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AN EXACT ALGEBRAIC CONSTRUCTION FOR ERDŐS'S INTERPOLATION MINIMUM

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Abstract. For distinct nodes in $[-1,1]$, consider the integral of the sum of the squares of their Lagrange cardinal polynomials. We give a finite algebraic construction of its minimum for every positive number of nodes. For $n \geq 2$, put $k = n-1$ and $d = 2n(n-1)$. A rational symmetric polynomial matrix of order $(d+1)^k$ determines a nonzero rational polynomial of degree at most $kd(d+1)^k$. The minimum is a specified real root of this polynomial; its index is obtained from signatures at rational sample points. The proof establishes attainment and endpoint inclusion, expresses the objective in squared gap coordinates, and converts its threshold into a real critical point count. A fixed monomial basis and the Hermite trace form give the required count at every parameter, including parameters with multiple critical points. All steps of the value rule use rational polynomial arithmetic, univariate real root isolation, and rational symmetric elimination.

1. Introduction and main result

Let $X = (x_1, \ldots, x_n)$ consist of distinct real nodes in $[-1,1]$. Its Lagrange cardinal polynomials are

$$\ell_i^X(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \qquad 1 \leq i \leq n. \tag{1.1}$$

We use the empty product convention when $n = 1$. Define

$$I(X) = \int_{-1}^{1} \sum_{i=1}^{n} \ell_i^X(x)^2 \, dx, \qquad m_n = \inf_{\substack{x_i \in [-1,1] \\ x_i \neq x_j \ (i \neq j)}} I(X). \tag{1.2}$$

The integral is taken with respect to ordinary Lebesgue measure. The value of $I(X)$ is invariant under a permutation of the nodes, so ordered nodes may be used throughout.

This minimization is among the classical interpolation problems of Erdős; see Erdős, Szabados, Varma, and Vértesi [4]. Brutman and Toledano [3] studied the finite minima numerically and their asymptotic behavior. The problem is also recorded as Erdős Problem 1131 [2]. Our result concerns the exact finite-n minimum.

Algebraicity and an effective value procedure follow from general real algebraic methods [1]. Indeed, after clearing the positive squared Vandermonde denominator on the ordered node region, the assertion that t is an attained minimum is a first-order formula over $\mathbb{Q}$. Quantifier elimination then describes the singleton $\{m_n\}$ by univariate polynomial sign conditions. The construction below gives explicit polynomial and matrix data for this particular objective. It uses the classical Hermite trace principle and its parametric form; see [1, 5]. We include the trace argument needed here, including the treatment of multiplicities.

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For a real symmetric matrix H, write

$$\operatorname{sig} H = n_+(H) - n_-(H),$$

where $n_+(H)$ and $n_-(H)$ count positive and negative eigenvalues with multiplicity. Zero eigenvalues contribute zero.

Theorem 1.1. *For every positive integer n, the infimum in (1.2) is attained. For $n = 1$ it equals 2. For $n \geq 2$, every minimizing node set contains both endpoints, and*

$$\frac{2}{n} \leq m_n < 2. \tag{1.3}$$

Fix $n \geq 2$ and put

$$k = n - 1, \qquad d = 2n(n-1), \qquad M = (d+1)^k.$$

There are an explicitly defined symmetric polynomial matrix and a nonzero polynomial satisfying

$$H_n(t) \in \operatorname{Sym}_M(\mathbb{Q}[t]), \qquad 0 \neq R_n(t) \in \mathbb{Q}[t], \qquad \deg R_n \leq kdM,$$

constructed in Sections 5 and 7, with the following property. Let $r_1 < \cdots < r_s$ be the distinct real roots of R_n in $(0,2)$, put $r_{s+1} = 2$, and choose rational numbers $\rho_j \in (r_j, r_{j+1})$ for $1 \leq j \leq s$. Then $s \geq 1$ and

$$\boxed{m_n = r_{1+c_n}, \qquad c_n = \#\{j : 1 \leq j \leq s,\ \operatorname{sig} H_n(\rho_j) = 1\}.} \tag{1.4}$$

The construction and the rule depend only on n and use finitely many exact rational operations and univariate real root isolations.

