hexagon:2611.00744v3

# Half-line density, spectral generation and lattice structure of crystalline measures

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23 November 2026

**Abstract**

A crystalline measure in $\mathbb{R}^p$ is a finite sum of translated and modulated full-rank lattice Dirac combs if and only if its support and spectrum each have finite lower density in one origin-centred half-ball. The two directions and the two sequences of radii may be chosen independently. The weights may be complex and the lattices may differ. The main step bounds the number of sites in every unit ball from an ordered Fourier rank inequality and a count of exact additive triples. On the line, a finite lower support density $d$ on one ray is the minimum frequency width of an exponential polynomial vanishing on the support. A minimizer divides every such polynomial and has only real zeros. Every open spectral interval longer than $2d$ generates the spectral group, as does every half-open interval of length $2d$. Outside the comb class, each closed spectral interval of length $T$ contains at least $k(k+1)/2$ atoms, where $k = \lfloor T/d \rfloor$. For prescribed carriers of lower half-line densities $a, b$, the sharp dimension bound is $\lfloor ab \rfloor$.

## 1 Counting criteria and main results

Poisson summation pairs a lattice Dirac comb with a comb on the reciprocal lattice. The second Hexagon version of this paper asked whether lower counts in two independently chosen half-balls characterize finite sums of translates and modulations of these combs [20, p. 3]. The following theorem answers that question.

We use $e(t) = \exp(2\pi i t)$ and the Fourier kernel $e(-x \cdot \xi)$. A crystalline measure on $\mathbb{R}^p$ is a tempered distribution $\mu$ such that, on compactly supported smooth test functions,

$$\mu = \sum_{x \in A} u_x \delta_x, \qquad \widehat{\mu} = \sum_{\xi \in B} v_\xi \delta_\xi.$$

The support $A$ and spectrum $B$ meet compact sets in finitely many points. The displayed coefficients are nonzero, so each count records distinct sites after cancellation. The weights may be complex. Temperedness is assumed in the distributional sense. The total variations can grow faster than every polynomial.

**Theorem 1.1** (Independent half-ball counts)**.** *Let $\mu$ be crystalline on $\mathbb{R}^p$, $p \geq 1$. For a unit vector $u$ write*

$$H_u(R) = \{x \in \mathbb{R}^p : |x| \leq R,\ u \cdot x \geq 0\}, \qquad d_{u,*}(S) = \liminf_{R \to \infty} \frac{\#(S \cap H_u(R))}{\mathrm{vol}_p(H_u(R))}.$$

*There are unit vectors $\omega, \eta$ with $d_{\omega,*}(A) < \infty$ and $d_{\eta,*}(B) < \infty$ if and only if $\mu$ has the form*

$$\mu = \sum_{j=1}^{m} c_j e(s_j \cdot x) \sum_{h \in H_j} \delta_{a_j + h},$$

*where each $H_j$ is a full-rank lattice, $a_j, s_j \in \mathbb{R}^p$ and $c_j \in \mathbb{C}$. The two lower limits may be witnessed by independent sequences of radii. The lattices $H_j$ may differ.*

**A planar example.** Write $\delta_L = \sum_{x\in L} \delta_x$. For $L = (\sqrt{2}\,\mathbb{Z}) \times \mathbb{Z}$, let $\mu = \delta_{\mathbb{Z}^2} - \delta_L$. The common vertical axis cancels in $\mu$. Poisson summation gives $\widehat{\mu} = \delta_{\mathbb{Z}^2} - 2^{-1/2}\delta_{(\mathbb{Z}/\sqrt{2})\times\mathbb{Z}}$. Along a horizontal line the irrational ratio of spacings gives arbitrarily close distinct sites in each carrier. Each carrier lies in two lattices, so every half-ball contains $O((1+R)^2)$ sites.

**Theorem 1.2** (Several-lattice structure under uniform discreteness)**.** *Let $\mu$ be crystalline on $\mathbb{R}^p$ with uniformly discrete support and spectrum. Then $\mu$ has the representation in Theorem 1.1. In particular, each carrier is covered by finitely many translates of full-rank lattices.*

*Proof.* Around the sites of either carrier take disjoint balls of a fixed positive radius. Those with centres in $B(0,R)$ fit in a ball of radius $R+O(1)$, so their number is $O((1+R)^p)$. Apply Theorem 1.1. The cover of the spectrum follows by Poisson summation. □

In dimensions $p > 1$, the physical cover answers the question of Lev and Olevskii [16, §11.2(1)], also posed for measures by Lev and Reti [17, §6.2]. Theorem 1.2 appeared as Theorem A.1 of the second Hexagon version [20]. Section 7 gives its independent proof using Alon and Kummer's uniformly discrete results and Laurent's theorem.

**What one count gives on the line.** On $\mathbb{R}$, put $N_S(R) = \#(S \cap [-R,R])$ for a locally finite set $S$, and define

$$d_+(S) = \liminf_{R\to\infty} \frac{\#(S\cap[0,R])}{R}, \qquad d_-(S) = \liminf_{R\to\infty} \frac{\#(S\cap[-R,0])}{R}.$$

Set $d_*(S) = \min\{d_+(S), d_-(S)\}$. The upper uniform density is $D^+(S) = \limsup_{T\to\infty} \sup_{x\in\mathbb{R}} \#(S\cap[x,x+T])/T$. A relatively separated set has uniformly bounded unit-interval counts. A uniformly discrete set has a positive minimum separation between distinct points. For an exponential polynomial $P(x) = \sum_s c_s e(sx)$ in reduced form, the frequencies are distinct real numbers and its width is the difference between its largest and smallest frequencies. Write $\langle E\rangle_{\mathbb{Z}}$ for the additive subgroup generated by $E$.

**Theorem 1.3** (Minimum width and spectral generation)**.** *Let $\mu \neq 0$ be crystalline on $\mathbb{R}$, and suppose $d = d_*(A) < \infty$. Then $d > 0$, $A$ is relatively separated, and the counting ratios on both rays converge to $d = D^+(A)$. Write $\Gamma = \langle B\rangle_{\mathbb{Z}}$.*

1. *Every bounded open interval $I$ of length $> 2d$ satisfies $\langle B\cap I\rangle_{\mathbb{Z}} = \Gamma$. The same equality holds for every closed or half-open interval of length $2d$. Thus $\Gamma$ is finitely generated. The integer comb gives sharpness for each endpoint convention.*

2. *There is a nonzero exponential polynomial $Q$ with frequencies in $\Gamma$, vanishing on $A$, of width exactly $d$. It divides every real-frequency exponential polynomial that vanishes on $A$. The minimum-width annihilator is unique up to multiplication by a nonzero constant and a character, and all its complex zeros are real.*

*In particular $d \in \Gamma$, and $\#(A\cap[x,x+T]) \le dT + C_Q$ for every $x\in\mathbb{R}$ and $T > 0$.*

Following Meyer [24], a generalized Dirac comb on $\mathbb{R}$ is a finite linear combination of measures $e(sx)\sum_{n\in\mathbb{Z}}\delta_{a+hn}$, with $h > 0$ and $a, s \in \mathbb{R}$. The spacings can vary between summands. For example, the sum of the combs on $\mathbb{Z}$ and $\sqrt{2}\,\mathbb{Z}$ has arbitrarily close atoms on both sides of the Fourier transform.

The earlier version identified $d$ as an infimum of widths [20, Theorem 1.1]. Proposition 2.5 attains that infimum. The extremal frequencies of a minimizer give a difference equation of width exactly $d$ for the spectral masses. This exact width yields the following lower bound even when $T$ is an integer multiple of $d$.

**Theorem 1.4** (Spectral count on every closed interval)**.** *Let $\mu \neq 0$ be crystalline on $\mathbb{R}$ with $d = d_*(A) < \infty$. If $\mu$ is not a generalized Dirac comb, then for every $x \in \mathbb{R}$, $T > 0$, and $k = \lfloor T/d \rfloor$,*

$$\#(B \cap [x, x+T]) \geq \frac{k(k+1)}{2}. \tag{1.1}$$

*Consequently*

$$\liminf_{T\to\infty} \inf_{x\in\mathbb{R}} \frac{\#(B \cap [x, x+T])}{T^2} \geq \frac{1}{2d^2}, \qquad \liminf_{R\to\infty} \frac{N_B(R)}{R^2} \geq \frac{2}{d^2}.$$

*The same conclusions hold with physical and Fourier sides exchanged.*

**Corollary 1.5** (Half-line counting criterion)**.** *A crystalline measure on $\mathbb{R}$ is a generalized Dirac comb if and only if $d_*(A)$ and $d_*(B)$ are both finite. The choices of rays and of subsequences in the two lower limits are independent.*

The finite-interval inequality explains what happens when the second count fails: every sufficiently long interval contains quadratically many spectral atoms. The Kurasov–Sarnak examples [10] show that the exponent is optimal. Section 5 also bounds the optimal centered coefficient between 2 and 4 after multiplication by $d^2$. Section 6 applies the gap argument to two prescribed carriers, obtaining a sharp dimension bound and, for a common carrier, an attained finite-hole formula.

**Earlier results.** Lev and Olevskii proved one-dimensional rigidity for uniformly discrete supports and spectra [15]. Their subsequent theorem allows bounded unit-interval spectral counts when the physical support is uniformly discrete [16, Theorem 2.2]. Baake, Strungaru and Terauds obtained a common-lattice representation for support in a Meyer set and a sparse Fourier spectrum along a van Hove sequence [4, Theorem 6.6]. Corollary 1.5 allows arbitrarily close sites on both sides. Gonçalves and Vedana parameterize summation formulas subject to weighted growth conditions [8]. The counting conditions above concern the locations of nonzero atoms and are independent of their sizes.

Meyer constructed convolution annihilators for the spectrum under uniform discreteness, and also treated finite unions of uniformly discrete sets [25, Theorem 6 and discussion following Lemma 15]. Here the Fourier transform of $Q$ in Theorem 1.3 is a finite sum of point masses in an interval of minimum length $d$. Olevskii and Ulanovskii identify unit-mass Fourier quasicrystals on the line with zero sets of appropriate exponential polynomials [26]. For arbitrary complex weights our conclusion is containment, which can be strict. Alon and Kummer's results on uniformly discrete crystalline measures include finite generation and algebraic vanishing for arbitrary weights, as well as a stronger Lee–Yang characterization for unit weights [3, Theorems 3.1 and 4.2]. Section 2 proves the one-dimensional assertions from a half-line count. Section 7 uses the higher-dimensional theorems of Alon and Kummer explicitly.

**Proof outline.** The physical half-ball count gives a rank bound for ordered frequency patches (Appendix A). Along the independently chosen spectral sequence, exact additive triples force a uniform local count in the measured half-space. Recurrence moves any finite cluster into that half-space. Applying the argument in the other Fourier orientation gives the corresponding count for the physical support. Appendix B then obtains finite lattice covers on both sides and recovers the complex weights.

On the line a Fourier gap detects vanishing from a single ray. Annihilators of width approaching $d$ yield spectral generation from a bounded interval. The vanishing ideal in the resulting Laurent

ring supplies a generator of width $d$. The real support sites already account for $dR + o(R)$ zeros on $[0, R]$. A nonreal zero would recur in a positive linear number of disjoint translated disks, exceeding the strip zero bound. Sections 3–5 use the difference equation to prove the spectral count, and Section 6 treats prescribed carriers. Section 8 states the remaining subquadratic conjecture and records the development of these results.

## 2 Half-line density and a minimum-width annihilator

We first turn a gap in frequency into a count of physical atoms. This gives a uniqueness interval for exponential multiples of $\mu$. A finite linear system on such an interval will produce an annihilator.

**Lemma 2.1** (A gap forces atoms on each ray)**.** *Let $T \neq 0$ be a tempered distribution that is locally a sum of point masses on a locally finite set $A \subset \mathbb{R}$. If $\widehat{T}$ vanishes on a bounded open interval $I$, then $d_+(A) \geq |I|$ and $d_-(A) \geq |I|$.*

*Proof.* Modulate $T$ to center the gap at the origin, writing it as $(-h, h)$. After multiplication by a constant phase, its real part $U$ is nonzero. Its Fourier transform retains the symmetric gap, since taking the real part averages a transform with its conjugate reflection. All atoms of $U$ lie in $A$.

We construct a real continuous function $f$ of polynomial growth with $D^m f = U$ for some $m \geq 2$ and with the same spectral gap. Temperedness gives integers $p, q \geq 0$ such that, on real Schwartz functions,

$$|U(\varphi)| \leq C \max_{0 \leq j \leq q} \sup_t (1 + t^2)^p |\varphi^{(j)}(t)|.$$

Apply Hahn–Banach to the map $\varphi \mapsto ((1 + t^2)^p \varphi^{(j)})_{j=0}^q$ into a finite product of copies of $C_0(\mathbb{R})$. By the Riesz representation theorem, $U = \sum_{j=0}^q D^j \nu_j$ with real measures satisfying $\int (1 + t^2)^{-p} \, d|\nu_j|(t) < \infty$. The cumulative mass of $\nu_j$ from the origin gives a locally bounded primitive of polynomial growth. One more integration makes it continuous. Taking $m = q + 2$ and integrating each $\nu_j$ to order $m - j$ therefore gives a continuous $F$ of polynomial growth with $D^m F = U$. On $(-h, h)$ the restriction of $\widehat{F}$ is killed by $\xi^m$, so it is a linear combination of $\delta_0, \dots, \delta_0^{(m-1)}$. Subtract from $F$ the inverse transform of this combination. Conjugate symmetry makes the subtracted polynomial real, and its degree is less than $m$. The resulting $f$ has the required properties.

