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A DIVISOR-CLOSED CRITERION FOR BEHREND SEQUENCES

WILL HOWE, GRANT HUFFMAN, JENSEN KOHLMEYER, AND LIAM KRUER

Abstract. We give a necessary and sufficient condition for a set of positive integers to be a Behrend sequence. The condition quantifies over sequences of finite divisor-closed sets supported on consecutive prime blocks. For each element at one level, its permitted extensions at the next level retain a prescribed reciprocal mass. We prove that the set of multiples of A has natural density one if and only if A meets every such family. The proof combines the classical Davenport–Erdős finite-approximation theorem with a finite pruning argument for independent geometric prime valuations.

1. Statement of the criterion

Write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For $A \subseteq \mathbb{N}$, let

$$M_A = \{n \in \mathbb{N} : a \mid n \text{ for some } a \in A\}. \tag{1.1}$$

For $B \subseteq \mathbb{N}$, its lower natural density is

$$\underline{d}(B) = \liminf_{x\to\infty} \frac{|B \cap [1, x]|}{x}.$$

When the corresponding limit exists, it is denoted by $d(B)$. A set A is called a *Behrend sequence* when $d(M_A) = 1$. Erdős asked for a necessary and sufficient condition for this property [2, p. 77].

Let $p_1 < p_2 < \cdots$ be the primes, and let $v_p(n)$ denote the exponent of p in n. For $\ell \in \mathbb{N}_0$, define the prime-prefix projection

$$\pi_\ell(n) = \prod_{i\le\ell} p_i^{v_{p_i}(n)}. \tag{1.2}$$

For integers $0 \le r \le s$, put

$$\mathcal{H}_{r,s} = \{n \in \mathbb{N} : v_{p_i}(n) = 0 \text{ whenever } i \notin (r, s]\}, \qquad Q_{r,s} = \prod_{r<i\le s} \left(1 - \frac{1}{p_i}\right). \tag{1.3}$$

Empty products equal one, so $\pi_0(n) = 1$, $\mathcal{H}_{r,r} = \{1\}$, and $Q_{r,r} = 1$. A set $S \subseteq \mathbb{N}$ is *divisor closed* if $n \in S$ and $d \mid n$ imply $d \in S$.

Theorem 1.1. *For every $A \subseteq \mathbb{N}$, the following are equivalent.*

(i) *The set M_A has natural density one.*

(ii) *For every admissible family $(\ell_k, S_k)_{k\ge0}$ in Definition 1.2,*

$$A \cap \bigcup_{k\ge0} S_k \neq \varnothing. \tag{1.4}$$

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howew@purdue.edu (W. Howe), huffma97@purdue.edu (G. Huffman), jenwin@purdue.edu (J. Kohlmeyer), and kruerl@purdue.edu (L. Kruer).

Definition 1.2. An *admissible family* consists of integers

$$0 \le \ell_0 < \ell_1 < \ell_2 < \cdots$$

and finite sets $S_k \subseteq \mathbb{N}$, $k \in \mathbb{N}_0$, satisfying $S_0 = \{1\}$ and the following conditions for every $k \ge 1$.

(i) S_k is divisor closed and $S_k \subseteq \mathcal{H}_{\ell_0,\ell_k}$.

(ii) $\pi_{\ell_{k-1}}(S_k) \subseteq S_{k-1}$.

(iii) For every $m \in S_{k-1}$, the finite set

$$F_k(m) = \{n \in \mathcal{H}_{\ell_{k-1},\ell_k} : mn \in S_k\} \tag{1.5}$$

satisfies

$$Q_{\ell_{k-1},\ell_k} \sum_{n \in F_k(m)} \frac{1}{n} \ge 1 - 2^{-k-2}. \tag{1.6}$$

The definition of an admissible family is independent of A. Each condition at a fixed level involves finite sets, prime factorization, and finite reciprocal sums. The multiplier 1 is permitted in (1.5). The fibers may depend on the parent m.