The proof proceeds by an endpoint reduction, a homogeneous polynomial encoding, and a coercive perturbation whose real critical points detect the minimum. A fixed quotient basis makes the trace matrix valid at every real parameter. The last step identifies the correct root by counting signatures on the intervals between consecutive roots.

2. Attainment and endpoint reduction

Write $S_X(x) = \sum_i \ell_i^X(x)^2$.

Lemma 2.1. *The minimum m_n is attained, and $m_n \geq 2/n$.*

Proof. Choose a basis $p_0, \ldots, p_{n-1}$ of the polynomials of degree at most $n-1$ that is orthonormal for the inner product $\langle f, g\rangle = \int_{-1}^{1} fg$. Let $V_{ij} = p_{j-1}(x_i)$. Distinctness of the nodes makes V invertible. The coefficient vector of ℓ_i^X in this basis is $V^{-1}e_i$. Consequently,

$$I(X) = \|V^{-1}\|_F^2 = \operatorname{tr}((V^{\mathsf{T}}V)^{-1}). \tag{2.1}$$

Here $\|\cdot\|_F$ is the Frobenius norm.

Suppose a sequence of ordered configurations tends to a collision in the closed ordered region $-1 \leq x_1 \leq \cdots \leq x_n \leq 1$. The limiting evaluation matrix has two equal rows, so its smallest singular value tends to zero. Since

$$I(X) \geq \sigma_{\min}(V)^{-2},$$

the objective tends to infinity along such a sequence. A minimizing sequence has bounded objective by comparison with any fixed distinct-node configuration. Compactness of the closed ordered region therefore gives a convergent subsequence with a distinct-node limit. Continuity of (2.1) at that limit proves attainment. For $n = 1$ the same conclusion follows directly from $\ell_1^X = 1$.

The interpolation identity $\sum_i \ell_i^X(x) = 1$ and the Cauchy–Schwarz inequality give $S_X(x) \geq 1/n$. Integration proves the lower bound. □

Lemma 2.2. *For every $n \geq 2$, there is a node configuration with $I(X) < 2$.*

Proof. Maximize the Vandermonde product

$$\Delta(X) = \prod_{a<b} (x_b - x_a)$$

on the closed ordered node region. The maximum is positive, hence the maximizing nodes are distinct. Moving the leftmost node to -1 strictly increases the product unless it is already there. The corresponding argument at the right gives $x_n = 1$.

For every interior maximizing node, logarithmic differentiation yields

$$0 = \sum_{j \neq i} \frac{1}{x_i - x_j} = (\ell_i^X)'(x_i). \tag{2.2}$$

The polynomial S_X equals 1 at each node. At an interior node,

$$S_X'(x_i) = 2 \sum_j \ell_j^X(x_i)(\ell_j^X)'(x_i) = 2(\ell_i^X)'(x_i) = 0.$$

Moreover, S_X has degree exactly $2n-2$, since its leading coefficient is a sum of positive squares. Counting the endpoint zeros and the double interior zeros of $1 - S_X$ now gives

$$1 - S_X(x) = c(1-x^2) \prod_{i=2}^{n-1} (x - x_i)^2, \qquad c > 0. \tag{2.3}$$

The sign of c follows from the leading coefficient. The right-hand side is nonnegative on $[-1,1]$ and is positive except at finitely many points. Thus $I(X) < 2$. □

Proposition 2.3. *Every minimizing node set with $n \geq 2$ contains -1 and 1.*

Proof. Let X be a minimizing configuration with hull $[a,b]$, and put $L = b - a > 0$. For $x < a$, every factor of

$$\ell_1^X(x) = \prod_{j>1} \frac{x - x_j}{a - x_j}$$

is greater than one. Thus $S_X(x) > 1$ on $[-1, a)$. The same argument with ℓ_n^X gives $S_X(x) > 1$ on $(b, 1]$.