Let $s_R(f)$ be the largest number of successive sign changes observed at ordered points in $(0, R)$ where $f$ is nonzero. On every bounded interval, $f$ is a spline. Writing $u_a$ for the coefficient of $U$ at $a$, we can express $f$ there as a polynomial of degree at most $m - 1$ plus a finite sum of terms $u_a (x - a)_+^{m-1} / (m - 1)!$. This follows by taking $m$ distributional derivatives. Thus its sign changes agree with the count used by Eremenko and Novikov, the smallest degree of a nonzero real polynomial $p_0$ such that $p_0 f \geq 0$ on $(0, R)$. Each transition requires a root and one root at each transition suffices. The oscillation theorem [5, Theorem 1] gives $\liminf_{R \to \infty} s_R(f)/R \geq 2h$, since the angular-frequency half-width is $2\pi h$. Choose $b \geq 0$ with $|f(t)| = O((1 + |t|)^b)$. The weight $\omega(t) = (b + 2) \log(2 + |t|)$ is uniformly continuous and both $\int \omega(t)/(1 + t^2) \, dt$ and $\int |f(t)| e^{-\omega(t)} \, dt$ are finite.

For an absolutely continuous function, alternating signs at $k + 1$ points give $k$ successive increments with alternating signs. On each intervening interval its derivative has the sign of that increment on a set of positive measure. Hence differentiation loses at most one of these changes. The derivatives of our spline through order $m - 2$ are locally absolutely continuous. Its derivative of order $m - 1$ is a step function with jumps only at atoms of $U$. Choosing the last testing points

away from the jumps gives

$$s_R(f) \le \#(A \cap [0,R]) + m - 1.$$

Division by $R$ proves the positive-ray estimate. Apply it to the reflection to obtain the negative-ray estimate. □

For the integer comb, a gap between consecutive spectral sites has length one and both ray densities equal one. For a related bound on a centered integrated counting density, see Lev and Reti [17, Theorem 3.1].

Fix a nonzero crystalline measure $\mu$ with physical support $A$, spectrum $B$, and $d = d_*(A) < \infty$. Lemma 2.1 gives the detection rule

$$|J| > d, \qquad \widehat{P\mu}\big|_J = 0 \quad \Longrightarrow \quad P\mu = 0 \tag{2.1}$$

for every exponential polynomial $P$ and bounded open interval $J$. Indeed, $P\mu$ is tempered and has physical atom set contained in $A$.

**Proposition 2.2** (Annihilators of width arbitrarily close to $d$)**.** *For every $\ell > d$ there is a nonzero exponential polynomial of width less than $\ell$ vanishing on $A$.*

*Proof.* Choose $J, L > 0$ with $d < 2J < 2L < \ell$. Of the spectral sites, only the finite set $B_0 = B \cap [-J-L, J+L]$ can enter $[-J,J]$ under shifts in $[-L,L]$. Take a $\mathbb{Z}$-basis $\beta_1, \dots, \beta_r$ of $\langle B_0 \rangle_{\mathbb{Z}}$ and choose $0 < \varepsilon < L - J$ outside its rational span. For $c > 0$ let

$$E_c(N) = \{\beta \cdot n + \varepsilon k : n \in \mathbb{Z}^r,\ \|n\|_\infty \le N,\ k \in \mathbb{Z},\ |\beta \cdot n + \varepsilon k| \le c\}.$$

The coordinates are unique. For each $n$ the admissible $k$ lie in an interval of length $2c/\varepsilon$, and therefore

$$(2N+1)^r(2c/\varepsilon - 1) \le |E_c(N)| \le (2N+1)^r(2c/\varepsilon + 1).$$

These bounds include $r = 0$. Let $H \ge 0$ bound the absolute values of the integer coordinates of $B_0$, taking $H = 0$ for the trivial group. Then $(B + E_L(N)) \cap [-J,J] \subset E_J(N+H)$. Since $2L/\varepsilon - 1 > 2J/\varepsilon + 1$, this set has fewer elements than $E_L(N)$ for all sufficiently large $N$.

Write $v(y) = \widehat{\mu}(\{y\})$, extended by zero off $B$. For such an $N$ the homogeneous system

$$\sum_{s \in E_L(N)} c_s v(y - s) = 0, \qquad y \in (B + E_L(N)) \cap [-J,J],$$

has a nonzero solution. The characters of distinct frequencies are linearly independent, so $P(x) = \sum_s c_s \mathrm{e}(sx)$ is nonzero. The system makes $\widehat{P\mu}$ vanish on $(-J,J)$, and (2.1) gives $P\mu = 0$. Every site of $A$ has nonzero mass, so $P$ vanishes on $A$. Its width is at most $2L < \ell$. □

**Lemma 2.3** (Zero count in a vertical strip)**.** *For a nonzero exponential polynomial $P$ of width $w$, the number of complex zeros, counted with multiplicity, whose real parts lie in any interval of length $T$ is at most $wT + C_P$. The constant is independent of the interval.*

*Proof.* We give the real-frequency case of Langer's zero count [11]. Write $P(z) = \sum_{j=1}^k c_j \mathrm{e}(s_j z)$ with $s_1 < \cdots < s_k$ and $c_j \ne 0$. For $\operatorname{Im} z \ge H$, with $H$ fixed sufficiently large, the first term dominates the sum of the others in modulus. For $\operatorname{Im} z \le -H$ the last term dominates. All zeros lie between these horizontal lines. Along segments of length $T$ directed to the right at heights $H$ and $-H$ the changes of argument are respectively $2\pi s_1 T + O(1)$ and $2\pi s_k T + O(1)$. The quotients by the dominant terms stay in a disk centered at 1 of radius strictly less than one.

On a zero-free vertical edge, rotate by a constant phase so the imaginary part is not identically zero. It is a real linear combination of the $k$ functions $\exp(-2\pi s_j y)$ of the height $y$ and has at most $k-1$ distinct zeros. This follows by division by the first exponential and induction using Rolle's theorem. Between consecutive zeros the rotated values stay in a half-plane, giving a uniform bound on the change of argument along the edge. Move both ends of the original real interval outward by less than one so that the new vertical edges avoid zeros. The argument principle on the resulting rectangle gives the asserted bound. □

It follows from Proposition 2.2 and Lemma 2.3 that $A$ is relatively separated and $D^+(A) \le d$. The upper counting ratio on each ray is at most $D^+(A)$, while the lower ratio on each ray is at least $d$. Both limits thus exist and equal $D^+(A) = d$. Applying Lemma 2.1 in any nonempty open gap of the locally finite set $B$ gives $d > 0$. The zero count also shows that every nonzero annihilator has width at least $d$. We next identify an intrinsic frequency group for annihilators whose widths approach $d$.

**Proposition 2.4** (A sharp window for spectral generation)**.** *For every bounded open interval $I$ of length greater than $2d$,*

$$\langle B \cap I \rangle_{\mathbb{Z}} = \langle B \rangle_{\mathbb{Z}} =: \Gamma.$$

*The same equality holds for every closed or half-open interval of length $2d$. For every $\ell > d$ a nonzero annihilator of width less than $\ell$ can be chosen with all frequencies in $\Gamma$.*

*Proof.* Put $G = \langle B \cap I \rangle_{\mathbb{Z}}$. By Proposition 2.2 choose an annihilator of width $W < \min\{\ell, |I| - d\}$. Multiply it by a character so that its frequencies lie in $[-W/2, W/2]$. Let $J$ be the concentric open interval of length $|I| - W$. At $y \in J$, every contributing spectral site $y - s$ lies in $I$, hence in $G$. Partition the terms of the annihilator by cosets of $G$ containing their frequencies. In $J$, the Fourier transforms of these blocks times $\mu$ have their atoms in disjoint cosets. Their sum is zero, and thus each vanishes there. Each block times $\mu$ vanishes globally by (2.1), since $|J| > d$.

Keep one nonempty block and subtract its smallest frequency by modulation. Its frequencies are in $G$. Write them as $0 = s_0 < \cdots < s_k = w \le W$, where $k \ge 1$ since a single character has no zeros. The function $v(y) = \widehat{\mu}(\{y\})$ satisfies $\sum_{j=0}^{k} c_j v(y - s_j) = 0$, where $c_0 c_k \ne 0$. From a point $b$ where $v(b) \ne 0$, the equation at $b$ gives a backward step of size $s_j > 0$ to another such point. The equation at $b + w$ gives a forward step of size $w - s_j > 0$. All these step sizes belong to $G$, are at most $w$, and have a positive minimum. Choose a closed interval of length $w$ inside $I$. Repeatedly stepping towards it from any $b \in B$ reaches a site of $B \cap I$, since a step cannot jump across the interval. Hence every $b \in B$ belongs to $G$.

For $I = [x, x + 2d)$, local finiteness gives $\varepsilon > 0$ with $B \cap (x - \varepsilon, x) = \varnothing$. The open interval $(x - \varepsilon, x + 2d)$ has length greater than $2d$ and contains exactly the same spectral sites as $I$. For $I = (x, x + 2d]$ extend the right endpoint in the same way. A closed interval of length $2d$ contains either half-open interval, so it too generates $\Gamma$. □

For the integer comb, $d = 1$, $B = \mathbb{Z}$, and $B \cap (-1, 1) = \{0\}$. Thus an open interval can fail at equality. For every $0 < T < 2$, the closed interval $[-T/2, T/2]$ also contains only zero, proving sharpness for all endpoint conventions.

**Proposition 2.5** (The minimizing generator)**.** *There is an exponential polynomial $P_*$ with frequencies in $\Gamma$, of width $d$, whose complex zeros are all real and whose real zero set contains $A$. Every exponential polynomial vanishing on $A$ is divisible by $P_*$ in the ring of exponential polynomials with real frequencies. The minimizer is unique up to multiplication by a nonzero scalar and a character.*

*Proof.* By Proposition 2.4 the group $\Gamma$ is finitely generated. It has positive rank, since gaps in $B$ have length at most $d$ by Lemma 2.1. Choose a $\mathbb{Z}$-basis $\beta_1, \dots, \beta_r$, and write $\phi(z) = (\mathrm{e}(\beta_1 z), \dots, \mathrm{e}(\beta_r z))$. Substitution of $\phi$ identifies the Laurent ring $\mathbb{C}[z_1^{\pm 1}, \dots, z_r^{\pm 1}]$ with the ring of $\Gamma$-frequency exponential polynomials. Indeed, different exponent vectors give different real frequencies. Let $X$ be the Zariski closure of $\phi(A)$ in $(\mathbb{C}^\times)^r$. By Proposition 2.4 a nonzero Laurent polynomial vanishes on $X$, so $\dim X \le r - 1$.

Each irreducible component has dimension $r - 1$. Component isolation and the counting of characters in frequency windows also occur in Alon–Kummer [3, §4.2]. Here the lower half-line count gives the required independence on each component. Fix a component $Y$ and choose a Laurent polynomial $g$ vanishing on every other component but not identically on $Y$. To obtain $g$, take a product of one such separating factor for each other component, using irreducibility of $Y$. Take the empty product $g = 1$ when $Y = X$. Zariski density of $\phi(A)$ in $X$ implies that $\nu = (g \circ \phi)\mu$ is nonzero. Its support $A'$ maps into $Y$. Its spectrum lies in finitely many translates of $B$ and is locally finite. Applying Lemma 2.1 in a spectral gap gives $d_*(A') > 0$.

For $n \in [-N, N]^r \cap \mathbb{Z}^r$, the $(2N+1)^r$ distinct frequencies $n \cdot \beta$ lie in a real interval of length $O(N)$. Partition the range into intervals of a fixed length $h$ with $0 < h < d_*(A')$. One interval contains at least $cN^{r-1}$ frequencies, with $c > 0$ independent of $N$. The corresponding Laurent monomials are linearly independent on $Y$ because a nontrivial relation would give a nonzero exponential polynomial of width at most $h$ vanishing on $A'$, contrary to Lemma 2.3. On the other hand all Laurent monomials in this box span a space of dimension $O(N^{\dim Y})$ on $Y$. Multiplication by $(z_1 \cdots z_r)^N$ embeds their restrictions in the space of ordinary polynomials of degree at most $2rN$ restricted to the affine closure of $Y$ in $\mathbb{C}^r$. The degree-filtration Hilbert function of an affine variety of dimension $t$ is $O(N^t)$ (equivalently, homogenize and use its projective Hilbert polynomial). The affine closure has the same dimension as $Y$. These bounds give $\dim Y \ge r - 1$.

*The generator and its width.* The vanishing ideal of $X$ is the intersection of the prime ideals of its irreducible components. These primes have height one in a Laurent polynomial ring, a unique factorization domain. Each is generated by an irreducible element, and their intersection is generated by the product $Q$ of the distinct irreducible generators. Thus $P_* = Q \circ \phi$ divides every $\Gamma$-frequency annihilator. Widths add under multiplication since the least and greatest frequencies of a product each arise from a unique pair of extreme terms. The intrinsic annihilators in Proposition 2.4 therefore give $\mathrm{width}(P_*) \le d$. Lemma 2.3 gives the opposite bound.

