# The $(3, 4, \infty)$ family of complex six-spheres

# V. Complex fourfolds from the family

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**Abstract**

We construct compact complex fourfolds diffeomorphic, with their complex orientation, to $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$. They form a family of complex three-tori over $\mathbb{P}^1$ with monodromy in $GL_6(\mathbb{Z})$: two multiple fibres, of multiplicities five and four, and one Kuhn cusp. Its periods come from the exterior square of the genus-two pencil $y^2 = x^5 + \lambda^3 x + \lambda^4$. The family is parametrized exactly by the half-plane $\operatorname{Im}\xi > c_*$, where $c_* \approx 1.09454$ is minus the minimum of an explicit real-analytic period function. We compute the Hodge numbers, the tangent cohomology (the Kuranishi germ is smooth of dimension one) and the automorphism group $\mathbb{C} \times \mathbb{Z}/2$, and we show that the family extends analytically across no boundary point of its parameter domain. We also construct a compact complex fourfold outside Fujiki's class $\mathcal{C}$ as a small crepant resolution of a Weierstrass model, bimeromorphic to the fibre product of a complex six-sphere with an elliptic surface; give an effective classification of the integral monodromy data of this kind that yield three-connected eight-manifolds, all of which are diffeomorphic to $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$; and describe shifted fibre products of complex dimension five.

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# 1 Results and their proofs in outline

## 1.1 The series and this paper

This paper is the fifth of a series on the holomorphic families built from the $(3, 4, \infty)$ family of torus-fibred complex structures on $S^6$ (Claude and L. Alpöge, 2026; Engel, 2026). The table gives the main result of each paper of the series and the papers it uses.

| Paper | Main result | Uses |
|---|---|---|
| I | Exact periods, moduli, fibre strata and the two ends | — |
| II | Torus-fibred threefolds with $b_1 = b_2 = 0$: the Campana–Demailly–Peternell argument and the sign theorem for all degenerations | I, IV |
| III | Maps from the six-spheres and the manifolds they give | I, II, IV, V |
| IV | Smooth monodromy in $\Theta_7$ | I, II, III, V |
| V | Complex fourfolds on $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$ and a resolved fibre product | I, III, IV |

## 1.2 The main theorems

**Definition 1.1** (The period parameter)**.** [tag PV14] We use the algebraic splitting $\sigma_b$, invariant vector $e$ and functional $\alpha$ with the exact normalization of definition 3.7 and theorem 3.6. For the rank-three bundle $P$ of theorem 3.5, put

$$F_\xi = (\sigma_b + \xi\alpha e)(P), \qquad c_* = -\min h_b, \qquad \mathcal{U} = \{\xi \in \mathbb{C} : \operatorname{Im}\xi > c_*\},$$

where $h_b$ is the real-intersection function of definition 3.7. We write $M_\xi$ for the compact filling of these periods with the base and fibre conjugation of remark 3.10. For the imaginary ray we write $M_c = M_{ic}$, with $c > c_*$. The positive Kuhn fan is fixed throughout this family.

**Theorem A** (The oriented diffeomorphism type)**.** [tag 03KU] For every $\xi \in \mathcal{U}$, the genus-two periods of theorem C, with the integral datum of definition 2.1, extend over the finite root charts and the positive marked Kuhn cusp. They give a compact complex fourfold $M_\xi$ and an orientation-preserving diffeomorphism

$$M_\xi \;\cong\; 3\mathbb{HP}^2\#(-\mathbb{HP}^2).$$

The fourfolds form a holomorphic family over $\mathcal{U}$. Their smooth middle lattice and spin characteristic vector are isometric to $\big(I_{3,1}, (1, 1, 1, 1)\big)$.

*Mechanism.* The middle-lattice isometry transports both Euler and Pontryagin clutching data, after which the cited absorption theorem removes the homotopy sphere. *Relation to literature.* The compact complex realization is new; the smooth classification and absorption results used to identify its underlying manifold are listed in §1.4. Outline of the proof.

**Theorem B** (The exact family and its two ends)**.** [tag 03KV] Put

$$c_* = -h_{b,\min} = \frac{96}{625\pi}\,\operatorname{Re}(XH_\infty)_{03}, \qquad \mathcal{U} = \{\xi \in \mathbb{C} : \Im\xi > c_*\},$$

where $X$ is the normalized integral intertwiner of theorem 3.6 and $H_\infty$ is its regular infinity-root fundamental-solution limit, normalized at $\lambda = 1$ as in theorem 7.1. Thus

$$c_* \approx 1.09454.$$

The marked family is parametrized exactly by $\mathcal{U}$: its members are pairwise nonisomorphic, have algebraic dimension one and automorphism group $\mathbb{C} \times \mathbb{Z}/2$, and admit no antiholomorphic isomorphism for this fan. The real-intersection function determines the parameter domain; the

invariant line and one half-period determine the automorphisms, while the reflected cusp map fails to preserve a ray of the chosen fan. On the effective root curve put $L = \mathcal{O}((4/5)[\infty] - (3/4)[0])$. Then

$$R^q h_* \mathcal{O} = \bigwedge^q (\mathcal{O} \oplus L^{-1} \oplus L^{-2}), \qquad K_M = h^* L^{-6}.$$

With $p$ indexing rows and $q$ indexing columns, the Hodge table is

| $h^{p,q}$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 2 | 1 | 0 | 0 |
| 2 | 0 | 1 | 6 | 1 | 0 |
| 3 | 0 | 0 | 1 | 2 | 0 |
| 4 | 0 | 0 | 0 | 1 | 1 |

The tangent tuple is $h^q(T_M) = (1, 1, 1, 2, 0)$ and its Kuranishi germ is smooth of dimension one. A full parameter disc across the transversality boundary has the proper-filling obstruction of theorem 9.3. A proper smooth family on a punctured disc whose fibres belong to $\mathcal{U}$ has the smooth filling after the explicit degree-two torsor cover of theorem 9.4.

*Mechanism.* The normalized Frobenius period minimum fixes the exact domain, and the marked period and cap maps determine the family invariants and isomorphisms. *Relation to literature.* The exact parameter domain and the family invariants are new; the cited direct-image and torus-deformation inputs are listed in §1.4. Outline of the proof.

**Theorem C** (The genus-two period construction)**.** [tag `03KT`] The pencil $y^2 = x^5 + \lambda^3 x + \lambda^4$ has one nodal fibre away from $0, \infty$, at $\lambda_* = -256/3125$. Its good root charts have orders eight and five, and on that root curve its Hodge bundle is

$$\mathcal{H} = T^3 \oplus T, \qquad T = \mathcal{O}((2/5)[\infty] - (3/8)[0]), \qquad \deg T = 1/40.$$

The meromorphic differential $dx/y$ generates $\ell_0 = T^3$. In the full rank-four de Rham bundle, projection of $\ell_0 \wedge \mathcal{H}_{\mathrm{dR}}$ to $\bigwedge^2 \mathcal{H}_{\mathrm{dR}}/\mathbb{C}\omega$ is a rank-three bundle $P$. Its holomorphic lift to the rank-six extension is $(\sigma_b + \xi\alpha e)(P)$, with the normalization of $e, \alpha, \sigma_b$ in theorem 3.6. It is transverse to its conjugate on the completed base exactly for the parameters $\xi \in \mathcal{U}$ of Theorem B. The exterior-square and lift maps descend the finite root orders to five and four; the cusp has period $Z(q) = (\log q/(2\pi i))I_3 + C(q)$ with $C$ holomorphic at zero (theorems 3.1, 3.2, 3.5, 3.9 and 7.1).

*Mechanism.* The wedge image of the genus-two Hodge line has rank three, and its split lift becomes transverse after the normalized imaginary shift. *Relation to literature.* The pencil, its integral exterior-square comparison and its normalized lifted planes are new. Outline of the proof.

**Theorem D** (The integral rank-six construction)**.** [tag `03KS`] The rank-six integral monodromy of definition 2.1, with free affine fillings of orders five and four and the index-one Kuhn cusp, gives a closed smooth eight-manifold that is three-connected, has free integral homology, and has middle intersection form $3\langle 1\rangle \oplus \langle -1\rangle$.

*Mechanism.* In the integral Leray spectral sequence over $\mathbb{P}^1$, each order-two suspension class from the order-four filling transgresses to the corresponding nonzero global coinvariant, so the resulting homology is free; Todd integrality and the signature bound then force the middle form. *Relation to literature.* The integral rank-six datum and its global homology calculation are new; the toroidal filling is the cited construction listed in §1.4. Outline of the proof.

**Theorem E** (The integral input classification)**.** [tag 03KW] Let $k \in \{2,3\}$ and $\Lambda = \mathbb{Z}^6$. Consider ordered integral tuples $(M_1, \dots, M_k, M_0)$ with each $M_i$ of finite order and having a nonzero fixed vector, with $M_0 = (M_1 \cdots M_k)^{-1}$, and with $N = M_0 - I$ satisfying $N^2 = 0$, rank three, and $\operatorname{im} N = \ker N$ as integral sublattices of $\Lambda$. Every such tuple has a computable canonical representative and an explicit simultaneous unimodular conjugator; equality of the representatives is equivalent to simultaneous integral conjugacy. For input entry height $H$, the finite stopping bound is $(2H+1)^{36(k+1)}$ candidates before the proved rejection filters. The ordered sector with both finite linear orders two has exactly 56 classes. Each has free affine lifts of affine order four, and every free affine filling of this sector has nontrivial fundamental group.

Specify the free affine actions $\tau_i(x) = M_i x + v_i/m_i$, with $v_i \in \Lambda$ and affine order $m_i$, an integral translation clutch $\kappa \in \Lambda$, and a primitive Kuhn cusp marking. The algorithms compute the full fundamental group and every integral homology group of the resulting marked gluing for every such input. Their constructive parametrization and finite tests identify exactly the inputs with $\pi_1 = 1$ and $H_2 = H_3 = 0$ (theorem 15.4). Every resulting three-connected manifold is orientation-preservingly diffeomorphic to $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$, by corollary 13.6.

On $B^* = \mathbb{P}^1 \setminus \{p_0, \dots, p_k\}$ with $k \geq 1$, every representation $\rho : \pi_1(B^*) \to GL^+(6,\mathbb{R})$ whose $p_0$ monodromy satisfies $(M_0 - I)^2 = 0$ and $\operatorname{rank}(M_0 - I) = 3$ has a holomorphic period family transverse to its conjugate everywhere on $B^*$. In the marking $M_0 = \left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$, its cusp period is

$$Z(t) = \frac{\log t}{2\pi i} I_3 + C(t), \qquad C \text{ holomorphic at } 0,$$

with any prescribed finite jet of $C$ at zero. An integral primitive cusp admits an integral marking. The other $p_i$ remain punctures in this assertion (theorem 16.5).

For a polarized algebraic abelian-surface family with toric-rank-two cusp, the construction that takes a fibrewise degree-zero line bundle and quotients its punctured bundle by a lifted translation cannot have $b_1 = b_2 = 0$ with the stipulated rank-three square-zero Kuhn cusp; this holds for both proportional and nonproportional source sections. The filtration and Euler-parity proof is theorem 16.1.

*Mechanism.* Exact intertwiner lattices decide simultaneous conjugacy, finite integral complexes compute each marked gluing, and the cusp-period construction realizes the punctured analytic data. *Relation to literature.* The classification and bounded-cusp period theorem are new; their cited lattice-isomorphism and Oka inputs are listed in §1.4. Outline of the proof.

**Theorem F** (The resolved transverse elliptic fibre product)**.** [tag 03KR] Let $X$ be a canonical marked member of the $(3,4,\infty)$ family, with reduced multiple fibres $S_1, S_2$ of multiplicities three and four, and put $L = \mathcal{O}_X(S_1 - S_2)$. Let $S \to \mathbb{P}^1$ be the elliptic surface in theorem 17.2, with its cusp over the cusp of $X$. The Weierstrass model $\mathscr{W} \to X$ of that theorem has a small crepant resolution $r : Z \to \mathscr{W}$. Its composition $\Pi : Z \to X$ is proper, and $Z$ is bimeromorphic to $X \times_{\mathbb{P}^1} S$. It is a compact complex fourfold with $K_Z = \Pi^* L^{-5}$, algebraic dimension two, $b_1(Z) = 0$, and $h^{0,1}(Z) = 1$. Its signature is two and its middle intersection form is $\langle 1 \rangle \oplus \langle 1 \rangle$. It lies outside Fujiki's class $\mathcal{C}$.

*Mechanism.* The explicit period comparison and a small crepant Kuhn resolution preserve the root canonical calculation and the integral characteristic pairings. *Relation to literature.* The global small crepant resolution and its computed invariants are new; the cited six-sphere, elliptic and toroidal constructions are the inputs listed in §1.4. Outline of the proof.

## 1.3 The proofs in outline

### Theorem A: The oriented diffeomorphism type

1. The lifted periods commute with the finite root actions and have the prescribed bounded cusp correction, so the same quotient and toroidal charts are holomorphic and proper. Their

collar identifications are the marked maps used in the smooth construction (theorem 3.11, theorem 2.4).

2. The preceding integral calculation gives the intersection lattice $I_{3,1}$. The spin characteristic vector $p_1/2$ has square two, by the Chern-number calculation (theorem 6.1).

3. The characteristic vector has odd coordinates and norm two in $I_{3,1}$. The integral reflection in $(1,1,1,1)$ strictly decreases the absolute negative coordinate until all four coordinates have absolute value one. The resulting explicit isometry is proved in theorem 6.1.

4. In the quaternionic clutching coordinates, the map is $(a,b) \mapsto (e, p_1/2) = (a-b, a+b)$, with inverse $((e+p_1/2)/2, (p_1/2-e)/2)$. Thus the pairing and characteristic vector determine the normal datum; the Poincaré-duality map transports it with the middle lattice isometry (theorem 6.2).

5. Each quaternionic projective-plane summand has $p_1 = 2u$. The oriented connected-sum pairing and characteristic vector are therefore the same pair $(I_{3,1}, (1,1,1,1))$. Wall's stated extended-form classification gives the oriented diffeomorphism up to a homotopy eight-sphere (theorem 6.2).

6. Kramer–Stolz's Theorem A absorbs that sphere in one positive $\mathbb{HP}^2$ summand. Its diffeomorphism has degree $+1$ because the square of a middle generator evaluates to 1 on both sides. Oriented connected-sum maps therefore give the required oriented diffeomorphism of the full fourfold (theorem 6.2).

**Theorem B: The exact family and its two ends**

1. The infinity-root symmetry makes the differential of the normalized height vanish at zero. Its positive Hessian proves a strict local minimum. Convergent root and cusp expansions, with interval continuation over finite closed covers, prove the height is strictly larger on their complements; the positive cusp graph treats the remaining inner disc. This proves the unique global minimum and the exact domain in theorem 7.1.

2. On the effective root curve of orders five and four put $L = \mathcal{O}((4/5)[\infty] - (3/4)[0])$. The direct images are $R^q h_* \mathcal{O} = \bigwedge^q (\mathcal{O} \oplus L^{-1} \oplus L^{-2})$, and $K_M = h^* L^{-6}$. The toric normalization complex proves the exterior-algebra assertion at the cusp, and the root-chart invariant cohomology proves it at the finite fibres (theorem 8.2).

3. The logarithmic cotangent sequence and its explicit residue matrices give the Hodge table. Serre duality applied to the twisted logarithmic and residue sequences gives $h^q(T_M) = (1,1,1,2,0)$. The normal tangent map and the period lift give a nonzero Kodaira–Spencer class; the local analytic-ring comparison makes the Kuranishi germ smooth of dimension one (theorem 8.3, theorem 8.4).

4. Restriction of a horizontal divisor to a regular torus fibre gives an invariant alternating class. The computed invariant space is zero, while positive pairing with a torus Kähler form rules out a nonzero effective divisor of zero class. Thus the meromorphic field is the base field $\mathbb{C}(\lambda)$. Every isomorphism therefore preserves the base and its three distinguished values. The simultaneous linear centralizer is $\mathbb{Q}I$; the order-five affine norm excludes its negative integral unit. The translation sheaf has identity component $\mathbb{C}$ and an integral parabolic quotient $\mathbb{Z}/2$, represented by the displayed half-period, proving the automorphism group and equality of parameters for isomorphic members (theorem 8.5).

5. The two integral anti-intertwiners are $\pm R$. One is excluded by the finite cap norm. For the other, the based cusp correction acts on rays by $-P$, and takes $e_2$ to $(0,-1,1)$, outside the

component's rays $\pm(\{0,1\}^3 \setminus \{0\})$. The corresponding monomial map contracts a divisor, contradicting a biholomorphism (theorem 8.6).

6. A proposed proper Hausdorff filling across a boundary point of a full parameter disc recovers the marked base from the moving pencil of $K_M^{-4}$, whose two sections have ratio $\lambda$. Closed relative forms give torus fibres outside a proper analytic subset. Their marked periods continue as $F_\xi$, while the real hypersurface $h_b + \operatorname{Im}\xi = 0$ cannot be contained in that analytic subset. A torus fibre there violates real transversality (theorem 9.3). For a family on a punctured disc whose fibres belong to $\mathcal{U}$, the unique classifying map is holomorphic; its Cayley transform extends and the maximum principle keeps its limit inside $\mathcal{U}$. After the explicit degree-two removal of the finite torsor, that family has a smooth filling (theorem 9.4).

**Theorem C: The genus-two period construction**

1. The pencil has one node at $\lambda_* = -256/3125$ and good reductions after root orders eight and five. Its Hodge bundle on that root curve is $T^3 \oplus T$, with $T = \mathcal{O}((2/5)[\infty] - (3/8)[0])$ and $\deg T = 1/40$ (theorem 3.1).

2. In the exterior-square lattice, projection induces $j : \omega^\perp \to (\bigwedge^2 \mathbb{Z}^4)/\mathbb{Z}\omega$ with Smith factors $(1,1,1,1,2)$. This exact index-two map identifies the rational systems, and the quotient is the dual of the primitive integral lattice. The lower five-dimensional blocks of the given rank-six tuple are the quotient blocks under this map (theorem 3.3).

3. Put $P = p(\ell_0 \wedge \mathcal{H}_{\mathrm{dR}})$, where $\ell_0 = \langle dx/y \rangle$ and $p$ is projection modulo $\omega$. The wedge map has rank three and its image meets $\mathbb{C}\omega$ trivially: a decomposable two-vector has square zero, whereas $\omega^2 \neq 0$. This proves the required rank before a lift is chosen (theorem 3.5, theorem 3.2).

4. The root-line decomposition of $P$ has $H^1(P^*) = 0$ and a one-dimensional space of functionals by theorem 3.5. Its extension therefore splits holomorphically. The exact contraction and period comparison in definition 3.7 identify the normalization of $\alpha$ and $\sigma_b$.

5. For the real-intersection line, the normalized lift has transversality exactly when $h_b + \operatorname{Im}\xi \neq 0$. The cusp has $Z = (\log q)/(2\pi i)I_3 + C(q)$ and $h_b \to +\infty$. Thus the complete parameter domain is the half-plane above $-\min h_b$, by theorems 3.9 and 7.1.

**Theorem D: The integral rank-six construction**

1. The matrices of definition 2.1 have orders five and four. The square-zero cusp has primitive rank-three image, and its explicit unimodular marking is proved in theorem 2.3. The invariant covector $x \mapsto x_0$ detects every nonidentity affine power, proving the freeness used in lemma 2.9.

2. The cusp and finite quotient charts give the marked smooth gluing. Its van Kampen relations reduce to $5a = h$, $4b = -h$ and $a + b = 0$; their integral relation matrix is unimodular. Hence the manifold is simply connected, by lemma 2.9.

3. The signed-orbit bases at the finite cores, their norm maps, and the primitive cusp collapse give the integral restriction sheaves and their point kernels. Determinant minors compute their cohomology in lemma 5.2, using the collapse of theorem 2.6.

4. For the order-four core, the loops $v = g^4h^{-1}$ and $t = g$ form an integral basis. A torsion representative pulls back to $(0, w)$, so its transgression is $\pm[w]$ by lemma 5.3. The three nonzero global images are proved in lemma 5.2.

5. The three torsion differentials are isomorphisms onto $\mathbb{Z}/2$. The remaining cusp kernel is free of rank one. The Leray filtration, universal coefficients, duality and Hurewicz then give free integral homology and three-connectivity in theorem 5.4.

6. The almost complex structure has $c_1 = c_3 = 0$. The index and signature identities of proposition 4.1 force $c_2^2 = 2$, $\sigma = 2$ and the middle form $I_{3,1}$. The Pontryagin identities and characteristic vector are computed in theorem 6.1.

**Theorem E: The integral input classification**

1. Primitive square-zero cusp matrices have the explicit integral normal form and centralizer $\left(\begin{smallmatrix} P & Q \\ 0 & P \end{smallmatrix}\right)$. Eigenvalue one is necessary for a free affine lift, leaving the finite cyclotomic list and its complex-compatibility filter (proposition 2.2, proposition 10.4).

2. The rational intertwiner algebra, its radical and its semisimple quotient reduce the simultaneous integral-conjugacy problem to the cited lattice-isomorphism algorithm over the resulting product of matrix algebras over number fields. Searching by height and lexicographic order terminates by the input height, yielding a canonical representative and an explicit unimodular conjugator (theorem 15.1, theorem 15.2).

3. For that conjugator $R$, the coordinate map $x \mapsto Rx$ transports $(M_i, v_i, \kappa, P)$ to $(RM_iR^{-1}, Rv_i, R\kappa, RP)$. Since $(RP)^{-1}(Rx) = P^{-1}x$, it intertwines the cusp charts as well as the affine caps and clutch. The resulting finite cellular cone and group presentation compute all integral homology and the full fundamental group (theorem 14.1, theorem 14.6, theorem 15.4).

4. The exact two-point tests and the three-point orbifold quotient give the complete constructive parametrization; the separate involution branch has 56 classes. The displayed unbounded trace family shows why the terminating bound depends on the input (corollary 13.6, theorem 15.3, theorem 15.4).

5. Nine explicitly given complex-line directions and a uniform singular-value bound put a spanning tube in the unpolarized period domain. The local-spray approximation theorem, with the relative extension-point argument proved here, produces a holomorphic transverse period section and retains any prescribed bounded cusp jet (theorem 16.2, lemma 16.4, theorem 16.5).

6. For a polarized abelian-surface source, semisimplicity and the maximal square-zero cusp force the middle rank-four system to have no invariant or coinvariant vectors. The filtration $0 \subset \mathbb{Q} \subset E \subset T$, with middle quotient that system and top quotient $\mathbb{Q}$, then forces both relevant invariant ranks to be one; the integral Euler-parity relation excludes $b_1 = b_2 = 0$ (theorem 16.1).

**Theorem F: The resolved transverse elliptic fibre product**

1. Pull back the Weierstrass family to the root curve of the threefold fibration. The integral matrices $A, B$ give the period comparison $AZ_3B^{-1} = \operatorname{diag}(Z_2, s + c_E)$. See theorem 17.2.

2. The unimodular Kuhn fan resolves the local fibre product without adding rays. Its period action is proper and its marked section extends face by face. The resolution is therefore small and crepant. See theorem 17.2.

3. The canonical formula on the root curve gives $K = \Pi^* L^{-5}$. The split relative elliptic complex and Leray give $(h^{0,q}) = (1, 1, 0, 0, 0)$; the explicit meromorphic product comparison gives algebraic dimension two. See theorem 17.5.

4. A Kähler modification in class $\mathcal{C}$ preserves $b_1$ and $h^{0,1}$. Here $b_1 = 0$ and $h^{0,1} = 1$, contradicting the Kähler equality $b_1 = 2h^{0,1}$. This proves the stated exclusion through these invariants. See theorem 17.5.

5. The universal Todd and Pontryagin formulas give $c_2^2 = 2$, $c_4 = 6$ and signature two. The middle lattice is positive definite unimodular of rank two, hence has the displayed form. See theorem 17.5.

### 1.4 Known and new

| Theorem | Status and principal cited inputs |
|---|---|
| A | New compact complex realization. Smooth classification: (Conway et al., 2023, Definition 3.6 and Theorem 3.7(1), arXiv v3, pp. 12–14); sphere absorption: (Kramer and Stolz, 2007, Theorem A, p. 178); connected-sum criterion and tangent classes: (Mandal, 2026, Corollary 1.4, p. 4, Lemma 2.1, p. 5, and Theorem 2.4, p. 7). |
| B | New parameter domain and invariants. Weighted-base and direct-image method: (Du et al., 2026, Lemmas 6.1–6.2, pp. 74–75, Proposition 6.10, pp. 79–80); torus deformations: (Catanese, 2004, Theorem 2.1, p. 264). |
| C | New genus-two pencil and integral lifted-period construction. The cusp model is (Yang, 2026, Theorem 8.7, arXiv v1, pp. 57–59). |
| D | New integral rank-six gluing and homology. Toroidal filling and Euler number: (Yang, 2026, Theorem 8.7 and Corollary 8.8, arXiv v1, pp. 57–59); integral specialization: (Claude and L. Alpöge, 2026, §7.7, pp. 60–63). |
| E | New integral classification and period theorem. Lattice isomorphism: (Bley et al., 2022, Theorem 3.4, p. 6, Algorithm 8.3, p. 21); local-spray approximation: (Alarcón and Forstnerič, 2025, Proposition 7.13 and preceding paragraph, arXiv v5, pp. 23–24). |
| F | New small crepant resolution and its invariants. Six-sphere: (Claude and L. Alpöge, 2026, Theorems 7.17 and 7.22, pp. 57 and 60); elliptic surface: (Engel, 2026, §2, arXiv v1, p. 3); characteristic identities: (Esole and Kang, 2018, (1.2)–(1.4), pp. 3–4); class-$\mathcal{C}$ criterion: (Jia and Meng, 2024, Theorem 1.1, Remark 1.2, p. 2). |

### 1.5 Methodology

This text is AI-produced; no part of it was written by a human. GPT-5.6 Sol (OpenAI), in Codex at reasoning effort ultra, wrote the working edition on 24 August–9 September 2026. GPT-6 Astra (OpenAI), in ChatGPT Pro mode, carried out the research of 12–16 November, the independent re-derivation of 16–18 November, and the writing of 16–20 November 2026. Claude Opus 5.5 (Anthropic), at effort max, revised the presentation on 20 November 2026, rewriting the abstract and Section 1; parts of both are in its wording. The publication revision and further checks of this paper were undertaken by GPT-6 Astra on 20–21 November 2026. The reasoning-effort setting of this publication session was inherited; the interface exposed no separate numerical setting.

Claude Fable 5.2 (Anthropic), at effort high and at effort max, and Claude Opus 5.5 (Anthropic), at effort max, undertook readings, intakes, checks and recomputations on 12–20 November 2026. Their objections and calculations were compared with the statements, proofs and cited sources, and the resulting corrections were incorporated into the text. The records specify the passages and calculations examined by each model.

For the statements printed in this paper, the record contains 334 examinations of proof steps, 45 independent re-derivations, and 166 computations or formal certificates. It contains another eight proof examinations and three re-derivations associated specifically with steps of the introductory outlines. Each total sums recorded links from a check to a statement, so one calculation applied to several statements can contribute several links. Each outcome states the part examined. The 340 readings associated with the current statements are recorded separately.

Numerical interval enclosures are accompanied by an independently implemented numerical comparison and exact checks of the finite covering and sign inequalities. Their respective scopes are given in the records. No human has checked the mathematics.

The historical sources, programs and check records are available in the project repository. The records and programs of this publication revision are in the same repository at commit `af034f2`. The text and accompanying work are dedicated to the public domain under CC0 1.0 Universal.

### 1.6 How the paper is organized

Part I constructs the marked rank-six family. The local monodromy and cusp in §2 precede the period planes in §3, the integral homology in §5, and the oriented smooth classification in §6. Sections 7–9 determine its parameter domain, invariants and boundary. Part II classifies the integral data and constructs punctured-base periods. Part III gives the resolved elliptic fibre product. Part IV uses the sphere fibrations to construct shifted fivefolds and determines their integral homotopy and smooth topology.

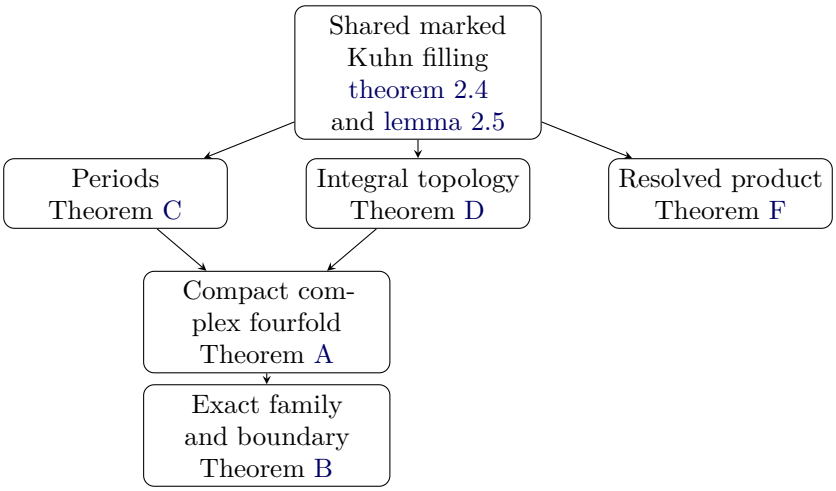



Figure 1: The shared marked cusp and the main construction. The general classification of Theorem E is developed separately in Part II.

#### Further results

- Theorems 18.2 and 18.4: the shifted fivefold has a torus-knot peripheral presentation and explicitly computed integral homology.
- Proposition 18.5 and theorem 18.7: all transverse shifts have one oriented smooth type and are framed null-bordant over their fundamental group.
- Theorems 19.3 and 21.2: the cusp and peripheral maps determine the full second-homotopy module and its coinvariants.
- Theorems 20.1 and 20.2: a circle fibration has an infinite-order return map given by the separating torus twist.
- Proposition 22.1 and theorem 22.3: the relative cusp model gives an explicit smooth framed-handle presentation.

# Part I

# The marked rank-six construction and its family

## 2 The monodromy and the marked cusp

**Definition 2.1** (The marked rank-six datum)**.** [tag PV02] Let $\Lambda = \mathbb{Z}^6$ be the marked homology lattice, with column vectors. We use the integral cohomology matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Put $M_1 = A^t$, $M_2 = B^t$, $M_0 = (M_1 M_2)^{-1}$ and

$$v_1 = (1, 3, -2, -1, 2, 0)^t, \qquad v_2 = (-1, 0, 0, 0, 0, 0)^t.$$

The affine orders are $(m_1, m_2) = (5, 4)$, and $\tau_i(x) = M_i x + v_i/m_i$ on $\mathbb{R}^6/\mathbb{Z}^6$. Positive based meridians satisfy $g_1 g_2 g_0 = 1$. The exterior-square basis of $\bigwedge^2 \mathbb{Z}^4$ is $(01, 02, 03, 12, 13, 23)$.

**Proposition 2.2** (The cusp normal form)**.** [tag 02MY]

*Let $N = M_0 - I$ satisfy $N^2 = 0$, $\operatorname{rank} N = 3$, and suppose that $\operatorname{im} N$ is primitive in $\Lambda = \mathbb{Z}^6$. Then $M_0$ is conjugate in $\mathrm{GL}_6(\mathbb{Z})$ to*

$$U = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}.$$

*Its integral centralizer consists exactly of $\left(\begin{smallmatrix} P & Q \\ 0 & P \end{smallmatrix}\right)$ with $P \in \mathrm{GL}_3(\mathbb{Z})$ and $Q \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$.*

*Proof.* 1. Let $N = M_0 - I$. The index-one condition says that the image is the saturated kernel lattice and that $\Lambda/\ker N \to \operatorname{im} N$ is an isomorphism. Choose a basis $a_1, a_2, a_3$ of this image and lifts $b_i$ with $N b_i = a_i$. Saturation shows that the six vectors form an integral basis. It gives the displayed matrix. Commuting a block matrix with $N$ forces its lower-left block to vanish and its two diagonal blocks to agree. Invertibility over $\mathbb{Z}$ is then equivalent to unimodularity of that block.

□

**Theorem 2.3** (The rank-six monodromy)**.** [tag 02QI] *The finite homology monodromies have orders $5, 4$, respectively, and characteristic polynomials $(T-1)^2\Phi_5(T)$ and $(T-1)^2(T+1)^2(T^2+1)$. For $N = M_0 - I$ we have $N^2 = 0$, $\operatorname{rank} N = 3$, and nonzero Smith factors $1, 1, 1$. Moreover*

$$\Lambda^{\langle M_1, M_2\rangle} = 0, \qquad \Lambda_{\langle M_1, M_2\rangle} = \mathbb{Z},$$

*where the quotient homomorphism is $\ell(x) = x_0$. On the cohomology lattice the corresponding groups are $\mathbb{Z}e_0$ and $\mathbb{Z}/2$. Every nonidentity power of each specified affine action $\tau_i$ is free.*

*Proof.* 1. Exact multiplication gives

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $S$ is unimodular and $S^{-1}M_0S = \left(\begin{smallmatrix} I_3 & I_3 \\ 0 & I_3 \end{smallmatrix}\right)$. The columns of $S$ are henceforth denoted $(a_1, a_2, a_3, b_1, b_2, b_3)$, so that $Nb_j = a_j$. This integral change of basis also proves the Smith assertion. The Hermite basis of $(M_1 - I)\Lambda + (M_2 - I)\Lambda$ is $(e_1, \dots, e_5)$; the stacked matrix of the two fixed-vector equations has rank six. Transposing the matrices gives a fixed lattice $\mathbb{Z}e_0$ and coinvariant Smith factors $(1, 1, 1, 1, 1, 2)$. The pairing $\Lambda^\vee \times \Lambda \longrightarrow \mathbb{Z}$ identifies the fixed cohomology vector $e_0$ with the covector $\ell$.

2. We have $M_i v_i = v_i$, $\ell(v_1) = 1$, and $\ell(v_2) = -1$. Thus $\tau_i^k(x) = M_i^k x + k v_i/m_i$, and $\tau_i^{m_i} = 1$ on the torus. A fixed point of $\tau_i^k$ would give $k\ell(v_i)/m_i = 0$ in $\mathbb{R}/\mathbb{Z}$. For $0 < k < m_i$ this is impossible.

□

**Theorem 2.4** (The Kuhn filling at the cusp)**.** [tag 02R0] *Let $C(q)$ be a holomorphic $3 \times 3$ matrix on a disc about zero, and put $s = \log q/(2\pi i)$ on its punctured universal cover. After shrinking the disc, the periodic Kuhn fan in the marking $S$ gives a smooth proper holomorphic filling of the periods $Z(q) = sI_3 + C(q)$. Its central fibre is reduced with normal crossings. Modulo the index-one period lattice it has respectively $1, 7, 12, 6$ strata of dimensions $3, 2, 1, 0$, each isomorphic to the corresponding complex torus $(\mathbb{C}^*)^d$, and Euler number $6$. An index-$d$ period sublattice gives $d$ times these stratum counts and Euler number $6d$.*

*Proof.* The fan consists of the cones over the tetrahedra

$$[m, m + e_{\pi_1}, m + e_{\pi_1} + e_{\pi_2}, m + (1, 1, 1)], \quad m \in \mathbb{Z}^3, \quad \pi \in S_3. \tag{2.1}$$

The period-lattice action is

$$(x, q) \longmapsto (e^{2\pi i C(q)\lambda} q^\lambda x, q), \qquad \lambda \in \mathbb{Z}^3, \tag{2.2}$$

1. Apply (Yang, 2026, Theorem 8.7 and Corollary 8.8, arXiv v1, pp. 57–59) with $n = 3$, $t = q$, $D(q) = C(q)$, $B = I_3$, and the displayed periodic Kuhn triangulation. These substitutions give exactly the action
$$(z, q) \longmapsto (q^\lambda \exp(2\pi i C(q)\lambda) z, q)$$
and the period order $[I_3 \mid sI_3 + C(q)]$, where $q = \exp(2\pi i s)$. Thus the identity on the dense torus, with the same integral monomial charts, identifies our quotient with the cited quotient. The source theorem supplies its smoothness, Hausdorffness, properness, and the reduced local equations $q = x_0 \cdots x_k$. Flatness follows since this nonconstant function is a nonzero divisor in each smooth chart.

2. For an index-$d$ period sublattice $\Gamma = B\mathbb{Z}^3$, the same substitutions give the source's quotient map induced by $\Gamma \subset \mathbb{Z}^3$. It has degree $d$, and Corollary 8.8 gives $\chi(W) = 3!d = 6d$ for this marked Kuhn fan.

□

**Lemma 2.5** (Smooth dependence on the bounded cusp correction)**.** [tag PV01] *Let $n \in \{2,3\}$ and let $C_t(q)$, $0 \le t \le 1$, be a smooth family of holomorphic $n \times n$ matrix functions on a disc, uniformly bounded on a smaller closed disc. After choosing a sufficiently small common disc, the marked Kuhn fillings with periods $[I_n \mid (\log q/(2\pi i))I_n + C_t(q)]$ are diffeomorphic over that common disc. The diffeomorphisms preserve the central-fibre strata and the marked angular and deck lattices. They can be chosen on the outer collar to extend a smooth marked trivialization of the regular torus bundles.*

*Proof.* For the linear deformation $C_t = tC$, (Yang, 2026, Theorem 8.7, arXiv v1, pp. 57–58) applies directly in dimension $n$ to $D(q,s) = sC(q)$ for $|s| < 2$. For a smooth bounded family the same local charts give the following comparison. The maximal toric charts have coordinates $(z_0, \dots, z_n)$ with $q = z_0 \cdots z_n$. Their changes of coordinates depend smoothly on $t$ and are toric monomials multiplied by the holomorphic units $\exp(2\pi i C_t(q)\lambda)$. They therefore define a smooth family of manifolds with the same coordinate-stratum charts. The parameter vector field in each chart lifts $\partial/\partial t$, fixes $q$, and is tangent to every coordinate stratum. Differences of these lifts are vertical fields with the same properties. A partition of unity gives a global lift; on an outer collar take the prescribed lift from the regular marked torus bundle and use a partition subordinate to that collar and the interior charts.

For $y = \operatorname{Im}(\log q/(2\pi i))$, uniform boundedness of the corrections gives, on a sufficiently small common disc,

$$\tfrac{1}{2}\|\lambda\| \le \|(I_n + y^{-1}\operatorname{Im} C_t(q))\lambda\| \le \tfrac{3}{2}\|\lambda\|, \qquad 0 \le t \le 1.$$

The same fundamental-domain estimate as in the local filling theorem therefore gives a proper family over the closed disc times $[0,1]$. The lifted field has a flow for the entire parameter interval. That flow fixes $q$, preserves each stratum, and has the specified collar restriction. Since the parameter interval is contractible and the lattice marking is constant there, its induced lattice map is the identity. □

**Theorem 2.6** (The central cusp fibre and its collapse map)**.** [tag 02QK] *The index-one central cusp fibre $C$ has torsion-free integral homology with ranks*

$$(b_0, \dots, b_6) = (1,3,9,9,10,3,1). \tag{2.3}$$

*The collapse $\kappa : T^6 \to C$ has primitive image in every degree; it is onto except in degree four, where its cokernel is $\mathbb{Z}$. Its image ranks are $(1,3,9,9,9,3,1)$. In every degree its kernel is the saturation of $(\bigwedge^k M_0 - I)\bigwedge^k \Lambda$. Dually, the specialization $\kappa^* : H^k(C;\mathbb{Z}) \to H^k(T^6;\mathbb{Z})$ has image equal to the full monodromy-invariant lattice. Its kernel is zero except in degree four, where it is $\mathbb{Z}$.*

*Proof.*

1. Apply lemma 2.5 to $C_t(q) = tC(q)$. The comparison preserves the central strata, the integral marking, and the real oriented logarithmic boundary; it therefore intertwines the collapse maps. We calculate with $C(q) = 0$.

2. Here is a concrete topological model. Put $G = 4I_3 - \mathbf{1}\mathbf{1}^t$. Its positive eigenvalues are $1, 4, 4$. The Voronoi polytope in dual coordinates is cut out by

$$|v^t y| \le \tfrac{1}{2} v^t G v \quad (0 \ne v \in \{0,1\}^3).$$

The value on a binary vector with $k$ nonzero entries is $k(4-k)$. These fourteen primitive facet normals give the star of a vertex in the Kuhn triangulation. One checks its Delaunay tetrahedra by the successive chains $0, e_i, e_i + e_j, e_1 + e_2 + e_3$ for the six permutations $(i,j,k)$; solving their three perpendicular-bisector equations and the fourteen inequalities verifies precisely their common faces. The translates of this Voronoi polytope by $G\mathbb{Z}^3$ tile $\mathbb{R}^3$.

3. For a simplex $\sigma$ with vertices $v_0, \dots, v_r$, put $L_\sigma = \sum_i \mathbb{Z}(v_i - v_0)$ and $D_\sigma = \bigcap_{v\in\sigma} P_v$, where $P_v = P + Gv$. The component torus has lattice $A = \mathbb{Z}^3$; the common-stratum marking is the explicit map

$$(A \oplus \mathbb{Z})/\sum_i \mathbb{Z}(v_i, 1) \longrightarrow A/L_\sigma, \qquad [(a,k)] \longmapsto a - kv_0.$$

The Kuhn unimodularity makes $L_\sigma$ primitive.

We choose compatible moment maps as follows. The polytope $2P_v$ is integral, since its vertex facet normals are unimodular and its facet constants are integral. Embed the component $X_v$ using the characters $\chi^{-m}$ for $m \in \mathcal{A}_v = 2P_v \cap A^\vee$, and define

$$\mu_v([Z_m]) = \frac{\sum_{m\in\mathcal{A}_v} m|Z_m|^2}{2\sum_{m\in\mathcal{A}_v} |Z_m|^2}.$$

The sign is specified: the ray $w - v$ tends to the face on which $\langle w - v, -m\rangle$ is minimal, hence to the upper common Voronoi face of $P_v$ and $P_w$. Thus the character polytope is $-2P_v$, and $\mu_v$ is minus one-half its usual moment map. The monomial moment theorem (Cox et al., 2011, equations (12.2.3)–(12.2.4) and Theorem 12.2.5, p. 571) identifies the nonnegative toric part with $P_v$. For the squared weights here, the intervening map on that part is $[a_m] \mapsto [a_m^2]$, with inverse $[b_m] \mapsto [\sqrt{b_m}]$: both preserve the homogeneous binomial relations and positive common projective rescaling. Write $s_v : P_v \to X_v$ for the resulting continuous positive section.

On a shared face put $\mathcal{A}_\sigma = 2D_\sigma \cap A^\vee$ and choose $m_0 \in \mathcal{A}_\sigma$. For $m \in \mathcal{A}_\sigma$, the difference $m - m_0$ annihilates $L_\sigma$. The ratio $Z_m/Z_{m_0}$ is consequently the restriction of the same character

$$\big(-(m - m_0), \langle m - m_0, v\rangle\big) \in (A \oplus \mathbb{Z})^\vee$$

for every vertex $v$ of $\sigma$. At $C = 0$ all deck unit multipliers are one, and the toric gluing identifies these character functions exactly. On either component the restricted moment map is therefore the identical expression

$$\frac{\sum_{m\in\mathcal{A}_\sigma} m|\chi^{-(m-m_0)}|^2}{2\sum_{m\in\mathcal{A}_\sigma} |\chi^{-(m-m_0)}|^2}.$$

The equality persists on every smaller face by restricting the index set. Positive character values also agree, so the sections $s_v$ glue on the entire shared orbit closure. The angular maps are the displayed quotient homomorphisms $A \to A/L_\sigma$.

The period deck map is $(a, k) \mapsto (a + k\nu, k)$, sending $v$ to $v + \nu$. Its component marking is unchanged because $a + k\nu - k(v + \nu) = a - kv$. Moreover $\mathcal{A}_{v+\nu} = \mathcal{A}_v + 2G\nu$; the common character factor cancels in projective ratios. Hence

$$\mu_{v+\nu}(T_\nu p) = \mu_v(p) + G\nu, \qquad s_{v+\nu}(y + G\nu) = T_\nu s_v(y).$$

These give an explicit well-defined map

$$\mathcal{Q} = \big((\mathbb{R}^3/G\mathbb{Z}^3) \times (\mathbb{R}^3/\mathbb{Z}^3)\big)/\sim \longrightarrow C,$$
$$[y, \theta] \longmapsto e^{2\pi i\theta}\, s_v(y) \quad (y \in P_v).$$

where over relint $D_\sigma$ the relation collapses the subtorus $L_\sigma \otimes (\mathbb{R}/\mathbb{Z})$. On each stratum the inverse takes its moment value and its angular character class in $T_A/T_{L_\sigma}$; the latter is obtained by $z \mapsto z/|z|$ on its nonzero characters. The map is a continuous bijection, and its source is compact and $C$ Hausdorff, so it is a homeomorphism. The angular identity $[y, 0]$ gives the continuous section, with the same face and deck compatibilities just proved.

4. The nearby-fibre collapse is the quotient map in this model. Give the semistable family its divisorial logarithmic structure. The local equations $q = x_0x_1x_2x_3$ and their face restrictions make it fine, saturated, log smooth and vertical over the standard logarithmic disc. The family is proper by theorem 2.4. In its logarithmic central fibre, use all arguments of the $x_j$ with their sum fixed by $\arg q$. The integral angular lattice over a face with vertices $v_0, \dots, v_r$ is the kernel of the height map on $\mathbb{Z}^3 \oplus \mathbb{Z}$. Forgetting the arguments of the zero coordinates induces exactly

$$\mathbb{Z}^3 \longrightarrow (\mathbb{Z}^3 \oplus \mathbb{Z}) / \sum_i \mathbb{Z}(v_i, 1) \;\cong\; \mathbb{Z}^3/L_\sigma.$$

The last isomorphism sends the class of $(a, k)$ to $a - kv_0 \pmod{L_\sigma}$; its inverse sends $a$ to $(a, 0)$.

The polar maps agree under the integral monomial overlaps. At $C(q) = 0$ the deck multipliers have positive real values, so the logarithmic fibre at angle one is the product of the base and angular three-tori in item 3, with their given markings.

The mapping-cylinder comparison for a proper vertical log-smooth family identifies this logarithmic forgetful map with the nearby-fibre collapse, compatibly with the central strata (Achinger and Ogus, 2020, formula (3-3-1), p. 478; Theorem 4.1.1 and Lemmas 4.1.2–4.1.3, pp. 480–482). Its angular cycles are $a_j$ and its base cycles are $b_j$. In these column-basis conventions monodromy fixes each $a_j$ and sends $b_j$ to $b_j + a_j$, as specified by $Nb_j = a_j$. The stratified deformation in item 1 transports this map back to the original bounded period matrix.

5. Filter $\mathcal{Q}$ by the dimension $d$ of its dual base cells. Over the interior of the dual cell of a $p$-simplex, $d = 3 - p$ and the angular quotient has lattice $Q_\sigma = \mathbb{Z}^3/L_\sigma$ of rank $d$. Excision for the cell and its boundary gives the relative chain groups of the resulting homology spectral sequence:

$$E^1_{d,q} = \bigoplus_{\dim \sigma = 3-d} \bigwedge^q Q_\sigma.$$

Orient dual cells by the orientation of the base and the ordered simplex. The differential is the signed quotient map for each codimension-one coface. More explicitly, with integral projection matrices $Q_\sigma$ and chosen right inverses $S_\sigma$,

$$D_{p,q}[\tau, \sigma] = \sum_{\sigma = \partial_j \tau \bmod \mathbb{Z}^3} (-1)^j \bigwedge^q (Q_\tau S_\sigma). \tag{2.4}$$

All $L_\sigma$ are primitive because the tetrahedra are unimodular. Thus these projections and right inverses are integral. Formula (2.4) specifies every integer entry, including multiple incidences under translation identifications. There are 26 simplex classes and 74 codimension-one incidences; the signed quotient formula above determines their integer matrices.

6. The nonzero $E^2_{d,q}$ ranks are

| $q \backslash d$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 3 | 3 | 1 |
| 1 | 0 | 6 | 8 | 3 |
| 2 | 0 | 0 | 7 | 3 |
| 3 | 0 | 0 | 0 | 1 |

and every group is free. The reconstructed incoming Smith matrices have only unit nonzero factors and reproduce every entry of this table. Support leaves only the possible higher

differential $d_2 : E^2_{3,0} \to E^2_{1,1}$. The angular identity section splits the base projection as a map of filtered spaces, so the base fundamental class survives. This kills that differential. All subsequent differentials have zero source or target. The filtration extensions are extensions of free abelian groups and therefore split, proving the asserted integral homology.

7. The product-to-quotient map induces on $E^1$ the exterior powers of the quotient projections. On integral cycle and quotient bases, the face projections and the Poincaré-dual Alexander–Whitney formula in the following item give the induced maps. Their images are primitive, and the only nonzero graded cokernel is $\mathbb{Z}$ at $(d, q) = (2, 2)$. The summed image ranks are

$$r_k = (1, 3, 9, 9, 9, 3, 1).$$

The same exact exterior-power calculation gives these ranks for $\ker(\bigwedge^k M_0 - I)$. The collapse satisfies $\kappa_*(\bigwedge^k M_0 - I) = 0$, because a monodromy circuit has the same retraction to the central fibre in the mapping-cylinder comparison. Therefore its rank is at most $r_k$. A filtered map has rank at least the sum of the ranks of its graded maps; hence the rank is exactly $r_k$.

Choose integral bases compatible with the source and target filtrations; their graded groups are free. In each graded block choose the unit Smith pivots of the computed map. The corresponding $r_k$ rows and columns of the filtered map form a block triangular submatrix with unimodular diagonal pivot blocks. Its determinant is $\pm 1$. Since the full map has rank $r_k$, these unit pivots prove that its image is primitive. Thus the map is onto in every degree except four, where its cokernel is the free group of rank $10 - 9 = 1$. This rank argument also rules out additional lower-filtration images of source classes in the kernels of the graded maps.

