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Half-line density and the structure of crystalline measures

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Abstract

Let μ be a crystalline measure on the real line. Both μ and its Fourier transform are locally finite sums of point masses. We prove that μ is a finite sum of translated and modulated lattice Dirac combs if and only if its support and its spectrum each have finite lower linear density on at least one half-line. The two half-lines and the sequences of radii may be chosen independently. If the support of a nonzero measure has lower half-line density $d < \infty$, then d is the infimum of the frequency widths of the nonzero exponential polynomials vanishing on the support. In particular, the number of atoms per unit interval is uniformly bounded. If μ is not a generalized Dirac comb, its spectrum contains at least $(1/(2d^2) - o(1))T^2$ points in every interval of length T, uniformly in the position of the interval as $T \to \infty$. The proof combines a spectral gap theorem with a finite difference equation for the spectral masses. For a prescribed set Λ of lower density $d < \infty$ on one half-line, we describe the space of measures with support and spectrum in Λ and prove the sharp upper bound $\lfloor d^2 \rfloor$ for its dimension.

1 The counting theorems

Poisson summation identifies the Fourier transform of a lattice Dirac comb with a comb on the reciprocal lattice. Translations, modulations and finite linear combinations give a larger family of summation formulas. How can one recognize this family from the locations of the atoms alone? The criterion below uses two counts. On each side of the Fourier transform it suffices to count from the origin in a single direction, and only along a sequence of radii.

We use $e(t) = \exp(2\pi i t)$ and $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e(-x\xi)\,dx$. A *crystalline measure* is a tempered distribution μ on $\mathbb{R}$ such that, on compactly supported smooth test functions,

$$\mu = \sum_{a \in A} u_a \delta_a, \qquad \widehat{\mu} = \sum_{b \in B} v_b \delta_b.$$

Here A and B meet every compact interval in finitely many points, and the displayed coefficients are nonzero. Thus A and B are the *actual* support and spectrum, with each site counted once after cancellation. The coefficients may be complex. Temperedness is understood in the distributional sense throughout, including when the total variation has no polynomial growth bound.

For a locally finite set S put $N_S(R) = \#(S \cap [-R, R])$. Its lower densities on the two half-lines are

$$d_+(S) = \liminf_{R \to \infty} \frac{\#(S \cap [0, R])}{R}, \qquad d_-(S) = \liminf_{R \to \infty} \frac{\#(S \cap [-R, 0])}{R}. \tag{1}$$

Write $d_*(S) = \min\{d_+(S), d_-(S)\}$. We say that S is *relatively separated* if $\sup_x \#(S \cap [x, x+1]) < \infty$. This allows distinct points to be arbitrarily close. We use

$$D^+(S) = \limsup_{L \to \infty} \sup_{x \in \mathbb{R}} \frac{\#(S \cap [x, x + L])}{L} \tag{2}$$

for the upper uniform density. An exponential polynomial here is a finite sum $P(x) = \sum_j c_j e(s_j x)$ with real frequencies. For $P \neq 0$ in reduced form, define its *frequency width* by

$$\operatorname{width}(P) = \max_j s_j - \min_j s_j. \tag{3}$$

Theorem 1.1 (Density on one half-line)**.** *Let $\mu \neq 0$ be a crystalline measure with support A. If $d = d_*(A) < \infty$, then $d > 0$. For every $\ell > d$ there is a nonzero exponential polynomial P of width less than ℓ which vanishes on A. In particular, A is relatively separated. The counting densities on the positive and negative half-lines both exist and equal $D^+(A) = d$. Moreover,*

$$d = \inf\{\operatorname{width}(P) : P \neq 0 \text{ is an exponential polynomial vanishing on } A\}.$$

Following Meyer [10], a *generalized Dirac comb* is a finite linear combination of measures of the form

$$e(sx) \sum_{n\in\mathbb{Z}} \delta_{a+hn}, \qquad h > 0, \quad a, s \in \mathbb{R}.$$

The summands may have different lattice spacings. For instance, the sum of the combs on $\mathbb{Z}$ and $\sqrt{2}\,\mathbb{Z}$ has arbitrarily close atoms. Its Fourier transform has the same property, since the two reciprocal spacings are 1 and $1/\sqrt{2}$.

Theorem 1.2 (Quadratic lower bound for the spectrum)**.** *Let $\mu \neq 0$ be a crystalline measure with support A and spectrum B. Suppose $d = d_*(A) < \infty$. If μ is not a generalized Dirac comb, then*

$$\liminf_{R\to\infty} \frac{N_B(R)}{R^2} \geq \frac{2}{d^2}. \tag{4}$$

The same statement holds with the support and spectrum interchanged.

Corollary 5.1 gives the corresponding estimate uniformly over all translates of the counting interval and, in particular, on each half-line separately.