For an annihilator with arbitrary real frequencies, group its terms by cosets of $\Gamma$. The Fourier transform of each block times $\mu$ has its atoms in its own coset, since $B \subset \Gamma$. The transforms are locally finite sums of point masses and their sum is zero. Each block thus annihilates $\mu$. After removal of one character, the block is a $\Gamma$-frequency annihilator and hence divisible by $P_*$. Adding the blocks gives divisibility in the full ring. If an annihilator has width $d$, its quotient by $P_*$ has width zero, and is a scalar multiple of a character.

*Location of the complex zeros.* Suppose $P_*$ has a nonreal zero. Choose a closed disk $D$ about it disjoint from the real axis, with $P_*$ nonzero on $\partial D$. There is a relatively dense set of real shifts $\tau$ for which

$$\sup_{\partial D} |P_*(z + \tau) - P_*(z)| < \min_{\partial D} |P_*(z)|.$$

Indeed, it suffices that all the finitely many phases $\mathrm{e}(s\tau)$ for frequencies $s$ of $P_*$ be sufficiently close to 1. The returns to an identity neighborhood are relatively dense by compactness of the closure of this one-parameter subgroup of the torus. Rouché's theorem then puts a nonreal zero in every $D + \tau$. From these disks with real projections in $[0, R]$ one can choose at least $cR$ pairwise disjoint disks for all large $R$, by taking one shift in each of regularly spaced intervals separated by more than the diameter of $D$. Here $c > 0$ is independent of $R$. The real sites of $A \cap [0, R]$ already

supply $dR + o(R)$ zeros. The additional nonreal zeros contradict the upper bound $dR + C_{P_*}$ of Lemma 2.3. □

Since $d$ is the difference of two frequencies of $P_*$, $d \in \Gamma$. Lemma 2.3 also gives the uniform estimate

$$\#(A \cap [x, x+T]) \le dT + C_{P_*} \qquad (x \in \mathbb{R},\ T > 0).$$

Together with Propositions 2.4 and 2.5 this completes the proof of Theorem 1.3.

The real-rooted conclusion concerns containment. For instance, if $\alpha$ is irrational, $(\mathrm{e}(\alpha x) - 1)\sum_{n\in\mathbb{Z}} \delta_n$ has support $\mathbb{Z} \setminus \{0\}$, whereas its width-one minimizer $\mathrm{e}(x) - 1$ also vanishes at zero. For the corresponding exact zero-set statements for unit-mass Fourier quasicrystals see Olevskii–Ulanovskii [26]. For the Lee–Yang representation of real-rooted exponential polynomials see Alon–Cohen–Vinzant [1].

## 3 Counting the support of a finite difference equation

Let $u : \mathbb{R} \to \mathbb{C}$ have locally finite support $S = \{t : u(t) \neq 0\}$ and satisfy

$$(Du)(t) = \sum_{j=0}^{s} a_j u(t - h_j) = 0 \quad (t \in \mathbb{R}), \qquad h_0 < \cdots < h_s, \qquad a_j \neq 0. \tag{3.1}$$

Write $W = h_s - h_0 > 0$. Translating the equation makes $h_0 = 0$ and $h_s = W$. For $t \in S$, the equations at $t$ and at $t + W$ supply two further points of $S$,

$$t - h_j \quad \text{for some } j > 0, \qquad t + W - h_i \quad \text{for some } i < s.$$

Each step has length at most $W$ and the finitely many possible positive step lengths have a positive minimum. Starting outside any closed interval of length $W$, successive steps toward it must enter it. In particular a solution of (3.1) is determined by its values on any closed interval of length exactly $W$. For an open interval of the same length this can fail: the indicator of $\mathbb{Z}$ satisfies $u(t) = u(t-1)$ and vanishes on $(0, 1)$.

Taking the interval $[0, W]$ shows that $S$ lies in the additive group $G$ generated by the shifts $h_j$ and the finite set $S \cap [0, W]$. Choose an isomorphism $n : G \to \mathbb{Z}^r$ and write $\ell : \mathbb{Z}^r \to G$ for its inverse. For $E = n(S)$, let $H_E(m)$ be the dimension of the restrictions to $E$ of polynomials in $r$ variables of total degree at most $m$.

**Lemma 3.1** (Restriction to a finite interval)**.** *For every integer $m \ge 0$ and every closed interval $J$ of length at least $(m+1)W$,*

$$H_E(m) \le \#(S \cap J).$$

*Proof.* Given a polynomial $p$, put $u_p(t) = p(n(t))u(t)$ on $G$ and set it equal to zero elsewhere. If $\deg p \le m$, then $D^{m+1}u_p = 0$. For $m = 0$ this follows from $Du = 0$. For the induction step, subtract $p(n(t))(Du)(t)$ from $(Du_p)(t)$ at $t \in G$. The coefficient of $a_j u(t - h_j)$ in the difference is

$$p(n(t - h_j)) - p(n(t - h_j) + n(h_j)).$$

It has degree at most $m - 1$ in $n(t - h_j)$. Thus $Du_p$ is a sum of translates of polynomial multiples of lower degree. Outside $G$ the identity has both sides zero. Translations commute with $D$, so induction gives the asserted equation.

The extreme shifts of $D^{m+1}$ are 0 and $(m+1)W$, with coefficients $a_0^{m+1}$ and $a_s^{m+1}$. The stepping argument therefore gives injectivity of restriction to $J$ on the space of these $u_p$. Since $u(t) \neq 0$ at every $t \in S$, this space has dimension $H_E(m)$, and its restrictions have at most $\#(S \cap J)$ coordinates. □

**Proposition 3.2** (Counting alternative)**.** *Under (3.1), either $S$ lies in a finite union of lattice translates, or every closed interval $J$ of length $T > 0$ satisfies*

$$\#(S \cap J) \geq \frac{k(k+1)}{2}, \qquad k = \lfloor T/W \rfloor.$$

*Proof.* We may assume $S \neq \varnothing$. Let $V \subset \mathbb{C}^r$ be the Zariski closure of $E$. If some irreducible component has dimension at least two, two of its coordinate functions are algebraically independent, since its coordinate functions generate its function field. The monomials in those two coordinates of total degree at most $m$ are linearly independent on $E$. Any relation on $E$ would persist on $V$ and on the chosen component. Hence $H_E(m) \geq (m+1)(m+2)/2$. If $k \geq 1$, take $m = k-1$ in Lemma 3.1. For $k = 0$ the estimate is immediate.

Suppose now that all components of $V$ have dimension at most one. For every $n_0 \in E$ the forward step gives $n_0 + v \in E$ for some

$$v \in F = \{n(W - h_j) : 0 \leq j < s\}, \qquad \ell(v) > 0.$$

It follows that $V \subset \bigcup_{v \in F}(V - v)$. Every irreducible curve component $Y$ is contained in $Y' - v$ for some component $Y'$ and $v \in F$. Then $Y'$ is a curve and $Y = Y' - v$, because a proper closed subset of an irreducible curve is finite. Following such translations through the finite set of components eventually revisits a component. That component is invariant under translation by a sum $v_*$ of steps in $F$ with $\ell(v_*) > 0$. If $z$ is a point on it, every polynomial vanishing on the component vanishes on $z + nv_*$ for all positive integers $n$, hence on $z + \mathbb{C}v_*$. The component equals this line. Translation back along the finite chain shows that $Y$ is a line as well.

A complex affine line containing two distinct integer points has a primitive integer direction. Its integer points are a translate of the cyclic group generated by that direction. A line with fewer integer points and a zero-dimensional component each contribute at most one point. Applying $\ell$ therefore covers $S$ by finitely many translates of lattices in $\mathbb{R}$. □

## 4 Recovering the weights on the line

Covering lattices with irrational ratios of spacings can approach one another arbitrarily closely. We first separate their coefficient sequences and then recover the complete measure from its behavior on a half-line.

**Proposition 4.1.** *Suppose $\mu$ is a crystalline measure on $\mathbb{R}$ and each of* $\operatorname{supp}\mu$ *and* $\operatorname{supp}\widehat{\mu}$ *lies in a finite union of lattice translates. Then $\mu$ is a generalized Dirac comb.*

*Proof.* Write $\nu = \widehat{\mu}$ and cover $\operatorname{supp}\mu$ by $a_j + h_j\mathbb{Z}$ with $h_j > 0$. The polynomial $\prod_j(\mathrm{e}(x/h_j) - \mathrm{e}(a_j/h_j))$ vanishes on $\operatorname{supp}\mu$. Fourier transformation gives an equation $Qv = 0$ for the coefficient function $v(y) = \nu(\{y\})$, where

$$Q = \prod_j(\tau_{t_j} - z_j), \qquad t_j = h_j^{-1}, \qquad z_j = \mathrm{e}(a_j/h_j), \qquad (\tau_t v)(y) = v(y - t).$$

Partition the lattices covering $\operatorname{supp}\nu$ into groups according to commensurability of their spacings. Assign the mass at each intersection to one of its groups and write $v = \sum_i v_i$. Each $v_i$ is supported on finitely many cosets of a lattice $h\mathbb{Z}$ depending on $i$. Finite translation sums preserve this property, so it holds for $Qv_i$ as well. Any two cosets from distinct groups intersect in at most one point, even after arbitrary translations. Since $\sum_i Qv_i = 0$, each $Qv_i$ is finitely supported. All operations to this point act on coefficient functions.

Fix a group. If $w$ is carried by finitely many cosets of $h\mathbb{Z}$, $t/h \notin \mathbb{Q}$, $z \neq 0$, and $(\tau_t - z)w$ is finitely supported, then $w$ is finitely supported. Indeed, one can choose $n \geq 1$ so that translating those cosets by $nt$ avoids every original coset. For each pair of cosets at most one integer $n$ is excluded. The factorization of $\tau_{nt} - z^n$ by $\tau_t - z$ shows that $(\tau_{nt} - z^n)w$ is finitely supported. Its two terms have disjoint supports, which proves the assertion. Apply this argument successively to remove from $Q$ all factors with $t_j/h \notin \mathbb{Q}$. At every step the remaining translation sum applied to $v_i$ still has support in finitely many $h\mathbb{Z}$-cosets.

Choose $h' > 0$ so that $h$ and all the remaining $t_j$ are integer multiples of $h'$. On each relevant coset $x_0 + h'\mathbb{Z}$ the sequence $v_i(x_0 + nh')$ obeys a constant-coefficient recurrence away from a finite set. With $T$ the backward shift on sequences, its operator is $\prod_j (T^{t_j/h'} - z_j)$ over the remaining indices. Its roots have modulus one. Consequently on the positive tail the sequence is a finite sum of $p(n)\lambda^n$, with $p$ a polynomial and $|\lambda| = 1$, by the elementary solution formula for linear recurrences. An empty product yields the zero tail.

Extend these expressions to all integers on every relevant coset and sum the resulting measures over the groups. The resulting measure $\sigma$ agrees with $\nu$ on a right half-line and has polynomially growing total variation. By Poisson summation, $\widehat{\sigma}$ is a finite sum of modulated derivatives of translated lattice combs. Multiplication by a polynomial differentiates the transformed comb, while the factor $\lambda^n$ translates it up to a constant phase. Thus $\widehat{\sigma}$ has locally finite support. So does $\widehat{\nu - \sigma} = \widehat{\nu} - \widehat{\sigma}$, and consequently it has an open spectral gap. The tempered measure $\nu - \sigma$ has only finitely many atoms on the positive half-line. Lemma 2.1 gives $\nu = \sigma$.

Finally, $\widehat{\nu}$ is the reflection of $\mu$ and consists of point masses. Expand the products of smooth exponentials with derivatives of Dirac masses in $\widehat{\sigma}$. At any given site the Dirac mass and its distinct positive-order derivatives are linearly independent, so all positive-order coefficients cancel. The order-zero contribution of each modulated comb derivative is a constant multiple of a modulated comb, since derivatives of a single exponential are scalar multiples of that exponential. Reflection proves the proposition. □

## 5 The spectral count and its coefficient

*Proof of Theorem 1.4.* Let $A = \operatorname{supp} \mu$, $B = \operatorname{supp} \widehat{\mu}$ and $d = d_*(A) < \infty$. By Theorem 1.3 the nonzero measure has $d > 0$, its support is relatively separated, and a polynomial of width $d$ vanishes on $A$. Fourier transformation of the product with $\mu$ gives (3.1) of width $d$ for $v(y) = \widehat{\mu}(\{y\})$.

If $B$ has a finite cover by $b_j + w_j\mathbb{Z}$, the nonzero polynomial $\prod_j (\mathrm{e}(\xi/w_j) - \mathrm{e}(b_j/w_j))$ annihilates $\widehat{\mu}$. Transforming this identity gives a finite difference equation of positive width for the coefficients of the reflected physical measure $\widehat{\widehat{\mu}}$. Relative separation of $A$ gives $\#((-A) \cap [-R, R]) = O(R)$. Proposition 3.2 forces a lattice cover of $-A$, and hence of $A$. Proposition 4.1 then gives the comb representation.

