hexagon:2611.00746v1

A ZFC COUNTEREXAMPLE TO TAYLOR'S CONJECTURE ON FINITE SUBGRAPHS

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Abstract. We construct, in ZFC, a graph G of chromatic number $\aleph_1$ such that every graph whose finite ordinary subgraphs embed into G has chromatic number at most $2^{\aleph_0}$. Taking the target chromatic number to be $(2^{\aleph_0})^+$ gives a negative answer to Taylor's conjecture on finite subgraphs. The construction uses hereditarily countable vertex data, finite coordinate inequalities, and a countable-closure reflection argument. A compactness argument transfers the resulting finite coloring properties to the entire finite age of G.

1. Introduction

All graphs in this paper are simple and undirected. An embedding of a graph F into a graph G is an injective map on vertices that preserves adjacency. Write $\mathrm{Age}(G)$ for the class of finite graphs that embed into G, up to isomorphism. The chromatic number $\chi(G)$ is the least cardinality of a set admitting a proper coloring of G.

A conjecture of Walter Taylor, recorded by Erdős [2], asks whether every graph G with $\chi(G) = \aleph_1$ has the following property. For every cardinal κ, there is a graph H with

$$\chi(H) = \kappa \qquad \text{and} \qquad \mathrm{Age}(H) \subseteq \mathrm{Age}(G). \tag{1.1}$$

Komjáth and Shelah [3] proved that a negative answer is consistent with ZFC. The theorem below gives a counterexample in ZFC itself. Throughout, put $\mathfrak{c} = 2^{\aleph_0}$.

Theorem 1.1. *There is a graph G with $\chi(G) = \aleph_1$ such that, for every graph H,*

$$\mathrm{Age}(H) \subseteq \mathrm{Age}(G) \quad \Longrightarrow \quad \chi(H) \leq \mathfrak{c}. \tag{1.2}$$

Consequently, (1.1) fails for this G and $\kappa = \mathfrak{c}^+$.

The host construction is based on Lambie-Hanson's work on the growth of chromatic numbers of finite subgraphs [4], in particular his construction using hereditarily countable structures [5]. We give the construction and all the properties needed for (1.2). The argument uses one chosen predecessor at each natural-number label and increasing bounds on the lengths of odd closed walks in the label tails.

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$. A closed walk of length ℓ is a sequence $v_0, v_1, \ldots, v_\ell = v_0$ with consecutive vertices adjacent. Vertices may repeat. A coloring makes a walk monochromatic when all its vertices receive the same color. We work throughout in ZFC.

2. A compactness criterion for the finite age

Proposition 2.1. *Suppose that, for each $j \in \mathbb{N}$, a graph G admits a coloring a_j into a finite nonempty set A_j such that no odd closed walk of length at most $2j + 3$ is monochromatic under a_j. If $\mathrm{Age}(H) \subseteq \mathrm{Age}(G)$, then $\chi(H) \leq \mathfrak{c}$.*

Date: November 10, 2026.

2020 *Mathematics Subject Classification.* 05C63, 05C15, 03E05.

Key words and phrases. infinite graphs, chromatic number, finite subgraphs, countable closure, compactness.

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Proof. Fix j. Give A_j the discrete topology and consider the compact space

$$X_j = A_j^{V(H)}.$$

For every odd closed walk Q of H of length at most $2j+3$, let U_Q be the set of colorings in X_j that make Q nonmonochromatic. Each U_Q is clopen because membership depends on finitely many coordinates.

A finite collection of these conditions involves a finite set $S \subseteq V(H)$. The finite graph $H[S]$ embeds into G. Pulling back a_j along such an embedding colors S and satisfies every condition in the collection. Indeed, an adjacency-preserving map sends each of the specified closed walks to a closed walk of the same length. Extend the coloring arbitrarily to the other vertices of H. Thus the sets U_Q have the finite intersection property. Compactness gives a coloring

$$b_j \colon V(H) \longrightarrow A_j$$

for which no odd closed walk of length at most $2j+3$ is monochromatic. This is the usual finite-color compactness principle underlying the de Bruijn–Erdős theorem [1].

Choose such a b_j for every j and set

$$b(v) = \langle b_j(v) : j \in \mathbb{N} \rangle.$$

Every fiber of b induces a graph with no odd closed walk. Otherwise a walk of odd length ℓ in a fiber would be monochromatic under b_j for any j with $\ell \leq 2j+3$.