Corollary 1.3 (A half-line counting criterion)**.** *A crystalline measure is a generalized Dirac comb if and only if $d_*(A) < \infty$ and $d_*(B) < \infty$.*

Proof. For a generalized Dirac comb both supports are contained in finite unions of lattice translates, by Poisson summation. Conversely, Theorem 1.1 applied to $\widehat{\mu}$ gives $N_B(R) = O(R)$. For a nonzero measure, Theorem 1.2 now gives the conclusion. The zero measure is the empty sum. □

Relation to earlier results. Lev and Olevskii proved rigidity for measures on the line with uniformly discrete support and spectrum [8]. Their later Theorem 2.2 in [9] already permits bounded unit-interval counts on the spectral side, assuming a positive minimum spacing on the physical side. Baake, Strungaru and Terauds [2, Theorem 6.6] derived a representation on translates of a common lattice when the support lies in a Meyer set and the spectrum has finite upper density along a van Hove sequence. In our one-dimensional setting, Corollary 1.3 allows both sets to have arbitrarily small spacings. Theorem 1.2 also allows the spectrum to have unbounded unit-interval counts.

For the spectral gap argument, Lev and Olevskii [9, Proposition 6.1] obtain a positive lower bound for the centered counting density of a set that supports a nonzero slowly increasing measure with a spectral gap. Their bound depends on the gap length and a uniform bound on the unit-interval counts. In Lemma 2.1 we adapt the oscillation theorem of Eremenko and

Novikov [3, Theorem 1] to count locally finite atoms. This gives the gap length itself as a lower bound on each half-line, assuming only distributional temperedness and local finiteness of the atom set.

Meyer [11, Theorem 6] showed that if the support is uniformly discrete, a nonzero compactly supported continuous function annihilates $\widehat{\mu}$ under convolution. The proof following Lemma 15 of [11] constructs such a function for finite unions of uniformly discrete sets by convolving annihilators for the individual sets. Theorem 1.1 gives an annihilator with finitely many point masses in an interval of any length greater than $d_*(A)$. Meyer's Lemma 14 studies the support of a solution to a particular finite-difference equation. Proposition 3.1 gives a counting alternative for arbitrary finite shifts with nonzero extreme coefficients.

Olevskii and Ulanovskii characterize supports of unit-mass Fourier quasicrystals as zero sets of exponential polynomials [12]. Alon and Kummer obtain a Lee–Yang description for uniformly discrete crystalline measures with unit weights. For arbitrary weights on uniformly discrete supports they also prove finite generation of the spectral group and polynomial vanishing [1, Theorems 3.1 and 4.2]. Their Lemma 4.7 estimates the dimension of a space of modulations by restricting the Fourier transform to a fixed window. Here a half-line count determines how long that interval must be. By adjoining one frequency to the group generated by the relevant spectral sites, we obtain a polynomial whose width is arbitrarily close to the lower half-line density. Its zero set contains the support and may contain additional points.

Gonçalves and Vedana parameterize summation formulas under weighted growth conditions by almost periodic holomorphic functions [4]. The counting conditions above concern the locations of nonzero atoms and are independent of their sizes. Section 6 shows that one finite union of lattice translates contains the supports and spectra of all measures carried on both sides by a prescribed set of finite lower half-line density. The points of that union missing from the prescribed set impose finitely many linear conditions. This gives a reduction of the dimension question raised by Meyer [10, Section 6] and a sharp upper bound in terms of the lower half-line density.

Outline of the argument. A gap in the Fourier transform forces a minimum density of atoms on each half-line. We use this fact to turn a relation visible on one bounded spectral interval into an identity on the whole line. The resulting finite difference equation constrains the spectral masses. Either their locations lie in a finite union of lattice translates, or multiplying the masses by polynomials produces enough independent solutions to force quadratic growth. We prove this alternative in Section 3. Section 4 recovers the weights on the lattices and Section 5 proves Theorem 1.2 and Corollary 5.1.

2 From a half-line count to an exponential polynomial

A gap in frequency forces many atoms in physical space, even if only one half-line is observed. We use the oscillation theorem of Eremenko and Novikov, who also discuss its extension to tempered distributions [3, Theorem 1 and the subsequent discussion]. The proof below avoids a growth assumption on the total variation of the measure.

Lemma 2.1 (Point counts in the presence of a spectral gap)**.** *Let $T \neq 0$ be a tempered distribution that is locally a sum of point masses on a locally finite set A. If $\widehat{T}$ vanishes on a bounded open interval I, then*

$$\liminf_{R\to\infty} \frac{\#(A \cap [0,R])}{R} \geq |I|, \qquad \liminf_{R\to\infty} \frac{\#(A \cap [-R,0])}{R} \geq |I|.$$

Proof. Modulate T to center the gap at the origin, and write it as $(-h,h)$. After multiplication by a constant of modulus one, its real part U is nonzero. Taking the real part replaces the Fourier transform by its average with its conjugate reflection, so the symmetric gap persists. The atom set of U is contained in A.

We need a real, continuous function f of polynomial growth with $f^{(m)} = U$ for some $m \geq 2$, whose Fourier transform still has the gap. Here is one construction. Temperedness gives integers p, q and a bound

$$|U(\varphi)| \leq C \max_{0\leq j\leq q} \sup_{t\in\mathbb{R}} (1+t^2)^p |\varphi^{(j)}(t)|.$$

Apply Hahn–Banach to the map $\varphi \mapsto ((1+t^2)^p \varphi^{(j)})_{j=0}^q$ into a finite product of copies of $C_0(\mathbb{R})$. The Riesz representation theorem then writes $U = \sum_{j=0}^q D^j \nu_j$, where the real measures ν_j satisfy $\int (1+t^2)^{-p}\, d|\nu_j|(t) < \infty$. Each ν_j has a locally bounded primitive of polynomial growth, obtained by taking its cumulative mass from the origin. One more integration gives a continuous primitive. Integrating the terms to a common order $m = q+2$ thus gives a real continuous F of polynomial growth with $D^m F = U$.