Suppose $L < 2$, and write

$$A = \int_a^b S_X(x)\,dx, \qquad B = \int_{[-1,1]\setminus[a,b]} S_X(x)\,dx.$$

Then $B > 2 - L$. Lemma 2.2 gives $A + B = m_n < 2$, so $A < L$. The affine map from $[a,b]$ onto $[-1,1]$ carries X to an admissible configuration Y. Its cardinal polynomials transform by composition, and change of variables gives $I(Y) = 2A/L$. Consequently,

$$I(X) - I(Y) = B - \frac{2-L}{L} A > (2-L)\left(1 - \frac{A}{L}\right) > 0,$$

contradicting minimality. Hence $L = 2$, which forces $a = -1$ and $b = 1$. □

The affine hull argument also occurs in the exact four-node analysis of White [6]. Proposition 2.3 gives the reduction for every $n \geq 2$.

3. A rational homogeneous encoding

Fix $n \geq 2$, and retain $k = n-1$ and $d = 2n(n-1)$. For $u = (u_1, \ldots, u_k)$ define

$$g_a = u_a^2, \qquad T = \sum_{a=1}^{k} g_a, \qquad Y_i = -T + 2\sum_{a<i} g_a \quad (1 \leq i \leq n). \tag{3.1}$$

In particular, $Y_1 = -T$, $Y_n = T$, and $Y_{i+1} - Y_i = 2u_i^2$. For an n-tuple y, let $y^{(i,w)}$ denote the tuple obtained by replacing its ith entry by w.

Define the $\mathbb{Q}[u]$-linear functional $\mathcal{L}_z$ on $\mathbb{Q}[u,z]$ by

$$\mathcal{L}_z(z^j) = \begin{cases} 2/(j+1), & j \text{ even}, \\ 0, & j \text{ odd}. \end{cases} \tag{3.2}$$

Thus $\mathcal{L}_z$ is integration in z over $[-1,1]$. Put

$$D(u) = \Delta(Y)^2, \qquad N(u) = \sum_{i=1}^{n} \mathcal{L}_z\Big(\Delta\big(Y^{(i,Tz)}\big)^2\Big). \tag{3.3}$$

These are rational homogeneous forms of degree d. Indeed, every entry of Y and the replacement Tz has u-degree two, and the Vandermonde has degree $n(n-1)/2$ in its entries.

Lemma 3.1. *For real u with every coordinate nonzero,*

$$\frac{N(u)}{D(u)} = I(Y_1/T, \ldots, Y_n/T). \tag{3.4}$$

Every ordered configuration with endpoints -1 and 1 arises in this way. For every real u, $D(u) \geq 0$ and $N(u) \geq 0$, and $D(u) = 0$ precisely when some $u_a = 0$.

Proof. If all coordinates are nonzero, then $T > 0$ and (3.1) gives strictly increasing normalized nodes with the required endpoints. For any distinct-node tuple x, the Vandermonde replacement identity is

$$\ell_i^x(z) = \frac{\Delta(x^{(i,z)})}{\Delta(x)}.$$

Scaling all entries by T shows that this ratio for $x = Y/T$ equals $\Delta(Y^{(i,Tz)})/\Delta(Y)$. Squaring, summing, and integrating gives (3.4).

Conversely, for ordered nodes with these endpoints choose

$$u_a = \sqrt{\frac{x_{a+1} - x_a}{2}}.$$

Then $T = 1$ and $Y_i = x_i$. Finally, D is a square, and its zero set is determined by the successive differences $2u_a^2$. Each summand defining N is the integral of a real square, which proves its nonnegativity even when several gaps vanish. □

For a real parameter t, define

$$P_t(u) = N(u) - tD(u). \tag{3.5}$$

Proposition 3.2. *For every real t,*

$$P_t(u) \geq 0 \text{ for every } u \in \mathbb{R}^k \quad \Longleftrightarrow \quad t \leq m_n. \tag{3.6}$$

Proof. If $t \leq m_n$ and $D(u) > 0$, Lemma 3.1 gives $P_t(u) = D(u)(I(Y/T) - t) \geq 0$. At every remaining real point, $D(u) = 0$ and $P_t(u) = N(u) \geq 0$.

