j}$, and therefore

$$\frac{b_j r^j}{j!} \leq \left(\frac{erW_j}{j}\right)^j \leq \left(\frac{eW_j}{\sqrt{j}}\right)^j.$$

By (6), for all sufficiently large r the last quantity is at most 2^{-j} for every $j > J$. The tail is at most $2^{-J} \leq 1$, which is at most the preceding exponential bound. This proves (7). Finally each coefficient law is nonatomic, so $\mathbb{P}(\xi_j = 0) = 0$ for each j. A countable union of null events still has probability zero. □

Normalize the derivatives by

$$G_n(z) = \frac{f^{(n)}(z)}{b_n} = \sum_{k=0}^{\infty} \xi_{n+k} B_{n,k} \frac{z^k}{k!}, \qquad B_{n,k} = \frac{b_{n+k}}{b_n} = \prod_{j=n+1}^{n+k} W_j. \tag{8}$$

The empty product is one. Normalization does not change any zero set.

Lemma 5 (A uniform upper bound for a derivative)**.** *For every fixed $H > 0$, for all sufficiently large n and all $0 \leq r \leq H$,*

$$S_n(r) := \sum_{k=0}^{\infty} B_{n,k} \frac{r^k}{k!} \leq e^{rW_{2n}} + 2^{-n} \leq 2e^{rW_{2n}}. \tag{9}$$

In particular $|G_n(z)| \leq 2e^{|z|W_{2n}}$ for $|z| \leq H$.

Proof. For $k \le n$, all factors in $B_{n,k}$ are at most W_{2n}. These terms sum to at most $e^{rW_{2n}}$. For $k > n$, we have $n + k \le 2k$, hence

$$B_{n,k}\frac{r^k}{k!} \le \left(\frac{eHw(2k)}{k}\right)^k.$$

Since $w(2k)/k \to 0$, for all sufficiently large n the last quantity is at most 2^{-k} simultaneously for every $k > n$. The sum of these terms is 2^{-n}. The last assertion uses $|\xi_j| \le 1$. □

4 One-coordinate anti-concentration

Lemma 6 (An inverse-amplitude moment)**.** *If ξ is uniform on the unit disk, $a \neq 0$ and $v \in \mathbb{C}$, then*

$$\mathbb{P}(|a\xi + v| < s) \le \min\{1, s^2/|a|^2\} \quad (s > 0), \qquad \mathbb{E}\frac{1}{|a\xi + v|} \le \frac{2}{|a|}. \tag{10}$$

Proof. The first inequality follows by comparing areas: the preimage of the disk of radius s has area at most $\pi s^2/|a|^2$ in the unit disk. The layer-cake formula and this inequality give

$$\mathbb{E}\frac{1}{|a\xi + v|} \le \int_0^{1/|a|} 1\,dt + \int_{1/|a|}^{\infty} \frac{1}{|a|^2 t^2}\,dt = \frac{2}{|a|}.$$

□

Lemma 7 (A near-peak coefficient)**.** *For $t \ge 1$ and $m = \lfloor t \rfloor$,*

$$\frac{t^m}{m!} \ge \frac{e^t}{e^2 t}. \tag{11}$$

Consequently, if $|z|W_n \ge 1$, the term with $m = \lfloor |z|W_n \rfloor$ in (8) has coefficient $a = B_{n,m}z^m/m!$ satisfying

$$|a| \ge \frac{e^{|z|W_n}}{e^2|z|W_n}.$$

Proof. For $m \ge 1$, comparison with an integral gives

$$\log(m!) \le m\log m - m + 1 + \log m.$$

Thus

$$\log\frac{t^m}{m!} \ge m\log(t/m) + m - 1 - \log m \ge t - 2 - \log t,$$

because $m \le t$, $m \ge t - 1$, and $\log m \le \log t$. Monotonicity of w gives $B_{n,m} \ge W_n^m$, proving the second assertion. □

Lemma 8 (Uniform inverse moments on a compact set)**.** *For any $H > 0$, put $H_+ = \max\{1, H\}$. For every $n \ge 1$ and $|z| \le H$,*

$$\mathbb{E}\left(\frac{e^{W_n|z|}}{|G_n(z)|}\right) \le 2e^2 H_+ W_n. \tag{12}$$

The value at $G_n(z) = 0$ is interpreted as $+\infty$.