On $(-h,h)$, the distribution $\widehat{F}$ is killed by ξ^m. Its restriction there is consequently a linear combination of $\delta_0, \ldots, \delta_0^{(m-1)}$. Subtract the inverse transform of this combination from F. The subtracted function is a real polynomial of degree less than m, by conjugate symmetry. The resulting f has the required gap and $D^m f = U$.

Let $s(f;[0,R])$ denote the greatest number of successive sign changes at ordered points where f is nonzero. Eremenko and Novikov use the minimum degree of a nonzero real polynomial p for which pf is nonnegative on $(0,R)$. For the continuous, piecewise polynomial function f here the counts agree. Each sign transition forces a root of p, and one root placed at each transition suffices. Continuity allows the testing points to be chosen in the open interval. The oscillation theorem of Eremenko and Novikov [3, Theorem 1] gives

$$\liminf_{R\to\infty} \frac{s(f;[0,R])}{R} \geq 2h.$$

Indeed, if $|f(t)| = O((1+|t|)^b)$ with $b \geq 0$, the uniformly continuous weight $\omega(t) = (b+2)\log(2+|t|)$ satisfies $\int_{\mathbb{R}} \omega(t)/(1+t^2)\, dt < \infty$ and $\int_{\mathbb{R}} |f(t)| e^{-\omega(t)}\, dt < \infty$. Their angular-frequency half-width is $2\pi h$ in our convention.

To bound the sign changes from above, consider an interval slightly larger than $[0,R]$. There f is the sum of a polynomial of degree at most $m-1$ and the functions $u_a (x-a)_+^{m-1}/(m-1)!$ for the finitely many atoms $u_a \delta_a$ in that interval. This follows by taking m distributional derivatives. If an absolutely continuous real function alternates sign at $k+1$ ordered points, its k successive increments alternate sign. Its derivative takes the respective signs on sets of positive measure in the intervening intervals. Each differentiation therefore costs at most one sign change. The derivatives through order $m-2$ are locally absolutely continuous. After $m-1$ differentiations we take the almost-everywhere step-function representative and choose the sign-testing points away from its jumps. Its jumps in $[0,R]$ occur only at atoms. Thus

$$s(f;[0,R]) \leq \#(A \cap [0,R]) + m - 1.$$

Division by R proves the first inequality. Reflection gives the second. □

In particular, any interval of length greater than $d_*(A)$ detects whether an exponential multiple $P\mu$ vanishes. If $\widehat{P\mu}$ vanishes there, then $P\mu = 0$. Indeed, multiplying by an exponential polynomial preserves temperedness and can only remove atoms from A. We now construct a nonzero polynomial P for which the Fourier transform of $P\mu$ vanishes on such an interval.

Proposition 2.2. *Suppose μ is a crystalline measure with support A, and $d_*(A) < \infty$. For each $\ell > d_*(A)$ there is a nonzero exponential polynomial P of frequency width less than ℓ which vanishes on A.*

Proof. Choose $J, L > 0$ with $d_*(A) < 2J < 2L < \ell$, and let $B = \operatorname{supp} \widehat{\mu}$. We seek frequencies in $[-L, L]$. Only the finitely many points of $B_0 = B \cap [-J - L, J + L]$ can be shifted by these frequencies into $[-J, J]$.

Choose a $\mathbb{Z}$-basis $\beta_1, \dots, \beta_r$ for the additive group generated by B_0. Choose $0 < \varepsilon < L - J$ outside their rational linear span. For $c > 0$ and an integer $N \geq 0$, put

$$E_c(N) = \{\beta \cdot n + \varepsilon k : n \in \mathbb{Z}^r,\ \|n\|_\infty \leq N,\ k \in \mathbb{Z},\ |\beta \cdot n + \varepsilon k| \leq c\}.$$

Each frequency has a unique representation. For fixed n, the eligible integers k lie in an interval of length $2c/\varepsilon$. Hence

$$(2N+1)^r(2c/\varepsilon - 1) \leq |E_c(N)| \leq (2N+1)^r(2c/\varepsilon + 1).$$

These bounds also hold for the trivial group, with $r = 0$. Choose an integer $H \geq 0$ bounding all coordinates of all points of B_0 in absolute value. Take $H = 0$ when there are no coordinates. All possible points in $(B + E_L(N)) \cap [-J, J]$ lie in $E_J(N + H)$. Since $2L/\varepsilon - 1 > 2J/\varepsilon + 1$, for large N the latter set contains fewer points than $E_L(N)$.

