THE SHARP DEFICIT IN PRIMITIVE SIGNED RECIPROCAL RELATIONS

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Abstract. Let $c(N)$ be the largest cardinality of a subset of $\{1,\dots,N\}$ admitting a signed reciprocal sum equal to zero with no nonempty proper zero subsum under the same signs. We prove

$$N - c(N) \sim \mathrm{e}^{-\gamma}\frac{N\log\log\log N}{\log\log N},$$

where γ is Euler's constant. The construction joins divisor blocks along a path of primes and completes their reciprocal sum while avoiding those primes. For the converse, we prove a sharp harmonic bound for sets whose reciprocals have no nonempty zero subsum modulo an arbitrary integer. Exact rational representation in a dense sign class then turns this modular bound into a lower bound for the omitted denominators.

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1. Introduction

For an integer $N \geq 1$, write $[N] = \{1,\dots,N\}$. A pair (A,δ), where $A \subseteq [N]$ and $\delta\colon A \longrightarrow \{-1,1\}$, is a *primitive signed reciprocal relation* if

$$\sum_{n\in A}\frac{\delta(n)}{n} = 0, \qquad \sum_{n\in B}\frac{\delta(n)}{n} \neq 0 \quad (\varnothing \neq B \subsetneq A). \tag{1.1}$$

The signs in the second condition are the restrictions of δ. Let $c(N)$ be the maximum cardinality of such a support, with value zero when no nonempty support exists. Erdős and Graham asked for the size of this extremal function [3, p. 43]. The question is recorded as Problem 319 in [1].

Theorem 1.1. *As $N \longrightarrow \infty$ through the positive integers,*

$$N - c(N) = \bigl(\mathrm{e}^{-\gamma} + o(1)\bigr)\frac{N\log\log\log N}{\log\log N}. \tag{1.2}$$

Throughout the paper, all logarithms are natural. We use the notation

$$L = \log N, \qquad \lambda = \log\log N, \qquad \mu = \log\log\log N, \qquad \Delta_N = \frac{N\mu}{\lambda}. \tag{1.3}$$

These quantities are used only for sufficiently large N. For a finite set E of positive integers, put

$$R(E) = \sum_{n\in E}\frac{1}{n}, \qquad E_q = \{n \in E : q \mid n\}, \qquad D_E = \operatorname{lcm}\{n : n \in E\},$$

and set $D_\varnothing = 1$. We write $\mathcal{Q}(E)$ for the set of all prime powers dividing at least one member of E. In particular, every positive power of a prime up to its exponent in D_E belongs to $\mathcal{Q}(E)$. An integer is *S-powersmooth* if every prime-power divisor is at most S. We take $P^+(1) = 1$ and use $\Omega(n)$ for the number of prime factors counted with multiplicity.

Croot's short-interval theorem [2] gives the earlier lower bound $c(N) \geq (1 - 1/\mathrm{e} + o(1))N$, as observed by Adenwalla [1]. A density-one construction was subsequently proposed by g-01234, also known as popular_12345 [4]. Its mechanism couples a positive family with a dense set of

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denominators avoiding a specified collection of primes. We use this mechanism with a path of pairwise connectors and quantitative denominator control. A complementary large-prime obstruction appears in [6]. The exact rational representation theorem of Liu and Sawhney [5] supplies the analytic input for our converse. We state the form needed here and give its divisor and Fourier argument in Appendix A.

2. Analytic preliminaries

We use the prime number theorem, partial summation, and the Mertens estimates in the forms

$$\sum_{p\le x}\frac{1}{p}=\log\log x+O(1),\qquad \prod_{p\le x}(1-p^{-1})^{-1}\sim \mathrm{e}^{\gamma}\log x,\qquad \sum_{p^j\le x}\log p\sim x. \tag{2.1}$$

Here and below a letter p in a prime sum denotes a prime, and $j\ge 1$. The usual prime-number-theorem error term also gives $\sum_{x<p\le x^2}p^{-1}=\log 2+o(1)$.

Lemma 2.1. *Put $S=N\exp(-6L^{4/5})$. Then*

$$\Theta(N,S):=\sum_{\substack{q\le N\\ q\ a\ prime\ power\\ q>S}}\frac{1}{q}=O(L^{-1/5}), \tag{2.2}$$

$$|\{n\le N: n\ is\ not\ S\text{-}powersmooth\}|\le N\Theta(N,S), \tag{2.3}$$

$$|\{n\le N:\Omega(n)>5\lambda\}|=O(NL^{-3/2}). \tag{2.4}$$

If $Y\longrightarrow\infty$ and $u\ge 1$, then

$$\sum_{\substack{n>Y^u\\ P^+(n)\le Y}}\frac{1}{n}\ll \mathrm{e}^{-u}\log Y. \tag{2.5}$$

The implied constants are absolute.

Proof. For primes in $(S,N]$, partial summation and $\pi(x)\ll x/\log x$ give

$$\sum_{S<p\le N}\frac{1}{p}\ll\frac{1+\log(N/S)}{\log S}\ll L^{-1/5}.$$

For powers of exponent at least two, the sum over bases $p\le\sqrt{S}$ is $O(S^{-1/2})$, since each geometric tail is at most $2/S$. The bases $p>\sqrt{S}$ contribute at most $2\sum_{n>\sqrt{S}}n^{-2}=O(S^{-1/2})$. This proves (2.2), and a union bound proves (2.3).

The multiplicative identity

$$(3/2)^{\Omega(n)}=\sum_{d\mid n}g(d),\qquad g(p^j)=\tfrac{1}{2}(3/2)^{j-1}$$

and its Euler product give

$$\sum_{n\le N}(3/2)^{\Omega(n)}\le N\prod_{p\le N}\left(1+\frac{1/2}{p-3/2}\right)\ll NL^{1/2}.$$

Markov's inequality proves (2.4), because $1/2-5\log(3/2)<-3/2$.

For (2.5), take $\sigma=1/\log Y$. Rankin's inequality gives

$$\sum_{\substack{n>Y^u\\ P^+(n)\le Y}}\frac{1}{n}\le \mathrm{e}^{-u}\prod_{p\le Y}(1-p^{-1+\sigma})^{-1}.$$

The product on the right is $O(\log Y)$. Indeed, its logarithm differs from that of the Mertens product by $O(1)$, using $p^\sigma - 1 \le \mathrm{e}\sigma \log p$ and $\sum_{p\le Y}(\log p)/p \ll \log Y$. The terms of degree at least two in the logarithmic expansions have bounded total. □

Lemma 2.2 (Exact rational representation)**.** *For all sufficiently large N, let $L^{-3/100} \le c \le 1$ and let $E \subseteq [cN, N] \cap \mathbb{N}$ satisfy*

$$|E| \ge N/L^2, \qquad n \text{ is } S\text{-powersmooth and } \Omega(n) \le 5\lambda \quad (n \in E), \tag{2.6}$$

$$qR(E_q) \ge L^{-1/25} \quad (q \in \mathcal{Q}(E)), \qquad S = N\exp(-6L^{4/5}). \tag{2.7}$$

If $r > 0$ is rational, its reduced denominator divides D_E, and

$$(1 + L^{-1})r \le R(E) \le Lr, \tag{2.8}$$

then $R(F) = r$ for some $F \subseteq E$. The threshold is uniform over c, E, r.