Proof. If $|z|W_n \geq 1$, choose m as in Lemma 7 and condition on all coefficients except ξ_{n+m}. The normally convergent series becomes $a\xi_{n+m}+v$, with v fixed under this conditioning. Independence, Lemma 6, and Lemma 7 give

$$\mathbb{E}\left(\frac{e^{W_n|z|}}{|G_n(z)|}\middle|\text{ all other coefficients}\right) \leq 2e^2|z|W_n \leq 2e^2 H_+ W_n.$$

Integration removes the conditioning. If $|z|W_n < 1$, condition instead on ξ_n, whose coefficient in $G_n(z)$ is one. The left side is then at most $2e$, which is no larger than the right side since $H_+, W_n \geq 1$. This includes $z = 0$. □

5 Summable probabilities of zero-free disks

Lemma 9 (Strict circular gain)**.** *For $c \in \mathbb{C}$ and $\eta > 0$,*

$$\gamma(c,\eta) = \frac{1}{2\pi}\int_0^{2\pi} |c+\eta e^{i\theta}|\,d\theta - |c| > 0.$$

Proof. Pair the angles θ and $\theta+\pi$. The triangle inequality gives $|c+\eta e^{i\theta}| + |c-\eta e^{i\theta}| \geq 2|c|$. For $c \neq 0$ the inequality is strict on an interval of angles near a direction perpendicular to c. For $c = 0$ the assertion is immediate. Integrating proves strict positivity. □

Fix $c \neq 0$ and $0 < R < |c|$, put $\eta = R/2$ and $H = |c| + \eta$, and let $\mathcal{H}_n(c,R)$ be the event that G_n has no zero in $B(c,R)$. Define the nonnegative random variable

$$Y_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp(W_n|c+\eta e^{i\theta}|)}{|G_n(c+\eta e^{i\theta})|}\,d\theta. \tag{13}$$

Tonelli's theorem and (12) imply

$$\mathbb{E}Y_n \leq 2e^2 H_+ W_n. \tag{14}$$

On $\mathcal{H}_n(c,R)$, $\log|G_n|$ is harmonic throughout $B(c,R)$, and therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \log|G_n(c+\eta e^{i\theta})|\,d\theta = \log|G_n(c)|. \tag{15}$$

Jensen's inequality for the exponential, applied to (13), and (15) give

$$\log Y_n \geq W_n(|c| + \gamma(c,\eta)) - \log|G_n(c)|.$$

For all sufficiently large n, Lemma 5 bounds the last term. Thus, on $\mathcal{H}_n(c,R)$,

$$Y_n \geq \frac{1}{2}\exp\left(\gamma(c,\eta)W_n - |c|(W_{2n}-W_n)\right). \tag{16}$$

Markov's inequality, (14), and (16) now yield

$$\mathbb{P}(\mathcal{H}_n(c,R)) \leq 4e^2 H_+ W_n \exp(-\gamma(c,\eta)W_n + |c|(W_{2n}-W_n)) \tag{17}$$

$$\leq 4e^2 H_+ W_n \exp\left(-\tfrac{1}{2}\gamma(c,\eta)W_n\right) \tag{18}$$

for all sufficiently large n. The second inequality uses $W_{2n}/W_n \to 1$ and $\gamma(c,\eta) > 0$.

For every $a > 0$, (6) implies

$$\sum_{n=1}^{\infty} W_n e^{-aW_n} < \infty. \tag{19}$$

For example, eventually $W_n \le \sqrt{n}$ and $aW_n \ge 4\log n$, so the summand is at most $n^{-7/2}$. Consequently the probabilities in (18) are summable. The first Borel–Cantelli lemma proves that, almost surely, only finitely many G_n are zero-free in this fixed disk. No independence between different derivative orders is used.

For completeness the events above are measurable. Normal convergence makes $(\omega, z) \mapsto G_n(\omega, z)$ jointly continuous when the sample space is the countable product of closed unit disks with its product topology. Zero-freeness on an open disk equals positivity of the minimum modulus on each of a countable increasing sequence of closed subdisks. Each minimum is a measurable function; it can also be computed as an infimum over a countable dense set. The nonnegative integrand in (13) is measurable, so Tonelli applies without any integrability assumption in advance.

Completion of Theorem 2. Take the countable family

$$\mathcal{B} = \{B(c, R) : c \in \mathbb{Q} + i\mathbb{Q}, \quad R \in \mathbb{Q}_{>0}, \quad R < |c|\}.$$

Every nonempty open subset of $\mathbb{C}$ contains a member of $\mathcal{B}$: choose a nonzero point in it, then a nearby rational complex center and a sufficiently small rational radius. The preceding Borel–Cantelli conclusion holds simultaneously for all disks in $\mathcal{B}$ outside a countable union of null events. Intersect with the probability-one event on which all coefficients are nonzero. Choose a sample in this intersection. The resulting deterministic transcendental entire f has (C); the quantifier lemma gives the original subsequence formulation as well.

Lemma 4 implies (1), since $w(\lceil r^2 \rceil) = O(\log r \log\log r)$. Thus $\rho(f) \le 1$. Every entire function of order less than one has finite exponential type: choose $\rho(f) < q < 1$, then eventually $\log M_f(r) \le r^q \le r$, and adjust a constant on the remaining compact disk. Proposition 3 therefore forces $\rho(f) \ge 1$. It also rules out finite exponential type for our chosen function. This proves all assertions of the theorem. □

6 A further check: limiting zero density

The same construction provides a quantitative version. This section is not needed for the existence proof above. Let ν_n be the locally finite measure that counts the zeros of $f^{(n)}$, with multiplicities.