Fix such an N, and write $b_y = \widehat{\mu}(\{y\})$, taking $b_y = 0$ off B. The equations

$$\sum_{s \in E_L(N)} c_s b_{y-s} = 0, \qquad y \in (B + E_L(N)) \cap [-J, J],$$

have a nonzero solution because there are fewer equations than unknowns. The distinct characters in $P(x) = \sum_s c_s e(sx)$ are linearly independent, so P is nonzero. The equations say exactly that the mass of $\widehat{P\mu}$ vanishes at every possible point of $[-J, J]$. Lemma 2.1 now gives $P\mu = 0$. Each atom of μ has nonzero mass, hence P vanishes on A. Its frequency width is at most $2L < \ell$. □

Lemma 2.3 (Zero count for an exponential polynomial)**.** *For a nonzero exponential polynomial P of frequency width w, its number of real zeros in any interval of length T, counted with multiplicity, is at most $wT + C_P$.*

Proof. This is the elementary real-frequency case of Langer's zero count [6]. Write $P(z) = \sum_{j=1}^{k} c_j e(s_j z)$, with $s_1 < \cdots < s_k$ and all $c_j \neq 0$. On $\operatorname{Im} z = H$, for fixed sufficiently large H, the first summand dominates all the others combined. On $\operatorname{Im} z = -H$, the last summand does. Along a horizontal segment of length T, oriented from left to right, the changes of argument are therefore $2\pi s_1 T + O(1)$ at the top and $2\pi s_k T + O(1)$ at the bottom. The error is uniform in the endpoints, since the quotient by the dominant term stays in a disk about 1 of radius less than 1.

On any zero-free vertical edge $z = x + iy$, rotate the values by a constant phase so that their imaginary part is not identically zero. It is a real linear combination of the k functions $\exp(-2\pi s_j y)$, so it has at most $k - 1$ zeros, by division by the first exponential and induction

using Rolle's theorem. Between consecutive zeros the argument changes by at most π. The vertical contributions to the argument principle are thus bounded independently of the edge. Enlarge the real interval by less than one at each end so that the vertical edges avoid zeros. The argument principle on the resulting rectangle bounds the number of enclosed zeros by $wT + O_P(1)$. □

Completion of the proof of Theorem 1.1. If $d_*(A) < \infty$, Proposition 2.2 and Lemma 2.3 give $D^+(A) \le \ell$ for every $\ell > d_*(A)$. The zero count with $T = 1$ also bounds the number of points of A in every unit interval. Conversely, each half-line counting ratio has limsup at most $D^+(A)$, since its interval is among those in the uniform supremum. Both ray ratios therefore converge to $d_*(A) = D^+(A)$. The two preceding results also identify this common value with the infimum of the widths of nonzero exponential polynomials vanishing on A. Finally, a nonzero crystalline measure has strictly positive lower density on each ray. Its locally finite spectrum leaves some nonempty open interval free of atoms, to which Lemma 2.1 applies. □

3 A counting alternative for a finite difference equation

The following proposition concerns a function on the real line. Its proof uses only the positions of its nonzero values and a linear relation among its translates. The constant comes from counting monomials in two variables.

Proposition 3.1. *Let $u : \mathbb{R} \to \mathbb{C}$ have locally finite support $S = \{t : u(t) \neq 0\}$. Suppose that*

$$(Du)(t) := \sum_{j=0}^{k} a_j u(t - s_j) = 0 \quad (t \in \mathbb{R}), \qquad s_0 < s_1 < \cdots < s_k, \quad a_j \neq 0.$$

If $W = s_k - s_0 > 0$, then either S is contained in a finite union of translates of lattices in $\mathbb{R}$, or

$$\liminf_{R\to\infty} \frac{\#(S \cap [-R, R])}{R^2} \ge \frac{2}{W^2}.$$

Proof. Translating the equation, we may take $s_0 = 0$ and $s_k = W$. We may also assume $S \neq \varnothing$.

Integer coordinates for the support. If $t \in S$, the equation at t shows that $t - s_j \in S$ for at least one $j > 0$. The equation at $t + W$ shows that $t + W - s_j \in S$ for at least one $j < k$. In either direction, the step is positive, at most W, and bounded away from zero. Starting at any $t > W$ in S and stepping to the left, we must therefore reach $S \cap [0, W]$. From $t < 0$ we reach the same interval by stepping to the right.

It follows that S is contained in the additive subgroup G generated by the shifts s_j and the finite set $S \cap [0, W]$. This group is finitely generated and torsion-free. Fix a basis over $\mathbb{Z}$ and write $n(t) \in \mathbb{Z}^r$ for the coordinates of $t \in G$. Denote the inverse coordinate map by $\ell : \mathbb{Z}^r \to G$. The same stepping argument also proves uniqueness from an interval. If a solution of a difference equation of width w vanishes on an open interval of length greater than w, it vanishes everywhere. A chain of nonzero values starting on either side would have to enter that interval.

Polynomial multiples and observation on an interval. For a polynomial p in r variables, set $u_p(t) = p(n(t))u(t)$ for $t \in G$, and set $u_p(t) = 0$ elsewhere. If $\deg p \le m$, then

$$D^{m+1} u_p = 0.$$

For constants this is the original equation. For the induction step, subtract $p(n(t))(Du)(t)$ from $(Du_p)(t)$ for $t \in G$. The summand corresponding to s_j becomes

$$a_j\,[p(n(t-s_j)) - p(n(t))]\,u(t-s_j).$$

The expression in brackets is a polynomial of degree at most $m-1$ in $n(t-s_j)$. Thus Du_p is a sum of translates of lower-degree polynomial multiples of u. The induction hypothesis applies since D commutes with translations. Outside G every summand is zero.

