BALANCED ADDITIVE COMPLEMENTS WITH BOUNDED REPRESENTATION FUNCTIONS

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Abstract. We construct infinite sets $A = \{a_1 < a_2 < \cdots\}$ and $B = \{b_1 < b_2 < \cdots\}$ of positive integers such that $|a_i - b_i| \le 5\sqrt{b_i}$ for every i, every integer at least 16384 belongs to $A + B$, and the number of ordered representations of any integer as $a + b$, with $a \in A$ and $b \in B$, is at most 47005. In particular, $a_i/b_i \to 1$, giving a negative answer to a conjecture of Erdős and Sárközy. The construction uses two nested prime-power scales and a uniform residue bound for the entire union of earlier blocks. A local cardinality matching preserves the points needed for coverage and provides the required control of the increasing enumerations.

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1. Introduction

For sets $A, B \subseteq \mathbb{N} = \{1, 2, \ldots\}$, define their sumset and ordered representation function by

$$A + B = \{a + b : a \in A,\ b \in B\}, \qquad r_{A,B}(n) = \#\{(a, b) \in A \times B : a + b = n\}.$$

Erdős and Sárközy asked whether, for infinite A, B and some $N \ge 1$,

$$\frac{a_i}{b_i} \longrightarrow 1, \qquad A + B \supseteq [N, \infty) \cap \mathbb{Z} \quad \Longrightarrow \quad \limsup_{n\to\infty} r_{A,B}(n) = \infty, \tag{1.1}$$

where a_i and b_i are the increasing enumerations of A and B. The question appears as Problem 1.17 in the 1999 collection [4] and as Problem 1145 in [1].

Theorem 1.1. *There are infinite sets $A = \{a_1 < a_2 < \cdots\}$ and $B = \{b_1 < b_2 < \cdots\}$ of positive integers satisfying*

$$|a_i - b_i| \le 5\sqrt{b_i} \qquad (i \ge 1), \tag{1.2}$$

$$[16384, \infty) \cap \mathbb{Z} \subseteq A + B, \tag{1.3}$$

$$r_{A,B}(n) \le 47005 \qquad (n \in \mathbb{Z}). \tag{1.4}$$

Consequently the implication in (1.1) is false.

The construction proceeds through finite blocks whose elements have size comparable to the square of their scale. Geometric block constructions and matched local configurations for this problem were considered in the public notes [2, 3]. Here we maintain a residue-class bound for the full union of preceding blocks. Powers of two control one component and powers of three control the other. The resulting invariant bounds all mixed representations at a fixed maximal scale simultaneously.

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2. Scales and residue bounds

Fix

$$H = 260, \qquad L = H + 1 = 261, \qquad M = 3L = 783, \qquad K = 2M + 1 = 1567. \tag{2.1}$$

For $k \geq 3$, put $q_k = 3^k$ and let p_k be the least power of two greater than q_k. Since distinct positive powers of two and three are unequal,

$$q_k < p_k < 2q_k, \qquad q_{k+1} = 3q_k, \qquad p_{k+1} = d_k p_k, \quad d_k \in \{2, 4\}. \tag{2.2}$$

Indeed, $p_k < 3q_k < 4p_k$, so the next selected power of two is either $2p_k$ or $4p_k$. In particular, $p_3 = 32$, $q_3 = 27$, and $\gcd(p_i, q_j) = 1$ for all $i, j \geq 3$. Each of the two modulus sequences is nested under divisibility.

For a finite set $S \subseteq \mathbb{Z}$ and a positive integer m, write

$$\rho_m(S; r) = \#\{x \in S : x \equiv r \pmod{m}\}.$$

At stage k, the preceding blocks will have unions

$$P_k^A = \bigcup_{3 \leq j < k} A_j, \qquad P_k^B = \bigcup_{3 \leq j < k} B_j.$$

The induction maintains

$$\rho_{q_k}(P_k^A; r) \leq K, \qquad \rho_{p_k}(P_k^B; r) \leq K \qquad (r \in \mathbb{Z}). \tag{2.3}$$