8. We represent a base cycle $b_J$ by Poincaré duality from the Alexander–Whitney cup cocycle on the complementary coordinates. On an ordered $p$-simplex its coefficient is

$$\operatorname{sgn}(J, J^c) \prod_{r=1}^{p} (v_r - v_{r-1})_{(J^c)_r}.$$

Tensoring with an angular exterior vector and applying (2.4) gives the marked matrices. Their source basis is the ordered exterior basis $(a_1, a_2, a_3, b_1, b_2, b_3)$; the output gives the integral target homology projections and all Smith basis maps. They annihilate $\bigwedge^k M_0 - I$ exactly. The collapse also annihilates that operator, since turning the smoothing parameter does not change its image in the central fibre. The computed ranks equal the ranks of the monodromy coinvariants; primitivity therefore identifies both kernels with the saturation of its image. Consequently these are the matrices of the collapse in suitable integral target bases.

9. Every homology group just computed is free, so the universal coefficient maps identify cohomology with the integral duals of homology. The transpose of the collapse has image

$$\operatorname{im} \kappa^* = \operatorname{Ann}(\ker \kappa_*) = \operatorname{Ann}\big((\bigwedge^k M_0 - I) \bigwedge^k \Lambda\big) = \ker\big((\bigwedge^k M_0)^t - I\big).$$

The first equality uses the primitive image of $\kappa_*$: an integral functional on that image extends to the free central-fibre group. Passing to saturation preserves the annihilator. The last group is the full integral cohomology invariant lattice; inverse transpose has the same fixed subgroup. The kernel of $\kappa^*$ is the integral dual of $\operatorname{coker} \kappa_*$, giving precisely the stated degree-four kernel.

□

**Remark 2.7** (The branch resolution of the Kuhn fibre)**.** [tag PV08] The branch resolution and the moment-cell complex have the following augmentation. Over a dual cell $D_\eta$ of the Kuhn decomposition, the angular lattice is $Q_\eta = \mathbb{Z}^3/L_\eta$ and the branch indices form the simplex on the vertices of $\eta$. Tensor its augmented integral simplex complex with $\bigwedge^q Q_\eta$ and sum over dual cells. The degree-zero augmentation adds branch coefficients; the higher branch terms map to zero. The incidence maps $Q_\eta \to Q_\tau$ for cofaces $\tau \supset \eta$ commute with this augmentation. Inserting the least vertex contracts each augmented simplex complex. Thus the augmentation is a quasi-isomorphism to the moment-cell complex of theorem 2.6. Taking cohomology of the compact toric branch intersections first gives the normalization table below.

The normalization is the three-dimensional permutohedral toric variety. Its paired boundary surfaces consist of four copies of $\mathrm{dP}_6$ and three copies of $\mathbb{P}^1 \times \mathbb{P}^1$. There are twelve normalized triple curves and six quadruple points. The normalization complex for the constant sheaf is the alternating branch complex. Local exactness follows from the augmented simplex on the branches of $z_0 z_1 z_2 z_3 = 0$. Its nonzero rows after taking cohomology are

| $q$ | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $\operatorname{rank} d^0$ | $\operatorname{rank} d^1$ or $d^2$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 7 | 12 | 6 | 0 | $4, 5$ |
| 2 | 11 | 22 | 12 | 0 | 5 | 9 |
| 4 | 11 | 7 | 0 | 0 | 4 | 0 |
| 6 | 1 | 0 | 0 | 0 | 0 | 0 |

(2.5)

Every nonzero Smith invariant in these matrices is one.

The integral matrices are as follows. The fourteen rays are $\pm v$, with $v \in \{0,1\}^3 \setminus \{0\}$. The twenty-four maximal cones are the translated Kuhn tetrahedra containing zero. A unit cube contains zero exactly when its lower vertex lies in $\{-1,0\}^3$, so these eight cubes give the complete enumeration. The Picard group of the normalization is the group of ray coefficients modulo the three character relations. We set the coefficients at $e_1, e_2, e_3$ to zero, giving an integral basis of rank eleven. On a boundary surface $D_v$, choose adjacent rays $u_1, u_2$ for which $(v, u_1, u_2)$ is unimodular and set the two remaining character coefficients to zero. Restriction is evaluated by subtracting the unique integral linear function with the prescribed values at these three rays. The paired restriction is $\mathrm{res}_{D_v} - \mathrm{res}_{D_{-v}}$, with the second adjacent ray corresponding to $u - v$.

For a triangle $(0, v, u)$ with $0 < v < u$ coordinatewise, the next map is the alternating sum of the degrees on the three boundary curves of $D_{u-v}, D_u, D_v$. In a surface fan, if the neighbours of a ray $w$ are $h_1, h_2$ and $h_1 + h_2 = bw$ modulo $v$, that degree is $a_{h_1} + a_{h_2} - b a_w$. In degree four, the restriction map is dual to the seven divisor differences $[D_v] - [D_{-v}]$. In degree zero the matrices are the cellular incidence matrices of the one-vertex Kuhn triangulation of $T^3$. These rules define (2.5) over $\mathbb{Z}$, and both compositions are zero. Its exact Smith reductions give the ranks and primitivity just stated.

**Lemma 2.8** (Integral degree-two classes on the Kuhn filling)**.** [tag PV13] *For the standard Kuhn filling with bounded correction zero, in dimension $n = 2$ or $3$, every integral symmetric matrix $S$ gives a line bundle whose Chern class on a smooth fibre is $\sum_{i,j} S_{ij} a_i \wedge b_j$. Here $a_i$ are the angular classes and $b_i$ the invariant deck classes in the marked product description. The corresponding classes on the central fibre specialize to these classes on the smooth fibre. The deck classes $b_i$ also lie in the specialization image, and the marked bounded-correction comparison preserves these specialization images.*

*Proof.* At a fan vertex $m \in \mathbb{Z}^n$ put

$$q_S(m) = \frac{m^t S m - \sum_i S_{ii} m_i}{2}.$$

This is integral because $m_i^2 - m_i$ is even. Interpolation on each unimodular Kuhn simplex gives an integral piecewise-linear support function. Translation by $\lambda$ changes it by the integral affine

function $m^t S\lambda + q_S(\lambda)$. The associated Cartier data consequently have deck linearization

$$f_\lambda(x, q) = q^{-q_S(\lambda)} x^{-S\lambda}.$$

The identity $q_S(\lambda+\mu) = q_S(\lambda) + q_S(\mu) + \lambda^t S\mu$ gives $f_\lambda(q^\mu x, q) f_\mu(x, q) = f_{\lambda+\mu}(x, q)$, so the line bundle descends to the period quotient.

For the Chern-class convention write $q = e^{2\pi i s}$, $y = \operatorname{Im} s > 0$, and $x_i = e^{2\pi i z_i}$ with $z = u + sv$. Then $a_i = du_i$ and $b_i = dv_i$. The Hermitian squared norm

$$h(z) = \exp\left( -\frac{2\pi}{y} (\operatorname{Im} z)^t S (\operatorname{Im} z) + 2\pi \sum_i S_{ii} \operatorname{Im} z_i \right)$$

satisfies $h(z + s\lambda)|f_\lambda(z)|^2 = h(z)$. Its Chern form is

$$\frac{i}{2y} \sum_{i,j} S_{ij}\, dz_i \wedge d\bar{z}_j = \sum_{i,j} S_{ij}\, du_i \wedge dv_j,$$

where symmetry of $S$ cancels the angular–angular and deck–deck terms. Restriction of the line bundle to the central fibre gives the stated specialization classes. The projection to the deck torus in the moment-cell model supplies the classes $b_i$. Finally lemma 2.5 identifies the marked collapse maps under a bounded correction, and hence transports these integral classes and their specialization images. □

**Lemma 2.9** (The marked smooth gluing)**.** [tag PV11] *The datum of definition 2.1, with its two affine caps, the primitive Kuhn cusp in the marking $S$, and zero additional translation clutch, gives a smooth map $f : M \to S^2$ from a closed connected oriented smooth eight-manifold $M$. It is simply connected, has Euler number six, and admits an almost complex structure with this orientation.*

*Proof.* 1. Use the matrices and affine vectors of definition 2.1. Theorem 2.3 gives their orders, the primitive cusp marking, the coinvariant homomorphism $\ell(x) = x_0$, and freeness of every nonidentity power of the affine actions. Consequently

$$(T^6 \times \Delta_u)/\langle (x, u) \mapsto (M_i x + v_i/m_i, \zeta_{m_i} u) \rangle, \qquad (m_1, m_2) = (5, 4),$$

are smooth oriented finite caps. On the universal angular cover $u = re^{2\pi i a}$ their marked collar maps are

$$\alpha_i(x, u) = x - a v_i, \qquad \alpha_i(M_i x + v_i/m_i, \zeta_{m_i} u) = M_i \alpha_i(x, u).$$

At the cusp we take the index-one Kuhn cap of Theorem 2.4 with $C(q) = 0$, in the explicit marking $S$ of Theorem 2.3. The three displayed boundary bundles therefore glue to the marked torus bundle over the intervening pair of pants, with zero additional clutch translation. Every boundary is paired. The result is a closed connected smooth eight-manifold, oriented by the cusp complex structure and the continued fibre/base orientation.

2. The small cusp cap retracts onto its central fibre $C$ through the mapping-cylinder homeomorphism (Achinger and Ogus, 2020, Theorem 4.1.1 and Lemmas 4.1.2–4.1.3, pp. 480–482). Indeed its divisorial log chart is $\mathbb{N} \to \mathbb{N}^r$, $1 \mapsto (1, \dots, 1)$; it is log smooth, saturated and vertical, and the cap is proper by Theorem 2.4. In the resulting cylinder the homotopy $[x, r] \mapsto [x, (1-t)r]$ fixes $C$ and retracts onto it.

   Before the period quotient, the components of $C$ and their nonempty intersections are smooth complete toric strata. Each is simply connected: a loop moves into its dense torus by general position, and the one-parameter loops along the rays of a smooth maximal cone bound the coordinate discs; those rays are an integral basis. Van Kampen sends component-crossing

paths to edges and triple intersections to triangles of the Kuhn nerve in $\mathbb{R}^3$. Thus the covering core has trivial fundamental group. Its period quotient has deck group $\mathbb{Z}^3$, so $\pi_1(C) = \mathbb{Z}^3$. The nearby-torus map is the quotient of $\Lambda$ by the saturated image of $M_0 - I$, by the $H_1$ collapse in Theorem 2.6. It is onto, and the cusp meridian dies because its zero section extends.

We also record the invariant normal closure needed below. Let $K$ be the smallest $\langle M_1, M_2\rangle$-invariant subgroup containing $(M_0 - I)\Lambda$, and let $J = (M_1 - I)\Lambda + (M_2 - I)\Lambda = \ker \ell$. The equality $\ell M_i = \ell$ gives $K \subseteq \ker \ell$. On the abelian quotient $\Lambda/K$, the induced maps $a, b$ satisfy $ab = 1$, $a^5 = 1$ and $b^4 = 1$. Thus $a^4 = a^5 = 1$, so $a = b = 1$ and $J \subseteq K$. Consequently $K = J = \ker \ell$ as integral subgroups.

Let $a_i$ be the two finite peripheral lifts. The universal covering deck transformation of the $i$th cap has $m_i$th power equal to translation by $v_i$, since $M_i v_i = v_i$. The collar map in Step 1 sends $a_i$ to that transformation. Van Kampen therefore gives

$$a_i \lambda a_i^{-1} = M_i \lambda, \qquad a_1^5 = v_1, \quad a_2^4 = v_2, \quad a_1 a_2 = 1.$$

Write $a_1 = t$ and $a_2 = t^{-1}$. Then $t = t^5 t^{-4} = v_1 + v_2$ lies in the abelian fibre subgroup. Conjugation is consequently trivial, and that subgroup is a quotient of $\Lambda_{\langle M_1, M_2\rangle} = \mathbb{Z}$, with coordinate $\ell$. Now $\ell(v_1 + v_2) = 0$, so $t = 0$, while $t^5 = v_1$ kills its generator $\ell(v_1) = 1$. Every generator vanishes, proving $\pi_1 = 1$. Equivalently, after this abelian reduction, the relations on $(\ell, a_1, a_2)$ have matrix

$$\begin{pmatrix} -1 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \det = -1.$$

3. Euler additivity over the base strata gives $\chi(M) = 6$: the regular tori and finite free torus quotients have Euler number zero, and theorem 2.4 gives six for the cusp.

4. Each finite monodromy admits an invariant real linear complex structure: its 1 and $-1$ real eigenspaces have even dimension, and each remaining conjugate eigenpair gives a real two-plane. Choose a complex structure on each summand. Reversing it on one fixed two-plane selects the required orientation. These structures commute with $M_i$, so their direct sums with the root-disc structure descend to the affine caps. The cusp cap already has its complex structure. Their vertical complex structures on the boundary collars are translation-invariant in the marked torus fibres.

   The space $\mathcal{J} = \mathrm{GL}_6^+(\mathbb{R})/\mathrm{GL}_3(\mathbb{C})$ of allowable vertical structures retracts to $\mathrm{SO}(6)/\mathrm{U}(3)$ by the explicit conjugation

$$J \longmapsto P_t J P_t^{-1}, \qquad P_t = \big((1-t)I + t(I + J^t J)\big)^{1/2}.$$

   At $t = 1$ the structure is orthogonal, since $J$ preserves the metric $I + J^t J$; an orthogonal $J$ is fixed throughout this homotopy. The inclusion $\mathrm{U}(3) \to \mathrm{SO}(6)$ sends $\mathrm{diag}(e^{i\theta}, 1, 1)$ to one-plane rotation and induces the surjection $\mathbb{Z} \to \mathbb{Z}/2$ on fundamental groups. Indeed the determinant fibration and $\mathrm{SU}(2) = S^3 \to \mathrm{SU}(3) \to S^5$ give the first group; the quaternion cover of $\mathrm{SO}(3)$ and the sphere fibrations for $\mathrm{SO}(n)$ give the second, with a full rotation lifting from 1 to $-1$. The homogeneous-space exact sequence therefore gives $\pi_0\mathcal{J} = \pi_1\mathcal{J} = 0$.

   Over the two-dimensional pair of pants the prescribed boundary sections of the associated $\mathcal{J}$-bundle extend over relative one-cells by connectedness and over relative two-cells by simple connectivity. Smooth approximation relative to the collars gives a smooth vertical structure. Horizontal splittings form an affine space, so a partition of unity extends the prescribed complex splittings from those collars. Combining the vertical structure with the pulled-back base structure and gluing the caps gives the required almost complex structure on $M$.

□

## 3 The genus-two pencil and the lifted period planes

**Theorem 3.1** (The genus-two pencil and its Hodge bundle)**.** [tag 02QV] *The smooth projective models of the pencil*

$$C_\lambda : \quad y^2 = x^5 + \lambda^3 x + \lambda^4 \tag{3.1}$$

*have one nodal fibre away from* $0, \infty$*, at* $\lambda_* = -256/3125$*. The minimal good root orders at* $0, \infty$ *are* $8, 5$*. On the root curve with reduced root divisors* $D_0 = [0]/8$ *and* $D_\infty = [\infty]/5$*, the Hodge bundle is*

$$\mathcal{E} = \mathcal{T}^3 \oplus \mathcal{T}, \qquad \mathcal{T} = \mathcal{O}_{\mathcal{B}'}(2D_\infty - 3D_0), \qquad \deg \mathcal{T} = \frac{1}{40}. \tag{3.2}$$

*Thus* $T = \mathcal{T} = \mathcal{O}((2/5)[\infty] - (3/8)[0])$*, and* $\ell_0 = T^3$ *is generated meromorphically by* $dx/y$*. The extension at* $\lambda_*$ *is the dualizing Hodge extension.*

*Proof.* 1. The discriminant of $x^5 + \lambda^3 x + \lambda^4$ is $\lambda^{15}(3125\lambda + 256)$. At $\lambda_*$ the double root is $64/625$; the second $x$-derivative and first $\lambda$-derivative there are nonzero. The remaining roots are simple. The odd-degree hyperelliptic compactification is smooth at its branch point at infinity. This gives one node, a smooth total surface at that node, and genus-one normalization.

2. At zero set $\lambda = u^8$, $x = u^6 X$, $y = u^{15} Y$; at infinity set $\lambda = u^{-5}$, $x = u^{-4} X$, $y = u^{-10} Y$. The resulting smooth equations and differentials are

| | equation | $dx/y$ | $x\,dx/y$ |
|---|---|---|---|
| $0$ | $Y^2 = X^5 + X + u^2$ | $u^{-9} dX/Y$ | $u^{-3} X\, dX/Y$ |
| $\infty$ | $Y^2 = X^5 + uX + 1$ | $u^6 dX/Y$ | $u^2 X\, dX/Y.$ |

   Their central polynomials are square-free. The deck maps at zero and infinity are $(X, Y) \mapsto (iX, \zeta_8 Y)$ and $(X, Y) \mapsto (\zeta_5^{-1} X, Y)$, respectively. The eigenvalues on $H^{1,0}$ are $(\zeta_8, \zeta_8^3)$ and $(\zeta_5^{-1}, \zeta_5^{-2})$. Including conjugates gives characteristic polynomials $T^4 + 1$ and $\Phi_5$, proving minimality of the root orders. The fourth power at zero is the hyperelliptic involution, acting as $-I$ on $H^1$ and as $I$ on its exterior square.

3. The two meromorphic differential generators have orbifold divisors $(6/5)[\infty] - (9/8)[0]$ and $(2/5)[\infty] - (3/8)[0]$. At the node they are independent sections of the dualizing Hodge extension: one has opposite nonzero residues at the two points of the normalization, and the linear combination vanishing at the node is a regular nonzero elliptic differential. Thus there is no additional divisor at $\lambda_*$. These computations prove the asserted direct sum and its degrees.

□

**Theorem 3.2** (The rational rank-five monodromy)**.** [tag 0207]

*For* $U = \mathbb{P}^1 \setminus \{0, \infty, \lambda_*\}$*, let* $\mathcal{H}_{\mathrm{dR}}$ *be the polarized rational cohomology system of the pencil in theorem 3.1. Its primitive exterior square is rationally isomorphic to the rank-five quotient of the system with matrices* $A, B$ *in definition 2.1 by its invariant first-coordinate line. The identification uses the explicit companion and quotient maps below and the stated based meridians.*

*Proof.* 1. By theorem 3.1, the cusp is an ordinary node with genus-one normalization. Cutting and regluing an annular neighborhood gives a Dehn twist in the primitive nonseparating vanishing circle. Its logarithm $N_H$ has square zero and rank one. The finite characteristic polynomials, from the same theorem, are $f(T) = T^4 + T^3 + T^2 + T + 1$ at infinity and $g(T) = T^4 + 1$ at zero.

2. We identify the representation. Choose positive peripheral generators $T_1, T_2$ at infinity and zero, in that order, with $T_1T_2$ inverse to nodal monodromy. Set $R = T_1^{-1}$ and $S = T_2$. Then $R - S$ has rank one and their characteristic polynomials are $f, g$. The companion presentation is Levelt's theorem in (Beukers and Heckman, 1989, Theorem 3.5, pp. 331–332; Corollary 3.6, p. 332). Its generators are related to the present pair by

$$h_\infty = R = T_1^{-1}, \qquad h_0 = S^{-1} = T_2^{-1}, \qquad h_1 = R^{-1}S = T_1T_2, \qquad h_\infty h_1 h_0 = I.$$

The following calculation supplies the rational change of basis for this presentation. If $u$ spans $\operatorname{im}(R - S)$, it is cyclic for $R$: the span of its iterates is stable under both $R$ and $S$, and the irreducibility of $f$ makes that span all of $\mathbb{Q}^4$. In the rational basis $u, Ru, R^2u, R^3u$, the matrix of $R$ is the companion matrix of $f$, and $S = R - u\varphi$ for a covector $\varphi$. The determinant identity

$$g(T) = f(T) + \varphi \operatorname{adj}(TI - R)u \tag{3.3}$$

determines $\varphi$ uniquely: the four coordinates of $\operatorname{adj}(TI - R)u$ are a basis of the polynomials of degree at most three, as is seen from their successively distinct leading degrees in the cyclic basis. It follows that any two such pairs are simultaneously rationally conjugate. Apply the lemma to the explicit companion pair

$$C_f = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad C_g = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.4}$$

Their difference has rank one and their characteristic polynomials are $f, g$. More explicitly, put

$$\begin{aligned} \mathsf{B} &= (u, Ru, R^2u, R^3u), \\ d &= (C_f - C_g)e_4 = (0, -1, -1, -1)^t, \\ \mathsf{B}_c &= (d, C_f d, C_f^2 d, C_f^3 d). \end{aligned}$$

Here $\det \mathsf{B}_c = 1$. Uniqueness of the covector in (3.3) gives the intertwiner $\mathsf{P} = \mathsf{B}\mathsf{B}_c^{-1}$, with $R\mathsf{P} = \mathsf{P}C_f$ and $S\mathsf{P} = \mathsf{P}C_g$. Thus $(T_1, T_2)$ is rationally conjugate to $(C_f^{-1}, C_g)$, with the chosen peripheral conventions.

3. Use the lexicographic basis in $\bigwedge^2 \mathbb{Q}^4$. Its common invariant vector is $(1, 1, 1, 1, 1, 1)^t$. Let $Q$ be the identity matrix of size six with first column replaced by this vector. It is unimodular. Direct multiplication gives the exact identities

$$\left(Q^{-1} \bigwedge\nolimits^2 (C_f^{-1}) Q\right)_{2:6,2:6} = A_{2:6,2:6}, \qquad \left(Q^{-1} \bigwedge\nolimits^2 (C_g) Q\right)_{2:6,2:6} = B_{2:6,2:6}. \tag{3.5}$$

The invariant line is the inverse of the polarization, up to a nonzero rational scalar. Projection along that line identifies the quotient with the primitive exterior square $\mathcal{W}$.

4. The differentials $\omega_1 = dx/y$ and $\omega_2 = x\,dx/y$ span the extended Hodge bundle everywhere. At the node they are independent dualizing differentials: $\omega_1$ has nonzero opposite residues on the normalization and $(x - r)\omega_1$ is a nonzero holomorphic differential there. The extension assertion can be checked on the nodal annulus. In coordinates $uv = t$ a relative dualizing differential has a convergent Laurent expansion $\sum_n a_n(t)u^n\,du/u$. Integration on the vanishing circle is $2\pi i a_0(t)$. Integration across the annulus has the single logarithmic term $a_0(t)\log t$; integration of every nonconstant Laurent monomial and of the complementary fixed arcs gives a holomorphic remainder. Thus the period vector has the form $\exp(sN_H)$ times a holomorphic vector, $s = \log t/(2\pi i)$. Untwisting by $\exp(-sN_H)$ is exactly the

nilpotent canonical extension used here. The two independent dualizing limits remain a basis of its Hodge subbundle. Their isotropic pairing, zero away from zero, remains zero at the limit by holomorphic continuation, so that limit is Lagrangian. The root-divisor calculation is (3.2). The nonzero residue of $\omega_1$ also shows

$$N_H \ell_\infty \neq 0, \qquad \ell = \mathcal{T}^3 \subset \mathcal{E}, \tag{3.6}$$

where $\ell_\infty$ denotes its limit at the nodal cusp. Indeed integration on the vanishing circle tends to $2\pi i$ times that residue; the transvection sends the cohomology class of a differential to that period times its vanishing cohomology vector.

The algebraic calculations are implemented in the programs in Table 1. □

**Theorem 3.3** (The integral exterior-square quotient)**.** [tag `02QT`] *The lower blocks of $A, B$ are precisely the quotient representations of $\bigwedge^2(C_f^{-1}), \bigwedge^2 C_g$ in the basis $Q$. The integral quotient is the dual of the primitive exterior-square lattice. The natural map from the primitive lattice into that quotient has cokernel $\mathbb{Z}/2$; these two integral representations are not isomorphic. Their rationalizations are isomorphic by that same map.*

*Proof.* 1. The two matrices obtained from their two-by-two minors are

$$Q^{-1} \bigwedge^2 (C_f^{-1}) Q = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix},$$

$$Q^{-1} \bigwedge^2 C_g Q = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Their lower five-by-five blocks are the asserted blocks, entry by entry. Put $L = \bigwedge^2 \mathbb{Z}^4$, let $(\ ,\ )$ be its unimodular wedge pairing, and set $\omega = (1,1,1,1,1,1)^t$. Then $(\omega,\omega) = 2$ and the primitive sublattice $P_\mathbb{Z} = \omega^\perp$ is the kernel of $(1,-1,1,1,-1,1)$. In the basis with columns

$$\begin{pmatrix} 1 & -1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the inclusion followed by projection $j : P_\mathbb{Z} \longrightarrow L/\mathbb{Z}\omega$ has matrix

$$j = \begin{pmatrix} 0 & 1 & 1 & -1 & 1 \\ -1 & 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & -1 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 & 2 \end{pmatrix}, \qquad \det j = 2.$$

Its Smith factors are $(1,1,1,1,2)$. The unimodular pairing on $L$ identifies $L/\mathbb{Z}\omega$ with $P_\mathbb{Z}^\vee$; under this identification $j$ is the pairing map.

2. We rule out every possible integral intertwiner. Denote the lower blocks above by $A_5, B_5$. The Krylov matrix $(e_3, A_5e_3, A_5^2e_3, A_5^3e_3, A_5^4e_3)$ has determinant $-1$, so $A_5$ is cyclic. Any rational endomorphism commuting with $A_5$ therefore has the form $u_0I + u_1A_5 + \cdots + u_4A_5^4$. The entries $(1,2), (1,4), (2,4), (3,1)$ of its commutator with $B_5$ give, on $(u_1, u_2, u_3, u_4)^t$, the coefficient matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \qquad \det = -1.$$

Thus $u_1 = u_2 = u_3 = u_4 = 0$, proving that the simultaneous rational centralizer is exactly $\mathbb{Q}I$. The natural map $j$ above is equivariant and rationally invertible. For any rational equivariant map $f : P_{\mathbb{Q}} \to (L/\mathbb{Z}\omega)_{\mathbb{Q}}$, the composite $fj^{-1}$ lies in this centralizer. Consequently $f = cj$ for some $c \in \mathbb{Q}$. An integral isomorphism would have determinant $\pm 1$, whereas $\det(cj) = 2c^5$. Its 2-adic valuation would give $1 + 5v_2(c) = 0$, impossible because $v_2(c) \in \mathbb{Z}$. Thus $j$ is the explicit rational isomorphism, and the determinant valuation excludes an integral isomorphism.

□

**Theorem 3.4** (Integral marking of the pencil)**.** [tag `02QY`] *The integral $H^1$ monodromy of the pencil has a companion marking with finite pair $(C_f^{-1}, C_g)$, and cusp product $C_fC_g^{-1}$. Its exterior-square quotient is consequently the integral quotient representation appearing in theorem 3.3.*

*Proof.* 1. Let $L = H^1(C_\lambda, \mathbb{Z})$ with its integral cup pairing, and write $T_1, T_2$ for the finite generators of theorem 3.2. Put $A = C_f$ and $B = C_g$. The rational intertwiner proved there is a map

$$q : L_{\mathbb{Q}} \longrightarrow \mathbb{Q}^4, \qquad qT_1q^{-1} = A^{-1}, \quad qT_2q^{-1} = B.$$

Its relation to the Levelt presentation is $(h_\infty, h_1, h_0) = (T_1^{-1}, T_1T_2, T_2^{-1})$; the ordered product is one. The known rigidity input and its explicit rational change of basis are recorded in theorem 3.2. The nodal peripheral is $c = (T_1T_2)^{-1}$. Its primitive Picard–Lefschetz class $\delta \in L$ and coefficient one are the geometric node input proved there: the nonseparating vanishing circle has a transverse integral cycle meeting it once, and the annulus $xy = t$ turns once. Thus $T_1T_2$ acts by $v \mapsto v + \epsilon\delta\langle\delta, v\rangle$, where $\epsilon \in \{1, -1\}$ records the fixed inverse and pairing conventions.

2. A $B$-invariant alternating form has upper entries $(u, v, u, u, v, u)$. Invariance under $A$, evaluated on the first and fourth basis vectors, gives $v = -u$. The common invariant forms are therefore exactly the multiples of

$$J = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The Pfaffian of $J$ is 1, and direct multiplication gives

$$A^tJA = B^tJB = J, \qquad A^{-1}B - I = de_4^t, \qquad d^tJ = e_4^t.$$

Here $e_4$ is the fourth standard vector. Write the transported cup form as $sJ$ and $q(\delta) = kd$, with $s, k \in \mathbb{Q}^*$. Comparison with the Picard–Lefschetz formula gives $\epsilon sk^2 = 1$.

3. The integral classes $\delta_i = T_1^{-i}\delta$, $0 \le i \le 3$, have rational images equal to the columns of $kD$, where

$$D = (d, Ad, A^2d, A^3d) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \det D = 1.$$

Their pairing is $\epsilon G$, with

$$G = D^t J D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \det G = 1.$$

These four classes are a basis of $L_{\mathbb{Q}}$. For any $v \in L$, write $v = \sum_i x_i \delta_i$ with $x_i \in \mathbb{Q}$. The vector $(\langle \delta_i, v \rangle)_i = \epsilon G x$ is integral. Since $G^{-1}$ is integral, so is $x$. Thus the four classes are an integral basis of $L$.

4. The required integral marking is the precise map $\mu = k^{-1} q : L \longrightarrow \mathbb{Z}^4$. Indeed $q(L) = kD\mathbb{Z}^4 = k\mathbb{Z}^4$. Its inverse is specified on standard vectors by

$$\mu^{-1}(e_j) = \sum_{i=0}^{3} (D^{-1})_{i+1,j} \delta_i, \qquad D^{-1} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

These two integral maps preserve the cup form $\epsilon J$. In this marking the finite pair is $(A^{-1}, B)$.

5. The exact simultaneous companion reversal is

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It satisfies $W^2 = I$, $\det W = 1$, $WAW = A^{-1}$, $WBW = B^{-1}$ and $W^t J W = -J$. These identities give the integral comparison of the two companion conventions, including the pairing sign. For the based cusp used in the statement take $c' = T_1^{-1} c T_1 = (T_2 T_1)^{-1}$. Then

$$\mu c \mu^{-1} = B^{-1} A, \qquad \mu c' \mu^{-1} = A(B^{-1}A)A^{-1} = AB^{-1}.$$

The homology matrices are the inverse transposes under the evaluation pairing with $L$.

6. Apply $\bigwedge^2 \mu$ to the integral wedge lattice. In the lexicographic basis of theorem 3.3, the invariant vector is $\omega = (1,1,1,1,1,1)^t$. Let $Q$ have first column $\omega$ and remaining columns the last five standard vectors, as in that result. The induced quotient is the map

$$[u \wedge v] \longmapsto \mathrm{pr}_{2,\dots,6}\, Q^{-1}(\mu(u) \wedge \mu(v)).$$

Its kernel before passage to the quotient is precisely the invariant line. Its inverse is induced by $(\bigwedge^2 \mu^{-1})Q$ on the last five standard vectors. The finite quotient matrices are the lower five-by-five blocks of

$$Q^{-1}(\bigwedge^2 A^{-1})Q, \qquad Q^{-1}(\bigwedge^2 B)Q.$$

They are exactly the integral quotient matrices of theorem 3.3, and the cusp follows by the same based word $(T_2 T_1)^{-1}$. The integral saturation argument in steps 2–4 supplies the map $\mu$ for the geometric lattice.

□

**Theorem 3.5** (The rank-three bundle $P$). [tag 02QW] *The wedge map from $\ell_0 \otimes (\mathcal{H}_{\mathrm{dR}}/\ell_0)$, followed by projection modulo the nondegenerate polarization bivector, is injective and has rank-three image. This image defines the bundle $P$ used in the period construction. Moreover $P \cong \mathcal{O} \oplus T^2 \oplus T^4$, $H^1(P^\vee) = 0$, and $H^0(P^\vee)$ is the one-dimensional space of contractions. Its inclusion into the rank-five quotient lifts holomorphically to the rank-six extension.*

*Proof.* On the root curve of orders five and eight, let $\mathcal{H}_D$ be the logarithmic de Rham bundle of the pencil. Its Hodge subbundle is $\mathcal{E} = T^3 \oplus T$, with $\ell = T^3$. The polarization identifies $\mathcal{H}_D/\mathcal{E}$ with $\mathcal{E}^\vee$. This remains true at the node: the dualizing differentials have independent limits, and their isotropic pairing extends there holomorphically. The primitive projection $p$ and the marked extension give

$$0 \longrightarrow \mathbb{C}e \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0, \tag{3.7}$$

$$P = p(\ell \wedge \mathcal{H}_D). \tag{3.8}$$

In logarithmic frames the first sequence extends over the cusp, since the nilpotent exponential commutes with its inclusion and quotient. It extends equivariantly over the finite root charts. Every nonzero vector in $\ell \wedge \mathcal{H}_D$ is decomposable, whereas the polarization bivector has nonzero square. Thus primitive projection is injective there. The Hodge sequence therefore gives

$$0 \longrightarrow T^4 \longrightarrow P \longrightarrow \mathcal{O} \oplus T^2 \longrightarrow 0. \tag{3.9}$$

For the coarse projection $\rho$, invariant sections in the two root charts give

$$\rho_* T^j = \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{2j}{5} \right\rfloor + \left\lfloor -\frac{3j}{8} \right\rfloor\right). \tag{3.10}$$

Finite-group averaging makes descent exact. For $j = \pm 2, \pm 4$ the degree is $-1$, so both cohomology groups vanish. Consequently $P = \mathcal{O} \oplus T^2 \oplus T^4$ and $H^1(P^\vee) = 0$. The pullback of (3.7) to $P$ splits holomorphically. Its splittings differ by $z\alpha e$, since $H^0(P^\vee) = \mathbb{C}\alpha$. □

### 3.1 The algebraic connection and its splitting

**Theorem 3.6** (The algebraic connection)**.** [tag 02QQ] *The rank-six flat extension, as a complex flat bundle with its invariant line, has the following explicit algebraic connection model. Set $\omega_k = x^k dx/y$, $0 \le k \le 3$, and use the quotient frame*

$$(w_2, w_1, w_0, w_{-1}, w_{-2}) = (\omega_{01}, \omega_{02}, \omega_{03}, \omega_{13}, \omega_{23}), \qquad \omega_{12} = -3\omega_{03}.$$

*These are meromorphic frames of $(L^2, L, \mathcal{O}, L^{-1}, L^{-2})$, where $L = T^2$ on the root orbifold of orders $5, 4$. If a frame is written as a row of vectors, its derivative is right multiplication by*

$$\Gamma_6 = \begin{pmatrix} 0 & b \\ 0 & \Gamma_W \end{pmatrix}, \qquad b = \frac{1}{\lambda - \lambda_*}\left(-\frac{25}{6\lambda^2}, -\frac{125}{48\lambda}, 0, 1, 0\right), \tag{3.11}$$

*where*

$$\Gamma_W = \frac{1}{3125\lambda + 256}\begin{pmatrix} -8(625\lambda+48)/\lambda & -8 & 10\lambda & 75\lambda^2/2 & 0 \\ -125/(2\lambda) & -4(625\lambda+48)/\lambda & 8 & 0 & 75\lambda^2/2 \\ 300/\lambda^2 & -375/\lambda & 0 & -48 & 60\lambda \\ 120/\lambda^3 & 0 & 125/(2\lambda) & 4(625\lambda+48)/\lambda & -8 \\ 0 & 120/\lambda^3 & 50/\lambda^2 & -125/(2\lambda) & 8(625\lambda+48)/\lambda \end{pmatrix}. \tag{3.12}$$

*This model specifies a canonical algebraic splitting, denoted $\sigma_b$. The compatible contraction used for the real parameter is $\alpha = (625\pi i/128)\alpha_{\mathrm{alg}}$, where $\alpha_{\mathrm{alg}}(w_0) = 4/3$ and it vanishes on $w_1, w_2$.*

*Proof.* 1. Reduction modulo exact differentials is completely explicit. To differentiate $\omega_k$, solve the polynomial identity

$$-\tfrac{1}{2}x^k \partial_\lambda f = f\sum_{j=0}^{3} G_{kj}x^j + R_k' f - \tfrac{1}{2}R_k f', \qquad \deg R_k \le 4,$$

where $f = x^5 + \lambda^3 x + \lambda^4$. The last two terms are the numerator of $d(R_k/y)$. Matching the nine coefficients gives the four-by-four Gauss–Manin matrix exactly. At the branch point at infinity, a local coordinate $z = x^{-1/2}$ gives $\omega_k = -2z^{2-2k}(1 + O(z^8))\,dz$. Residues of a primitive times a differential give the alternating pairings $J_{03} = 4/3$, $J_{12} = 4$, with their negatives in reverse order and all other entries zero. This form is nondegenerate. The invariant bivector is therefore $3\omega_{03} + \omega_{12}$. Taking the exterior-square connection and this quotient gives (3.12).

2. The root-chart powers computed from $x^k dx/y$ are $u^{6k-9}$ at zero and $u^{6-4k}$ at infinity. Their nondegenerate pairing shows that the four resulting regular frames form the full de Rham bundle on those charts. At the node the computed connection has a simple pole and square-zero residue; hence these are the nilpotent logarithmic extension frames there. Thus the four line bundles are $T^3, T, T^{-1}, T^{-3}$, and their quotient exterior-square lines are the stated powers of $L$.

3. For the coarse projection $\rho$ on the effective root curve,

$$\rho_* L^j = \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{4j}{5} \right\rfloor + \left\lfloor -\frac{3j}{4} \right\rfloor\right).$$

The degree is zero for $j = 0$ and minus one for $j = \pm 1, \pm 2$. Finite stabilizer invariants are exact, so all five $H^1(L^j)$ vanish. Consequently every holomorphic extension of this $W$ by $\mathcal{O}$ splits as a bundle. In such a splitting its flat connection is an upper block extension by a row $b$. Regularity at the finite root charts and a logarithmic pole at the cusp require, entry by entry,

$$b = \frac{1}{\lambda - \lambda_*}(a\lambda^{-2}, b'\lambda^{-1}, 0, c, d\lambda).$$

For example the $L^j$ entry, after $\lambda = u^4$, has exponent $3j + 3 + 4p$ for a term $\lambda^p/(\lambda - \lambda_*)$; the infinity chart requires $-4j - 1 - 5p \geq 0$. These two inequalities leave exactly the four displayed terms. The equation that the full cusp residue have square zero is $b_{\mathrm{res}}(\Gamma_W)_{\mathrm{res}} = 0$. Its solution space has basis

$$(-25/6, -125/48, 1, 0), \qquad (5/3, -25/3, 0, 1).$$

The holomorphic splitting changes are the constant multiples of the row $r_0 = (0, 0, 1, 0, 0)$ selecting $w_0$. Its connection coboundary is the third row $r_0\Gamma_W$, whose parameter vector is $(-48b_1 + 60b_2)/3125$. The quotient by this exact gauge direction is one-dimensional, with coordinate $s = c + 4d/5$. The first vector has $s = 1$ and rank-three square-zero cusp residue.

The extension is nontrivial, since its dual has no common invariant vector by theorem 2.3; consequently its coordinate $s$ is nonzero. Set $t = 3125d/60$. Its full row is $\beta = sb + tr_0\Gamma_W$. The constant bundle maps

$$C = \begin{pmatrix} 1 & -tr_0 \\ 0 & sI_5 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & (t/s)r_0 \\ 0 & s^{-1}I_5 \end{pmatrix}$$

are holomorphic in the stated root frames and fix the invariant section $e$. Direct multiplication gives

$$C^{-1}\Gamma_6 C + C^{-1}dC = \begin{pmatrix} 0 & \beta \\ 0 & \Gamma_W \end{pmatrix}.$$

This proves the required equivalence of the complete flat bundles with their fixed invariant line. Evaluation at the marked basepoint gives a common intertwiner for both specified meridians. Thus its existence follows from the exact connection maps.

4. For the last normalization, the kernel of the quotient cusp residue on $(w_2, w_1, w_0)$ is generated by $v = (-4096/1171875, 64/1875, 1)^t$. It satisfies $b_{\mathrm{res}}(v) = 625/192$ and $\alpha_{\mathrm{alg}}(v) = 4/3$. The logarithmic monodromy is $2\pi i$ times this residue. We fix the contraction on this kernel line by the exact companion identity $N_V(\widetilde{v}) = \alpha(v)e$. Since the space of global contraction forms is one-dimensional, the ratio $(625/192)/(4/3) = 625/256$ fixes its scalar everywhere and gives the displayed $625\pi i/128$.

□

**Definition 3.7** (The normalized height function)**.** [tag PV03] Let $e = e_0$ be the fixed real section of the rank-six system, and let $\sigma_b : P \to V$ be its algebraic splitting. The invariant bivector, primitive projection and contraction are normalized in the companion marking as in theorem 3.6. For each regular base point, including the finite root points, let $n$ be the real vector in $P \cap \bar{P}$ with $\alpha(n) = 1$. We define $h_b$ by

$$\sigma_b(n) - \overline{\sigma_b(n)} = 2ih_b e. \tag{3.13}$$

*Well-definedness.* Let $u$ be a local nonzero section of $\ell_0$. Its Hodge metric is positive, so $\mathrm{span}_{\mathbb{C}}\{u, \bar{u}\}$ is a nondegenerate symplectic plane. The inverse polarization bivector $\omega$ has a nonzero component on the exterior square of its symplectic orthogonal complement. Consequently an equality $u \wedge v - \bar{u} \wedge w = t\omega$ forces $t = 0$; with $t = 0$ its common value belongs to $\mathbb{C}(u \wedge \bar{u})$. The primitive projection therefore gives

$$P \cap \bar{P} = p(\ell_0 \wedge \bar{\ell}_0). \tag{3.14}$$

We make the scalar in the primitive decomposition explicit. In the algebraic de Rham frame put $\omega_{\mathrm{alg}} = 3\omega_{03} + \omega_{12}$ and let $\alpha_{\mathrm{alg}}(v)$ be the contraction of its unique wedge lift $j(v)$ with the residue pairing $J_{03} = 4/3$, $J_{12} = 4$. Then $\mathrm{contr}_J(\omega_{\mathrm{alg}}) = 8$. The raw primitive coefficient is therefore $\alpha_0 = \alpha_{\mathrm{alg}}/8$, whereas theorem 3.6 fixes

$$\alpha = \frac{625\pi i}{128}\alpha_{\mathrm{alg}} = \frac{625\pi i}{16}\alpha_0, \qquad \omega_{\mathrm{norm}} = \frac{16}{625\pi i}\omega_{\mathrm{alg}}.$$

For the primitive representative $v$ of a quotient vector, its unique wedge lift is exactly

$$j(v) = v + \alpha_0(v)\omega_{\mathrm{alg}} = v + \alpha(v)\omega_{\mathrm{norm}}. \tag{3.15}$$

Here $\omega_{\mathrm{norm}}$ is real in the marked flat system, although the algebraic formula contains $i$. To see the precise comparison, use the companion cup matrix $\varepsilon J_c$ of theorem 3.4, with $\varepsilon \in \{1, -1\}$, and its invariant bivector $\omega_c = (1, 1, 1, 1, 1, 1)^t$. Contraction with $J_c$ gives $\mathrm{contr}_{J_c}(\omega_c) = 2$. For the rank-four period comparison $\Pi$, the residue formula is $\Pi^t(\varepsilon J_c)\Pi = 2\pi i J$. Indeed, cut the curve along a symplectic polygon and small circles around the poles; the boundary integral of a primitive times the second differential is the alternating period pairing on the polygon and $2\pi i$ times the sum of residues on the circles. It follows by inversion of the two alternating forms that $(\bigwedge^2 \Pi)\omega_{\mathrm{alg}} = (8\pi i/\varepsilon)\omega_c$.

The scaled exterior-square map $\Psi = (625\varepsilon/256)\bigwedge^2 \Pi$ induces the quotient comparison in the fixed rank-six extension. Its scalar is forced by the cusp contraction identity: on the kernel vector of theorem 3.6 the extension residue is $625/192$, while $\alpha_{\mathrm{alg}}$ has value $4/3$; the logarithmic monodromy contributes $2\pi i$. The ratio is $625\pi i/128$, exactly the pullback of $\mathrm{contr}_{J_c}$ under this scaled comparison. The one-dimensional extension calculation in theorem 3.6 then lifts it uniquely to the normalized rank-six comparison fixing $e$. Consequently $\Psi(\omega_{\mathrm{norm}}) = \omega_c/2$, and the induced primitive comparison sends $\alpha$ to $\mathrm{contr}_{J_c}$. Both are real in the companion marking. For $r = ip(u \wedge \bar{u})$, the value $\alpha(r)$ is real and nonzero by the positive polarization pairing. Thus $n = r/\alpha(r)$ is the unique real vector specified above. Multiplying $u$ by a nonzero complex number multiplies both $r$ and $\alpha(r)$ by its squared modulus, and leaves $n$ unchanged.

The image of $\sigma_b(n)$ in the quotient is real. Its difference from its conjugate is therefore an imaginary multiple of the real vector $e$, which gives a unique real $h_b$. Every marked monodromy is real and fixes $e$, so the same equality defines one function on the base. The construction is real analytic on each root chart; it is invariant under its finite deck action and hence descends.

For its coordinate expression, let $H' = H\Gamma_6$, $H(1) = I$. Take the positive zero meridian with stem from 1 to 1/32 and a counterclockwise circle of radius 1/32, and the infinity meridian $\lambda = \exp(-2\pi it)$, $0 \le t \le 1$. Let $H_0, H_\infty^{\mathrm{mon}}$ be their based continuations. The comparison matrix $X$ is specified by

$$XH_0 = BX, \qquad XH_\infty^{\mathrm{mon}} = AX, \qquad Xe_0 = e_0.$$

Its existence is the marked flat-bundle identification in theorem 3.6. The equations $RA = AR$, $RB = BR$ for the explicit integral matrices force $R$ to be scalar. Thus any two such comparisons differ by a scalar, and $Xe_0 = e_0$ fixes that scalar. This proves uniqueness with the based meridians specified.

Put $Y = XH$ and $k = 625\pi/96$. Since $\alpha(w_0) = ki$ and $\alpha(w_1) = \alpha(w_2) = 0$, the vector $kn$ has coordinates $(0, v_2, v_1, -i, 0, 0)^t$ in the quotient lift for a unique pair $(v_2, v_1)$. Equivalently, the lower five coordinates of its image under $Y$ are real. Equation (3.13) gives

$$h_b(\lambda) = \frac{96}{625\pi}\,\mathrm{Im}\big(Y_{01}v_2 + Y_{02}v_1 - iY_{03}\big).$$

This proves the existence, uniqueness and normalization of the real system used in the height computation. □

**Remark 3.8** (Change of splitting)**.** [tag PV04] Every holomorphic splitting of the pullback extension over $P$ has the form $\sigma_b - a\alpha e$, $a \in \mathbb{C}$, because $H^0(P^\vee) = \mathbb{C}\alpha$. Its parameter and height are related to the ones fixed here by the exact equalities

$$(\sigma_b - a\alpha e + \xi_{\mathrm{alt}}\alpha e)(P) = (\sigma_b + (\xi_{\mathrm{alt}} - a)\alpha e)(P), \qquad h_{\mathrm{alt}} = h_b - \mathrm{Im}\, a.$$

Thus $\xi = \xi_{\mathrm{alt}} - a$. We use $\sigma_b$ and $\xi$ throughout.

## 3.2 The lifted periods and compactification

**Theorem 3.9** (Lifted periods and their exact parameter domain)**.** [tag 02QZ] *Let $P$ be the rank-three wedge bundle of theorem 3.5, and use its normalized contraction $\alpha$ and the algebraic splitting $\sigma_b$. The planes*

$$F_\xi = (\sigma_b + \xi\alpha e)(P), \qquad \xi \in \mathbb{C}, \tag{3.16}$$

*are holomorphic rank-three subbundles of the marked rank-six system. They extend over its finite root charts of orders five and four. The function $h_b$ of definition 3.7 has an attained finite minimum $h_{b,\min}$, tends to $+\infty$ at the cusp, and satisfies*

$$F_\xi \cap \overline{F_\xi} = 0 \quad \Longleftrightarrow \quad h_b + \mathrm{Im}\,\xi \neq 0 \tag{3.17}$$

*at each nonsingular base point, including the finite root points. Thus the parameters transverse on the completed base are exactly $\mathrm{Im}\,\xi > -h_{b,\min}$. In an integral marking at the cusp their periods are*

$$Z(q) = \frac{\log q}{2\pi i} I_3 + C_\xi(q), \qquad C_\xi \text{ holomorphic at } q = 0. \tag{3.18}$$

*Proof.* The holomorphic splitting and the one-dimensional space of its changes are given by theorem 3.5. We use the algebraic choice $\sigma_b$ of theorem 3.6.

The real-intersection calculation and well-definedness proof of definition 3.7 give $P \cap \bar{P} = \mathbb{C}n$ with $\alpha(n) = 1$ and $\sigma_b(n) - \overline{\sigma_b(n)} = 2ih_b e$. The lifted vectors for parameter $\xi$ differ from their

conjugates by $2i(h_b + \operatorname{Im}\xi)e$. An arbitrary complex multiple of $n$ gives no further intersection direction. This proves (3.17).