The extreme shifts of D^{m+1} are 0 and $(m+1)W$, with nonzero coefficients a_0^{m+1} and a_k^{m+1}. Consequently, if I is any open interval of length greater than $(m+1)W$, the values of u_p on I determine u_p everywhere. Put $E = n(S)$, and let $H(m)$ be the dimension of the space of restrictions to E of polynomials of total degree at most m. Multiplication by u preserves this dimension, since u is nonzero at every point of S. Restriction to I is injective on these multiples, so

$$H(m) \le \#(S \cap I). \tag{5}$$

Quadratic growth or a union of lines. Let $V \subset \mathbb{C}^r$ be the Zariski closure of E, the common zero set of all polynomials vanishing on E. If an irreducible component of V has dimension at least two, two coordinate functions on it are algebraically independent. Indeed, the coordinate functions generate its function field, whose transcendence degree is its dimension. The monomials of degree at most m in these two coordinates are therefore linearly independent on E. Indeed, a relation on E would hold on V and hence on this component. We obtain $H(m) \ge (m+1)(m+2)/2$. In (5), take $I = (-R, R)$ and $m = \lfloor 2R/W \rfloor - 2$ for all sufficiently large R. Division by R^2 gives the asserted lower bound.

It remains to treat the case in which every component is a curve or a point. By the forward step above, for every $n \in E$ there is a vector

$$v \in F := \{n(W - s_j) : 0 \le j < k\} \qquad \text{with } n + v \in E.$$

Here F is finite and $\ell(v) > 0$ for every $v \in F$. Taking Zariski closures gives $V \subset \bigcup_{v \in F}(V - v)$. If C is an irreducible curve component of V, it lies in $C' - v$ for some component C' and some $v \in F$. The component C' must itself be a curve, and the inclusion is equality, since every proper closed subset of an irreducible algebraic curve is finite.

Starting at C we may thus repeatedly pass to a component that is a translate of the preceding one by a vector from F. There are finitely many components, so one repeats. The repeated curve is invariant under translation by the sum v_* of the intervening vectors. Since $\ell(v_*) > 0$, this sum is nonzero. For any point z on the repeated curve, it contains $z + nv_*$ for every positive integer n. Every polynomial vanishing on the curve therefore vanishes identically on the complex line $z + \mathbb{C}v_*$. The curve equals that line. Translating back along the chain shows that C was a line as well.

Finally, the integer points on any complex affine line are empty, a singleton, or a translate of a cyclic subgroup of $\mathbb{Z}^r$. Indeed, if there are two such points, their difference defines a primitive integer direction for the line. Applying ℓ sends these integer points into a translated lattice in $\mathbb{R}$. The finitely many point components can also be covered by translated lattices. This proves the other alternative. □

4 From lattice supports to lattice combs

Support on several lattices allows their spacings to have irrational ratios. Consequently the different lattices can come arbitrarily close. The following argument works directly with the

coefficients at their points.

Proposition 4.1. *Let μ be a crystalline measure on $\mathbb{R}$. Suppose that each of* $\operatorname{supp}\mu$ *and* $\operatorname{supp}\widehat{\mu}$ *is contained in a finite union of translates of lattices. Then μ is a finite linear combination of translated and modulated lattice Dirac combs.*

Proof. Put $\nu = \widehat{\mu}$, and cover $\operatorname{supp}\mu$ by the sets $a_j + h_j\mathbb{Z}$, with $h_j > 0$. The function

$$P(x) = \prod_j \bigl(e(x/h_j) - e(a_j/h_j)\bigr)$$

vanishes on $\operatorname{supp}\mu$. Write $\tau_t v(x) = v(x-t)$ for translation of the coefficient function of a point measure. Fourier transformation of $P\mu = 0$ gives

$$Q\nu = 0, \qquad Q = \prod_j (\tau_{t_j} - z_j), \qquad t_j = h_j^{-1}, \qquad z_j = e(a_j/h_j).$$

Group the lattices covering $\operatorname{supp}\nu$ according to the rational ratios of their spacings. Write $\nu = \sum_i v_i$ accordingly, assigning an atom at an intersection to one group. For now we use each v_i only through its coefficients. For each i, its support is contained in finitely many cosets of $h\mathbb{Z}$ for some $h > 0$ depending on i. The same holds for Qv_i. Lattices from different groups still intersect in at most one point after arbitrary translations, and only finitely many translations occur in Q. Since $\sum_i Qv_i = 0$, every nonzero coefficient of Qv_i occurs at one of these finitely many intersections. Thus Qv_i has finite support.

Fix i. We can remove from Q every factor with $t_j/h \notin \mathbb{Q}$ while preserving this conclusion. Indeed, suppose w is supported on finitely many cosets of $h\mathbb{Z}$, and $(\tau_t - z)w$ has finite support, where $t/h \notin \mathbb{Q}$ and $z \neq 0$. The translates of these cosets by kt are disjoint from the original cosets for some positive integer k. For each pair of cosets at most one k can give an intersection. By telescoping, $(\tau_{kt} - z^k)w$ has finite support. Its two terms have disjoint supports, so w has finite support. Applying this observation successively is legitimate because each remaining product of translations preserves containment in finitely many cosets of $h\mathbb{Z}$.