Lemma 2.1. *Let m, m' be positive integers with $m \mid m'$, and suppose $\rho_m(S; r) \leq K$ for every r. Call a residue class modulo m' safe if it contains at most $K - M = 784$ elements of S. Each residue class modulo m contains at most one unsafe class modulo m'. Moreover, if a finite set C uses only safe classes and $\rho_{m'}(C; r) \leq M$ for every r, then $\rho_{m'}(S \cup C; r) \leq K$ for every r.*

Proof. Two unsafe classes in the same class modulo m would contain at least

$$2(K - M + 1) = 1570 > K$$

elements of S. In a safe class, the union contains at most $(K - M) + M = K$ elements. An unsafe class receives no element of C, and its old count is bounded by the count of its parent class modulo m. □

At stage k, safety for the A-component is defined from P_k^A modulo q_{k+1}. Safety for the B-component is defined from P_k^B modulo p_{k+1}. Thus every class modulo q_k has at most one unsafe child among its three classes modulo q_{k+1}, and every class modulo p_k has at most one unsafe child among its d_k classes modulo p_{k+1}.

3. Construction of a block

Assume that the earlier blocks have been constructed, satisfy (2.3), and are pairwise disjoint within each component. Throughout this section abbreviate $p = p_k$ and $q = q_k$.

3.1. **Separating the blocks.** We choose integers

$$U_k = p^2 + \alpha_k, \quad 0 \leq \alpha_k < p, \qquad V_k = p^2 + \beta_k, \quad 0 \leq \beta_k < q. \tag{3.1}$$

Write $U = U_k$ and $V = V_k$. The candidate sets are

$$\mathcal{A}_k = \{U + pX + sq : 0 \leq X < Lq,\ 0 \leq s < 3\}, \tag{3.2}$$

$$\mathcal{B}_k = \{V + qY + up : 0 \leq Y < Lp,\ 0 \leq u < 2\}. \tag{3.3}$$

All parameters in these formulas are integers.

Lemma 3.1. *The anchors in (3.1) can be chosen so that $\mathcal{A}_k \cap A_j = \varnothing$ and $\mathcal{B}_k \cap B_j = \varnothing$ for every $3 \le j < k$.*

Proof. For each earlier j, impose

$$U + sq \not\equiv U_j + s'q_j \pmod{p_j} \qquad (0 \le s, s' < 3), \tag{3.4}$$

$$V + up \not\equiv V_j + u'p_j \pmod{q_j} \qquad (0 \le u, u' < 2). \tag{3.5}$$

Each condition in (3.4) excludes one residue class for α_k modulo p_j. Because $p_j \mid p$, this removes exactly a fraction $1/p_j$ of its p possible values. Also,

$$9\sum_{j=3}^{k-1}\frac{1}{p_j} \le \frac{9}{32}\sum_{h=0}^{\infty}2^{-h} = \frac{9}{16} < 1.$$

The union of the excluded choices therefore has size less than p. Similarly, the excluded fraction of the q possible values of β_k is at most

$$4\sum_{j=3}^{k-1}\frac{1}{q_j} \le 4\sum_{j=3}^{\infty}3^{-j} = \frac{2}{9} < 1.$$

Both anchors exist.

A point of $\mathcal{A}_k$ is congruent to $U + sq$ modulo p_j, since $p_j \mid p$. Every point of $A_j \subseteq \mathcal{A}_j$ is congruent to $U_j + s'q_j$ for some s'. Thus (3.4) gives the required disjointness on the A-side. The divisibility $q_j \mid q$ and (3.5) give the same conclusion on the B-side. □

The parameters within each candidate set are unique. Indeed, an equality between two expressions in (3.2), reduced modulo p, gives $p \mid (s - s')q$. Coprimality and $|s - s'| \le 2 < p$ imply $s = s'$, and then $X = X'$. On the B-side, reduction modulo q gives $q \mid (u - u')p$, whence $u = u'$ and $Y = Y'$.

Every candidate satisfies

$$p^2 \le z \le 263p^2. \tag{3.6}$$

For the upper bound in (3.2), use $q < p$ and $p \ge 32$ to obtain

$$U + pX + sq < p^2 + p + Lpq + 2q < (L+1)p^2 + 3p \le (L+2)p^2 = 263p^2.$$

The B-bound follows from $V + qY + up < p^2 + q + Lpq + p < (L+1)p^2 + 2p \le 263p^2$.