We calculate its cusp behavior in the fixed companion marking. For a local nonzero $u \in \ell_0$, put $r = ip(u \wedge \bar{u})$ and $a_r = \alpha(r)$, so that $n = r/a_r$. Put $D = C_f C_g^{-1} - I$, $N_V = (BA)^{-1} - I$ and let $N_W$ be its quotient. The only nonzero column of $D$ is $(0,1,1,1)^t$, and the top row of $N_V$ is $(0,0,0,-1,1,-1)$. The wedge projection and normalized contraction satisfy, for all $a, b \in \mathbb{C}^4$,

$$(N_V)_{0,*}(0, p(a \wedge Db)) = \operatorname{contr}(a \wedge Db) = a_0 b_0, \qquad N_W p(a \wedge Db) = 0.$$

These identities follow by expanding the two-by-two minors in the companion basis. Write $u = \exp(sD)a(q)$ with $s = \log q/(2\pi i)$ and $y = \operatorname{Im} s$. The vector $a$ is holomorphic, and $a_0(0) \neq 0$ because $dx/y$ has nonzero nodal residue. In the logarithmic frame,

$$e^{-sN_W} r = ip(a \wedge \bar{a}) + 2y\, p(a \wedge D\bar{a}), \qquad a_r = i \operatorname{contr}(a \wedge \bar{a}) + 2y|a_0|^2.$$

The splitting is bounded in that frame. On returning by $e^{sN_V}$, its imaginary invariant-coordinate contribution is $2|a_0|^2 y^2 + O(y)$. Division by $a_r = 2|a_0|^2 y + O(1)$ yields $h_b(q) = y + O(1)$, uniformly with $\operatorname{Re} s$ in a unit interval. Thus $h_b(q) \to +\infty$ at the cusp.

The same identities give the logarithmic graph. At the limiting plane choose a symplectic basis $a, b, d_1, d_2$ with $N_H b = a$ and $\ell = \mathbb{C}b$. Then

$$P_0 = \langle p(b \wedge a), b \wedge d_1, b \wedge d_2 \rangle, \quad N_W(b \wedge d_i) = a \wedge d_i, \quad N_W p(b \wedge a) = 0. \tag{3.19}$$

The first generator maps under $N_V$ to the nonzero invariant line, while the other two map to independent quotient directions. Therefore $N_V : F_{\xi,0} \to \ker N_V$ is an isomorphism for every $\xi$. In the primitive integral Jordan marking, this makes the untwisted plane a graph with holomorphic coefficient $C_\xi(q)$. Restoring $e^{sN_V}$ adds $sI_3$, giving (3.18).

The height extends continuously over both finite root charts. Its cusp growth and compactness away from that cusp give an attained minimum. The completed punctured base is connected, so its image is $[h_{b,\min}, \infty)$. Equation (3.17) gives the exact parameter half-plane. The fourth power of the order-eight action on $\mathcal{H}_D$ is $-I$; it acts trivially on $\mathcal{W}, P, \alpha$ and $\mathcal{V}$. The equivariant lifted plane therefore descends to the order-four chart. The other chart has order five. □

**Remark 3.10** (The parameter under base and fibre conjugation)**.** [tag PV05] Put $k = 625\pi/96$. In the normalized integral frame the lifted period columns are $W_\xi = (Y_{*,1}, Y_{*,2}, Y_{*,3} + ki\xi e_0)$. For the based meridians fixed in definition 3.7, complex conjugation reverses each meridian. Since the connection is rational and $H(1) = I$, its based continuation satisfies $\overline{H_\gamma} = H_\gamma^{-1}$ for either meridian. Consequently $K_0 = X\bar{X}^{-1}$ satisfies $K_0 A = A^{-1} K_0$, $K_0 B = B^{-1} K_0$ and $K_0 e_0 = e_0$. Direct multiplication gives the same identities for

$$K = \begin{pmatrix} 1 & 2 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad K^2 = I.$$

The common commutant of $A, B$ is scalar by the calculation in definition 3.7. Hence $K^{-1}K_0$ is scalar, and its value on $e_0$ makes that scalar one. Thus $K_0 = K$ is real. Along conjugate continuation paths, $Y(\bar{\lambda}) = K\overline{Y(\lambda)}$, so

$$\overline{W_{-\bar{\xi}}(\bar{\lambda})} = K W_\xi(\lambda).$$

This is the exact parameter map for simultaneous base and fibre conjugation. It preserves the half-plane of theorem 3.9 and fixes every $\xi = ic$, $c \in \mathbb{R}$. The conjugated Hodge plane, parametrized in this way, is holomorphic jointly in $\lambda$ and $\xi$ because it equals the fixed linear image $KF_\xi$.

**Theorem 3.11** (Compact holomorphic filling)**.** [tag 0209]

*In the notation of definition 1.1, for every $\xi \in \mathcal{U}$ the marked gluing of lemma 2.9 has a compact smooth complex realization $M_\xi$ and a proper holomorphic map to $\mathbb{P}^1$. This includes the subfamily $M_c$, $c > c_*$. Its regular fibres are complex three-tori; its two finite reduced fibres are the free affine quotients of orders five and four in definition 2.1; its remaining fibre is the index-one Kuhn degeneration. With its complex orientation its fundamental group is trivial and its Euler number is six.*

*Proof.* 1. We fix the dual convention together with the parameter. The preceding construction has cohomology monodromies $A, B$; definition 2.1 uses homology monodromies $A^t, B^t$. For each admissible $\xi$ we start with the preceding member at parameter $-\bar{\xi}$, pull it back by conjugation of the base, and conjugate its complex fibres. In the fixed marking the resulting Hodge plane is

$$\widehat{F}_\xi(\lambda) = \overline{F_{-\bar{\xi}}(\bar{\lambda})} = KF_\xi(\lambda),$$

by remark 3.10. Thus this family is holomorphic in the coordinate $\xi$. The real based stems of definition 3.7 are fixed by conjugation and their peripheral meridians are reversed. Its cohomology monodromies are therefore $A^{-1}, B^{-1}$ and its homology monodromies are exactly $A^t, B^t$. Put $M_1 = A^t$, $M_2 = B^t$, $M_o = M_2M_1$, $M_0 = (M_1M_2)^{-1}$ and $J = \operatorname{diag}(I_3, -I_3)$. The based peripheral word is $\kappa(g_0) = g_1^{-1}g_0^{-1}g_1$, so $M_0 = M_1M_o^{-1}M_1^{-1}$. For the target marking $S$, direct integral multiplication gives

$$S^{-1}M_0S = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}, \qquad S_o = M_1^{-1}SJ, \qquad S_o^{-1}M_oS_o = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}.$$

Any two markings of this positive standard form differ by $G = \left(\begin{smallmatrix} P & Q \\ 0 & P \end{smallmatrix}\right)$, where $P \in \mathrm{GL}_3(\mathbb{Z})$ and $Q$ is integral: this follows by multiplying $G$ with the displayed unipotent matrix. The period change is

$$(I_3, Z)G = (P, Q + ZP), \qquad P^{-1}(P, Q + ZP) = (I_3, sI_3 + P^{-1}Q + P^{-1}CP).$$

Thus the bounded holomorphic logarithmic form holds in $S_o$. In the real-normalized cusp coordinate, $q_o = \bar{q}$ and $s_o = -\bar{s}$. For the member just defined, base and fibre conjugation give $(I_3, -sI_3 + C_\xi^\kappa(q))$, where

$$C_\xi^\kappa(q) = \overline{C_{-\bar{\xi}}(\bar{q})}.$$

The period-cycle change $J$ gives the target marking $M_1S_oJ = S$ and the positive periods $(I_3, sI_3 - C_\xi^\kappa(q))$.

2. For a holomorphic Hodge subbundle $\mathcal{F}$ with $\mathcal{F} \cap \bar{\mathcal{F}} = 0$, restriction of integral homology functionals embeds the homology lattice discretely in $\mathcal{F}^*$. Real transversality makes its real span the whole real vector space. The quotient $\mathcal{F}^*/\mathcal{V}_\mathbb{Z}^*$ is consequently a holomorphic family of compact complex three-tori over $U$, with its zero section. This also proves local properness, by a smoothly varying fundamental parallelepiped.

3. On the root disc at the finite point of order $m_i$, the family extends smoothly and its deck map has linear part $M_i$. The invariant vector $v_i$ of definition 2.1 gives a holomorphic torsion section $v_i/m_i$. Use the diagonal affine deck action

$$(z, s) \longmapsto (M_iz + v_i/m_i, \zeta_{m_i}s). \tag{3.20}$$

Here $M_iz$ denotes the induced complex-linear map $\mathcal{F}_s^* \to \mathcal{F}_{\zeta_{m_i}s}^*$ from the marked lattice action, not multiplication of a complex three-vector by a six-by-six matrix. Likewise $v_i/m_i$ means its evaluation as a point of the target torus. It has order $m_i$, since the $m_i$th

translation is integral. Every nonidentity power is fixed-point-free on the central torus: the invariant integral covector $\ell$ takes $v_i$ to 1 or $-1$, so a fixed point would require the noninteger $\pm j/m_i$ to vanish in $\mathbb{R}/\mathbb{Z}$. Away from the central fibre the base rotation is already free. The quotient is therefore smooth and proper over the coarse disc, with precisely the required multiple fibre.

4. Its marked punctured family is holomorphically identified with the original torus family by logarithmic translation. On the root cover use $r(s) = (\log s/(2\pi i))v_i$. Then $r(\zeta_{m_i} s) - M_i r(s) = v_i/m_i$, modulo the integral lattice; the change of branch of $\log s$ is itself an integral vector. Thus translation by $r$ conjugates the linear deck action to (3.20). This fixes the affine clutching, including its sign.

5. At the cusp use the integral marking of (3.18), the fan (2.1) and the action (2.2) with $C_3 = C$. Its maximal charts are $\mathbb{C}^4$ with map $t = z_0 z_1 z_2 z_3$, and its transition maps are the toric changes of coordinates multiplied by the holomorphic units $\exp(2\pi i C(t)\lambda)$. The multiplicative comparison is

$$q = \bar{q}_o, \qquad x = (\bar{x}_o)^{-1}, \qquad \nu = -\nu_o, \qquad C_{\mathrm{new},\xi}(q) = -C^{\kappa}_{\xi}(q).$$

   It carries the old period action to $x \mapsto \exp(2\pi i C_{\mathrm{new},\xi}(q)\nu) q^{\nu} x$. On cocharacters it is $\Phi = \mathrm{diag}(-I_3, 1)$ together with conjugation. The Kuhn simplex at $v$ with ordering $\sigma$ is carried to the simplex at $-v-(1,1,1)$ with reversed ordering. If $N_o$ and $N$ are their height-one generator matrices and $R$ reverses four columns, then $N = \Phi N_o R$ and $N^{-1}\Phi = R N_o^{-1}$. Consequently the full affine-chart map is $z_j = \overline{z_{o,3-j}}$. These character identities also give equality on every monomial overlap; the product of the four coordinates is the conjugate base coordinate. The already displayed period-action equality makes the map descend to the analytic period quotient. Hence the fan and marking in this comparison are precisely the prescribed ones.

   Theorem 2.4 applies to this bounded holomorphic correction. It gives smoothness, proper discontinuity and properness in the prescribed marking. We glue with zero cusp translation.

6. The three punctured overlap maps are holomorphic isomorphisms with the same marked torus family, so their cocycle condition is literal. Proper local models and a compact complementary base give a Hausdorff compact total space. Every chart is smooth. The affine vectors, cusp fan and clutching are exactly those in lemma 2.9. Varying their bounded corrections and varying the fibre complex structures along a smooth path does not change these marked smooth fillings: on the finite root models the underlying real torus family is the fixed local-system bundle; on the cusp use the marked, stratum-preserving comparison of lemma 2.5. Thus the resulting underlying smooth manifold is the gluing of lemma 2.9. Its fundamental group is trivial and its Euler number is six by that lemma.

   The finite actions and all gluing maps are holomorphic in $\xi$. The cusp correction $C_{\mathrm{new},\xi}$ is jointly holomorphic; on each relatively compact parameter neighbourhood the cusp disc can be chosen uniformly by theorem 2.4. These constructions therefore give a holomorphic family over the parameter half-plane.

□

# 4 Characteristic-number conventions

**Proposition 4.1** (The characteristic numbers and middle form)**.** [tag 034M] *Let $M$ be a closed connected almost complex manifold of real dimension eight, with its complex orientation, with $H^2(M;\mathbb{Q}) = H^6(M;\mathbb{Q}) = 0$, $b_4(M) = 4$, and $\chi(M) = 6$. Then its middle integral pairing modulo*

*torsion is $I_{3,1}$ and*

$$H^4(M;\mathbb{Z})/\operatorname{tors} \cong I_{3,1}, \qquad c_4[M] = 6, \qquad c_2^2[M] = 2, \qquad \sigma(M) = 2, \qquad \operatorname{Td}[M] = 0. \quad (4.1)$$

*Proof.* The classes $c_1, c_3$ vanish rationally. The index and signature polynomials of (Esole and Kang, 2018, equations (1.2)–(1.4), pp. 3–4) therefore reduce to

$$\operatorname{Td} = \frac{3c_2^2 - c_4}{720}, \qquad \chi^1 = \frac{3c_2^2 - 31c_4}{180}, \qquad \sigma = \frac{3c_2^2 + 14c_4}{45}. \quad (4.2)$$

Here the first two numbers are elliptic-symbol indices of the almost complex tangent bundle; their integrality follows from (Hirzebruch, 1978, Theorem 24.5.4 and its corollary, p. 184). Substitution of $c_4[M] = 6$ gives $\sigma(M) = 16\operatorname{Td}[M] + 2$. The inequality $|\sigma(M)| \le b_4(M) = 4$ forces $\operatorname{Td}[M] = 0$ and $\sigma(M) = 2$, and then $c_2^2[M] = 2$. Poincaré duality makes the middle pairing modulo torsion unimodular. Its signature excludes an even unimodular form, whose signature is divisible by eight. The classification of indefinite odd unimodular forms (Peters and Sterk, 2024, Theorem 2.2.1, p. 61) gives $I_{3,1}$. □

**Remark 4.2** (The orientation in the connected-sum criterion)**.** [tag PV06] We use the orientation in (Mandal, 2026, Corollary 1.4, p. 4, and its proof in §5): $\ell\mathbb{HP}^2 \# m(-\mathbb{HP}^2)$ admits an orientation-compatible almost complex structure exactly for $\ell = 2m + 1$ and odd $m$. The tangent-class conventions are those of (Mandal, 2026, Lemma 2.1, p. 5, and Theorem 2.4, p. 7).

## 5 The integral eight-manifold

**Theorem 5.1** (Homology of the finite fillings)**.** [tag 02QU] *The two finite filling pieces have first integral homology $\mathbb{Z}^2$ and $\mathbb{Z}^2 \oplus \mathbb{Z}/2$.*

*Proof.* 1. For $K_i = T^6/\langle\tau_i\rangle$, the universal covering deck group has commuting translations $x^u$, $u \in \Lambda$, and a generator $a_i$, with the complete relations

$$a_i x^u a_i^{-1} = x^{M_i u}, \qquad a_i^{m_i} = x^{v_i}.$$

Indeed the affine lifts satisfy these relations, and they reduce every word uniquely to $x^u a_i^r$, $0 \le r < m_i$. The abelianization relation matrix has columns $((M_i - I)e_j, 0)$ and $(-v_i, m_i)$. Smith reduction of these explicit seven-by-seven matrices gives the groups above. The disk quotient $(T^6 \times \Delta)/\langle(\tau_i, \zeta_{m_i})\rangle$ retracts onto $K_i$ by equivariant radial contraction. Freeness supplies ordinary smooth quotient charts by disjoint translates of small balls.

□

**Lemma 5.2** (The integral restriction complex)**.** [tag PV12] *Let $f : M \to S^2$ be the map of the marked smooth gluing in lemma 2.9, and let $j : U \hookrightarrow S^2$ be the complement of its three special values $p_1, p_2, p_0$. Put $\mathcal{R}_q = R^q f_*\mathbb{Z}$, $\mathcal{L}_q = \mathcal{R}_q|_U$, $\mathcal{S}_q = \operatorname{im}(\mathcal{R}_q \to j_*\mathcal{L}_q)$, and $\mathcal{K}_q = \ker(\mathcal{R}_q \to j_*\mathcal{L}_q)$. The fibres are connected, so $\mathcal{R}_0 = \mathbb{Z}$. For $1 \le q \le 4$ the image sheaf has $H^2(S^2, \mathcal{S}_q) = \mathbb{Z}/2$; its only nonzero groups in base degrees zero and one are $H^0(S^2, \mathcal{S}_1) = \mathbb{Z}$ and $H^1(S^2, \mathcal{S}_3) = \mathbb{Z}^3$. Writing $i_a : \{p_a\} \hookrightarrow S^2$, its kernels are*

$$\mathcal{K}_1 = 0, \qquad \mathcal{K}_2 = \mathcal{K}_3 = (i_2)_*(\mathbb{Z}/2), \qquad \mathcal{K}_4 = (i_2)_*(\mathbb{Z}/2) \oplus (i_0)_*\mathbb{Z}.$$

*For $q = 2, 3, 4$, the torsion suspension generator at $p_2$ has nonzero image in the global coinvariant group of $\mathcal{L}_{q-1}$.*

*Proof.* 1. Use the marked smooth gluing of lemma 2.9, the matrices and affine vectors of definition 2.1, and the integral cusp-collapse maps of theorem 2.6. The matrix $S$ is the explicit integral marking in the first proof item of theorem 2.3. Write $m_1 = 5$, $m_2 = 4$, and put

$$C_1 = \begin{pmatrix} -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the translation vectors $v_i$, the matrices $P_i = [v_i, (0; C_i)]$ are unimodular. Direct multiplication gives

$$P_i^{-1} M_i P_i = 1 \oplus D_i, \qquad Q_i = P_i^{-t},$$

where $D_1$ sends $e_j$ to $e_{j+1 \bmod 5}$, and $D_2$ sends $e_j$ to $e_{j+1 \bmod 4}$ for $0 \le j < 4$ and $e_4$ to $-e_4$. These equalities give the integral coordinate maps. In these coordinates the affine action is

$$(u, y, z) \longmapsto (u + 1/m_i, D_i y, e^{2\pi i/m_i} z).$$

Thus its core $F_i$ is the mapping torus of $D_i : T^5 \to T^5$. The function $W = e^{-2\pi i u} z$ trivializes its normal disc bundle. With $\theta = m_i u \bmod 1$, the coarse-base function on that bundle is

$$q = W^{m_i} e^{2\pi i \theta}.$$

In particular the boundary is $F_i \times S^1_W$ with this specified map to the base circle.

2. We compute the integral restriction images on this model. Let $V = \mathbb{Z}^5$ be the vertical cohomology lattice, let $T_{i,k} = \bigwedge^k D_i^t$, and let $N_{i,k} = \sum_{a=0}^{m_i - 1} T_{i,k}^a$. The signed permutations preserve the product cell structure with one vertex and one edge on each circle. Its cellular differential is zero. Fibre cells and their prisms give the mapping-torus complex

$$C^q(F_i) = \bigwedge^q V \oplus \bigwedge^{q-1} V, \qquad d(a, b) = (0, (T_{i,q} - I)a).$$

The $m_i$-fold covering by the generic six-torus pulls back $(a, b)$ to $(a, N_{i,q-1} b)$: the horizontal cell is the union of the $m_i$ consecutive prisms, with their transported coefficients. Consequently its image in the marked stalk $L_q = \bigwedge^q \mathbb{Z}^6$ of $\mathcal{L}_q$ is

$$J_{i,q} = \bigwedge^q Q_i \left( \operatorname{Inv} T_{i,q} \;\oplus\; e_0 \wedge N_{i,q-1} \bigwedge^{q-1} V \right).$$

Here vertical coordinates occupy slots $1, \dots, 5$ in the six-lattice. This formula specifies both the lattice and its inclusion.

We fix its column basis as follows. Exterior coordinates are in increasing lexicographic order, with indices starting at zero. For each signed-permutation orbit, start at its least coordinate and sum its successive signed images until it returns there. A positive orbit of length $l$ gives a primitive invariant column; the norm image has the column $m_i/l$ times that column. A negative orbit has zero invariant lattice and one $\mathbb{Z}/2$ in its coinvariant group. Indeed the successive relations identify all its coordinates with signed copies of the first, and the last

relation is twice that first coordinate. Order the positive orbits by their least coordinates, with vertical columns before horizontal columns. For $D_1$ there is no torsion in vertical degrees 1 through 4. For $D_2$ there is one negative orbit in each such degree; the positive-orbit counts are $1, 2, 2, 1$.

3. At the cusp, theorem 2.6 supplies a primitive collapse image and free homology, with the only collapse cokernel equal to $\mathbb{Z}$ in degree four. Dualizing those integral maps therefore gives

$$J_{0,q} = \mathrm{Inv}(\bigwedge^{q} M_0^t),$$

with cohomological restriction kernel zero for $q = 1, 2, 3$ and $\mathbb{Z}$ for $q = 4$. Here are explicit primitive bases, which also fix the column order in the calculation below. In the cusp marking $S$, write the cohomology basis as $x_0, x_1, x_2, y_0, y_1, y_2$; its action sends $x_i$ to $x_i + y_i$ and fixes $y_i$. In degree one take $(y_0, y_1, y_2)$. In degree two take, in order,

$$y_i \wedge y_j \ (i < j), \qquad x_i \wedge y_i \ (0 \le i \le 2), \qquad x_i \wedge y_j + x_j \wedge y_i \ (i < j).$$

In degree three take $y_0 \wedge y_1 \wedge y_2$, then $x_i \wedge y_j \wedge y_k$ in lexicographic $(j, k)$ order with $j < k$ and $i = j, k$, and then

$$x_0 \wedge y_1 \wedge y_2 + x_1 \wedge y_0 \wedge y_2,$$
$$x_0 \wedge y_1 \wedge y_2 - x_2 \wedge y_0 \wedge y_1.$$

Their coordinate equations show that these bases are primitive and have ranks $3, 9, 9$. For degree four set $J = \left(\begin{smallmatrix} 0 & I_3 \\ -I_3 & 0 \end{smallmatrix}\right)$, and let $H(e_I)$ be the complementary exterior basis vector, with the sign of the permutation $(I, I^c)$. Apply $H \bigwedge^2 J$ to the displayed degree-two basis. The identity $JU = U^{-t}J$ gives the invariant equation, and the signed-permutation maps $H$ and $\bigwedge^2 J$ preserve primitivity. Transport all these columns by $\bigwedge^q S^{-t}$ to obtain $J_{0,q}$ in the original marking. Thus its degree-four rank is nine.

4. Since $M$ is compact and the base is Hausdorff, the proper-base-change map identifies its stalks with fibre cohomology (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1; chapter PDF p. 30). Its three stalk images are exactly the lattices above. The kernel $\mathcal{K}_q$ is supported at those points. It is zero for $q = 1$, since the degree-zero norm on the finite suspension group is injective and the cusp restriction is injective. For $q = 2, 3$ it is the order-four $\mathbb{Z}/2$, and for $q = 4$ it is that $\mathbb{Z}/2$ together with the cusp $\mathbb{Z}$. Point-supported sheaves have no positive cohomology, so $H^p(\mathcal{R}_q) = H^p(\mathcal{S}_q)$ for $p > 0$.

Put $A_q = \bigwedge^q M_1^t$ and $B_q = \bigwedge^q M_2^t$. The based marking gives $\bigwedge^q M_0^t = (B_q A_q)^{-1}$. Oriented cellular duality on the punctured sphere and its two-loop spine gives the compact-support complex $L_q^2 \to L_q$ in degrees one and two. Attaching the three stalk lattices gives the full complex

$$J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q,$$
$$d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0),$$
$$\Delta_q(a, b) = (A_q - I)a + (B_q - I)b.$$

For the cusp column, the two consecutive peripheral paths carry $j_0$ and $A_q j_0$; their boundary is $(B_q A_q - I)j_0 = 0$. The reverse cusp orientation gives their negatives. The two finite columns are the corresponding single peripheral loops. This is the relative cellular sequence for extension by zero and the three stalk inclusions, so it preserves their integral indices. It also verifies $\Delta_q d_0 = 0$ with the displayed transports.

We compute its cohomology by determinants in the ordered bases above. Every matrix is specified by the marked $M_i, S, C_i$ and exterior minors. For each $q$ select the following columns

$\mathcal{P}_q$ of $\Delta_q$ and call their square matrix $E_q$:

$$\begin{aligned}
\mathcal{P}_1 &= (1,2,5,8,9,10),\\
\mathcal{P}_2 &= (0,1,2,8,9,11,12,13,16,17,23,24,25,26,28),\\
\mathcal{P}_3 &= (1,2,3,5,6,10,11,12,13,15,16,18,19,\\
&\qquad\qquad 26,29,31,32,34,35,37),\\
\mathcal{P}_4 &= (0,1,5,7,9,10,11,13,16,17,20,23,26,28,29).
\end{aligned}$$

Direct integer multiplication gives

$$\det E_q = (2,2,2,-2)_q, \qquad E_q^{-1}\Delta_q \in \mathrm{Mat}(\mathbb{Z}).$$

Hence $E_q\mathbb{Z}^{\binom{6}{q}}$ is the entire image of $\Delta_q$ and its cokernel is $\mathbb{Z}/2$. Let $\mathcal{F}_q$ be the increasing complement of $\mathcal{P}_q$ in $\{0,\dots,2\binom{6}{q}-1\}$. These free-coordinate rows identify $\ker\Delta_q$ integrally: the pivot coordinates of a kernel vector are $-E_q^{-1}(\Delta_q)_{\mathcal{F}_q}$ times its free coordinates. The relation matrix is therefore $R_q = (d_0)_{\mathcal{F}_q}$. Its sizes, ranks and primitive minors are

| $q$ | size of $R_q$ | $\operatorname{rank} R_q$ | maximal minors |
|---|---|---|---|
| 1 | $6\times 7$ | 6 | $-20,\ -5,\ 4$ |
| 2 | $15\times 15$ | 15 | $\det R_2 = 1$ |
| 3 | $20\times 17$ | 17 | $-1$ |
| 4 | $15\times 15$ | 15 | $\det R_4 = -1$ |

For $q=1$ the three column sets are respectively $(0,1,2,3,5,6)$, $(0,1,2,4,5,6)$, $(0,2,3,4,5,6)$, using all rows. For $q=3$ use all columns and rows $(0,1,\dots,13,15,16,17)$. These determinants have greatest common divisor one, so the quotient by the relations is free. The stalk column counts are $(2,2,3)$, $(3,3,9)$, $(4,4,9)$ and $(3,3,9)$. Thus the kernel dimensions and the quotient ranks give exactly

| $q$ | $H^0(\mathcal{S}_q)$ | $H^1(\mathcal{S}_q)$ | $H^2(\mathcal{S}_q)$ |
|---|---|---|---|
| 1 | $\mathbb{Z}$ | 0 | $\mathbb{Z}/2$ |
| 2 | 0 | 0 | $\mathbb{Z}/2$ |
| 3 | 0 | $\mathbb{Z}^3$ | $\mathbb{Z}/2$ |
| 4 | 0 | 0 | $\mathbb{Z}/2$. |

5. For $k=1,2,3$ the negative signed orbits have representatives $e_4$, $e_0\wedge e_2$, $e_0\wedge e_2\wedge e_4$. Map them by the vertical inclusion followed by $\bigwedge^k Q_2$; call the resulting columns $w_k \in L_k$. The following rows $\rho_k$, read modulo two, annihilate $\Delta_k$:

$$\begin{aligned}
\rho_1 &= (0,1,1,1,1,1),\\
\rho_2 &= (1,1,1,1,1,1,1,0,0,1,0,1,0,1,1),\\
\rho_3 &= (0,0,0,0,0,0,0,0,0,0,0,1,1,0,1,0,1,1,1,0,0).
\end{aligned}$$

The explicit representatives and integer witnesses are

$$\begin{aligned}
w_1 &= (0,0,1,1,1,0)^t,\\
w_2 &= (0,0,0,0,0,0,0,0,-1,0,0,-1,0,-1,-1)^t,\\
w_3 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0)^t,\\
z_1 &= (0,0,0,-1,-1,-1)^t,\\
z_2 &= (-1,0,1,-2,-2,-2,4,2,-1,-3,2,0,-2,0,2)^t,\\
z_3 &= (-2,-2,-3,0,0,-1,-1,-1,2,2,-1,3,-1,4,3,\\
&\qquad -3,-2,1,-4,1)^t.
\end{aligned}$$

They satisfy $E_k z_k = 2w_k$ and $\rho_k w_k = 1 \bmod 2$. The nonzero local torsion class therefore maps to the nonzero global class in each degree $q = 2, 3, 4$.

□

**Lemma 5.3** (The torsion transgression at the order-four filling)**.** [tag 02QF]

*Use the order-four filling and the marked matrices of definition 2.1. Let $D$ be the bottom-right $5 \times 5$ block of $M_2$, and put $T = \bigwedge^{q-1} D^t$ on the integral vertical cochains of the five-torus. If $w$ represents a torsion element of $\operatorname{coker}(T - I)$, its point-supported degree-$q$ class has Leray differential equal, up to the one boundary-orientation sign, to the image of $w$ in the global degree-$(q-1)$ coinvariants. For each of $q = 2, 3, 4$, the unique nonzero local torsion class maps to the nonzero class in the global $\mathbb{Z}/2$ quotient.*

*Proof.* 1. Use the explicit matrix $C_2$ in the proof of lemma 5.2. In the original vertical marking $D$ is the bottom-right block of $M_2$; multiplication gives $DC_2 = C_2 D_2$, where $D_2$ is the four-cycle followed by the sign on its fifth coordinate. The change of cochain coordinates is $\bigwedge^k C_2^t$, and the inverse is $\bigwedge^k C_2^{-t}$. Thus the product cell structure of the signed-permutation torus, transported by this integral torus isomorphism, has zero cellular differential and the stated action $T = \bigwedge^{q-1} D^t$. The fibre cells and their prisms give the mapping-torus cochain complex. The degree-$q$ suspension classes are consequently the coinvariants of $T$ on vertical degree-$q-1$ cochains.

2. Write $u = -x_0$ in the original marking. The affine action is $(u, y, z) \mapsto (u + 1/4, Dy, iz)$, and $W = e^{-2\pi i u} z$ is invariant. With core angle $a = 4u \bmod 1$ and normal angle $k = \arg W/(2\pi)$, the coarse angle is $a + 4k$. Let $g, h$ denote the two corresponding loops. The integral basis

$$v = g^4 h^{-1}, \qquad t = g, \qquad [v, t]_{[g,h]} = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}$$

has determinant one. Its coarse degrees are zero and one; its vertical monodromies are $T^4 = I$ and $T$ respectively. It specifies the generic six-torus fibre and the primitive base loop in the boundary product.

3. A suspension cochain representing $[w]$ has values $(w, 0)$ on $(g, h)$. Following the four consecutive core prisms transports their values by $I, T, T^2, T^3$, giving

$$(w, 0)_{(g,h)} \longmapsto (N_T w, w)_{(v,t)}, \qquad N_T = I + T + T^2 + T^3. \tag{5.1}$$

For a torsion coinvariant there are an integer $n \neq 0$ and an integral cochain $b$ with $nw = (T - I)b$. Therefore $nN_T w = N_T(T - I)b = 0$, and $N_T w = 0$ in the free cochain lattice. The fibre restriction vanishes; its base-circle component is $[w]$. Replacing $w$ by $w + (T - I)b$ changes that component by its cellular coboundary. The formula is consequently a well-defined map of the stated integral coinvariant groups.

4. The base cover by a disc at the finite point and its complement gives the relative Čech connecting map. On their annular intersection the degree-zero fibre restriction is zero and the next component is the one-cocycle just computed. The connecting map sends it to $\pm[w]$ in the global coinvariants; the two ordinary peripheral loops impose the relations $\bigwedge^{q-1} M_1^t - I$ and $\bigwedge^{q-1} M_2^t - I$. These are the two successive boundaries in the same base–fibre complex, so their composition is the Leray $d_2$ for $f$. The one boundary-orientation choice accounts for the sign. Its coefficient is one, as witnessed by the degree-one loop $t$ above. Point-supported kernels or cokernels in the comparison of restriction sheaves have no positive cohomology, so they preserve this degree-two base quotient.

5. In signed-permutation coordinates the negative orbits in vertical degrees $1, 2, 3$ are represented respectively by $e_4$, $e_0 \wedge e_2$ and $e_0 \wedge e_2 \wedge e_4$. The orbit relations give exactly one local $\mathbb{Z}/2$ in each case. Transport by $\bigwedge^k C_2^{-t}$ and then include vertically in the six-lattice. This is the same map as the vertical inclusion followed by $\bigwedge^k Q_2$ used in lemma 5.2, because $P_2 = (-1) \oplus C_2$. The explicit $w_k, \rho_k, E_k, z_k$ there satisfy

$$\rho_k \Delta_k = 0 \pmod 2, \qquad \rho_k w_k = 1 \pmod 2, \qquad E_k z_k = 2w_k, \qquad |\det E_k| = 2.$$

Thus the unique nonzero local class maps to the unique nonzero global class in each of the three degrees. Formula (5.1) sends the zero representative to zero as well.

□

**Theorem 5.4** (Integral homology and the middle form)**.** [tag 02NX] *The marked smooth gluing $M$ of lemma 2.9 is three-connected and has torsion-free integral homology with Betti numbers*

$$(1, 0, 0, 0, 4, 0, 0, 0, 1). \tag{5.2}$$

*It admits an almost complex structure, has signature two, and its middle integral intersection form is $3\langle 1\rangle \oplus \langle -1\rangle$. The compact holomorphic realization of theorem 3.11 has this same marked smooth type.*

*Proof.* The marked identification in theorem 3.11 compares its finite affine actions, cusp fan and overlap maps with the gluing of lemma 2.9. The bounded-correction comparison lemma 2.5 preserves its collars. We compute the integral groups of that gluing. By lemma 5.2 and lemma 5.3, the point-supported torsion classes have differential

$$d_2[w] = \pm[w] \in L_{q-1}/\operatorname{im} \Delta_{q-1}. \tag{5.3}$$

The three images are the nonzero global classes, as shown by the integer witnesses in lemma 5.2. Since $H^0(\mathcal{S}_q) = 0$ in these three degrees, the point-supported kernels give

$$H^0(\mathcal{R}_2) = \mathbb{Z}/2, \quad H^0(\mathcal{R}_3) = \mathbb{Z}/2, \quad H^0(\mathcal{R}_4) = \mathbb{Z} \oplus \mathbb{Z}/2.$$

The Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4; chapter PDF p. 22) has only base columns zero, one and two. The maps from $(0, 2)$ to $(2, 1)$ and from $(0, 3)$ to $(2, 2)$ are the isomorphisms just proved. The restriction of the map from $(0, 4)$ to $(2, 3)$ to its torsion summand is also an isomorphism. Writing the image of a free cusp generator as $\epsilon \in \mathbb{Z}/2$, its kernel is the free group generated by $(1, \epsilon)$. All total-degree-three terms consequently vanish. In total degree four the remaining terms are $\mathbb{Z}$ and $\mathbb{Z}^3$. Their extension is free, so

$$H^3(M; \mathbb{Z}) = 0, \qquad H^4(M; \mathbb{Z}) = \mathbb{Z}^4.$$

The degree-one row of the same Leray sequence is

$$H^0(\mathcal{R}_1) = \mathbb{Z} \xrightarrow{d_2^{0,1}} H^2(\mathcal{R}_0) = \mathbb{Z}.$$

Its kernel is $H^1(M; \mathbb{Z}) = 0$ by lemma 2.9. Its degree is nonzero, so it becomes an isomorphism over $\mathbb{Q}$. The image-sheaf groups in lemma 5.2 give $H^2(M; \mathbb{Q}) = H^3(M; \mathbb{Q}) = 0$. Universal coefficients (Hatcher, 2002, Theorem 3.2 and Corollary 3.3, pp. 195–196) identify the torsion of $H_2$ and $H_3$ with the corresponding Ext subgroups of $H^3$ and $H^4$; both are zero. Their rational ranks also vanish, so $H_2 = H_3 = 0$. Integral Poincaré duality (Hatcher, 2002, Theorem 3.30, p. 241) gives the remaining groups. The successive Hurewicz homomorphisms (Hatcher, 2002, Theorem 4.32, p. 366) give $\pi_2 = \pi_3 = 0$. Finally the almost complex structure and Euler number in lemma 2.9 satisfy proposition 4.1, which gives the stated signature and middle form. □

## 6 The oriented diffeomorphism type

**Theorem 6.1** (Characteristic numbers and the middle lattice)**.** [tag 02QM] *With the complex orientation the fourfold has*

$$\sigma(M_c) = 2, \qquad H^4(M_c;\mathbb{Z}) \cong I_{3,1}, \qquad w_2(M_c) = 0,$$

*and there is a diagonal integral basis in which* $p_1/2 = (1,1,1,1)$*. Its Chern numbers are* $c_4 = 6$*,* $c_2^2 = 2$*, and* $c_1^4 = c_1^2 c_2 = c_1 c_3 = 0$*; its Pontryagin numbers are* $p_1^2 = 8$*,* $p_2 = 14$*.*

*Proof.* 1. By theorem 5.4, the integral groups in degrees two and six vanish, so $c_1 = c_3 = 0$. Proposition 4.1 gives $c_4[M_c] = 6$, $c_2^2[M_c] = 2$, $\sigma(M_c) = 2$, and the integral middle form $I_{3,1}$. The Chern-root identities $p_1 = -2c_2$ and $p_2 = c_2^2 + 2c_4$ then give $p_1^2[M_c] = 8$ and $p_2[M_c] = 14$.

2. Since $H^2(M_c;\mathbb{Z}/2) = 0$, its $w_2$ is zero. For the complex tangent bundle, $w_4 = c_2 \bmod 2 = (p_1/2) \bmod 2$. The Wu identity in degree four, with $w_1 = w_2 = 0$, says that $w_4$ is characteristic for the middle pairing. Thus the vector $p_1/2$ is characteristic and has square 2.

3. We give the required integral isometry of this pair explicitly as a terminating procedure. In a diagonal basis a characteristic vector has four odd coordinates. Change their signs to write it as $(a,b,c,d)$ with all positive and $a^2+b^2+c^2 = d^2+2$. For $d > 1$,

$$d < a + b + c < 3d.$$

The first inequality follows by squaring and using the positive cross terms; the second follows from Cauchy–Schwarz and $d > 1$. Reflection in the integral norm-two vector $(1,1,1,1)$ sends this vector to itself minus $(a+b+c-d)(1,1,1,1)$. Its fourth coordinate is $2d-a-b-c$, whose absolute value is strictly smaller than $d$. The reflection preserves the integral lattice and characteristic parity. Repeating, with sign changes at each step, terminates at $d = 1$, when the norm equation forces $a = b = c = 1$. This gives an integral isometry taking $p_1/2$ to $(1,1,1,1)$.

□

**Theorem 6.2** (A complex structure on the stated connected sum)**.** [tag 020C]

*The compact complex fourfold of Theorem 3.11 is three-connected, has Euler number six, and is orientation-preservingly diffeomorphic to*

$$3\mathbb{HP}^2 \,\#\, (-\mathbb{HP}^2). \tag{6.1}$$

*Its integral middle lattice is* $I_{3,1}$ *and, in a diagonal integral basis,* $p_1/2 = (1,1,1,1)$*.*

*Proof.* 1. Theorem 5.4 gives three-connectivity, Euler number six, and free homology of ranks $1,4,1$ in degrees $0,4,8$. The complex orientation is the one in theorem 3.11. By theorem 6.1, its middle pairing is $Q = I_{3,1}$ and $p = p_1/2 = -c_2$ is characteristic with square two.

2. Use the integral isometry supplied by the terminating descent in theorem 6.1. In diagonal coordinates it is a matrix $U$ with $Up = (1,1,1,1)^t$. We use this specific isometry in the comparison of normal data below.

3. We compute the complete normal clutching datum in fixed coordinates. Orient $\mathbb{H}$ by $(1,i,j,k)$ and use

$$\mathrm{Spin}(4) = S^3_L \times S^3_R, \qquad (u,w)h = uhw^{-1}.$$

The covering induces an isomorphism on $\pi_3$. Let $(a, b)$ be the degrees of the two clutching maps. Evaluation on 1 gives $uw^{-1}$, of degree $a-b$, so $e = a-b$, with the sphere orientation fixed by the left degree-one Euler generator. The left action is complex linear for right multiplication by $i$, whose real complex basis $(1, i, j, -k)$ has the opposite quaternion orientation. Its degree-one generator therefore has $c_2 = -1$, and hence $p_1 = 2$. The right inverse action is complex linear for left multiplication by $i$, with real basis $(1, i, j, k)$. Its Euler number and $c_2$ are both $-1$, so its $p_1$ is again 2. The determinant-one complex actions have $c_1 = 0$. Consequently the exact homomorphism and its inverse on its image are

$$(a, b) \longmapsto (e, p_1/2) = (a-b, a+b), \qquad (a, b) = \left(\frac{e + p_1/2}{2}, \frac{p_1/2 - e}{2}\right).$$

The image consists of pairs of the same parity, and the map is injective. The tangent clutching class of $S^4$ has $(e, p_1) = (2, 0)$, hence coordinates $(1, -1)$. The homotopy sequence of $SO(4) \to SO(5) \to S^4$ identifies its span with the kernel of stabilization and gives surjectivity on $\pi_3$. Subsequent stabilizations are isomorphisms there, since $\pi_3(S^n) = \pi_4(S^n) = 0$ for $n \geq 5$. The stable coordinate is therefore $a + b = p_1/2$, with the left generator sent to 1.

4. For $x \in H_4(M; \mathbb{Z})$, use the embedded sphere representative in the setup of (Conway et al., 2023, §3.3, pp. 11–12). Orient its normal bundle $\nu_x$ by the ambient orientation. Self-intersection and the tangent splitting give

$$e(\nu_x) = x \cdot x, \qquad \langle p_1(\nu_x), [S^4] \rangle = \langle p_1(TM), x \rangle,$$

because $TS^4$ is stably trivial. Thus its two clutching coordinates are exactly

$$\left(\frac{x \cdot x + \langle p, x \rangle}{2}, \frac{\langle p, x \rangle - x \cdot x}{2}\right), \qquad S\alpha_M(x) = \langle p, x \rangle.$$

Their integrality is the characteristic congruence in step 1. This is the stable normal homomorphism of (Conway et al., 2023, Definition 3.6, p. 12) with its generator. For an oriented middle-cohomology isometry $\psi$ from $M$ to a comparison manifold $N$, the corresponding homology map is

$$\Phi = \mathrm{PD}_N \circ \psi \circ \mathrm{PD}_M^{-1}.$$

When $\psi(p_M) = p_N$, the cup-product evaluation identity gives $\langle p_N, \Phi(x) \rangle = \langle p_M, x \rangle$. The map $\Phi$ preserves the intersection pairing and both displayed normal coordinates, so it intertwines the complete extended forms.

5. Take $N = 3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$. On each summand choose the generator $u$ with $p_1 = 2u$, as in (Mandal, 2026, Lemma 2.1, p. 5). Reversing orientation changes the evaluation of $u^2$ and leaves the tangent Pontryagin class unchanged. The collapse maps of the oriented connected sum give

$$H^4(N; \mathbb{Z}) = \bigoplus_{i=1}^{4} \mathbb{Z} u_i, \qquad Q_N = I_{3,1}, \qquad p_N = u_1 + u_2 + u_3 + u_4;$$

the tangent formula is (Mandal, 2026, Theorem 2.4, p. 7). Van Kampen, the connected-sum homology sequence and Hurewicz show that $N$ is three-connected. The isometry $U$ of step 2 is an isometry $\psi : H^4(M; \mathbb{Z}) \to H^4(N; \mathbb{Z})$ carrying $p_M$ to $p_N$. Step 4 supplies the isomorphism of their smooth extended forms. Wall's classification in (Conway et al., 2023, Theorem 3.7(1), pp. 13–14) now gives an orientation-preserving diffeomorphism $M \cong N \# \Sigma$ for an oriented homotopy eight-sphere $\Sigma$.

6. The theorem of Kramer and Stolz (Kramer and Stolz, 2007, Theorem A, p. 178) applies to $\mathbb{HP}^2$: it is smooth, simply connected, has dimension eight, and has total integral homology $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. It gives $\mathbb{HP}^2\#\Sigma\cong\mathbb{HP}^2$. This diffeomorphism preserves orientation: its map on $H^4$ sends a generator to that generator or its negative, whose square still evaluates to 1, forcing its degree to be $+1$. The oriented associativity and permutation maps for connected sum place $\Sigma$ beside one positive summand. Choose the punctured balls compatibly with that diffeomorphism and use the identity on the remaining punctured summands. These maps give
$$\begin{aligned} N\#\Sigma &\cong (\mathbb{HP}^2\#\Sigma)\#\mathbb{HP}^2\#\mathbb{HP}^2\#(-\mathbb{HP}^2)\\ &\cong N. \end{aligned}$$
Composing with step 5 supplies the required oriented smooth diffeomorphism. Transporting the integrable structure of theorem 3.11 along it gives the asserted complex structure.

□

# 7 The normalized connection and the exact parameter domain

**Theorem 7.1** (The minimum of the height function)**.** [tag 02QR] *The function $h_b$ has its unique minimum at $\lambda=\infty$. For the normalized fundamental solution $H(1)=I$ of the algebraic connection and its regular infinity limit $H_\infty$, this minimum is*
$$h_{b,\min}=-\frac{96}{625\pi}\operatorname{Re}(XH_\infty)_{03}.$$
*Its negative $c_*=-h_{b,\min}$ satisfies the exact bounds*
$$\begin{aligned} &1.094540250311402780116268263054945703725 < c_*,\\ &c_* < 1.094540250311402780116268263054945703726. \end{aligned}$$
*The full parameter domain is $\mathcal{U}=\{\xi:\operatorname{Im}\xi>c_*\}$.*

*Proof.* Put $a=256/3125$ and $c_0=96/(625\pi)$. We use the rational connection of theorem 3.6, its integral matrices $A,B$, and the normalization $Y=XH$, $H(1)=I$, $Xe_0=e_0$. The finite-root gauges are
$$\lambda=u^4,\quad D_0=\operatorname{diag}(1,u^6,u^3,1,u^{-3},u^{-6}),\qquad \lambda=u^{-5},\quad D_\infty=\operatorname{diag}(1,u^{-8},u^{-4},1,u^4,u^8).$$
For either gauge the regular connection is $D^{-1}\lambda'\Gamma_6D+D^{-1}D'$. Let $U_0,U_\infty$ be its solutions with value $I$ at zero. The deck characters $T_0,T_\infty$ give the exact based monodromies $U_0(1)^{-1}T_0^{-1}U_0(1)$ and $U_\infty(1)^{-1}T_\infty^{-1}U_\infty(1)$. The unique common intertwiner fixing $e_0$ is $X$, by theorem 3.6 and definition 3.7. In these gauges the exact identities are
$$H(u^4)D_0(u)=U_0(1)^{-1}U_0(u),\qquad H(u^{-5})D_\infty(u)=U_\infty(1)^{-1}U_\infty(u).$$
Thus the regular infinity limit in the statement means precisely
$$H_\infty=\lim_{u\to0}H(u^{-5})D_\infty(u)=U_\infty(1)^{-1}.$$
The regular limiting frames are consequently $F_0=XU_0(1)^{-1}$ and $F_\infty=XU_\infty(1)^{-1}$.

The height is obtained from a real linear system. Write $q=(0,v_2,v_1,-i,0,0)^t$ and impose that the lower five coordinates of $Yq$ be real. The four real unknowns are the real and imaginary parts of $v_2,v_1$; existence and uniqueness are those of definition 3.7. In these coordinates $h_b=c_0\operatorname{Im}(Yq)_0$. Every monodromy is real and fixes $e_0$, so the imaginary top coordinate is unchanged when the lower coordinates are real. This proves that the calculation gives one function on the base. The

same height is preserved by every constant real invertible change of frame $T$ fixing $e_0$. Indeed $T = \left(\begin{smallmatrix} 1 & r \\ 0 & S \end{smallmatrix}\right)$ with $S$ real and invertible: the lower coordinates of $Yq$ are real exactly when those of $TYq$ are real, and the top coordinates differ by a real number. The differential equation of $TY$ is still $(TY)' = (TY)G$. Such changes of frame may therefore be used in the rectangle enclosures below. The real normalized conjugation matrix $K = X\bar{X}^{-1}$ satisfies $Y(\bar{\lambda}) = K\overline{Y(\lambda)}$ and $Ke_0 = e_0$; using $-\bar{q}$ gives $h_b(\bar{\lambda}) = h_b(\lambda)$.

We give the enclosure method and the resulting whole-domain inequalities. For an analytic right connection $G$ on a disc $|z - z_0| \le R$, with a submultiplicative norm bound $\|G\| \le M$, its normalized solution satisfies $\|U\| \le e^{MR}$. Gronwall's estimate (Teschl, 2012, Lemma 2.7, pp. 42–43) and Cauchy's coefficient formula (National Institute of Standards and Technology, 2026, §1.9(iii), (1.9.30)–(1.9.31)) therefore bound the tail after degree $N$, on $|z - z_0| \le r < R$, by

$$e^{MR}\frac{(r/R)^{N+1}}{1 - r/R}. \tag{7.1}$$

For rational $G = P/Q$, the positive lower bound $|Q_0| - \sum_{j\ge1} |Q_j|R^j$ gives $M$ directly. These estimates, with outward interval operations, enclose the root transports, the intertwiner and the four-equation real system. Existence of the exact intertwiner follows from theorem 3.6; an invertible square subsystem encloses it. A matrix inverse on a rectangle and differentiation of $Mv = b$ give

$$v_s = M^{-1}(b_s - M_s v), \qquad v_{st} = M^{-1}(b_{st} - M_{st}v - M_s v_t - M_t v_s).$$

Together with $Y_x = YG$, $Y_y = iYG$ and $Y_{xx} = Y(G^2 + G')$, these enclose the real Hessian of $h_b$ on whole rectangles. For the value on a rectangle of half-diagonal $\rho$, an entrywise bound $|G| \le \mathcal{A}$ gives $|Y(z) - Y(z_0)| \le |Y(z_0)|_+(e^{\rho\mathcal{A}} - I)$. All inputs to these enclosures are the displayed rational connection, its gauges, the integral monodromies and the specified normalization. The endpoint, Hessian, cusp and outer-cover enclosures are given by the executable certificate in Table 1.