Choose a finer lattice $h'\mathbb{Z}$, $h' > 0$, so that h and all remaining t_j are integer multiples of h'. Along each of the finitely many relevant cosets of $h'\mathbb{Z}$ the coefficients of v_i now satisfy, outside a finite set, a constant-coefficient linear recurrence. Write $v(n)$ for the coefficient at $x_0 + nh'$ on a fixed coset and let $(Tv)(n) = v(n-1)$ be the backward shift. With $m_j = t_j/h'$, the remaining factors give

$$Q_i(T) = \prod_{j \text{ remaining}} (T^{m_j} - z_j).$$

For a solution $v(n) = \lambda^n$ of the homogeneous recurrence, $Q_i(\lambda^{-1}) = 0$. Since all roots of Q_i have modulus one, so do the exponential bases λ. The elementary solution formula for linear recurrences shows that the positive tail is a sum of terms $p(n)\lambda^n$, with p a polynomial and $|\lambda| = 1$. An empty product gives the zero tail.

Extend these formulas to all $n \in \mathbb{Z}$ on every coset and sum over the groups. The resulting measure σ is tempered, by polynomial growth, and agrees with ν on a right half-line. By Poisson summation, $\widehat{\sigma}$ is a finite sum of modulated derivatives of translated lattice combs. Put $\eta = \nu - \sigma$. Its transform $\widehat{\eta} = \widehat{\nu} - \widehat{\sigma}$ has locally finite support, hence vanishes on some open interval. On the positive ray the support of η is finite. Lemma 2.1 gives $\eta = 0$, so $\nu = \sigma$.

Finally, $\widehat{\nu}$ is the reflection of μ and consists of point masses. Expand the smooth modulation factors multiplying the derivatives of Dirac masses by the product rule. At each site the derivatives of different orders are linearly independent, so all positive-order terms cancel. The order-zero terms are modulated lattice combs, since every derivative of an exponential is a constant multiple of that exponential. Reflection gives the asserted form of μ. □

5 Proof of the quadratic bound and its consequences

Proof of Theorem 1.2. Write $d = d_*(A)$ and suppose that the left side of (4) is strictly smaller than $2/d^2$. Choose $\ell > d$ so close to d that

$$\liminf_{R\to\infty} N_B(R)/R^2 < 2/\ell^2.$$

By Theorem 1.1, some nonzero exponential polynomial P of width $W < \ell$ vanishes on A. A single exponential has no zeros, so $W > 0$. Taking Fourier transforms of $P\mu = 0$ gives a finite difference equation of width W for the function $b \mapsto v_b$, extended by zero off B. Proposition 3.1 forces B to lie in a finite union of lattice translates.

We also need a lattice cover on the physical side. Choose a cover $B \subset \bigcup_{j=1}^{m}(b_j + w_j\mathbb{Z})$, and take the product of the factors $e(\xi/w_j) - e(b_j/w_j)$. This nonzero exponential polynomial kills $\widehat{\mu}$. Its transform gives a difference equation of positive width for the coefficients of $\widehat{\widehat{\mu}}$ on $-A$. By Theorem 1.1, A is relatively separated, hence $N_{-A}(R) = O(R)$. A second application of Proposition 3.1 covers $-A$, and thus A, by finitely many lattice translates. Proposition 4.1 now identifies μ as a generalized Dirac comb. Reflection preserves centered counts and interchanges the two half-line densities, so the argument applies in the opposite direction as well. □

Corollary 5.1 (Quadratic growth in every interval)**.** *Under the hypotheses of Theorem 1.2, if μ is not a generalized Dirac comb, then*

$$\liminf_{T\to\infty} \inf_{x\in\mathbb{R}} \frac{\#(B \cap [x, x+T])}{T^2} \;\geq\; \frac{1}{2d^2}.$$

In particular, each spectral half-line satisfies

$$\liminf_{R\to\infty} \frac{\#(B \cap [0,R])}{R^2} \geq \frac{1}{2d^2}, \qquad \liminf_{R\to\infty} \frac{\#(B \cap [-R,0])}{R^2} \geq \frac{1}{2d^2}.$$

The conclusions also hold with the support and spectrum interchanged.

Proof. The second paragraph of the proof of Theorem 1.2 shows that a finite lattice cover of B would force μ to be a generalized Dirac comb. Thus B has no such cover. For any $\ell > d$, choose an annihilator of width $0 < W < \ell$ as in Theorem 1.1. The resulting recurrence for the spectral coefficients falls in the quadratic case of Proposition 3.1. For any sufficiently large T put $m = \lfloor T/W \rfloor - 2$, so that $T > (m+1)W$. The restriction estimate (5) and the monomial count in that proof give, for every $x \in \mathbb{R}$,

$$\#(B \cap [x, x+T]) \;\geq\; \#(B \cap (x, x+T)) \;\geq\; \frac{(m+1)(m+2)}{2}.$$

The right side is independent of x. Dividing by T^2, taking the infimum over x and then the lower limit in T gives at least $1/(2W^2)$, hence at least $1/(2\ell^2)$. Let $\ell \downarrow d$. Taking $x = 0$ and $x = -T$ yields the half-line estimates. Reflection gives the assertion in the opposite Fourier orientation. □

The quadratic scale is attained. Kurasov and Sarnak [5, Theorems 1 and 2] construct a positive unit-mass crystalline measure whose support is uniformly discrete and which is not a generalized Dirac comb. In their example the spectrum is contained in

$$\{0\} \cup \{\pm(mu + nv) : m, n \in \mathbb{Z}_{\geq 0},\ m + n > 0\}$$

for two positive frequencies u, v, after choosing the Fourier normalization. Counting the integer pairs with $mu + nv \le R$ gives $N_B(R) = O(R^2)$. Theorem 1.2 gives the matching lower order bound for the actual spectrum. Counting all semigroup points gives an upper bound even if some Fourier coefficients vanish. The example therefore establishes optimality of the exponent. The optimal universal coefficient in (4) remains to be determined.

