A COUNTEREXAMPLE TO ERDŐS PROBLEM 1068

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Abstract. We construct in ZFC a simple graph on $2^{\aleph_0}$ vertices with chromatic number $\aleph_1$ such that every infinite induced subgraph is disconnected by deleting finitely many vertices. Consequently, the graph contains no infinitely vertex-connected subgraph, giving a negative answer to the unrestricted formulation of Erdős problem 1068. The construction uses independent vertex sets indexed by a tree of countable sequences. Stabilization of the colors occurring above a node and a recursive choice of a vertex rule out countable colorings. Finite initial histories determine finite separators, which exclude all infinite candidates for the required connectivity.

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1. Introduction

All graphs considered here are simple and undirected, and all paths are finite. For a graph H and $S \subseteq V(H)$, write $H - S = H[V(H) \setminus S]$. A graph is connected when each pair of vertices is joined by a finite path. An infinite graph H is *infinitely vertex-connected* if $H - S$ is connected for every finite $S \subseteq V(H)$.

Erdős problem 1068 asks whether every graph of chromatic number $\aleph_1$ contains a countable infinitely vertex-connected subgraph [1]. We prove the following stronger negative statement.

Theorem 1.1. *There is a graph G such that*

$$|V(G)| = 2^{\aleph_0}, \qquad \chi(G) = \aleph_1,$$

and, for every infinite $W \subseteq V(G)$, there is a finite $S \subseteq W$ for which $G[W \setminus S]$ is disconnected.

The path formulation of infinite connectivity leads to the same obstruction needed here. We record the implication explicitly.

Lemma 1.2. *Suppose that H has at least two vertices and that every two distinct vertices of H are joined by infinitely many distinct simple paths with pairwise disjoint interiors. Then H is infinite and $H - S$ is connected for every finite $S \subseteq V(H)$.*

Proof. A finite graph has only finitely many simple paths between any fixed pair of vertices, so H is infinite. Fix a finite set S and distinct vertices $u, v \in V(H) \setminus S$. In a family of internally disjoint u–v paths, each vertex of S lies on at most one path, since neither endpoint belongs to S. Thus at most $|S|$ paths meet S. A remaining path joins u to v in $H - S$. □

Our construction and proof use only ZFC. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$, let ω_1 be the first uncountable ordinal, and put $\mathfrak{c} = 2^{\aleph_0}$.

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2. The graph

Let

$$A = \{0,1\}^{\mathbb{N}}, \qquad L = A^{<\omega} = \bigcup_{n<\omega} A^n, \qquad T = \bigcup_{\alpha<\omega_1} L^\alpha.$$

Thus an element of L is a finite word whose letters belong to A, and a node $t \in T$ is a sequence of such words with countable ordinal length. Write $\mathrm{ht}(t)$ for its length. Order T by extension, denoted by $\preceq$. The empty sequence is the root. For finite words, and for their relation to an infinite sequence in $A^{\mathbb{N}}$, write $\sqsubseteq$ for the initial-segment relation. If $s \in L$, then $t \frown \langle s \rangle$ is an immediate successor of t.

At each node t place the vertex set

$$B_t = \{(t,b) : b \in A^{\mathbb{N}}\}.$$

The sets B_t are pairwise disjoint. Set

$$V = \bigcup_{t\in T} B_t, \qquad C(t) = \bigcup_{t\preceq u} B_u, \qquad C(t,s) = C\big(t \frown \langle s \rangle\big).$$

We call $C(t)$ the cone above t, and $C(t,s)$ a child cone of t. The word s forms a single entry of the node $t \frown \langle s \rangle$. In particular, distinct words label disjoint child cones, including when one word is an initial segment of the other. We have the disjoint partition

$$C(t) = B_t \,\dot\cup\, \dot\bigcup_{s\in L} C(t,s). \tag{1}$$

Lemma 2.1. *The sets V, $C(t)$, and $C(t,s)$ all have cardinality $\mathfrak{c}$.*

Proof. We have $|A| = |L| = |A^{\mathbb{N}}| = \mathfrak{c}$. For every countable ordinal α,

$$|L^\alpha| \le \mathfrak{c}^{\aleph_0} = \big(2^{\aleph_0}\big)^{\aleph_0} = \mathfrak{c}.$$

There are $\aleph_1 \le \mathfrak{c}$ possible lengths. Hence $|T| \le \mathfrak{c}$ and $|V| \le \mathfrak{c}\cdot\mathfrak{c} = \mathfrak{c}$. Every B_t has cardinality $\mathfrak{c}$, so equality holds. Each cone contains a whole vertex set B_u and is contained in V, giving the remaining equalities. □

By the axiom of choice, fix, simultaneously for every $t \in T$ and $s \in L$, a bijection

$$\phi_{t,s} : A \longrightarrow C(t,s). \tag{2}$$

These maps are chosen from the already specified vertex sets. Define the graph G on V by the edge set

$$E(G) = \Big\{ \big\{(t,b), \phi_{t,b\restriction n}\big(b(n)\big)\big\} : t \in T,\ b \in A^{\mathbb{N}},\ n \in \mathbb{N} \Big\}. \tag{3}$$

Here $b \restriction n$ is the word consisting of the first n entries of b. Every displayed edge joins B_t to a strict descendant of t, so its endpoints are distinct.