This is the logarithmic-parameter specialization of the reciprocal-sum criterion of Liu and Sawhney [5, Proposition 5.2]. Appendix A supplies the specialization and the details of the divisor extraction used in its proof.

Lemma 2.3 (Cleaning)**.** *For all sufficiently large N and every $E \subseteq [N]$ there is $F \subseteq E$ such that all members of F are S-powersmooth, $\Omega(n) \le 5\lambda$ on F, and $qR(F_q) \ge L^{-1/25}$ for every $q \in \mathcal{Q}(F)$. Moreover,*

$$|E \setminus F| \ll N\big(L^{-1/5} + L^{-3/2} + L^{-1/25}\lambda\big). \tag{2.9}$$

After the first two restrictions, the further loss in reciprocal mass is $O(L^{-1/25}\lambda)$.

Proof. Apply the first two restrictions using Lemma 2.1. Whenever a prime power q divides a member of the current set and $qR(E_q) < \eta := L^{-1/25}$, remove all its multiples. A processed prime power never reappears. The total reciprocal mass removed is at most

$$\eta \sum_{\substack{q \le N \\ q \text{ a prime power}}} \frac{1}{q} = O(\eta\lambda).$$

The number of removed denominators is at most N times this mass. The finite process terminates with the required properties. □

3. Reciprocal sums modulo an integer

For $q \ge 2$, a finite set V of positive integers coprime to q is *inverse-zero-free modulo q* if

$$\sum_{n\in U} n^{-1} \ne 0 \pmod q \qquad (\varnothing \ne U \subseteq V).$$

The inverses in this definition are taken in $\mathbb{Z}/q\mathbb{Z}$. The partial sums along any ordering of V, including the initial zero, are distinct. Consequently,

$$|V| \le q - 1. \tag{3.1}$$

Lemma 3.1 (Prime inverse sums)**.** *There is an absolute constant C with the following property. For sufficiently large q, write $\ell = \log q$ and $Y = \ell^3(1 + \log \ell)^2$. If $Y \le X \le q$ and a set contains at least*

$$k_X = \left\lceil C\ell X^{2/3}(\log X)^{-2/3} \right\rceil$$

distinct primes in $[X, 2X)$ coprime to q, its inverse subset sums cover every residue class modulo q.

Proof. Use exactly $k = k_X$ primes. Fix $0 < h < q$. For each selected prime p, choose $u_p \in (-q/2, q/2]$ with $u_p \equiv hp^{-1} \pmod q$ and write $pu_p = h + j_p q$. Put $J = \lfloor k \log X/(32\ell) \rfloor$. Uniformly in the stated range, $J \longrightarrow \infty$ and $J < q - 1$. Each nonzero integer $h + jq$ with $|j| \le J$ has magnitude less than q^2 and has at most $2\ell/\log X$ prime divisors in $[X, 2X)$. Thus fewer than $k/2$ selected primes have $|j_p| \le J$. For all the others,

$$\frac{|u_p|}{q} \ge \frac{J-1}{2X} \gg \frac{k \log X}{\ell X}.$$

The modulus of the hth Fourier coefficient of a uniformly random inverse subset sum is

$$\prod_p \left| \cos \frac{\pi u_p}{q} \right| \le \exp\left(-c_0 \frac{k^3 (\log X)^2}{\ell^2 X^2} \right) < q^{-2}$$

when C is sufficiently large. We used $|\cos \pi t| \le \exp(-2t^2)$ for $|t| \le 1/2$. Fourier inversion now gives probability at least $q^{-1}(1 - (q-1)q^{-2}) > 0$ at each residue. This is the prime-inverse Fourier argument of Croot [2, Lemma 1 of the preprint], with a varying prime interval. □

Lemma 3.2. *Uniformly over inverse-zero-free V modulo q,*

$$\sum_{\substack{n \in V,\ n \le q \\ P^+(n) > Y}} \frac{1}{n} = O(1), \qquad Y = (\log q)^3 (1 + \log\log q)^2.$$

Proof. Write $n = mp$, where $p = P^+(n)$, and partition $p > Y$ into dyadic intervals $[X, 2X)$. For a fixed m, at most k_X primes can occur. Otherwise one can reserve a prime, express its negative inverse using k_X of the remaining primes by Lemma 3.1, and multiply by m^{-1} to obtain a forbidden zero subsum. Since $P^+(m) \le 2X$, the contribution of this interval is at most

$$\frac{k_X}{X} \prod_{p \le 2X} (1 - p^{-1})^{-1} \ll (\log q) X^{-1/3} (\log X)^{1/3}.$$

The geometric sum over the intervals is $O((\log q) Y^{-1/3} (\log Y)^{1/3}) = O(1)$. □

Lemma 3.3. *For every $\beta > 0$, all sufficiently large q have the following property, uniformly for integers $t \ge \log q$ and inverse-zero-free V modulo q,*

$$|\{n \in V : t/2 < n \le t,\ P^+(n) > \log q\}| < \beta t. \tag{3.2}$$

Proof. We may assume $0 < \beta < 1$. Suppose the set in (3.2) has at least βt members. Apply Lemma 2.3 with ambient parameter t. The resulting set $W \subseteq (t/2, t]$ has $|W| \ge \beta t/2$ for all sufficiently large t, and

$$\beta/2 \le R(W) \le 1.$$

Put $D = D_W$ and $\ell = \log q$. Every member of W has a prime factor greater than ℓ. If $\mathcal{P}$ is the collection of such primes dividing D, a union bound gives $\sum_{p \in \mathcal{P}} p^{-1} \ge \beta/2$. Therefore

$$\log D \ge \sum_{p \in \mathcal{P}} \log p \ge \ell \log \ell \sum_{p \in \mathcal{P}} \frac{1}{p} \ge \tfrac{\beta}{2} \ell \log \ell. \tag{3.3}$$

In particular $q/D \longrightarrow 0$, uniformly in t. Choose a positive integer x divisible by q such that $\beta/8 \le x/D \le \beta/4$. The hypotheses of Lemma 2.2, with ambient parameter t and $c = 1/2$, hold for W and $r = x/D$. Its cardinality condition and target inequalities hold once t is sufficiently large in terms of β. It gives a nonempty $U \subseteq W$ with $R(U) = x/D$. Since $(D, q) = 1$ and $q \mid x$, reduction modulo q contradicts inverse zero-freeness. Every threshold in this argument is uniform for $t \ge \log q$, proving the assertion. □

Theorem 3.4 (Sharp modular harmonic bound)**.** *Uniformly over all finite inverse-zero-free sets V modulo q,*

$$R(V) \le \big(\mathrm{e}^{\gamma} + o(1)\big) \log\log q \qquad (q \longrightarrow \infty). \tag{3.4}$$

The coefficient e^{γ} is sharp, even when q is restricted to primes.