Consequently, if $\mu$ is not a generalized Dirac comb, $B$ has no finite lattice cover. Proposition 3.2 applied to $v$ now gives, for every $x \in \mathbb{R}$ and $T > 0$,

$$\#(B \cap [x, x + T]) \geq \frac{\lfloor T/d \rfloor (\lfloor T/d \rfloor + 1)}{2}.$$

At $T = kd$ with $k \geq 1$, the attained width $d$ gives the full lower bound $k(k+1)/2$ because Lemma 3.1 applies to the closed interval with $m = k-1$. For every $W > d$, the integer $\lfloor kd/W \rfloor$ is at most $k-1$. Dividing by $T^2$ and taking the infimum over $x$ yields the asymptotic lower coefficient $1/(2d^2)$. In particular each spectral half-line has this quadratic lower coefficient. Taking $T = 2R$ and $x = -R$ gives the centered coefficient $2/d^2$ in the normalization $N_B(R) = \#(B \cap [-R, R])$. Applying the argument to $\widehat{\mu}$ interchanges $A$ and $B$ up to reflection. □

*Proof of Corollary 1.5.* For a generalized Dirac comb, both supports have bounded counts per unit interval by Poisson summation. Conversely, a finite lower linear count on either spectral half-line excludes the quadratic alternative of Theorem 1.4 whenever $d_*(A) < \infty$. This proves the criterion. The zero measure is the empty comb sum. □

**Proposition 5.1** (Range of the optimal centered coefficient)**.** *Let $C_*$ be the infimum of*

$$d_*(A)^2 \liminf_{R\to\infty} \frac{N_B(R)}{R^2}$$

*over all crystalline measures on $\mathbb{R}$ which are not generalized Dirac combs and satisfy $d_*(A) < \infty$. Then $2 \le C_* \le 4$.*

*Proof.* The lower bound follows from Theorem 1.4. For the upper bound use the two-frequency family of Kurasov and Sarnak [10, Section 3, formulas (41)–(44) and Theorem 2]. In the Fourier convention of this paper put $u = \xi_1/(2\pi)$ and $v = 2\xi_2/(2\pi)$, where $\xi_1, \xi_2$ are their positive angular frequencies. For each irrational $u/v$ their measure has unit masses and is not a generalized Dirac comb. Their secular equation is

$$3\sin\bigl(\pi(u+v)x\bigr) + \sin\bigl(\pi(u-v)x\bigr) = 0.$$

Its real zero set is $A$. Its exponential polynomial has width $u+v$. At successive points $(j+\frac{1}{2})/(u+v)$ the first summand alternates between 3 and $-3$, and the second has absolute value at most 1. Every intervening open interval therefore contains a zero. Lemma 2.3 supplies the reverse asymptotic bound, giving $d_*(A) = u + v$.

Their summation formula puts $B$ in the two signed copies of $\{mu+nv : m, n \in \mathbb{Z}_{\ge 0}\}$. For fixed $u, v$, summing $\lfloor (R-mu)/v \rfloor + 1$ over $0 \le m \le \lfloor R/u \rfloor$ counts $R^2/(2uv) + O_{u,v}(R+1)$ pairs. Hence

$$\limsup_{R\to\infty} \frac{N_B(R)}{R^2} \le \frac{1}{uv}.$$

This counts all candidate sites, including any at which the actual coefficient vanishes. Taking $u/v$ to 1 through positive irrational values gives $C_* \le \lim (u+v)^2/(uv) = 4$. □

This family and Theorem 1.4 show that quadratic growth is attained in order. Determining the exact value of $C_*$ remains open.

**Two subquadratic counts.** Suppose $N_A(R) + N_B(R) = o(R^2)$. If any one of the four lower half-line densities is finite, Theorem 1.4 in the appropriate Fourier orientation gives the comb representation. Thus a noncomb example with these two subquadratic counts would have to satisfy, for both $S = A$ and $S = B$,

$$\frac{\#(S \cap [0,R])}{R} \longrightarrow \infty, \qquad \frac{\#(S \cap [-R,0])}{R} \longrightarrow \infty.$$

The subquadratic conjecture in Section 8 (Conjecture 8.1) asserts that no such example exists.

## 6 Prescribed carriers and the dimension of the measure space

For locally finite sets $\Lambda, \Sigma \subset \mathbb{R}$, put

$$\mathcal{M}(\Lambda, \Sigma) = \{\mu : \mu \text{ is crystalline, } \operatorname{supp}\mu \subset \Lambda, \ \operatorname{supp}\widehat{\mu} \subset \Sigma\}, \qquad \mathcal{M}_\Lambda = \mathcal{M}(\Lambda, \Lambda).$$

The first argument is the physical carrier. A carrier can include inactive sites. Meyer [24, §6] asked when $\mathcal{M}_\Lambda$ is nonzero and how its dimension depends on $\Lambda$. The bound below needs only the gap estimate. The description of the space on a common carrier also uses the comb structure.

**Theorem 6.1** (Two-carrier dimension bound)**.** *If $a = d_*(\Lambda) < \infty$ and $b = d_*(\Sigma) < \infty$, then every member of $\mathcal{M}(\Lambda, \Sigma)$ is a generalized Dirac comb, and*

$$\dim_{\mathbb{C}} \mathcal{M}(\Lambda, \Sigma) \le \lfloor ab \rfloor.$$

*The two densities may be witnessed on different rays. For every prescribed pair $a, b > 0$, there are carriers with these densities for which equality holds.*

*Proof.* The actual supports inherit the ray bounds, so Corollary 1.5 gives the representation. For every bounded open interval $I$ of length $T > a$ the map

$$\mu \longmapsto \big(\widehat{\mu}(\{y\})\big)_{y \in \Sigma \cap I}$$

is injective. A vector in its kernel has a spectral gap of length $T$, whereas its physical support has lower ray density at most $a$; Lemma 2.1 forces it to vanish. Local finiteness of $\Sigma$ therefore gives $k = \dim_{\mathbb{C}} \mathcal{M}(\Lambda, \Sigma) < \infty$, and every such interval contains at least $k$ points of $\Sigma$. Each of the intervals $[0, R]$ and $[-R, 0]$ contains $\lfloor R/T \rfloor$ pairwise disjoint open intervals of length $T$. Thus $b \ge k/T$. Letting $T \downarrow a$ yields the stated integer bound, including when $a = 0$ or $b = 0$.

For sharpness, fix $a, b > 0$ and put $q = \lfloor ab \rfloor$ and $h = 1/a$. If $q = 0$, take $\Lambda = h\mathbb{Z}$ and $\Sigma = b^{-1}\mathbb{Z}$. The bound just proved gives dimension zero. For $q \ge 1$, take $\Lambda = h\mathbb{Z}$ and first use $\Sigma_0 = (qh)^{-1}\mathbb{Z}$. A measure in $\mathcal{M}(\Lambda, \Sigma_0)$ has period $qh$, since $\mathrm{e}(qh\,\xi)$ fixes its Fourier transform. Its masses on $h\mathbb{Z}$ are determined by $q$ successive values. Conversely, every $q$-periodic coefficient pattern gives a measure in this space by Poisson summation. Thus its dimension is $q$.

The density of $\Sigma_0$ on each ray is $qh = q/a$. If $b = q/a$, take $\Sigma = \Sigma_0$. If $r = b - q/a > 0$, take $\Sigma = \Sigma_0 \cup (t + r^{-1}\mathbb{Z})$ with $t \notin \Sigma_0 - r^{-1}\mathbb{Z}$. The excluded set is countable, and the two disjoint cosets contribute densities $q/a$ and $r$ on each ray. In either case $d_*(\Sigma) = b$. The enlarged space contains the original $q$ independent measures, and the upper bound gives equality. □

**Theorem 6.2** (An attained finite-hole formula)**.** *Let $d = d_*(\Lambda) < \infty$. Consider all finite unions $L$ of full lattice translates in $\mathbb{R}$ such that $F_L = L \setminus \Lambda$ is finite. Include $L = \varnothing$, with $\mathcal{M}_\varnothing = 0$. Define, on $\mathcal{M}_L$,*

$$E_{F_L}\mu = \big(\mu(\{x\}), \widehat{\mu}(\{x\})\big)_{x \in F_L}.$$

*There is one such $L$ containing the supports and spectra of all members of $\mathcal{M}_\Lambda$. For this $L$, $\mathcal{M}_\Lambda = \ker E_{F_L}$. In particular*

$$\dim_{\mathbb{C}} \mathcal{M}_\Lambda = \max_{L:\, \#(L \setminus \Lambda) < \infty} \big(\dim_{\mathbb{C}} \mathcal{M}_L - \operatorname{rank} E_{F_L}\big) \le \lfloor d^2 \rfloor.$$

*Proof.* Theorem 6.1 gives finite dimension, both for $\mathcal{M}_\Lambda$ and for each $\mathcal{M}_L$. We first show that the actual support of a generalized comb is a finite union of full lattice translates with finitely many points removed. Group summands according to whether their spacings have rational ratio.

Within each group take a common positive period that is an integer multiple of every spacing. Split the lattices into cosets of this period. On each coset the coefficients have the form $\sum_j c_j \lambda_j^n$ with $|\lambda_j| = 1$.

The Skolem–Mahler–Lech theorem [14] describes the zeros of this recurrence sequence on a half-line as a finite union of arithmetic progressions together with a finite set. A progression of zeros on a tail extends to the entire two-sided progression because the recurrence for the subsequence is invertible. Applying the theorem in both directions and taking a common period for the zero progressions leaves a finite union of full subprogressions, with finitely many further deletions. Cosets from distinct groups meet at only finitely many sites in total, so further cancellations remove only finitely many points. Poisson summation gives the same description for the spectrum.

Take a basis of $\mathcal{M}_\Lambda$ and, for the support and spectrum of each basis vector, form the full lattice union before its finite deletions. Let $L$ be the union of these finitely many sets. It carries every basis vector on both sides and $L \setminus \Lambda$ is finite. Thus $\mathcal{M}_\Lambda \subset \mathcal{M}_L$, and the kernel on $\mathcal{M}_L$ consists exactly of the measures carried on both sides by $\Lambda$. For any other eligible $L'$, that kernel is a subspace of $\mathcal{M}_\Lambda$. Rank–nullity now gives the maximum and its attainment. For the zero space one may take $L = \varnothing$. □

The formula is a finite linear system once an attaining $L$ is known. Its proof selects $L$ from a basis of the unknown space, so a construction of $L$ from an arbitrary description of $\Lambda$ is a separate question. The next two examples distinguish the effect of a missing point from that of unused carrier sites.

**Example 6.3** (One missing point, two independent conditions)**.** For $L = q^{-1/2}\mathbb{Z}$, Theorem 6.1 gives $\dim_{\mathbb{C}} \mathcal{M}_L = q = d_*(L)^2$ by taking $h = q^{-1/2}$ in its equality example. Take $q = 3$ and $\Lambda = L \setminus \{0\}$. Write $c_0, c_1, c_2$ for the three repeated coefficients. The physical mass at zero is $c_0$; the Fourier mass there is $(c_0 + c_1 + c_2)/\sqrt{3}$. These independent conditions give $\dim_{\mathbb{C}} \mathcal{M}_\Lambda = 1$. Its generator has repeated weights $(0, 1, -1)$, the character modulo three used by Meyer [24, Lemma 4.2].

**Example 6.4** (A carrier with unbounded counts on one side)**.** Let $\Lambda = \mathbb{Z} \cup \{-\sqrt{n} : n = 1, 2, \ldots\}$. The positive ray has density 1, while $[-m-1, -m]$ contains $2m+2$ sites for every positive integer $m$. The integer comb belongs to $\mathcal{M}_\Lambda$, and the dimension bound gives $\mathcal{M}_\Lambda = \mathbb{C} \sum_{n\in\mathbb{Z}} \delta_n$. Here $L = \mathbb{Z}$ attains the maximum with no holes.

Meyer also asks about the shifted square-root family $\{\pm\sqrt{n+\theta} : n = 0, 1, \ldots\}$, $0 < \theta < 1$. Its ray counts are quadratic, beyond the hypotheses above. For this indexing, the separate Martin–Shen classification gives dimension 2 for $0 < \theta \le 1/8$, 1 for $1/8 < \theta \le 3/8$, and 0 for $3/8 < \theta < 1$ [23, 19].

## 7 Uniformly discrete support and spectrum in Euclidean space

Uniform discreteness gives the half-ball bounds in Theorem 1.1 by a packing estimate. Disjoint balls of a fixed positive radius around the sites in $B(0, R)$ lie in a ball of radius $R + O(1)$. Here we give the independent proof of Theorem 1.2 from the earlier manuscript [20]. We first pull an algebraic cover of the torus image back to physical space and eliminate its lower-dimensional pieces. We then recover the weights by averaging over the dual lattices. In this section $p$ denotes ambient dimension.

For positive or positive-definite measures, Lev and Olevskii obtained a common lattice [16, Theorem 1.1]. Favorov's signed example shows why several lattices may be needed [6]. His later comb theorem permits a spectrum of bounded density under a uniform positive lower bound on the

magnitudes of the physical coefficients [7, Theorem 10]. Here the coefficients may approach zero and the spectrum is uniformly discrete.

**Lemma 7.1** (Lattice covers from the torus image)**.** *Let $\mu \neq 0$ be crystalline on $\mathbb{R}^p$. If its support $A$ and spectrum $B$ are uniformly discrete, then each is contained in a finite union of translates of full-rank lattices.*

*Proof.* By Alon and Kummer's finite-generation theorem [3, Theorem 3.1], applied to $\mu$ and to $\nu = \widehat{\mu}$, the additive groups generated by $B$ and by $A$ are both finitely generated. Both measures are Bohr almost periodic by the same theorem. A nonzero Bohr almost periodic measure has relatively dense support. Indeed, convolve it with a compactly supported smooth function to obtain a function with a nonzero value at some point. All sufficiently good almost periods repeat a value bounded away from zero and hence locate the support within a fixed distance. In particular $B$ spans $\mathbb{R}^p$ over $\mathbb{R}$.