Lemma 1.3. *Every admissible family satisfies $S_{k-1} \subseteq S_k$ for $k \ge 1$. Each fiber $F_k(m)$ is nonempty and divisor closed. If $m', m \in S_{k-1}$ and $m' \mid m$, then $F_k(m) \subseteq F_k(m')$.*

Proof. The right-hand side of (1.6) is positive, so $F_k(m)$ is nonempty. If $n \in F_k(m)$ and $d \mid n$, then $md \mid mn$. Divisor closure of S_k gives $md \in S_k$, and d has the required prime support. Thus $d \in F_k(m)$. In particular $1 \in F_k(m)$, so $m \in S_k$. Finally, if $m' \mid m$ and $n \in F_k(m)$, then $m'n \mid mn$, giving $n \in F_k(m')$. □

2. Prime valuations and the density transfer

Let $V_1, V_2, \ldots$ be independent random variables with

$$\mathbf{P}(V_i = e) = \left(1 - \frac{1}{p_i}\right) p_i^{-e}, \qquad e \in \mathbb{N}_0. \tag{2.1}$$

We work on the product probability space $\Omega = \mathbb{N}_0^{\mathbb{N}}$ with these coordinate laws. For $a \in \mathbb{N}$, write $a \mid V$ when $v_{p_i}(a) \le V_i$ for every i. This event depends on finitely many coordinates. Define

$$C = C_A = \{V : a \nmid V \text{ for every } a \in A\}. \tag{2.2}$$

The event C is measurable and is downward closed in the coordinatewise order on Ω.

For $\ell \in \mathbb{N}_0$ and $0 \le r \le s$, let

$$X_\ell = \prod_{i \le \ell} p_i^{V_i}, \qquad Y_{r,s} = \prod_{r < i \le s} p_i^{V_i}.$$

Independence gives the exact distributions

$$\mathbf{P}(X_\ell = m) = \frac{Q_{0,\ell}}{m} \quad (m \in \mathcal{H}_{0,\ell}), \qquad \mathbf{P}(Y_{r,s} = n) = \frac{Q_{r,s}}{n} \quad (n \in \mathcal{H}_{r,s}). \tag{2.3}$$

In particular, every finite-prefix cylinder has positive probability. We denote the second probability in (2.3) by $\nu_{r,s}(n)$. Then

$$\sum_{n \in \mathcal{H}_{r,s}} \nu_{r,s}(n) = 1. \tag{2.4}$$

Lemma 2.1. *For every finite $F \subseteq \mathbb{N}$,*

$$\mathbf{P}(a \mid V \text{ for some } a \in F) = d(M_F). \tag{2.5}$$

Proof. For $e \in \mathbb{N}_0$, summing (2.1) gives $\mathbf{P}(V_i \geq e) = p_i^{-e}$. Consequently, for every nonempty finite $J \subseteq \mathbb{N}$,

$$\mathbf{P}(a \mid V \text{ for every } a \in J) = \prod_i p_i^{-v_{p_i}(\operatorname{lcm}(J))} = \frac{1}{\operatorname{lcm}(J)}.$$

The positive integers divisible by every member of J are precisely the multiples of $\operatorname{lcm}(J)$, and their density is the same quantity. Applying finite inclusion-exclusion to both sides gives

$$\mathbf{P}(a \mid V \text{ for some } a \in F) = \sum_{\varnothing \neq J \subseteq F} \frac{(-1)^{|J|+1}}{\operatorname{lcm}(J)} = d(M_F).$$

The assertion for $F = \varnothing$ is immediate. □

We use the classical finite-approximation theorem of Davenport and Erdős [1].