At infinity use the regular frame $Y(u^{-5})D_\infty(u) = F_\infty U_\infty(u)$. The normalized vector in that frame is $D_\infty(u)^{-1}q$; its two variable coefficients are $\widetilde{v}_2 = u^8 v_2$ and $\widetilde{v}_1 = u^4 v_1$. The deck action sends this pair to $(\zeta^8\widetilde{v}_2, \zeta^4\widetilde{v}_1)$, where $\zeta = e^{2\pi i/5}$. Its value at the root point is therefore zero. The height there is $h_b(\infty) = -c_0 \operatorname{Re}(F_\infty)_{03}$, and the same deck action makes its gradient zero. In the scaled coordinate $w = a^{1/5}u$, so $\lambda = aw^{-5}$, the enclosures on the closed square $[-1/10, 1/10]^2$ give

$$\min(h_{xx}, h_{yy}) - |h_{xy}| > \frac{1}{4}.$$

The real Hessian is consequently at least $I_2/4$ there. Integrating along the segment from zero gives

$$h_b(aw^{-5}) - h_b(\infty) \ge \frac{1}{8}|w|^2.$$

The normalized endpoint enclosure has radius below $4 \cdot 10^{-76}$ and lies between the negatives of the two endpoints in the statement.

For the rest of the base use $v = a^{-1/4}u_0$, so $\lambda = av^4$, and put $R = 49/50$, $S = 103/100$. The first-quadrant $w$-disc $|w| \le R$ covers $|\lambda| \ge aR^{-5}$ under its finite deck maps; the first-quadrant $v$-disc covers $|\lambda| \le aR^4$. The collar $R \le |w| \le S$ covers the intervening annulus, since $R^4 \ge S^{-5}$. The cusp coordinate is $q = \lambda/\lambda_* - 1 = -w^{-5} - 1$. The same enclosure formulas on rational rectangular partitions give the following strict lower bounds away from the central square and the inner cusp disc:

| region | lower bound for $h_b$ |
|---|---|
| infinity-root disc | $-1.094216$ |
| zero-root disc | $-1.083056$ |
| cusp annulus $1/100 \le \lvert q\rvert \le 1/4$ | $-1.083777$ |
| collar | $-1.030556$ |

Every endpoint in this table is an exact terminating decimal strictly greater than $h_b(\infty)$. Rectangles meeting a boundary are kept closed. Those lying in another of the stated regions are handled there; those outside a specified radius are excluded by their exact squared-distance inequality. The deck maps and conjugation give the remaining sectors. Thus the bounds cover all points outside $|q| < 1/100$ and the central square.

For the inner cusp disc, the exact gauge is $D_\lambda = \operatorname{diag}(1, \lambda^2, \lambda, 1, \lambda^{-1}, \lambda^{-2})$. Its connection is $R_c/q + A_0/(1+q)$ with $R_c^2 = 0$. Writing $YD_\lambda = H_c(I + R_c \log q)P(q)$, $P(0) = I$, the coefficient recurrence is

$$(nI - \operatorname{ad})P_n = \sum_{j<n} (-1)^{n-1-j} P_j A_0, \qquad \operatorname{ad}(Q) = QR_c - R_cQ.$$

Here $\operatorname{ad}^3 = 0$, so the inverse is $n^{-1}(I + \operatorname{ad}/n + \operatorname{ad}^2/n^2)$. A scalar majorant $(1 - |q|)^{-\gamma}$ gives the same Cauchy tail control. Continuation to the positive cusp anchor fixes $H_c$ and the logarithm branch. In the unimodular marking $J$ with $J^{-1}NJ = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$, the lower graph block is invertible throughout the closed inner disc. The marking used for this anchor and the initial cusp marking are related by $N = AN'A^{-1}$; their change has block form $\left(\begin{smallmatrix} D & E \\ 0 & D \end{smallmatrix}\right)$ and sends $Z$ to $DZD^{-1} + ED^{-1}$. It preserves the real-intersection criterion.

At the parameter $\xi = ic_*$ write the resulting graph as $Z(q) = \log q\, I_3/(2\pi i) + C(q)$, and put $y_0 = \log 100/(2\pi)$ and $t = -\log|q|/(2\pi) - y_0 \geq 0$. The coefficients, in increasing degree, of $\det(tI_3 + y_0 I_3 + \operatorname{Im} C(q))$ are bounded below on $|q| \leq 1/100$ by $3/2, 9/2, 15/4, 1$, respectively. The determinant is therefore positive for every nonzero $q$ in that disc. A graph vector $(Zv, v)$ is real precisely when $v$ is real and $\operatorname{Im} Z\, v = 0$. Thus the $c_*$ plane is transverse there. The height at $q = 1/100$ lies between $-0.451$ and $-0.450$; connectedness and (3.17) give $h_b(q) > h_b(\infty)$ on the entire punctured inner disc.

The strict central Hessian inequality, the four outer bounds and the inner cusp inequality prove that infinity is the unique minimum. The exact domain now follows from theorem 3.9. □

**Theorem 7.2** (The height at the marked basepoint)**.** [tag `02QP`] *In the canonical splitting $\sigma_b$ of theorem 3.6,*

$$-1.034276622416321799494248845378346052594964777902188211320 < h_b(1),$$
$$h_b(1) < -1.034276622416321799494248845378346052594964777902188211319.$$

*In particular $h_b(1) < -1$. The value $c = 5/(3\pi)$, for which the determinant $\det C_1$ in theorem 8.3 vanishes at $\xi = ic$, lies outside the global transversality domain.*

*Proof.* Use the root transports, normalized intertwiner and interval linear system in theorem 7.1. At the marked basepoint $H(1) = I$, so the real system in definition 3.7 uses $Y = X$. Its enclosure has radius less than $7 \cdot 10^{-76}$ and is contained in the following wider interval:

$$h_b(1) \in \big[ -1.034276622416321799494248845378346052594964777902188211320 \pm 8.11 \cdot 10^{-57} \big].$$

This proves the two strict bounds. For the coefficient column $q = (0, v_2, v_1, -i, 0, 0)^t$ of definition 3.7, the normalized vector at zero in the regular frame is $D_0(u)^{-1}q$; its variable coefficients are $u^{-6}v_2, u^{-3}v_1$. The nontrivial deck characters force their limiting values to vanish. With the regular limiting frame $F_0$ of theorem 7.1, the same enclosure method gives

$$h_b(0) = -\frac{96}{625\pi} \operatorname{Re}(F_0)_{03}$$
$$= -0.931793376356523505430795105796011114990828\ldots.$$

The connection of this calculation with based-loop continuation is an exact gauge identity. Put $a = 256/3125$ and

$$D(\lambda) = \operatorname{diag}(1, \lambda^2, \lambda, 1, \lambda^{-1}, \lambda^{-2}).$$

Then $\widetilde{Y} = YD$ satisfies the Fuchsian equation

$$\widetilde{\Gamma} = D^{-1}\Gamma_6 D + D^{-1}D' = \frac{R_0}{\lambda} + \frac{R_*}{\lambda + a},$$
$$R_0 = \lim_{\lambda \to 0} \lambda\widetilde{\Gamma}, \qquad R_* = \lim_{\lambda \to -a} (\lambda + a)\widetilde{\Gamma}.$$

Since $D(1) = I$, the based meridians of definition 3.7 retain their matrices $B, A$. The regular-root deck calculation in theorem 7.1 represents those same meridians. Both descriptions therefore give the same normalized intertwiner $X$, by its uniqueness in definition 3.7. If $T_{12}$ is the Fuchsian transition along $[1, 2]$ and $u = 2^{-1/5}$, uniqueness of the normalized differential equation gives the exact equality

$$XT_{12}D(2)^{-1} = F_\infty U_\infty(u) D_\infty(u)^{-1}.$$

Thus the loop and regular-root calculations refer to the same period plane with the same scalar and based marking.

Finally $5/(3\pi) < 1$ and $h_b(1) < -1$, whereas $h_b(\lambda) \to +\infty$ at the cusp. Along a path from 1 to that region the function $h_b + 5/(3\pi)$ has a zero. The exact transversality criterion of theorem 3.9 therefore excludes this parameter. □

# 8 Cohomology, deformations and isomorphisms

**Definition 8.1** (Root-orbifold notation)**.** [tag PV15] Use the complex orientation and the integral marking of theorem 2.3. Let $\widehat{B}$ be the root orbifold of orders $5, 4$, let $h : M \to \widehat{B}$ be the map given by the two multiple divisors, and let $\pi : \widehat{B} \to \mathbb{P}^1$ be its coarse map. Write $L = T^2$; in the pencil coordinate it is

$$L = \mathcal{O}_{\widehat{B}}(\tfrac{4}{5}[\infty] - \tfrac{3}{4}[0]), \qquad \deg L = \tfrac{1}{20}.$$

The cusp point has line bundle $L^{20}$, and $T_{\widehat{B}} = L^9$. These identities include the stabilizer characters: their quotients have integral divisors of degree zero on $\mathbb{P}^1$. Denote the cusp by $W$, its normalization by $Y$, and its reduced singular locus by $D$.

**Theorem 8.2** (Direct images and the canonical bundle)**.** [tag 02R1] *For every admissible member, its vertical translation bundle on $\widehat{B}$ and its first coherent direct image are*

$$E_0 = F^\vee = \mathcal{O} \oplus L^{-1} \oplus L^{-2}, \qquad G = R^1 h_* \mathcal{O}_M = \mathcal{O} \oplus L^{-1} \oplus L^{-2}.$$

*Moreover $R^q h_* \mathcal{O}_M = \bigwedge^q G$ for $0 \le q \le 3$, $K_M = h^* L^{-6}$, and*

$$(h^0(\mathcal{O}_M), \dots, h^4(\mathcal{O}_M)) = (1, 1, 0, 0, 0).$$

*Proof.* 1. The evaluation of the marked lattice on the lifted plane $F$ identifies the Lie bundle of the smooth tori with $F^\vee$. Theorem 3.9 gives $F = \mathcal{O} \oplus L \oplus L^2$; its lift as a bundle has the same splitting. The logarithmic toric vector fields extend this Lie bundle over the cusp, so the identification holds on the whole orbifold, including both finite root charts.

2. On the smooth locus the constant-cocycle description of torus cohomology gives $G = V/F$. In the rank-five orthogonal quotient, $K = \ker(\alpha : P \to \mathcal{O}) = L \oplus L^2$ is the radical of the restricted pairing on $P$. Indeed every wedge containing the fixed line pairs to zero with another such wedge except through their contraction components; removing that scalar

component leaves exactly $K$. The nondegenerate pairing consequently identifies $W/P$ with $K^\vee$. Thus

$$0 \longrightarrow \mathcal{O}e \longrightarrow V/F \longrightarrow L^{-1} \oplus L^{-2} \longrightarrow 0.$$

For every integer $j$, descent by invariant local Laurent series gives

$$\pi_* L^j = \mathcal{O}_{\mathbb{P}^1}(d(j)), \qquad d(j) = \lfloor 4j/5 \rfloor + \lfloor -3j/4 \rfloor. \tag{8.1}$$

The extension group $H^1(L \oplus L^2)$ therefore vanishes, so this sequence splits as a holomorphic vector-bundle sequence.

3. We verify that this quotient is the coherent extension at the cusp. The normalization resolution of $\mathcal{O}_W$ has one toric threefold, seven toric surfaces, twelve toric curves and six points. All positive coherent cohomology of those smooth complete toric strata vanishes (Cox et al., 2011, Theorem 9.2.3, p. 410). Their constants form the cellular cochain complex of the periodic Kuhn triangulation modulo $\mathbb{Z}^3$: its simplices are the faces of $(0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, (1,1,1))$ for $\pi \in S_3$, identified by integral translations. The quotient map $x \mapsto x \bmod \mathbb{Z}^3$ identifies its realization with the dual three-torus and respects every face map. Its cohomology is $\bigwedge^q \mathbb{C}^3$, with the exterior cup product. The invariant flat cohomology $\ker N$ maps to these classes by constant Cech cocycles on the toroidal covering. A component translation in that cover evaluates the corresponding lattice coordinate. In the nilpotent logarithmic frame, $\ker N$ maps isomorphically to $V/F$, since $N : F \to \ker N$ is an isomorphism. Thus these are the extending quotient classes.

4. Flatness over each smooth root disc follows because its base is a DVR and the smooth total space is torsion free over it. Properness was proved in theorems 2.4 and 3.11. The coherent fibre dimensions are the constant binomial dimensions on smooth tori, finite root fibres, and the cusp. (Grauert, 1960, §7, Satz 5, pp. 63–64), with these hypotheses, gives local freeness and base change. The cup-product maps are isomorphisms on every fibre, including the preceding central cellular model, so they identify the remaining direct images with $\bigwedge^q G$.

5. There is an exact sequence of locally free logarithmic bundles

$$0 \longrightarrow h^* L^{11} \longrightarrow \Omega^1_M(\log W) \longrightarrow h^* F \longrightarrow 0. \tag{8.2}$$

In a chart $q = x_0 \cdots x_r$, its first arrow sends $dq/q$ to $\sum dx_i/x_i$ and its quotient is the height-zero logarithmic cotangent bundle of the torus. These formulas also prove exactness on every stratum. Taking determinants and cancelling $\mathcal{O}_M(W) = h^* L^{20}$ yields $K_M = h^* L^{11+3-20} = h^* L^{-6}$.

6. For any bundle which is a sum of powers of $L$, projection and the exact finite averaging projector compute its direct images by tensoring with $\bigwedge^q G$ and applying (8.1). All exponents $-1, \ldots, -5$ have coarse degree $-1$, while exponent zero has degree zero. The Leray sequence on the curve then gives the asserted structure-sheaf cohomology. This includes the nonreduced coarse fibres, since the computation was made before exact finite descent.

□

**Theorem 8.3** (The Hodge numbers)**.** [tag 02R2] *The complete Hodge table of every admissible fourfold is the following, with rows indexed by $p$ and columns by $q$:*

| $h^{p,q}$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 2 | 1 | 0 | 0 |
| 2 | 0 | 1 | 6 | 1 | 0 |
| 3 | 0 | 0 | 1 | 2 | 0 |
| 4 | 0 | 0 | 0 | 1 | 1 |

*Proof.* 1. All maps needed for the two nontrivial rows can be evaluated at the cusp. We give their matrices and the reason evaluation determines their global ranks. In the algebraic splitting of theorem 3.6 write the three generators of $F$ as $(f_0, f_1, f_2)$ of weights $(0, 1, 2)$, and those of $G$ as $(g_0, g_1, g_2)$ of weights $(0, -1, -2)$. Put $z = (625\pi i/128)\xi$. The residue of the quotient connection is

$$R : F_0 \longrightarrow G_0, \qquad R = \begin{pmatrix} 0 & A & B \\ C & 0 & D_0 \\ E & D_0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} A &= -125(192z - 3125)/12288, & B &= -78125(96z + 3125)/393216, \\ C &= -125/512, & E &= 78125/32768, \\ D_0 &= -146484375/2097152. \end{aligned}$$

Here the lifted first vector is $w_0 + (4/3)ze$. These entries result from the exact Gauss–Manin reduction of theorem 3.6, with the quotient projection taken after lifting $F$.

2. The connecting map of (8.2), on relative cohomology, sends $f \otimes \eta$ to $R(f) \wedge \eta$ at the cusp. To see the extension class, trivialize logarithmic forms on the toric covering. A deck translation changes a relative logarithmic form by its period derivative times $dq/q$. The constant Cech class of this derivative is exactly the residue of the quotient connection. This proves the formula with the indicated frames. It also proves its exterior power version, with the usual alternating signs.

3. The floor formula gives the following absolute cohomology:

| | $h^0$ | $h^1$ | $h^2$ | $h^3$ | $h^4$ |
|---|---|---|---|---|---|
| $h^*F$ | 1 | 3 | 2 | 0 | 0 |
| $h^*L^{11}$ | 0 | 2 | 3 | 1 | 0 |
| $h^*(\bigwedge^2 F)$ | 0 | 2 | 3 | 1 | 0 |
| $h^*(L^{11} \otimes F)$ | 2 | 6 | 7 | 3 | 0. |

Every group in this display comes from global sections of the indicated direct image; none comes from $H^1$ of a line on the coarse base. Every contributing summand has coarse degree zero, so evaluation at the cusp is injective on its space of sections. For the equal-dimensional source and target spaces used below it is an isomorphism onto the displayed weight subspaces. Thus fibrewise evaluation computes the ranks of the absolute connecting maps.

4. For one-forms, the three middle source classes are $f_i \otimes g_i$. In the ordered target $(g_0g_1, g_0g_2, g_1g_2)$, the matrix is

$$C_1 = \begin{pmatrix} -C & A & 0 \\ -E & 0 & B \\ 0 & -D_0 & D_0 \end{pmatrix}, \qquad \det C_1 = \frac{476837158203125}{281474976710656}(384z + 3125).$$

The degree-zero connecting map is nonzero because $R(f_0)$ has nonzero $g_1, g_2$ coefficients. The degree-two connecting map, from the classes $f_1g_0g_1, f_2g_0g_2$ to $g_0g_1g_2$, has rank one because $D_0 \neq 0$.

5. We must exclude the sole possible zero of $\det C_1$ on the whole admissible parameter domain. For $\xi = ic$ the preceding coordinate relation gives $z = -(625\pi/128)c$. The unique exceptional parameter is therefore $\xi = 5i/(3\pi)$. By theorem 7.1, the admissible half-plane

is $\operatorname{Im}\xi > -h_{b,\min} > 1$, whereas $5/(3\pi) < 1$. Thus this zero lies outside the whole admissible parameter domain. Consequently the three connecting ranks are $1, 3, 1$. The long exact sequence of (8.2) gives

$$h^q(\Omega^1_M(\log W)) = (0, 1, 1, 0, 0).$$

6. The residue sequence

$$0 \to \Omega^1_M \to \Omega^1_M(\log W) \to \nu_*\mathcal{O}_Y \to 0$$

is exact even at self-intersections: in a local branch chart it records independently the coefficient of each $dx_i/x_i$ on $x_i = 0$. The normalization $Y$ is smooth complete toric, so its only coherent structure cohomology is $H^0 = \mathbb{C}$ (Cox et al., 2011, Theorem 9.2.3, p. 410). The one-form row follows immediately.

7. For two-forms take the second exterior power of (8.2). Its short exact sequence has left term $h^*(L^{11} \otimes F)$ and right term $h^*(\bigwedge^2 F)$. In degree two the three source classes are $f_i f_j \otimes g_i g_j$, $i < j$. The target is $F \otimes g_0 g_1 g_2$. Its connecting matrix is

$$C_2 = \begin{pmatrix} -D_0 & D_0 & 0 \\ E & 0 & -B \\ 0 & -C & A \end{pmatrix}, \qquad \det C_2 = -\det C_1.$$

Thus it is an isomorphism, and $h^3(\Omega^2_M(\log W)) = 1$.

8. The ordinary/logarithmic comparison in this degree is the exact residue resolution

$$0 \to \Omega^2_M \to \Omega^2_M(\log W) \to \nu_*\Omega^1_Y(\log \partial Y) \to \nu^{[2]}_*\mathcal{O}_{W^{[2]}} \to 0.$$

The last space is the disjoint normalization of the seven double surfaces. Local exactness is verified by writing a logarithmic two-form as a sum of $a_{ij}\, dx_i/x_i \wedge dx_j/x_j$ and terms with ordinary differentials. The two residues of an $ij$ term have opposite sign; their cancellation is the sole compatibility. Every local double-surface coefficient lifts to one branch, which proves surjectivity at the last term. Both final sheaves have zero positive cohomology: the first is the pushforward of the trivial rank-three toric logarithmic bundle, and the second has complete toric surface components. Writing $Q = \Omega^2_M(\log W)/\Omega^2_M$, we obtain $H^q(Q) = 0$ for $q \geq 2$. Hence $h^3(\Omega^2_M) = 1$. In degree zero the last residue map is zero: the fourteen toric boundary divisors of $Y$ occur in seven opposite pairs; their primitive normal vectors have opposite signs and their gluing is a torus translation. This gives the zero 7-by-3 residue matrix on global logarithmic one-forms. Thus $H^0(Q) = \mathbb{C}^3$. The two global logarithmic two-forms come from the $L^{12}, L^{13}$ summands of $L^{11} \otimes F$; their residues are the independent $f_1, f_2$ directions at the cusp. Therefore $h^0(\Omega^2_M) = 0$.

9. Serre duality (Serre, 1955, Theorem 4, p. 22) on this compact smooth complex fourfold gives $h^{2,1} = h^{2,3} = 1$. Finally the analytic Riemann–Roch theorem (Hirzebruch, 1978, Theorem 25.3.1, p. 187; §25.4(a), pp. 188–190) applies to the Dolbeault elliptic operator of $\Omega^p_M$. Using $c_1 = c_3 = 0$, the Chern-root product gives

$$\chi_y(M) = \frac{(1+y)^4}{240}\, c_2^2 + \left( \frac{(-1+10y-y^2)^2}{144} - \frac{(1+y)^4}{120} \right) c_4 = -y + 4y^2 - y^3.$$

The displayed polynomial gives $\chi(\Omega^2_M) = 4$, hence $h^{2,2} = 6$. Theorem 8.2 and Serre duality supply all other entries of the table.

□

**Theorem 8.4** (Tangent cohomology and the Kuranishi germ)**.** [tag 02R3] *For every admissible fourfold,*

$$(h^0(T_M), h^1(T_M), h^2(T_M), h^3(T_M), h^4(T_M)) = (1, 1, 1, 2, 0).$$

*The complex parameter family has a nonzero Kodaira–Spencer map at every member and supplies a smooth one-dimensional miniversal Kuranishi germ.*

*Proof.* 1. Put $a = -256/3125$, and denote the whole reduced cusp fibre by $W$. The logarithmic sequence established in the proof of theorem 8.3, tensored with $K_M = h^*L^{-6}$ from theorem 8.2, is

$$0 \longrightarrow h^*L^5 \longrightarrow \Omega^1_M(\log W) \otimes K_M \longrightarrow h^*(F \otimes L^{-6}) \longrightarrow 0.$$

Here $K_{\widehat{B}}(a) = L^{11}$. More precisely, in the meromorphic $L$-frames of theorem 3.6, $\kappa = d\lambda/(\lambda - a)$ is $\lambda^9$ times the $L^{11}$-frame: both sides have divisor $3D_0 - D_\infty$. The logarithmic sequence is exact in each chart $q = z_0 \cdots z_r$, where its first arrow sends $dq/q$ to $\sum_{j=0}^{r} dz_j/z_j$; the quotient is the relative translation cotangent bundle $h^*F$. These arrows agree on the monomial and deck transitions and descend through the free finite root-chart actions.

2. For a summand $L^n$, the coarse degree is $d(n) = \lfloor 4n/5 \rfloor + \lfloor -3n/4 \rfloor$. Projection formula and $R^j h_* \mathcal{O}_M = \bigwedge^j G$ compute the two end terms. For $h^*L^5$ the weights in relative degrees $j = 0, 1, 2, 3$ are

$$\{5\}, \quad \{5, 4, 3\}, \quad \{4, 3, 2\}, \quad \{2\}.$$

Weights $4, 5$ have degree zero and weights $2, 3$ have degree $-1$. Thus $h^q(h^*L^5) = (1, 2, 1, 0, 0)$. The three weights of $F \otimes L^{-6}$ are $-4, -5, -6$. After tensoring with $\bigwedge^j G$, every weight is between $-9$ and $-4$. We have $d(-9) = -2$ and $d(n) = -1$ for $-8 \leq n \leq -4$. Weight $-9$ occurs once in each of relative degrees 2 and 3, so

$$h^q\big(h^*(F \otimes L^{-6})\big) = (0, 0, 0, 1, 1).$$

The base has cohomological dimension one, so the two-column Leray spectral sequences have no further differential. Every connecting map for the displayed logarithmic sequence has zero target. Consequently

$$h^q(\Omega^1_M(\log W) \otimes K_M) = (1, 2, 1, 1, 1).$$

3. Let $\nu : Y \to W$ be the branch normalization. The residue sequence, with its branchwise maps, is

$$0 \longrightarrow \Omega^1_M \otimes K_M \longrightarrow \Omega^1_M(\log W) \otimes K_M \longrightarrow \nu_* \mathcal{O}_Y \otimes K_M|_W \longrightarrow 0.$$

Since $K_M$ is pulled back from the base, its restriction to $W$ is a constant line. The complete smooth toric variety $Y$ has $H^0(\mathcal{O}_Y) = \mathbb{C}$ and $H^{>0}(\mathcal{O}_Y) = 0$ (Cox et al., 2011, Theorem 9.2.3, p. 410, with divisor zero). The sole global section of the logarithmic middle term comes from $h^*L^5$. Its meromorphic expression is $\lambda^{-5}\kappa$ tensored with the $L^{-6}$-frame: the unique $L^5$ section is $\lambda^4$ times its frame, and $\kappa$ contributes $\lambda^9$. Its residue at $a$ is $a^{-5} \neq 0$. Hence the map on global sections is an isomorphism onto the residue line, and

$$h^q(\Omega^1_M \otimes K_M) = (0, 2, 1, 1, 1).$$

The pointwise perfect evaluation pairing $T_M \otimes (\Omega^1_M \otimes K_M) \to K_M$ induces analytic Serre duality (Serre, 1955, §13, Theorem 4, p. 22; §15, p. 23). It identifies $H^q(T_M)^\vee$ with $H^{4-q}(\Omega^1_M \otimes K_M)$ and gives the stated tangent cohomology dimensions.

4. We check the class of the parameter family on the smooth fibre $A = h^{-1}(1)$. In the fixed frame its period plane has basis

$$w_2, \quad w_1, \quad w_0 + \frac{625\pi i}{96}\xi e.$$

Differentiation in $\xi$, followed by projection to $G = H/F$, gives the rank-one map $P : F \to G$ with sole nonzero matrix entry $P_{13} = 625\pi i/96$ in the ordered bases above and $(e, w_{-1}, w_{-2})$ of the quotient. Differentiation in the base coordinate gives, by projecting the connection in theorem 3.6, the matrix

$$Q = \frac{1}{3125(1-a)} \begin{pmatrix} -78125/6 + 300r & -390625/48 - 375r & 0 \\ 120 & 0 & 125/2 \\ 0 & 120 & 50 \end{pmatrix}, \qquad r = \frac{625\pi}{96}(-i\xi).$$

Its determinant is $3/[625(1-a)^3] \neq 0$. The translation-valued Čech cocycles obtained by differentiating the lattice gluing identify $H^1(T_A) = \mathrm{Hom}(F, G)$; contraction with a relative one-form sends the two cocycles to the displayed projected period derivatives, in the same convention.

5. The normal tangent sequence on this fibre is

$$0 \longrightarrow T_A \longrightarrow T_M|_A \longrightarrow \mathcal{O}_A \otimes T_{\widehat{B},1} \longrightarrow 0.$$

Its connecting map sends a base tangent vector to its Kodaira–Spencer class $Q$. As $H^0(\mathcal{O}_A) = \mathbb{C}$, the kernel of $H^1(T_A) \to H^1(T_M|_A)$ is exactly $\mathbb{C}Q$. The nonzero matrix $P$ has determinant zero; every nonzero scalar multiple of $Q$ has nonzero determinant. Therefore the image of $P$ in $H^1(T_M|_A)$ is nonzero. The global parameter Kodaira–Spencer cocycle restricts to this image, so its class in $H^1(T_M)$ is nonzero. The finite affine actions are fixed, and the cusp multipliers $\exp(2\pi i C(q,\xi)\nu)$ vary holomorphically in $\xi$. The same finite fan charts, with $C$ bounded on a small parameter neighbourhood, give the proper local family of theorem 2.4, theorem 3.11. Thus the cocycle just computed is the derivative of this compact family.

6. Douady's versality and semiuniversality theorems (Douady, 1974, §VII.8, pp. 598–599; §VIII.1, pp. 599–600) give a miniversal germ with tangent space $H^1(T_M) = \mathbb{C}$. Its local ring has a minimal presentation $\mathbb{C}\{u\}/I$, where $I \subset (u)^2$. The parameter disc classifies by $u \mapsto g(s)$ with $g'(0) \neq 0$, by the preceding step. Substitution $\mathbb{C}\{u\} \to \mathbb{C}\{s\}$ is therefore an isomorphism of convergent power-series rings. Since this map factors through $\mathbb{C}\{u\}/I$, it annihilates $I$; its injectivity gives $I = 0$. The miniversal germ is reduced and smooth of dimension one, and the classifying map is a local isomorphism. This proves miniversality of the given family at each member.

□

**Theorem 8.5** (Algebraic dimension and automorphisms)**.** [tag 02R4] *The fourfold has algebraic dimension one and*

$$\mathrm{Aut}^0(M) = \mathbb{C}, \qquad \mathrm{Aut}(M) = \mathbb{C} \times \mathbb{Z}/2.$$

*Two members $M_\xi, M_\eta$ of the family in definition 1.1 are biholomorphic exactly when $\xi = \eta$.*

*Proof.* 1. An effective divisor dominating the base would restrict to a nonempty effective divisor on a general complex torus. Pairing its Chern class with the square of a Kähler form on that torus is strictly positive. The restriction class is an alternating integral form $J$ on $\Lambda$ fixed by both monodromies:

$$M_1^t J M_1 = J, \qquad M_2^t J M_2 = J.$$

The exact linear system in its fifteen alternating entries has rank fifteen, so $J = 0$. This contradicts the positive pairing, and every divisor is vertical. The pole divisor of a meromorphic function is vertical; its restriction to a general compact fibre is holomorphic and therefore constant. The compact reduced graph of the function maps properly to $\mathbb{P}^1 \times \mathbb{P}^1$ by $h$ on its first factor. Remmert's proper-image theorem, in (Grauert, 1960, §7, Satz 1, p. 60), makes its image analytic. It is generically a single graph and therefore defines the descended meromorphic function on the normal base curve, including the three special values. Conversely pullbacks from the base occur. Hence the meromorphic function field is $\mathbb{C}(t)$ and $a(M) = 1$. Every automorphism consequently induces a Möbius map on the base. The two distinct multiplicities and the cusp identify all three values separately, so this map is the identity.

2. On smooth fibres an automorphism is affine. Its integral linear part is locally constant and commutes with $M_1, M_2$. The two simultaneous commutator equations have rational solution space $\mathbb{Q}I$ by rational row reduction. Their integral invertible elements are therefore $\pm I$. At the order-five filling an affine lift with linear part $\epsilon I$ would have to conjugate the same finite affine generator. For its translation $b$ this requires

$$(I - M_1)b = (1 - \epsilon)v_1/5 \pmod{\Lambda}.$$

Applying the invariant integral covector $\ell$ makes the left side zero. For $\epsilon = -1$ the right side is $2/5 \pmod{}Z$, which is nonzero. Thus every automorphism is a translation.

3. Let $\mathcal{A}$ be the sheaf of translations extending to the cusp model and finite root charts. There is an exact analytic sheaf sequence

$$0 \to j_*\Lambda \to E_0 \to \mathcal{A} \to 0$$

on the root orbifold. Here $j$ omits the cusp and the stalk of $j_*\Lambda$ there is $\ker N$. On a finite root cover the torus evaluation sequence is exact and equivariant for the prescribed affine action; its translation part cancels under conjugation. We descend this sequence on the orbifold, with its finite stabilizer cohomology.

To check the last surjectivity at the cusp, lift a translation to the toroidal cover. It preserves the fan precisely when its valuation is an integral translation of the periodic fan. Subtract the corresponding period; the remaining characters are nonvanishing holomorphic functions on a disc and possess logarithms. These give a local section of $E_0$. The kernel consists exactly of the invariant integral periods. This also proves that the sequence accounts for all extendable translations, including possible component permutations.

4. Write $A = M_1$, $B = M_2$ and $N_1 = I + A + \cdots + A^4$, $N_2 = I + B + B^2 + B^3$. The orbifold fundamental group away from the cusp has generators $g_1, g_2$ with relations $g_1^5 = g_2^4 = 1$. A lattice cocycle is therefore a pair $(x_1, x_2)$ with $N_1 x_1 = N_2 x_2 = 0$. The degree-one exact sequence for $j_*$ says that it extends across the cusp precisely when its value on $g_1 g_2$ is a coboundary:

$$x_1 + Ax_2 = (AB - I)u, \qquad u \in \Lambda.$$

Subtract the coboundary of $u$. The resulting pair is $(-Ay, y)$, where $N_1 y = N_2 y = 0$. The ambiguity in $u$ is $t \in \ker(AB - I)$, which changes $y$ by $(B - I)t$. Thus the cohomology group, with the finite-root torsion, is

$$H^1(\widehat{B}, j_*\Lambda) = \frac{\ker N_1 \cap \ker N_2}{(B - I)\ker(AB - I)}.$$

Primitive integral column bases for its numerator and for $\ker(AB - I)$ are respectively

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Direct multiplication gives

$$(B - I)T = KZ, \qquad Z = \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The determinant of $Z$ has absolute value two; an entry and a 2-by-2 minor have absolute value one. Its Smith factors are $1, 1, 2$, proving $H^1(\widehat{B}, j_*\Lambda) = \mathbb{Z}/2$.

5. Now $H^0(j_*\Lambda) = 0$ by theorem 2.3, $H^0(E_0) = \mathbb{C}$, and $H^1(E_0) = 0$ by the floor calculation. The exact sequence gives
$$0 \to \mathbb{C} \to H^0(\mathcal{A}) \to \mathbb{Z}/2 \to 0.$$
The half-period $\tau = (0, 1, 1, 1, 1, 1)^t/2$ is fixed modulo $\Lambda$ by every monodromy matrix. It extends over the finite quotients because the translations commute there. At the cusp $N\tau$ is integral; primitiveness of $N$ provides $\lambda \in \Lambda$ with $N\lambda = N\tau$. The representative $\tau - \lambda \in \ker N$ is a bounded angular half-period and extends on every toric chart. It is a nonzero order-two global section. Since the additive group $\mathbb{C}$ has no nonzero torsion, it splits the displayed exact sequence. Translations commute, giving the asserted full group.

6. For an isomorphism between two members, the same intrinsic base argument applies. Its integral linear part still lies in the common centralizer. Either scalar $\pm I$ preserves a complex period plane exactly by equality of the two planes. The graphs $(\sigma_b + \xi\alpha e)(P)$ are equal precisely when their parameters $\xi$ agree, since $e \notin F$ and $\alpha$ is nonzero. Translation does not change this condition. This proves the parameter assertion.

□

**Theorem 8.6** (No antiholomorphic isomorphisms)**.** [tag 02QQ] *For the cusp marking $S$ fixed in theorem 2.3 and the corresponding Kuhn fan, none of the completed fourfolds in this family admits an anti-holomorphic automorphism. In particular none admits a real structure. More strongly, two members with this same cusp marking are never anti-biholomorphic.*

*Proof.* 1. By theorem 8.5, the meromorphic function field of each member is $\mathbb{C}(\lambda)$, with the given base map as its algebraic reduction. An anti-biholomorphism therefore induces an anti-Möbius map of the base. The two distinct multiplicities $5, 4$ and the singular cusp support distinguish the three special values, so this base map fixes $\infty, 0, \lambda_*$. As those points are real, the map is precisely $\lambda \mapsto \bar{\lambda}$.

2. A holomorphic map between compact complex tori is affine: after lifting to their vector spaces, its derivative is periodic and therefore constant, since it descends to holomorphic functions on a compact torus. Applying this to an anti-holomorphic map after conjugating its source gives the same assertion for its anti-linear lift. Its induced integral map on $H_1$ is constant along the regular base. Complex conjugation reverses the two finite meridians, so

this matrix must simultaneously conjugate $M_i$ to $M_i^{-1}$. Solving these exact integral linear equations gives only $R$ and $-R$ among unimodular matrices, where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad R^2 = I.$$

Indeed the simultaneous intertwiner space over $\mathbb{Q}$ has dimension one and this primitive matrix is unimodular; determinant one in absolute value permits only the scalar factors $\pm 1$.

3. The finite affine cap excludes $G = R$. Normalize the pullback by $t = u^{m_i}$; any base-preserving isomorphism lifts to this canonical root cover and conjugates its deck generator to the inverse. At its central torus the translation part $t_i$ would satisfy

$$(I - M_i^{-1})t_i = -(v_i + Gv_i)/m_i \pmod{\Lambda}.$$

Applying $\ell$, and using $\ell R = \ell$, gives $0 = -2\ell(v_i)/m_i$ in $\mathbb{R}/\mathbb{Z}$, impossible already for $m_i = 5$. Hence $G = -R$ is necessary.

4. We now use the cusp marking, keeping track of the conjugation of based meridians. Conjugation sends $g_0 = (g_1 g_2)^{-1}$ to $g_2 g_1 = g_2 g_0^{-1} g_2^{-1}$. Thus the matrix in a fixed local cusp frame is $H = M_2^{-1} G$, and it anti-conjugates $M_0$. Exact multiplication in the marking $S$ gives

$$S^{-1}HS = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} P & Q \\ 0 & -P \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}. \tag{8.3}$$

Changing the based cusp path by a cusp loop multiplies by a power of $M_0$ and does not change $P$.

5. We explain why an arbitrary translation cannot remove the obstruction in (8.3). In the additive torus coordinates whose first periods are the $a_j$, the anti-linear map must have the form $\zeta' = P\bar{\zeta} + t(q)$, because its values on these three real periods already determine its linear part. In multiplicative coordinates it is $z' = u(q)\bar{z}^{-P}$. The source zero section limits to a smooth point of the central fibre. An isomorphism takes it to another smooth point. The central fibre is irreducible, and its smooth locus is exactly the dense torus of its toric normalization: every toric boundary divisor is a normal-crossings self-identification. In a chart about this image, the translation therefore has the form $q^v$ times nonvanishing holomorphic units, with $v \in \mathbb{Z}^3$. Translating by the period $v$ changes the selected component chart; the units act by torus translation. Neither changes the linear action $-P$ on the fan.

6. The primitive rays of the normal fan of that component are

$$\{\pm v : 0 \neq v \in \{0,1\}^3\}.$$

An extending torus monomial isomorphism must permute these rays. This can be seen directly on a one-parameter arc $z = t^v$ meeting the generic point of its boundary divisor:

if the image vector lies in the interior of a cone of dimension at least two, that divisor is contracted, which is impossible for an isomorphism. Complex conjugation itself preserves the toric charts, so the same criterion applies to the anti-monomial map above. But $-Pe_2 = (0,-1,1)^t$ is not one of the displayed rays. This is a contradiction. We have exhausted both possible integral linear parts, so no anti-biholomorphism exists.

□

# 9 The boundary of the family

**Definition 9.1** (Analytic extension at the parameter boundary)**.** [tag PV16]

We use the canonical splitting $\sigma_b$, height $h_b$, parameter domain $\mathcal{U}$ and family $M_\xi$ of definition 1.1. The exact coordinate change for another splitting is remark 3.8. By an analytic extension across a boundary point we mean a proper Hausdorff complex-analytic family on a full complex disc whose restriction to the nonempty open part lying in $\mathcal{U}$ is the prescribed family. At infinity we use the coordinate $u = 1/\xi$. This definition permits singular central spaces, multiplicities, and additional central components.

**Remark 9.2** (Deformations of complex tori)**.** [tag PV17] The stability of complex tori under deformation in the large is established in (Catanese, 2004, definition, p. 261; Theorem 2.1, p. 264). The relative-form calculation below writes the torus quotient on the regular analytic open set used in the boundary argument.

**Theorem 9.3** (No analytic extension across the boundary)**.** [tag 02R5] *There is no such analytic extension across any finite boundary point of $\mathcal{U}$, or across infinity. The same conclusion holds after any finite ramified change of the local boundary coordinate. In particular neither end of the real interval $c_* < c < +\infty$ has a proper Hausdorff analytic extension in this sense.*

*Proof.* 1. We first identify the moving pencil on each known member. By theorem 8.2, $K_M^{-4} = h^*L^{24}$ and the fibres of $h$ are connected. For the coarse map $\rho : \widehat{B} \to \mathbb{P}^1$, the root-divisor formula gives

$$\rho_* L^{24} = \mathcal{O}_{\mathbb{P}^1}\big(\lfloor 96/5 \rfloor\, \infty + \lfloor -72/4 \rfloor\, 0\big) = \mathcal{O}_{\mathbb{P}^1}(19\infty - 18\,0).$$

Its sections are spanned by $\lambda^{18}, \lambda^{19}$. On a root chart the invariant sections have a common local monomial factor, and their ratio cancels that factor. Thus $h^0(M, K_M^{-4}) = 2$, and the moving part of this complete system is exactly the marked base map, up to a projective coordinate change. In particular its base ideal is invertible on every member of the prescribed family.

2. Suppose that a proper Hausdorff analytic extension over a full parameter disc exists. Take the reduced component containing the given nonempty open family and resolve it. Analytic resolution applies on a relatively compact neighbourhood of the compact central fibre and can be kept unchanged over the given smooth locus (Bierstone and Milman, 1997, Theorem 1.6 and its following analytic-neighbourhood paragraph, preprint p. 5). Write $p : Z \to \Delta$ for the resulting proper map from a smooth irreducible space. The proper images of the exceptional and critical loci are analytic (Grauert, 1960, §7, Satz 1, p. 60); they omit the given open parameter set. After shrinking the disc they are supported at its centre. Hence $p$ is a proper holomorphic submersion over $\Delta^*$, with compact fourfold fibres.

Put $\mathcal{J} = \omega_{Z/\Delta}^{-4}$. The sheaves $R^j p_* \mathcal{J}$ are coherent (Grauert, 1960, §6, Hauptsatz I, p. 59). Multiplication by a nonzero parameter germ is injective on $\mathcal{J}$ and on $p_*\mathcal{J}$, so $p_*\mathcal{J}$ is locally free over the disc and free after shrinking. The exact sequence for a fibre over $t$ gives

$$0 \longrightarrow (p_*\mathcal{J})_t/\mathfrak{m}_t(p_*\mathcal{J})_t \longrightarrow H^0(Z_t, \mathcal{J}_t) \longrightarrow \ker\big((t'-t) : (R^1p_*\mathcal{J})_t \longrightarrow (R^1p_*\mathcal{J})_t\big) \longrightarrow 0.$$

The last term is supported in the torsion support of the coherent sheaf $R^1p_*\mathcal{J}$, a closed discrete analytic set. At a point of the given open family outside this support, the displayed comparison shows that $p_*\mathcal{J}$ has rank two. Shrinking makes the support empty away from the centre. Thus a frame of $p_*\mathcal{J}$ gives the complete two-dimensional linear system on each punctured fibre.

3. The two evaluation sections define a coherent base ideal and a meromorphic map $Z \dashrightarrow \mathbb{P}^1$ over the full disc. The ideal is locally invertible on the given open family by the first step. Its non-invertibility locus is analytic; its proper image in $\Delta$ is therefore discrete and can be removed away from the centre. Blowing up that ideal and resolving the graph consequently changes only the central fibre. It gives a proper holomorphic map to $\mathbb{P}^1 \times \Delta$. The construction uses the graph of the coherent evaluation map, so its coefficients have finite meromorphic order in the parameter.

   Its critical image is proper analytic. Every component dominating the parameter disc meets the given open family, where there are precisely three critical values. Their degrees over the parameter are one. The multiplicities five and four distinguish two of them, and the third is the cusp. After shrinking, their normalizations give three distinct labelled meromorphic sections, and all vertical exceptional components lie over the centre. The projective transformation taking these sections to $\infty, 0, \lambda_*$ has meromorphic coefficients and is invertible on $\Delta^*$. Resolving its graph gives the marked map $H$ over the disc, still unchanged on punctured fibres. Its regular restriction is a proper holomorphic submersion

$$q : W \longrightarrow B = (\mathbb{P}^1 \setminus \{0, \infty, \lambda_*\}) \times \Delta^*.$$

   Ehresmann transport makes the number of components of a fibre locally constant. The base is connected and the known fibres are connected, so every fibre of $q$ is connected.

4. We need the torus structure on an analytically open dense part of this regular family. The coherent sheaf $\mathcal{E} = q_*\Omega^1_{W/B}$ has generic rank three. Generic cohomology and base change for this flat bundle gives a proper analytic exceptional set outside which $\mathcal{E}$ is locally free of rank three and its fibre is the space of holomorphic one-forms. The constant-dimension comparison is (Grauert, 1960, §7, Satz 5, pp. 63–64). Use the global coherent evaluation map

$$q^*\mathcal{E} \longrightarrow \Omega^1_{W/B}.$$

   Its cokernel has analytic support. The proper image of that support is analytic in $B$ and omits the nonempty part of the known torus open set outside the generic exceptional set. Enlarge the exceptional set by this image. On its complement evaluation is an isomorphism of rank-three bundles.

   Relative exterior differentiation defines an $\mathcal{O}_B$-linear map $\mathcal{E} \to q_*\Omega^2_{W/B}$. It vanishes on the known torus open set. The target is torsion-free, since it is the proper direct image of a vector bundle under a submersion. The identity theorem therefore makes this map zero. Each remaining compact connected fibre has three everywhere independent closed holomorphic one-forms. Their dual vector fields commute and are complete by compactness. Their flows give a transitive holomorphic $\mathbb{C}^3$-action: every orbit is open and the fibre is connected. Its stabilizer $\Gamma$ is discrete, and compactness of $\mathbb{C}^3/\Gamma$ makes $\Gamma$ a full real lattice. This is the complex-torus structure on each such fibre.

5. Let $E \subset B$ be the proper analytic exceptional set just removed. On the universal cover of $B \setminus E$, use Gauss–Manin transport from the given open family to mark integral $H^1$. Integration of the closed relative forms gives a holomorphic map to $\mathrm{Gr}(3, 6)$. On the known open set it is $\widehat{F}_\xi = KF_\xi$, by remark 3.10. Applying the fixed real integral change of marking $K^{-1}$ identifies it with the canonical plane $F_\xi = (\sigma_b + \xi\alpha e)(P)$. The identity theorem on

the connected cover then identifies these two maps everywhere. The explicit graph is a rank-three holomorphic plane for every finite $\xi$, including parameters for which it is not real-transverse. At infinity the same assertion is used with $\xi = 1/u$ on the punctured parameter disc. All changes of continuation marking are real integral and preserve the condition $F \cap \bar{F} = 0$.

6. Let $\xi_0$ be a finite boundary point, so $\operatorname{Im} \xi_0 = c_* = -\min h_b$. The minimum occurs at $\lambda = \infty$ by theorem 7.1. Continuity on its root chart gives regular $\lambda$ with $h_b(\lambda)$ arbitrarily close to $\min h_b$. Every sufficiently small full parameter disc about $\xi_0$ therefore contains noncentral points of

$$\operatorname{Im} \xi + h_b(\lambda) = 0.$$

This equation defines nonempty smooth real hypersurface pieces in the complex surface $B$, since its derivative in $\operatorname{Im} \xi$ is one. A proper complex analytic subset of $B$ has real dimension at most two; it cannot contain those three-dimensional pieces. Choose such a point outside $E$. Its fibre is a complex torus by the fourth step, so its period plane is real-transverse. The fifth step identifies that plane with $F_\xi$, whereas the exact criterion of theorem 3.9 gives $F_\xi \cap \overline{F_\xi} \neq 0$ at the chosen point. This is a contradiction.

At infinity fix a regular $\lambda$ and set

$$\xi = R - i h_b(\lambda), \qquad u = \frac{1}{R - i h_b(\lambda)}, \qquad R \in \mathbb{R}.$$

These are noncentral points of the same failure locus and tend to $u = 0$ as $|R| \to \infty$. Varying $\lambda$ and $R$ again gives real hypersurface pieces in every punctured $u$-disc, so one can choose a point outside $E$ and obtain the same contradiction. Under a finite ramified change of the boundary coordinate, the parameter map has nonzero derivative off the centre after shrinking. The inverse images of these hypersurface pieces remain nonempty real hypersurfaces there, and the argument is unchanged.

The contradiction occurs on a noncentral compact smooth torus fibre of the recovered marked map. The proper reductions and modifications preserve the prescribed open family. Thus the argument applies to every extension in definition 9.1, proving both assertions.

□

**Theorem 9.4** (Families over a punctured disc)**.** [tag 02R6] *Let $p : \mathcal{M} \to \Delta^*$ be a proper holomorphic submersion whose fibres are members of this fourfold family. Its classifying parameter has an interior limit. After a base change of degree at most two, it is isomorphic to the pullback of the constructed family by a holomorphic map of the full disc to $\mathcal{U}$, and therefore has a proper smooth filling. Such a family cannot approach either parameter end.*

*Proof.* 1. At every punctured parameter, theorem 8.4 supplies local miniversality and a local holomorphic classifying map to $\mathcal{U}$. Theorem 8.5 makes this map unique on each geometric point. Thus these local maps agree on overlaps and define one holomorphic function $\xi : \Delta^* \to \mathcal{U}$. There is no quotient by parameter translations, by theorem 8.5.

2. The Cayley coordinate

$$z = \frac{\xi - i(c_* + 1)}{\xi - i(c_* - 1)}$$

has $|z| < 1$. It extends holomorphically across the puncture by the removable-singularity theorem. Its value there has modulus strictly less than one, since a boundary value would force $z$ to be constant by the maximum-modulus principle. The inverse Cayley map therefore extends $\xi$ with value in $\mathcal{U}$.

