hexagon:2611.00749v1

A counterexample to the Erdős triangle–four-cycle asymptotic

Will Howe, Grant Huffman, Jensen Kohlmeyer, and Liam Kruer

November 2026

Abstract

We construct an infinite family of finite simple graphs containing neither a triangle nor a four-cycle. Their edge counts exceed the conjectured leading asymptotic in Erdős Problem 573 by a fixed positive factor. For every positive integer h, put $q = 659^h$. The graph G_q has

$$n_q = \frac{1307}{659}q^2 - q, \qquad e_q = \frac{428875}{659^2}q^3 - \frac{658}{659}q^2.$$

Its limiting ratio to $(n_q/2)^{3/2}$ is $1.0000452143352195\ldots$. The construction joins 47 affine three-part pieces along a common root part. A multiplicative partition of reciprocal field labels excludes four-cycles across the pieces, and an additive restriction retains enough internal edges while excluding triangles.

1. Introduction

Write $\mathrm{ex}(n, \{C_3, C_4\})$ for the maximum number of edges of a simple graph on n vertices containing no subgraph isomorphic to either C_3 or C_4. Subgraphs throughout are ordinary subgraphs. Erdős Problem 573 asks whether

$$\mathrm{ex}(n, \{C_3, C_4\}) \sim (n/2)^{3/2}.$$

Theorem 1.1. For every positive integer h, put $q = 659^h$. There is a finite simple $\{C_3, C_4\}$-free graph G_q with

$$|V(G_q)| = \frac{1307}{659}q^2 - q, \qquad e(G_q) = \frac{428875}{659^2}q^3 - \frac{658}{659}q^2.$$

Consequently,

$$\limsup_{n\to\infty} \frac{\mathrm{ex}(n, \{C_3, C_4\})}{(n/2)^{3/2}} \;\geq\; L > 1,$$

where

$$L = \frac{428875}{659^2}\left(\frac{1318}{1307}\right)^{3/2}.$$

The original problem is recorded in [1]. Let $z(n, C_4)$ denote the maximum number of edges in a bipartite C_4-free graph on n vertices. The relation $z(n, C_4) \sim (n/2)^{3/2}$ makes the conjecture above equivalent to $\mathrm{ex}(n, \{C_3, C_4\})/z(n, C_4) \to 1$, the formulation in [3, Conjecture 1.1]. Theorem 1.1 disproves this limit. Determining the exact extremal function remains open.

2. Finite-field labels

Set $p = 659$, a prime. Trial division by the primes at most $\sqrt{659} < 26$ suffices. For divisors $2, 3, 5, 7, 11, 13, 17, 19, 23$, the respective remainders are $1, 2, 4, 1, 10, 9, 13, 13, 15$.

Let H be the subgroup of seventh powers in $\mathbb{F}_p^\times$. Its order is 94, and it is the kernel of the map $x \mapsto x^{94}$. It contains -1. Put

$$I = \{1, 2, \dots, 7\}, \qquad J = \{1, 2, \dots, 10\} \subset \mathbb{F}_p.$$

The values of i^{94} for $i = 1, \dots, 7$, reduced modulo 659, are

$$1,\ 307,\ 144,\ 12,\ 389,\ 55,\ 410.$$

They are distinct, so I represents the seven distinct cosets of H. Choose a set M containing one representative from each pair $\{\mu, -\mu\}$ in H. Then $|M| = 47$, and every element of $\mathbb{F}_p^\times$ has a unique representation

$$\epsilon\mu i, \qquad \epsilon \in \{1, -1\}, \quad \mu \in M, \quad i \in I.$$

An explicit choice of M is recorded in Appendix A.

Fix $h \geq 1$, put $q = p^h$, and let $F = \mathbb{F}_q$. The absolute trace is

$$\tau(x) = x + x^p + \cdots + x^{p^{h-1}}.$$

This is an $\mathbb{F}_p$-linear map from F to $\mathbb{F}_p$. Its values lie in $\mathbb{F}_p$ because $\tau(x)^p = \tau(x)$. Its defining polynomial is nonzero and has degree $p^{h-1} < q$, so it does not vanish on every element of F. It is therefore surjective, and each fiber has q/p elements. Define

$$U = \{v \in F : \tau(v) \in I\}, \qquad R = \{a \in F^\times : \tau(a^{-1}) \notin J\}.$$

For each $\mu \in M$, put

$$B_\mu = \{(\mu v)^{-1} : v \in U\}, \qquad C_\mu = \{-(\mu w)^{-1} : w \in U\}.$$

Every element of U is nonzero. The reciprocal trace of a member of B_μ belongs to μI, and that of a member of C_μ belongs to $-\mu I$. The multiplicative partition above shows that these 94 slope sets are pairwise disjoint.