The case of two subquadratic counts. Suppose that $N_A(R) + N_B(R) = o(R^2)$. If any one of the four half-line lower densities is finite, Theorem 1.2, applied in the appropriate direction, forces μ to be a generalized Dirac comb. Thus a counterexample to rigidity under this joint bound would have to satisfy

$$\frac{\#(S \cap [0,R])}{R} \longrightarrow \infty \qquad \text{and} \qquad \frac{\#(S \cap [-R,0])}{R} \longrightarrow \infty \qquad (S = A, B).$$

Whether such a crystalline measure exists is open.

6 Measures with prescribed support and spectrum

For a locally finite set $\Lambda \subset \mathbb{R}$, let M_Λ be the complex vector space of crystalline measures μ such that $\operatorname{supp}\mu \subset \Lambda$ and $\operatorname{supp}\widehat{\mu} \subset \Lambda$. Meyer [10, Section 6] asked which sets admit a nonzero measure and how the dimension depends on the geometry of the set. For $d_*(\Lambda) < \infty$, Corollary 1.3 identifies the measures as generalized Dirac combs. We show that their supports and spectra have a common cover by finitely many lattice translates, with only finitely many points of the cover outside Λ. This reduces the dimension question to the space on that cover and the linear conditions at its missing points. Meyer also asked about the sets $\{\pm\sqrt{k+\theta} : k \in \mathbb{N}\}$ for $0 < \theta < 1$. Their half-line counts grow quadratically, so that question lies outside the hypothesis below.

Let L range over finite unions of full lattice translates for which $F_L = L \setminus \Lambda$ is finite. Include the empty set, with $M_\varnothing = 0$. On M_L define the map

$$E_{F_L} \colon \mu \longmapsto \big(\mu(\{x\}), \widehat{\mu}(\{x\})\big)_{x \in F_L}. \tag{6}$$

Its rank counts independent conditions. In particular, a single missing point may impose two independent conditions.

Theorem 6.1. *If $d = d_*(\Lambda) < \infty$, then every member of M_Λ is a generalized Dirac comb and $\dim_{\mathbb{C}} M_\Lambda \le \lfloor d^2 \rfloor$. One finite union L of lattice translates contains the supports and spectra of all members of M_Λ, and $F_L = L \setminus \Lambda$ is finite. For this L, the space M_Λ is the kernel of (6) in M_L. In particular,*

$$\dim_{\mathbb{C}} M_\Lambda = \max_{L\colon \#(L\setminus\Lambda)<\infty} \big(\dim_{\mathbb{C}} M_L - \operatorname{rank} E_{F_L}\big), \tag{7}$$

where L ranges over the unions specified above.

Proof. Both actual supports of every $\mu \in M_\Lambda$ inherit the half-line bound. Corollary 1.3 makes μ a generalized Dirac comb. For every bounded open interval I with $|I| > d$, the map

$$\mu \longmapsto \big(\widehat{\mu}(\{b\})\big)_{b \in \Lambda \cap I}$$

is injective by Lemma 2.1. Thus $k = \dim_{\mathbb{C}} M_\Lambda$ is finite and every open interval of length $T > d$ contains at least k points of Λ. In either $[0,R]$ or $[-R,0]$ one can fit $\lfloor R/T \rfloor$ disjoint open

intervals of length T. Taking the lower density on each ray gives $d \geq k/T$. Hence $k \leq dT$ for every $T > d$. Letting $T \downarrow d$ proves $k \leq \lfloor d^2 \rfloor$. The same reasoning gives finite dimension for M_L.

We record why finitely many missing points suffice. The support of any generalized Dirac comb is a finite union of full lattice translates with at most finitely many points removed. To see this, group its terms by commensurable spacings. Within a group, take a positive common multiple of the finitely many spacings and split each lattice into cosets of that common period. On each coset the coefficient sequence has the form $\sum_j c_j \lambda_j^n$, with $|\lambda_j| = 1$. By the Skolem–Mahler–Lech theorem [7], its zeros on the integers consist of finitely many full arithmetic progressions and a finite set. For the two-sided formulation, apply the theorem on each half-line. Vanishing on a tail of a progression implies vanishing on the entire progression by the invertible recurrence for its subsequence. Removing the full zero progressions from the coset leaves a finite union of full subprogressions. The remaining zeros remove only finitely many points. Between different groups only finitely many intersection sites can undergo further cancellation. Poisson summation gives the same description for the spectrum.