Conversely, Lemma 2.1 and Proposition 2.3 provide a minimizing configuration with both endpoints. Apply the converse parametrization in Lemma 3.1. Its denominator is positive, so $P_t(u) \geq 0$ implies $m_n \geq t$. □

4. Detecting the threshold by critical points

Set

$$Q_t(u) = P_t(u) + \sum_{a=1}^{k} u_a^{d+2}. \tag{4.1}$$

Since d is even, the added form is strictly positive away from the origin.

Proposition 4.1. *The polynomial Q_t is coercive for every real t, and*

$$\{u \in \mathbb{R}^k : \nabla Q_t(u) = 0\} = \{0\} \quad \Longleftrightarrow \quad t \le m_n. \tag{4.2}$$

Proof. On the unit sphere, $\sum_a u_a^{d+2}$ has a positive minimum, while $|P_t(u)|$ has a finite maximum. Homogeneity therefore gives constants $c > 0$ and $C_t \ge 0$ such that

$$Q_t(u) \ge c\|u\|^{d+2} - C_t\|u\|^d.$$

This proves coercivity. In particular, Q_t attains a global minimum. The origin is a critical point because $d \ge 4$.

If $t > m_n$, Proposition 3.2 gives v with $P_t(v) < 0$. For sufficiently small $s > 0$,

$$Q_t(sv) = s^d \left(P_t(v) + s^2 \sum_a v_a^{d+2} \right) < 0.$$

A global minimizer of Q_t is then a nonzero real critical point.

Conversely, at a nonzero real critical point v, Euler's identity gives

$$0 = v \cdot \nabla Q_t(v) = dP_t(v) + (d+2) \sum_a v_a^{d+2}. \tag{4.3}$$

The last sum is positive, so $P_t(v) < 0$. By Proposition 3.2, this implies $t > m_n$. □

Remark 4.2. At a nonzero real critical point, all gap coordinates are nonzero. Indeed, $P_t(v) < 0$ is incompatible with the nonnegativity of N on $D = 0$. Thus the critical-point threshold also respects every collision face of the node region.

5. A fixed quotient basis and its trace matrix

In the coefficient ring $\mathbb{Q}[t]$, write $P(u,t) = N(u) - tD(u)$ and set

$$h_i(u,t) = -\frac{1}{d+2}\frac{\partial P}{\partial u_i}(u,t), \qquad f_i(u,t) = u_i^{d+1} - h_i(u,t). \tag{5.1}$$

Thus $f_i = (d+2)^{-1}\partial Q_t/\partial u_i$. Each h_i is homogeneous of u-degree $d-1$ and has coefficients affine in t. For a multi-index, write $|\alpha| = \sum_i \alpha_i$. Fix the lexicographic order on the set

$$\mathcal{B} = \{u^\alpha : \alpha \in \{0,1,\dots,d\}^k\}, \qquad |\mathcal{B}| = M = (d+1)^k. \tag{5.2}$$

The reduction rule is

$$u_i^{d+1} \longmapsto h_i(u,t). \tag{5.3}$$

Lemma 5.1. *Reduction by (5.3) terminates and gives a unique normal form in the span of $\mathcal{B}$. The residue classes of $\mathcal{B}$ form a basis of*

$$\mathcal{A} = \mathbb{Q}[t, u_1, \dots, u_k]/(f_1, \dots, f_k) \tag{5.4}$$

as a $\mathbb{Q}[t]$-module. For every real τ, the same monomials form a basis of

$$\mathcal{A}_\tau = \mathbb{R}[u_1, \dots, u_k]/(f_1(u,\tau), \dots, f_k(u,\tau)),$$

and normal forms commute with specialization $t = \tau$.

Proof. Every replacement reduces total u-degree by two, so reduction terminates. We prove uniqueness by induction on the total degree of a monomial. If two possible first replacements use the same index, their results agree. If they use distinct indices i and j, write the monomial as $u_i^{d+1}u_j^{d+1}v$. The two replacements give $h_iu_j^{d+1}v$ and $u_i^{d+1}h_jv$. In the first expression replace the displayed u_j^{d+1} factor, and in the second replace the displayed u_i^{d+1} factor. Both paths reach h_ih_jv. Every intermediate monomial has smaller degree than the original one. The induction hypothesis makes all subsequent normal forms equal. Linearity extends the conclusion to polynomials.