Take as the rows of $L \in \mathbb{R}^{n\times p}$ a $\mathbb{Z}$-basis of $\langle B\rangle_{\mathbb{Z}}$, with the first $p$ rows linearly independent over $\mathbb{R}$. Set

$$\phi(x) = \big(\mathrm{e}((Lx)_1), \dots, \mathrm{e}((Lx)_n)\big) \in (\mathbb{C}^\times)^n, \qquad X = \overline{\phi(A)}^{\mathrm{Zar}}.$$

The first assertion of [3, Theorem 4.2] bounds the dimension of each irreducible component of $X$ by $n-p$. This assertion applies to arbitrary complex weights. The group $\langle \phi(A)\rangle = \phi(\langle A\rangle_{\mathbb{Z}})$ is finitely generated. Laurent's theorem on finitely generated subgroups of algebraic tori [12] therefore covers $\phi(A)$ by finitely many translates of algebraic subtori contained in $X$. In a related use of torus intersections, Kurasov and Sarnak applied Liardet's theorem in a two-dimensional torus to analyze an explicit example [10, §3].

Consider a coset in this finite cover meeting $\phi(A)$, and choose $x_0 \in A$ mapping into it. Let the rows of an integer $k \times n$ matrix $C$ be a basis of the lattice of characters trivial on its subtorus. The codimension of the subtorus is $k \geq p$. With $M = CL$, the inverse image of the coset is

$$x_0 + \{v \in \mathbb{R}^p : Mv \in \mathbb{Z}^k\}.$$

The rows of $M$ are independent over $\mathbb{Z}$, since the rows of $C$ and of $L$ are each independent over $\mathbb{Z}$. If $M$ has real rank $p$, select $p$ independent rows to form a square matrix $M_0$. The inverse image lies in $x_0 + M_0^{-1}\mathbb{Z}^p$. If $M$ has real rank $t < p$, its image $W$ in $\mathbb{R}^k$ has dimension $t < k$. If $W \cap \mathbb{Z}^k$ had rank $t$, it would span $W$. A nonzero integer row would then annihilate $W$, contradicting the row independence of $M$. Thus $\operatorname{rank}(W \cap \mathbb{Z}^k) < t$. Lifting a basis of $W \cap \mathbb{Z}^k$ through $M$ and adding $\ker M$ shows that the span of $M^{-1}(\mathbb{Z}^k)$ has dimension at most $p - t + \operatorname{rank}(W \cap \mathbb{Z}^k) < p$. These cosets contribute only proper affine hyperplanes to the cover.

We have covered $A$ by lattice cosets $a_i + H_i$ and finitely many proper affine hyperplanes. Suppose $x_* \in A$ lies outside the lattice cosets. Write $H^* = \{t \in \mathbb{R}^p : t \cdot H \subset \mathbb{Z}\}$ for a dual lattice. Since $(H_i^*)^* = H_i$, there exists $u_i \in H_i^*$ with $u_i \cdot (x_* - a_i) \notin \mathbb{Z}$. Then

$$Q(x) = \prod_i \big(\mathrm{e}(u_i \cdot (x - a_i)) - 1\big)$$

vanishes on every $a_i + H_i$ and satisfies $Q(x_*) \neq 0$. We take the empty product to be one. The physical support of $Q\mu$ inherits uniform discreteness from $A$. Its spectrum is locally finite because it lies in finitely many translates of $B$. Thus $Q\mu$ is crystalline, and [3, Theorem 3.1] makes its nonempty support relatively dense. Yet that support lies in the remaining hyperplanes. A ray whose direction is parallel to none of them has points arbitrarily far from their union. This contradiction gives the lattice cover of $A$. Applying the same argument to $\nu$, whose spectrum is $-A$, gives the cover of $B$. □

For a locally finite set $E$ write $\delta_E = \sum_{x\in E}\delta_x$. A measure $\sigma$ is translation bounded if $\sup_{y\in\mathbb{R}^p}|\sigma|(y+B(0,1))<\infty$.

**Lemma 7.2** (Recovery of the weights from two lattice covers)**.** *Let $\mu$ and $\nu=\widehat{\mu}$ be translation-bounded measures on $\mathbb{R}^p$. Suppose the support of each is contained in a finite union of translates of full-rank lattices. Then each measure is a finite linear combination of translated and modulated full-rank lattice combs.*

*Proof.* First restrict $\mu$ to a single coset $D=a+H$ of a full-rank lattice. In a basis of $H^*$, let $F_N$ be the image of $[-N,N]^p\cap\mathbb{Z}^p$. Define

$$f_N(x)=\frac{1}{|F_N|}\sum_{t\in F_N}\mathrm{e}(t\cdot(x-a)).$$

The one-variable geometric sums in each basis direction show that $f_N(x)\to\mathbf{1}_D(x)$, and $|f_N|\le 1$. Since $|\mu|$ is translation bounded, it integrates the absolute value of every Schwartz test function. Hence $f_N\mu\to\mu|_D$ on Schwartz tests. With $(\tau_t\nu)(E)=\nu(E-t)$, the transforms are

$$\rho_N=\frac{1}{|F_N|}\sum_{t\in F_N}\mathrm{e}(-t\cdot a)\,\tau_t\nu.$$

Their variations have a common translation bound. Distributional convergence therefore gives vague convergence to the translation-bounded measure $\rho_D=\widehat{\mu|_D}$. Indeed, on each compact set the variations are uniformly bounded, so smooth-test convergence extends to continuous compactly supported tests by uniform approximation.

Write $\operatorname{supp}\nu\subset\bigcup_j(b_j+K_j)$, and put $c=\sup_b|\nu(\{b\})|<\infty$. The cover is locally finite and $\nu$ is pure point. Consequently

$$|\rho_N|\le c\sum_j\frac{1}{|F_N|}\sum_{t\in F_N}\delta_{b_j+t+K_j}.$$

Let $\pi_j:\mathbb{R}^p\to\mathbb{R}^p/K_j$ be the quotient map, and set $G_j=\overline{\pi_j(H^*)}$ in the compact torus $\mathbb{R}^p/K_j$. The projected box averages converge weakly to Haar probability on $G_j$. Indeed, their Fourier coefficients converge to one on the characters trivial on $H^*$ and to zero on all remaining characters, by the finite geometric-sum formula. A continuous compactly supported test $\psi$ periodizes to the continuous function $\pi_j(s)\mapsto\sum_{\ell\in K_j}\psi(b_j+s+\ell)$. Thus the $j$th average of combs has vague limit $\eta_j$ given by integrating this function against Haar probability on $G_j$. Passing to the limit in

$$|\rho_N(\psi)|\le c\sum_j\frac{1}{|F_N|}\sum_{t\in F_N}\delta_{b_j+t+K_j}(|\psi|)$$

gives $|\rho_D|\le c\sum_j\eta_j$.

If $G_j$ is infinite, its Haar probability gives zero mass to singletons. Under periodization a point of $\mathbb{R}^p$ can receive mass only from its single projected class (shifted by $-\pi_j(b_j)$), so $\eta_j$ has no atoms. If $G_j$ is finite, then $H^*\cap K_j$ has finite index in $H^*$. It has full rank and hence also finite index in $K_j$. The support of $\eta_j$ is contained in $b_j+H^*+K_j$, a finite union of $H^*$-cosets. Domination shows that all atoms of $\rho_D$ lie in the finitely many $H^*$-cosets obtained in this way. For every $t\in H^*$,

$$\tau_t\rho_D=\mathrm{e}(t\cdot a)\rho_D,\qquad \rho_D(\{\xi+t\})=\mathrm{e}(-t\cdot a)\rho_D(\{\xi\}).$$

The first identity follows by Fourier transformation from $\mathrm{e}(t\cdot x)\mu|_D=\mathrm{e}(t\cdot a)\mu|_D$. Thus the atomic part of $\rho_D$ is a finite sum of modulated combs on those cosets.

It remains to treat intersections in a finite lattice cover of $\operatorname{supp}\mu$. A nonempty intersection of cosets is $a+H$, where $H$ is the intersection of their lattices. If $H$ has full rank the preceding argument applies. For smaller rank, take the product of the functions $f_N$ associated with the intersecting cosets. These products are bounded by one, converge pointwise to $\mathbf{1}_{a+H}$, and their exponential coefficients have absolute sum at most one. The same convergence argument shows that $\widehat{\mu|_{a+H}}$ is a translation-bounded measure. For every $t \in (\operatorname{span}_{\mathbb{R}} H)^{\perp}$ it obeys the same translation identity with phase $\mathrm{e}(t\cdot a)$. An atom would therefore give infinitely many atoms of equal magnitude on a bounded line segment, contrary to local finiteness of the variation. Its atomic part is zero.

Apply inclusion–exclusion to the physical cover, Fourier transform the resulting finite sum of restrictions, and take its atomic parts. Since $\nu$ is pure point, this recovers all of $\nu$ as a finite sum of translated and modulated lattice combs. Inverse Poisson summation gives the stated representation for $\mu$. □

The transform of an individual restriction can have a diffuse part. For example, restricting $\delta_{\mathbb{Z}}$ to $\sqrt{2}\,\mathbb{Z}$ gives $\delta_0$, whose Fourier transform is Lebesgue measure. This is why the proof takes atomic parts in the inclusion–exclusion identity.

*Proof of Theorem 1.2.* The zero measure is the empty sum. For a nonzero measure, Lemma 7.1 supplies full-rank lattice covers on both sides. Alon and Kummer's translation bound [3, Corollary 3.5] applies to $\mu$ and $\widehat{\mu}$ separately because their physical supports are uniformly discrete in the respective orientations. Lemma 7.2 gives the desired formula. □

## 8 The remaining conjecture and the history of the results

The higher-dimensional question in the earlier version of this section asked whether two independent origin-centred half-ball counts suffice for a comb decomposition. Theorem 1.1 answers it in every positive dimension, with independently chosen directions and sequences of radii. A different counting question on the line remains open.

**Conjecture 8.1** (Subquadratic joint count)**.** Let $\mu$ be a nonzero crystalline measure on $\mathbb{R}$, with support $A$ and spectrum $B$. If

$$N_{A\cup B}(R) = \#\bigl((A\cup B)\cap[-R,R]\bigr) = o(R^2) \qquad (R\to\infty),$$

then $\mu$ is a generalized Dirac comb.

The union counts each distinct site carrying a nonzero mass on either side of the Fourier transform once. Since $N_{A\cup B} \le N_A + N_B \le 2N_{A\cup B}$, its subquadratic growth is equivalent to subquadratic growth of both counts separately. Equivalently, the conjecture asks whether every non-comb satisfies $\limsup_{R\to\infty} N_{A\cup B}(R)/R^2 > 0$. Theorem 1.4 proves this when any one of the four lower half-line densities is finite. The case still to be decided is

$$d_+(A) = d_-(A) = d_+(B) = d_-(B) = \infty.$$

Section 5 leaves a second quantitative question open. What is the optimal centred coefficient $C_*$ of Proposition 5.1? The finite-interval bound and the Kurasov–Sarnak examples place it in $[2, 4]$.

**Development of the one-dimensional results.** The author developed the one-dimensional finite-window generation and polynomial-vanishing arguments independently. The following dates describe what was recorded in private research notes in 2026.

- **24 September.** For a relatively separated physical support, the notes recorded that $B \cap I$ generates the full spectral group for every bounded open interval of length greater than $2D^+(A)$. The existence of an annihilator in higher spectral rank was then open.
- **28–29 September.** Under a spectral cone hypothesis the notes obtained vanishing for relatively separated support, and also treated uniformly discrete support without a positive lower bound on the magnitudes of the masses. The Alon–Kummer citation there concerns their earlier work on periodic hypersurfaces [2].
- **1 October.** For relatively separated physical support and arbitrary nonzero complex masses, the notes obtained a nonzero vanishing polynomial with frequencies in the original spectral group, without a spectral cone hypothesis.
- **22 October.** The notes obtained an annihilator under a finite lower count on one ray, and a comb decomposition when the opposite centred quadratic lower density vanishes.

**Related public work.** Kummer's seminar on 24 March 2026 announced joint work with Alon on toral type for uniformly discrete Fourier quasicrystal sets [9]. Lawton's 31 May preprint attributes the toral-type result for the Fourier quasicrystals studied there to Alon and Kummer [13, §4.2]. Wei's 7 October preprint treats positive integer weights [27]. On 18 November, Alon and Kummer posted their preprint and described a two-year development [3]. Their Theorem 3.1 gives generation from a ball with radius controlled by the separation constant. On the line, Proposition 2.4 gives the sharp translated-window threshold $2d$ for complex weights under a finite lower half-line count. In higher dimensions the uniformly discrete argument in Section 7 uses Alon and Kummer's finite-generation, dimension and translation-boundedness theorems together with Laurent's theorem [12].

**Versions of this paper.** The 20 November Hexagon deposit proved the one-dimensional half-line criterion. The 21 November version added the uniformly discrete several-lattice classification, then Theorem A.1 and here Theorem 1.2, and explicitly posed the independent half-ball question [20]. The 22 November manuscript gave the attained minimum width in Theorem 1.3, the finite-interval spectral count, the bracket $2 \le C_* \le 4$, and the two-carrier dimension bound. Its Section 8 retained the half-ball question. The 23 November revision incorporated the proof of Theorem 1.1 from the research drafts [30, 31].

**Formal results and AI assistance.** The two one-dimensional Palomar records [21, 22] refer to fixed Lean statements. Their declarations, source and build metadata are available at a pinned public revision. The 23 November Palomar record [18] concerns the uniformly discrete classification. Table 1 gives the scope of the three records and the analytical results in this paper.