Lemma 2.2. *The graph G has the following properties.*

1. *If an edge joins B_t to B_u, then t and u are distinct comparable nodes.*
2. *Distinct child cones of a node have no edges between them.*
3. *For $v = (t,b)$, its neighbors in $C(t) \setminus B_t$ are precisely*

$$\phi_{t,b\restriction n}\big(b(n)\big), \qquad n \in \mathbb{N}.$$

Consequently, v has at most one neighbor in each child cone of t.

Proof. The first assertion follows from the range in (2) and the definition (3). Nodes in distinct child cones are incomparable, which proves the second assertion. If an edge from (t,b) has its other endpoint at greater height, then (t,b) is the lower endpoint in (3). This proves the displayed description of its upper neighbors. The words $b \restriction n$ have distinct lengths and therefore label distinct child cones, proving the last assertion. □

3. The chromatic number

The function $(t, b) \mapsto \mathrm{ht}(t)$ is a proper coloring by ω_1, by Lemma 2.2. Thus $\chi(G) \leq \aleph_1$. To obtain equality, we show that every function $f : V \to \mathbb{N}$ assigns the same color to the endpoints of some edge.

For a fixed $f : V \to \mathbb{N}$, put

$$P(t) = f[C(t)] \subseteq \mathbb{N}.$$

Each $P(t)$ is nonempty, and $t \preceq u$ implies $P(u) \subseteq P(t)$.

Lemma 3.1. *There is a node $t_* \in T$ such that*

$$P(u) = P(t_*) \qquad \text{for every } u \succeq t_*. \tag{4}$$

Proof. Starting with the root t_0, construct nodes t_n recursively. At stage n, if there is an extension $u \succeq t_n$ with $n \notin P(u)$, choose one and set $t_{n+1} = u$. If there is no such extension, set $t_{n+1} = t_n$. The nodes form an increasing sequence. Their union

$$t_* = \bigcup_{n<\omega} t_n$$

is a node of T, since its domain is the supremum of countably many countable ordinals and is therefore countable.

Let $u \succeq t_*$. Monotonicity gives $P(u) \subseteq P(t_*)$. Conversely, suppose that $n \in P(t_*)$. Color n was not removed at stage n, since removal would imply $n \notin P(t_{n+1})$ and hence $n \notin P(t_*)$. Therefore every extension of t_n has color n in its cone. As $u \succeq t_* \succeq t_n$, we have $n \in P(u)$. □

Proposition 3.2. *The graph G has chromatic number $\aleph_1$.*

Proof. Suppose that $f : V \to \mathbb{N}$ is a proper coloring, and take t_* as in Lemma 3.1. Choose a sequence $(k_n)_{n\in\mathbb{N}}$ whose range is $P(t_*)$. If this set is finite, repeat its elements.

We construct $b \in A^{\mathbb{N}}$ recursively. Once $b \restriction n$ is known, (4) applied to $t_* \frown \langle b \restriction n \rangle$ gives a vertex $w_n \in C(t_*, b \restriction n)$ with $f(w_n) = k_n$. Define

$$b(n) = \phi^{-1}_{t_*, b\restriction n}(w_n).$$

The resulting vertex $v = (t_*, b)$ belongs to $C(t_*)$, so $f(v) \in P(t_*)$. Choose m with $k_m = f(v)$. By construction,

$$\{v, w_m\} \in E(G) \quad \text{and} \quad f(w_m) = k_m = f(v),$$

contradicting properness. Hence G has no countable proper coloring. Together with the height coloring, this gives $\chi(G) = \aleph_1$. □

4. Finite separators inside a cone

Fix $t \in T$ and a nonempty word $p \in L$ of length k. Define

$$D(t, p) = \{(t, b) : p \sqsubseteq b\} \ \cup \bigcup_{\substack{s\in L \\ p \sqsubseteq s}} C(t, s), \tag{5}$$

$$F(t, p) = \left\{\phi_{t,p\restriction j}(p(j)) : 0 \leq j < k\right\}. \tag{6}$$

For $D \subseteq C(t)$, its external vertex boundary in $G[C(t)]$ is

$$\partial_{C(t)} D = \{y \in C(t) \setminus D : \text{some } x \in D \text{ is adjacent to } y\}.$$

Lemma 4.1. *For the sets in* (5) *and* (6),

$$\partial_{C(t)} D(t,p) \subseteq F(t,p), \qquad F(t,p) \cap (D(t,p) \cup B_t) = \varnothing, \qquad |F(t,p)| = k.$$

Proof. For $j < k$, the vertex $\phi_{t,p\restriction j}(p(j))$ lies in the child cone $C(t, p \restriction j)$. Its label has length less than k, so this cone is disjoint from every child cone whose label extends p. It also misses B_t. The labels $p \restriction j$ are distinct, and therefore the k displayed vertices are distinct. This proves the disjointness and cardinality claims.