Theorem 2.2 (Davenport–Erdős)**.** *For every $A \subseteq \mathbb{N}$, with $A_t = A \cap \{1, \dots, t\}$,*

$$\underline{d}(M_A) = \lim_{t \to \infty} d(M_{A_t}). \tag{2.6}$$

Corollary 2.3. *For every $A \subseteq \mathbb{N}$,*

$$\mathbf{P}(C_A) = 1 - \underline{d}(M_A). \tag{2.7}$$

In particular,

$$d(M_A) = 1 \quad \Longleftrightarrow \quad \mathbf{P}(C_A) = 0. \tag{2.8}$$

Proof. The events $\{V : a \mid V \text{ for some } a \in A_t\}$ increase to C_A^c. Continuity from below, Lemma 2.1, and Theorem 2.2 give (2.7). A subset of $\mathbb{N}$ with lower natural density one has natural density one, because its upper natural density is at most one. This proves (2.8). □

3. Positive avoidance from an admissible family

Proposition 3.1. *Suppose $(\ell_k, S_k)_{k \geq 0}$ is admissible and $A \cap S_k = \varnothing$ for every k. Then*

$$\mathbf{P}(C_A) \geq \frac{3}{4} Q_{0,\ell_0} > 0. \tag{3.1}$$

Proof. Set $E_k = \{V : X_{\ell_k} \in S_k\}$. Since $S_0 = \{1\}$, we have $\mathbf{P}(E_0) = Q_{0,\ell_0}$. The projection condition in Definition 1.2 gives $E_k \subseteq E_{k-1}$. For each $m \in S_{k-1}$, the new block Y_{ℓ_{k-1},ℓ_k} is independent of $X_{\ell_{k-1}}$. Equations (2.3) and (1.6) therefore give

$$\begin{aligned} \mathbf{P}(E_k) &= \sum_{m \in S_{k-1}} \mathbf{P}(X_{\ell_{k-1}} = m) \sum_{n \in F_k(m)} \nu_{\ell_{k-1},\ell_k}(n) \\ &\geq (1 - 2^{-k-2}) \mathbf{P}(E_{k-1}). \end{aligned} \tag{3.2}$$

For $u_j \in [0,1]$, induction gives $\prod_{j=1}^k (1 - u_j) \geq 1 - \sum_{j=1}^k u_j$. Hence

$$\mathbf{P}(E_k) \geq Q_{0,\ell_0} \prod_{j=1}^{k} (1 - 2^{-j-2}) \geq \frac{3}{4} Q_{0,\ell_0}.$$

Continuity from above yields

$$\mathbf{P}\left(\bigcap_{k \geq 0} E_k \right) \geq \frac{3}{4} Q_{0,\ell_0}. \tag{3.3}$$

We claim that $\bigcap_{k\geq 0} E_k \subseteq C_A$. Indeed, let V belong to this intersection and suppose $a \in A$ divides V. The integer a has finitely many prime divisors, and $\ell_k \to \infty$. Thus $a \mid X_{\ell_k}$ for all sufficiently large k. Since $X_{\ell_k} \in S_k$ and S_k is divisor closed, it follows that $a \in S_k$. This contradicts $A \cap S_k = \varnothing$. The claim and (3.3) prove (3.1). □

4. Finite-prefix extraction

For $m \in \mathcal{H}_{0,\ell}$, define

$$h_\ell(m) = \mathbf{P}(C_A \mid X_\ell = m). \tag{4.1}$$

The conditioning is legitimate by (2.3). Equivalently, fix the first ℓ prime exponents to those of m and sample the remaining coordinates according to (2.1). Let

$$C_\ell = \{V : a \nmid X_\ell \text{ for every } a \in A\}. \tag{4.2}$$

Then $C_\ell \downarrow C_A$. Indeed, any integer dividing V divides X_ℓ once the prefix contains all of its prime divisors.

Lemma 4.1. *For every $\ell \in \mathbb{N}_0$,*

(i) $0 \leq h_\ell(m) \leq 1$ *for* $m \in \mathcal{H}_{0,\ell}$.
(ii) *If* $m', m \in \mathcal{H}_{0,\ell}$ *and* $m' \mid m$, *then* $h_\ell(m') \geq h_\ell(m)$.
(iii) *If some* $a \in A$ *divides* m, *then* $h_\ell(m) = 0$.
(iv) $\mathbf{E}[h_\ell(X_\ell)] = \mathbf{P}(C_A)$.