3. Local identifications with the pulled-back constructed family differ by its automorphisms. These form the constant holomorphic group $\mathbb{C} \times \mathbb{Z}/2$ over the parameter half-plane: the translation field and the half-period in theorem 8.5 extend over the entire family. The finite component torsor on $\Delta^*$ is killed by the double cover. For a coordinate $z$ on $\Delta^*$, the functions $z$ and $z^{-1}$ confine the holomorphic hull of each compact set to a compact annulus; $z$ also separates points and supplies a local coordinate. Thus $\Delta^*$ is Stein. Cartan's Theorem B (Cartan, 1952, Exposé 19, §5, pp. 19-14–19-15), applied to its coherent structure sheaf, gives $H^1(\Delta^*, \mathcal{O}) = 0$, so the remaining additive holomorphic torsor is a coboundary. Thus after that cover the local identifications glue to a global isomorphism. Pulling back the already constructed proper smooth family along the extended classifying map supplies the stated filling.

□

# Part II
# Integral monodromy and marked gluings

## 10 Admissible data and local restrictions

**Definition 10.1** (Admissible data)**.** [tag PV18] Let $\Lambda = \mathbb{Z}^6$ and $k \in \{2, 3\}$. An *admissible linear tuple* is a tuple $(M_1, \dots, M_k, M_0)$ of integral matrices with each $M_i$ of finite order and with a nonzero fixed vector, $M_0 = (M_1 \cdots M_k)^{-1}$, and $N = M_0 - I$ satisfying $N^2 = 0$, $\operatorname{rank} N = 3$ and $\operatorname{im} N = \ker N$ as integral sublattices of $\Lambda$ (the *index-one condition*). An *admissible datum* is an admissible linear tuple together with free affine lifts $\tau_i(x) = M_i x + v_i/m_i$ of affine orders $m_i$, an integral translation clutch $\kappa \in \Lambda$ and a primitive marking of the Kuhn cusp. Gluing the affine caps, the index-one Kuhn cap and the torus bundle over the punctured sphere gives a closed smooth eight-manifold, the *gluing* of the datum.

**Theorem 10.2** (Euler number of a Kuhn cusp)**.** [tag 02MU]

*Let $g : M^8 \to S^2$ be a proper triangulable map, a $T^6$-bundle off finitely many values. Suppose its unique non-quotient special fibre is the rank-three Kuhn toroidal fibre with period-lattice index $d$, and all its other special reduced fibres are free finite quotients of $T^6$. Then $\chi(M) = 6d$.*

*Proof.* 1. The rank-three Kuhn fibre is the case $n = 3$ of (Yang, 2026, Corollary 8.8, arXiv v1, p. 59): the identical height-one fan and the inclusion $\Gamma \subset \mathbb{Z}^3$ identify its index-$d$ quotient, and give central Euler number $3!d = 6d$.

2. The regular fibre has Euler number $\chi(T^6) = 0$. Each other reduced fibre is a free finite quotient of $T^6$, so its Euler number is also zero. Euler additivity for the stated proper triangulable map therefore gives $\chi(M) = 6d$.

□

**Proposition 10.3** (The diagonal obstruction)**.** [tag 02MX]

*If $A \in \mathrm{GL}_n(\mathbb{Z})$ has no eigenvalue one, every affine map $x \mapsto Ax + b$ of $\mathbb{R}^n/\mathbb{Z}^n$ has a fixed point. In particular, finite monodromies without eigenvalue one, such as the diagonal elliptic ones of orders four and six, admit no free affine lifts.*

*Proof.* 1. The real map $I - A$ is invertible. A real solution of $(I - A)x = b$ maps to a fixed point in the torus. The diagonal copies of the order-four and order-six elliptic rotations have no eigenvalue one, so the conclusion applies to every translation part.

□

**Proposition 10.4** (A complete finite-order filter)**.** [tag 02N9]

*The linear order of a finite matrix in* $\mathrm{GL}_6(\mathbb{Z})$ *which admits a free affine lift belongs to*

$$\{1, 2, 3, 4, 5, 6, 8, 10, 12\}. \tag{10.1}$$

*There are 38 possible degree-six cyclotomic characteristic polynomials with an eigenvalue one, and 19 of them commute with some real complex structure.*

*Proof.* 1. Proposition 10.3 requires a nonzero one-eigenspace. The other cyclotomic factors have total degree at most five. The elementary identity

$$\frac{\phi(n)^2}{n} = \prod_{p^a \| n} p^{a-2}(p-1)^2 \geq \frac{1}{2}$$

implies $n \leq 50$ when $\phi(n) \leq 5$. The complete enumeration through 50 gives $n = 1, 2, 3, 4, 5, 6, 8, 10, 12$. Enumerating their multiplicities with total degree six and at least one factor $T - 1$ gives 38 polynomials. A commuting complex structure exists on a real finite-order module exactly when the multiplicities of the real eigenvalues 1 and $-1$ are even; nonreal pairs already carry such a structure. Imposing those two parity conditions leaves 19. Taking least common multiples of their root orders gives exactly (10.1), also for the 38-polynomial list.

□

**Theorem 10.5** (The self-dual parity obstruction)**.** [tag 0201]

*Suppose a compact smooth eight-manifold has the torus fibration and the Kuhn cusp of 10.2 with index one, and its other reduced fibres are free quotients of complex three-tori by finite holomorphic affine actions. If its rational* $H^1$ *monodromy representation is self-dual, then it cannot satisfy* $b_1 = b_2 = 0$*. In particular no admissible datum with symplectic monodromy has a three-connected gluing, for either* $k = 2$ *or* $k = 3$ *(or any number* $k$ *of finite points).*

*Proof.* 1. Let $\mathcal{V}$ be the rank-six cohomology local system on the punctured sphere and let $j$ be its inclusion. On a finite quotient fibre, averaging gives $R^1 g_* \mathbb{Q} = (j_* \mathcal{V})$ at that point. At the Kuhn cusp, its normalization components are simply connected and the degree-one branch complex is the cochain complex of its dual $T^3$. Specialization takes its three deck classes to the covectors annihilating the angular lattice, exactly the three-dimensional monodromy invariant subspace. Hence the same equality holds at the cusp. It holds everywhere.

2. Put $a = \dim H^0(B, j_* \mathcal{V})$ and $b = \dim H^2(B, j_* \mathcal{V})$. The only differential that can kill a class in $H^0(R^1)$ or in $H^2(R^0)$ is $d_2 : H^0(R^1) \to H^2(R^0) = \mathbb{Q}$. If $b_1 = b_2 = 0$, it must be an isomorphism, so $a = 1$. Moreover $H^1(B, j_* \mathcal{V})$ must vanish because its position $(1, 1)$ has no possible incoming or outgoing differential. Self-duality and compactly supported duality give $b = a = 1$.

3. At the $i$th finite point, let $r_i$ be the dimension of the fixed subspace in $H^1$. The derivative is complex linear, so its one-eigenspace has even real dimension. Each $r_i$ is therefore even. The cusp fixed dimension is three. Euler additivity gives

$$\chi(B, j_* \mathcal{V}) = 6(1 - k) + 3 + \sum_{i=1}^{k} r_i,$$

an odd integer. Consequently

$$\dim H^1(B, j_* \mathcal{V}) = a + b - \chi(B, j_* \mathcal{V})$$

is odd and cannot vanish. This is the required contradiction. A preserved nondegenerate alternating form identifies a representation with its dual, so symplectic monodromy is included. The explicit form $(6\gamma\wedge\delta+u\wedge w)\oplus\omega_E$ checks this hypothesis for the fibre-product example of Part III.

□

**Proposition 10.6** (The remaining rank patterns)**.** [tag 0204]

*If a possibly non-self-dual holomorphic datum with the same local models had $b_1 = b_2 = 0$, put $b = \dim H^0(B, \mathcal{V}^*)$. For $k = 2$, the only possible unordered pairs $(r_1, r_2; b)$ are*

$$(2, 2; 0), \qquad (2, 4; 2).$$

*For $k = 3$, they are*

$$(2, 2, 6; 0),\ (2, 4, 4; 0),\ (2, 4, 6; 2),\ (4, 4, 4; 2). \tag{10.2}$$

*Proof.* 1. The preceding Leray argument still gives $a = 1$ and vanishing $H^1(B, j_*\mathcal{V})$. Euler additivity therefore gives $b = \sum r_i + 3 - 1 - 6(k-1)$. Each $r_i$ belongs to $\{2, 4, 6\}$ because a free affine lift requires an eigenvalue one and complex linearity makes its multiplicity even. Also $0 \le b \le \min(3, r_1, \dots, r_k)$, by inclusion in each local invariant subspace. Enumerating these three-value lists gives exactly (10.2).

□

# 11 Finite spectra and rational extensions

**Theorem 11.1** (The orbifold quotient)**.** [tag 02NY]

*For the torus fillings of an admissible datum, with affine multiplicities $m_1, \dots, m_k$, the fundamental group of the total space has a canonical surjection onto*

$$\langle x_1, \dots, x_k \mid x_i^{m_i} = 1,\ x_1 \cdots x_k = 1 \rangle. \tag{11.1}$$

*Consequently a simply connected example with $k = 2$ requires $\gcd(m_1, m_2) = 1$. There is no simply connected example with $k = 3$ and $m_1, m_2, m_3 \ge 2$. An affine multiplicity-one point is an ordinary smooth point and reduces to the problem with one fewer multiple fibre.*

*Proof.* 1. Over the complement of small discs at the special values, the torus bundle projects surjectively on fundamental groups to the punctured base. Quotient its fundamental group by the normal closure of all regular-fibre loops. At a finite filling its root cover is a smooth torus bundle over a disc, and its free deck group is cyclic of order $m_i$. The covering exact sequence shows that the quotient of the filling group by its fibre subgroup is precisely this cyclic group. The boundary base meridian maps to its generator. Thus adjoining that filling imposes exactly $x_i^{m_i} = 1$ after the fibre loops have been killed. At the index-one Kuhn cusp the fundamental group is generated by the three deck loops of its dual torus; these are regular fibre loops. Its base meridian contracts. Hence its filling imposes the ordinary meridian relation and leaves no extra generator in this quotient. The base relation gives the product relation in (11.1). Van Kampen proves that (11.1) is the resulting quotient, in particular the asserted surjection.

2. For two points, substitute $x_2 = x_1^{-1}$; the quotient is $\mathbb{Z}/\gcd(m_1, m_2)$. For three points of orders at least two it is a nontrivial triangle group. Nontriviality can be seen directly. Take a geodesic triangle with angles $\pi/m_1, \pi/m_2, \pi/m_3$ in the sphere, Euclidean plane or hyperbolic plane according as their sum is greater than, equal to or less than $\pi$. In the spherical case the possible orders are, after sorting, $(2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$, and

the corresponding spherical triangles exist. The Euclidean and hyperbolic angle conditions give the other triangles directly. If $r_1, r_2, r_3$ are reflections in the opposite sides, the rotations $r_2r_3, r_3r_1, r_1r_2$ have the stated orders and product one. They therefore give a nontrivial representation of (11.1) into the orientation-preserving isometry group. Its quotient cannot be trivial. Finally a free cyclic action of order one is the identity construction on a root disc of order one, hence an ordinary torus bundle filling and not a multiple fibre.

□

**Proposition 11.2** (Automatic local complex compatibility)**.** [tag 02N4]

*Every admissible datum with simply connected gluing and two finite affine fillings automatically has finite linear actions commuting with real complex structures.*

*Proof.* 1. Let $m_i$ be the order of the free cyclic affine action defining the $i$th filling. Its linear order divides $m_i$. By 11.1 the integers $m_1, m_2$ are coprime, so at least one of them is odd. The corresponding linear determinant is one, because a finite real matrix of odd order cannot have determinant minus one. The relation $M_1M_2M_0 = I$ and the unipotence of $M_0$ imply $\det M_1 \det M_2 = 1$, so both determinants are one.

2. A finite-order real matrix is semisimple. Its nonreal eigenvalues occur in conjugate pairs, and its only possible real eigenvalues are 1 and $-1$. Determinant one says that the $-1$ eigenspace has even real dimension. Since the total dimension is six, the 1 eigenspace also has even real dimension. Choose a complex structure on each of these real eigenspaces by pairing a real basis. On each nonreal isotypical real summand, multiplication by $i$ in its complex eigenspace supplies a commuting complex structure. Their direct sum commutes with $M_i$. The fixed eigenspace is nonzero because the affine action is free, as in 10.3, so its even dimension is at least two.

3. Each chosen structure makes $\mathbb{R}^6/\mathbb{Z}^6$ a complex torus and the given affine map holomorphic; thus it supplies precisely the local complex-compatibility hypothesis used in the subsequent reductions.

□

**Proposition 11.3** (The only first-cohomology pattern for two points)**.** [tag 0200]

*For a simply connected two-multiple-fibre datum with $b_2 = 0$, the rational cohomology system $V$ has one invariant, no coinvariants, and fixed rank two at each finite generator. Its quotient $W = V/\mathbb{Q}e$ by the invariant line has neither invariants nor coinvariants. If one finite characteristic polynomial on $W$ is $(T-1)\Phi_d(T)$ with $\deg \Phi_d = 4$, then $W$ is irreducible over $\mathbb{Q}$.*

*Proof.* 1. Write $a = \dim V^\Gamma$, $b = \dim(V^*)^\Gamma$ and $f_i$ for the finite fixed ranks. The Leray argument in the proof of Theorem 10.5 gives $a = 1$, vanishing first parabolic cohomology, and $f_1 + f_2 = 4 + b$. By 11.2 each $f_i$ is positive and even, so $b$ is even. The maximal square-zero cusp has $\ker N = \operatorname{im} N$. Therefore every globally fixed vector belongs to $\operatorname{im} N$, whereas every globally fixed covector kills $\operatorname{im} N$. The global invariant and coinvariant spaces pair to zero.

2. Restriction of a globally invariant covector to the fixed space of a finite generator is injective: averaging that generator is the projection onto its fixed space, and the covector is unchanged by averaging. Its restriction kills the $a$ global invariants in that space. Hence $a + b \le f_i$ for each $i$. If $b \ge 2$ is even, the even ranks satisfy $f_i \ge b + 2$, so $4 + b = f_1 + f_2 \ge 2b + 4$, a contradiction. Thus $b = 0$ and $f_1 = f_2 = 2$.

3. The quotient $W$ has no invariant covectors, since its dual embeds into $V^*$. If it had a fixed line, its preimage in $V$ would be an extension of the trivial line by the trivial line.

Every finite generator on this two-dimensional space would be unipotent of finite order, hence identity. This would give two invariants in $V$, a contradiction. Thus $W$ also has no invariant vectors.

4. If one finite characteristic polynomial is $(T-1)\Phi_d(T)$ with the second factor irreducible of degree four, every proper nonzero rational submodule has dimension one or four. In dimension one that generator acts as the identity. Since the product of the two finite generators is unipotent, their induced one-dimensional characters are inverse; hence the other generator also acts trivially. This would be an invariant line. In dimension four the one-dimensional quotient is trivial by the same argument, giving a coinvariant. Both have just been excluded.

□

**Proposition 11.4** (Exclusion of an involutory finite part)**.** [tag 02NZ]

*A simply connected datum with $b_2 = 0$, $k = 2$ and the stated finite affine fillings has neither finite linear monodromy of order one or two. Its unordered pair of finite linear orders must therefore belong to*

$$\{(3,4),(3,5),(3,8),(3,10),(4,5),(5,6),(5,8),(5,12)\}. \tag{11.2}$$

*The affine multiplicities may still be larger than these linear orders.*

*Proof.* 1. The linear orders divide the affine multiplicities and are therefore coprime by 11.1. The fixed-rank calculation of 11.3, which uses only 10.5 and the evenness in 11.2, gives fixed rank two at each finite generator. An identity generator would make the product finite and unipotent, hence the identity, contrary to the rank-three cusp.

2. Suppose the cohomology generators are $A, B$ with $A^2 = I$. The order of $B$ is odd. The $(-1)$-eigenspace of $A$ has dimension four, whereas $\ker(AB - I)$ has dimension three. Their intersection in the six-dimensional space contains a nonzero vector $v$. Then $ABv = v$ and $Av = -v$ imply
$$Bv = Av = -v,$$
which contradicts the odd order of $B$. Interchanging the two generators proves the same assertion for either finite site.

3. Delete one and two from the complete finite-order list (10.1) and enumerate its coprime unordered pairs. They are exactly the eight pairs in (11.2). Only the divisibility of the linear orders into the affine multiplicities has been used.

□

**Lemma 11.5** (Primitive exterior-square comparison)**.** [tag PV07] *Let $E = \mathbb{C}^4$ have symplectic basis $e_1, e_2, f_1, f_2$, put $\omega = e_1 \wedge f_1 + e_2 \wedge f_2$, and give $E_0 = \omega^{\perp} \subset \bigwedge^2 E$ the wedge pairing. The morphism $\rho : \mathrm{Sp}(E) \to \mathrm{SO}(E_0)$ is surjective with kernel $\{I, -I\}$ (Borcherds et al., 2016, §7.3, p. 129). A unipotent element of Jordan type $(2,2,1)$ has as its two lifts a symplectic transvection and its negative. For a semisimple lift with eigenvalues $x, x^{-1}, y, y^{-1}$, the eigenvalues on $E_0$ are*

$$1, \quad xy, \quad x/y, \quad y/x, \quad (xy)^{-1}.$$

*Proof.* We use the cited primitive exterior-square morphism and compute the two comparisons needed here. The transvection fixing $e_1, e_2, f_2$ and sending $f_1$ to $f_1 + e_1$ acts on the primitive basis

$$e_1 \wedge e_2, \quad e_1 \wedge f_2, \quad e_2 \wedge f_1, \quad f_1 \wedge f_2, \quad e_1 \wedge f_1 - e_2 \wedge f_2.$$

Its difference from the identity sends the third vector to the negative of the first and the fourth to the second, and kills the others. Its Jordan type is $(2,2,1)$. Conversely, an orthogonal unipotent

$I + N$ of this type has $N^2 = 0$ and $N^* = -N$, so its image is an isotropic plane. Choose that plane, its dual isotropic plane, and the remaining nondegenerate line. In this decomposition $N$ is determined by a nonzero alternating form on the dual plane. All such forms are equivalent under its general linear group; extending the change by the inverse transpose on the image plane gives an orthogonal change of basis. Its determinant is one, after the choice of sign on the residual line. Thus these unipotents form one special-orthogonal conjugacy class. Surjectivity of $\rho$ lifts this conjugacy, and its kernel gives the two stated lifts. Finally, taking pairwise products of the four eigenvalues and removing the invariant bivector gives the displayed five eigenvalues. □

**Theorem 11.6** (Exclusion of three further order pairs)**.** [tag 02MZ]

*No simply connected datum with $k = 2$, $b_2 = 0$, and finite linear orders $(3, 5)$, $(3, 8)$ or $(3, 10)$ has the required maximal square-zero cusp. Among such data, the remaining necessary unordered linear order pairs are*

$$\{(3, 4), (4, 5), (5, 6), (5, 8), (5, 12)\}. \tag{11.3}$$

*When one order is five, the other finite characteristic polynomial cannot be $(T - 1)^2\Phi_4(T)^2$ or $(T - 1)^2\Phi_6(T)^2$.*

*Proof.* 1. For each proposed pair, 11.3 gives an irreducible rational rank-five quotient $W$. Its cusp has Jordan type $(2, 2, 1)$. One finite polynomial is $(T - 1)\Phi_3(T)^2$ and the other is $(T - 1)\Phi_d(T)$ for $d = 5, 8, 10$. For the two additional assertions, replace $\Phi_3^2$ by $\Phi_4^2$ or $\Phi_6^2$, with $d = 5$. In all these cases a commuting endomorphism of $W$ restricts to a rational scalar on the one-dimensional fixed space of the second generator. This restriction is injective: a nonzero endomorphism of an irreducible representation is invertible and cannot vanish on that line. Consequently

$$\mathrm{End}_\Gamma(W) = \mathbb{Q}.$$

2. Consider $\mathcal{H} = \mathrm{Hom}(W, W^*)$. The local fixed dimensions are $9, 5, 13$. At the repeated finite site, pairing reciprocal eigenspaces of dimensions $1, 2, 2$ gives $1 + 4 + 4 = 9$. At the distinct finite site it gives five. For the cusp, the ordered pairs of blocks of sizes $2, 2, 1$ contribute the sum of their minimum sizes, namely thirteen. The invariant-sheaf Euler formula therefore gives

$$\chi(\mathbb{P}^1, j_*\mathcal{H}) = -25 + 9 + 5 + 13 = 2. \tag{11.4}$$

Here degree zero is $\mathrm{Hom}_\Gamma(W, W^*)$, and degree two is dual to $\mathrm{Hom}_\Gamma(W^*, W)$. At least one is nonzero, so irreducibility gives a nondegenerate invariant bilinear form $J : W \longrightarrow W^*$, inverting the map if necessary. Its alternating part is equivariant; if nonzero it would be invertible, which is impossible in odd dimension. Thus $J$ is symmetric.

3. An invariant bivector is an alternating equivariant map $K : W^* \longrightarrow W$. The comparison map

$$K \longmapsto KJ \in \mathrm{End}_\Gamma(W)$$

makes it a scalar, so $K$ is a scalar multiple of the symmetric map $J^{-1}$. Hence $K = 0$. Applying the same argument to an alternating map $W \longrightarrow W^*$ gives

$$(\Lambda^2 W)^\Gamma = (\Lambda^2 W^*)^\Gamma = 0.$$

Both degree zero and degree two of $j_*\Lambda^2 W$ vanish. Its Euler characteristic must therefore be nonpositive.

4. The repeated finite spectrum has four fixed bivectors and the distinct spectrum has two. At the cusp, write the unipotent module as two blocks of size two and one block of size one. Their exterior square has two trivial blocks from the individual wedges, blocks of sizes

three and one from the product of the two size-two blocks, and two further size-two blocks. Its fixed rank is six. Consequently

$$\chi(\mathbb{P}^1, j_*\Lambda^2 W) = -10 + 4 + 2 + 6 = 2,$$

contradicting the preceding step. This proves all five stated exclusions. Removing the three excluded order pairs from (11.2) leaves exactly (11.3).

□

**Proposition 11.7** (Rational uniqueness of the extension)**.** [tag 02N0]

*Let $W$ be a rational rank-five representation of a twice-punctured affine line, with two finite generators having fixed rank one and product with unipotent Jordan type $(2, 2, 1)$. Suppose $W$ has neither invariants nor coinvariants. Up to rescaling the invariant line, there is exactly one rational extension $0 \to \mathbb{Q}e \to V \to W \to 0$ in which the two finite generators retain their finite orders and the cusp has square-zero logarithm of rank three. Thus, for the two-point data of 11.3, the remaining rational representation problem is exactly the problem of the rank-five quotients.*

*Proof.* 1. Write the finite generators of $W$ as $S_1, S_2$ and the extension generators as

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & S_i \end{pmatrix}.$$

If $S_i$ has order $m_i$, retaining that finite order is equivalent to $a_i(I + S_i + \cdots + S_i^{m_i-1}) = 0$. Averaging identifies this condition with $a_i \in \operatorname{row}(S_i - I)$, a space of dimension four. A change of splitting by $b \in W^*$ sends the two rows to $a_i + b(S_i - I)$. This coboundary map is injective because $W$ has no invariant covectors. Thus the extension space subject to the two finite conditions has dimension $8 - 5 = 3$.

2. Put $U = S_1S_2 = I + N$, so $N^2 = 0$ and $\operatorname{rank} N = 2$. The product $A_1A_2$ has upper row $c = a_1S_2 + a_2$. Changing the splitting sends $c$ to $c + bN$. Hence cusp restriction is the explicit map

$$[(a_1, a_2)] \longmapsto [c] \in W^*/\operatorname{row} N.$$

It is an isomorphism. Indeed, a class in its kernel can be represented with $c = 0$, so $a_2 = -a_1S_2$. The finite conditions then say

$$a_1 \in \operatorname{row}(S_1 - I) \cap \operatorname{row}(S_2 - I).$$

This intersection has dimension three: the two four-dimensional row spaces span $W^*$ because $W$ has no common fixed vector. The rows $b$ satisfying $bN = 0$ also form a three-dimensional space. Their map $b \mapsto b(S_1 - I)$ lands in the displayed intersection, since $bU = b$ gives $b(S_1 - I)S_2 = b - bS_2$. It is injective, because $bS_1 = b$ and $bU = b$ together give $bS_2 = b$, forcing $b = 0$. It is therefore onto. Every pair with $c = 0$ is a coboundary, proving injectivity of cusp restriction. Both its source and target have dimension three, so it is an isomorphism.

3. The square of the lifted product minus identity vanishes exactly when $cN = 0$. Under this restriction isomorphism the square-zero extensions form

$$\frac{\operatorname{ann}(\operatorname{im} N)}{\operatorname{row} N}, \tag{11.5}$$

of dimension $3 - 2 = 1$. The lifted nilpotent has the rows of $N$ together with the row $c$, so its rank is three exactly when $c \notin \operatorname{row} N$. These are precisely the nonzero classes in (11.5). Rescaling the invariant line rescales this one-dimensional rational space and is transitive on its nonzero classes. This proves the asserted existence and uniqueness with the stated finite orders and maximal cusp.

□

**Theorem 11.8** (An integral obstruction for a rational $(5,12)$ family)**.** [tag 02N1]

*Let a rational rank-six cohomology representation have generators*

$$A=\begin{pmatrix}1&0\\0&A_5\end{pmatrix},\qquad B=\begin{pmatrix}1&c\\0&B_5\end{pmatrix},\qquad c=(0,0,0,1,0),$$

*where*

$$A_5=\begin{pmatrix}1&0&1&0&0\\1&0&0&0&0\\-1&0&0&1&1\\1&-1&0&-1&0\\0&1&0&0&0\end{pmatrix},\qquad B_5=\begin{pmatrix}0&1&0&0&0\\0&0&0&0&1\\1&-2&0&0&0\\0&1&0&-1&-1\\0&0&1&1&2\end{pmatrix}. \tag{11.6}$$

*For every invariant integral lattice with an index-one maximal square-zero cusp, and every free affine filling of multiplicities $m_1,m_2$ divisible respectively by five and twelve, a simply connected total space has*

$$\operatorname{Tor}_3 H_2\simeq\mathbb{Z}/3^{v_3(m_2)}. \tag{11.7}$$

*In particular none of these data is three-connected. The standard lattice nevertheless has a primitive cusp, free affine fillings of orders five and twelve, and a simply connected total space with rational Betti vector* $(1,0,0,0,4,0,0,0,1)$.

*Proof.* 1. Put $R=\mathbb{Z}_{(3)}$ and use the rational coordinates of (11.6). Let $\mathcal{P}_R$ be the algebra of blocks $\left(\begin{smallmatrix}a&b\\0&D\end{smallmatrix}\right)$, where $a\in R$, $b\in R^{1\times5}$ and $D\in\operatorname{Mat}_5(R)$. We first determine the integral algebra generated by $A,B$. The following ordered words have 31 coordinate columns in the order $(0,j)$, $0\le j\le5$, followed by $(i,j)$, $1\le i,j\le5$:

$$I,A,B,A^2,AB,BA,B^2,A^3,A^2B,ABA,AB^2,BA^2,$$
$$BAB,B^2A,B^3,A^3B,A^2BA,A^2B^2,ABA^2,AB^2A,$$
$$AB^3,BA^3,BA^2B,BAB^2,B^2A^2,B^3A,A^3BA,$$
$$A^3B^2,A^2BA^2,BA^3B,B^2A^3.$$

Their coordinate determinant is $-32$. Its adjugate divided by $-32$ therefore expresses every elementary matrix of $\mathcal{P}_R$ as an $R$-linear combination of these words. Since $A,B\in\mathcal{P}_R$, this proves $R[A,B]=\mathcal{P}_R$ with the coordinate map.

For a stable lattice $L_R$, the elementary idempotents split its six coordinate fractional ideals; the matrices $E_{ij}$ for $i,j\ge1$ make the last five equal. Thus

$$L_R=3^aRe_0\oplus3^bR^5,\qquad b\ge a,$$

where the inequality follows from the matrices $E_{0j}$. The matrix $N=AB-I$ has square zero and nonzero Smith factors $1,1,1$, while $A_5B_5-I$ has factors $1,1$. In this basis of $L_R$, its upper-right row is multiplied by $3^{b-a}$. For $b>a$ its reduction modulo three has rank two, contradicting the primitive rank-three cusp. Hence $b=a$. The scalar map $3^{-a}I$ identifies $L_R$ with $R^6$ and intertwines every exterior power of the two generators. The given cusp operator has the same elementary divisors: $BA-I=A^{-1}(AB-I)A$, and $(BA)^{-1}-I=-(BA)^{-1}(BA-I)$. This proves the asserted local marking for every stable lattice.

2. Write $v_i=g_i^{m_i}$ for the finite power vectors. They are fixed by the homology matrices $M_i$. The base-orbifold quotient in theorem 11.1 shows that simple connectivity forces $\gcd(m_1,m_2)=1$. After the cusp meridian relation the two base generators are inverse up to a fibre translation. Their actions on the abelian fibre lattice are therefore inverse. Their linear orders five and twelve force both actions to be the identity in this quotient.

The exact Smith factors of $[A^t - I\ B^t - I]$ are five units, with first row zero, so its fibre coinvariant module over $R$ is $R$ through the primitive character $\ell = x_0$.

Set $a_i = \ell(v_i)$. An arbitrary cusp clutch contributes a scalar $\delta$ to $g_1 + g_2 = \delta z$. The remaining relation matrix is

$$\begin{pmatrix} m_1 & 0 & -a_1 \\ 0 & m_2 & -a_2 \\ 1 & 1 & -\delta \end{pmatrix}.$$

Its determinant is a unit for a simply connected total space. Since $3 \mid m_2$ and $3 \nmid m_1$, reduction modulo three makes $a_2$ a unit in $R$. This conclusion includes arbitrary clutch translations and twists.

3. We determine the second finite-cap restriction over $R$. Choose a primitive integral invariant character $\alpha$ whose $R$-span is $\ell$. The stable sublattice

$$\Lambda' = \mathbb{Z}v_2 \oplus \ker \alpha$$

has index $|\alpha(v_2)|$, prime to three. It gives a finite torus cover; the affine action lifts because it preserves $v_2$, preserves $\ker \alpha$, and translates by $v_2/m_2$. The corresponding reduced cap $F_2'$ covers $F_2$ with the same degree. In its basis beginning with $v_2$, the action is

$$(u, y) \longmapsto (u + 1/m_2, Dy).$$

Thus $F_2'$ is the mapping torus of $D : T^5 \to T^5$, and its regular six-torus is the $m_2$-fold cover in the circle direction. The coefficient change has determinant a unit in $R$; under the scalar marking of item 1, the vertical cohomology action $D^t$ is the displayed $B_5$.

The matrices $D^t - I$ and $\bigwedge^2 D^t - I$ have no nonunit three-primary Smith factors. Their Wang sequences therefore make $H^q(F_2'; R)$ free for $q \le 3$. Pullback to the generic torus is injective in these degrees, because transfer kills its kernel by $m_2$ and the source is free. Deck transformations of $F_2' \to F_2$ act by translations on the generic torus cover. Their action there is trivial on cohomology, so the injectivity just proved makes their action on $H^q(F_2'; R)$ trivial as well. Transfer for this prime-to-three cover now identifies $H^q(F_2; R)$ with $H^q(F_2'; R)$, compatibly with their restriction maps. In particular the degree-two and degree-three restrictions have zero kernel.

Put $T = D^t$. Its polynomial is $(x - 1)\Phi_{12}(x)$, and

$$E = \Phi_{12}(T) = T^4 - T^2 + I, \qquad E^2 = E, \qquad \sum_{j=0}^{11} T^j = 12E.$$

The image of this integral rank-one idempotent is the primitive invariant line. Hence the norm for every stated $m_2$ is $m_2 E$. The Wang sequence in degree two has quotient $\mathrm{Inv}(\bigwedge^2 T)$ and kernel $\mathrm{coker}(T - I)$. Restriction to the $m_2$-fold cover projects identically onto the former group. On the latter it is the horizontal wedge with the norm: a horizontal cell lifts to $m_2$ successive transported prisms. Consequently the restriction image $J_2$ projects onto the vertical invariants, and its intersection with the horizontal invariant summand is $e_0 \wedge m_2 \,\mathrm{Inv}(T)$. The horizontal inclusion induces the isomorphism

$$\frac{\mathrm{Inv}(T)}{m_2 \,\mathrm{Inv}(T)} \xrightarrow{\ \sim\ } \frac{\mathrm{Inv}(\bigwedge^2 V_R)}{J_2} \ \cong\ R/3^r R.$$

At the first finite cap, $m_1$ is a unit in $R$, so transfer and averaging give the full invariant image and zero kernel. At the cusp, theorem 2.6 gives injective restrictions onto the invariant lattices in degrees one, two and three, by dualizing its saturated-kernel collapse.

4. The integral specialization-sheaf approach is used in (Claude and L. Alpöge, 2026, Lemma 7.25 and Propositions 7.26–7.27, pp. 61–63) and (Engel, 2026, §7, pp. 17–21). Here are the coefficient maps and calculations for the present lattice. Put $L_q = \bigwedge^q V_R$, $A_q = \bigwedge^q A$, $B_q = \bigwedge^q B$, and let $j$ include the thrice-punctured base. For the full invariant-extension sheaf $j_*L_q$ the oriented cellular complex is

$$\begin{gathered} J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q, \\ J_{1,q} = \ker(A_q - I), \quad J_{2,q} = \ker(B_q - I), \quad J_{0,q} = \ker(B_q A_q - I), \\ d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0), \\ \Delta_q(a, b) = (A_q - I)a + (B_q - I)b. \end{gathered}$$

These are the base attachment maps: the two cusp paths carry $j_0$ and $A_q j_0$, whose combined boundary is $(B_q A_q - I)j_0 = 0$. Primitive invariant bases and a unimodular kernel basis of $\Delta_q$ give the following Smith forms:

| $q$ | $\operatorname{rank} H^0$ | $\operatorname{rank} H^1$ | $\mathrm{SNF}(d_0 \text{ in } \ker \Delta_q)$ | $\mathrm{SNF}(\Delta_q)$ |
|---|---|---|---|---|
| 1 | 1 | 0 | $1^6$ | $1^5, 2$ |
| 2 | 0 | 0 | $1^{15}$ | $1^{15}$ |
| 3 | 0 | 3 | $1^{16}, 2$ | $1^{19}, 2.$ |

Here $1^s$ denotes $s$ entries equal to one. The row-coordinate matrices and their unimodular HNF transformations are specified by the preceding formulas. In particular $H^0(j_*L_2) = H^1(j_*L_2) = 0$, $H^2(j_*L_1) = 0$ over $R$, and $H^0(j_*L_3) = 0$.

Write $\mathcal{R}_q = R^q f_* R$. The stalk/fibre identification is proper base change (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1; chapter PDF p. 30). The local maps of item 3 give

$$0 \longrightarrow \mathcal{R}_2 \longrightarrow j_*L_2 \longrightarrow Q \longrightarrow 0, \qquad Q_{p_2} = R/3^r R,$$

with no other support. Hence its connecting homomorphism gives $H^1(\mathcal{R}_2) = R/3^r R$. The degree-one quotient is also point supported, so $H^2(\mathcal{R}_1) = H^2(j_*L_1) = 0$. The degree-three sheaf map is injective, so $H^0(\mathcal{R}_3) = 0$.

5. The only total-degree-three term in the Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4; chapter PDF p. 22) is therefore $E_2^{1,2} = R/3^r R$. It has no possible differential because the base has cohomological dimension two. Thus $H^3(M; R) = R/3^r R$. Universal coefficients (Hatcher, 2002, Theorem 3.2, p. 195) identify this torsion module with $\operatorname{Ext}_R(H_2(M; R), R)$: the quotient $\operatorname{Hom}_R(H_3(M; R), R)$ is free and must vanish. The elementary divisor decomposition over the PID $R$ proves (11.7).

6. For the standard lattice take

$$v_1 = (3, 0, 0, 0, 0, 0)^t, \qquad v_2 = (-7, 7, -7, 7, 0, 7)^t, \qquad (m_1, m_2) = (5, 12).$$

They are fixed by $A^t, B^t$ respectively. Their $\ell$-values $3, -7$ are units modulo the respective multiplicities; applying $\ell$ to a fixed-point equation proves that every nonidentity affine power is free. The primitive cusp in item 1 has the integral marking of proposition 2.2. Glue the two finite quotient caps and the index-one Kuhn cap to the marked torus bundle over a pair of pants, with zero additional clutch translation. The finite boundary identification is the prescribed logarithmic fibre translation; the cusp is attached in that integral marking. These local models give a closed smooth oriented manifold. Its remaining van Kampen matrix is

$$\begin{pmatrix} 5 & 0 & -3 \\ 0 & 12 & 7 \\ 1 & 1 & 0 \end{pmatrix},$$

with determinant of absolute value one. It is simply connected.

Over $\mathbb{Q}$ the finite images are full invariants. Item 4 gives vanishing $H^1(j_*V_{\mathbb{Q}})$, $H^0(j_*\bigwedge^2 V_{\mathbb{Q}})$, $H^1(j_*\bigwedge^2 V_{\mathbb{Q}})$ and $H^2(j_*V_{\mathbb{Q}})$, and the degree-three invariant group is zero. The five-term Leray sequence and $H^1(M;\mathbb{Q}) = 0$ make the map from the one-dimensional $H^0(\mathcal{R}_1)$ to $H^2(\mathbb{P}^1;\mathbb{Q})$ an isomorphism. Thus $b_2 = b_3 = 0$. Euler additivity gives $\chi(M) = 6$ from the Kuhn cusp; all other fibres have Euler number zero. Poincaré duality therefore gives $(1,0,0,0,4,0,0,0,1)$, as claimed.

□

**Theorem 11.9** (Two further rational families excluded integrally)**.** [tag 02N2]
*Let $C_f, C_g$ denote the degree-four companion matrices. Take $f = \Phi_5$ and either*

$$\textit{case A: } g = \Phi_{12}, \qquad \textit{case B: } g = \Phi_3\Phi_6. \tag{11.8}$$

*Let $W$ be the quotient of $\bigwedge^2(C_f^{-1}, C_g)$ by its invariant line, and let $V$ be its unique rational extension of 11.7. For every invariant integral lattice in $V$ with primitive maximal square-zero cusp, every simply connected free affine filling, with multiplicities $m_1, m_2$ divisible by five and six respectively, fails three-connectivity. More precisely, put $r = v_3(m_2) \geq 1$. In case A, $\operatorname{Tor}_3 H_3$ has order $3^{2r-1}$. In case B, $\operatorname{Tor}_3 H_2 \simeq \mathbb{Z}/3^r$. For $m_2 = 6$ the respective obstructing group is $\mathbb{Z}/3$.*

*Proof.* 1. We first give rational models, including the quotient maps. Use the exterior basis $(01, 02, 03, 12, 13, 23)$ for the companion pair. Its common primitive invariant vectors are

$$\omega_{\mathrm{A}} = (1,2,1,2,2,1)^t, \qquad \omega_{\mathrm{B}} = (1,0,1,0,0,1)^t.$$

Let $T$ have columns $\omega, e_1, \dots, e_5$; its determinant is one. The last five rows of $T^{-1}$ give the quotient map. In that quotient the two matrices $a, b$, and an extension row $c$, are

| | $a$ | $b$ |
|---|---|---|
| A | $\begin{pmatrix} 2 & -1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 \\ 3 & 0 & -2 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix}$ |
| B | $\begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix}$ |

$$c_{\mathrm{A}} = (-2,-3,1,2,0), \qquad c_{\mathrm{B}} = (0,-1,1,0,0), \qquad A = \operatorname{diag}(1,a), \quad B = \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix}.$$

Exterior eigenvalue products give characteristic polynomial $(x-1)\Phi_5$ for $a$, and $(x-1)(x+1)^2\Phi_3$ or $(x-1)(x+1)^2\Phi_6$ for $b$. Each has fixed rank one. Direct multiplication gives $A^5 = B^6 = I$, $(AB-I)^2 = 0$, and nonzero Smith factors $1,1,1$ for $AB-I$ and $1,1$ for $ab-I$. The row conditions are $c\sum_{j=0}^{5} b^j = 0$ and $c(ab-I) = 0$. The full quotient matrix algebra verified in the next item implies that this rank-five quotient is irreducible, with no invariant vectors or covectors. Thus proposition 11.7 identifies these models with the rational extensions, up to rescaling their invariant line.

2. Put $R = \mathbb{Z}_{(3)}$ and let $\mathcal{P}_R$ be the algebra of blocks $\left(\begin{smallmatrix} u & v \\ 0 & D \end{smallmatrix}\right)$, with $u \in R$, $v \in R^{1\times 5}$ and $D \in \operatorname{Mat}_5(R)$. In the coordinate order $(0,j)$, $0 \leq j \leq 5$, followed by $(i,j)$, $1 \leq i,j \leq 5$, the

following 31 words have coordinate determinants 512 and $-512$ in cases A and B:

$$I, A, B, A^2, AB, BA, B^2, A^3, A^2B, ABA, AB^2, BA^2, BAB, B^2A,$$
$$A^3B, A^2BA, A^2B^2, ABA^2, AB^2A, BA^3, BA^2B, BAB^2, B^2A^2, B^2AB,$$
$$A^3BA, A^3B^2, A^2BA^2, ABA^3, ABA^2B, BA^3B, A^3BA^2.$$

Their adjugates divided by the displayed determinants express every elementary matrix of $\mathcal{P}_R$ as an $R$-linear combination of these words. Consequently $R[A, B] = \mathcal{P}_R$. Its lower block also proves the irreducibility used in item 1.

For an invariant lattice, the elementary projectors and lower-block matrix units give

$$L_R = 3^\alpha Re_0 \oplus 3^\beta R^5, \qquad \beta \geq \alpha;$$

the inequality is forced by $E_{0j}$. In this basis the upper nilpotent row is multiplied by $3^{\beta-\alpha}$. If $\beta > \alpha$, reduction modulo three has rank two, contradicting the primitive rank-three cusp. Thus $\beta = \alpha$ and the map $3^{-\alpha}I$ identifies $L_R$ with $R^6$, intertwining every exterior power. The other product conventions are related by

$$BA - I = A^{-1}(AB - I)A, \qquad (BA)^{-1} - I = -(BA)^{-1}(BA - I).$$

3. We record the consequence of simple connectivity for the affine translations. Write a finite affine generator, after a translation of the torus, as $x \mapsto M_i x + v_i/m_i$, where $M_i v_i = v_i$ and $v_i$ is its integral power vector. This change has an explicit map. If $t$ is a lift of the original translation and $d$ is the linear order, put $t_0 = d^{-1} \sum_{j=0}^{d-1} M_i^j t$ and $b_0 = t - t_0$. Then

$$h = \frac{1}{d} \sum_{j=0}^{d-1} (d - 1 - j) M_i^j b_0, \qquad (I - M_i)h = b_0.$$

For $T_h(x) = x + h$, the conjugate $T_h^{-1} g T_h$ has translation $t_0$, and $v_i = m_i t_0$ is integral because the given affine action has period $m_i$.

The base-orbifold quotient in theorem 11.1 shows that $\gcd(m_1, m_2) = 1$. In particular $m_1$ is a unit in $R$. The homology augmentation matrix $[A^t - I \mid B^t - I]$ has first row zero and rank-five minors $-4$ and $-1$ in its last five rows, respectively. Its coinvariant quotient over $R$ is therefore $R$ through $\ell(x) = x_0$. With $a_i = \ell(v_i)$, the finite power relations and cusp clutch give the quotient relation matrix

$$\begin{pmatrix} m_1 & 0 & -a_1 \\ 0 & m_2 & -a_2 \\ 1 & 1 & -\delta \end{pmatrix}.$$

Here $\delta$ is the character value of the cusp clutch. The character annihilates the collapsed angular cycles, so these maps on the finite caps and cusp agree on the boundary and define a quotient of the abelianized fundamental group. Simple connectivity makes its determinant a unit. Reduction modulo three forces $a_2$ to be a unit. Arbitrary clutch translations are included in $\delta$.

4. Let $\alpha$ be the primitive integral invariant character whose $R$-span is $\ell$, and let $K = \ker \alpha$. The invariant sublattice $\Lambda' = \mathbb{Z}v_2 \oplus K$ has index $|\alpha(v_2)|$, prime to three. It gives a torus cover and a cover $F_2' \to F_2$ of the second reduced finite fibre. In coordinates beginning with $v_2$, its affine map is

$$(\theta, y) \longmapsto (\theta + 1/m_2, Dy).$$

Thus $F_2'$ is a five-torus mapping torus, and its generic six-torus is the $m_2$-fold cover in the circle direction. Restriction to $K$ is the dual quotient $(x_0, x') \mapsto x'$ from $V_R$ to $W_R$,

intertwining $B$ with $b$. Reversing the mapping-torus circle replaces $b$ by $b^{-1}$; the identities $b^{-1} - I = -b^{-1}(b - I)$ and $\sum_{j=0}^{5} b^{-j} = \sum_{j=0}^{5} b^{j}$ give the exact comparison.

Put $b_k = \bigwedge^k b$. The nonzero Smith factors of $b_k - I$ for $0 \le k \le 4$ are units in $R$. The Wang sequences make $H^q(F_2'; R)$ free for $q \le 4$. Pullback to its generic torus is injective, since transfer kills its kernel by $m_2$ and the source is free. Deck transformations of $F_2' \to F_2$ act by translations on that generic cover. Their action on its cohomology is trivial, and injectivity makes it trivial on $H^q(F_2'; R)$ as well. Transfer for this prime-to-three cover therefore identifies the restriction maps with those for $F_2$.

The covering map on the horizontal term of the Wang sequence is the norm

$$\mathcal{N}_{m_2,k} = \sum_{j=0}^{m_2-1} b_k^j : \operatorname{coker}(b_k - I) \longrightarrow \operatorname{Inv}(b_k).$$

Indeed, the lift is the union of the $m_2$ successive prisms with their transported coefficients. On the vertical invariant quotient pullback is the identity. Consequently horizontal inclusion induces the isomorphism

$$\frac{\operatorname{Inv}(b_{q-1})}{\mathcal{N}_{m_2,q-1} \bigwedge^{q-1} R^5} \xrightarrow{\sim} \frac{\operatorname{Inv}(\bigwedge^q V_R)}{J_{2,q}}.$$

Surjectivity follows by subtracting a restriction with the same vertical component; its kernel is exactly the horizontal norm image.

5. Write $\lambda = m_2/6$. Since $b^6 = I$, $\mathcal{N}_{m_2,k} = \lambda \mathcal{N}_{6,k}$. In case A a primitive basis of $\operatorname{Inv}(b_2)$, in lexicographic exterior order, is the column matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & 4 \\ 2 & 4 \\ 0 & 2 \\ 0 & 3 \\ 1 & 4 \\ 0 & 3 \\ 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

The norm coordinates in this basis are

$$\begin{pmatrix} 0 & 3 & 0 & -3 & 0 & 0 & -4 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \end{pmatrix}.$$

Their image has basis $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 6 \end{smallmatrix}\right)$, and multiplication by $\left(\begin{smallmatrix} 1 & 0 \\ -4 & 1 \end{smallmatrix}\right)$ gives $\operatorname{diag}(1, 6)$. This gives the explicit quotient map $(x, y) \mapsto (x, y - 4x)$ modulo $(\lambda, 6\lambda)$. In case B the primitive invariant vector of $b$ is $(1, 0, 0, 1, 0)^t$, and the norm coordinates are $(3, 0, 0, 3, 0)$. After multiplication by the remaining units of $R$, these give

$$Q_{2,3}^{\mathrm{A}} \cong R/3^{r-1}R \oplus R/3^r R, \qquad Q_{2,2}^{\mathrm{B}} \cong R/3^r R.$$

6. We use the integral specialization and base-sheaf method of (Claude and L. Alpöge, 2026, Lemma 7.25 and Propositions 7.26–7.27, pp. 61–63) and (Engel, 2026, §7, pp. 17–21), with

the following explicitly computed coefficients. Put $L_q = \bigwedge^q R^6$, $A_q = \bigwedge^q A$, $B_q = \bigwedge^q B$. The based-sphere complex for $j_* L_q$ is

$$J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q,$$
$$J_{1,q} = \ker(A_q - I), \quad J_{2,q} = \ker(B_q - I), \quad J_{0,q} = \ker(B_q A_q - I),$$
$$d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0),$$
$$\Delta_q(x, y) = (A_q - I)x + (B_q - I)y.$$

The cusp paths carry $j_0$ and $A_q j_0$, so their combined boundary is zero. Inverse generator conventions give a chain isomorphism: use cusp basis $A_q J_{0,q}$ for the inverse pair, and maps $-I$, $\mathrm{diag}(-A_q, -B_q)$, $I$ in degrees zero, one and two. Both chain identities follow by multiplication.

Unimodular HNF transformations give primitive invariant bases and kernel coordinates for $d_0$ in $\ker \Delta_q$. In both cases the exact Smith records are

| $q$ | $\operatorname{rank} H^0$ | $\operatorname{rank} H^1$ | $\operatorname{rank} H^2$ | SNF($d_0$ in $\ker \Delta_q$) | SNF($\Delta_q$) |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | $1^6$ | $1^5, 2$ |
| 2 | 0 | 0 | 0 | $1^{15}$ | $1^{14}, 2$ |
| 3 | 0 | 3 | 0 | $1^{17}$ | $1^{19}, 2$ |
| 4 | 0 | 0 | 0 | $1^{15}$ | $1^{14}, 2$ |

All displayed groups are therefore free over $R$.