Table 1: Hypotheses and conclusions covered by the formalizations and by the analytical proofs.

| **Source** | **Mathematical scope** |
|---|---|
| LowerRay [21] | A finite lower count on one ray of a carrier gives an annihilator. For an actual counted support, a vanishing opposite centred quadratic liminf gives combs in either Fourier orientation. |
| WholeSpace [22] | For bounded unit-cell counts, an attained finite-hole formula imposes both physical and spectral conditions. The record also gives combs for two relatively separated carriers on the line and a common annihilator for a prescribed carrier. |
| Uniformly discrete case [18] | The Lean theorem `theoremA1` gives the comb decomposition and both lattice covers in every positive dimension for uniformly discrete support and spectrum. Its corollary `latticeCoverQuestion_holds` gives the physical cover of Lev and Olevskii. |
| Analytical results on the line | Sharp generation windows, minimum width and divisibility, real zeros, finite-interval counts, coefficient bounds, the sharp two-carrier dimension bound, and the one-ray finite-hole formula. |
| Independent half-balls (analytical) | Theorem 1.1 gives the comb classification in every positive dimension from two independent lower half-ball counts and allows arbitrary complex atom weights. Its proof is in Appendices A and B. |

The original Hexagon deposit describes the text as primarily AI-generated and lists the tools used [20]. AI assistance also contributed to the subsequent proofs and exposition, including the minimum-width argument and the analytical half-ball proof in Appendices A and B.

# A From half-ball counts to uniform local counts

The two half-ball counts enter at different stages. The physical count supplies a bound on the total volume of certain frequency patches. The spectral count and that volume bound then imply a uniform local count on the spectral half-space. Approximate returns of finite patterns extend this count to all of $\mathbb{R}^p$. The geometric argument is from [31]; the patch estimate and the recurrence of patterns are from [30].

Write $|E|$ for Lebesgue volume and $\#X$ for cardinality. The zero measure is immediate. For $\mu \neq 0$, choose physical radii $R_j \to \infty$ with

$$\frac{\#(A \cap H_\omega(R_j))}{|H_\omega(R_j)|} \longrightarrow a < \infty.$$

For the lattice comb $\mu = \delta_H$ one has $B = H^*$ and $a = 1/\operatorname{covol}(H) = \operatorname{covol}(H^*)$. The patches in (A.1) project injectively to $\mathbb{R}^p/H^*$ and their images are pairwise disjoint. Their total volume is therefore at most the volume of a dual fundamental cell. In general the difference set $B - s$ can distinguish the two signs of a displacement. The order of the patches in the following lemma retains this distinction.

**Lemma A.1** (Ordered frequency patches)**.** *At every $s \in B$, let $E_1, \dots, E_m$ be nonempty compact finite unions of boxes of positive volume satisfying*

$$(s + E_i - E_i) \cap B = \{s\}, \qquad (s + E_i - E_k) \cap B = \varnothing \quad (i > k). \tag{A.1}$$

*Then $\sum_i |E_i| \leq a$.*

*Proof.* Choose $0 \le \theta \le 1$ smooth with compact support in the interior of $H_\omega(1)$. Put $h = \widehat{\theta}$ and $\nu_R = \theta(\cdot/R)\widehat{\mu}$. On $\bigoplus_i L^2(E_i)$ use the block kernel $\nu_R(s+x-y)$ for $x \in E_i$, $y \in E_k$. Each contributing physical atom gives one rank-one kernel across the entire sum, by separation of the character in $x$ and $y$. The rank is at most $\#(A \cap H_\omega(R))$.

On the compact difference windows the kernel is the sum of the terms $v_b R^p h(R(\cdot - b))$ from a fixed finite spectral neighbourhood, up to $O(R^{-M})$ for every $M$. Indeed, cut off $\widehat{\mu}$ by a smooth function equal to one on a neighbourhood of the windows. The compact part is a finite point measure. Off that neighbourhood the test $R^p h(R(z-\cdot))$ times the complementary cutoff tends rapidly to zero in every Schwartz seminorm, uniformly for $z$ in the windows. This uses temperedness of the distribution alone. Atoms a positive distance from a window contribute equally small errors there.

It follows from (A.1) that the lower blocks are $o(1)$ in operator norm. Each diagonal block is $v_s K_{i,R} + o(1)$, where $K_{i,R}$ is convolution by $R^p h(R\cdot)$ compressed to $E_i$. All upper blocks have bounded norm: the full convolution is a Fourier multiplier of norm at most one, and only a fixed finite number of translated kernels enter. The compressed operator satisfies $0 \le K_{i,R} \le I$, and

$$\operatorname{tr} K_{i,R} = R^p |E_i| \int \theta, \qquad R^{-p} \operatorname{tr} K_{i,R}^2 = \int |h(u)|^2 |E_i \cap (E_i + u/R)|\, du \longrightarrow |E_i| \int \theta^2.$$

The first identity follows by factoring through the square root of the multiplier, and the second by integration of the squared kernel, translation continuity in $L^1$, and Plancherel. The number of eigenvalues at least $\varepsilon$ is at least $\operatorname{tr} K_{i,R}^2 - \varepsilon \operatorname{tr} K_{i,R}$. Compress all blocks to these spectral subspaces. For fixed $\varepsilon > 0$ the upper triangular part with diagonal $v_s K_{i,R}$ has a uniformly bounded inverse: its diagonal inverses have norm at most $(|v_s|\varepsilon)^{-1}$, and its strictly upper triangular part after factoring the diagonal is nilpotent of order at most $m$. The $o(1)$ perturbation preserves invertibility. The total rank therefore dominates the sum of the indicated eigenvalue counts. Divide by $R_j^p$, let $j \to \infty$ and $\varepsilon \downarrow 0$, and choose cutoffs approaching the half-ball indicator in $L^2$. The common factor $|H_\omega(1)|$ cancels, giving the claim. □

**Lemma A.2** (Recurrence of a finite pattern)**.** *For finite nonempty $F \subset B$ and $\epsilon > 0$, the set $T$ of translations $t$ with $\operatorname{dist}(f+t, B) < \epsilon$ for all $f \in F$ is relatively dense.*

*Proof.* Choose a closed cube $K$ of positive volume and diameter less than $\epsilon$ and than the isolation distance of every $f \in F$. Choose an integer $m \ge 2$ with $m|K| > a$. If an ordered list of centres has no forward difference in $T$, colour each pair $i > k$ by an $f \in F$ witnessing failure. On a monochromatic $m$-sublist the patches $x_i + K$ satisfy (A.1) at this $f$, contradicting Lemma A.1. Finite Ramsey theory bounds the length of every such list. Extend a list until maximal. Every possible new centre then belongs to some $x_i + T$, so finitely many translates of $T$ cover the space. □

**Theorem A.3** (Geometric local count)**.** *Let $B \subset \mathbb{R}^p$ be locally finite and let $\eta$ be a unit vector. Assume that a finite constant $a$ bounds $\sum_i |E_i|$ for every $s \in B$ and every finite ordered family of positive-volume compact finite unions of boxes satisfying (A.1) at $s$. Suppose also that for some $S_j \to \infty$, $\#(B \cap H_\eta(S_j)) = O(S_j^p)$. Then*

$$\sup_{y \in \mathbb{R}^p} \#\{s \in B \cap B(y,1) : s \cdot \eta \ge 0\} < \infty.$$

The statement concerns a locally finite set. Its only Fourier input in the application is the patch bound from Lemma A.1. From here until the end of the geometric proof, all scales come from the sequence $(S_j)$; its choice is independent of $(R_j)$.

For a witnessing radius $S$ put $X = B \cap H_\eta(S)$, let $\Omega$ be a centred axis-parallel cube of diameter $S/3$, and set $N = |\Omega|$. There is a fixed $C \ge 1$ such that $\#X \le CN$ at all late scales. Write $\operatorname{pack}_1(Y)$ for the largest size of a subset of $Y$ with mutual distances strictly greater than one.

**Lemma A.4** (Exact triples at a fixed site)**.** *Under the hypotheses of Theorem A.3, suppose* $X = X_0 \sqcup X_1$ *at each scale, with* $\mathrm{pack}_1(X_0) = o(N)$*. For a fixed* $s \in B$ *with* $s \cdot \eta \geq 0$*, let*

$$F_i = \{\pm(x - s) : x \in X_i \setminus \{s\},\ (x - s) \cdot \eta \geq 0,\ x - s \in \Omega - \Omega\}, \qquad F = F_1.$$

*There is a* $\tau = \tau(a, C, p) > 0$ *common to all such partitions and all anchors. For each fixed* $s$*, there is a starting index* $j_0(s)$*, for the given partitions, such that for every* $j \geq j_0(s)$*,*

$$\#\{(h, k, l) \in F^3 : h + k + l = 0\} > \tau N^2. \tag{A.2}$$

*Proof.* Put $C_* = \max(1, 2 \cdot 3^p C)$. Choose $\Lambda$ large, $D = \lceil 8C_*\Lambda \rceil$ and $k_D = \lfloor (\log_2 D)/2 \rfloor$, with

$$(1 + \sqrt{D})^{-1} \geq k_D/(4D), \qquad \Lambda k_D/(16D) > a + 1.$$

This is possible since $D/\Lambda$ stays bounded. Fix $\tau = (16 \cdot 3^{2p}\Lambda^2)^{-1}$. Suppose (A.2) fails on a subsequence. Let $d_s > 0$ be a lower bound on the distance to all other sites.

Tile $\Omega$ by congruent coarse cubes of side between $\delta/2$ and $\delta$, where $\delta = \min(1, d_s/(4\sqrt{p}))$. For each pair of cells declare an edge when translation of one closure by some vector of $F_0$ meets the other. There are no loops by the diameter bound. A maximal 1-packing covers $X_0$ by $o(N)$ closed unit balls. Thus $F_0$ is also covered by $o(N)$ unit balls, and shifts of any one cell by vectors from any one of these balls can meet at most $O_{p,\delta}(1)$ cells. The maximum coarse degree is $o(N)$, uniformly in the coarse cell for the fixed anchor. Occupy coarse cells independently with probability $q = \Lambda/N$, and delete occupied cells having an occupied $F_0$-neighbour. The expected volume lost is at most $Nq^2 o(N) = o(1)$.

Subdivide each coarse cell equally into fine cubes of common diameter $d'$ so small that $3d' < |h + k + l|$ for every nonzero such sum from $F^3$. Define fine edges by contact of closures under an $F$-shift. An edge joins distinct coarse cells: every label has length at least $d_s$, whereas a coarse cell has diameter at most $d_s/4$. For each fine cell and fixed shift, at most $3^p$ partner cells can meet it. The degree is at most $3^p \# F \leq C_* N$. Put $w$ equal to a fine-cell volume and $n = N/w$. Around a triangle choose oriented edge labels in $F$. Each differs from the oriented centre displacement by at most $d'$. The three centre displacements sum to zero. Thus the label sum has length at most $3d'$ and the choice of mesh forces it to be exactly zero. Each starting cell and ordered label triple admit at most $3^{2p}$ choices of the next two cells. There are at most $n3^{2p}\tau N^2$ triangles, each using three distinct coarse cells.

In the initially occupied graph the expected vertex volume is $\Lambda$. Its expected degree sum times $w$ is at most $NC_* Nq^2 = C_*\Lambda^2$, and its expected triangle count times $w$ is at most $3^{2p}\tau\Lambda^3$. The three vertices of a triangle belong to pairwise different coarse cells, since each pair is joined by an edge. Its occupation probability is $q^3$, by independence of the three coarse cells. An edge likewise survives with probability $q^2$. Besides the coarse deletions, delete all vertices of initial occupied degree $> D$ and one chosen vertex from each initially occupied triangle. The expected loss is at most $o(1) + \Lambda/8 + \Lambda/16$. Some outcome therefore leaves vertex volume at least $\Lambda/2$ in a triangle-free graph of maximum degree at most $D$.

Such a graph on $v$ vertices has an independent set of size at least $vk_D/(8D)$. To see this, choose an independent set uniformly among all independent sets. Given its choices outside the closed neighbourhood of a vertex, let $z$ be the number of unblocked neighbours. They have no mutual edges. Of the $1 + 2^z$ equiprobable extensions, one contains the central vertex, and the others choose any subset of the $z$ neighbours. The conditional probability of selecting the vertex is $1/(1 + 2^z)$; the expected number of selected neighbours is at least $z/4$. When $z \leq k_D$ the former is at least $k_D/(4D)$; when $z > k_D$ the latter divided by $D$ is at least this large. Average and sum over vertices.

The sum of the two expectations, the second divided by $D$, is at most twice the expected size of the set. This gives the claim.

Let $W$ be the union of the open fine cubes of this independent set. Its volume is at least $\Lambda k_D/(16D) > a+1$, and $W-W$ avoids $F_0 \cup F$. At this anchor and scale the offsets in $(B-s)\cap(\Omega-\Omega)$ are finite, including offsets on the uncounted side. Slice $W$ by open slabs normal to $\eta$ whose width is less than the smallest nonzero absolute projection of these offsets (any positive width if all are zero). Discard the null dividing planes. In each nonempty slice retain a compact finite union of boxes, keeping total volume $> a$. Order the slices by increasing height. If a site offset joins two points in one patch, its projection has absolute value less than the slab width and hence is zero. For $i > k$, every displacement in $E_i - E_k$ has strictly positive projection. In particular the diagonal test includes sites on the bounding hyperplane when $s \cdot \eta = 0$. For every such nonzero offset $h$ with $s + h \in B$, the site is in $X$ for late $S$, since $(s + h) \cdot \eta \geq 0$ and $|s + h| \leq |s| + S/3 \leq S$. The offset belongs to $F_0 \cup F$, already excluded. Zero cannot connect different slabs. The patches therefore satisfy (A.1) and have total volume greater than $a$, contrary to the assumed patch bound. □

The points $\{(k/n^2)e_1 : 1 \leq k \leq n\}$ form a cluster in a unit ball. Its difference set has only $2n - 1$ elements, while its 1-packing has size one. The next lemma extracts pieces that have both small difference sets and 1-packings of size comparable to $N$.