Proof. The first assertion follows from the definition as a conditional probability. For the second, use the same future prime exponents with prefixes m' and m. The first complete valuation vector is coordinatewise no larger than the second. Downward closure of C_A shows that survival for the second implies survival for the first. For the third, every completion of the prefix m remains divisible by a. The last assertion is the law of total probability applied to the countable partition $\{X_\ell = m\}$. □

Lemma 4.2. *If $c = \mathbf{P}(C_A) > 0$, then $h_\ell(1) \to 1$ as $\ell \to \infty$.*

Proof. By Lemma 4.1, the function h_ℓ is at most $h_\ell(1)$ and vanishes on prefixes outside C_ℓ. Consequently,

$$c = \mathbf{E}[h_\ell(X_\ell)] \leq h_\ell(1)\mathbf{P}(C_\ell).$$

Since $\mathbf{P}(C_\ell) \downarrow c > 0$,

$$1 \geq h_\ell(1) \geq \frac{c}{\mathbf{P}(C_\ell)} \longrightarrow 1. \qquad \square$$

Lemma 4.3. *Let $r \in \mathbb{N}_0$, let $G \subseteq \mathcal{H}_{0,r}$ be finite, and let $0 < \delta < 1$. There exist $\ell > r$ and, for each $m \in G$, a finite set $N_m \subseteq \mathcal{H}_{r,\ell}$ such that*

$$h_\ell(mn) \geq 1 - \delta \qquad (n \in N_m) \tag{4.3}$$

and

$$\sum_{n \in N_m} \nu_{r,\ell}(n) > h_r(m) - 2\delta. \tag{4.4}$$

The cutoff ℓ is common to all $m \in G$.

Proof. Fix $m \in G$ and write $\mathbf{P}_m$ and $\mathbf{E}_m$ for probability and expectation conditional on $X_r = m$. For $\ell \geq r$, we have $X_r = \pi_r(X_\ell)$. The tower property and Lemma 4.1(iii) give

$$\begin{aligned} D_\ell(m) &:= \mathbf{E}_m\big[(1 - h_\ell(X_\ell))\mathbf{1}_{C_\ell}\big] \\ &= \mathbf{P}_m(C_\ell) - h_r(m). \end{aligned} \tag{4.5}$$

Indeed, $h_\ell(X_\ell)$ vanishes off C_ℓ and has conditional expectation $h_r(m)$ under $\mathbf{P}_m$. Continuity from above gives $\mathbf{P}_m(C_\ell) \downarrow \mathbf{P}_m(C_A) = h_r(m)$, and therefore $D_\ell(m) \to 0$. Since G is finite, choose a common $\ell > r$ such that

$$D_\ell(m) < \delta^2 \qquad (m \in G). \tag{4.6}$$

On $C_\ell \cap \{h_\ell(X_\ell) < 1 - \delta\}$, the integrand in (4.5) is greater than δ. It follows that

$$\mathbf{P}_m(C_\ell \cap \{h_\ell(X_\ell) < 1 - \delta\}) < \delta.$$

The event $\{h_\ell(X_\ell) \geq 1 - \delta\}$ is contained in C_ℓ, because $\delta < 1$ and h_ℓ vanishes on unsafe prefixes. Thus

$$\mathbf{P}_m(h_\ell(X_\ell) \geq 1 - \delta) > \mathbf{P}_m(C_\ell) - \delta \geq h_r(m) - \delta. \tag{4.7}$$

Under $\mathbf{P}_m$, the new multiplier $Y_{r,\ell}$ has the law $\nu_{r,\ell}$, and $X_\ell = mY_{r,\ell}$. Consequently, the probability on the left of (4.7) is the sum of $\nu_{r,\ell}(n)$ over

$$\{n \in \mathcal{H}_{r,\ell} : h_\ell(mn) \geq 1 - \delta\}.$$

This is a convergent series of nonnegative terms. Choose a finite subset N_m whose sum is greater than that series minus δ. Equations (4.3) and (4.4) follow. □