7. Write $\mathcal{R}_q = R^q f_* R$. Proper base change (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1; chapter PDF p. 30) identifies the stalks with the fibres. At the first finite cap, $m_1$ is a unit in $R$, so transfer gives the full invariant restriction image and zero kernel. At the second finite cap, item 4 gives injectivity and the norm quotient. At the prescribed cusp, dualizing the primitive-image collapse of theorem 2.6 gives the invariant restriction image, with zero kernel in degrees one through three and a free kernel $R$ in degree four. The latter is dual to the free degree-four cokernel of the collapse.

   For $q = 2, 3$ there is consequently an exact sequence

$$0 \longrightarrow \mathcal{R}_q \longrightarrow j_* L_q \longrightarrow Q_q \longrightarrow 0,$$

   where $Q_q$ is supported at the second finite point. Since $H^0(j_* L_q) = 0$, its connecting map embeds $Q_q$ in $H^1(\mathcal{R}_q)$, with the free quotient $H^1(j_* L_q)$ computed above. Its torsion is exactly $Q_q$. Also $H^2(\mathcal{R}_1) = H^2(\mathcal{R}_2) = H^2(\mathcal{R}_3) = 0$. Injectivity gives $H^0(\mathcal{R}_3) = 0$. The degree-four free cusp kernel, together with $H^0(j_* L_4) = 0$, gives $H^0(\mathcal{R}_4) = R$.

8. In the Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4; chapter PDF p. 22), total degree three has only $H^1(\mathcal{R}_2)$. Its middle base column has no incoming or outgoing differential. Thus case B gives $H^3(M; R) = R/3^r R$. In total degree four, the surviving terms are $H^1(\mathcal{R}_3)$ and $H^0(\mathcal{R}_4) = R$; the degree-two base term is zero and the possible differential from the latter term lands in $H^2(\mathcal{R}_3) = 0$. The extension by the free module $R$ splits. Hence in case A

$$\operatorname{Tor} H^4(M; R) \cong R/3^{r-1} R \oplus R/3^r R.$$

   Universal coefficients (Hatcher, 2002, Theorem 3.2, p. 195) identify this torsion with the torsion of $H_3(M; R)$, and similarly identify the case-B torsion with that of $H_2(M; R)$. Localization retains the finite three-primary subgroups by the identity on each $\mathbb{Z}/3^s$ summand. This proves the stated orders and the case-B isomorphism. At $m_2 = 6$ the first case-A summand vanishes and the second is $\mathbb{Z}/3$; case B is also $\mathbb{Z}/3$. These groups obstruct three-connectivity.

9. The two displayed rational models have explicit smooth marked realisations with the required free caps and simply connected filling. Use their displayed standard cohomology lattice $\mathbb{Z}^6$ and its dual homology basis, so $M_1 = A^t$, $M_2 = B^t$. Direct multiplication gives $A^5 = B^6 = I$, exact orders five and six, and determinant one. Put $K = BA - I$. Then $K^2 = 0$, and the third minor on rows $(0, 3, 4)$ and columns $(1, 2, 3)$ is one in both models. Hence $K$ has rank three and all its nonzero Smith factors are one. The marked cusp is

$$M_0 - I = ((M_1 M_2)^{-1} - I) = -K^t.$$

Set $b_1 = e_0, b_2 = e_3, b_3 = e_4$, $a_j = (M_0 - I)b_j$ and $S = (a_1, a_2, a_3, b_1, b_2, b_3)$. Direct multiplication gives $S \in \mathrm{GL}_6(\mathbb{Z})$ and

$$(M_0 - I)S = S \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}.$$

This is the exact integral marking used to transport the index-one Kuhn cap and its collapse maps.

The affine power vectors in these same homology coordinates are

| | $v_1$ | $v_2$ |
|---|---|---|
| A | $(1, 0, 0, 0, 0, 0)$ | $(-1, 2, 0, -1, 0, 0)$ |
| B | $(1, 0, 0, 0, 0, 0)$ | $(-1, 0, 0, 0, 0, 1)$. |

They satisfy $M_i v_i = v_i$. The first-coordinate covector $\ell$ is fixed by both $M_i$ and has values $1, -1$ on $v_1, v_2$. Consequently every nonidentity power of $x \mapsto M_1 x + v_1/5$ or $x \mapsto M_2 x + v_2/6$ changes the $\ell$-circle by $j/5$ or $-j/6$, respectively. These are free cyclic actions of the exact stated orders. Their diagonal actions on the product with the rotating disc therefore give the smooth finite quotient caps.

Take the affine torus bundle over the three-holed sphere with these two boundary holonomies and their inverse product on the third boundary. Glue the two finite caps and the proper index-one Kuhn cap of theorems 2.4 and 2.6 by the specified marked mapping-torus collars and zero cusp clutch. For an affine cusp monodromy $x \mapsto M_0 x + c$, a path of translations $d(s)$ with $d(0) = 0$, $d(1) = c$ gives the collar map $(x, s) \mapsto (x + d(s), s)$ from the linear mapping torus. The prescribed zero clutch fixes its integral lift. The lattice map on this collar is the displayed $S$.

The cap fundamental-group maps retain these choices. On the finite cap the universal covering $\mathbb{R}^6 \times \Delta$ has the deck transformation $(x, t) \mapsto (M_i x + v_i/m_i, e^{2\pi i/m_i} t)$; its $m_i$-th power is the lattice translation $v_i$. Thus the boundary generators $a, b$ satisfy $a^5 = v_1$, $b^6 = v_2$. On the Kuhn cap the polar projection onto its base three-torus and angular-identity section induce inverse maps on fundamental groups. In its dual-cell model the one-skeleton consists of the base edges, and each angular loop contracts after transport to a dual vertex where that angular fibre is collapsed. The fibre map is therefore exactly $\Lambda \to \Lambda / \operatorname{im}(M_0 - I)$, expressed by the last three rows of $S^{-1}$. The meridian bounds the toric coordinate disc at a smooth point of a central component; the zero clutch identifies it with $(ab)^{-1}$. Hence the cusp imposes $ab = 1$.

For $J = [M_1 - I \mid M_2 - I]$, the first row is zero. Its lower five rows have a unit minor on columns $(1, 2, 3, 6, 10)$ in case A and $(1, 2, 3, 4, 6)$ in case B, with determinants $-1$ and $1$. The inverse of that minor supplies integral preimages of $e_1, \ldots, e_5$. Thus $\operatorname{im} J = \ker \ell$ integrally. Van Kampen now gives simple connectivity at the group level: from $b = a^{-1}$ and the two power relations,

$$a^{-1} = a^5 a^{-6} = v_1 v_2 \in \Lambda.$$

So $a, b$ lie in the abelian image of the fibre lattice, their conjugation actions become trivial, and $J$ is killed. The exact remaining fibre quotient is $\mathbb{Z}$ via $\ell$, with generator $h$ and images $v_1 = h$, $v_2 = -h$. The displayed identity forces $a = 1$, and $a^5 = v_1$ then kills $h$. Every generator is trivial. Equivalently, after this nonabelian reduction the residual relation matrix on $(h, a, b)$ is

$$\begin{pmatrix} -1 & 5 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \det = 1.$$

Use the rational invariant-extension table above for these models. The finite covering restrictions identify rational cap cohomology with invariants. The primitive cusp collapse of 2.6 gives the same identification through degree three and a one-dimensional kernel in degree four. Hence $R^q p_* \mathbb{Q} = j_* \Lambda^q V$ for $1 \le q \le 3$, and

$$0 \longrightarrow \mathbb{Q}_{\mathrm{cusp}} \longrightarrow R^4 p_* \mathbb{Q} \longrightarrow j_* \Lambda^4 V \longrightarrow 0.$$

The table therefore leaves, in total degree at most four, only $E_2^{0,0} = E_2^{2,0} = E_2^{0,1} = \mathbb{Q}$, $E_2^{1,3} = \mathbb{Q}^3$ and $E_2^{0,4} = \mathbb{Q}$. Simple connectivity forces the $d_2 : E_2^{0,1} \to E_2^{2,0}$ to be an isomorphism. There are no other possible nonzero differentials on these terms. Thus $(b_0, \dots, b_4) = (1, 0, 0, 0, 4)$; oriented duality on the closed smooth eight-manifold gives the full tuple $(1, 0, 0, 0, 4, 0, 0, 0, 1)$.

□

**Theorem 11.10** (Four order pairs and eight local spectral pairs)**.** [tag 02N3]

*Every simply connected two-finite-point datum with $b_2 = 0$ has one finite linear order equal to five and the other equal to four, six, eight or twelve. After suppressing the common factor $(T-1)^2$, its unordered pair of finite characteristic polynomials is exactly one of the following necessary possibilities:*

| *orders* | *order-five factor* | *other factor* |
|---|---|---|
| $(5, 4)$ | $\Phi_5$ | $\Phi_2^2\Phi_4$ |
| $(5, 6)$ | $\Phi_5$ | $\Phi_2^2\Phi_3,\ \Phi_2^2\Phi_6,\ \Phi_3\Phi_6$ |
| $(5, 8)$ | $\Phi_5$ | $\Phi_8$ |
| $(5, 12)$ | $\Phi_5$ | $\Phi_{12},\ \Phi_3\Phi_4,\ \Phi_4\Phi_6.$ |

(11.9)

*Its rank-five quotient is rationally irreducible. In particular all order-three pairs, including $(3, 4)$, are impossible.*

*Proof.* 1. By propositions 11.2 and 11.3, the cohomology system $V$ has a one-dimensional invariant space $I$, no invariant covectors, and two-dimensional finite fixed spaces $F_1, F_2$. The finite orders are among the coprime pairs of proposition 11.4. At a finite quotient, averaging identifies the degree-two specialization with the fixed part of $\Lambda^2 V$. At the cusp, theorem 2.6 gives the surjective homology collapse with kernel the saturation of $(\Lambda^2 M_0 - I)\Lambda^2\Lambda$. Its dual has image $\ker(\Lambda^2 M_0^t - I)$ under the evaluation pairing. Consequently the natural specialization morphism identifies

$$R^2 g_* \mathbb{Q} = j_* \Lambda^2 V.$$

The term $H^0(j_*\Lambda^2 V)$ has no incoming Leray differential, and its only possible outgoing differential targets $H^2(j_* V) = H^0(V^*)^* = 0$. It survives to $H^2(M; \mathbb{Q}) = 0$. Thus $a_2' := \dim(\Lambda^2 V)^\Gamma = 0$.

2. Let $\beta : V \to V^*$ be an invariant alternating form. Since $\beta(I) \subset (V^*)^\Gamma = 0$, its radical contains $I$. For the finite averaging projection $P_i : V \to F_i$, invariance gives $\beta(u, x) =$

$\beta(u, P_i x)$ when $u \in F_i$. The alternating restriction to the two-dimensional space $F_i$ has $I$ in its radical and is therefore zero. Hence $F_1 + F_2 \subset \operatorname{rad}\beta$. This sum has dimension three because $F_1 \cap F_2 = I$. The alternating rank is even, so every such form has rank at most two.

Suppose $\beta, \gamma$ were independent invariant forms. They and their sum have rank two, so $\beta \wedge \gamma = 0$. Their image planes in $V^*$ are distinct invariant two-planes meeting in a rational line. Each finite generator acts on that line by a sign. At least one finite order is odd, and the unipotent product acts trivially on the line; both signs are therefore $+1$. This contradicts $(V^*)^\Gamma = 0$. It follows that $b_2' := \dim(\Lambda^2 V^*)^\Gamma \le 1$.

3. At the cusp, the exterior square of three size-two unipotent blocks has fixed rank nine: three fixed vectors come from their individual exterior squares and two from each of the three pairwise tensor products. If $d_i$ denotes a finite fixed rank on $\Lambda^2 V$, the invariant-sheaf Euler identity gives

$$-15 + d_1 + d_2 + 9 = a_2' - h^1(j_*\Lambda^2 V) + b_2' \le 1. \tag{11.10}$$

Here the degree-two group is dual to $(\Lambda^2 V^*)^\Gamma$, by compact-support duality and the extension-by-zero sequence.

4. The two-dimensional unit eigenspace leaves four dimensions for the nonunit cyclotomic factors. Local complex compatibility requires even multiplicity of $-1$. Partitioning four into the cyclotomic degrees gives the following complete list:

| | |
|---|---|
| 3 | $\Phi_3^2$ |
| 4 | $\Phi_4^2,\ \Phi_2^2\Phi_4$ |
| 5 | $\Phi_5$ |
| 6 | $\Phi_6^2,\ \Phi_2^2\Phi_3,\ \Phi_2^2\Phi_6,\ \Phi_3\Phi_6$ |
| 8 | $\Phi_8$ |
| 10 | $\Phi_{10}$ |
| 12 | $\Phi_{12},\ \Phi_3\Phi_4,\ \Phi_4\Phi_6.$ |

The unit eigenspace contributes one fixed bivector. A repeated nonreal quadratic pair contributes four further fixed bivectors, so $d_i = 5$ for $\Phi_3^2, \Phi_4^2, \Phi_6^2$. Every other listed factor contributes two, so $d_i = 3$. Inequality (11.10) excludes every occurrence of $d_i = 5$. In particular all order-three pairs disappear. The coprime order list of proposition 11.4 therefore reduces to $(5,4), (5,6), (5,8), (5,12)$; their remaining factors are exactly the eight entries of (11.9). Each pair contains the order-five factor, so proposition 11.3 proves irreducibility of $V/I$.

□

## 12 Integral lattices and extension classes

**Theorem 12.1** (An unbounded integral family)**.** [tag 02N5]

*There are infinitely many simultaneous rational conjugacy classes of integral rank-six tuples of the linear type* $(5,4)$ *satisfying definition 10.1, with primitive maximal square-zero cusp. An explicit infinite subfamily has free affine lifts of orders five and four. Apart from the parameter zero, the rank-five irreducible quotient in the family is not self-dual. Thus no finite bound on rational conjugacy classes is valid for the unrestricted input tuples.*

*Proof.* 1. Let $v \in \mathbb{Z}$, put $q = v^2 - v + 1$, and let $Ae_j = e_{j+1}$ on $\mathbb{Q}^5$, with indices modulo five. Define

$$X = (e_0,\ e_2 + ve_4), \qquad Y = \begin{pmatrix} 0 & -q & -v^2 & -1 & v \\ 0 & 1-vq & -vq & v^2 - v & q \end{pmatrix}, \qquad N = XY. \tag{12.1}$$

Multiplying $Y$ by the two columns of $X$ gives $YX = 0$, so $N^2 = X(YX)Y = 0$. Consequently

$$C = (I + N)A, \qquad B = A^4(I - N) = C^{-1}.$$

The cycle gives $A^5 = I$. Exact multiplication and determinant expansion give

$$N^2 = 0, \qquad A^5 = C^4 = I_5, \qquad \chi_C(x) = (x-1)(x+1)^2(x^2+1). \tag{12.2}$$

For the determinant calculation one may substitute

$$(xI - A)^{-1} = \frac{x^4 I + x^3 A + x^2 A^2 + xA^3 + A^4}{x^5 - 1}$$

in the rank-two determinant identity

$$\det(xI - C) = \det(xI - A)\det\big(I_2 - YA(xI - A)^{-1}X\big).$$

Direct multiplication gives $C^4 = I$. The factor $x^2 + 1$ shows that $B$ and $C$ have exact order four.

2. Let $P$ have columns $\mathbf{1}/q, e_1, e_2, e_3, e_4$, and put $A' = P^{-1}AP$, $B' = P^{-1}BP$. Since $q > 0$ and $\det P = 1/q$, these are rational changes of coordinates. Define

$$s = v^2 + v - 1, \qquad r = -\frac{(v+1)(v^2-2)}{2}, \qquad \beta = (s,\ s - vr,\ v^2 s + vr,\ r,\ -vs - r) \tag{12.3}$$

and

$$\widehat{A} = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} 1 & \beta \\ 0 & B' \end{pmatrix}. \tag{12.4}$$

The lattice $P\mathbb{Z}^5$ has basis $f_0 = \mathbf{1}/q, f_1 = e_1, \ldots, f_4 = e_4$. Its images under $A$ are $f_0, f_2, f_3, f_4$ and $qf_0 - f_1 - f_2 - f_3 - f_4$, respectively; $A^{-1} = A^4$ therefore also preserves this lattice. The columns of $X$ are $qf_0 - f_1 - f_2 - f_3 - f_4$ and $f_2 + vf_4$, while

$$Y' := YP = \begin{pmatrix} -2 & -q & -v^2 & -1 & v \\ 2(1-v) & 1 - vq & -vq & v^2 - v & q \end{pmatrix}.$$

Thus $X$ and $Y$ are integral maps between the lattices. The already proved $YX = 0$ shows that $I + N$ and its inverse $I - N$ preserve $P\mathbb{Z}^5$. Since $B = A^{-1}(I - N)$, both $A', B'$ belong to $\mathrm{GL}_5(\mathbb{Z})$. For $r$, odd $v$ makes $v + 1$ even, and even $v$ makes $v^2 - 2$ even; hence $r$ and all entries of $\beta$ are integers. The hatted matrices are therefore integral automorphisms.

The exact unit needed in the hatted lattice is

$$z = (-v^3 + v - 1, 0, 0, 2, 2(1 - v^2))^t.$$

Writing $a = -v^3 + v - 1$, the first coordinate of $Y'z$ is $-2a - 2 + 2v(1 - v^2) = 0$. Half its second coordinate is $(1 - v)a + (v^2 - v) + q(1 - v^2) = 0$, since the first two terms sum to $v^4 - v^3 + v - 1$ and the last is its negative. Furthermore

$$\beta z = s(v^3 - v - 1) + 2v^2 r = 1. \tag{12.5}$$

Indeed the two summands expand to $v^5 + v^4 - 2v^3 - 2v^2 + 1$ and $-v^5 - v^4 + 2v^3 + 2v^2$. Since $AB = I - N$ and $NPz = X(Y'z) = 0$, the six-dimensional map satisfies

$$(\widehat{A}\widehat{B} - I)(0, z)^t = e_0.$$

This establishes the unit in the displayed hatted coordinates.

The row identity $\beta(I + B' + B'^2 + B'^3) = 0$ proves $\widehat{B}^4 = I$; also $\widehat{A}^5 = I$. Their quotient actions show that these orders are exact.

Use the lattice change

$$L = \frac{1}{4}\begin{pmatrix} -2 & 3 & -5 & -1 & 5 & -3 \\ 0 & 2 & -2 & 2 & -2 & 2 \\ 0 & -4 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -4 & 4 & -4 \\ 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & -4 & 4 & 0 \end{pmatrix}, \qquad A_v = L^{-1}\widehat{A}L, \quad B_v = L^{-1}\widehat{B}L. \tag{12.6}$$

Here $\det L = -1/4$. Expanding these products gives entries in $\mathbb{Z}[v]$ for both matrices, with $LA_v = \widehat{A}L$ and $LB_v = \widehat{B}L$. Thus the defined matrices act on the stated $\mathbb{Z}^6$ for every integer parameter. Their determinants are one and their inverses are the integer powers $A_v^4, B_v^3$. Their characteristic polynomials are

$$(x-1)^2\Phi_5(x), \qquad (x-1)^2(x+1)^2(x^2+1),$$

respectively.

3. Put $K = B_vA_v - I$. Multiplication gives $K^2 = 0$. The following are third minors of $K$, in zero-based indices:

| rows | columns | determinant |
|---|---|---|
| $(0,1,4)$ | $(1,2,3)$ | $v(v+1)$ |
| $(0,1,4)$ | $(1,3,4)$ | $v^2$ |
| $(0,3,4)$ | $(1,2,3)$ | $(v+1)^2$. |

Their integer combination $(v+1)^2 - 2v(v+1) + v^2$ equals one. Hence $K$ has rank at least three, while $K^2 = 0$ in dimension six gives rank at most three. The gcd of its third minors is one, so its three nonzero Smith factors are all one. This proves primitivity directly in the lattice, for every $v$. For $M_1 = A_v^t$ and $M_2 = B_v^t$ the cusp map is

$$M_0 = (M_1M_2)^{-1} = (I + K^t)^{-1} = I - K^t.$$

Thus $M_0 - I = -K^t$, which has the same integral Smith factors and the required maximal square-zero type.

4. Both $A_v, B_v$ fix $e_0$. The quotient morphism to the original rank-five pair is

$$R = P\ \mathrm{pr}_5\, L : \mathbb{Q}^6 \longrightarrow \mathbb{Q}^5, \qquad RA_v = AR, \quad RB_v = BR.$$

It is surjective and has kernel $\mathbb{Q}e_0$, since $Le_0 = -e_0/2$. We prove that this quotient is irreducible over $\mathbb{Q}$. The coprime factors $x-1$ and $\Phi_5(x)$ of $\chi_A$ give the projector $(I + A + A^2 + A^3 + A^4)/5$. Its image is $\mathbb{Q}\mathbf{1}$ and its kernel is $S = \{\sum x_i = 0\}$. Irreducibility of $\Phi_5$ shows that these are the only nonzero proper rational $A$-invariant subspaces.

Neither is $C$-invariant. Indeed,

$$N\mathbf{1} = -2qX\begin{pmatrix} 1 \\ v-1 \end{pmatrix}$$

has nonzero first entry and zero entries in positions one and three, so $C\mathbf{1} = \mathbf{1} + N\mathbf{1}$ is not a multiple of $\mathbf{1}$. Write $\epsilon = (1,1,1,1,1)$. If $S$ were $C$-invariant, then $\epsilon N$ would be a multiple of $\epsilon$; its first entry is zero, so that multiple would vanish. But its entries in columns three and four are $v^3 - v - 1$ and $v^3 + v + 1$. Their simultaneous vanishing would give $v = -1$, at which the first is $-1$. This is impossible. Since $B = C^{-1}$, the pair $(A, B)$ is irreducible, as claimed.

5. The rank-two update gives exact traces

$$\operatorname{tr}(AC) = \operatorname{tr}(YA^2X) = 1 - v^2q, \qquad \operatorname{tr}((AC)^{-1}) = -\operatorname{tr}(YA^3X) = 1 + v^2q.$$

Here $\operatorname{tr}(A^2) = \operatorname{tr}(A^3) = 0$ and $B = A^4(I - N)$ give the displayed equalities. An equivariant isomorphism $F$ to the contragredient representation would satisfy $FA = A^{-t}F$ and $FB = B^{-t}F$, and therefore identify the character of the word $AC = AB^{-1}$ with the character of its contragredient matrix. The two displayed traces would be equal. Their difference $-2v^2q$ is nonzero when $v \neq 0$, proving the required nonselfduality.

At $v = 0$ the symmetric matrix $Q_{ij} = 1$ for $j - i \equiv 1, 4 \pmod 5$ and $Q_{ij} = 0$ otherwise has determinant two and satisfies $A^tQA = Q$, $B_0^tQB_0 = Q$. It gives the equivariant isomorphism in that case.

For the rank-six matrices, the same word has trace

$$h(v) := \operatorname{tr}(A_vB_v^{-1}) = 2 - v^2q, \qquad h(v+1) - h(v) = -4v^3 - 3v^2 - 3v - 1.$$

Thus $h$ is strictly decreasing on the nonnegative integers, in particular on the nonnegative multiples of twenty. A simultaneous rational conjugacy of the ordered matrices would preserve this trace. There are consequently infinitely many such rational conjugacy classes already in that subfamily.

6. For $v \in 20\mathbb{Z}$, put $a = (1, 3, -2, -1, 2, 0)^t$, $e = (1, 0, 0, 0, 0, 0)^t$, and define

$$w_1 = \frac{1}{5}\sum_{j=0}^{4}(A_v^t)^j a, \qquad w_2 = -\frac{1}{4}\sum_{j=0}^{3}(B_v^t)^j e. \tag{12.7}$$

The polynomial matrices reduce to their parameter-zero values modulo five and four, respectively. At zero, $M_1a = a$ and $M_2e = e$. The two sums are therefore divisible by their indicated denominators, so $w_1, w_2$ are integral. Their defining averages give $M_iw_i = w_i$. The common first-coordinate covector $\ell$ is fixed by both $M_i$ and satisfies $\ell(w_1) = 1$, $\ell(w_2) = -1$.

For $(m_1, m_2) = (5, 4)$ let $g_i(x) = M_ix + w_i/m_i$ on $\mathbb{R}^6/\mathbb{Z}^6$. Then $g_i^j(x) = M_i^jx + jw_i/m_i$, so $g_i^{m_i}$ is the identity. Every power with $0 < j < m_i$ changes the $\ell$-circle coordinate by $j/5$ or $-j/4$, respectively. It has no fixed point. These are free actions of the exact orders five and four, with the affine lifts.

7. Substitution of $v = 0$ in the displayed construction gives exactly the marked pair of definition 2.1, and the averages give $w_1(0) = a$, $w_2(0) = -e$. This identifies both the matrices and the affine numerators in the fixed marking.

□

**Proposition 12.2** (An odd torsion obstruction in this lattice)**.** [tag 02N6]

*For the integral tuple $(A_v, B_v)$ in (12.6), any simply connected filling with free finite affine caps and the fixed primitive Kuhn cusp has a subgroup*

$$\mathbb{Z}/(v^2 - v + 1) \ \subset \operatorname{Tor} H_2(M; \mathbb{Z}). \tag{12.8}$$

*In particular, in this specified family of lattices only $v = 0$ and $v = 1$ can be three-connected, regardless of the affine multiplicities and twists. The stronger exclusion for every stable lattice in each rational class is proved in Theorem 12.4 below.*

*Proof.* 1. First compute the global coinvariants in the hatted lattice. In the rank-five quotient, the $A'$ relations identify $f_1 = f_2 = f_3 = f_4 = t$ and $qf_0 = 5t$. The number $q = v^2 - v + 1$ is

relatively prime to five, as its five residue values show. Thus these coinvariants are infinite cyclic, with $f_0 = 5h$ and $t = qh$. The $B'$ relations become $q\kappa_j h$, where

$$(\kappa_0, \dots, \kappa_4) = (2v^2,\ v(v^3+v-1),\ v(v^3+v+1), -v^3+v+1,\ -v^3-v-1). \tag{12.9}$$

Their gcd is one: it divides $2v$ by subtracting the second and third entries, and it divides $v^3 - v - 1$ by the fourth; the latter is odd and is $-1$ modulo $v$. Hence the rank-five global coinvariants are $\mathbb{Z}/q$. The extension line is killed in rank-six coinvariants by (12.5), since $(\widehat{A}\widehat{B} - I)(0, z)^t = e$. Consequently the hatted rank-six coinvariants are also $\mathbb{Z}/q$. The lattice change (12.6) is an isomorphism over every $\mathbb{Z}_{(p)}$ with $p$ odd, so the odd part of the rank-six global coinvariants is precisely $\mathbb{Z}/q$.

2. Let $V$ be the degree-one fibre cohomology local system. The only difference between $R^1 g_* \mathbb{Z}$ and $j_* V$ is a finite cokernel at finite caps; the cusp degree-one specialization is onto. Thus

$$H^2(\mathbb{P}^1, R^1 g_* \mathbb{Z}) = H^2(\mathbb{P}^1, j_* V) = V_\Gamma,$$

including its $q$-primary part just computed. The space of global degree-two invariants is zero: the irreducible odd rank-five quotient has no invariant alternating tensor, and the extension filtration $0 \to W \to \bigwedge^2 V \to \bigwedge^2 W \to 0$ gives $(\bigwedge^2 V)^\Gamma = 0$.

3. It remains to check every possible source of the differential $d_2 : H^0(\mathbb{P}^1, R^2 g_* \mathbb{Z}) \to H^2(\mathbb{P}^1, R^1 g_* \mathbb{Z})$. The cusp has no kernel in degree two. At a finite cap the kernel is torsion in $H^2$ of that cap, namely torsion in its first homology. Simple connectivity of the total filling forces the distinguished affine numerator to be prime to the affine multiplicity, by the determinant argument of 11.8. The cap is therefore a mapping torus of a torus automorphism whose order divides the linear order, five or four. Its first-homology torsion is supported only at the primes five or two, respectively: averaging over the finite order kills the torsion of the vertical coinvariants. Hence the source of this $d_2$ has no torsion at any prime dividing $q$, since $q$ is odd and prime to five.

4. The subgroup $\mathbb{Z}/q$ in bidegree $(2, 1)$ survives. There are no longer differentials on a two-dimensional base, and this terminal filtration term is a subgroup of $H^3(M; \mathbb{Z})$. Universal coefficients identify its torsion with a subgroup of $\operatorname{Tor} H_2(M; \mathbb{Z})$. This proves (12.8). Finally, $v^2 - v + 1 = 1$ for an integer $v$ exactly when $v = 0$ or $v = 1$.

□

**Theorem 12.3** (A three-connected datum with non-self-dual quotient)**.** [tag 02N7]

*The parameter $v = 1$ in (12.6), with affine orders five and four and twists*

$$w_1 = (1, 3, -2, -1, 2, 0)^t, \qquad w_2 = (-1, 0, 0, 0, 0, 0)^t,$$

*gives a three-connected smooth eight-manifold with $H_4 = \mathbb{Z}^4$ and all other reduced homology zero except $H_8 = \mathbb{Z}$. It is almost complex and is oriented diffeomorphic to $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$. Its irreducible rank-five quotient has no invariant bilinear form, so integral three-connectivity does not force that quotient to be orthogonal. By theorem 16.5 it has holomorphic periods with bounded cusp correction on the punctured base.*

*Proof.* 1. The first cohomological matrix is the same $A$ as in definition 2.1. The second is

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 1 & -3 \\ 0 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}. \tag{12.10}$$

Take $M_1 = A^t$, $M_2 = B_1^t$ and $M_0 = (M_1M_2)^{-1}$. The displayed twists are fixed integral vectors with $\ell(w_1) = 1$, $\ell(w_2) = -1$. They therefore give free affine actions. The cusp is primitive by 12.1. Exact Hermite reduction gives

$$\operatorname{col}_{\mathbb{Z}}[\, M_0 - I \mid M_1(M_0 - I) \mid M_2(M_0 - I)\,] = \langle e_1, \dots, e_5 \rangle.$$

Van Kampen leaves only $5a = h$, $4b = -h$, $a + b = 0$, whose relation matrix is unimodular. This proves $\pi_1 = 1$ for the gluing, including zero cusp clutching.

2. We give the integral data needed to reuse the differential calculation of lemma 5.3. The image-sheaf cohomology ranks are those of lemma 5.2, as the following integral data verify. The finite cap first homology groups are $\mathbb{Z}^2$ and $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. On integral invariant two-form bases their suspension periods divided by the covering orders are, respectively, $(-1, 0, -1)$ and $(0, 0, 1)$. The determinants of the invariant degree-three skew pairings are $5^4$ and $4^4$, while the degree-two/degree-four pairings have absolute determinants $5^3$ and $4^3$. Thus all the finite covering indices used in degrees two, three and four are one.

3. For exterior coefficient degrees two, three and four, the global coboundary Smith form has one factor two and all other nonzero factors one. Its torsion class restricts to zero at the order-five boundary and at the cusp, and to a nonzero class at the order-four boundary. Thus the parabolic groups are torsion free, of ranks $0, 3, 0$. The degree-one coboundary is primitive. The common invariant covector maps to $(1 \bmod 5, -1 \bmod 4)$ in the degree-one finite covering cokernels. Its map from $\mathbb{Z}$ onto $\mathbb{Z}/5 \oplus \mathbb{Z}/4$ is surjective by the Chinese remainder theorem. The global coinvariants in degrees one through four are each $\mathbb{Z}/2$.

4. Write $M_2 = \operatorname{diag}(1, D)$. This time use the unimodular marking

$$P_1 = \begin{pmatrix} -1 & 1 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 \end{pmatrix}, \qquad \det P_1 = -1.$$

It satisfies $DP_1 = P_1 \operatorname{diag}(\text{four-cycle}, -1)$. The cap therefore has the same cellular mapping-torus model used to prove (5.3). With the unchanged twist $w_2 = -e_0$, the boundary coordinates remain $\theta = a + 4k$, $h = k$; the suspension representative restricts to $w$ in the base direction and to $-\operatorname{Norm}_4(w) = 0$ in the auxiliary fibre direction. Consequently the coefficient in the torsion differential is again one.

5. The new vertical representatives in lexicographic exterior bases are

$$\begin{aligned} q = 2 &: (0, -1, -1, -1, 0), \\ q = 3 &: (0, 0, 1, 0, 0, 0, 0, 0, 0, 1), \\ q = 4 &: (0, 0, 0, 0, 0, 0, 0, -1, -1, 0). \end{aligned}$$

Each has residue one in its global coinvariant $\mathbb{Z}/2$. Thus all three torsion differentials in (5.3) are isomorphisms. The integral Leray table and its free degree-one differential are therefore exactly those of 5.4. They give $H^2 = H^3 = 0$ and $H^4 = \mathbb{Z}^4$. Universal coefficients, Hurewicz and Poincaré duality give the stated integral homology and three-connectivity.

6. The relative almost-complex extension over the pair of pants in lemma 2.9 applies: the finite matrices have the same even real eigenspace dimensions and the cusp is the same marked Kuhn type. The purely smooth characteristic-vector and normal-invariant argument in 6.2 then gives the stated oriented diffeomorphism type.

7. Finally, Theorem 12.1 gives nonselfduality of the irreducible rank-five quotient at $v = 1$. These calculations use the explicit monomial marking displayed above.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 12.4** (A residual obstruction for every lattice)**.** [tag 02N8]

*Let a rational rank-six datum have finite linear orders five and four, a primitive maximal cusp on a stable integral lattice, and invariant line $L$, with irreducible rank-five quotient $W$ and no rational global coinvariants. Suppose that at a prime $p \neq 2, 5$ the semisimplified reduction of a stable lattice in $W$ is the sum of the trivial module and an absolutely irreducible module of dimension four. Then no stable integral lattice in that rational datum, with primitive maximal cusp, and no free finite affine twists can give $\pi_1 = 1$ and $H_2 = 0$ simultaneously.*

*For every $v \notin \{0, 1\}$, the rational family in 12.1 satisfies this hypothesis at each prime dividing $v^2 - v + 1$. Consequently the exclusion of those parameters holds for every stable lattice and every twist. The two remaining parameters give the three-connected manifolds of theorems 5.4 and 12.3.*

*Proof.* 1. We first give the lattice-independent mechanism. For any stable lattice $V_{\mathbb{Z}}$, use its primitive invariant line $L_{\mathbb{Z}}$ and quotient lattice $W_{\mathbb{Z}}$. The semisimplified reduction of a rational representation is independent of its stable lattice. One elementary way to see this is to compare two commensurable lattices by their finite quotient $T$. The exact sequence obtained by reduction modulo $p$ has end terms $T[p]$ and $T/pT$, which have the same composition factors by the filtration of $T$ by powers of $p$. Hence the two lattice reductions have equal classes in the Grothendieck group of finite-length modules. Chaining such comparisons proves the assertion for any two lattices.

2. Thus $W_{\mathbb{Z}}/pW_{\mathbb{Z}}$ has length two, with one trivial constituent and one irreducible four-dimensional constituent. It has either a trivial submodule or a trivial quotient (and possibly both).

3. Suppose first that it has a trivial submodule. Its inverse image in $V_{\mathbb{Z}}/pV_{\mathbb{Z}}$ is an extension of a trivial module by a trivial module. Each finite generator acts on this two-dimensional space by a unipotent matrix. Since its order is five or four, prime to $p$, it acts as the identity. There are therefore at least two global invariant vectors in degree-one fibre cohomology modulo $p$. Write their dimension as $a_p \geq 2$; this is also the dimension of the global coinvariants of fibre homology over $\mathbb{F}_p$, by the perfect pairing between fibre homology and cohomology.

4. The abelianized van Kampen presentation now shows

$$b_1(M; \mathbb{F}_p) \geq a_p - 1. \tag{12.11}$$

Indeed, after the monodromy relations the fibre contributes $a_p$ dimensions, and the two finite meridians contribute two more. The two finite power relations and the single base relation impose at most three linear relations. The cusp introduces no further fibre relation in this abelianization. On the dual homology lattice its nilpotent operator is still primitive and maximal, so its angular lattice is $\ker(M_0 - I) = \operatorname{im}(M_0 - I)$ by primitivity, and the latter image already vanishes among the global monodromy relations. Its base meridian is the base relation just counted. Changing affine twists changes coefficients of these three relations, not their number. Thus (12.11) contradicts simple connectivity.

5. Suppose instead that the reduction of $W$ has a trivial quotient. Its integral global coinvariants have nonzero $p$-primary torsion, since they have rational rank zero. The natural map from the global coinvariants of $V$ to those of $W$ is an isomorphism. To check its kernel, a primitive generator $e$ of $L_{\mathbb{Z}}$ lies in $\ker(M_0 - I) = \operatorname{im}(M_0 - I)$, and hence is already zero

among global coinvariants. Therefore $H^2(R^1)$ has nonzero $p$-primary torsion. If the total space were simply connected, the finite-cap argument of 12.2 would make every possible torsion source in $H^0(R^2)$ supported only at two and five. The invariant part of $\bigwedge^2 V$ is zero by irreducibility of the odd-dimensional quotient. Consequently the nonzero $p$-primary subgroup survives to $H^3(M;\mathbb{Z})$ and then to $\operatorname{Tor} H_2(M;\mathbb{Z})$, again a contradiction. This proves the general criterion.

6. For its application to 12.1, let $p \mid q = v^2 - v + 1$. We have $p \neq 2, 5$. In the original cycle lattice of (12.1), $Y\mathbf{1} = 2q(-1, 1-v)^t$, so $\mathbf{1}$ gives a trivial submodule after reduction. On the four-dimensional quotient the first finite generator has characteristic polynomial $\Phi_5$, and $C = B^{-1}$ has polynomial $(x+1)^2(x^2+1)$. This quotient is absolutely irreducible. Here is a characteristic-independent proof over $\overline{\mathbb{F}}_p$.

7. A one-dimensional composition factor would make the eigenvalues of $A$ and $C$ equal, because $CA^{-1}$ is unipotent. An eigenvalue of order dividing both five and four must be one, which is absent from $\Phi_5$. If the quotient were reducible, its only remaining possible composition-factor dimensions would be two and two. On each factor, the determinants of $A$ and $C$ must agree and have orders dividing five and four, so both determinants are one. The two $C$ eigenvalue pairs must therefore be $(-1,-1)$ and $(i,-i)$ in some order. On the former factor $C = -I$, since its order is prime to $p$. Then the unipotence of $CA^{-1} = -A^{-1}$ would force the eigenvalues of $A$ to be $-1$, impossible for $\Phi_5$. This proves absolute irreducibility and establishes the required residual composition factors at every $p \mid q$. When $v \notin \{0, 1\}$, the integer $q$ is greater than one and has such a prime divisor. The criterion applies.

□

**Theorem 12.5** (The complete rational $(5,4)$ normal form)**.** [tag 02NA]

*Consider ordered rational rank-five pairs $(A, C)$ with*

$$\chi_A = (x-1)\Phi_5, \qquad \chi_C = (x-1)(x+1)^2(x^2+1).$$

*Put*

$$N = CA^{-1} - I, \qquad N^2 = 0, \quad \operatorname{rank} N = 2,$$

*where $A^5 = C^4 = I$ and the pair is irreducible over $\mathbb{Q}$. Their simultaneous rational conjugacy classes are parametrized bijectively by*

$$\{[r : s : t] \in \mathbb{P}^2(\mathbb{Q}) : D(r,s,t) \neq 0\} \;\sqcup\; \mathbb{A}^1(\mathbb{Q}), \tag{12.12}$$

*where*

$$\begin{aligned} D = {} & r^3 - 4r^2s - r^2t + 7rs^2 + rst - 3rt^2 \\ & - 7s^3 + 11s^2t - 11st^2 + 7t^3. \end{aligned} \tag{12.13}$$

*The second summand lies over the single exceptional point $[4 : 1 : 3]$ of $D = 0$. The construction below is explicit and uses only a ten-variable linear system. By 11.7 it also parametrizes the rational rank-six maximal-cusp extensions of all these pairs, with $B = C^{-1}$. This is a rational classification; the stable integral lattices, the affine twists and the homology of the gluings are treated below.*

*Proof.* 1. Write $K = \mathbb{Q}(\zeta)$, $\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$, and identify the rational space with $\mathbb{Q} \oplus K$. The matrix of $A$ is multiplication by $(1, \zeta)$:

$$A = \operatorname{diag}(1, J), \qquad J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad R = (A+I)^{-1}. \tag{12.14}$$

Its rational centralizer is $\mathbb{Q}^* \times K^*$. Let $I_N = \operatorname{im} N$. If $X$ is any basis matrix for this plane and $N = XY$, then $YX = 0$. The identity

$$C + I = (A + I) + XYA$$

shows that its two-dimensional kernel has the same dimension as the kernel of $I_2 + YARX = I_2 - YRX$. Hence

$$YRX = I_2, \qquad I_N \cap RI_N = 0. \tag{12.15}$$

Conversely the two conditions in (12.15), together with $YX = 0$, give precisely that two-dimensional $-1$ eigenspace.

2. The plane $I_N$ projects nontrivially to $\mathbb{Q}$: otherwise the first-coordinate covector is invariant under both $A$ and $C$, contrary to irreducibility. It contains no nonzero vector of $\mathbb{Q} \oplus 0$, because $R$ acts as $1/2$ on that line and (12.15) would fail. Write a basis of the plane as $(0, u), (1, v)$, where $u \in K^*$. Multiplication on $K$ by $u^{-1}$ puts it into

$$I_N = \langle (0, 1), (1, \beta) \rangle_{\mathbb{Q}}.$$

Changing the second basis vector by a rational multiple of the first changes $\beta$ by a rational constant. Two such planes are in the same centralizer orbit exactly when $\beta' = a\beta + b$, with $a \in \mathbb{Q}^*$ and $b \in \mathbb{Q}$. Indeed a centralizer element carrying one to the other must preserve the kernel of the projection to $\mathbb{Q}$, so its $K^*$ component is itself rational; the displayed affine change follows, and every such change is realized. We may therefore take

$$\beta = r\zeta + s\zeta^2 + t\zeta^3, \qquad [r : s : t] \in \mathbb{P}^2(\mathbb{Q}), \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & r \\ 0 & s \\ 0 & t \end{pmatrix}. \tag{12.16}$$

The triple is nonzero; a rational $\beta$ would make the plane contain the forbidden first-coordinate line. The stabilizer of such a plane in the centralizer consists only of common scalar matrices. Thus it introduces no further identifications of the resulting nilpotent operators.

3. Write $Y$ as an unknown two-by-five matrix. It is determined by the following ten linear equations:

$$YX = 0, \qquad YRX = I_2, \qquad \operatorname{tr}(YAX) = -1, \qquad \operatorname{tr}(YA^{-1}X) = 1. \tag{12.17}$$

The first eight are $YX = 0$ and $YRX = I_2$. The last two are exactly $\operatorname{tr} C = \operatorname{tr} C^{-1} = -1$, since $\operatorname{tr} A = \operatorname{tr} A^{-1} = 0$ and $C^{-1} = A^{-1}(I - XY)$. They suffice to obtain the required characteristic polynomial: the two $-1$ roots and determinant one leave a monic cubic of constant term $-1$, and the two traces fix that cubic to $(x-1)(x^2+1)$. These remaining three roots are distinct, and the $-1$ eigenspace already has dimension two, so $C$ is semisimple and $C^4 = I$. Also $YRX = I_2$ ensures that $N$ has rank two. Thus every solution of (12.17) gives all the asserted local linear conditions.

4. In row-major order on the ten entries of $Y$, the determinant of the coefficient matrix in (12.17) is $D/4$. For $D \neq 0$, the column of the ten row-ordered entries of $Y$ is $E^{-1}b$, where $E$ is the coefficient matrix of the displayed linear equations and $b$ is their right-hand side. This gives the unique solution.

5. We now treat the exceptional locus. Set

$$Q_1 = r^2 - rs - rt + s^2 + t^2, \qquad Q_2 = r^2 - 2rs - 3rt + 4s^2 - 4st + 4t^2.$$

The first column of the unique generic solution is

$$y_{10} = \frac{2Q_1Q_2}{D}, \qquad y_{20} = \frac{2(r-s-t)Q_1}{D}. \tag{12.18}$$

Here $y_{10}, y_{20}$ are the first column of $Y$. The two equalities after multiplication by $D$ are polynomial linear combinations of (12.17), by its adjugate matrix. These polynomial identities remain necessary when $D = 0$. Moreover

$$2Q_1 = (r-s)^2 + (r-t)^2 + s^2 + t^2 > 0$$

for every nonzero rational triple. Consequently consistency on $D = 0$ forces $r = s + t$ and $Q_2 = 0$. Substitution gives

$$D(s+t, s, t) = (t - 3s)(2t - s)^2, \qquad Q_2(s+t, s, t) = (3s - t)(s - 2t). \tag{12.19}$$

There are precisely two resulting projective points, $[3 : 2 : 1]$ and $[4 : 1 : 3]$. At the former, $[X \mid RX]$ has rank less than four, so (12.15) is impossible. At the latter the coefficient and augmented matrices of (12.17) both have rank nine, and every solution is

$$Y_z = \begin{pmatrix} 4z + 16/5 & 0 & -2z - 1 & 4/5 - 2z & 2z \\ 2z - 2/5 & 0 & -z & 2/5 - z & z \end{pmatrix}, \qquad z \in \mathbb{Q}. \tag{12.20}$$

Direct substitution proves both necessity and sufficiency. This establishes the exceptional affine line in (12.12).

6. Every pair just constructed is irreducible over $\mathbb{Q}$. An $A$-stable proper rational subspace has dimension one or four. A one-dimensional subrepresentation or quotient must have trivial character, since its finite eigenvalues of orders dividing five and four must agree under the unipotent product. The only $A$-invariant covector is the first-coordinate covector. Its product with $N$ is the second row of $Y$, which is nonzero by $YRX = I_2$. The only $A$-invariant vector is the first-coordinate vector; its image under $N$ vanishes exactly if both entries in (12.18) vanish. For $D \neq 0$, positivity of $Q_1$ and (12.19) rule this out. On the exceptional line the two entries $4z + 16/5$ and $2z - 2/5$ never vanish simultaneously. There are therefore neither invariant vectors nor covectors, which excludes both possible dimensions.

7. Finally, rescaling $(r, s, t)$ is exactly the centralizer change already described; it gives a conjugate pair. Conversely any conjugacy of two normalized pairs centralizes $A$ and hence identifies their projective parameters as proved above. At the exceptional point its stabilizer is scalar, so distinct $z$ give distinct pairs. This proves the stated bijection, including completeness and uniqueness.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Corollary 12.6** (All good-prime lattices are forced)**.** [tag 02NB]

*For a $(5, 4)$ datum with $\pi_1 = 1$ and $H_2 = 0$, the rank-five quotient has absolutely irreducible reduction at every prime $p \neq 2, 5$. At each such prime its stable $\mathbb{Z}_p$-lattice is unique up to homothety. The same is true of the rank-six extension among lattices with primitive maximal cusp. Thus, once a rational pair in 12.5 admits one three-connected datum, all remaining integral-lattice ambiguity is supported at the primes two and five.*

*Proof.* 1. Work first over an algebraic closure of $\mathbb{F}_p$. Both finite generators are semisimple because $p \neq 2, 5$. A one-dimensional composition factor must be trivial: its $A$ and $C$ eigenvalues agree by the unipotent product, and their orders divide five and four. There can be at most one such factor, since the 1 eigenvalue of $A$ occurs once. If it occurs, the remaining four-dimensional module is absolutely irreducible by the spectral argument in 12.4. The only other possible reducible composition pattern is $2 + 3$. On each factor, determinants of $A$ and $C$ agree and have orders dividing five and four, hence are one. The two-dimensional $C$ eigenvalue pair is therefore $(-1, -1)$ or $(i, -i)$. The first would give $C = -I$ on that factor, contradicting the unipotence of $CA^{-1}$ and the absence of $-1$ among the eigenvalues of $A$. In the second case the three-dimensional factor has $C$ eigenvalues $1, -1, -1$, so $C^2 = I$ there; it has an $A$-fixed vector because the other factor has determinant one and cannot contain the simple 1 eigenvalue of $A$.

2. Here is the required elementary involutory identity in the present characteristic. Put $B = A^{-1}$ on this factor. From $(CB - I)^2 = 0$ and $C^2 = I$ one obtains

$$BC + CB^{-1} = 2I.$$

For $Bv = v$ this says $(B + I)Cv = 2v$. The operator $B + I$ is invertible, since $B^5 = I$ and $p \neq 2$, so $Cv = v$. The resulting common fixed line contradicts simplicity of the three-dimensional factor. We have proved that any reducible residual representation has exactly the constituents 1 and an absolutely irreducible four-dimensional module. Theorem 12.4 excludes that possibility for a three-connected datum, proving absolute irreducibility at every indicated prime.

3. Let $R = \mathbb{Z}_p$ and $K_p = \mathbb{Q}_p$. For two stable quotient lattices, multiply one by a power of $p$ until it is contained in the other but not in its $p$-multiple. Its nonzero residual image must then be the whole quotient by absolute irreducibility. Nakayama's lemma, or the Smith form of the inclusion, makes the lattices equal. This proves quotient-lattice uniqueness.

4. For completeness the extension-lattice comparison is also integral. Fix a primitive-cusp lattice $V_R$, with invariant line $Re$ and quotient $W_R$. The coboundary map

$$W_R^* \longrightarrow W_R^* \oplus W_R^*$$

is primitive, because its reduction has trivial kernel. Thus its cokernel, the degree-one cohomology of the free group on the two generators, is torsion free, and an integral coboundary over $K_p$ is already integral over $R$. The extension class $\alpha$ is primitive in this cokernel. Indeed, divisibility by $p$ would make the extra cusp map

$$\ker N_W / \operatorname{im} N_W \longrightarrow Re$$

divisible by $p$, contrary to the three unit Smith factors of the full cusp. Here $N_W$ has primitive image: it is the quotient of the primitive image of $N_V$ by $Re \subset \operatorname{im} N_V$.