Let NF denote the resulting $\mathbb{Q}[t]$-linear normal-form map. For a monomial v, the defining replacement shows $\mathrm{NF}(vf_i) = 0$. Hence NF annihilates the ideal $(f_1, \dots, f_k)$. Each reduction changes a polynomial by an element of this ideal, so $p - \mathrm{NF}(p)$ belongs to the ideal for every p. These two facts show that $\mathcal{B}$ spans the quotient and that a linear combination of $\mathcal{B}$ belonging to the ideal must be zero.

This proof works over $\mathbb{R}$ after any specialization. Moreover, the reduction uses only addition and multiplication of coefficients and the fixed rational factor $(d+2)^{-1}$. Specialization therefore commutes with the entire reduction. □

For $g \in \mathbb{Q}[t, u]$, let $M_g(t)$ be the matrix of multiplication by g in the ordered basis $\mathcal{B}$. Its column indexed by u^β consists of the coefficients of $\mathrm{NF}(gu^\beta)$. Define

$$H_n(t)_{\alpha,\beta} = \mathrm{tr}\big(M_{u^{\alpha+\beta}}(t)\big), \qquad u^\alpha, u^\beta \in \mathcal{B}. \tag{5.5}$$

This is a symmetric matrix over $\mathbb{Q}[t]$. Lemma 5.1 identifies $H_n(\tau)$ with the trace-form matrix of $\mathcal{A}_\tau$ for every real τ.

6. The real trace identity, including multiplicities

We give the finite-dimensional trace argument to account explicitly for multiple zeros. For a finite-dimensional real algebra A, the trace form is the symmetric bilinear form

$$(a, b) \longmapsto \mathrm{tr}(L_{ab}),$$

where L_c denotes multiplication by c on A.

Proposition 6.1 (Hermite trace identity)**.** *Let $A = \mathbb{R}[u_1, \dots, u_k]/J$ be a nonzero finite-dimensional algebra. The signature of its trace form is the number of distinct real common zeros of J.*

Proof. Let $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$. The multiplication operators on $A_{\mathbb{C}}$ commute. They are simultaneously upper triangular in a suitable basis. For completeness, a finite commuting family over $\mathbb{C}$ has a common eigenvector: restrict the family to an eigenspace of one member and continue, decreasing dimension whenever necessary. Its common invariant line gives a commuting family on the quotient. Induction on dimension then gives simultaneous upper triangularization. Applying this to the multiplication operators of a vector-space basis of $A_{\mathbb{C}}$ suffices.

Every diagonal position defines a unital $\mathbb{C}$-algebra homomorphism $\chi : A_{\mathbb{C}} \to \mathbb{C}$, since diagonal entries add and multiply under upper triangular matrix operations. The values $\chi(u_i)$ give a common zero of J. Let Z be the finite set of zeros obtained in this way, and let m_ζ be the number of diagonal positions yielding ζ. Then m_ζ is a positive integer and

$$\mathrm{tr}(L_g) = \sum_{\zeta \in Z} m_\zeta g(\zeta) \qquad (g \in A_{\mathbb{C}}). \tag{6.1}$$

Every complex common zero is in Z. To prove this, suppose w is a common zero outside Z. There is a polynomial q over $\mathbb{C}$ with $q(w) = 1$ and $q(\zeta) = 0$ for all $\zeta \in Z$. Such a polynomial is obtained by multiplying, for each ζ, a normalized coordinate difference separating w and ζ. The matrix L_q is then strictly upper triangular, so $L_q^{\dim A} = 0$. Applying this operator to 1 gives $q^{\dim A} = 0$ in $A_{\mathbb{C}}$, contradicting evaluation at w.

Complex conjugation permutes Z. It also preserves the multiplicities. Indeed, choose a polynomial q_ζ that equals one at ζ and zero at the other points of Z. Conjugation of coefficients gives

$$\operatorname{tr}(L_{\overline{q_\zeta}}) = \overline{\operatorname{tr}(L_{q_\zeta})} = m_\zeta.$$

By (6.1), the left side is $m_{\bar\zeta}$.