Take a basis of M_Λ. For the support and spectrum of each basis vector choose the finite lattice union just obtained. Their union L contains all these supports and spectra, and $L \setminus \Lambda$ is finite. It follows that M_Λ is precisely the kernel of E_{F_L} in M_L. For every other eligible L' the corresponding kernel is a subspace of M_Λ. Rank–nullity proves (7), including attainment. For the zero space take $L = \varnothing$. □

Once an attaining L is known, the two coefficient conditions at each point of F_L form a finite linear system on M_L. Finding such an L from a description of Λ is a separate problem. The proof obtains L from the supports and spectra of a basis of M_Λ.

Example 6.2 (Deleting the origin)**.** The dimension bound is sharp for $L = q^{-1/2}\mathbb{Z}$, where q is a positive integer. For $\mu \in M_L$, multiplication by $e(\sqrt{q}\,\xi)$ fixes $\widehat{\mu}$, so μ is periodic with period $\sqrt{q}$. Its coefficients on L are specified by q numbers repeated periodically. Conversely, every such choice belongs to M_L by Poisson summation. Thus $\dim_{\mathbb{C}} M_L = q = d_*(L)^2$.

Now take $q = 3$ and $\Lambda = L \setminus \{0\}$. Write (c_0, c_1, c_2) for the three repeated coefficients. The physical mass at zero is c_0, and the Fourier mass there is $(c_0 + c_1 + c_2)/\sqrt{3}$. The two conditions are independent. Thus $\dim_{\mathbb{C}} M_L = 3$ and $\dim_{\mathbb{C}} M_\Lambda = 1$. The kernel is spanned by the measure with repeated weights $(0, 1, -1)$, the character modulo three used by Meyer [10, Lemma 4.2].

A carrier with unbounded unit-interval counts. Take $\Lambda = \mathbb{Z} \cup \{-\sqrt{n} : n = 1, 2, \ldots\}$. Its positive half-line has density one, while $[-m-1, -m]$ contains $2m+2$ distinct points of Λ for each integer $m \geq 1$. In particular, $d_*(\Lambda) = 1$ and the unit-interval counts are unbounded. The integer comb belongs to M_Λ by Poisson summation. Theorem 6.1 now gives $M_\Lambda = \mathbb{C} \sum_{n \in \mathbb{Z}} \delta_n$. Here $L = \mathbb{Z}$ attains (7) and F_L is empty.

Formalisation. The one-ray formalisation [13] proves that a finite lower half-line count on one carrier gives a nonzero exponential polynomial annihilating the measure. When the counted carrier is the actual support and the opposite count satisfies $\liminf_{R \to \infty} N_B(R)/R^2 = 0$, it proves the comb conclusion on both sides, in either Fourier orientation. The common-carrier formalisation [14] proves the attained finite-hole formula, including the joint rank of the hole conditions, for bounded unit-interval counts. Theorems 1.1 and 1.2 here give the optimal width and the bound $2/d^2$, with the uniform estimate in Corollary 5.1. Theorem 6.1 gives (7) and the sharp bound $\lfloor d^2 \rfloor$ under the one-ray hypothesis.

References

[1] L. Alon and M. Kummer, A characterization of uniformly discrete crystalline measures with unit weights, arXiv:2611.24286 (2026).

[2] M. Baake, N. Strungaru and V. Terauds, Pure point measures with sparse support and sparse Fourier–Bohr support, *Trans. Lond. Math. Soc.* **7** (2020), 1–32.

[3] A. Eremenko and D. Novikov, Oscillation of Fourier integrals with a spectral gap, *J. Math. Pures Appl.* (9) **83** (2004), 313–365.

[4] F. Gonçalves and G. Vedana, A complete classification of Fourier summation formulas on the real line, arXiv:2504.02741 (2025).

[5] P. Kurasov and P. Sarnak, Stable polynomials and crystalline measures, *J. Math. Phys.* **61** (2020), 083501.

[6] R. E. Langer, On the zeros of exponential sums and integrals, *Bull. Amer. Math. Soc.* **37** (1931), 213–239.

[7] C. Lech, A note on recurring series, *Ark. Mat.* **2** (1953), 417–421.

[8] N. Lev and A. Olevskii, Quasicrystals and Poisson's summation formula, *Invent. Math.* **200** (2015), 585–606.

[9] N. Lev and A. Olevskii, Fourier quasicrystals and discreteness of the diffraction spectrum, *Adv. Math.* **315** (2017), 1–26.

[10] Y. Meyer, Measures with locally finite support and spectrum, *Rev. Mat. Iberoam.* **33** (2017), 1025–1036.

[11] Y. Meyer, Crystalline measures and mean-periodic functions, *Trans. R. Norw. Soc. Sci. Lett.* (2021), no. 2.

[12] A. Olevskii and A. Ulanovskii, Fourier quasicrystals with unit masses, *C. R. Math. Acad. Sci. Paris* **358** (2020), 1207–1211.

[13] Jamie Martin, Annihilation and finite-comb classification from finite lower density on one ray, Palomar Registry, PALOMAR-2026-11-20-000023, version 1 (2026). Registry record.

[14] Jamie Martin, Finite-comb and finite-hole classification for relatively separated carriers, Palomar Registry, PALOMAR-2026-11-20-000019, version 1 (2026). Registry record.