5. Normalize any other stable lattice by the already proved quotient homothety so that its quotient is $W_R$. Its invariant line is $p^r Re$ for some integer $r$. The extension class in that basis is $p^{-r}\alpha$; primitivity makes it integral only when $r \leq 0$. Primitivity of the coboundary map eliminates any additional rational change of splitting. The extra cusp map is then multiplied by $p^{-r}$, so its Smith factor is a unit exactly when $r = 0$. The lattices coincide after the common homothety. This proves the assertion.

□

**Theorem 12.7** (A terminating lattice algorithm for each rational pair)**.** [tag 02NC]

*Let $(A, C)$ be any irreducible rational rank-five pair in one of the eight spectral cases of 11.10, with $A^5 = I$, $B = C^{-1}$, and $(AB - I)^2 = 0$ of rank two. There is an explicit terminating algorithm which decides whether the pair preserves an integral lattice, and, if so, lists all such lattices up to simultaneous integral conjugacy. It then lists all rank-six extensions with the same finite orders and primitive maximal square-zero cusp, up to simultaneous integral conjugacy. The bounds below depend on the input rational pair.*

*Proof.* 1. We give the algorithm and its termination proof. The fixed line of $A$ has dimension one. Choose a nonzero vector $e$ on it and a nonzero fixed covector $\varphi$. Irreducibility implies that the vectors $ge$, with $g$ a word in $A, C$, span the whole rational space; the analogous statement holds for the covectors $\varphi h$. In fact words of length at most four suffice. The spans of words of lengths at most $j$ increase from dimension one; if one step does not increase the span, it is invariant under both generators and hence already the whole space. There are only four possible increases.

2. Choose words $g_1, \dots, g_5$ and $h_1, \dots, h_5$ giving bases in this way, and put

$$P = I + A + \cdots + A^4, \qquad T_{ij} = g_i P h_j. \tag{12.21}$$

The rank-one operator $P$ is a nonzero scalar multiple of $e\varphi$. Hence the 25 matrices $T_{ij}$ are a rational basis of $\mathrm{Mat}_5(\mathbb{Q})$. Each belongs to $\mathcal{R} = \mathbb{Z}[A, C]$. Let $\mathcal{R}_0$ be their integral span and set

$$\mathcal{R}_0^{\#} = \{X : \mathrm{tr}(XT) \in \mathbb{Z} \text{ for all } T \in \mathcal{R}_0\}. \tag{12.22}$$

The trace pairing on the full matrix algebra is nondegenerate, as is seen from the matrix units, so this is another full rational lattice, computed by inverting the 25-by-25 Gram matrix.

3. If any stable lattice exists, every element of $\mathcal{R}$ acts integrally on it. Consequently $\mathrm{tr}(XY) \in \mathbb{Z}$ for all $X, Y \in \mathcal{R}$, and necessarily

$$\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_0^{\#}. \tag{12.23}$$

A nonintegral Gram entry therefore proves nonexistence. Otherwise let $d = [\mathcal{R}_0^{\#} : \mathcal{R}_0]$, the absolute Gram determinant. Start with the lattice generated by $\mathcal{R}_0$ and $I$, and repeatedly replace a current lattice $L$ by $L + AL + CL$, using exact Hermite reduction. If it leaves $\mathcal{R}_0^{\#}$, (12.23) proves that there is no stable lattice. While it remains inside, every strict enlargement reduces its index by a factor at least two. There can be at most $\lfloor \log_2 d \rfloor$ such enlargements starting from $\mathcal{R}_0$, so a bound of $1 + \lfloor \log_2 d \rfloor$ tests after adjoining $I$ is sufficient. Stabilization gives exactly $\mathcal{R}$: the stable lattice contains $I$ and all words in $A, C$, and every element introduced belongs to their ring. It is then an order, and $\Lambda_0 = \mathcal{R}\mathbb{Z}^5$ is an explicitly computed stable lattice. This proves the decision step, including termination in the nonexistence case.

4. Use a basis of $\Lambda_0$ from now on, so $\mathcal{R} \subset \mathrm{Mat}_5(\mathbb{Z})$. Smith reduction gives an integer $c \geq 1$ with

$$c\,\mathrm{Mat}_5(\mathbb{Z}) \subset \mathcal{R}; \tag{12.24}$$

one may take the largest Smith factor of the inclusion of the order in the full matrix lattice. Normalize any stable lattice by a rational homothety so that it is contained in $\mathbb{Z}^5$ and the gcd of all coordinates of a lattice basis is one. The operators $cE_{ij}$ preserve it. The coordinate values of its vectors generate $\mathbb{Z}$, so Bezout's identity, applied to $cE_{ij}$, shows that it contains every $ce_i$. Therefore every normalized lattice lies in the finite interval

$$c\mathbb{Z}^5 \subset \Lambda \subset \mathbb{Z}^5. \tag{12.25}$$

Conversely every lattice in this interval stable under $A, C$ is included by the algorithm. The only rational operators commuting with the pair are scalars: they act by a scalar on the one-dimensional fixed line of $A$, and then by the same scalar on the spanning vectors $g_i e$. Thus rational equivalence of two stable lattices is precisely homothety. The coordinate-gcd normalization is unique, so (12.25) gives the integral conjugacy classification without further identifications.

5. The enumeration has the following finite bound. For each $p^a \| c$, enumerate all subspaces of $\mathbb{F}_p^5$ in reduced echelon form. Beginning with $\mathbb{Z}^5$, pass through $a$ levels of stable lattices between $pH$ and $H$, retaining only those containing $p^a\mathbb{Z}^5$ and identifying equal Hermite forms. Every desired lattice occurs: for any target $\Lambda$, the chain $\Lambda + p^j\mathbb{Z}^5$, $0 \le j \le a$, is such a chain and ends at $\Lambda$. If $s_p$ denotes the number of all subspaces of $\mathbb{F}_p^5$, there are at most $s_p^a$ branches before deduplication. Intersect the resulting lattices for the different primes and discard those of coordinate gcd greater than one. The Chinese remainder theorem proves that this enumerates (12.25) completely.

6. For each quotient lattice $W_\mathbb{Z}$ so obtained, put $N_W = AB - I$. Its image must be primitive: for a primitive full cusp, $\operatorname{im} N_V = \ker N_V$ contains the primitive invariant line, and quotienting that image by the line gives a primitive image in $W_\mathbb{Z}$. Reject a quotient whose two nonzero Smith factors are not one. For the remaining quotients use extension rows $c_1, c_2 \in \mathbb{Z}^{1\times 5}$ and impose

$$\begin{gathered} c_1(I + A + \cdots + A^4) = 0, \qquad c_2(I + B + \cdots + B^{m-1}) = 0, \\ (c_2 + c_1 B)N_W = 0, \qquad m = \operatorname{ord}(B). \end{gathered} \tag{12.26}$$

These are integral linear equations. Their integral kernel modulo the coboundaries

$$z \longmapsto \big(z(A - I), z(B - I)\big), \qquad z \in \mathbb{Z}^{1\times 5}, \tag{12.27}$$

is a finitely generated group $G$ of rank one by 11.7. Compute its free and torsion factors by Smith reduction. Choose an integral generator $w$ of the free rank-one quotient $\ker N_W / \operatorname{im} N_W$. The homomorphism

$$\kappa : G \longrightarrow \mathbb{Z}, \qquad [(c_1, c_2)] \longmapsto (c_2 + c_1 B)w \tag{12.28}$$

is well defined and nonzero: (12.26) kills the image, and (12.27) kills $w$; rational nonvanishing is 11.7. The full cusp has unit Smith factors exactly when $\kappa = \pm 1$. Indeed the quotient rank-two image is already primitive, and the remaining invariant-line image is generated by that one integer. If $\operatorname{im}\kappa \neq \mathbb{Z}$, reject the lattice. Otherwise the inverse image of 1 is a finite coset of $\operatorname{Tor} G$, and lists all extensions. Changing the sign of the invariant-line generator identifies $\kappa = -1$ with $\kappa = 1$. There are no further identifications: an integral isomorphism on the quotient is $\pm I$, on the invariant line is $\pm 1$, and, after $\kappa = 1$ is fixed, their ratio is one. The remaining row is exactly an integral change of splitting (12.27). This proves completeness and termination of the extension step as well.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 12.8** (Lattice classes of the two three-connected members)**.** [tag `02ND`]

*For each of the two rational representations $v = 0$ and $v = 1$ of 12.1, there are exactly two homothety classes of stable rank-five quotient lattices, and exactly three integral conjugacy classes of rank-six extensions with primitive maximal cusp. Exactly one of those three can have simply connected finite affine fillings. It is the integral class already realized in 5.4, respectively 12.3. Thus arbitrary stable-lattice changes produce no further linear classes with three-connected gluing anywhere in the family 12.1. The affine twists and multiplicities are treated in theorem 12.9.*

*Proof.* 1. Apply the preceding algorithm in the original five-cycle lattice (12.1). For both parameters the order is obtained after four strict enlargements. Its Smith factors in the full matrix lattice are 21 factors one and four factors two; thus the conductor $c = 2$ in (12.24) is exact. Enumerate all 374 subspaces of $\mathbb{F}_2^5$, including zero and the whole space. After imposing stability and coordinate gcd one, the two lattice representatives are

$$\mathbb{Z}^5, \qquad \Lambda_* = 2\mathbb{Z}^5 + \mathbb{Z}(1,1,1,1,1)^t. \tag{12.29}$$

The latter has index 16.

2. For $\mathbb{Z}^5$, the extension group in (12.26)–(12.27) is $\mathbb{Z}$ and $\kappa$ is a unit on its generator. For $\Lambda_*$, that group is $\mathbb{Z} \oplus \mathbb{Z}/2$, again with primitive $\kappa$. Hence there are exactly one and two primitive extensions, respectively. For each extension, restrict the primitive global invariant homology covector $\ell$ to the two finite fixed lattices. The resulting positive divisibilities are

| quotient lattice | extension class | $(\mathrm{div}_1, \mathrm{div}_2)$ |
|---|---|---|
| $\mathbb{Z}^5$ | unique | $(1,2)$ |
| $\Lambda_*$ | first torsion coset | $(1,1)$ |
| $\Lambda_*$ | second torsion coset | $(1,2)$. |

(12.30)

The table is identical for $v = 0$ and $v = 1$.

3. For a simply connected filling the determinant quotient of 11.8 requires $\gcd(\ell(v_i), m_i) = 1$ at each finite cap. The second affine multiplicity is divisible by four. A divisibility of two in (12.30) therefore excludes that class for every choice of twists and larger multiplicities. For the sole remaining class, solve the integral intertwiner equations with the corresponding pair in (12.6). They have a solution of determinant $\pm 1$. The fillings of theorems 5.4 and 12.3 for those pairs are three-connected. 12.4 excludes every other rational parameter for all lattices. Together these facts prove all the claims, including the absence of any additional linear class with three-connected gluing in the unbounded family.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 12.9** (All three-connected twists in the unbounded family)**.** [tag `02NE`]

*Use the fixed Kuhn cusp and the affine translation clutching convention of 11.8. In the entire rational family 12.1, allow every stable lattice, every free affine finite lift, and every pair of finite multiplicities. The data with $\pi_1 = 1$ and $H_2 = H_3 = 0$ are precisely the following, up to the integral linear classification already proved and the translation identifications described below:*

$$\begin{aligned} v \in \{0,1\}, &\qquad \textit{the unique eligible integral class of 12.8,} \\ (m_1, m_2) = (5,4), &\qquad 20\kappa + 4r_1 + 5r_2 = \pm 1. \end{aligned} \tag{12.31}$$

*Here $r_i = \ell(v_i)$ is the invariant affine numerator, and $\kappa \in \mathbb{Z}$ is the invariant coordinate of the cusp translation clutching, with signs fixed by the displayed van Kampen relation below. The remaining fixed-direction components of the twists impose no condition. Every manifold in (12.31) is oriented diffeomorphic to its corresponding known example and hence to $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$.*

*Proof.* 1. 12.4 excludes all other rational parameters, and 12.8 excludes all other integral linear classes. We therefore work with either known integral pair, in the marking of (12.6). Its primitive invariant homology covector is $\ell(x) = x_0$. Each finite fixed lattice has an integral basis $u_i, \lambda_i$, where $\ell(u_i) = 1$, $\ell(\lambda_i) = 0$, and $\lambda_i$ is primitive in $\ker \ell$. We may write the affine numerator as $v_i = r_i u_i + s_i \lambda_i$. This covers arbitrary finite affine lifts: translating the origin makes their translation vector invariant by averaging, and its $m_i$th iterate is then the integral vector $v_i$. The resulting boundary translation change is included in the specified affine clutching.

2. For both linear pairs, the normal closure of the cusp angular lattice in the fibre lattice is exactly $\ker \ell$, by lemma 2.9 and 12.3. Van Kampen therefore leaves the abelian presentation

$$m_1 a = r_1 h, \qquad m_2 b = r_2 h, \qquad a + b + \kappa h = 0. \tag{12.32}$$

This also defines the sign of $\kappa$. Its determinant is, up to sign,

$$p = m_1 m_2 \kappa + m_2 r_1 + m_1 r_2. \tag{12.33}$$

Thus simple connectivity is exactly $p = \pm 1$. It forces $\gcd(m_1, m_2) = 1$ and $\gcd(r_i, m_i) = 1$ by reduction modulo each multiplicity.

3. We first prove that a three-connected datum has the minimal multiplicities. Set $n_1 = 5$, $n_2 = 4$ and $m_i = n_i k_i$. In the splitting $\Lambda = \mathbb{Z}u_i \oplus \ker \ell$, write the finite linear map as $1 \oplus D_i$. The reduced finite fibre is a mapping torus of $T^5$ over the circle $m_i \ell$; its fibre is connected because $r_i$ is prime to $m_i$. Its vertical monodromy is a power of $D_i$ prime to $n_i$, possibly followed by a translation, which changes no integral Wang group.

4. The primitive fixed vertical loop $\lambda_i$ gives the only additional free two-cycle beyond vertical cycles. Consequently an invariant integral two-form $\omega$ on the covering torus descends integrally exactly when

$$\frac{\omega(v_i, \lambda_i)}{m_i} = \frac{r_i \omega(u_i, \lambda_i)}{m_i} \quad \text{is an integer.} \tag{12.34}$$

For precision, the torus made from the affine generator and $\lambda_i$ projects with degree $r_i$ to the mapping- torus base, whereas the ordinary $u_i$ suspension projects with degree $m_i$. A Bezout combination is a primitive suspension. Its integrality condition is equivalent to (12.34), because the ordinary covering periods are already integral and $\gcd(r_i, m_i) = 1$. The Wang sequence shows that these vertical cycles and one primitive suspension generate the free second homology.

5. The integral invariant two-form computations in lemma 5.2 and 12.3 give

$$\gcd_{\omega} \omega(u_i, \lambda_i) = n_i. \tag{12.35}$$

Hence the covering-image cokernel in degree two at the $i$th finite point is exactly $\mathbb{Z}/k_i$. The global invariant group of $\bigwedge^2 \mathcal{V}$ and its integral first parabolic group both vanish. The skyscraper cokernel sequence therefore gives

$$H^1(B, R^2 g_* \mathbb{Z}) = \mathbb{Z}/k_1 \oplus \mathbb{Z}/k_2. \tag{12.36}$$

Point-supported torsion kernels do not change this degree-one base cohomology, and the cusp specialization is onto. The term (12.36), in bidegree $(1, 2)$, survives every Leray differential on the two-dimensional base. All rational terms of total degree three vanish, independently of the multiplicities and twists, by the same exterior local-system table as 5.4. Thus $H^3$ is finite. If either $k_i > 1$, its surviving nonzero filtration quotient makes $H^3$ nonzero; universal coefficients then give nonzero torsion in $H_2$. This proves $m_1 = 5$, $m_2 = 4$ for every three-connected datum.

6. We now identify all twists with these multiplicities. The following integral fact permits removal of each $s_i$:

$$\mathrm{Norm}_{n_i} : \ker \ell \longrightarrow \mathbb{Z}\lambda_i \quad \text{is onto.} \tag{12.37}$$

Here the norm is $I + D_i + \cdots + D_i^{n_i - 1}$. One can read (12.37) directly from (12.35). Let $\varphi_i$ generate the primitive invariant covector lattice on $\ker \ell$. The splitting by $u_i$ shows that

the gcd in (12.35) is $|\varphi_i(\lambda_i)|$: its forms are $\ell \wedge \varphi_i$ plus invariant vertical forms. Thus that value is $n_i$. The norm has the form $x \mapsto d_i\varphi_i(x)\lambda_i$ for an integer $d_i$; evaluating $\varphi_i$ gives $|d_i|n_i = n_i$, hence $|d_i| = 1$. This proves surjectivity. Choose integral preimages $q_i \in \ker \ell$ with $\mathrm{Norm}_{n_i} q_i = \lambda_i$.

7. Put
$$w_i = -\frac{s_i}{n_i} \sum_{j=0}^{n_i-1} jM_i^j q_i.$$
Then
$$(I - M_i)w_i = -s_i\lambda_i/n_i + s_iq_i. \tag{12.38}$$
Translation by $w_i$ conjugates the finite affine action with numerator $r_iu_i + s_i\lambda_i$ to that with numerator $r_iu_i$: the remaining $s_iq_i$ is integral. It gives a diffeomorphism of the entire finite cap. In the standard linear boundary marking, its translation section is $w_i + \theta s_i\lambda_i/n_i$. Its endpoint mismatch with the linear deck map is exactly $s_iq_i$ by (12.38), so its integral boundary residue has $\ell$-coordinate zero.

8. We explain why these local changes extend simultaneously, including the prescribed cusp marking. Let $P$ be the complementary pair of pants and $\mathcal{T} \to P$ its flat torus group bundle. A section on the boundary extends over the relative one-skeleton because the torus is connected. The only remaining obstruction is the two-cell obstruction with values in the fibre fundamental-group local system:
$$H^2(P, \partial P; \Lambda) \simeq H_0(P; \Lambda) = \Lambda_\Gamma. \tag{12.39}$$
The equality follows from the oriented relative cellular complex, equivalently integral Poincaré–Lefschetz duality with local coefficients. Its boundary residues map to the indicated global coinvariants. For both linear pairs the horizontal monodromy relation matrix has five unit Smith factors, so $\Lambda_\Gamma = \mathbb{Z}$, detected exactly by $\ell$. Thus the residues $s_iq_i$, with the identity section at the cusp, have zero obstruction and extend over $P$. A section may be smoothed relative to its boundary collars; translation by it is a fibre-preserving diffeomorphism. Together with the finite-cap maps it removes all $s_i$ without changing the cusp. The same argument removes all components of translation clutching whose $\ell$-coordinate is zero.

9. It remains to handle the integer numerators and the remaining cusp coordinate. If $p = -1$, the congruences from (12.33) give
$$r_1 = 1 + 5a, \qquad r_2 = -1 + 4b, \qquad a + b + \kappa = 0. \tag{12.40}$$
Adding $5au_1$ or $4bu_2$ changes no finite affine action; in its linear boundary marking it changes the translation residue by $au_1$ or $bu_2$. The difference of the cusp marking has residue of $\ell$-coordinate $\kappa$, by (12.32). Their total obstruction in (12.39) is therefore $a+b+\kappa = 0$. The same extension argument identifies the glued manifold with the original twists $(u_1, -u_2)$ and zero cusp clutching. If $p = 1$, the identical calculation starts from $(-u_1, u_2)$.

10. Finally simultaneous fibre inversion identifies these two sign choices by an orientation-preserving diffeomorphism. It conjugates every affine translation to its negative. It extends over the Kuhn cusp: the lattice involution $(x, h) \mapsto (-x, h)$ preserves its toric fan, since a simplex given by a chain in a unit cube is taken to the chain with reversed coordinate order in the opposite cube. It normalizes the cusp lattice action and therefore descends to its quotient, while fixing the base parameter. The resulting toric map is holomorphic, and in the regular real six-torus inversion has determinant $(-1)^6 = 1$. All translation maps also preserve orientation. Thus every datum in (12.31) gives the stated oriented smooth type and has $H_2 = H_3 = 0$, proving sufficiency as well as necessity.

The corresponding exact calculations are implemented in the programs in Table 1. □

## 13 Three-connectivity and affine twists

**Theorem 13.1** (The universal degree-two suspension obstruction)**.** [tag 02NF]

*Every admissible datum whose gluing has $\pi_1 = 1$ and $H_2 = H_3 = 0$ has exactly two genuine finite fillings and has minimal affine multiplicities. After ordering the order-five point first, its finite characteristic polynomials are one of precisely the following two remaining necessary possibilities:*

$$\begin{array}{c|c|c} (m_1, m_2) & \chi_{M_1} & \chi_{M_2} \\ \hline (5,4) & (x-1)^2\Phi_5 & (x-1)^2\Phi_2^2\Phi_4 \\ (5,6) & (x-1)^2\Phi_5 & (x-1)^2\Phi_2^2\Phi_3. \end{array} \tag{13.1}$$

*In particular all order-eight and order-twelve spectral loci, and the other two order-six spectral loci of 11.10, are excluded for every rational relative position, every stable lattice and every free affine twist.*

*Proof.* 1. Theorem 11.1 excludes three genuine finite fillings. For two fillings, 11.3 and 11.10 give a primitive global invariant homology covector $\ell$, finite fixed rank two, and an irreducible rank-five cohomology quotient. In particular

$$H^0(B, j_* \bigwedge^2 \mathcal{V}) = 0. \tag{13.2}$$

This is also immediate from the exterior filtration and irreducibility in odd dimension, as in 12.2. Simple connectivity gives the determinant quotient of 11.8, so each invariant affine numerator $r_i = \ell(v_i)$ is prime to the multiplicity $m_i$, and the two multiplicities are coprime. As before one may conjugate any finite affine lift by a translation so that its numerator $v_i$ is integral and fixed by its linear part.

2. Consider one finite point. Let $M$ have linear order $n$, so $n \mid m$, and let its complementary degree-four factor be $f(x)$, with $\chi_M = (x-1)^2 f(x)$. The image of $\ell$ on the finite fixed lattice is $g\mathbb{Z}$ for some $g \mid n$: averaging $I + M + \cdots + M^{n-1}$ on any vector of $\ell$-value one proves this divisibility. Since $g \mid r$ and $\gcd(r, m) = 1$, we have $g = 1$. Choose a fixed integral vector $u$ with $\ell(u) = 1$. It gives an integral invariant splitting

$$\Lambda = \mathbb{Z}u \oplus H, \qquad H = \ker \ell, \qquad M = 1 \oplus D, \qquad \chi_D = (x-1)f(x). \tag{13.3}$$

The order of $D$ is $n$. Let $\lambda$ generate its primitive fixed lattice and let $\varphi$ generate its primitive invariant covector lattice. Choose their signs so that

$$h = \varphi(\lambda) > 0. \tag{13.4}$$

Their rational pairing is nonzero, since the finite map is semisimple and its averaging projection has rank one.

3. Both $\mathrm{Norm}_n(D)$ and $f(D)$ are integral operators, have image on the line $\mathbb{Q}\lambda$, and have invariant defining covector proportional to $\varphi$. Primitivity of $\lambda$ and $\varphi$ therefore implies

$$\mathrm{Norm}_n(D) = \frac{n}{h}\lambda\varphi, \qquad f(D) = \frac{f(1)}{h}\lambda\varphi, \qquad h \mid n, \quad h \mid f(1). \tag{13.5}$$

For example, the integer coefficient in the second formula is seen by applying the integral operator to a vector on which $\varphi$ is one; evaluating on $\lambda$ determines that coefficient.

4. Write $v = ru + s\lambda$. The quotient finite fibre is a connected $T^5$ mapping torus over the circle $m\ell$, because $r$ is prime to $m$. Its vertical cycles already have integral periods for an invariant integral two-form on the covering torus. The Wang sequence leaves one

additional primitive free suspension cycle, from $\lambda$. Exactly as in (12.34), descent of an invariant form $\omega$ is equivalent to $m \mid r\omega(u,\lambda)$, and hence to $m \mid \omega(u,\lambda)$. In the integral splitting (13.3), every invariant two-form is an integral multiple of $\ell \wedge \varphi$ plus an invariant vertical two-form. Its possible values on $(u,\lambda)$ are therefore exactly $h\mathbb{Z}$. It follows that the degree-two specialization image at this point has cokernel

$$C_p^2 \simeq \mathbb{Z}/(m/h). \tag{13.6}$$

The integer $m/h$ is defined because $h \mid n \mid m$. Torsion in the finite fibre cohomology is a specialization kernel. The primitive suspension and vertical cycles generate the free second homology, so (13.6) is the full cokernel.

5. Let $I^2$ be the image sheaf of $R^2 g_* \mathbb{Z} \to j_* \bigwedge^2 \mathcal{V}$. The cusp specialization is onto by theorem 2.6, and all kernels and cokernels at finite points are skyscraper sheaves. Thus $H^1(R^2 g_* \mathbb{Z}) = H^1(I^2)$. Equation (13.2) and the cokernel exact sequence give an injection

$$\bigoplus_{p \text{ finite}} C_p^2 \longhookrightarrow H^1(B, R^2 g_* \mathbb{Z}). \tag{13.7}$$

The Leray term of bidegree $(1,2)$ has no incoming or outgoing differential on a real two-dimensional base. If $H_2 = H_3 = 0$, the universal coefficient theorem gives $H^3 = 0$, so that term must vanish. Every cyclic cokernel in (13.7) is therefore zero. Combining (13.5)–(13.6) yields the universal necessary condition

$$m = h = n, \qquad n \mid f(1). \tag{13.8}$$

This both fixes the multiplicity and tests the spectrum.

6. Apply (13.8) to the eight-case list of 11.10. The order-five factor gives $\Phi_5(1) = 5$. At the other point the exact values are

| $n$ | $f$ | $f(1)$ | $n/\gcd(n, f(1))$ |
|---|---|---|---|
| 4 | $\Phi_2^2\Phi_4$ | 8 | 1 |
| 6 | $\Phi_2^2\Phi_3$ | 12 | 1 |
| 6 | $\Phi_2^2\Phi_6$ | 4 | 3 |
| 6 | $\Phi_3\Phi_6$ | 3 | 2 |
| 8 | $\Phi_8$ | 2 | 4 |
| 12 | $\Phi_{12}$ | 1 | 12 |
| 12 | $\Phi_3\Phi_4$ | 6 | 2 |
| 12 | $\Phi_4\Phi_6$ | 2 | 6. |

(13.9)

The last column is a lower bound for the nonzero local covering-cokernel order already at the smallest allowed multiplicity. Only the first two rows pass. This proves (13.1) and the stated universal exclusions.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 13.2** (The complete rational remaining $(5,6)$ locus)**.** [tag `02NG`]

*The irreducible rational rank-five pairs in the second row of* (13.1)*, with square-zero rank-two cusp, are classified up to simultaneous rational conjugacy by*

$$\{[r:s:t] \in \mathbb{P}^2(\mathbb{Q}) : D(r,s,t) \neq 0,\ Q_-(r,s,t) \neq 0\} \sqcup \mathbb{A}^1(\mathbb{Q}), \qquad Q_- = r^2 - rs + st, \tag{13.10}$$

*where $D$ is the same cubic as in* (12.13)*. The exceptional line again lies over $[4:1:3]$. Thus the two rational spectral loci left by 13.1 both have complete explicit normal forms, including their entire rational exceptional loci.*

*Proof.* 1. Use precisely the space $\mathbb{Q} \oplus \mathbb{Q}(\zeta_5)$, matrices $A, R, X$ and projective plane normalization in (12.14)–(12.16). The centralizer and image-plane arguments in 12.5 require only irreducibility and the two-dimensional $-1$ eigenspace of $C$; both hold here. Consequently every pair has a unique projective image-plane parameter, with only scalar stabilizer. The unknown $Y$ now satisfies

$$YX = 0, \qquad YRX = I_2, \qquad \operatorname{tr}(YAX) = -2, \qquad \operatorname{tr}(YA^{-1}X) = 2. \tag{13.11}$$

Its coefficient matrix is unchanged from (12.17), so has determinant $D/4$. The two traces and the $-1$ eigenspace imply

$$\chi_C = (x-1)(x+1)^2(x^2+x+1).$$

Its other three roots are distinct, and the $-1$ eigenspace has its full multiplicity, so $C^6 = I$. The cusp $XY$ still has square zero and rank two. For $D \neq 0$, the ten linear equations give the unique explicit solution.

2. The two first-column adjugate identities become

$$Dy_{10} = 2Q_-Q_2, \qquad Dy_{20} = 2(r-s-t)Q_-, \tag{13.12}$$

with the same $Q_2$ as in (12.18). For $D \neq 0$, these entries vanish simultaneously exactly when $Q_- = 0$. Indeed, if $Q_- \neq 0$ they would instead force $r = s + t$ and $Q_2 = 0$, and (12.19) would give $D = 0$. The invariant-covector argument of 12.5 is unchanged. Thus the pair is rationally irreducible exactly off the conic $Q_- = 0$ in the generic locus.

3. No nonzero rational triple satisfies $D = Q_- = 0$. If $s = 0$, the conic forces $r = 0$, and then $D = 7t^3$. If $s \neq 0$, rescale to $s = 1$ and put $r = u$, $t = u - u^2$. Direct substitution gives

$$D(u, 1, u-u^2) = -(7u^2 - 11u + 7)(u^4 - u^3 + u^2 - u + 1). \tag{13.13}$$

The first factor has discriminant $-75$ and positive leading coefficient. The second is strictly positive, as the identity

$$u^4 - u^3 + u^2 - u + 1 = \left(u^2 - \frac{u}{2}\right)^2 + \frac{3}{4}\left(u - \frac{2}{3}\right)^2 + \frac{2}{3}$$

shows. In particular neither has a rational zero. On the exceptional cubic the necessary identities (13.12) therefore force the same two points $[3:2:1]$ and $[4:1:3]$ as before. The first is again impossible because $[X \mid RX]$ has rank less than four. At the second, the complete solution is

$$Y_z = \begin{pmatrix} 4z + 26/5 & 0 & -2z - 2 & -2z - 1/5 & 2z + 1 \\ 2z - 2/5 & 0 & -z & 2/5 - z & z \end{pmatrix}, \qquad z \in \mathbb{Q}. \tag{13.14}$$

Its first-column entries cannot vanish simultaneously, so each pair on this line is irreducible. There are no further identifications, by the scalar stabilizer argument already proved in 12.5. This establishes the claimed bijection.

4. The finite lattice and extension algorithm of 12.7 enumerates the integral lattices and extensions over every point of (13.10).

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 13.3** (The entire remaining order-six locus is excluded)**.** [tag 02NH]

*Every simply connected datum in the second row of* (13.1)*, at every rational relative position and every stable integral lattice with primitive maximal cusp, has*

$$\operatorname{Tor}_3 H_3(M;\mathbb{Z}) \neq 0$$

*if its degree-two suspension cokernels vanish. Consequently no such datum is three-connected. Together with 13.1 this leaves only the minimal order pair* $(5,4)$ *as a possible three-connected admissible datum.*

*Proof.* 1. We use cohomology with coefficients in the discrete valuation ring $R = \mathbb{Z}_3$. Suppose the degree-two suspension cokernels vanish; otherwise 13.1 already gives nonzero $H_2$ or $H_3$. That theorem forces $m = n = h = 6$ at the order-six point, in the notation (13.3)–(13.5). We first determine its local integral representation over $R$.

2. The splitting by the fixed vector $u$ is integral: $\Lambda_R = Ru \oplus H_R$. On $H_R$, the finite map $D$ has characteristic polynomial $(x-1)(x+1)^2\Phi_3(x)$. Since two is a unit in $R$, the idempotents $(I \pm D^3)/2$ give an integral direct sum

$$H_R = L_+ \oplus L_-, \qquad \operatorname{rank} L_+ = 3, \quad \operatorname{rank} L_- = 2.$$

On $L_-$ the map is exactly $-I$, by its rational spectrum. On $L_+$ it has order three. The primitive fixed vector $\lambda$ lies in $L_+$, and the primitive invariant covector $\varphi$ vanishes on $L_-$, so their restrictions remain primitive. Equation (13.5) with $h = 6$ gives

$$2(I + D + D^2)|_{L_+} = \lambda\varphi|_{L_+}. \tag{13.15}$$

This operator is nonzero modulo three. But modulo three $I + D + D^2 = (D - I)^2$ and $(D - I)^3 = 0$. The reduction of $D$ on the rank-three space therefore has one Jordan block of length three. Choose $q \in L_+$ whose image has $(D - I)^2 q \neq 0$ modulo three. Then $q, Dq, D^2q$ are a basis modulo three and hence an $R$-basis. In this basis $D$ is the three-cycle permutation matrix, since $D^3 = I$. Thus the full finite representation over $R$ is

$$1 \oplus \operatorname{Perm}_3 \oplus(-I_2). \tag{13.16}$$

Its dual has the same form, by reversing the three-cycle.

3. Let $u, a, b, c, d, e$ be a cohomology basis for (13.16), with $a, b, c$ cycled and $d, e$ negated. A full integral basis of the invariant degree-three lattice is

$$u(ab + bc + ca), \qquad abc, \qquad ude, \qquad (a + b + c)de. \tag{13.17}$$

Here juxtaposition denotes exterior multiplication. To check completeness, terms with an odd number of $d, e$ are negated by the cube of the finite map and have no invariants. The terms with zero or two such factors give exactly the four orbit sums in (13.17), each primitive. With volume $uabcde$, their skew pairing has one block with coefficient three and one block with coefficient a unit. Thus its determinant has three-adic valuation two. Exterior multiplication in the displayed integral basis gives Smith factors $(1, 1, 3, 3)$.

4. Let $B_6$ be the reduced finite fibre. Its torus cover has degree six. Its free middle cohomology has rank four and unimodular skew pairing, by integral Poincaré duality. If $I$ denotes the index of its pullback lattice in the invariant lattice, multiplication of pairings by the covering degree gives

$$2v_3(I) + 2 = 4v_3(6) = 4, \qquad v_3(I) = 1. \tag{13.18}$$

Consequently the three-primary degree-three covering cokernel is $R/3R$. Translations of the affine action do not enter this calculation: the cover degree, linear invariant lattice and oriented pairings are those of the free quotient for every allowed affine numerator.

5. We explain why this local group must appear in total homology even when the global integral lattice varies. Let $\mathcal{V}$ be the rational rank-six cohomology local system. It is an extension of the irreducible rank-five system $W$ by a trivial line. The determinant of $W$ is trivial. Irreducibility in odd dimension implies that neither $\bigwedge^2 W$ nor its dual has a nonzero invariant: an invariant alternating form would have an invariant radical and would have to be nondegenerate, impossible in odd dimension. Also $\bigwedge^3 W \simeq \bigwedge^2 W^*$ by its trivial determinant. The exterior filtration therefore gives

$$(\bigwedge^3 \mathcal{V})^\Gamma = 0, \qquad ((\bigwedge^2 \mathcal{V})^*)^\Gamma = 0. \tag{13.19}$$

These rational vanishing statements remain true over $\mathbb{Q}_3$, since they are kernels of rational matrices.

6. As in (13.7), take the image of $R^3 g_* R \to j_* \bigwedge^3 \mathcal{V}_R$. Its kernel and cokernel are point-supported sheaves. The first vanishing in (13.19) makes the local group $R/3R$ inject into $H^1(B, R^3 g_* R)$. Bidegree $(1,3)$ has no incoming or outgoing Leray differential on the two-dimensional base, so this injection survives to $E_\infty^{1,3}$.

7. The second vanishing in (13.19) says that $H^2(B, R^2 g_* R)$ has rational rank zero. Point-supported changes leave degree-two base cohomology unchanged. Put $\mathcal{L} = \bigwedge^2 \mathcal{V}_R$ on the punctured base $B^\circ$. The quotient $j_*\mathcal{L}/j_!\mathcal{L}$ is supported at the punctures, so its exact sequence and integral Poincaré duality give

$$H^2(B, j_*\mathcal{L}) = H^2_c(B^\circ, \mathcal{L}) = H_0(B^\circ, \mathcal{L}) = (\mathcal{L}_b)_\Gamma, \qquad b \in B^\circ.$$

Thus the group is a finitely generated torsion $R$-module and hence a finite group. Therefore $E_\infty^{2,2}$ is finite as well. In the filtration of $H^4(M; R)$, take the inverse image of the injected $R/3R \subset E_\infty^{1,3}$. It is an extension of $R/3R$ by the finite group $E_\infty^{2,2}$. It is thus a nonzero finite three-primary subgroup of $H^4(M; R)$. Universal coefficients identify this torsion with the three-primary torsion of $H_3(M; \mathbb{Z})$. This proves the claim at every rational position and every integral lattice, and completes the exclusion.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 13.4** (Universal fundamental group and twist normalization)**.** [tag `02NI`]

*In the remaining $(5,4)$ locus, fix the marked standard Kuhn cap and allow affine translation clutching. Let $\Lambda = \mathbb{Z}^6$ be fibre homology and let $\Lambda_\Gamma$ be its global monodromy coinvariants. Write $\Lambda_\Gamma = \mathbb{Z}h \oplus T$, with $T$ finite, and let $\ell$ be the primitive invariant covector detecting the free factor. The following are complete reductions for any datum that could satisfy $\pi_1 = 1$ and $H_2 = H_3 = 0$.*

*The multiplicities must be $(5,4)$, and the local vertical fixed pairings must be $h_1 = 5, h_2 = 4$, as in 13.1. For any affine twists and clutching the fundamental group is abelian. If its free presentation determinant is nonzero, its order is*

$$|\pi_1| = |20\kappa + 4r_1 + 5r_2|\,|T|, \qquad r_i = \ell(v_i). \tag{13.20}$$

*If that determinant is zero the group is infinite. In particular simple connectivity is equivalent to*

$$T = 0, \qquad 20\kappa + 4r_1 + 5r_2 = \pm 1. \tag{13.21}$$

*Whenever these conditions and $h_1 = 5, h_2 = 4$ hold, every such twist choice is oriented diffeomorphic to the normalized filling with twists $(u_1, -u_2)$ and zero cusp clutching, where $u_i$ is any integral fixed vector with $\ell(u_i) = 1$. Thus each eligible integral linear datum has exactly one normalized smooth filling to test for the remaining homology conditions.*

*Proof.* 1. The image of the fibre fundamental group is a normal abelian subgroup in the van Kampen presentation. After quotienting by it, the two finite meridians satisfy $a^5 = b^4 = ab = 1$. That quotient is trivial, because five and four are coprime. The entire fundamental group is therefore equal to the image of the fibre group and is abelian.

2. In abelianization the monodromy relations give $\Lambda_\Gamma$. The primitive cusp angular lattice is the image of its nilpotent operator, and this image is already zero among those monodromy relations. The remaining presentation is

$$5a = v_1, \qquad 4b = v_2, \qquad a + b + v_0 = 0 \quad \text{in } \Lambda_\Gamma \oplus \mathbb{Z}a \oplus \mathbb{Z}b, \tag{13.22}$$

where $\ell(v_0) = \kappa$ fixes the sign convention. Use any Smith presentation of $T$ and split the free part $\mathbb{Z}h$. The full relation matrix is block triangular, with the presentation of $T$ as one diagonal block and the three-by-three free matrix of determinant $\pm(20\kappa + 4r_1 + 5r_2)$ as the other. Its determinant gives (13.20); when the latter free matrix has rank less than three the group has positive rank and is infinite. For determinant $\pm 1$, elimination of the three free variables leaves precisely $T$, even if the twists have nonzero torsion components. This proves the exact simple-connectivity criterion.

3. We give the global twist comparison to make its scope precise. Under (13.21), the finite fixed covector image is one at each point, so choose integral fixed $u_i$ with $\ell(u_i) = 1$. The primitive vertical fixed vector $\lambda_i$ and invariant covector $\varphi_i$ satisfy $\varphi_i(\lambda_i) = m_i$, by the assumed maximal local pairings. Formula (13.5) makes the finite norm onto $\mathbb{Z}\lambda_i$. Consequently every vertical numerator component $s_i\lambda_i$ is removed by the explicit cap translation (12.38), whose boundary residue has $\ell$-coordinate zero.

4. For the complementary pair of pants the full relative obstruction group for extending boundary translation sections is $H^2(P, \partial P; \Lambda) = \Lambda_\Gamma$. Since $T = 0$, this group is $\mathbb{Z}$, detected by $\ell$. Prescribing the identity translation at the cusp therefore allows all the vertical finite changes to extend over $P$. The same argument removes the zero-$\ell$ component of any cusp translation clutching. For determinant $-1$, write $r_1 = 1 + 5a$, $r_2 = -1 + 4b$; then (13.21) says $a + b + \kappa = 0$. The two integer changes of finite boundary marking and the remaining cusp change thus also have zero total relative obstruction and extend. This gives the normalized twists $(u_1, -u_2)$. For determinant 1 the same argument gives their negatives. The orientation-preserving fibre inversion extending over the Kuhn fan, proved in 12.9, identifies the two choices. All maps used in the comparison are cap translations or translations by smooth torus-bundle sections over $P$; each comparison respects the prescribed cusp marking. This proves the claimed complete twist normalization.

□

**Theorem 13.5** (A finite integral test for three-connectivity)**.** [tag 02NJ]

*Take any rational pair in 12.5 and any of its integral, primitive-cusp rank-six extensions given by 12.7. Write $M_1, M_2$ for geometric transport on fibre homology $\Lambda = \mathbb{Z}^6$, in the peripheral convention $M_0M_1M_2 = I$. First require the preliminary conditions*

$$\Lambda_\Gamma = \mathbb{Z}, \qquad \ell(\Lambda^{M_i}) = \mathbb{Z}, \qquad h_1 = 5, \quad h_2 = 4. \tag{13.23}$$

*Here $\ell$ is the primitive global invariant covector, and $h_i$ is the integral fixed-vector/fixed-covector pairing on* $\ker \ell$ *from 13.1. Failure of any condition excludes the datum. If they hold, use the unique normalized affine filling of 13.4, with twists $(u_1, -u_2)$ and zero cusp clutching. The following six finite integer-matrix tests are necessary and sufficient for this filling to have $H_2 = H_3 = 0$.*

*Let*

$$T_i = M_i^{-t}, \quad T_0 = (T_1T_2)^{-1}, \qquad G_q = \operatorname{coker}[\textstyle\bigwedge^q T_1 - I \mid \bigwedge^q T_2 - I\,] \quad (1 \le q \le 4). \tag{13.24}$$

*All $G_q$ are finite. Let $P_q$ be the integral parabolic cohomology group of the exterior local system $\bigwedge^q \mathcal{V}$, for $q = 2, 4$. Let $t_q \in G_{q-1}$, $2 \le q \le 5$, be the natural global coinvariant image of the unique nonzero torsion suspension class at the order-four cap, as specified below. Let $c_4 \in G_3$ be the natural image of the unique nonzero torsion class in* $\operatorname{coker}(\bigwedge^3 T_0 - I)$. *Then the six tests are*

| | *first test* | *second test* | *third test* |
|---|---|---|---|
| $H^3 = 0$ | $G_1 = \langle t_2 \rangle$ | $P_2 = 0$ | $t_3 \neq 0$ |
| $H^5 = 0$ | $G_3 = \langle t_4, c_4 \rangle$ | $P_4 = 0$ | $t_5 \neq 0.$ |

(13.25)

*Every group, class, and equality here is computed by finitely many integer kernels, Smith forms, and exterior minors. This is a decision criterion for every rational parameter and every integral lattice in the normal form.*

*Proof.* 1. We establish the uniform local calculations for the criterion. The fixed vectors $u_i$ split the homology lattice as $\mathbb{Z}u_i \oplus H_i$, where $H_i = \ker \ell$. Write $D_i = M_i|_{H_i}$. The maximal pairing $h_i = m_i$ gives

$$\operatorname{Norm}_{m_i}(D_i) = \lambda_i \varphi_i, \tag{13.26}$$

with both factors primitive.

2. At five, work over $\mathbb{Z}_5$. Modulo five the norm is $(D_1 - I)^4$ and is nonzero. Choose $q$ with $\operatorname{Norm}_5 q$ primitive. The five vectors $q, D_1 q, \dots, D_1^4 q$ are independent modulo five: otherwise the nilpotent Jordan block generated by $q$ would have length less than five. They are therefore a $\mathbb{Z}_5$ basis. Thus $D_1$ is the regular five-cycle permutation locally.

3. At four, work over $\mathbb{Z}_2$. The norm is $(D_2 - I)^3$ modulo two and is nonzero. The vectors $q, D_2 q, D_2^2 q, D_2^3 q$ similarly generate a primitive regular four-cycle submodule $Q$ of rank four. The quotient is rank one with $D_2 = -1$, by the characteristic polynomial. For a lift $z$ of its generator, put $a = (D_2 + I)z \in Q$. Since $D_2^4 = I$,

$$(D_2^3 - D_2^2 + D_2 - I)a = 0.$$

On the regular four-cycle module the kernel of the displayed alternating norm is exactly $(D_2 + I)Q$, integrally. One can see this by writing its four coefficients; equivalently $D_2 + I$ has three nonzero Smith factors, all one, and the indicated rational image. Subtracting a suitable element of $Q$ from $z$ makes $D_2 z = -z$. Consequently

$$M_1 \sim 1 \oplus \operatorname{perm}_5 \quad \text{over } \mathbb{Z}_5, \qquad M_2 \sim 1 \oplus \operatorname{perm}_4 \oplus (-1) \quad \text{over } \mathbb{Z}_2. \tag{13.27}$$

These local comparisons induce isomorphisms on cap cohomology, as follows. Indeed the integral lattice of intertwiners is the kernel of an integer matrix and its tensor with $\mathbb{Z}_p$ is the lattice of local intertwiners. Approximate a local invertible intertwiner by an integral one whose determinant is a $p$-adic unit. Keeping the $u_i$ summand fixed produces an equivariant torus isogeny of degree prime to $p$, preserving the normalized translation $\pm u_i/m_i$. The resulting cap and boundary coverings induce isomorphisms on $\mathbb{Z}_p$ cohomology: the vertical torus maps are cohomology isomorphisms over $\mathbb{Z}_p$, so the mapping-torus Wang sequences give the assertion. At primes not dividing $m_i$, transfer already makes finite-cover pullback an isomorphism onto invariants.

4. Here are the full required integral results for these two monomial models. The ranks of their invariant exterior lattices in degrees zero through six are $(1, 2, 3, 4, 3, 2, 1)$. The order-five cap has no cohomology torsion in degrees two through five. The order-four cap has exactly $\mathbb{Z}/2$ in each of those four degrees. This follows from the Wang sequence for the mapping torus of the vertical $D_i$: its torsion in degree $q$ is

$$\operatorname{Tor} \operatorname{coker}(\bigwedge^{q-1} D_i^t - I). \tag{13.28}$$

In particular it introduces no additional torsion primes.

5. Integral pullback from either finite cap is onto the invariant lattice in degrees two through five. For degree two this is precisely the $m_i/h_i = 1$ suspension calculation of 13.1. The determinants of the pairings between the invariant lattices in complementary degrees $(1,5),(2,4),(3,3)$ are respectively $m_i, m_i^3, m_i^4$ in absolute value. The degree-one cover image has index $m_i$. The free downstairs complementary pairings are unimodular, and pullback multiplies each pairing by the covering degree $m_i$. The middle pairing first gives degree-three index one. The degree-two index one then gives degree-four index one; the degree-one index $m_i$ gives degree-five index one. For both local normal forms, exterior multiplication gives these pairing determinants, and integer row and column operations give the Wang Smith forms in exterior degrees zero through six.

6. We next identify the boundary maps. Choose any unimodular matrix $P_2 = [u_2, H_2]$ realizing the splitting at four, and put $J = P_2^{-t}|_{\mathbb{Z}^5}$ for the inclusion of vertical cohomology. Choose the unique nonzero torsion representative

$$w_q \in \operatorname{coker}(\bigwedge^{q-1} D_2^t - I), \qquad t_q = [(\bigwedge^{q-1} J)w_q] \in G_{q-1}. \tag{13.29}$$

   A Smith transformation supplies an integral $w_q$. Its norm is exactly zero, since it lies rationally in the image of $\bigwedge^{q-1} D_2^t - I$. The finite boundary coordinates of 5.4 satisfy $\theta = a + 4k$, $h = k$, hence $a = \theta - 4h$. The pullback of the suspension cocycle has $\theta$ component $w_q$ and $h$ component $-\operatorname{Norm}_4(w_q) = 0$. This holds in every exterior degree, including $q = 5$. It is an integral cellular assertion for the monomial model, where the torus cellular differentials vanish; the prime-to-two isogeny comparison transports it to every eligible cap. The base connecting map therefore gives exactly the class $t_q$ in (13.29). Different representative choices change it by a global monodromy relation. Any sign change is immaterial for an order-two class.

7. There is one further boundary map, from the free degree-four cusp kernel. The primitive cusp is integrally conjugate to $\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$. Its degree-three exterior difference has ten Smith factors one and one Smith factor two; its cokernel is $\mathbb{Z}^9 \oplus \mathbb{Z}/2$. The other exterior differences in degrees one, two, four and five have torsion-free cokernels. Let $C$ be the smooth cusp cap and $Y = \partial C$. The boundary Wang sequence gives $\operatorname{Tor} H^4(Y) = \mathbb{Z}/2$. In the Poincaré–Lefschetz exact sequence its image in

$$H^5(C, Y) \cong H_3(C) = \mathbb{Z}^9$$

   is zero, so this torsion class lifts to $H^4(C)$. Its restriction to an ordinary fibre is zero. By theorem 2.6 the kernel of integral specialization is a primitive copy $K = \mathbb{Z}$ in degree four. The lift is in $K$, and the image of this cyclic group in $H^4(Y)$ contains the nonzero torsion class. A cyclic image containing nonzero torsion cannot also have a nonzero free part. Its primitive generator therefore restricts to the unique nonzero torsion class of $H^4(Y)$. Consequently the cusp contribution to $d_2$ is exactly $c_4$ as defined above.