Write r for the number of real points of Z and choose one point from each of its c nonreal conjugate pairs. Evaluation defines a surjective real linear map

$$A \longrightarrow \mathbb{R}^r \times \mathbb{C}^c. \tag{6.2}$$

To see surjectivity, first interpolate arbitrary values at the finite set Z, assigning conjugate values at conjugate points. Polynomial interpolation at finitely many distinct points follows from the same separating coordinate products. Averaging the interpolating polynomial with its coefficientwise conjugate gives real coefficients and preserves the assigned values.

For real a, b, equation (6.1) now expresses the trace form as the pullback under (6.2) of

$$\sum_{\zeta \text{ real}} m_\zeta a(\zeta)b(\zeta) + \sum_{\{\zeta,\bar\zeta\}} 2m_\zeta \operatorname{Re}(a(\zeta)b(\zeta)). \tag{6.3}$$

Each real summand has one positive direction. On $\mathbb{C} \cong \mathbb{R}^2$, the form $2m\operatorname{Re}(zw)$ has matrix $\operatorname{diag}(2m, -2m)$ and hence one positive and one negative direction. Choose a real vector-space complement to the kernel of evaluation. Surjectivity identifies the trace form on that complement with (6.3); the kernel contributes zero directions. The signature is therefore r. □

Corollary 6.2. *For every real t,*

$$\operatorname{sig} H_n(t) = \#\{u \in \mathbb{R}^k : \nabla Q_t(u) = 0\}, \qquad \operatorname{sig} H_n(t) = 1 \iff t \le m_n. \tag{6.4}$$

Proof. Lemma 5.1 gives a quotient algebra of dimension M at every parameter. Its defining equations are nonzero scalar multiples of the partial derivatives of Q_t. Apply Proposition 6.1 and then Proposition 4.1. □

7. The boundary polynomial and the selected root

Expand the characteristic polynomial as

$$C_n(t,\lambda) = \det(\lambda I_M - H_n(t)) = \sum_{j=0}^{M} c_j(t)\lambda^j. \tag{7.1}$$

Since $c_M = 1$, the following definition is finite and unambiguous.

Definition 7.1. Let e be the smallest index for which $c_e(t)$ is a nonzero polynomial, and put $R_n(t) = c_e(t)$.

Lemma 7.2. *Every entry of $H_n(t)$ has degree at most kd in t, and*

$$\deg R_n \le kdM. \tag{7.2}$$

Proof. Consider a diagonal coefficient of multiplication by $u^{\alpha+\beta}$ on the basis element u^γ. Reduction starts at total degree $|\alpha| + |\beta| + |\gamma|$. Every step decreases this degree by exactly two. A term contributing to the coefficient of u^γ therefore uses exactly

$$\frac{|\alpha| + |\beta|}{2} \le kd$$

steps. If this number is not an integer, there is no such term. The coefficient introduced by each step has t-degree at most one. Thus every contributing coefficient has t-degree at most kd. Taking the trace preserves the bound.

Each coefficient of the determinant in (7.1) is a sum of products of at most M matrix entries. Its degree is at most kdM, which proves (7.2). □

Lemma 7.3. *The signature of $H_n(t)$ is constant on each interval containing no real root of R_n, and $R_n(m_n) = 0$.*

Proof. If $R_n(t) \neq 0$, equation (7.1) shows that zero is an eigenvalue of algebraic multiplicity exactly e. Symmetry identifies this multiplicity with the nullity. The eigenvalues of a continuously varying real symmetric matrix vary continuously. Near a given parameter where $R_n \neq 0$, the nonzero eigenvalues retain their signs, and the nullity remains e. The signature is consequently locally constant there, and hence constant on every such interval.

If $R_n(m_n) \neq 0$, there would be an interval about m_n on which the signature is constant. Corollary 6.2 gives signature one to the left of m_n and a strictly larger signature to its right, a contradiction. □

Proof of Theorem 1.1. Attainment and the lower bound follow from Lemma 2.1. The one-node integral is 2. Lemma 2.2 and Proposition 2.3 prove the remaining assertions about minimizers.