5. Construction of an avoiding family

Proposition 5.1. *If $\mathbf{P}(C_A) > 0$, then there exists an admissible family $(\ell_k, S_k)_{k\geq 0}$ such that $A \cap S_k = \varnothing$ for every k.*

Proof. Set

$$\delta_k = 2^{-k-5} \qquad (k \geq 0). \tag{5.1}$$

By Lemma 4.2, choose ℓ_0 with $h_{\ell_0}(1) \geq 1 - \delta_0$, and put $S_0 = \{1\}$. Lemma 4.1(iii) gives $A \cap S_0 = \varnothing$. We construct the family inductively, maintaining

$$h_{\ell_k}(s) \geq 1 - \delta_k \qquad (s \in S_k). \tag{5.2}$$

Suppose that ℓ_{k-1} and the finite divisor-closed set $S_{k-1} \subseteq \mathcal{H}_{\ell_0,\ell_{k-1}}$ have been chosen. Apply Lemma 4.3 with

$$r = \ell_{k-1}, \qquad G = S_{k-1}, \qquad \delta = \delta_k.$$

It gives $\ell_k > \ell_{k-1}$ and finite sets $N_m \subseteq \mathcal{H}_{\ell_{k-1},\ell_k}$ for $m \in S_{k-1}$. Let

$$T_k = \bigcup_{m\in S_{k-1}} \{mn : n \in N_m\}, \qquad S_k = \mathrm{Div}(T_k) = \{d \in \mathbb{N} : d \mid t \text{ for some } t \in T_k\}. \tag{5.3}$$

The set S_k is finite and divisor closed, and its elements use only primes in $(\ell_0, \ell_k]$.

If $d \mid mn$ with $m \in S_{k-1}$ and $n \in N_m$, then $\pi_{\ell_{k-1}}(d) \mid m$, because n uses only primes with indices greater than ℓ_{k-1}. Divisor closure of S_{k-1} therefore gives

$$\pi_{\ell_{k-1}}(d) \in S_{k-1}.$$

This verifies the projection condition. Moreover, Lemma 4.1(ii) and (4.3) imply

$$h_{\ell_k}(d) \geq h_{\ell_k}(mn) \geq 1 - \delta_k.$$

Thus (5.2) holds at level k. Since $1 - \delta_k > 0$, Lemma 4.1(iii) shows that no element of S_k has a divisor in A. In particular $A \cap S_k = \varnothing$.

For every $m \in S_{k-1}$, the inclusion $mN_m \subseteq S_k$ gives $N_m \subseteq F_k(m)$. Using (4.4) and the induction hypothesis, we obtain

$$\begin{aligned} Q_{\ell_{k-1},\ell_k} \sum_{n \in F_k(m)} \frac{1}{n} &\geq \sum_{n \in N_m} \nu_{\ell_{k-1},\ell_k}(n) \\ &> h_{\ell_{k-1}}(m) - 2\delta_k \\ &\geq 1 - \delta_{k-1} - 2\delta_k = 1 - 2^{-k-3} \geq 1 - 2^{-k-2}. \end{aligned} \tag{5.4}$$

The lower bound is positive, so every N_m is nonempty. All conditions of Definition 1.2 have now been verified, completing the induction. □

Proof of Theorem 1.1. Suppose $d(M_A) = 1$. By Corollary 2.3, $\mathbf{P}(C_A) = 0$. If an admissible family were disjoint from A at every level, Proposition 3.1 would give $\mathbf{P}(C_A) > 0$. Hence every admissible family meets A.

Conversely, suppose A meets every admissible family. If M_A failed to have natural density one, Corollary 2.3 would give $\mathbf{P}(C_A) > 0$. Proposition 5.1 would then produce an admissible family disjoint from A at every level, a contradiction. Therefore $d(M_A) = 1$. □

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