Theorem 10. *For almost every sample in (5),*

$$\frac{\nu_n}{W_n} \longrightarrow \frac{1}{2\pi|z|}\, dA(z)$$

locally weakly as $n \to \infty$. The limiting measure has no atom at zero; its displayed density is locally integrable. In particular, for every fixed $r > 0$,

$$\nu_n(B(0, r)) \sim r W_n.$$

Proof. For a fixed $H > 0$, write

$$V_n(z) = \big(W_n|z| - \log|G_n(z)|\big)_+, \qquad Q_n = \frac{1}{\pi H^2} \int_{|z|<H} V_n(z)\, dA(z).$$

The value $+\infty$ at a zero causes no difficulty. Nonzero analytic functions have locally integrable logarithms. Moreover $e^{x+} \le 1 + e^x$ and Lemma 8 give

$$\mathbb{E}e^{V_n(z)} \le 1 + 2e^2 H_+ W_n \qquad (|z| \le H).$$

Jensen's inequality for normalized area measure and Tonelli's theorem imply

$$\mathbb{E}e^{Q_n} \le 1 + 2e^2 H_+ W_n.$$

Hence, for every $\varepsilon > 0$,

$$\mathbb{P}(Q_n > \varepsilon W_n) \le (1 + 2e^2 H_+ W_n) e^{-\varepsilon W_n}.$$

These probabilities are summable by (19). Apply Borel–Cantelli for integer $H \ge 1$ and positive rational ε to obtain, almost surely, $Q_n/W_n \to 0$ on every compact disk.

Set $u_n(z) = W_n^{-1} \log |G_n(z)|$. Lemma 5 gives

$$(u_n(z) - |z|)_+ \le H \left(\frac{W_{2n}}{W_n} - 1 \right) + \frac{\log 2}{W_n} \longrightarrow 0$$

uniformly for $|z| \le H$. The opposite part is exactly $(|z| - u_n(z))_+ = V_n(z)/W_n$. Thus

$$u_n \longrightarrow |z| \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C}) \quad \text{almost surely.} \tag{20}$$

For a nonzero analytic function h, the elementary distributional identity

$$\Delta \log |h| = 2\pi \sum_{h(a)=0} \mathrm{mult}_a(h) \delta_a$$

follows from local factorization at each zero and $\Delta \log |z - a| = 2\pi \delta_a$; the nonvanishing local factor has harmonic log modulus. Therefore $\nu_n/W_n = (2\pi)^{-1} \Delta u_n$. Taking distributional Laplacians in (20) gives

$$\frac{\nu_n}{W_n} \longrightarrow \frac{1}{2\pi} \Delta |z| = \frac{1}{2\pi |z|} \, dA(z).$$

For the last equality, away from zero the radial Laplacian is $1/|z|$. Its flux through a circle of radius s is $2\pi s$, which tends to zero, so there is no delta mass at the origin.

Distributional convergence of these positive measures implies local weak convergence: a smooth compactly supported upper cutoff bounds their masses on any fixed compact set, and continuous test functions can then be uniformly approximated by smooth ones. Each disk boundary has zero limiting measure, so the masses of any fixed disk converge to the mass prescribed by the limit. Polar integration gives $\int_{B(0,r)} (2\pi|z|)^{-1} \, dA = r$. □

7 General weights and verification notes

The proof uses no special identity for (4). It works for any nondecreasing sequence $W_n \ge 1$ with

$$W_{2n}/W_n \to 1, \qquad W_n/\log n \to \infty, \qquad W_n/\sqrt{n} \to 0,$$

using $b_n = \prod_{j=1}^{n} W_j$. The deterministic bound becomes $\log M_f(r) \le r W_{\lceil r^2 \rceil} + \log 2$ eventually, and the limiting zero measure has normalization W_n. For example, slowly varying factors growing more slowly than $\log \log n$ can replace that factor in (4), as long as the three displayed conditions hold. The exact least possible growth rate inside the order-one infinite-type class is not determined by this argument.

The main proof is analytic and has no numerical assumptions. An accompanying Python script checks the coefficient peak inequality and the derivative majorant at selected parameter values using high precision arithmetic. Those checks are sanity checks, not substitutes for the proofs and not a Lean formalization.

AI assistance and authorship disclosure

Jensen Kohlmeyer (GitHub: JENW1N) and Liam Kruer (GitHub: lkruer) are the designated authors of this AI-assisted research manuscript. OpenAI Codex assisted with the mathematical construction, proof development, coefficient checks, literature comparison, and manuscript preparation. The public submission is made from the JENW1N account. No independent human verification or distinct individual author roles are asserted by this disclosure; attribution and correctness are submitted for review.

The principal points for independent review are the conditional inverse moment, the uniform tail estimate for all $k > n$, and the summability for every positive disk-dependent constant. The full proof above addresses the original infinite-subsequence quantifier. It has no unproved mathematical premise. First-solution credit is expressly not claimed; novelty of the sharper growth statements remains unverified.

References

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