**Lemma A.5** (Signed representations in pieces with large packings)**.** *Under the hypotheses of Theorem A.3 there is a single subsequence of the $S_j$ and, at each of its scales, a disjoint family of pieces $Y_i \subset X$ from a fixed finite list of labels, with the following properties. For each retained label $i$, $\#(Y_i - Y_i) \leq K\#Y_i$ and $\mathrm{pack}_1(Y_i) \geq \delta_i N$ at all sufficiently late scales, where $K$ and $\delta_i > 0$ are fixed. There exists $\beta > 0$ such that, for each fixed $s \in B$ with $s \cdot \eta \geq 0$, all sufficiently late scales admit three retained labels $i_1, i_2, i_3$ and signs $\sigma_t \in \{-1, 1\}$ for which*

$$\#\{(x_1, x_2, x_3) \in \textstyle\prod_{t=1}^3 Y_{i_t} : \sum_{t=1}^3 \sigma_t x_t = (\sum_{t=1}^3 \sigma_t)s\} \geq \beta N^2.$$

*The labels may repeat and may depend on the anchor and the scale.*

*Proof.* For a finite set $Y$ let $E(Y) = \#\{(u, v, w, z) \in Y^4 : u + v = w + z\}$. First fix $\varepsilon > 0$ with $24C^{5/4}\varepsilon^{1/4} < \tau/2$, using $\tau$ from Lemma A.4. At each scale write

$$X = U \sqcup Y_1 \sqcup \cdots \sqcup Y_L, \qquad E(U) < \varepsilon N^3, \qquad \#Y_i \geq cN, \qquad \#(Y_i - Y_i) \leq K\#Y_i.$$

Indeed, while a remainder $V$ has energy at least $\varepsilon N^3$, it has $\#V \geq \varepsilon^{1/3}N$ and $E(V) \geq (\varepsilon/C^3)(\#V)^3$. The Balog–Szemerédi–Gowers theorem in difference-set form [28] gives a piece of the indicated size and doubling bound. Removing such pieces takes at most $C/c$ steps; $c, K$ depend only on $C, \varepsilon$.

Pass once to a subsequence with fixed $L$ and a limit for each $\mathrm{pack}_1(Y_i)/N$. Retain a label, and call it spread, if its limit is positive. At late indices a spread piece contains $\delta_i N$ mutually more than one apart sites, for a fixed $\delta_i > 0$. Use the union of the other pieces for $X_0$ and all spread pieces together with $U$ for $X_1$. Then $\mathrm{pack}_1(X_0) = o(N)$.

Apply Lemma A.4 to this partition. An ordered triple counted in (A.2) gives a signed representation $\sum_{j=1}^3 \sigma_j x_j = ms$, with $x_j \in X_1$, $\sigma_j \in \{-1, 1\}$ and $m = \sum_j \sigma_j$. Distinct offset triples require distinct signed site triples. For each prescribed sign pattern and remainder position the number involving $U$ in that position is at most $C^{5/4}\varepsilon^{1/4}N^2$. Indeed the count is at most $\|\mathbf{1}_{\sigma_1 U} * \mathbf{1}_{\sigma_2 X}\|_2 (\#X)^{1/2}$ after permutation, with counting measure on the additive group. The square of the convolution norm is bounded by $\sqrt{E(U)E(X)}$ by Cauchy–Schwarz on difference multiplicities, and $E(X) \leq (CN)^3$. Summing over eight signs and three positions leaves more than $\tau N^2/2$ representations using spread pieces only. Some three signed spread labels therefore represent

$ms$ at least $\beta N^2$ times, where $\beta = \tau/(16\max(1,L)^3)$. This proves the assertion on the common subsequence. □

**Lemma A.6** (Local packing of represented anchors)**.** *On the subsequence of Lemma A.5, there is an integer $M$ such that at any sufficiently late scale and in any open unit ball, at most $M$ distinct anchors admit the representation bound of that lemma.*

*Proof.* Fix the three labels and signs at a scale. Write $A_t = \sigma_t Y_{i_t}$, $m = \sum_t \sigma_t$ and $Z = A_1 + A_2 + A_3$. Call an anchor qualifying for this tuple if $ms$ has at least $\beta N^2$ ordered representations. If any anchor qualifies, fixing the coordinate other than $1, t$ by averaging gives, for $t = 2, 3$, a set $I_t = A_1 \cap (z_t - A_t)$ of size at least $\beta N/C$. Ruzsa's triangle inequality gives

$$\#(A_1 + A_t) \le \frac{\#(A_1 - I_t)\,\#(I_t + A_t)}{\#I_t} \le \frac{\#(A_1 - A_1)\,\#(A_t - A_t)}{\#I_t} = O(N).$$

For the last inclusion use $I_t + A_t \subset z_t + (A_t - A_t)$. A maximal family of disjoint translates $z + A_1$, $z \in A_t$, has $O(1)$ members. Hence $A_t$ is covered by $O(1)$ translates of $A_1 - A_1$. Thus $Z + A_1$ lies in $O(1)$ translates of $4A_1 - 2A_1$, of total size $O(N)$ by the signed Plünnecke–Ruzsa inequality applied to $A_1, -A_1$ [29, Theorem 1.2]. The integer coefficients of sets here denote iterated sums.

Thin the 1-packing in $A_1$ greedily to a subset $P$ with mutual distances greater than 7 and $\#P \ge c_p \delta_i N$. A radius-7 ball contains only boundedly many mutually more than one apart points. For any open unit ball $D_0$, the set $mD_0$ has diameter at most 6 since $m \in \{-3, -1, 1, 3\}$. The translates of $Z \cap mD_0$ by the sites of $P$ are disjoint in $Z + A_1$. Consequently

$$\#P\,\#(Z \cap mD_0) \le \#(Z + A_1) = O(N).$$

The map $s \mapsto ms$ is injective. Thus a constant independent of the ball and of all late scales bounds the number of qualifying anchors in that ball. Summing over the finitely many labels and signs gives one integer cap $M \ge 1$. It can depend on the positive packing fractions $\delta_i$ of the chosen subsequence, as well as on the preceding constants. It is uniform in the ball, its anchors and the late scale. □

*Proof of Theorem A.3.* Choose the subsequence of Lemma A.5 and the cap $M$ of Lemma A.6. Suppose one unit ball contained distinct sites $s_1, \dots, s_{M+1}$ in the counted half-space. Each has a starting index on this same subsequence after which it admits the required representations. Choose a single index beyond all $M + 1$ starting indices and the onset of the cap. All $M + 1$ sites qualify in the same ball at that scale, a contradiction. □

**Global relative separation.** If an open unit ball anywhere contained $M+1$ spectral sites, choose $\epsilon$ below their distances to its boundary and half their minimum mutual distance. Lemma A.2 gives an approximate return sufficiently far in the $\eta$ direction that the translated ball lies wholly in the counted half-space: a relatively dense set meets a ball of fixed radius about $T\eta$ for arbitrarily large $T$. The returned sites are distinct, contradicting the half-space cap. This proves relative separation of $B$. Apply the same reasoning to $\widehat{\mu}$, reflecting the appropriate direction under the double transform, to obtain relative separation of $A$. Appendix B finishes the proof.

## B From two locally bounded site counts to lattice combs

We prove the structural implication used in Theorem 1.1. Throughout this appendix $\mu \neq 0$ and both actual carriers $A, B$ are relatively separated. This hypothesis is available in both orientations by

Appendix A. In particular, it allows a shorter route through Laurent's theorem than the one-sided structural result of [30]. The measurement estimate, frequency-window argument and recovery of the coefficients below adapt that manuscript and the finite-generation and window arguments of Alon and Kummer [3].

**Lemma B.1** (Detection in a fixed frequency ball)**.** *Fix a uniform bound on the number of sites of $E \subset \mathbb{R}^p$ in an open unit ball. There exist a bounded open ball $D$ centred at zero and a Schwartz seminorm $q$ depending only on this bound and $p$ with the following property. If $T$ is a tempered distribution locally given by point masses on $E$ and $\widehat{T}$ is locally a point measure on a locally finite set, then*

$$|\langle T, \psi\rangle| \le q(\psi) \sum_{b \in D \cap \operatorname{supp} \widehat{T}} |\widehat{T}(\{b\})|, \qquad \psi \in \mathcal{S}(\mathbb{R}^p).$$

*Proof.* Partition $E$ into $m$ sets which are $\delta$-separated for fixed $0 < \delta < 1$. Colour the graph joining sites at distance less than $\delta$ greedily: its degree is uniformly bounded. For arbitrary $y$ we construct a Schwartz function $\rho_y$, of fixed bandwidth, equal to one at $y$ and zero on $E \setminus B(y, \delta/2)$. Its Schwartz seminorms in coordinates centred at $y$ are uniformly bounded.

Take $k \in \mathcal{S}$ with $\widehat{k} \in C_c^\infty$ and $k(0) = 1$. For a $\delta/2$-separated set $E'$, packing in concentric annuli and rapid decay give

$$\sup_{a \in E'} \sum_{b \in E',\, b \ne a} |k(N(a-b))| < \tfrac{1}{2}$$

for a common $N$. Thus the matrix $G_{ab} = k(N(a-b))$ is invertible on $\ell^\infty(E')$ with inverse norm at most two. For each colour discard its sites in $B(y, \delta/2)$ and adjoin $y$. On the resulting $E'$ solve $Gc = \mathbf{1}_{\{y\}}$. The function $h(z) = \sum_{a \in E'} c_a k(N(z-a))$ takes the prescribed values. Packing bounds all derivatives uniformly. The series converges in tempered distributions, so the Fourier support of $h$ lies in $N \operatorname{supp} \widehat{k}$. Multiply the $m$ functions $h$ and one bandlimited Schwartz function $\chi(z-y)$ with $\chi(0) = 1$. The product $\rho_y$ has bandwidth in a fixed compact set (convolution of compactly supported Fourier distributions). Uniform derivative bounds give $0 < r < \delta/2$ with $|\rho_y| \ge 1/2$ on $B(y, r)$.

For $f \in C_c^\infty(B(y, r))$, set $g = f/\rho_y$ there and extend by zero. The open ball $B(y, \delta/2)$ contains at most one site of each colour. At its sites $a_1, \dots, a_s$ interpolate $g$ by a polynomial $P$ of degree at most $s-1$. The coefficients in coordinates about $y$ have bounds uniform even as the sites approach each other. Indeed, start from

$$g(z) = g(a_1) + (z - a_1) \cdot \int_0^1 \nabla g(a_1 + t(z - a_1))\, dt.$$

Apply this identity componentwise to the coefficient functions in the remainder, at $a_2$, and continue through $a_s$. The final remainder has in each term a factor from each $z - a_j$, so vanishes at every node. The integral formulas bound the coefficients of $P$ by derivatives of $g$ up to order $s-1$ on the fixed-radius ball. Take $P = 0$ when $s = 0$. Then $P\rho_y$ agrees with $f$ on $E$, is Schwartz with Fourier support strictly inside a fixed ball $D = -D$, and has $L^1$ norm bounded by a constant times $\max_{|\alpha| \le m} \|\partial^\alpha f\|_\infty$. Its action on $T$ equals that of $f$: multiply the difference by compact cutoffs tending to one in the Schwartz multiplier sense. The compactly supported differences vanish on $E$ and pair to zero. Fourier duality bounds the action on $P\rho_y$ by its $L^1$ norm times the spectral sum in the lemma. A smooth partition of unity on a fixed grid subordinate to translates of $B(0, r)$ completes the proof. The sum over its pieces of the derivative suprema is controlled by a Schwartz seminorm with weight $(1 + |z|)^{p+1}$. □

**Proposition B.2** (Finite generation of both carrier groups)**.** *For a nonzero crystalline measure whose two carriers are relatively separated, $\langle A\rangle_{\mathbb{Z}}$ and $\langle B\rangle_{\mathbb{Z}}$ are finitely generated, and both carriers span $\mathbb{R}^p$ over $\mathbb{R}$. For the spectral group, one may take as generators $B \cap D$, where the radius of $D$ depends only on $p$ and the local site bound for $A$.*

*Proof.* Take the fixed detection ball for twice the local site bound of $A$, so it applies to every difference $T_t = \mu - \tau_t\mu$, where $\tau_t\mu$ translates each site by $t$. If $b \in B$ lies outside $\langle B \cap D\rangle_{\mathbb{Z}}$, a circle character on the group generated by $b$ and $B \cap D$ equals one on the latter sites and differs from one at $b$. Indeed the quotient is finitely generated abelian and its characters separate its points, including torsion points. Kronecker's approximation theorem on the finite coordinate torus realizes this character as a limit of $s \mapsto \mathrm{e}(-t_j \cdot s)$ on these sites: the integer relations defining the closure of the evaluation subgroup are exactly $\sum n_s s = 0$. The right side of Lemma B.1 for $T_{t_j}$ tends to zero on every fixed test, since $B \cap D$ is finite. But the Fourier test supported near $b$ and taking value one there detects a nonzero limiting coefficient. This contradiction gives $\langle B\rangle_{\mathbb{Z}} = \langle B \cap D\rangle_{\mathbb{Z}}$. Applying this to $\widehat{\mu}$ gives finite generation of $\langle A\rangle_{\mathbb{Z}}$ as well. The real span of $B$ is $\mathbb{R}^p$: a missing normal would give a continuous line of translation periods for $\mu$, impossible for its nonempty locally finite support. Apply the same argument to $\widehat{\mu}$ for the real span of $A$. □

**Proposition B.3** (Full-rank lattice covers for both carriers)**.** *If both carriers of a nonzero crystalline measure are relatively separated, each is covered by finitely many translates of full-rank lattices.*

*Proof.* Using Proposition B.2, choose a $\mathbb{Z}$-basis of $\langle B\rangle_{\mathbb{Z}}$ as the rows of an $r \times p$ matrix $L$ with the first $p$ rows independent over $\mathbb{R}$. Put $T = L^{\mathsf{t}}$, injective on $\mathbb{Z}^r$, and write $\phi(x) = (\mathrm{e}((Lx)_1), \dots, \mathrm{e}((Lx)_r))$. Let $X$ be the Zariski closure of $\phi(A)$ in the algebraic torus $(\mathbb{C}^\times)^r$. The group $\phi(\langle A\rangle_{\mathbb{Z}})$ is finitely generated. Laurent's theorem [12] applied to this group and $X$ shows that $X$ is a finite union of cosets of connected algebraic subtori: intersect with the finitely generated group and take Zariski closures, splitting any disconnected algebraic subgroup into its finitely many components.