A graph with no odd closed walk is bipartite. To see this, choose a root in each connected component and color a vertex by the parity of a path from the root. Two such paths have the same parity, since concatenating one with the reverse of the other gives a closed walk, which must have even length. Adjacent vertices have opposite parity. Choose a binary coloring on each fiber of b, and denote the resulting bit at v by $\varepsilon(v)$.

The map $v \mapsto (b(v), \varepsilon(v))$ is a proper coloring of H. Its color set has cardinality at most $\mathfrak{c}$. In fact, each finite A_j embeds into $\mathbb{N}$, so

$$\left| \left(\prod_{j \in \mathbb{N}} A_j \right) \times \{0,1\} \right| \leq |\mathbb{N}^{\mathbb{N}} \times \{0,1\}| = \mathfrak{c}.$$

□

3. A finite coordinate obstruction

For a positive integer s, define

$$B_s = (2s+1)(s+1), \qquad n_s = 2B_s + 2s + 3 = 4(s+1)^2 + 1. \tag{3.1}$$

If a and b are strictly increasing n_s-tuples of ordinals, write $a \; R_s \; b$ when, for every integer i with $s+1 \leq i \leq n_s - s - 2$,

$$a(i) < b(i-s) \qquad \text{and} \qquad b(i) < a(i+s+1). \tag{3.2}$$

These are coordinate inequalities supplied by the usual Specker disjoint types. The proof below uses the inequalities directly.

Lemma 3.1. *Suppose that $a_0, a_1, \ldots, a_\ell = a_0$ are strictly increasing n_s-tuples of ordinals and, for every $t < \ell$, either $a_t \; R_s \; a_{t+1}$ or $a_{t+1} \; R_s \; a_t$. If ℓ is odd, then $\ell > 2s+1$.*

Proof. Suppose that $\ell \leq 2s+1$. Choose one of the two available orientations for each step. Reversing the walk if necessary, arrange that the number u of forward steps exceeds the number d of reverse steps. This is possible because $u + d = \ell$ is odd. In particular, $d \leq s$.

Start at index $i_0 = B_s + s + 1$. At a forward step decrease the index by s, and at a reverse step increase it by $s+1$. Thus

$$i_{t+1} = \begin{cases} i_t - s, & a_t \ R_s \ a_{t+1} \text{ is the chosen orientation,} \\ i_t + s + 1, & a_{t+1} \ R_s \ a_t \text{ is the chosen orientation.} \end{cases}$$

At most ℓ changes occur, each of absolute value at most $s+1$. Since $\ell(s+1) \le B_s$, every index lies in

$$[s+1,\, 2B_s + s + 1] = [s+1,\, n_s - s - 2].$$

The corresponding inequality in (3.2) therefore applies at every step and gives

$$a_0(i_0) < a_1(i_1) < \cdots < a_\ell(i_\ell).$$

On the other hand,

$$i_\ell - i_0 = -su + (s+1)d = -s(u-d) + d \le -s + d \le 0.$$

As $a_\ell = a_0$ is increasing, $a_\ell(i_\ell) \le a_0(i_0)$, a contradiction. □

For later use put

$$s_j = j + 1, \qquad n_j = n_{s_j}, \qquad b_0 = 0, \qquad b_{j+1} = b_j + n_j. \tag{3.3}$$

The jth block of an increasing sequence $C\colon \mathbb{N} \to \omega_1$ is the tuple

$$C[j] = \langle C(b_j + i) : 0 \le i < n_j \rangle.$$

4. The host graph

4.1. **Vertex data.** Let $\mathcal{H} = H(\aleph_1)$ be the set of hereditarily countable sets. We use the von Neumann representation of ordinals and fix the Kuratowski encoding of ordered pairs. A vertex is a quadruple

$$v = \langle M_v, \delta_v, h_v, C_v \rangle \tag{4.1}$$

with the following properties.

(1) $M_v \in \mathcal{H}$ is transitive and $M_v \cap \omega_1 = \delta_v$, where δ_v is a nonzero limit ordinal.
(2) $h_v\colon M_v \to \mathbb{N}$ is a function.
(3) $C_v\colon \mathbb{N} \to \delta_v$ is strictly increasing and cofinal in δ_v.

Let V be the set of all such vertices. Each vertex is identified with the code

$$\langle M_v, \langle \delta_v, \langle h_v, C_v \rangle \rangle \rangle,$$

where functions are represented by their graphs. This code is hereditarily countable, and distinct vertex data have distinct codes. We consequently regard V as a subset of $\mathcal{H}$.