3.2. **Selection and matching.** Only safe candidates will be retained. This suffices to preserve the next instance of (2.3). For a fixed s and a fixed residue modulo q, exactly L indices $0 \le X < Lq$ give that residue, because p is invertible modulo q. Consequently

$$\rho_q(\mathcal{A}_k; r) \le 3L = M.$$

Every class modulo q_{k+1} is contained in a class modulo q, so its candidate count is also at most M. Likewise $\rho_p(\mathcal{B}_k; r) \le 2L \le M$, which bounds every candidate count modulo p_{k+1}. Lemma 2.1 now applies to any subsets of the safe candidates.

The choice of those subsets is arranged to give a close matching. For an integer $0 \le x < q$, put

$$Y_x = \{y \in \mathbb{Z} : 0 \le y < p,\ \lfloor qy/p \rfloor = x\}, \qquad d_x = |Y_x|.$$

The interval description

$$\frac{px}{q} \le y < \frac{p(x+1)}{q} \tag{3.7}$$

shows that

$$1 \le d_x \le 2. \tag{3.8}$$

Indeed, the half-open interval in (3.7) has length $p/q \in (1, 2)$ and lies in $[0, p]$.

For integers $0 \le t \le H$ and $0 \le x < q$, form the two stocks

$$S^A_{x,t} = \{U + p(x + qt) + sq : 0 \le s < 3\},$$
$$S^B_{x,t} = \{V + q(y + pt) + up : y \in Y_x,\ 0 \le u < 2\}.$$

Let $T^A_{x,t}$ and $T^B_{x,t}$ denote their safe subsets. The three A-points occupy the three different children of one class modulo q. Hence $|T^A_{x,t}| \ge 2$. For each $y \in Y_x$, the two B-points occupy different children of one class modulo p, because $p_{k+1} \ge 2p$. At least one is safe. Parameter uniqueness gives

$$|T^A_{x,t}| \ge 2 \ge d_x, \qquad |T^B_{x,t}| \ge d_x. \tag{3.9}$$

The mandatory points in each stock are its safe unshifted points, obtained by setting $s = 0$ or $u = 0$. Their counts, say m_A and m_B, satisfy $m_A \le 1$ and $m_B \le d_x$. Set $r = \max(m_A, m_B)$. By (3.9), on each side we can choose exactly r safe points containing all its mandatory points. Match the two chosen subsets by any bijection.

Let A_k and B_k be the unions of these choices over all (x, t). The uniqueness of $X = x + qt$ and $Y = y + pt$, together with parameter uniqueness, makes the stocks pairwise disjoint on each side. We have therefore constructed finite sets and a bijection

$$f_k : A_k \longrightarrow B_k. \tag{3.10}$$

Every safe unshifted point in (3.2) or (3.3) belongs to the corresponding selected set.

Proposition 3.2. *The sets A_k, B_k are disjoint from the preceding blocks in their respective components. They satisfy (3.6), and $P^A_k \cup A_k$, $P^B_k \cup B_k$ satisfy (2.3) with k replaced by $k + 1$. Moreover,*

$$|a - f_k(a)| \le 5p_k \qquad (a \in A_k). \tag{3.11}$$

Proof. Disjointness follows from Lemma 3.1, and the range bound holds for all candidates. The residue bounds follow from the safety argument above.

For points in one matched group, the term pqt cancels and their difference is

$$(U - V) + (px - qy) + sq - up.$$

We have $|U - V| < p$, $0 \le qy - px < p$, $sq \le 2q < 2p$, and $up \le p$. The triangle inequality proves (3.11). □

4. Coverage of each scale interval

Lemma 4.1. *Suppose U, V satisfy (3.1) and n is an integer with $16p^2 \le n \le 256p^2$. There are integers $0 \le x < q$, $0 \le y < p$, and $12 \le R \le 507$ such that*

$$n = U + V + px + qy + pqR. \tag{4.1}$$

Proof. Coprimality gives unique x and y in the stated ranges satisfying

$$px \equiv n - U - V \pmod q, \qquad qy \equiv n - U - V \pmod p.$$

Thus $n - U - V - px - qy$ is divisible by pq. Put $S = U + V + px + qy$. The parameter ranges imply

$$2p^2 \le S \le 2p^2 + 2pq - 2. \tag{4.2}$$

In particular, $S < n$, so $R = (n - S)/(pq)$ is a nonnegative integer. If $R \le 11$, then

$$n \le 2p^2 + 13pq - 2 < 15p^2,$$

contrary to $n \ge 16p^2$. If $R \ge 508$, then $p < 2q$ gives

$$n \ge 2p^2 + 508pq > 256p^2.$$

This proves the bounds on R. □

Proposition 4.2. *Every integer in* $[16p_k^2, 256p_k^2]$ *belongs to* $A_k + B_k$.