3. Definition of the graph

The vertices are the disjoint, tagged parts $R \times F$, $B_\mu \times F$, and $C_\mu \times F$, over $\mu \in M$. Tags distinguish root vertices from leaf vertices even when their first coordinates coincide.

There are two adjacency rules. A root vertex (a, x) is adjacent to a leaf vertex (d, y) whenever

$$y - x = ad.$$

Within the pair indexed by μ, a positive leaf (b, y) is adjacent to a negative leaf (c, z) whenever

$$z - y = \mu bc, \qquad \tau(v + w) \in J,$$

where $v = (\mu b)^{-1}$ and $w = -(\mu c)^{-1}$. These are exactly the parameters satisfying $b = (\mu v)^{-1}$ and $c = -(\mu w)^{-1}$. The parts are independent. Leaf edges occur only within a pair with the same index. These rules define an undirected simple graph, since every edge joins different tagged parts.

4. Excluding triangles and four-cycles

Lemma 4.1. The graph G_q contains no triangle.

Proof. A triangle must use one root and both leaf parts with the same index μ. Its root edges give $z - y = a(c - b)$, while its leaf edge gives $z - y = \mu bc$. Substitution and multiplication by the nonzero denominators yield

$$a(c - b) = \mu bc \quad \Longrightarrow \quad a(v + w) = 1.$$

Thus $\tau(a^{-1}) = \tau(v + w) \in J$, contrary to $a \in R$. This proves the lemma.

Lemma 4.2. The graph G_q contains no four-cycle.

Proof. Consider the graph on part labels. It has one root label adjacent to every leaf label, and a matching between the paired leaf labels. This graph has no four-cycle on four distinct labels. Adjacent vertices of G_q always have different labels, so any four-cycle in G_q would have two opposite vertices in the same part. We prove that each pair of distinct vertices in a single part has at most one common neighbor. The argument allows all internal matchings before the trace restriction on leaf edges.

For two root vertices (a, x) and (a', x'), a common leaf neighbor of slope d requires

$$(a - a')d = x' - x.$$

When $a = a'$, their intercepts are distinct and the equation is impossible. Otherwise it determines d uniquely. Pairwise disjointness of the leaf slope sets determines its part, and the adjacency equation determines its intercept.

For two vertices (b, y), (b', y') of $B_\mu \times F$, a common root neighbor and a common negative leaf neighbor respectively require

$$y - y' = a(b - b'), \qquad y - y' = -\mu c(b - b').$$

When $b = b'$, neither can occur. When $b \neq b'$, each equation determines its neighbor's type and intercept uniquely. If both kinds of neighbor occurred, then $a = -\mu c = 1/w$ for $w \in U$. This would give $\tau(a^{-1}) \in I \subseteq J$, contrary to $a \in R$.

For two vertices (c, z), (c', z') of $C_\mu \times F$, the corresponding equations are

$$z - z' = a(c - c'), \qquad z - z' = \mu b(c - c').$$

Equal types make both impossible. Distinct types determine at most one neighbor of either kind, and having both would force $a = \mu b = 1/v$ for $v \in U$, again excluded by the root condition. This exhausts the same-part pairs and proves the lemma.

5. Exact counts

There are $(p-10)q/p-1$ root types. Indeed, the allowed reciprocal trace classes contain $(p-10)q/p$ field elements, including zero, and zero is removed before taking inverses. Each leaf slope set has $7q/p$ members. The total number of leaf types is $94 \cdot 7q/p = (p-1)q/p$. Each type has q intercepts. Hence

$$n_q = \left(2 - \frac{11}{p}\right) q^2 - q = \frac{1307}{659} q^2 - q.$$

Every root-type and leaf-type pair defines a matching with exactly q edges. For an internal leaf pair, the 49 ordered pairs in I^2 have integer sums from 2 to 14. Exactly $4 + 3 + 2 + 1 = 10$ pairs

have sums 11, 12, 13, or 14. The remaining 39 have sum in J. There is no reduction ambiguity modulo 659. Each retained pair of trace values has $(q/p)^2$ lifts in U^2, and each lifted pair defines q edges. Thus

$$e_q = q\left(\frac{(p-10)q}{p} - 1\right)\frac{(p-1)q}{p} + 47 \cdot 39\frac{q^3}{p^2}.$$