The explicit construction of H_n is (5.5), using the rational forms (3.3) and the reduction (5.3). Definition 7.1 specifies R_n, and Lemma 7.2 gives its degree bound. Since $0 < m_n < 2$, Lemma 7.3 shows that $m_n = r_h$ for some $1 \leq h \leq s$.

For $j < h$, the rational sample ρ_j is less than m_n; for $j \geq h$, it is greater than m_n. By Corollary 6.2,

$$\operatorname{sig} H_n(\rho_j) = 1 \quad \Longleftrightarrow \quad j < h.$$

Exactly $h - 1$ samples therefore have signature one, proving (1.4).

Finally, all polynomials and matrices in the construction are obtained by finite rational arithmetic. The index e is determined by testing the finitely many rational coefficient polynomials for equality to zero. Univariate real root isolation orders the roots of R_n in $(0, 2)$ and supplies rational samples between them. The signatures of the rational symmetric matrices at those samples are obtained by exact symmetric elimination. These operations terminate for every positive integer n. □

Corollary 7.4. *For $n \geq 2$, the number m_n is algebraic over $\mathbb{Q}$ of degree at most*

$$2n(n-1)^2(2n(n-1)+1)^{n-1}.$$

Proof. The minimal polynomial of m_n divides the nonzero polynomial R_n. Apply (7.2). □

Remark 7.5. Exact symmetric elimination requires only rational operations. At a nonzero diagonal entry, a congruence and its Schur complement remove a one-dimensional block. If all diagonal entries vanish and some off-diagonal entry b is nonzero, use the block $\left(\begin{smallmatrix} 0 & b \\ b & 0 \end{smallmatrix}\right)$, which has one positive and one negative eigenvalue, and remove it by a block Schur complement. An all-zero remaining matrix contributes only zero eigenvalues. Dimension decreases at every step, including at singular parameter values.

8. The two- and three-node cases

The first two nontrivial cases illustrate both the encoding and the selected-root rule.

For $n = 2$, one has $k = 1$, $d = 4$, and

$$D(u) = 4u^4, \qquad N(u) = \frac{16}{3}u^4.$$

Put $a = \frac{8}{3}(t - \frac{4}{3})$. The normalized critical equation is $u^5 - au^3 = 0$, so reduction uses $u^5 = au^3$. On the basis $1, u, u^2, u^3, u^4$, the trace matrix is

$$H_2(t) = \begin{pmatrix} 5 & 0 & 2a & 0 & 2a^2 \\ 0 & 2a & 0 & 2a^2 & 0 \\ 2a & 0 & 2a^2 & 0 & 2a^3 \\ 0 & 2a^2 & 0 & 2a^3 & 0 \\ 2a^2 & 0 & 2a^3 & 0 & 2a^4 \end{pmatrix}. \tag{8.1}$$

The trace moments follow directly from the multiplication matrix: $\operatorname{tr}(M_1) = 5$, $\operatorname{tr}(M_{u^{2j}}) = 2a^j$ for $j \geq 1$, and all odd moments vanish. Expanding the characteristic polynomial gives

$$R_2(t) = -12a^3(1+a^2)^2.$$

Its only real root is $t = 4/3$, and (1.4) yields $m_2 = 4/3$.

For $n = 3$, Proposition 2.3 reduces the configuration to $(-1, y, 1)$ with $-1 < y < 1$. Its cardinal polynomials are

$$\frac{(x-y)(x-1)}{2(1+y)}, \qquad \frac{1-x^2}{1-y^2}, \qquad \frac{(x+1)(x-y)}{2(1-y)}.$$

Their squared integrals sum to

$$I(-1, y, 1) = \frac{4}{3} - \frac{4}{3(1-y^2)} + \frac{8(3+y^2)}{15(1-y^2)^2}. \tag{8.2}$$

As a function of $s = y^2 \in [0, 1)$, its derivative is

$$\frac{36 + 28s}{15(1-s)^3} > 0.$$

Thus the unique ordered minimizing triple is $(-1, 0, 1)$ and $m_3 = 8/5$. Theorem 1.1 selects this same value from R_3.

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