Proof. Use Lemma 4.1, and consider

$$a(t) = U + p(x + qt),$$
$$b(t) = V + q(y + p(R - t)),$$

where t ranges over the integer interval

$$I = [\max(0, R - H), \min(H, R)] \cap \mathbb{Z}. \tag{4.3}$$

Both t and $R - t$ lie between 0 and H. The bounds $12 \le R \le 507$ show that I contains at least thirteen consecutive integers.

All the $a(t)$ lie in a single class modulo q. Modulo $3q$, changing t by one changes $a(t)$ by pq. Since p is relatively prime to 3, the three residues of t modulo 3 give the three different child classes. Lemma 2.1 implies that at most one residue of t modulo 3 makes $a(t)$ unsafe.

Write $p_{k+1} = d_k p$ with $d_k \in \{2, 4\}$. All the $b(t)$ lie in one class modulo p. Changing t by one changes $b(t)$ by $-pq$, and q is relatively prime to d_k. The residues of t modulo d_k therefore give the different child classes. At most one of those residues makes $b(t)$ unsafe.

In any thirteen consecutive integers, a fixed residue modulo 3 occurs at most five times, and a fixed residue modulo d_k occurs at most seven times. At most twelve positions are excluded. Choose a remaining $t \in I$. Then $a(t)$ and $b(t)$ are safe unshifted candidates, so both were retained. Finally (4.1) gives $a(t) + b(t) = n$. □

5. The infinite sets and their representation function

Start at $k = 3$ with empty preceding unions, which satisfy (2.3). Lemma 3.1 and Proposition 3.2 allow the construction to continue for every $k \ge 3$. Define

$$A = \bigcup_{k \ge 3} A_k, \qquad B = \bigcup_{k \ge 3} B_k. \tag{5.1}$$

Proposition 4.2 makes each block nonempty. Disjointness within each component therefore makes both unions infinite. All their elements are positive by (3.6).

The intervals covered in Proposition 4.2 overlap, since

$$16p_{k+1}^2 \le 16(4p_k)^2 = 256p_k^2.$$

Their endpoints tend to infinity. Their first lower endpoint is $16p_3^2 = 16 \cdot 32^2 = 16384$, which proves (1.3).

Lemma 5.1. *For any positive integer* n, *at most five indices* $j \ge 3$ *satisfy*

$$p_j^2 \le n \le 526p_j^2. \tag{5.2}$$

Proof. Successive values of p_j^2 increase by a factor of at least four. If six indices $j_0 < \cdots < j_5$ satisfied (5.2), then

$$1024p_{j_0}^2 = 4^5 p_{j_0}^2 \le p_{j_5}^2 \le n \le 526p_{j_0}^2,$$

which is impossible. □

Proposition 5.2. *The sets in* (5.1) *satisfy* $r_{A,B}(n) \le 47005$ *for every* $n \in \mathbb{Z}$.

Proof. There are no representations when $n \le 0$, so let $n > 0$. Each representation $n = a + b$ has unique block indices i, j with $a \in A_i$ and $b \in B_j$. Put $\ell = \max(i, j)$. Monotonicity of the scales and (3.6) give

$$p_\ell^2 \le n \le 526p_\ell^2.$$

Lemma 5.1 leaves at most five possibilities for ℓ.

Fix one such ℓ, and abbreviate $p = p_\ell$, $q = q_\ell$, $U = U_\ell$, $V = V_\ell$. Representations with maximal index ℓ fall into the following three disjoint cases.