Proof. Write $\ell = \log q$ and $Y = \ell^3(1+\log\ell)^2$. The terms exceeding q have total at most 1 by (3.1). Lemma 3.2 treats the terms at most q with $P^+(n) > Y$. For fixed $A > 1$, Lemma 2.1 gives

$$\sum_{\substack{n > Y^A \\ P^+(n) \le Y}} \frac{1}{n} \ll \mathrm{e}^{-A} \log Y.$$

There are $O(A \log Y)$ dyadic intervals between ℓ and Y^A. By Lemma 3.3, each contributes at most 2β to the reciprocal mass of terms with $P^+(n) > \ell$, for sufficiently large q in terms of β. Thus this entire part has mass

$$O(1) + O(\mathrm{e}^{-A} \log Y) + O(\beta A \log Y).$$

Divide by $\log\ell$. First let q tend to infinity with A, β fixed, then let A tend to infinity and βA tend to zero. The contribution is $o(\log\ell)$, uniformly in V. The remaining terms have total at most

$$\sum_{P^+(n) \le \ell} \frac{1}{n} = \prod_{p \le \ell} (1 - p^{-1})^{-1} = \big(\mathrm{e}^{\gamma} + o(1)\big) \log \ell.$$

For sharpness, let q be prime and put $D = \operatorname{lcm}(1, \dots, \lfloor \frac{1}{4} \log q \rfloor)$. The set of positive divisors of D is coprime to q for large q. Every nonempty reciprocal subtotal is a/D with $0 < a \le DR(\{d : d \mid D\}) < q$, using $\log D \sim \frac{1}{4} \log q$ and the Mertens product. It is consequently inverse-zero-free, and its reciprocal mass is $(\mathrm{e}^{\gamma} + o(1)) \log\log q$. □

Corollary 3.5. *For every $\varepsilon > 0$, all sufficiently large N satisfy the following uniform assertion. If integers q, t satisfy $2 \le q \le N$ and $t \ge \exp(\lambda^3)$, and $V \subseteq [t]$ is inverse-zero-free modulo q, then $|V| \le \varepsilon t$.*

Proof. For q above a sufficiently large fixed threshold, Lemma 3.3 and a dyadic partition bound the number of members with $P^+(n) > \log q$ by $\varepsilon t/2$. All remaining members are L-smooth. Rankin's inequality, with $\sigma = 1/\log L = 1/\lambda$, gives

$$|\{n \le t : P^+(n) \le L\}| \le t^{1-\sigma} \prod_{p \le L} (1 - p^{-1+\sigma})^{-1} \ll t\lambda \mathrm{e}^{-\lambda^2} = o(t).$$

For bounded q, use (3.1). □

4. A dense primitive construction

For a positive integer G, write

$$\mathbb{Z}_{(G)} = \{a/b \in \mathbb{Q} : (b, G) = 1\}.$$

Membership is understood after reduction, or equivalently through any representation with denominator coprime to G.

Lemma 4.1 (A path of prime blocks)**.** *Let $D \geq 1$, put $H = \sum_{d|D} d^{-1}$ and $K = DH$, and let $p_1, \ldots, p_k$ be distinct primes, where $k \geq 2$, $p_i > K + 1$, and $p_i \nmid D$. Suppose*

$$0 < B \leq p_i \leq M, \qquad \rho = \sum_{i=1}^{k} \frac{1}{p_i} \leq \frac{1}{2}, \qquad G = \prod_{i=1}^{k} p_i.$$

There is a finite set E of positive integers such that

$$(e, G) > 1 \quad (e \in E), \qquad R(E) \in \mathbb{Z}_{(G)}, \tag{4.1}$$

$$R(U) \in \mathbb{Z}_{(G)},\ U \subseteq E \quad \Longrightarrow \quad U = \varnothing \text{ or } U = E, \tag{4.2}$$

$$\max E \leq \max\{MD, 4kM^4\}, \qquad H\rho \leq R(E) \leq H\rho + \rho/B. \tag{4.3}$$

Proof. Place the block $\{p_i d : d \mid D\}$ at vertex i of a path. For a denominator divisible by p_i exactly once, its residue at p_i means the reduction of p_i times its reciprocal modulo p_i. Let $\mathcal{S}_i \subseteq \mathbb{F}_{p_i}$ be the set of block subsums of these residues, and let s_i be the full-block residue. After multiplication by D, every such subsum is an integer between 0 and $K < p_i$. Hence residue zero occurs only for the empty block, residue s_i occurs only for the full block, and

$$|\mathcal{S}_i| \leq K + 1 < p_i, \qquad \mathcal{S}_i = s_i - \mathcal{S}_i.$$

At each internal vertex choose a_i such that $-a_i \notin \mathcal{S}_i$. Prescribe residue a_i for the incoming edge and $-s_i - a_i$ for the outgoing edge. Both are nonzero. At an endpoint prescribe residue $-s_i$ for its unique edge. For an edge joining p_i and p_{i+1}, the Chinese remainder theorem provides a unit residue class u modulo $p_i p_{i+1}$ so that the denominator $p_i p_{i+1} u$ has the two prescribed residues.

Among the first $4k$ positive representatives of this class, a different marked prime p excludes at most $4k/p+1$ representatives. The total excluded is at most $4k\rho + k - 2 < 4k$. Choose an unexcluded representative $u_i < 4kp_i p_{i+1}$ and add the edge denominator $p_i p_{i+1} u_i$. Each edge denominator is divisible by exactly its two incident marked primes, each to exponent one. The block and edge denominators are therefore all distinct.

Let E be their union. The residue of its sum is zero at every marked prime, proving (4.1). Suppose $R(U) \in \mathbb{Z}_{(G)}$ for a subset $U \subseteq E$. At an internal vertex, selecting exactly the incoming edge would require $-a_i \in \mathcal{S}_i$. Selecting exactly the outgoing edge would require $s_i + a_i \in \mathcal{S}_i$, which is equivalent to the same forbidden condition by complement symmetry. Thus the two edge indicators agree at each internal vertex. All edges are selected together or all are absent. In the latter case every block is empty, and in the former every block is full. The same conclusion holds when $k = 2$. This proves (4.2).