Fix an irreducible component $Y$. There is a Laurent polynomial $g$ vanishing on the other components and not identically on $Y$. For each other component choose a polynomial in its ideal which does not vanish identically on $Y$, and multiply them. The measure $(g \circ \phi)\mu$ is nonzero by Zariski density, has physical support in $\phi^{-1}(Y)$, and has locally finite spectrum because the polynomial has only finitely many frequencies. A nonzero crystalline measure cannot be supported in a proper affine hyperplane: multiplication by every character in the normal direction is scalar on its support, so its nonempty spectral carrier would be invariant under every translation parallel to that normal, contradicting local finiteness. Therefore $\phi^{-1}(Y)$ spans $\mathbb{R}^p$ affinely.

Let $K \subset \mathbb{Z}^r$ be the subgroup of characters trivial on the subtorus of $Y$, and set $M = T(K)$. At any $x_0 \in \phi^{-1}(Y)$ its inverse image equals

$$x_0 + \{x : m \cdot x \in \mathbb{Z} \text{ for all } m \in M\}.$$

The group in braces contains $p$ independent vectors by the affine-span conclusion. Thus $M$ is contained in the dual lattice of the lattice generated by these vectors. In particular $M$ is discrete.

For bounded $U \subset \mathbb{R}^p$ write $W(U, n)$ for the space on $X$ spanned by restrictions of monomials $Z^k$ with $\|k\|_\infty \le n$ and $Tk \in U$. We claim

$$\dim W(U, n) \le C_D(1 + n + n_U)^{r-p}, \tag{B.1}$$

with the same $C_D$ for every bounded $U$ and every $n$. The index allowance $n_U$ may depend on $U$, but is fixed as $n$ varies. Use the ball from Lemma B.1 for support $A$. A linear combination $P$ of

these monomials vanishes on $X$ precisely when $(P \circ \phi)\mu = 0$. This measure is determined by its Fourier coefficients in $D$ by detection. Every candidate spectral site in $D$ has index $k + j$ with $Tj \in B \cap (D - U)$; this latter set is finite and the norms of its unique integral indices are bounded by some $n_U$. After fixing the last $r - p$ integer coordinates in the enlarged box of radius $n + n_U$, only a bounded number of first $p$ coordinates map into $D$, uniformly over translations of $D$, since the first $p$ basis rows generate a lattice. This proves (B.1).

Choose a small open cube $U_0$ about zero of diameter less than the separation of $M$ (any fixed small cube if $M = 0$). Distinct indices in this window have distinct classes modulo $K$, since $T$ is injective on $\mathbb{Z}^r$. Their restrictions are independent on $Y$: different characters on the connected subtorus are linearly independent, and its translation only rescales each one. For all large $n$ the index box of radius $\lfloor n/4 \rfloor$ has $\gg n^r$ sites, with images covered by $O(n^p)$ cubes so small that each difference set lies in $U_0$. In a most populated cube subtract one index from all the indices, including itself. This gives at least $cn^{r-p}$ distinct indices of norm at most $n/2$ mapping to $U_0$, with $c > 0$ independent of $n$.

Suppose $M$ has a nonzero normal $u$. Choose $k_0 \in \mathbb{Z}^r$ with $u \cdot Tk_0$ larger than the width of the projection $u \cdot U_0$, using the full real rank of $T$ and integer multiples. For any fixed integer $l$ translate the preceding indices by $jk_0$, $1 \le j \le l$. All have norm at most $n$ for large $n$. Their classes modulo $K$ remain distinct between translates by the disjoint projected windows. Thus at least $lcn^{r-p}$ restrictions are independent on $Y$, hence on $X$, in the fixed bounded window $U = \bigcup_{j=1}^{l}(jTk_0 + U_0)$. For this fixed $l$, the window $U$ and hence $n_U$ are fixed. Dividing (B.1) by $n^{r-p}$ gives

$$lc \le C_D(1 + (1 + n_U)/n)^{r-p}.$$

First let $n \to \infty$ to obtain $lc \le C_D$. Both $c$ and $C_D$ are independent of $l$, so letting $l$ grow yields a contradiction. The discrete group $M$ thus has full rank. The inverse image of $Y$ is a coset of its full-rank dual lattice. The finitely many components cover $A$, and Fourier duality gives the same conclusion for $B$. □

**Proposition B.4** (Complex weights on two finite lattice covers)**.** *If both actual carriers of a crystalline measure admit finite covers by cosets of full-rank lattices, the measure is a finite sum of translated, modulated full-rank lattice combs.*

*Proof.* Write $A \subset \bigcup_i (a_i + H_i)$ and set $v(t) = \widehat{\mu}(\{t\})$. Group the lattices covering $B$ by commensurability (finite index of their intersection in both). Within each group use their intersection $K$ as common sublattice. Assign any overlapping site to exactly one group and write $v = \sum_K v_K$. Each $v_K$ is a function supported in finitely many $K$-cosets. We use these pieces only as coefficient functions. The cone construction below will yield polynomially weighted lattice combs, to which Fourier transformation applies.

Call an error thin if it is supported in finitely many affine hyperplanes. For distinct groups $K, K'$, the intersection of two cosets, even after arbitrary translations, is empty or a coset of $K \cap K'$, of rank at most $p - 1$. Fix once and for all a hyperplane direction containing $\operatorname{span}_{\mathbb{R}}(K \cap K')$ for each such pair. All thin errors below can use just these finitely many directions. Finite sums and finite shifts preserve this property.

Fix $K$. For any $t_i \in H_i^*$, let

$$D = \prod_i (\tau_{t_i} - e(t_i \cdot a_i)), \qquad \tau_t w(y) = w(y - t).$$

Then $Dv = 0$. Each $Dv_K$ is supported in finitely many $K$-cosets, and it must vanish outside the union of the supports of the other groups' $Dv_{K'}$. Thus $Dv_K$ is thin.

When $H_i^*$ is incommensurate with $K$, choose $t_i \in H_i^*$ for which no positive integer multiple lies in $K$. Such a choice is possible since otherwise clearing denominators of a basis would give a common full-rank sublattice. To remove this factor, suppose that $w$ is supported on finitely many $K$-cosets and $(\tau_t - z)w$ is thin, with $z \neq 0$ and $nt \notin K$ for $n \geq 1$. Choose $n$ such that these cosets and their translates by $nt$ are disjoint. For any pair of cosets at most one $n$ is excluded. Factoring $\tau_{nt} - z^n$ through $\tau_t - z$ shows that $(\tau_{nt} - z^n)w$ is thin. Disjointness shows that $w$ itself is thin, with the same permitted hyperplane directions. Apply this successively to remove all incommensurate factors.

Let $J_K = \{i : H_i^* \text{ is commensurate with } K\}$ and choose a common finite-index sublattice $K_0$ of $K$ and the lattices $H_i^*$ for $i \in J_K$ (use $K$ for the empty list). For each $h \in K_0$, taking $t_i = h$ for all $i \in J_K$ gives

$$\prod_{i \in J_K} (\tau_h - e(h \cdot a_i))v_K \quad \text{supported on a thin error.} \tag{B.2}$$

If $J_K$ is empty, $v_K$ itself is thin.

*A common cone of exact recurrences.* Choose a unit direction $w$ outside all permitted hyperplane directions. Choose a small open convex cone about $w$ on which every defining normal has the same strict sign as at $w$. For each nonempty $J_K$, choose $p$ independent $h_1, \dots, h_p \in K_0$ in that cone whose positive hull contains $w$ in its interior. To see this, surround $w$ by a small simplex of directions, scale its vertices far out, and round them to the lattice. These vectors generate a finite-index sublattice of $K$. Apply (B.2) for each $h_i$; there are now only finitely many actual error hyperplanes.

On each of the finitely many relevant cosets of the sublattice generated by the $h_i$, choose a point $z$ far enough in direction $w$ that it lies beyond every error hyperplane in the chosen sign. For example take a lattice point of that coset within bounded distance of $Tw$, for large $T$. Then the entire grid $z + \sum_{i=1}^{p} \mathbb{N}_0 h_i$ avoids all these hyperplanes. On this grid the one-variable recurrence in each coordinate has nonzero first and last coefficient. Its roots have modulus one. The values on a rectangular initial block, whose side lengths are the degrees of the recurrences, therefore determine an extension to the whole grid coset. Explicitly, extend the block in each variable using the corresponding one-dimensional recurrence. The operations commute because they act on different indices. The resulting solution is a sum of polynomials in the lattice coordinates times unitary characters, by the one-dimensional formula for repeated roots. Successive forward recurrences establish equality with $v_K$ on the positive grid. For an empty $J_K$ use the zero extension beyond its thin error.

The intersection of the interiors of all the positive hulls contains $w$. A smaller closed angular neighborhood of $w$, sufficiently far from the origin, is contained in every translated real cone $z + \sum_i \mathbb{R}_{\geq 0} h_i$ and avoids the thin errors for all empty groups. On this region the sum of the extensions is exactly $v$. In particular $\widehat{\mu}$ agrees on a solid open cone outside a ball with a measure $\sigma$ that is a finite sum of polynomially weighted and modulated translates of lattice combs. Its inverse transform is locally a finite sum of derivatives of point masses on finitely many lattice cosets, by Poisson summation.

*Why cone agreement gives global agreement.* If a tempered distribution $T$ vanishes in such a cone and $\widehat{T}$ is supported on a locally finite set, then $T = 0$. At a site $b$ of $\widehat{T}$, write its local expansion as $\sum_\alpha c_\alpha \partial^\alpha \delta_b$. For any smooth $\phi$ compactly supported inside the cone and away from the origin, test $T$ against $R^{-p}\phi(x/R)e(-b \cdot x)$. This is zero for large $R$. Fourier duality tests $\widehat{T}$ against $\widehat{\phi}(R(b - \cdot))$. Outside a fixed neighborhood isolating $b$ its contribution is $O(R^{-N})$ for every $N$, by the Schwartz seminorm bounds after a smooth cutoff. The expansion at $b$ gives a polynomial in $R$, with coefficient at $R^k$ equal to $\sum_{|\alpha|=k} c_\alpha \partial^\alpha \widehat{\phi}(0)$. All these coefficients vanish. As $\phi$ varies, this says the homogeneous polynomial $\sum_{|\alpha|=k} c_\alpha x^\alpha$ vanishes on an open cone, so all $c_\alpha$ are zero.

Apply this to $T = \widehat{\mu} - \sigma$; the double transform of $\mu$ is its reflection. Hence $\widehat{\mu} = \sigma$ globally. On taking inverse transforms, collect the derivatives of each point mass. The positive-order terms cancel at every site because $\mu$ is a measure. The order-zero part of each transformed summand is a constant multiple of a translated, modulated comb: moving a character through a derivative of a point mass only differentiates the character, whose derivatives are constant multiples of itself. Taking these order-zero parts proves the proposition. □

*Conclusion of Theorem 1.1.* Appendix A gave relative separation on both sides. Proposition B.3 gives two finite lattice covers, and Proposition B.4 gives the asserted formula. Conversely, Poisson summation places the support and the spectrum of any such finite sum in finite unions of full-rank lattice cosets. Each coset contributes $O((1 + R)^p)$ sites to any radius-$R$ ball by lattice packing. This implies the half-ball counts for every direction and for all radii tending to infinity. The weights on both sides are bounded by the sum of the absolute coefficients of their respective comb representations. □

**Archived proof sources.** The PDF includes three text attachments. The two parent source files record the ordered Fourier rank estimate and the recurrence argument at revision `1c6441d` [30]. The third contains the standalone geometric proof at revision `822f994` [31]. The same files are reproduced at the end of the accompanying LaTeX source. Appendices A and B give the complete argument used in this paper.

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