Every member of $\mathcal{H}$ has countable rank. This follows by well-founded induction from

$$\mathrm{rk}(x) = \sup\{\mathrm{rk}(y) + 1 : y \in x\}$$

and the fact that a countable supremum of countable ordinals is countable. In particular, $\mathcal{H}$ is a set, contained in the cumulative-hierarchy level V_{ω_1}. The coding also gives

$$\mathrm{rk}(M_v) < \mathrm{rk}(v) < \omega_1. \tag{4.2}$$

For vertices u, v, write $u \prec v$ when $u \in M_v$. Then

$$u \prec v \quad \Longrightarrow \quad \mathrm{rk}(u) < \mathrm{rk}(M_v) < \mathrm{rk}(v). \tag{4.3}$$

4.2. **Edges and their layers.** For $v \in V$ and $k \in \mathbb{N}$, a vertex u is *eligible at* (v, k) if

$$u \in M_v, \qquad h_v(u) = k, \qquad C_u[j]\ R_{s_j}\ C_v[j] \quad (0 \le j \le k). \tag{4.4}$$

When an eligible vertex exists, choose one and denote it by $p_k(v)$. Put

$$E_k = \{\{v, p_k(v)\} : v \in V \text{ and } p_k(v) \text{ is defined}\}, \qquad E = \bigcup_{k\in\mathbb{N}} E_k,$$

and let $G = (V, E)$. Formula (4.3) excludes loops and shows that rank is a proper coloring of G. Hence

$$\chi(G) \le \aleph_1. \tag{4.5}$$

Lemma 4.1. *For every k, the graph (V, E_k) is a forest. For every j, the graph*

$$G_{\ge j} = \left(V, \bigcup_{k\ge j} E_k\right)$$

has no odd closed walk of length at most $2j + 3$.

Proof. A finite cycle in (V, E_k) would have a vertex v of maximum rank. Its two neighbors on the cycle have smaller rank, so each must be $p_k(v)$. This contradicts their distinctness. Thus (V, E_k) is a forest and admits a proper binary coloring.

For the second assertion, map a vertex v to the increasing tuple $C_v[j]$. An edge in $G_{\ge j}$ has a label $k \ge j$. Its two image tuples satisfy R_{s_j} in one of the two orientations by (4.4). An odd closed walk of length at most $2j + 3 = 2s_j + 1$ would therefore contradict Lemma 3.1. The images of its vertices may repeat, which is allowed in that lemma. □

Choose a binary coloring c_k of (V, E_k) for each k. For $j \in \mathbb{N}$, let

$$a_j(v) = \langle c_0(v), \dots, c_{j-1}(v)\rangle \in \{0, 1\}^j. \tag{4.6}$$

Every edge whose endpoints have the same a_j-value belongs to the tail $G_{\ge j}$. Lemma 4.1 implies that no odd closed walk of length at most $2j + 3$ is monochromatic under a_j.

5. Countable closures and reflection

We prove that G admits no coloring into $\mathbb{N}$. The countable closures below supply the reflection argument for the construction.

Lemma 5.1. *Let X be a set and let $\mathcal{F}$ be a countable collection of functions, each of the form $X^r \to X$ for some finite r. Every countable subset of X is contained in a countable subset closed under every member of $\mathcal{F}$.*

Proof. Starting with the given countable set A_0, let A_{m+1} contain A_m and all values of functions in $\mathcal{F}$ on tuples from A_m. Each A_m is countable. The union $A = \bigcup_{m<\omega} A_m$ is countable and closed, since every finite tuple from A is contained in some A_m. □

Fix an arbitrary function $f\colon V \to \mathbb{N}$. The functions used for closure will depend on f. For a finite sequence $\bar\alpha$ of countable ordinals and $k, r \in \mathbb{N}$, define predicates $P^k_r(\bar\alpha)$ recursively. The predicate $P^k_0(\bar\alpha)$ means that some $v \in V$ satisfies

$$f(v) = k \qquad \text{and} \qquad C_v(i) = \alpha_i \quad (i < |\bar\alpha|).$$

For $r \ge 0$, let

$$P^k_{r+1}(\bar\alpha) \iff \forall\beta < \omega_1\ \exists\gamma > \beta\ P^k_r(\bar\alpha \frown \langle\gamma\rangle). \tag{5.1}$$

Here and below the quantified ordinals are less than ω_1.