The block denominators are at most MD, and the edge denominators are less than $4kM^4$. Finally,

$$0 \leq R(E) - H\rho = \sum_{i=1}^{k-1} \frac{1}{p_i p_{i+1} u_i} \leq \frac{1}{B} \sum_{i=1}^{k-1} \frac{1}{p_i} \leq \frac{\rho}{B}.$$

□

Lemma 4.2 (Uniform rational completion)**.** *For every fixed $\eta > 0$ there is a constant C_η such that the following holds for all sufficiently large integers m. Let $G \geq 1$ and let $r \in [\eta, 2] \cap \mathbb{Z}_{(G)}$. Suppose every prime-power component of the reduced denominator of r is at most $m/(\log m)^{14}$, and every prime divisor of G exceeds both C_η and $(\log m)^{13}$. Then*

$$r = R(C), \qquad C \subseteq [m], \qquad (c, G) = 1 \quad (c \in C).$$

Proof. For a sufficiently large prime power q, put $X = (\log q)^{12}$ and $k = \lceil (\log q)^{10} \rceil$. There are at least k primes in $[X, 2X)$ coprime to q, by the prime number theorem. Lemma 3.1 applies to any k of them, since its required number is $o(k)$. Consequently every residue modulo q is a sum of inverses of a subset of these primes. The corresponding denominators qp have total reciprocal mass at most

$$\frac{k}{qX} \ll \frac{1}{q(\log q)^2}. \tag{4.4}$$

Choose $q_0 \geq 5$ so large that this assertion holds above q_0 and the sum of the right side of (4.4) over all integers $q > q_0$ is less than $\eta/2$.

Starting from r, choose the largest prime-power component $q > q_0$ of its current reduced denominator, and write the current fraction as $a/(qv)$, where $(q, v) = 1$. Select distinct primes $p \in [X, 2X)$ with

$$\sum_p p^{-1} \equiv av^{-1} \pmod{q},$$

and subtract $\sum_p 1/(qp)$. The new denominator divides $\operatorname{lcm}(v, \{p\})$. To see this, multiply the difference by $q \operatorname{lcm}(v, \{p\})$ and use the congruence to cancel the full factor q. All its prime-power components are strictly less than q. The processed components thus form a strictly decreasing finite sequence. The total subtracted mass is less than $\eta/2$, so the residual remains positive.

Every chosen denominator has largest prime-power component exactly its processed q. Thus different steps use disjoint denominators. All introduced primes are at most $2(\log m)^{12} < (\log m)^{13}$, and every chosen denominator is at most $2m/(\log m)^2 < m$ for large m. The denominator remains coprime to G at every step.

At termination the residual has denominator dividing $D_0 = \operatorname{lcm}(1, \dots, \lfloor q_0 \rfloor)$ and lies in $(0, 2]$. For each of $2D_0$ successive disjoint intervals, Croot's theorem [2] supplies a finite set of distinct denominators with reciprocal sum 1. Scaling all these sets by D_0 gives disjoint representations of $1/D_0$. Their denominators are fixed and divisible by 30. They represent every j/D_0 with $1 \leq j \leq 2D_0$ by taking unions. This finite reservoir is disjoint from the previously used denominators, which have at most two distinct prime factors. Choose C_η larger than every reservoir denominator. For large m the reservoir lies in $[m]$ and is coprime to G, completing the proof. □

Proposition 4.3. *There are primitive signed reciprocal relations in $[N]$ with*

$$N - |A| \leq \big(\mathrm{e}^{-\gamma} + o(1)\big)\Delta_N.$$

Proof. Put

$$s = \lfloor L/1000 \rfloor, \quad D = \operatorname{lcm}(1, \dots, s), \quad H = \sum_{d \mid D} \frac{1}{d}, \quad B = \left\lfloor N^{1/20} \right\rfloor.$$

The prime number theorem gives $D \leq N^{1/100}$ for large N. Moreover,

$$H = \prod_{p \leq s} \frac{1 - p^{-1-\lfloor \log s / \log p \rfloor}}{1 - p^{-1}} = \big(\mathrm{e}^{\gamma} + o(1)\big)\lambda. \tag{4.5}$$

Indeed, the sum of the omitted numerator factors is at most $\pi(s)/s = O(1/\log s)$, and the denominator is the Mertens product.

Choose the shortest initial segment $p_1, \dots, p_k$ of the primes in $(B, B^2]$ for which

$$\frac{\log H + 1}{H} \le \rho := \sum_i \frac{1}{p_i} < \frac{\log H + 1}{H} + \frac{1}{B}. \tag{4.6}$$

The choice is possible because the reciprocal mass of the band tends to $\log 2$. Eventually $k \ge 2$, $\rho \le 1/2$, $DH + 1 < B$, and all the chosen primes are coprime to D. Apply Lemma 4.1 and put $G = \prod_i p_i$. It gives a positive family E with

$$\max E \le N^{3/4}, \qquad R(E) = \log H + 1 + o(1). \tag{4.7}$$

For the first inequality, the block bound is $B^2 D \le N^{11/100}$, and the edge bound follows from $k \le B^2$ and $4k(B^2)^4 \le 4N^{1/2}$.

Let $n = \lfloor N/H \rfloor$ and let F consist of the S-powersmooth integers in $(n, N]$ coprime to G, where $S = N \exp(-6L^{4/5})$. By a union bound and Lemma 2.1,

$$N - |F| \le n + N\rho + O(NL^{-1/5}). \tag{4.8}$$

We also have

$$R(F) = \log H + o(1). \tag{4.9}$$

In fact, the unrestricted interval has reciprocal mass $\log(N/n) + o(1) = \log H + o(1)$. The powersmooth restriction removes mass at most $O(NL^{-1/5}/n) = O(HL^{-1/5}) = o(1)$. For a marked prime $p \le B^2 = o(n)$, the reciprocal mass of its multiples in $(n, N]$ is at most $p^{-1}(1 + \log(2N/n))$. Summing this bound removes at most $O(\rho(1 + \log H)) = o(1)$.

The difference $r = R(E) - R(F)$ tends to 1 and belongs to $\mathbb{Z}_{(G)}$. Its reduced denominator is S-powersmooth, because this holds for all denominators in $E \cup F$ and is preserved by taking an lcm and reducing a fraction. For large N,

$$S \le \frac{n}{(\log n)^{14}}, \qquad \min_i p_i > \max\{C_{1/2}, (\log n)^{13}\}.$$

Lemma 4.2 gives $C \subseteq [n]$, coprime to G, with $R(C) = r$. The sets E, F, C are disjoint. Assign positive signs to E and negative signs to $F \cup C$.

If a subfamily has signed sum zero, its positive reciprocal subtotal belongs to $\mathbb{Z}_{(G)}$. Lemma 4.1 forces its positive part to be empty or all of E. Positivity of the unsigned summands then forces its negative part to be empty or all of $F \cup C$, respectively. The relation is primitive.