Suppose that $x \in D(t,p)$ is adjacent to $y \in C(t) \setminus D(t,p)$. First consider $x = (t,b)$ with $p \sqsubseteq b$. By Lemma 2.2,

$$y = \phi_{t,b\restriction n}(b(n))$$

for some n. If $n \geq k$, the word $b \restriction n$ extends p, putting y in $D(t,p)$. Hence $n < k$, and $b \restriction n = p \restriction n$ and $b(n) = p(n)$. Thus $y \in F(t,p)$.

It remains to consider $x \in C(t,s)$ for a word s extending p. The vertex y lies outside this child cone. Distinct child cones have no edges between them, so the partition (1) implies that $y \in B_t$. Write $y = (t,b)$. An edge from this vertex into $C(t,s)$ requires $s = b \restriction n$ for some n, by Lemma 2.2. It follows that $p \sqsubseteq s \sqsubseteq b$, placing y in $D(t,p)$, a contradiction. This accounts for every possible boundary edge. □

Corollary 4.2. *Any two distinct vertices of B_t can be separated in $G[C(t)]$ by a finite vertex set that avoids both vertices.*

Proof. Let (t,b) and (t,b') be distinct. Choose j with $b(j) \neq b'(j)$, and set $p = b \restriction (j+1)$. Then

$$(t,b) \in D(t,p), \qquad (t,b') \notin D(t,p) \cup F(t,p).$$

The set $F(t,p)$ also avoids (t,b). By Lemma 4.1, no path in $G[C(t)] - F(t,p)$ can leave $D(t,p)$, proving the assertion. □

5. Excluding infinitely connected subgraphs

Lemma 5.1. *Let $W \subseteq V$ be nonempty, and suppose that $G[W]$ is connected. If $B_t \cap W \neq \varnothing$ and* $\mathrm{ht}(t)$ *is minimum among the heights of nodes whose vertex sets meet W, then $W \subseteq C(t)$.*

Proof. Choose $v \in B_t \cap W$. We show that every finite path in $G[W]$ starting at v stays in $C(t)$. Suppose that a vertex on the path lies in B_u with $t \preceq u$, and the next vertex lies in B_w. The nodes u and w are comparable, by Lemma 2.2. If $u \preceq w$, then $t \preceq w$. If $w \preceq u$, then w and t are both initial segments of u. Minimality of $\mathrm{ht}(t)$ gives $\mathrm{ht}(w) \geq \mathrm{ht}(t)$, so again $t \preceq w$. Induction along the path proves the claim. Connectedness now gives $W \subseteq C(t)$. □

Proof of Theorem 1.1. Lemma 2.1 gives $|V(G)| = \mathfrak{c}$, and Proposition 3.2 gives $\chi(G) = \aleph_1$.

Suppose, towards a contradiction, that $W \subseteq V$ is infinite and $H = G[W]$ remains connected after every finite vertex deletion. Choose a node t of minimum height with $B_t \cap W \neq \varnothing$. Lemma 5.1 gives $W \subseteq C(t)$.

There is exactly one vertex of W in B_t. Indeed, if there were two, Corollary 4.2 would supply a finite separator $F(t,p)$ avoiding both. Deleting $F(t,p) \cap W$ from H would still separate them, since every remaining path in H is a path in $G[C(t)] - F(t,p)$. This contradicts the assumed connectivity. Write $B_t \cap W = \{v\}$.

The graph $H - \{v\}$ is connected and infinite. Its vertices lie in the child cones of t, by (1). Since distinct child cones have no edges between them, all of $W \setminus \{v\}$ lies in a single child cone $C(t,s)$. Lemma 2.2 now shows that v has at most one neighbor in H.

Let S be the set of neighbors of v in H. This set is finite and does not contain v. In $H - S$, the vertex v is isolated, while infinitely many other vertices remain. Thus $H - S$ is disconnected, giving the final contradiction. □

Corollary 5.2. *There is a graph of chromatic number $\aleph_1$ with no infinitely vertex-connected subgraph. In particular, it has no countable subgraph of the kind required in Erdős problem 1068. The same conclusion holds for the path formulation on graphs with at least two vertices, requiring infinitely many internally vertex-disjoint paths between every two distinct vertices.*

Proof. Let G be the graph in Theorem 1.1. If a subgraph H on an infinite vertex set W remains connected after every finite deletion, then $G[W]$ has the same property, since it contains every edge of H. This contradicts the theorem. The path formulation implies the finite-deletion property by Lemma 1.2. □

References

[1] *Erdős Problems*, Problem 1068, `https://www.erdosproblems.com/1068`. Accessed November 11, 2026.

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