For every k, r and finite $\bar\alpha$, choose the following values, using zero as a default whenever the specified choice is unavailable.

(1) If $P^k_{r+1}(\bar\alpha)$ fails, choose a bound $B_{k,r}(\bar\alpha) < \omega_1$ such that

$$\gamma > B_{k,r}(\bar\alpha) \quad \Longrightarrow \quad \neg P^k_r(\bar\alpha \frown \langle\gamma\rangle). \tag{5.2}$$

(2) Whenever a witness exists above β, choose $W_{k,r}(\bar\alpha,\beta) > \beta$ satisfying

$$P^k_r(\bar\alpha \frown \langle W_{k,r}(\bar\alpha,\beta)\rangle).$$

(3) If $P^k_0(\bar\alpha)$ holds, choose a vertex $S_k(\bar\alpha)$ witnessing it.

The vertex in the third item is represented by its code in $\mathcal{H}$.

These choices yield countably many finitary functions on $\mathcal{H}$. For precision, for each possible input length take the corresponding function, and decode an arbitrary input $x \in \mathcal{H}$ as the countable ordinal $\mathrm{rk}(x)$. For an ordinal input α, this decoding returns α. Ordinal outputs and vertex codes belong to $\mathcal{H}$.

Include also the following functions in the closure family. For each $x \in \mathcal{H}$, fix a sequence $e_n(x)$ whose range contains every member of x and whose values lie in $x \cup \{\varnothing\}$. Include all functions e_n, the constant $\varnothing$, and the function

$$x \longmapsto \mathrm{rk}(x) + 1,$$

with its ordinal value regarded as an element of $\mathcal{H}$.

Lemma 5.2. *There are increasing countable subsets $M_i \subseteq \mathcal{H}$ for $i < \omega$, closed under all the chosen functions, such that, on writing*

$$\delta_i = M_i \cap \omega_1,$$

each M_i is transitive, each δ_i is a nonzero countable limit ordinal, and

$$\delta_i \in M_{i+1}, \qquad \delta_i < \delta_{i+1}. \tag{5.3}$$

The union $M_\infty = \bigcup_i M_i$ is closed and transitive, belongs to $\mathcal{H}$, and satisfies

$$M_\infty \cap \omega_1 = \delta_\infty = \sup_i \delta_i.$$

Proof. Apply Lemma 5.1 to obtain M_0 containing $\varnothing$. Closure under the functions e_n makes M_0 transitive. Its ordinals form an initial segment of ω_1. Closure under $x \mapsto \mathrm{rk}(x)+1$ makes that segment closed under successors, and it contains all finite ordinals. Since M_0 is countable, its ordinal part is a nonzero countable limit ordinal δ_0.

Having constructed M_i, apply Lemma 5.1 to $M_i \cup \{\delta_i\}$ to obtain M_{i+1}. The same argument gives transitivity and the required properties of its ordinal part. Inclusion of δ_i gives (5.3).

The union is countable and transitive. Every finite tuple from it lies in one stage, so it is closed under the chosen functions. A countable transitive set of hereditarily countable sets is hereditarily countable. Its ordinal part is the union of the δ_i, which is their supremum. □

Define $C_\tau(i) = \delta_i$, and define h_τ on M_∞ by

$$h_\tau(x) = \begin{cases} f(v), & x \text{ is the code of a vertex } v \in V, \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

Uniqueness of the vertex code makes this definition unambiguous. Lemma 5.2 shows that

$$\tau = \langle M_\infty, \delta_\infty, h_\tau, C_\tau\rangle \tag{5.5}$$

is a vertex of G. Put $k = f(\tau)$.

Lemma 5.3. *For every $r, i \in \mathbb{N}$,*

$$P^k_r(\langle\delta_0,\dots,\delta_{i-1}\rangle) \tag{5.6}$$

holds. In particular, $P^k_r(\varnothing)$ holds for every r.

Proof. Induct on r, simultaneously for all i. For $r = 0$, the vertex τ witnesses (5.6).