Finally, (4.8) and (4.6) imply

$$N - |E \cup F \cup C| \le \frac{N}{H}\big(\log H + 2 + o(1)\big) + O(NL^{-1/5}).$$

Now $H \sim \mathrm{e}^{\gamma}\lambda$ and $\log H \sim \mu$. The last error is $o(\Delta_N)$, proving the proposition. □

5. The lower bound for arbitrary primitive relations

5.1. The smaller sign class.

Lemma 5.1. *Let $A = A_+ \sqcup A_-$ be a primitive signed reciprocal relation in $[N]$ and put $d = N - |A|$. Uniformly over these relations,*

$$\min\{|A_+|, |A_-|\} \le (1 + o(1))Nr + \frac{\sqrt{r}}{1 - \sqrt{r}}\, d, \qquad r \sim 200\frac{\mu}{\lambda}. \tag{5.1}$$

In particular, if $d = O(\Delta_N)$, the smaller sign class has $O(\Delta_N)$ elements.

Proof. Fix $b = 1/100$ and put

$$\mathcal{P} = \{p : \lambda^2 \le p \le L^b\}, \qquad r = \prod_{p\in\mathcal{P}}(1-p^{-1}), \qquad \tau = L^{-b}.$$

The Mertens product gives $r \sim (2/b)\mu/\lambda$. We may suppose both sign classes have at least $Nr/2$ elements. No $p \in \mathcal{P}$ can satisfy

$$|(A_+)_p| \ge \tau N/p, \qquad |(A_-)_p| \ge \tau N/p. \tag{5.2}$$

To prove this, restrict either sign class to $[NL^{-3b}, N]$ and apply Lemma 2.3. The lower cutoff removes at most $NL^{-3b}/p + 1$ multiples of p. The powersmooth and Ω restrictions remove $O(NL^{-1/5} + NL^{-3/2}) = o(NL^{-2b})$ integers. Hence at least $\tau N/(2p)$ multiples of p remain before pruning. Their reciprocal mass is at least $\tau/(2p) \ge L^{-2b}/2$. The pruning removes mass $O(L^{-4b}\lambda) = o(L^{-2b})$, so p still divides the lcm of the resulting set U. Also $|U| \ge Nr/3$ and

$$r/3 \le R(U) \le 3b\lambda + O(1).$$

Apply Lemma 2.2 to U with target $1/p$. Its mass inequalities follow from $p \ge \lambda^2$ and $p \le L^b$. Doing this for both signs produces two subsets of equal mass $1/p$. Their union is a nonempty proper zero subsum, since the full mass of either sign class is at least $r/2 > 1/p$ for large N. This contradiction proves the assertion.

Assign each prime in $\mathcal{P}$ to a sign with the greater number of multiples. A term bearing a prime assigned to its opposite sign lies in an exceptional set of size at most

$$E = \tau N \sum_{p\in\mathcal{P}} \frac{1}{p} = o(Nr).$$

Write r_+, r_- for the products of $1 - 1/p$ over the two assigned collections, so $r_+r_- = r$. Relabel the signs so $r_- \le \sqrt{r}$. The product $Q = \prod_{p\in\mathcal{P}} p$ is $N^{o(1)}$. Periodicity modulo Q counts the integers bearing a prime from each collection as

$$N(1-r_+)(1-r_-) + O(Q) \le d + E + O(Q).$$

The positive terms outside the exceptional set avoid all minus-assigned primes. Their number is at most

$$\begin{aligned} |A_+| &\le Nr + N(1-r_+)r_- + E + O(Q) \\ &\le Nr + \frac{r_-}{1-r_-}(d + E + O(Q)) + E + O(Q). \end{aligned}$$

Since $Q = o(Nr)$, this proves (5.1). □

5.2. **Denominator excess and reciprocal mass.** Fix a positive integer D and put $\nu_p = v_p(D)$ and

$$s_p = p^{\nu_p+1}, \qquad \mathcal{S}_D = \{s_p : s_p \le N\}, \qquad \rho_D = \sum_{s\in\mathcal{S}_D} \frac{1}{s}. \tag{5.3}$$

For $n \ge 1$ define

$$b(n) = \prod_p p^{\max\{v_p(n)-\nu_p,0\}}, \qquad Q_b = \prod_{p\mid b} p^{\nu_p+v_p(b)}. \tag{5.4}$$

If $b(n) = b$, then $n = Q_b m$, where $m \mid D_b$ and $(m, b) = 1$, with $D_b = D/\prod_{p\mid b} p^{\nu_p}$. For any subset of this group,

$$D\sum_n \frac{1}{n} = \frac{1}{b}\sum_m \frac{D_b}{m}. \tag{5.5}$$

Thus its reciprocal sum has reduced denominator dividing D exactly when the inverse sum of its cofactors is zero modulo b.

Lemma 5.2. *For every $\varepsilon > 0$ there are constants C_ε and $\delta_\varepsilon > 0$ such that, for sufficiently large N, the following holds uniformly in D and $V \subseteq [N]$. Suppose V has no member dividing D, and the cofactors in each group $b(n) = b > 1$ are inverse-zero-free modulo b. If $\rho_D \le \delta_\varepsilon$, then*

$$R(V) \le C_\varepsilon + (\mathrm{e}^\gamma + \varepsilon)\lambda\rho_D. \tag{5.6}$$

Proof. Let $h(b)$ be the supremum of reciprocal masses of inverse-zero-free sets modulo b. Theorem 3.4 gives $h(b) \le (\mathrm{e}^\gamma + \varepsilon')\log\log b$ for sufficiently large b, and (3.1) gives $h(b) \le 1 + \log b$ for every $b \ge 2$. The contribution of group b is at most $h(b)/Q_b$.

Fix a large constant Y. The sum over groups supported entirely on primes at most Y is bounded by

$$\sum_{\substack{b>1\\ P^+(b)\le Y}} \frac{1+\log b}{b} < \infty,$$

uniformly in D, N. For all other groups, choose Y so large that the sharp estimate for $h(b)$ applies. Since $b \le N$ for a nonempty group, their total mass is at most $(\mathrm{e}^\gamma + \varepsilon')\lambda$ times the sum of Q_b^{-1} over groups containing a prime greater than Y. For a prime with $s_p \le N$, put

$$a_p = \sum_{j\ge 1} p^{-\nu_p - j} = \frac{1}{s_p(1-1/p)}.$$

The required sum of weights is bounded by

$$\left(\sum_{p>Y} a_p\right) \prod_{s_p \le N} (1+a_p) \le \left(1 + \frac{1}{Y-1}\right)\rho_D \exp(2\rho_D).$$

First choose ε' small and Y large, and then choose δ_ε small. This proves (5.6). □

5.3. **A dense sign class and the removal of small subsums.** Consider a sequence of primitive relations with $d = O(\Delta_N)$. By Lemma 5.1, orient the signs so that the negative class B has $N - O(\Delta_N)$ elements and the positive class P has $O(\Delta_N)$ elements. Their common reciprocal mass satisfies

$$R(P) = R(B) \ge H_N - H_{d+|P|} \ge \mu - \log\mu - O(1) = (1-o(1))\mu, \tag{5.7}$$

where $H_j = \sum_{i=1}^{j} 1/i$ and $H_0 = 0$.