The two edge families are disjoint, and distinct index pairs give disjoint leaf edges. Simplifying at $p = 659$ gives

$$e_q = \frac{428875}{659^2}q^3 - \frac{658}{659}q^2.$$

6. The leading-term contradiction

As $h \to \infty$, both q and n_q tend to infinity. Dividing the exact expressions in Section 5 gives

$$\lim_{h\to\infty} \frac{e_q}{(n_q/2)^{3/2}} = \frac{428875}{659^2}\left(\frac{1318}{1307}\right)^{3/2} = L.$$

All quantities are positive. Squaring and clearing denominators gives the exact comparison

$$8 \cdot 428875^2 - 659 \cdot 1307^3 = 133054063 > 0.$$

Equivalently,

$$L^2 = \frac{1471470125000}{1471337070937} = 1 + \frac{133054063}{1471337070937} > 1.$$

Therefore $L = 1.0000452143352195\ldots > 1$. Lemmas 4.1 and 4.2 imply $\mathrm{ex}(n_q, \{C_3, C_4\}) \geq e_q$. Taking the limsup proves Theorem 1.1.

7. Relation to prior constructions

Lv, Lu and Fang construct dense balanced three-partite graphs using affine matching equations [2, proof of Theorem 2, pp. 3-4]. For one of our leaf pairs, make the substitutions

$$A = a/\mu, \quad B = -b, \quad C = c, \qquad X = x/\mu, \quad Y = y/\mu, \quad Z = z/\mu.$$

The three edge equations become

$$AB = X - Y, \qquad BC = Y - Z, \qquad CA = Z - X,$$

which are the equations used in their construction. Their remark on page 4 extends the construction to prime powers by restricting a linear coordinate over the prime field. These affine equations and coordinate restrictions provide the local ingredients for the present construction.

Here a single root part serves 47 leaf pairs. The multiplicative partition makes all leaf slope sets disjoint, and the reciprocal-label restrictions prevent the possible short cycles through the common root. Retaining 39 additive label pairs in each block gives the strict leading comparison in Section 6.

Ma and Yang prove $\mathrm{ex}(n, \{C_3, C_4\}) \geq z(n, C_4) + cn^{5/4}$ for an absolute constant $c > 0$ and all $n \geq 7$ [3, Theorem 1.3]. Theorem 1.1 establishes a fixed multiplicative excess over the leading bipartite asymptotic along the orders n_q.

Appendix A. Explicit finite certificate

One explicit choice of the 47 multipliers is as follows.

$$1, 14, 15, 24, 31, 41, 43, 44, 46, 47, 50, 53$$

$$57, 63, 73, 80, 83, 85, 91, 108, 118, 128, 136, 139$$

$$145, 156, 173, 185, 190, 194, 196, 198, 207, 210, 214, 223$$

$$225, 232, 262, 274, 286, 296, 299, 302, 304, 323, 325.$$

Direct modular exponentiation gives $\mu^{94} = 1$ for each listed multiplier. They are distinct and all lie between 1 and 325, so no two are negatives modulo 659. They therefore select exactly one element from every sign pair in H.

Acknowledgment of AI assistance

OpenAI Codex assisted with the construction search, proof development, formalization, computational checks, and exposition.

References

[1] Erdős Problems, Problem 573. https://www.erdosproblems.com/573. Statement consulted November 12, 2026.

[2] Zequn Lv, Mei Lu and Chunqiu Fang. Density of Balanced 3-Partite Graphs without 3-Cycles or 4-Cycles. The Electronic Journal of Combinatorics 29(4) (2022), P4.44. https://doi.org/10.37236/10958.

[3] Jie Ma and Tianchi Yang. On extremal numbers of the triangle plus the four-cycle. Forum of Mathematics, Sigma 13 (2025), e154, 1-7. https://doi.org/10.1017/fms.2025.10100. See Conjecture 1.1.

Will Howe. howew@purdue.edu.

Grant Huffman (corresponding author). huffma97@purdue.edu.

Jensen Kohlmeyer. jenwin@purdue.edu.

Liam Kruer. kruerl@purdue.edu.