The A-point lies in A_ℓ and the B-point lies in P_ℓ^B. The A-point belongs to one of the three residues $U + sq$ modulo p. For a fixed s, its partner belongs to the residue $n - U - sq$ modulo p. The invariant (2.3) bounds the number of such partners in the entire set P_ℓ^B by K. Each partner determines the A-point uniquely, giving at most $3K$ representations.

The B-point lies in B_ℓ and the A-point lies in P_ℓ^A. There are two possible residues $V + up$ modulo q for the B-point. The same argument, using the A-side of (2.3), gives at most $2K$ representations.

Both points lie in block ℓ. Fix their shifts $0 \le s < 3$ and $0 \le u < 2$. The remaining equation is

$$pX + qY = n - U - V - sq - up.$$

Modulo q, this determines X uniquely as a residue class. Among the Lq indices $0 \le X < Lq$, that class contains L choices. Each choice determines at most one Y. The six shift pairs therefore contribute at most $6L$ representations.

Summing first over the three cases and then over the eligible maximal indices yields

$$r_{A,B}(n) \le 5(3K + 2K + 6L) = 5\big(5 \cdot 1567 + 6 \cdot 261\big) = 47005.$$

□

6. Increasing enumerations

The block bijections (3.10) combine to a bijection $f : A \to B$, because the blocks are disjoint within each component. For $a \in A_k$, both a and $f(a)$ are at least p_k^2. Equation (3.11) thus gives

$$|a - f(a)| \le 5\sqrt{a}, \qquad |a - f(a)| \le 5\sqrt{f(a)}. \tag{6.1}$$

Lemma 6.1. *Suppose infinite sets $A, B \subseteq \mathbb{N}$ admit a bijection satisfying (6.1). Their increasing enumerations satisfy $|a_i - b_i| \le 5\sqrt{b_i}$ for every $i \ge 1$.*

Proof. The images of $a_1, \dots, a_i$ are i distinct points of B. Each is at most $a_i + 5\sqrt{a_i}$, since the function $x \mapsto x + 5\sqrt{x}$ is increasing. Hence

$$b_i \le a_i + 5\sqrt{a_i}.$$

Apply the same argument to f^{-1} to obtain $a_i \le b_i + 5\sqrt{b_i}$. If $a_i \le b_i$, the first inequality gives $b_i - a_i \le 5\sqrt{a_i} \le 5\sqrt{b_i}$. If $a_i > b_i$, the second gives the required bound directly. □

Proof of Theorem 1.1. The construction in Section 5 gives infinite positive sets satisfying (1.3). Proposition 5.2 proves (1.4), and Lemma 6.1 proves (1.2). Since $b_i \to \infty$,

$$\left|\frac{a_i}{b_i} - 1\right| \le \frac{5}{\sqrt{b_i}} \longrightarrow 0.$$

The representation function has finite upper limit, which disproves (1.1). □

7. The everywhere-covering formulation

The same construction also gives a counterexample when the covering condition is imposed on every nonnegative integer, with zero allowed in the summand sets.

Corollary 7.1. *There are infinite sets $A^*, B^* \subseteq \mathbb{N}_0$ with $A^* + B^* = \mathbb{N}_0$, whose increasing enumerations satisfy $a_i^*/b_i^* \to 1$ and whose ordered representation function is uniformly bounded.*

Proof. Let A, B be the sets in Theorem 1.1 and put $N = 16384$. The complement of B in $\mathbb{N}_0$ is infinite. Otherwise, for any 47006 distinct elements a of A, every sufficiently large n would have $n - a \in B$ for all of them. This would give $r_{A,B}(n) \geq 47006$, contradicting (1.4).

Set $F = \{0, \dots, N-1\} \setminus A$ and $m = |F|$. We have $m \geq 1$ because $0 \notin A$. Choose a set $G \subseteq \mathbb{N}_0 \setminus B$ with $0 \in G$ and $|G| = m$, and put $A^* = A \cup F$, $B^* = B \cup G$. Every $n < N$ is represented by $n + 0$, and every $n \geq N$ retains an original representation.

For a fixed target, the representations using an element of F number at most m, and those using an element of G number at most m. Thus $r_{A^*,B^*}(n) \leq 47005 + 2m$. Beyond the finitely many added elements, the increasing enumerations are $a_i^* = a_{i-m}$ and $b_i^* = b_{i-m}$. Their ratio therefore tends to one. □

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