Fix $0 < a < \mathrm{e}^{-3}$. Apply Lemma 2.3 to $B \cap (aN, N]$, and call the resulting set F. Put $D = D_F$ and $\rho = \rho_D$. Then

$$|(B \cap (aN, N]) \setminus F| = o(\Delta_N), \qquad R(F) = \log(1/a) + o(1), \tag{5.8}$$

$$|(aN, N] \setminus F| = o(N), \qquad \rho = o(1). \tag{5.9}$$

Only the last assertion needs explanation. Every integer in $(aN, N]$ divisible by a member of $\mathcal{S}_D$ is absent from F. Let $Z(n)$ count these divisors. They have pairwise coprime bases, and $a < 1/4$. Hence

$$\sum_{aN<n\le N} Z(n) \ge \frac{N\rho}{4}, \qquad \sum_{aN<n\le N} Z(n)^2 \le N(\rho + \rho^2).$$

For the first inequality use $\lfloor x\rfloor - \lfloor ax\rfloor \ge x/4$ for $x \ge 1$. Cauchy–Schwarz gives at least $N\rho/(16(1+\rho))$ integers in this union. Its size is $o(N)$ by (5.9)'s first assertion, so $\rho = o(1)$.

Put $u_N = (\log(1/a)+1)/L$. For large N, Lemma 2.2 and (5.8) imply

$$\begin{aligned} &r \in \mathbb{Q}, \quad u_N \le r \le 2, \quad \operatorname{den}(r) \mid D \\ \implies \quad &r = R(U) \text{ for some } U \subseteq F. \end{aligned} \tag{5.10}$$

The lower mass inequality uses $\log(1/a) > 3$, and the upper one follows from $Lr \ge \log(1/a)+1$.

In each positive group with a fixed $b > 1$, successively remove inclusion-minimal nonempty subsets whose reciprocal sum has denominator dividing D. By (5.5) and the cyclic partial-sum argument, each has at most b elements and therefore mass at most $b/Q_b \le 1$. Remove members of the group $b = 1$ as singletons. Let $Z \subseteq P$ be the union of all removed blocks. If $R(Z) \ge u_N$, a first collection of whole blocks crossing u_N would have mass in $[u_N, u_N+1)$ and denominator dividing D. By (5.10), it would have an equal-mass representation in B. This would give a nonempty proper zero subsum, since $R(P) > u_N + 1$ by (5.7). Consequently

$$R(Z) < u_N, \qquad |Z| < Nu_N = o(\Delta_N). \tag{5.11}$$

The remaining positive groups are inverse-zero-free on their cofactors. Lemma 5.2 now gives, for every fixed $\varepsilon > 0$,

$$(1-o(1))\mu \le C_\varepsilon + (\mathrm{e}^\gamma + \varepsilon)\lambda\rho + u_N.$$

Since $\mu \longrightarrow \infty$, we obtain

$$\liminf_{N\to\infty} \frac{\lambda\rho}{\mu} \ge \mathrm{e}^{-\gamma}. \tag{5.12}$$

5.4. **Counting omitted denominators.** Let $\mathcal{S} = \{s \in \mathcal{S}_D : s \le S\}$, with $S = N\exp(-6L^{4/5})$. Let T be the set of S-powersmooth integers in $(aN, N]$ divisible by exactly one member of $\mathcal{S}$. Then

$$|T| \ge (1-2a)N\rho - N\rho^2 - 4S - 2N\Theta(N,S). \tag{5.13}$$

Indeed, let $Z_0(n)$ count the divisors from $\mathcal{S}$. The pointwise inequality $\mathbf{1}_{\{Z_0=1\}} \ge 2Z_0 - Z_0^2$, the first-moment bound

$$\sum_{aN<n\le N} Z_0(n) \ge (1-a)N \sum_{s\in\mathcal{S}} \frac{1}{s} - 2|\mathcal{S}|,$$

and the second-moment bound

$$\sum_{aN<n\le N} Z_0(n)^2 \le N\left(\sum_{s\in\mathcal{S}} \frac{1}{s} + \left(\sum_{s\in\mathcal{S}} \frac{1}{s}\right)^2\right)$$

prove the estimate before the powersmooth restriction. Use $|\mathcal{S}| \le S$, $\rho - \sum_{s\in\mathcal{S}} s^{-1} \le \Theta(N,S)$, and (2.3) to obtain (5.13).

A member of T cannot divide D. Since it is S-powersmooth, every first missing power that divides it belongs to $\mathcal{S}$. Its excess therefore has the form $b = p^j$, and $Q_b = p^{\nu_p+j} \le S$. For a group of $P \setminus Z$ with this property, its cofactors lie in $[\lfloor N/Q_b \rfloor]$ and are inverse-zero-free modulo b. Moreover,

$$\lfloor N/Q_b \rfloor \ge \tfrac{1}{2}\exp(6L^{4/5}) \ge \exp(\lambda^3)$$

for large N. Corollary 3.5 shows that for each fixed $\varepsilon > 0$ its cardinality is at most $\varepsilon N/Q_b$. Summing over all such groups yields

$$|T \cap (P \setminus Z)| \le \varepsilon N \sum_{s_p \le N} \sum_{j\ge 1} p^{-\nu_p - j} \le 2\varepsilon N\rho. \tag{5.14}$$

Every negative member of T lies in $(B \cap (aN, N]) \setminus F$. Every other member is omitted from A or lies in Z or in $T \cap (P \setminus Z)$. Combining (5.8), (5.11), (5.13), and (5.14) gives

$$d \ge (1-2a-2\varepsilon)N\rho - N\rho^2 - o(\Delta_N). \tag{5.15}$$

Here $S = o(\Delta_N)$ and $N\Theta(N,S) = o(\Delta_N)$ by Lemma 2.1. Since $\rho = o(1)$, equations (5.12) and (5.15) imply

$$\liminf \frac{d}{\Delta_N} \geq (1 - 2a - 2\varepsilon)\mathrm{e}^{-\gamma}.$$

Let a and ε tend to zero after taking the lower limit in N. This proves the lower bound for every sequence with $d = O(\Delta_N)$. For extremizers this size restriction follows from Proposition 4.3. That proposition and the lower bound complete the proof of Theorem 1.1.