Suppose the statement holds for r, and put $\bar{\delta}_i = \langle \delta_0, \dots, \delta_{i-1} \rangle$. The induction hypothesis at $i+1$ gives

$$P_r^k(\bar{\delta}_i \frown \langle \delta_i \rangle).$$

If $P_{r+1}^k(\bar{\delta}_i)$ failed, the chosen bound $B_{k,r}(\bar{\delta}_i)$ would belong to M_i. Indeed, every entry δ_j with $j < i$ belongs to M_i by (5.3), and M_i is closed under the bound function. As the bound is an ordinal, it is less than δ_i. Equation (5.2) would then exclude the displayed instance of P_r^k, a contradiction. □

Lemma 5.4. *Let $N \in \mathbb{N}$, and let $(g_t)_{t<N}$ be a nondecreasing sequence of natural numbers. There is a vertex $v \in M_\infty \cap V$ with $f(v) = k$ such that, for every $t < N$,*

$$C_v(t) < \delta_{g_t}, \qquad g_t > 0 \implies \delta_{g_t - 1} < C_v(t). \tag{5.7}$$

Proof. Starting with $P_N^k(\varnothing)$, construct countable ordinals $\gamma_0, \dots, \gamma_{N-1}$. At stage t, retain

$$P_{N-t}^k(\langle \gamma_0, \dots, \gamma_{t-1} \rangle).$$

Set $\beta_t = 0$ if $g_t = 0$, and $\beta_t = \delta_{g_t - 1}$ otherwise. The ordinal β_t belongs to M_{g_t}. Every previously chosen γ_j belongs to $M_{g_j} \subseteq M_{g_t}$. Closure under the witness function permits the choice

$$\gamma_t = W_{k,N-t-1}(\langle \gamma_0, \dots, \gamma_{t-1} \rangle, \beta_t) \in M_{g_t}.$$

It satisfies $\gamma_t > \beta_t$ and preserves the displayed predicate with $t+1$ in place of t. Since γ_t is an ordinal in M_{g_t}, it is less than δ_{g_t}.

At the end, $P_0^k(\langle \gamma_0, \dots, \gamma_{N-1} \rangle)$ holds. Closure of M_∞ under the vertex-selection function gives a witnessing vertex $v \in M_\infty$. Its first N ladder entries are the γ_t, so it satisfies (5.7). □

6. The chromatic number and the conclusion

Proposition 6.1. *The graph G has chromatic number $\aleph_1$.*

Proof. The upper bound is (4.5). Suppose that $f\colon V \to \mathbb{N}$ is a proper coloring, and construct the sets M_i, the vertex τ, and the color $k = f(\tau)$ as above. Set $N = b_{k+1}$. For $0 \le j \le k$ and $0 \le i < n_j$, prescribe

$$g_{b_j + i} = b_j + \max\{i - s_j, 0\}. \tag{6.1}$$

This sequence is nondecreasing within every block. Its values in the jth block are at least b_j and strictly less than b_{j+1}, so it is nondecreasing across the blocks as well.

Apply Lemma 5.4 to obtain $v \in M_\infty \cap V$ of color k. Fix $j \le k$ and $s_j + 1 \le i \le n_j - s_j - 2$. The upper inequality in (5.7) and (6.1) give

$$C_v(b_j + i) < \delta_{b_j + i - s_j} = C_\tau(b_j + i - s_j).$$

Apply the lower inequality in (5.7) at the coordinate $b_j + i + s_j + 1$. Its prescribed gap index is $b_j + i + 1 > 0$, so

$$C_\tau(b_j + i) = \delta_{b_j + i} < C_v(b_j + i + s_j + 1).$$

Thus $C_v[j]\ R_{s_j}\ C_\tau[j]$ for every $j \le k$. Since $v \in M_\infty = M_\tau$ and $h_\tau(v) = f(v) = k$, the vertex v is eligible at (τ, k).

The chosen predecessor $u = p_k(\tau)$ therefore exists. Its eligibility gives $h_\tau(u) = k$, and (5.4) gives $f(u) = k = f(\tau)$. But $\{u, \tau\} \in E_k$, contradicting properness of f. Hence G has no countable coloring, and (4.5) yields $\chi(G) = \aleph_1$. □

Proof of Theorem 1.1. Proposition 6.1 gives $\chi(G) = \aleph_1$. The finite colorings in (4.6) satisfy the hypotheses of Proposition 2.1, so every H with $\mathrm{Age}(H) \subseteq \mathrm{Age}(G)$ has $\chi(H) \le \mathfrak{c}$. Such an H cannot have chromatic number $\mathfrak{c}^+$. This proves both assertions. □

Acknowledgment. The authors acknowledge assistance from OpenAI's ChatGPT in the development and checking of the argument and in preparing the exposition.

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