Appendix A. Exact rational representation

We prove Lemma 2.2 by the divisor extraction and Fourier method of Liu and Sawhney [5, Section 5]. Throughout this appendix put

$$L = \log N, \quad \lambda = \log L, \quad z = L^{4/5}, \quad \eta = L^{-1/25}, \quad S = N\mathrm{e}^{-6z},$$
$$M = \lfloor cN/20000 \rfloor, \quad K = M\mathrm{e}^{-z}, \quad w = \log(N/M),$$
$$\Gamma = \max\left\{ \frac{\eta}{L^{1/5}\lambda^3}, \frac{\eta^2 L^{3/5}}{w^2\lambda^5} \right\}.$$

Here $c \geq L^{-3/100}$, so $w = O(\lambda)$ uniformly. In particular,

$$\Gamma \gg \frac{L^{13/25}}{\lambda^7}, \qquad \frac{\Gamma^2}{L^{1/50}(w+z)} \gg \frac{L^{11/50}}{\lambda^{14}} \longrightarrow \infty. \tag{A.1}$$

Also $N^{.99} \leq S \leq K \leq M \leq N/10^4$ for all sufficiently large N. All thresholds below are uniform over the stated range of c.

We use the following standard upper-bound sieve consequence, also in [5, Lemma 2.4]. If $\mathcal{P}$ is a collection of primes at most y, and $y \leq \exp(\log X/\sqrt{\log\log X})$, then

$$|\{n \in [X, 2X) \cap \mathbb{N} : p \nmid n \ (p \in \mathcal{P})\}| \ll X \prod_{p \in \mathcal{P}} (1 - p^{-1}). \tag{A.2}$$

The constant is absolute, and the same bound applies to a subinterval by enlarging to the displayed interval.

A.1. **Divisor extraction.**

Lemma A.1. *Let $T \subseteq [M, N] \cap \mathbb{N}$ satisfy $\Omega(n) \leq 5\lambda$ for every $n \in T$. Suppose $q \leq K$ is a prime power and $qR(T_q) \geq \eta/2$. Put*

$$H_* = \exp\left(\frac{\eta z}{2\lambda^3 w}\right).$$

There are a positive integer d and a nonempty $T^* \subseteq T_{qd}$ such that

$$qd \geq K, \qquad \min T^* \geq H_* qd, \qquad qdR(T^*) \gg \frac{\eta}{L^{1/5}\lambda}. \tag{A.3}$$

Proof. Set $y = \exp(z/(10\lambda))$. We have $2 \leq H_* \leq y$ for large N. For $n \in T_q$, define

$$d_n = \prod_{\substack{p \mid n/q \\ p > y}} p^{v_p(n/q)}.$$

Then $qd_n \mid n$ and

$$qd_n \geq n/y^{\Omega(n)} \geq M\mathrm{e}^{-z/2} \geq K.$$

Call n poor if it has fewer than two distinct prime divisors in $[H_*, y]$. If n is poor, so is n/q.

We count poor integers in $[M/q, N/q]$ using doubling intervals $[X, 2X)$. For integers with no prime in $[H_*, y]$, (A.2) and the Mertens estimates give $O(X \log H_*/\log y)$. For those with exactly one such prime p, divide by p and apply the sieve to the other primes of the interval. Leaving p unsieved

changes the product by at most a factor 2, and permits all its higher powers. Summation over p bounds their number by $O(X\lambda \log H_*/\log y)$. All sieve applications are valid, since $X \geq M/q \geq \mathrm{e}^z$ and $\log y = z/(10\lambda)$, including after division by $p \leq y$.

There are $O(w)$ doubling intervals, so their reciprocal mass gives

$$qR(\{n \in T_q : n \text{ poor}\}) \ll w\lambda \frac{\log H_*}{\log y} \ll \frac{\eta}{\lambda}.$$

Discard the poor elements. The remaining set has q times reciprocal mass at least $\eta/4$. Each of its members has two distinct primes in $[H_*, y]$. Division by the prime power q removes at most one of them, and division by d_n removes none. Consequently $n \geq H_* q d_n$.

Partition these members according to $d_n = d$. The possible d have all prime divisors in $(y, N]$, and

$$\sum_d \frac{1}{d} \leq \prod_{y<p\leq N} (1-p^{-1})^{-1} \ll \frac{L}{\log y} \ll L^{1/5}\lambda.$$

Since $qR(T_q \setminus \{\text{poor elements}\})$ is the sum of $d^{-1}qdR(T^*)$ over the fibers, one fiber satisfies (A.3). □

The quotient valuation $v_p(n/q)$ in this proof ensures $qd_n \mid n$. The use of two distinct intermediate primes ensures that one survives division by the prime power q. Both details are needed for (A.3).

A.2. **Fourier inversion and small frequencies.** Let E satisfy Lemma 2.2, put $Q = D_E$, and set $\tau = r/R(E)$. The target inequalities give

$$L^{-1} \leq \tau \leq \frac{L}{L+1}, \qquad \tau(1-\tau) \geq \frac{1}{2L}.$$

Independently select every $n \in E$ with probability τ and call the resulting subset U. Write $\mathrm{e}(t) = \exp(2\pi i t)$ and

$$\Phi(h) = \prod_{n\in E} (1 - \tau + \tau \mathrm{e}(h/n)).$$

Since Qr is an integer, Fourier orthogonality gives

$$\mathbb{P}(R(U) - r \in \mathbb{Z}) = \frac{1}{Q} \sum_{-Q/2<h\leq Q/2} \mathrm{Re}\big(\mathrm{e}(-hr)\Phi(h)\big). \tag{A.4}$$

The small-frequency estimate of [5, Lemma 3.1] gives

$$\sum_{|h|\leq M/2} \mathrm{Re}\big(\mathrm{e}(-hr)\Phi(h)\big) \geq \frac{3}{4}. \tag{A.5}$$

Its hypotheses hold because $M \geq N^{.95}$, $|E| \geq N/L^2 \geq N^{.95}$, and $\tau \in [L^{-2}, 1-L^{-2}]$ for large N. For completeness, the estimate follows by expanding at $h = 0$ after the linear term is canceled by $r = \tau R(E)$. The leading term is the positive Gaussian

$$\exp\left(-2\pi^2\tau(1-\tau)h^2 \sum_{n\in E} n^{-2}\right).$$

The cubic remainder is uniform for $|h| \leq M^{3/5}$ and is negligible on the Gaussian scale. For $M^{3/5} < |h| \leq M/2$, the inequality

$$|1 - \tau + \tau \mathrm{e}(t)| \leq \exp(-8\tau(1-\tau)\,\|t\|^2)$$

bounds the total contribution by $o(1)$. Here $\|t\|$ denotes distance to the nearest integer. These estimates give (A.5) uniformly in the present parameter range.

A.3. **Localization of large frequencies.** For an integer h, let $h_n \in (-n/2, n/2]$ be its least absolute residue modulo n. Put

$$T_0 = \frac{50N^2L\lambda}{\tau(1-\tau)K^2}, \qquad I_h = (h - K/2, h + K/2),$$

and define

$$\mathcal{D}_h = \{q \in \mathcal{Q}(E) : |\{n \in E_q : |h_n| \geq K/2\}| < T_0\}.$$

Each n lies in exactly $\Omega(n)$ prime-power divisibility classes. The preceding bound for a Fourier factor therefore gives

$$|\Phi(h)| \leq N^{-10|\mathcal{Q}(E)\setminus\mathcal{D}_h|}. \tag{A.6}$$

Indeed, the product over all pairs (q, n) with $q \mid n$ is at least $|\Phi(h)|^{5\lambda}$. For every q outside $\mathcal{D}_h$, at least T_0 factors have $|h_n| \geq K/2$, and their product is at most $\exp(-2\tau(1-\tau)K^2T_0/N^2) = \exp(-100L\lambda)$. Taking the $1/(5\lambda)$ power proves an even stronger version of (A.6).

We claim that there is an integer in I_h divisible by every member of $\mathcal{D}_h$. Fix $q \in \mathcal{D}_h$ and let $T_q = \{n \in E_q : |h_n| < K/2\}$. The definitions and the parameter bounds give

$$R(T_q) \geq \eta/q - T_0/M \geq \eta/(2q).$$

For the last inequality it suffices to note that

$$\frac{ST_0}{M\eta} \ll c^{-3}L^2\lambda\eta^{-1}\mathrm{e}^{-4z} \longrightarrow 0.$$

Apply Lemma A.1. It produces d_q and $T_q^* \subseteq T_q$ with $qd_q \geq K$. Every $n \in T_q^*$ has a multiple in I_h, which must be the unique multiple of qd_q in this interval. Call that integer x_q.

Let $E_q^* = \{n/(qd_q) : n \in T_q^*\}$. Then

$$\min E_q^* \geq H_*, \qquad \frac{\max E_q^*}{\min E_q^*} \leq \mathrm{e}^w, \qquad R(E_q^*) \gg \frac{\eta}{L^{1/5}\lambda}. \tag{A.7}$$

Partition E_q^* into dyadic intervals $[2^j, 2^{j+1})$, including the one containing its minimum. If $\sigma = \log H_*$, the reciprocal weights of these indices satisfy

$$\sum_j \frac{1}{j+1} \ll \min\{\lambda, w/\sigma\}.$$

Weighted pigeonholing in (A.7) gives a dyadic interval $[y_q, 2y_q)$ with

$$R(E_q^* \cap [y_q, 2y_q)) \geq \frac{\Gamma}{\log y_q}. \tag{A.8}$$

To check the constants, the lower bound obtained before absorbing them is an absolute positive multiple of

$$\frac{1}{\log y_q}\max\left\{\frac{\eta}{L^{1/5}\lambda^2}, \frac{\eta^2L^{3/5}}{w^2\lambda^4}\right\} = \frac{\lambda\Gamma}{\log y_q}.$$

Thus (A.8) follows for large N. It also implies $\log y_q \gg \Gamma$, since the reciprocal mass of a dyadic interval is bounded. Finally $y_q \leq N/(qd_q)$.

For $q_1, q_2 \in \mathcal{D}_h$, set $y = \min(y_{q_1}, y_{q_2})$ and take the prime interval

$$\mathcal{P} = \{p : \exp(L^{1/100}) \leq p \leq \exp((\log y)L^{-1/100})\}.$$

It is nonempty for large N by (A.1). Let $\mathcal{P}_q$ consist of the primes in $\mathcal{P}$ dividing none of the integers in $E_q^* \cap [y_q, 2y_q)$. The sieve (A.2), applied to the omitted primes, and (A.8) imply

$$\sum_{p\in\mathcal{P}_q}\frac{1}{p} \leq \log\left(\frac{C_0\log y_q}{\Gamma}\right)$$

for an absolute C_0. The sieve cutoff holds because $L^{-1/100} \le 1/\sqrt{\log\log y_q}$ for large N. The Mertens prime estimate gives

$$\sum_{p\in\mathcal{P}\setminus(\mathcal{P}_{q_1}\cup\mathcal{P}_{q_2})} \frac{1}{p} \ge \log\left(\frac{\Gamma^2}{2C_0^2 L^{1/50}\log\max(y_{q_1},y_{q_2})}\right)$$
$$\ge \log\left(\frac{\Gamma^2}{2C_0^2 L^{1/50}(w+z)}\right) \ge 1.$$

Every prime counted here divides both x_{q_1} and x_{q_2}. Indeed it divides a cofactor $n/(qd_q)$, and x_q is a multiple of n. There are at least $\exp(L^{1/100})$ such primes, all at least that large. Their product exceeds N, whereas $|x_{q_1} - x_{q_2}| < K \le N$. Hence $x_{q_1} = x_{q_2}$. This proves the claim. An empty $\mathcal{D}_h$ imposes no divisibility condition.

A.4. **Summing the large frequencies.** For $M/2 < |h| \le Q/2$ the set $\mathcal{D}_h$ is a proper subset of $\mathcal{Q}(E)$. Otherwise its common multiple in I_h would be a multiple of Q, at distance less than $K/2$ from such a centered h, which is impossible. For fixed $\mathcal{D} \subsetneq \mathcal{Q}(E)$, the number of residue classes h for which I_h contains a multiple of $\operatorname{lcm}(\mathcal{D})$ is at most

$$(K+1)\frac{Q}{\operatorname{lcm}(\mathcal{D})} \le N \prod_{q\in\mathcal{Q}(E)\setminus\mathcal{D}} q \le N^{1+|\mathcal{Q}(E)\setminus\mathcal{D}|}.$$

Using (A.6) and at most N^s choices of a complement of size s gives

$$\frac{1}{Q}\sum_{M/2<|h|\le Q/2} |\Phi(h)| \le \frac{1}{Q}\sum_{s\ge 1} N^{1+s}N^s N^{-10s} \le \frac{2}{QN}. \tag{A.9}$$

Equations (A.5) and (A.9) show that (A.4) is at least $1/(2Q)$.

Finally, $\mathbb{E}R(U) = r$ and each selected summand is at most $1/M$. The elementary bounded-summand concentration inequality gives

$$\mathbb{P}(|R(U) - r| \ge 1) \le 2\exp(-M^2/(2N)).$$

Since every prime-power component of Q is at most S, the prime number theorem gives $\log Q = O(S)$. Also $S = o(M^2/N)$. The last probability is therefore less than $1/(4Q)$ for large N. With positive probability $R(U) - r$ is an integer of absolute value less than 1, so it is zero. This proves Lemma 2.2.

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