8. It remains to prove that these are all the tests. For the irreducible odd-dimensional quotient $W$ of $\mathcal{V}$, invariant alternating forms and bivectors vanish: a nonzero one would give an isomorphism $W \cong W^*$ with an odd-dimensional nondegenerate alternating pairing. The exact exterior sequences for $0 \to \mathbb{Q} \to \mathcal{V}_{\mathbb{Q}} \to W \to 0$, together with $\det W = 1$, give

$$H^0(B, j_* \bigwedge^q \mathcal{V}) = 0 \quad (2 \le q \le 5), \qquad G_q \otimes \mathbb{Q} = 0 \quad (1 \le q \le 4). \tag{13.30}$$

   For the endpoint assertions, the extension cannot split: on $\mathbb{Q} \oplus W$ a square-zero nilpotent has rank at most two, whereas the stipulated rank is three. An invariant covector of $\mathcal{V}_{\mathbb{Q}}$ would split this extension, since $W$ has no trivial quotient. Thus $(\mathcal{V}_{\mathbb{Q}}^*)^{\Gamma} = 0$, which gives

both $H^0(\bigwedge^5 \mathcal{V}_{\mathbb{Q}}) = 0$ and $G_1 \otimes \mathbb{Q} = 0$. Here and below the first assertion is integral as well, since the coefficient lattices are torsion free. The local invariant ranks in exterior degrees two and four are $3, 3, 9$ at the three punctures. The Euler formula for the rank-fifteen local system therefore gives $P_2 \otimes \mathbb{Q} = P_4 \otimes \mathbb{Q} = 0$. In degree one it similarly gives rational parabolic cohomology zero. The only global degree-one section is the invariant line. Simple connectivity and the universal coefficient theorem make its free transgression onto $H^2(B, \mathbb{Z})$; equivalently the rational second Betti number is zero, as in 5.4. Thus the normalized filling has rational $b_2 = b_3 = 0$.

9. All specializations in exterior degrees two through five are onto, by the local calculations and theorem 2.6. Their only point-supported kernels are

$$\begin{array}{c|cccc} q & 2 & 3 & 4 & 5 \\ \hline \text{kernel} & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z} \oplus \mathbb{Z}/2 & \mathbb{Z}/2. \end{array} \tag{13.31}$$

   The free summand in degree four is at the cusp; the others are at four. Consequently $H^1(B, R^q g_* \mathbb{Z}) = P_q$ in the degrees used here, while $H^2(B, R^q g_* \mathbb{Z}) = G_q$. The latter equality also holds in degree one: changing a sheaf at isolated points changes no degree-two cohomology. The equality with coinvariants follows from compactly supported top cohomology on the punctured sphere and the sequence $j_! \to j_*$.

10. The base has real dimension two. Thus the only possible nonzero differential in the integral Leray spectral sequence is $d_2 : E_2^{0,q} \to E_2^{2,q-1}$. The complete associated graded groups in total degrees three and five are therefore

$$\begin{aligned} &\operatorname{gr} H^3 : G_1/\langle t_2 \rangle, \quad P_2, \quad \ker(\mathbb{Z}/2 \xrightarrow{t_3} G_2), \\ &\operatorname{gr} H^5 : G_3/\langle t_4, c_4 \rangle, \quad P_4, \quad \ker(\mathbb{Z}/2 \xrightarrow{t_5} G_4). \end{aligned} \tag{13.32}$$

    These are all finite. A finite filtered group vanishes exactly when every associated graded group vanishes, regardless of extension classes. Since $\pi_1 = 1$ and rational $b_2 = b_3 = 0$, the universal coefficient theorem gives $H^3 = \operatorname{Ext}(H_2, \mathbb{Z})$, so $H_2 = 0$ exactly when $H^3 = 0$. Integral Poincaré duality gives $H^5 \cong H_3$. This proves (13.25).

11. Given $A = \bigwedge^q T_1$, $B = \bigwedge^q T_2$ on a free lattice $L$, form the saturated integer kernel

$$K_q = \ker[\, A - I \mid A(B - I) \mid -(AB - I)\,] \subset L^3. \tag{13.33}$$

    Its image under $(x, y, z) \mapsto ((A - I)x, (B - I)y)$ is precisely the integral parabolic cocycle lattice. Quotient by the columns of $\left(\begin{smallmatrix} A-I \\ B-I \end{smallmatrix}\right)$ to obtain $P_q$. Use the integral images of the peripheral differences. Smith transformations also supply every $w_q$, the cusp order-two representative, and the tests for their images and generated subgroups in $G_q$.

12. For the parameters $v = 0, 1$ of 12.1, all six conditions hold; $G_1, \dots, G_4$ are each $\mathbb{Z}/2$, $P_2 = P_4 = 0$, all four finite-cap images are nonzero, and the cusp image is zero. For $v = 2$ they are each $\mathbb{Z}/6$, $P_2 = P_4 = \mathbb{Z}/3$, and the first, second, fourth and fifth conditions fail.

13. For the family matrices $A, B$ in (12.6), which are geometric cohomology pullbacks, the geometric homology inputs are $(A^t, B^t)$. Their left cohomology transport in (13.24) is $(A^{-1}, B^{-1})$. For comparison with computations written using $(A, B)$, simultaneous reversal of the two peripheral generators sends

$$g_1 \mapsto g_1^{-1}, \qquad g_2 \mapsto g_2^{-1}, \qquad g_0 = (g_1 g_2)^{-1} \mapsto g_2 g_1 = g_1^{-1}(g_1 g_2) g_1. \tag{13.34}$$

    This is an orientation-reversing marking of the pair of pants fixing its three ends: each peripheral conjugacy class is inverted. The image of the cusp generator is $g_1^{-1} g_0^{-1} g_1$. Inversion preserves invariant and coinvariant lattices, and the indicated conjugating generator

acts trivially on global coinvariants. All boundary classes in (13.25) have order two, so reversal signs do not change the tests.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Corollary 13.6** (Classification of the three-connected data)**.** [tag 02NL]

*For the marked Kuhn cap and affine translation clutchings used throughout this section, the following is a complete constructive classification of the data with $\pi_1 = 1$ and $H_2 = H_3 = 0$, up to integral conjugacy of their linear tuples and oriented diffeomorphism of their fillings.*

*There are no data with three genuine finite fillings. With two fillings their affine orders are five and four. Order the order-five point first. The complete, unbounded parameter set is the rational projective normal form of 12.5 together with its exceptional rational affine line. For each parameter, run the terminating order, conductor, stable-lattice, and primitive-extension algorithm of 12.7. Discard precisely the integral classes failing (13.23) or one of (13.25). The classes that pass these tests are exactly all integral linear tuples with three-connected gluing. For each such class all simply connected affine twists are exactly the choices satisfying (13.21); their fillings are all oriented diffeomorphic to its normalized filling. If ordered labels with the order-four point first are desired, interchange the finite labels and recompute $M_0 = (M_1 M_2)^{-1}$.*

*Every such datum has*

$$\pi_1 = 1, \qquad H_j = \begin{cases} \mathbb{Z}, & j = 0, 8, \\ \mathbb{Z}^4, & j = 4, \\ 0, & \text{otherwise}, \end{cases} \tag{13.35}$$

*and its oriented smooth manifold is $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$. It has bounded-cusp holomorphic periods in the sense of 16.5.*

*The fundamental group and the integral homology of every other gluing are computed by theorems 14.1 and 14.6.*

*Proof.* 1. The exclusions 11.1, 11.10, 13.1, and 13.3 leave exactly the stated minimal orders and rational quotient problem. The rational normal form is injective on conjugacy classes, including the exceptional line. The finite interval and extension-coset enumeration in 12.7 gives every integral class over each rational point, with all bounds proved for that input. No distinct rational points can become integrally conjugate, since an integral conjugacy induces a rational quotient conjugacy. Theorem 13.4 and the necessary-and-sufficient criterion 13.5 prove both directions of the retention rule and account for every affine twist. This is also an exhaustive enumeration if desired: enumerate primitive projective integer triples with first nonzero coordinate positive by increasing height, and enumerate reduced rational exceptional parameters by increasing numerator and denominator height. Every finite stage has finitely many inputs, each terminates by 12.7, and every rational input occurs.

2. Hurewicz makes these simply connected manifolds three-connected. The only singular contribution to Euler number is the Kuhn cusp, whose value is six by theorem 2.6; torus bundles and the free finite caps have Euler number zero. Integral Poincaré duality and the universal coefficient theorem then give (13.35), including torsion freeness.

3. For the smooth classification, 11.2 supplies a compatible complex structure at each finite cap, and the cusp already has its complex model. Choose their common fibre orientation to be the one induced by the cusp. Commuting complex structures at the finite caps can be chosen in that component: their two-dimensional fixed real summands allow the orientation to be reversed when necessary. The fibre of the bundle of orientation-compatible complex structures retracts to $\mathbb{CP}^3$. Its connectedness and simple connectivity extend the prescribed boundary structures across the two-dimensional pair of pants, exactly as in lemma 2.9. A

compatible horizontal distribution gives an almost complex structure on the total manifold. Proposition 4.1 therefore gives signature two, intersection lattice $I_{3,1}$, and $c_2^2 = 2$. The vanishing $H^2 = 0$ gives $c_1 = w_2 = 0$, so $p_1/2 = -c_2$. The spin Wu identity makes this class characteristic, and its square is two. A nontrivial integral multiple of a lattice vector has square divisible by the square of that integer; hence this norm-two class is primitive. The complete integral reflection, Wall-form and Kramer–Stolz argument in 6.2 depends only on these almost complex characteristic numbers and three-connectivity. It applies without alteration and gives the claimed exact oriented diffeomorphism. Finally 16.5 applies to the resulting determinant-one monodromy and primitive cusp. It gives the punctured-base periods with bounded cusp correction.

□

## 14 Fundamental groups and cellular models

**Theorem 14.1** (Complete fundamental-group data for every input)**.** [tag 02NM]

*Let $k = 2$ or $3$, and let an integral rank-six tuple satisfy the finite-order and primitive maximal-cusp conditions of definition 10.1. Specify each finite affine action as*

$$\tau_i(x) = M_i x + v_i/m_i, \qquad v_i \in \Lambda = \mathbb{Z}^6, \quad \tau_i^{m_i} = 1 \text{ on } \Lambda_{\mathbb{R}}/\Lambda, \tag{14.1}$$

*and require its cyclic action to be free. The numerator need not be fixed by $M_i$. Define its integral power vector and the cusp lattice by*

$$w_i = \frac{1}{m_i} \sum_{j=0}^{m_i-1} M_i^j v_i \in \Lambda^{M_i}, \qquad U = \operatorname{im}\big((M_1 \cdots M_k)^{-1} - I\big). \tag{14.2}$$

*Use left conjugation transport for the peripheral generators $s_i$, and record the integral translation clutching by $\kappa \in \Lambda$ in $s_1 \cdots s_k x^\kappa = 1$. Then the full fundamental group is*

$$\left\langle x^\lambda, s_1, \dots, s_k \;\middle|\; \begin{array}{l} x^\lambda x^\mu = x^{\lambda+\mu}, \quad x^u = 1 \ (u \in U), \\ s_i x^\lambda s_i^{-1} = x^{M_i\lambda}, \quad s_i^{m_i} = x^{w_i}, \quad s_1 \cdots s_k x^\kappa = 1 \end{array} \right\rangle. \tag{14.3}$$

*Only a basis of $\Lambda$ and of $U$ is needed in this finite presentation. The formulas below determine the injected quotient of the fibre subgroup.*

*Put*

$$L = \sum_{g \in \langle M_1, \dots, M_k \rangle} gU, \qquad A = \Lambda/L. \tag{14.4}$$

*Every element of $L$ is a finite integral sum of elements of the subgroups $gU$. For two points, put $d = \gcd(m_1, m_2)$, $M = M_1$, and $S_j(M) = I + M + \cdots + M^{j-1}$. Choose integers $a, b$ with $am_1 + bm_2 = d$, and set, in $A$,*

$$\begin{aligned} w_2' &= -w_2 - S_{m_2}(M)\kappa, \\ e &= (m_2/d)w_1 - (m_1/d)w_2', \qquad z = aw_1 + bw_2', \qquad \overline{A} = A/\langle e \rangle. \end{aligned} \tag{14.5}$$

*There is an exact sequence and a complete marked extension description*

$$1 \longrightarrow \overline{A} \longrightarrow \pi_1 \longrightarrow C_d \longrightarrow 1, \qquad txt^{-1} = M(x), \quad t^d = z. \tag{14.6}$$

*Thus its order is $d|\overline{A}|$ if $\overline{A}$ is finite; otherwise it is infinite. In particular $d = 1$ gives precisely the abelian group $\overline{A}$.*

*For three genuine points, so $m_i \geq 2$, let*

$$\Gamma = \langle q_1, q_2, q_3 \mid q_i^{m_i} = 1, \ q_1 q_2 q_3 = 1 \rangle, \qquad \epsilon = \sum_i 1/m_i - 1. \tag{14.7}$$

*The action $q_i \mapsto M_i$ is well defined on $A$. If $\epsilon \leq 0$, the exact sequence is*

$$1 \longrightarrow A \longrightarrow \pi_1 \longrightarrow \Gamma \longrightarrow 1. \tag{14.8}$$

*If $\epsilon > 0$, $\Gamma$ is the finite spherical triangle group. Using its action on $A$, form the exact transfer vector*

$$e = \sum_{g \in \Gamma} g\kappa + \sum_{i=1}^{3} \sum_{g \in \Gamma/\langle q_i \rangle} g w_i. \tag{14.9}$$

*The second sums use left coset representatives and are independent of their choice. In this case the exact sequence has kernel $A/\langle e \rangle$, with the same lifted relations (14.3). Consequently (14.3), (14.6), (14.8), and (14.9) give every such fundamental group by finite operations, including the precise injected abelian kernel and all extension data.*

*For every input, the full first integral homology is the Smith cokernel of the integer matrix*

$$\begin{pmatrix} M_1 - I & \cdots & M_k - I & -w_1 & & \cdots & & -w_k & \kappa \\ 0 & \cdots & 0 & & \operatorname{diag}(m_1, \ldots, m_k) & & & & \mathbf{1} \end{pmatrix}. \tag{14.10}$$

*Here the bottom left zero block has $k$ rows.*

*Proof.* 1. *Finite and cusp caps.* The affine lift on $\mathbb{R}^6$ has $m_i$th power translation by $w_i$, as the displayed sum shows. It commutes with that power, so $M_i w_i = w_i$. The group generated by lattice translations and this lift is the fundamental group of the reduced flat six-manifold. Its presentation consists of the lattice, its $s_i$ conjugation relations, and $s_i^{m_i} = w_i$. The finite cap deformation retracts onto that reduced manifold by the equivariant radial contraction of its root disc. Thus these are the cap relations, even for a numerator which is not an invariant lattice vector.

2. For the primitive Kuhn cap, the compact toric strata in its normalization and all their intersection components are simply connected. Applying van Kampen successively to their tubular neighborhoods identifies the fundamental group of the central fibre with that of its dual three-torus. The nearby fibre map kills the angular lattice $U$ and sends the long-period lattice $\Lambda/U$ isomorphically onto this dual-torus group. This is also the explicit toric collapse used in theorem 2.6. The primitive index-one condition gives $U = \ker(M_0 - I)$ and makes $\Lambda/U$ free of rank three. A section through a smooth central stratum bounds the base meridian by its parameter disc. The smooth cap retracts onto the central fibre, so its filling imposes exactly these relations.

3. Over the punctured sphere the bundle has a section up to homotopy, because the base retracts to a graph. Its group is the appropriate semidirect product. Choose the cap sections and retain their relative translation clutching as $\kappa$. Van Kampen with the boundary maps just described gives exactly (14.3). The opposite transport convention is related by the following explicit isomorphism: reverse the ordered list and replace each affine map by its inverse, and replace $\kappa$ by $-\kappa$. The new linear matrices are $M_k^{-1}, \ldots, M_1^{-1}$, the power vectors are $-w_k, \ldots, -w_1$, and the cusp image lattice is unchanged. The substitutions $s_i \mapsto s_{k+1-i}^{-1}$ give the explicit isomorphism between these two presentations. This supplies the dictionary when geometric transport is recorded with opposite path-composition order.

4. *Integral closure and two-point elimination.* In a normal abelian fibre subgroup, the normal closure of $U$ is precisely the integer sum (14.4). Start with its column Hermite basis and repeatedly append its images under every $M_i$ and $M_i^{-1}$, then take Hermite normal form again. If the rational rank increases, it can do so at most three times. At the first nonincrease the rational span is invariant under every generator and its inverse, hence never increases again. If the current lattice has saturation index $D$ in that span, every subsequent

strict enlargement divides this index by an integer at least two. There can therefore be at most $\lfloor \log_2 D \rfloor$ such enlargements. This proves termination with a computable bound. At every step the Hermite basis represents the integer sum of the generated columns.

5. On $A$ one has $M_2 = M^{-1}$ in the two-point case. The product relation eliminates $s_2 = t^{-1}x^{-\kappa}$, where $t = s_1$. Multiplying this word $m_2$ times gives

$$(t^{-1}x^{-\kappa})^{m_2} = t^{-m_2}x^{-S_{m_2}(M)\kappa}.$$

Thus the two power relations become $t^{m_1} = w_1$ and $t^{m_2} = w_2'$. Both right sides are fixed by $M$ on $A$. Also $M^{m_1} = M^{m_2} = 1$, hence $M^d = 1$ there. Bezout elimination gives exactly $e = 0$ and $t^d = z$ in (14.5). Replacing the Bezout pair changes $z$ by a multiple of $e$ and therefore changes no extension datum.

6. To prove the asserted injection, construct $\overline{A} \rtimes_M \mathbb{Z}$ and quotient by its central element $t^d z^{-1}$. This cyclic subgroup has nonzero projection $d\mathbb{Z}$ to the infinite cyclic factor, so it intersects $\overline{A}$ trivially. The quotient has precisely the presentation obtained by elimination. This proves (14.6).

7. *The three-point kernel and the finite transfer.* We compute the fibre injection by the homotopy sequence of a fibration over the orbifold classifying space. Let $\mathcal{B}$ be the orbifold classifying space obtained from the three-holed sphere by attaching, at its three boundary circles, the mapping cylinders of $S^1 \to BC_{m_i}$. Over the punctured sphere take the bundle with fibre $BA = K(A, 1)$ and the action. For each boundary, the cyclic extension

$$E_i = \langle A, s_i \mid s_i a s_i^{-1} = M_i(a),\ s_i^{m_i} = w_i \rangle$$

contains $A$ injectively: the same semidirect-product construction as above applies because $M_i^{m_i} = 1$ and $M_i w_i = w_i$. The associated fibration $BA \to BE_i \to BC_{m_i}$ restricts on $S^1$ to the required mapping-torus fibration. Replace the maps by their mapping-path fibrations before homotopy gluing, so that the fibre is preserved on every attaching cylinder. Glue these three fibrations, using the relative clutching $\kappa$ on the punctured sphere. Van Kampen gives (14.3) for the resulting total space. Thus the homotopy sequence of this fibration over $\mathcal{B}$ calculates precisely the image of $A$ in the group under discussion.

8. When $\epsilon \leq 0$, the universal orbifold cover is the plane or hyperbolic plane. Its contractibility identifies $\mathcal{B}$ with $B\Gamma$: indeed its Borel construction has the just-described boundary-cylinder model and contractible universal fibre. The Euclidean and hyperbolic alternatives are the oriented two-orbifold classification in (Scott, 1983, Theorems 2.3–2.4, p. 425; Euler-characteristic comparison, pp. 427–428). Since $\pi_2(B\Gamma) = 0$, the homotopy sequence proves (14.8).

9. When $\epsilon > 0$, the universal orbifold cover is $S^2$. Pulling the constructed $BA$ fibration to this finite cover trivializes its local system. Its transgression $\pi_2(S^2) \to A$ is its integral clutching vector. There is one lifted ordinary base cell for each $g \in \Gamma$, contributing $g\kappa$, and one lifted cone cap for each left coset of $\langle q_i \rangle$, contributing $gw_i$. To check the sign, the base relation has fibre term $+\kappa$ and each cone relation has term $-w_i$; the oriented sphere chain subtracts the cone cells from the base cell. The result is therefore the plus sign in both sums of (14.9). Changing a coset representative by $q_i^j$ does not change its contribution because $w_i$ is fixed. The result is $\Gamma$ invariant. The long homotopy sequence now says that the only additional kernel is exactly $\langle e \rangle$. This proves the spherical assertion, with the orientation fixed by this cell calculation.

10. After sorting its orders, the spherical possibilities are $(2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$, with groups of orders $2n, 12, 24, 60$, respectively. These follow directly from the positive

spherical-triangle angle condition. Construct the dihedral, tetrahedral, octahedral, or icosahedral permutation group and find the marked peripheral generators in that finite group. Breadth-first words supply their actions on $A$; the generator-edge equalities are evaluated modulo the lattice $L$. Compute the two finite sums in (14.9) and their integer Smith form.

11. An order-one site is not a genuine cone. Its matrix is the identity and its relation is $s_i = x^{w_i}$. Eliminate it from (14.3); moving this vector to the right of the remaining peripheral product replaces the clutching by

$$\kappa + (M_{i+1} \cdots M_k)^{-1} w_i. \tag{14.11}$$

This reduces the three-point case with such a site to the two-point computation. A primitive rank-three cusp cannot remain with only one finite linear monodromy: a finite-order unipotent matrix is the identity.

12. *Freeness and abelianization.* Freeness of (14.1) is itself decidable exactly. A nontrivial stabilizer in a cyclic group contains a subgroup of prime order, so it is enough to check $\tau_i^{m_i/p}$ for every prime $p \mid m_i$. For an affine map $Rx + b$ of a torus, a fixed point exists precisely when every integral covector in $\ker(R^t - I)$ takes an integral value on $b$. Smith form proves this criterion: the image of $R - I$ on the torus is the connected subtorus annihilated by those primitive covectors. Compute the powers in the rational affine matrix group and apply these finite tests.

13. Abelianizing (14.3) yields (14.10). The inclusion

$$\operatorname{im}(M_0 - I) \subset \sum_i \operatorname{im}(M_i - I)$$

follows by telescoping the product on the right; inversion preserves the image of its difference from the identity. The Smith form therefore computes every free and torsion summand of $H_1$ exactly.

The bounds in this construction depend on the input.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 14.2** (A complete three-point logarithmic family)**.** [tag 02NN]

*Start with the compact complex fourfold of 3.11 and its normalized order-five and order-four affine fillings. Let $\ell$ be the primitive invariant covector on fibre homology, with the sign chosen in 13.4. At any additional ordinary fibre choose $v \in \mathbb{Z}^6$ and an integer $m \geq 2$ such that*

$$r = \ell(v), \qquad \gcd(r, m) = 1. \tag{14.12}$$

*Perform the marked logarithmic transformation with translation $v/m$ described below. This gives a smooth compact complex fourfold $X_{m,v}$ with Euler number six and three genuine finite affine fillings, whose linear orders are five, four and one and whose affine orders are five, four and $m$. Its fundamental group is*

$$\pi_1(X_{m,v}) = \langle a, b, c, h \mid h \text{ central},\ a^5 = h,\ b^4 = h^{-1},\ c^m = h^r,\ abc = 1 \rangle. \tag{14.13}$$

*This is the group of the oriented Seifert manifold with invariants $(0; (5,-1), (4,1), (m,-r))$, in the convention $x_i^{\alpha_i} h^{\beta_i} = 1$ and product $x_i = 1$. Its base orbifold is hyperbolic, and its central fibre $h$ has infinite order.*

*Put $C_{m,r} = \mathbb{Z}/(20r-m)\mathbb{Z}$, with $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$, and let $\varepsilon = 1$ when $20r = m$ and zero otherwise. All the integral homology groups are*

| $j$ | $H_j(X_{m,v};\mathbb{Z})$ |
|---|---|
| $0, 8$ | $\mathbb{Z}$ |
| $1, 6$ | $C_{m,r}$ |
| $2$ | $\mathbb{Z}^{\varepsilon} \oplus (\mathbb{Z}/m)^5$ |
| $3$ | $(\mathbb{Z}/m)^{10}$ |
| $4$ | $\mathbb{Z}^4 \oplus (\mathbb{Z}/m)^{10}$ |
| $5$ | $(\mathbb{Z}/m)^5$ |
| $7$ | $\mathbb{Z}^{\varepsilon}$. |

(14.14)

*Under (14.12) the exceptional free-rank case is exactly $(m, r) = (20, 1)$. In particular $(m, r) = (19, 1)$ and $(21, 1)$ have $H_1 = 0$ but nontrivial, infinite fundamental groups. More generally this occurs whenever $|20r - m| = 1$. The result holds for every vertical component of $v$ in $\ker \ell$.*

*Proof.* 1. We first specify the complex construction and its marking. Over a small disc at the ordinary value use a holomorphic period marking $T_t = \mathbb{C}^3/\Lambda_t$. Let $\omega_v(t)$ be the holomorphic period vector corresponding to $v$. On the root disc $t = s^m$ quotient the pulled-back family by

$$(s, Z) \longmapsto (\zeta_m s,\ Z + \omega_v(s^m)/m). \tag{14.15}$$

Condition (14.12) makes $v$ have exact order $m$ in $\mathbb{Z}^6/m\mathbb{Z}^6$. Every nontrivial power therefore acts freely, including over $s = 0$. The quotient is smooth, has a proper map to the disc, and has central fibre of multiplicity $m$. On the punctured disc its holomorphic identification with the original family is

$$(s, Z) \longmapsto \left(s^m,\ Z - \frac{\log s}{2\pi i}\omega_v(s^m)\right). \tag{14.16}$$

Changing the branch of $\log s$ changes the second coordinate by a lattice period. Changing $s$ by $\zeta_m s$ changes its logarithm by $2\pi i/m$ and cancels the translation in (14.15). Thus (14.16) is a well-defined holomorphic isomorphism of the quotient with the original torus family off zero. Gluing on a fixed annulus supplies the asserted compact complex manifold: its charts are the original charts and the free finite quotient charts (14.15), and it is the union of compact pieces over closed base regions. The gluing is an isomorphism on collars, so the total space is Hausdorff and smooth. The Euler number is unchanged, since the removed ordinary torus cap, the new cap, and their boundary all have Euler number zero. This marking fixes the new relation $c^m = v$ and zero extra cusp clutching.

2. In the original datum the invariant normal closure of the cusp angular lattice is exactly $\ker \ell$, as proved in lemma 2.9. Adding the identity linear generator does not change that lattice. The fibre image in van Kampen therefore reduces to the central generator $h$, while the original two finite relations remain $a^5 = h$, $b^4 = h^{-1}$. The new relation is $c^m = h^{\ell(v)}$ and the base relation is $abc = 1$. This proves (14.13); the presentation depends on $v$ only through $r = \ell(v)$.

3. One may verify the Seifert identification directly: take a product circle bundle on a three-holed sphere and attach its three solid tori with primitive meridian slopes $(5, -1), (4, 1), (m, -r)$. They are primitive by (14.12), and van Kampen gives exactly (14.13). Its orbifold base has $1/5 + 1/4 + 1/m < 1$ for every $m \geq 2$. Pulling back the circle fibration to its contractible hyperbolic orbifold cover gives a circle bundle on the plane, hence the trivial circle bundle. The subsequent universal cover is the plane times $\mathbb{R}$, and the fibre subgroup is infinite cyclic. This also proves directly that none of these groups is trivial. The rational Euler number in these conventions is $r/m - 1/20$.

4. In abelianization the relations $5a = h$, $4b = -h$ give $a = 4x$, $b = -5x$, $h = 20x$ because five and four are coprime. The product relation gives $c = x$. The remaining relation is $(m - 20r)x = 0$. Thus $H_1 = C_{m,r}$ exactly, including its integral generator and the exceptional infinite-cyclic case.

5. We now compute the higher integral groups, including all extensions. Write $\mathcal{R}^q$ for the higher direct-image sheaf of the original fourfold and $\mathcal{R}^q_m$ for that of $X_{m,v}$, after identifying the bases and their ordinary local systems. At the new point the reduced finite quotient is a torus with lattice $\mathbb{Z}^6 + \mathbb{Z}(v/m)$. Its covering by the original torus is cyclic of degree $m$. The degree-one pullback has Smith factors $1, 1, 1, 1, 1, m$, so its degree-$q$ cokernel is

$$C^q = (\mathbb{Z}/m)^{\binom{5}{q-1}}, \qquad 1 \le q \le 6. \tag{14.17}$$

   The integral cohomology of this torus cap is free. The covering pullback at the new point gives the sheaf comparison

$$0 \longrightarrow \mathcal{R}^q_m \longrightarrow \mathcal{R}^q \longrightarrow C^q_{\{p\}} \longrightarrow 0. \tag{14.18}$$

   It is the identity away from the new point and the covering image at that point.

6. For $2 \le q \le 5$, all original global sections of $\mathcal{R}^q$ are supported at the original finite point or the cusp, by (13.30)–(13.31). Their map to $C^q_{\{p\}}$ is zero. Consequently their $H^0$ groups are unchanged, their $H^2$ groups are unchanged, and

$$0 \longrightarrow C^q \longrightarrow H^1(\mathcal{R}^q_m) \longrightarrow H^1(\mathcal{R}^q) \longrightarrow 0. \tag{14.19}$$

   The original degree-one base cohomology of these sheaves is zero for $q = 2, 4, 5$ and is $\mathbb{Z}^3$ for $q = 3$. The assertion for $q = 5$ also follows immediately from the original $H^6 = 0$: its bidegree $(1, 5)$ has no incoming or outgoing differential. Thus (14.19) gives $C^q$ in degrees two, four and five, and $\mathbb{Z}^3 \oplus C^3$ in degree three.

7. The original four order-two torsion transgressions remain isomorphisms. Indeed their boundary suspension representatives and their images in global coinvariants are the ones in 13.5; adding the identity monodromy changes none of those coinvariant quotients. These point-supported boundary maps do not depend on the free degree-one transgression, which changes under the new logarithmic gluing. The free degree-four cusp column likewise remains zero, since its boundary torsion representative has zero image in that same unchanged global quotient. It follows that the full filtrations in degrees three and five have just one nonzero term:

$$H^3 = C^2 = (\mathbb{Z}/m)^5, \qquad H^5 = C^4 = (\mathbb{Z}/m)^{10}. \tag{14.20}$$

   In degree four there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \oplus (\mathbb{Z}/m)^{10} \longrightarrow H^4 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

   It splits as an abelian-group extension because its quotient is free, giving the stated middle group.

8. For the remaining free ranks, $H^1(X_{m,v}) = \mathrm{Hom}(C_{m,r}, \mathbb{Z})$ has rank $\varepsilon$. The degree-one global section group has rank one, and the only rational differential involving it is $d_2 : H^0(\mathcal{R}^1_m) \to H^2(\mathcal{R}^0_m)$. Its kernel has rank $\varepsilon$, hence its cokernel also has rank $\varepsilon$. The degree-one parabolic group is rationally zero, and there are no rational degree-two global sections. Therefore $b_2 = \varepsilon$. The finite sheaf cokernel in this degree-one row is $\mathbb{Z}/\gcd(20, m)$: the original invariant section restricts as $20\ell$, and its value at the new covering cokernel is $20r$ modulo $m$. This finite group changes no rational rank.

9. Poincaré duality gives $H_3 = H^5$, $H_4 = H^4$, and $H_5 = H^3$. The universal coefficient theorem now identifies the torsion of $H_2$ with $H^3$ and its free rank with $\varepsilon$. It also gives $H^2 = C_{m,r}$: when $20r \neq m$ this is $\mathrm{Ext}(H_1, \mathbb{Z})$; when $20r = m$ it is the free rank-one group. Duality gives $H_6 = C_{m,r}$ and $H_7 = \mathbb{Z}^{\varepsilon}$. This proves the complete table (14.14).

Exact calculations for the displayed finite examples appear in the programs in Table 1. □

**Theorem 14.3** (Arbitrary finite affine caps)**.** [tag 02NO]

*Let $g(x) = Mx + v/m$ be a free cyclic affine action of affine order $m \geq 2$ on $T^r$, with $M \in \mathrm{GL}(r, \mathbb{Z})$ and $v \in \mathbb{Z}^r$. For every specified degree bound $d$, there is an explicit terminating integer algorithm producing chains and boundary maps through degree $d+1$ for the pair*

$$B = (D^2 \times T^r)/\langle (s, x) \mapsto (e^{2\pi i/m} s, g(x)) \rangle, \qquad Y = \partial B. \tag{14.21}$$

*The cap complex has at most $2^r$ generators in each degree and the boundary complex at most $2^{r+1}$. These are complexes of finite rank in each degree; only their indicated finite prefixes are required to compute $H_j(B)$, $H_j(Y)$ and $H_j(B, Y)$ for $j \leq d$. Their maps include the affine translations and all integral higher coherence terms.*

*Proof.* 1. We give the complete recursive algorithm. All chains below are finite integer dictionaries, so the displayed operations are effective over $\mathbb{Z}$.

2. First consider the normalized bar resolution of $\mathbb{Z}^r$ over $R = \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ and its Koszul resolution $K = R \otimes \bigwedge^* \mathbb{Z}^r$. The latter differential is contraction with $(t_i - 1)_i$. Write $t^a = t_1^{a_1} \cdots t_r^{a_r}$ and put $S_n(t) = (t^n - 1)/(t - 1)$, interpreted as its finite integral Laurent sum for positive, zero or negative $n$. The following $\mathbb{Z}$-linear contraction of the augmented Koszul complex fixes a completely explicit choice:

$$s_K(t^a e_I) = \sum_{j < \min I} t_{j+1}^{a_{j+1}} \cdots t_r^{a_r} S_{a_j}(t_j) e_j \wedge e_I, \tag{14.22}$$

where $\min \varnothing = r + 1$ and indices in $I$ are increasing. The bar contraction is $s_B(t^a[g_1|\cdots|g_q]) = [a|g_1|\cdots|g_q]$; terms with a zero bar entry are zero. Set $p_0 = 1$ and define the $R$-linear projection and homotopy on coefficient-one bar generators recursively by

$$p_q = s_K p_{q-1} d_B, \qquad h_q = s_B(1 - i p_q - h_{q-1} d_B). \tag{14.23}$$

Here $i(e_I)$ is the alternating sum of the bars of the ordered unit vectors in $I$. The recursion decreases degree. The identities for the two contractions give, by induction, $pi = 1$, $dh + hd = 1 - ip$, $ph = hi = h^2 = 0$. For example $s_K^2 = s_B^2 = 0$, $ps_B = s_K p$, and the alternating unit-vector formula gives $pi = 1$; substituting these into (14.23) proves the three side identities in the same induction. Augmenting $R \to \mathbb{Z}$ gives a strong chain contraction of the ordinary torus bar complex to $\bigwedge^* \mathbb{Z}^r$.

3. Translations must enter before this contraction. Let $b_j = \sum_{a=0}^{j-1} M^a v/m$ and let $p_j$ be its coordinatewise fractional part. The $m$ points $p_j$ are distinct because the action is free. Use the full fundamental groupoid of $T^r$ on these points: an arrow from object $a$ to object $b$ is an integer vector $\lambda$, represented by the affine path from $p_a$ to $p_b + \lambda$. Composition adds the integer vectors. The generator acts strictly by shifting objects and, for its $s$th power, by

$$\lambda \longmapsto M^s \lambda + z_s(b) - z_s(a), \qquad z_s(a) = M^s p_a + b_s - p_{a+s} \in \mathbb{Z}^r. \tag{14.24}$$

The subscripts on objects are taken modulo $m$. The nerve of this groupoid is equivariantly homotopy equivalent to the torus: the simplex with successive arrow labels $\lambda_i$ maps affinely to the torus with lifted vertices $p_{a_j} + \sum_{i \leq j} \lambda_i$. Forgetting the objects contracts its chains to the ordinary torus bar complex. An explicit homotopy is the negative alternating prism

which, at each possible position, inserts the zero-labelled arrow to object zero and replaces all subsequent objects by zero. Combining this with (14.23) gives a contraction $(i, p, h)$ of the groupoid chains to $\bigwedge^* \mathbb{Z}^r$ with the same side identities.

4. Use the normalized cyclic bar construction on this strict groupoid action. Its differential is $d_v + \delta$, where $\delta$ is the ordinary group-bar boundary, multiplied by $(-1)^q$ on vertical degree $q$; its first face applies (14.24). The vertical contraction transfers this to cyclic bar chains with exterior coefficients by the finite formulas

$$I = (1 + h\delta)^{-1} i, \qquad P = p(1 + \delta h)^{-1}, \qquad D = p\delta I. \tag{14.25}$$

Inverses mean their alternating geometric series. On a chain of cyclic bar degree $p$ each series has at most $p + 1$ terms, since $\delta$ lowers that degree. Multiplication and the contraction identities directly give $(d_v + \delta)I = ID$ and $P(d_v + \delta) = DP$, and a chain homotopy inverse. Equivalently these are the finite homological perturbation identities, here derived by expanding the finite series. The first term of $D$ is the cyclic bar differential for the exterior action of $M$. Every remaining term lowers cyclic bar degree by at least two.

5. The second contraction used in the algorithm is as follows. The cyclic periodic resolution has one $\mathbb{Z}[C_m]$ generator in each degree, with differential $g-1$ in odd degree and $1+g+\cdots+g^{m-1}$ in even degree. On $g^a$, $0 \le a < m$, its augmented contraction is $\sum_{j=0}^{a-1} g^j$ in even degree and is 1 precisely when $a = m - 1$ in odd degree, zero otherwise. Apply (14.23) with this contraction to compare it with the cyclic bar resolution. Tensor its coefficients with the exterior action of $M$. Transferring the higher terms of $D$ once more by (14.25) terminates: each higher term followed by the horizontal homotopy decreases cyclic degree by at least one. The resulting basis in total degree $n$ is $\{(p, I) : p + |I| = n,\ I \subseteq \{1, \dots, r\}\}$. This proves the claimed $2^r$ bound and termination in every specified degree.

6. **Explicit comparison with the periodic cyclic resolution.** Write $C_m = \langle g \rangle$, $R_m = \mathbb{Z}[C_m]$, and let $e_n$ generate the periodic resolution in degree $n$. Let $B_*$ be the normalized $R_m$-bar resolution, with the insertion contraction $s_B$. Besides the periodic contraction $s_P$ just specified, fix the inclusion recursively by

$$i_0(e_0) = [], \qquad i_n(e_n) = s_B\big(i_{n-1}(d_P e_n)\big), \quad n > 0,$$

and extend it $R_m$-linearly. This is a finite formula. More explicitly, with every summation index between 1 and $m - 1$,

$$i_{2j}(e_{2j}) = \sum_{a_1,\dots,a_j} [g^{a_1}|g|\cdots|g^{a_j}|g],$$
$$i_{2j+1}(e_{2j+1}) = \sum_{a_1,\dots,a_j} [g|g^{a_1}|g|\cdots|g^{a_j}|g].$$

The empty sum of indices gives $i_0(e_0) = []$ and $i_1(e_1) = [g]$. The contraction identity for $s_B$ proves $d_B i = i d_P$ by induction: the chain to which $s_B$ is applied is a cycle, since the preceding comparison commutes with differentials and $d_P^2 = 0$.

7. Define $p$ and $h$ on coefficient-one bar generators by the recursion already used for the torus comparison, substituting $s_P$ for $s_K$, and extend them $R_m$-linearly. Their values are given by the following formulas. For $1 \le a_i < m$, put $\epsilon(a, b) = \lfloor (a + b)/m \rfloor$. The projection is

$$p_{2j}[g^{a_1}|\cdots|g^{a_{2j}}] = \prod_{\nu=1}^{j} \epsilon(a_{2\nu-1}, a_{2\nu})\, e_{2j},$$
$$p_{2j+1}[g^{a_1}|\cdots|g^{a_{2j+1}}] = \left(\sum_{b=0}^{a_1-1} g^b\right) \prod_{\nu=1}^{j} \epsilon(a_{2\nu}, a_{2\nu+1})\, e_{2j+1}.$$

These expressions follow by induction from $p_n = s_P p_{n-1} d_B$: the terms other than the first bar face are annihilated after the preceding contraction identity is used. In $pi(e_{2j})$ only the summand with every $a_\nu = m - 1$ survives. The same observation, with initial factor 1, proves $pi(e_{2j+1}) = e_{2j+1}$. Thus $pi = 1$ in every degree.

8. The recursion $h = s_B(1 - ip - hd_B)$ gives $d_B h + h d_B = 1 - ip$ by induction. Moreover $ps_B = s_P p$ follows from the displayed projection formulas, while $s_P^2 = s_B^2 = 0$ and $s_B i = 0$ follow directly from their definitions. Substitution in the same recursion gives $ph = hi = h^2 = 0$. We therefore have the strong comparison required for the second finite perturbation series, including its inclusion, projection, homotopy and side identities. After tensoring with the exterior coefficient module, the same identities hold. Every higher perturbation followed by the horizontal homotopy lowers cyclic degree, proving termination with the $2^r$ generator bound.

9. The free action makes the Borel construction equivalent to $T^r/C_m$, and radial contraction identifies this with $B$. For $Y$ use exactly the same algorithm on $T^{r+1}$ with matrix $\operatorname{diag}(1, M)$ and numerator $(1, v)$. The projection forgetting its first circle is equivariant and represents $Y \to B$ under the radial contraction. More generally an equivariant integral matrix $L$ acts on groupoid arrows as $\lambda \mapsto L\lambda + k_b - k_a$, where $k_a = Lp_a - p'_a \in \mathbb{Z}^{r'}$. Transferring this strict bar map by $P'LI$ gives the required integer chain map. Its mapping cone computes the pair. Every series is finite with the proved degree bound, and integer kernels and Smith normal forms finish the calculation.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 14.4** (The marked Kuhn cusp pair)**.** [tag 02NP]

*The fixed primitive rank-three Kuhn cusp admits an explicit finite integer chain complex for its cap $W$, its boundary $Y$, and the inclusion $Y \to W$. It is obtained from a category with 26 objects, 170 nonidentity morphisms, and respectively $26, 170, 288, 144$ strict flags of lengths $0, 1, 2, 3$. The construction records the full integral collapse map. In particular it gives*

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\operatorname{rank} H_j(W)$ | 1 | 3 | 9 | 9 | 10 | 3 | 1 | 0 | 0 |
| $\operatorname{rank} H_j(Y)$ | 1 | 4 | 12 | 18 | 18 | 12 | 4 | 1 | 0 |
| $\operatorname{rank} H_j(W, Y)$ | 0 | 0 | 1 | 3 | 10 | 9 | 9 | 3 | 1 |

(14.26)

*The only torsion in these three rows is* $\operatorname{Tor} H_3(Y) = \mathbb{Z}/2$.

*Proof.* 1. Let $\mathcal{K}$ be the periodic Kuhn triangulation of $\mathbb{R}^3$. Normalize every simplex by translating its lexicographically first vertex to zero. The objects are its simplex orbits; a morphism $\tau \to \sigma$ is every strict translated face inclusion $\tau + w \subset \sigma$. Keep distinct translated inclusions as distinct morphisms. Composition adds the translation vectors. Enumerating all nonempty faces of the six chains in the unit cube gives $(1, 7, 12, 6)$ simplex orbits in dimensions zero through three, and the displayed morphism and flag counts. The category nerve is the barycentric cell model of the dual decomposition of $\mathbb{R}^3/\mathbb{Z}^3$; the direction of the face order is reversed when interpreted as dual cells.

2. For an object $\tau$ let $L_\tau \subset \mathbb{Z}^3$ be the primitive span of its vertex differences. Its primitivity follows from unimodularity of every Kuhn simplex. Choose by Smith reduction a quotient map $Q_\tau : \mathbb{Z}^3 \to \mathbb{Z}^{3-\dim \tau}$ with kernel $L_\tau$, and an integral section $R_\tau$. Define two strict diagrams of tori on this category:

| | object at $\tau$ | map for $\tau + w \subset \sigma$ |
|---|---|---|
| $\mathcal{W}$ | $T^{3-\dim \tau}$ | $Q_\sigma R_\tau$ |
| $\mathcal{Y}$ | $T^4$ | $S_w = \begin{pmatrix} 1 & 0 \\ w & I_3 \end{pmatrix}$. |

(14.27)

The first maps compose because $Q_\sigma$ vanishes on $L_\tau$, and the second compose because translations add. The maps $[\,0\ Q_\tau\,]$ form a natural transformation $\mathcal{Y} \to \mathcal{W}$: the only additional term is $Q_\sigma w = 0$, since $w$ is a vertex of $\sigma$.

3. Here these diagrams describe the local topology. The toric moment quotient on the dual cell to $\tau$ has angular torus $T^3/(L_\tau \otimes \mathbb{R}/L_\tau)$; passing to a face collapses exactly the next angular subtorus. Barycentrically subdividing the dual cells gives the homotopy colimit of $\mathcal{W}$ for the central fibre. For its real oriented logarithmic boundary, retain the argument $\theta_0$ of the base parameter and the three angular coordinates $\theta$ of the toric chart. Translating the chart by $w$ changes them to $(\theta_0, \theta + w\theta_0)$, which is precisely $S_w$. Thus the logarithmic boundary is the homotopy colimit of $\mathcal{Y}$, and forgetting the logarithmic phases on each stratum is exactly the natural transformation just given. This can also be read directly in the unimodular charts $x_0 \cdots x_k = t$: the moment coordinates form the dual simplex, the four logarithmic phases remain, and the ordinary stratum retains their quotient by the vertex differences. The finite face identifications give (14.27).

4. The Kuhn map with its divisor log structures is proper, log smooth and vertical after restricting to a small disc. The logarithmic comparison theorem (Achinger and Ogus, 2020, Theorem 4.1.1 and Lemmas 4.1.2–4.1.3, pp. 480–482) identifies its topological germ with the mapping cylinder of this logarithmic collapse. Consequently the preceding homotopy-colimit diagrams and natural transformation model the pair $(W, Y)$ itself. Fix the marking by letting $b_i$ be the category loop with translation label $e_i$. A homotopy-colimit edge with map $S_{e_i}$ has the relation $b_i^{-1}\theta_0 b_i = \theta_0 + a_i$; equivalently $\theta_0 b_i \theta_0^{-1} = b_i + a_i$. This specifies the required primitive cusp $U = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}$.

5. **Hypotheses and marking of the logarithmic comparison.** Give the small disc its standard divisorial log structure and the Kuhn total space the divisorial structure of the reduced central fibre. In a unimodular chart the characteristic monoid map is

$$\mathbb{N} \longrightarrow \mathbb{N}^{k+1}, \qquad 1 \longmapsto (1, \ldots, 1).$$

   Both monoids are fine and saturated. The group cokernel is free, and the underlying chart is the monomial semistable map $x_0 \cdots x_k = t$ with additional smooth coordinates; this is log smooth. The quotient monoid is a group, since the negative of each coordinate generator equals the sum of the other generators after the diagonal generator is killed. The morphism is therefore vertical. The periodic toric construction is proper over the sufficiently small disc.

6. These are precisely the hypotheses of (Achinger and Ogus, 2020, Theorem 4.1.1, pp. 480–482). Its proof also notes that exactness is automatic over the standard log disc. It identifies the germ with the mapping cylinder of the logarithmic collapse. This homeomorphism depends on a trivialization; Lemma 4.1.3 fixes that trivialization on the logarithmic central fibre. We keep the resulting radial boundary identification and its stated angular and tropical generators. This suffices to model the pair and its inclusion up to marked homotopy.

7. We specify finite integer chains. Replace every torus by its ordinary normalized bar complex. The affine simplex map from the bar construction to the torus is natural for all the integer homomorphisms in (14.27), so this replacement does not change either homotopy colimit or its collapse map. Take the simplicial replacement over all strict flags of the finite category. Its vertical differential is the torus bar boundary; its horizontal differential composes adjacent arrows, with the first face applying the diagram map, and has the sign $(-1)^q$ in vertical degree $q$. Apply the first contraction and the finite series (14.25) object by object.

There are no flags longer than three, so every series has at most four terms. The resulting finite basis consists of a strict flag beginning at $\tau$ and an exterior basis vector of its torus. Transfer the natural transformation by $P_{\mathcal{W}} F I_{\mathcal{Y}}$. This proves both the chain-map identity and completeness of the collapse map, including all integral coherence corrections.

8. The raw cap chain ranks in degrees zero through six are

$$(26, 199, 688, 1231, 1226, 648, 144),$$

and the raw boundary ranks in degrees zero through seven are

$$(26, 274, 1124, 2420, 3010, 2186, 864, 144).$$

Enumerate the objects, translated incidences and flags. Composition and the alternating boundary formulas give $d^2 = 0$ and the collapse-chain identity on their generators. Integral cancellation of a unit pivot is a chain homotopy equivalence: clear its row by integer column operations, then remove the resulting contractible two-generator summand. The adjacent differential updates are forced by $d^2 = 0$. After cancelling unit pairs for the cap, boundary and collapse cone, Smith reduction gives (14.26). The reduced cap and relative differentials vanish, and the boundary has one nonzero Smith factor, equal to two in degree four to three. This computes the integral groups.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Remark 14.5** (Cusp quotient and global chains)**.** [tag PV19] The finite-pair and cusp-pair chain models enter the global gluing cone of theorem 14.6. Its maps to the common torus-bundle complement include the peripheral word and clutching translation.

The periodic toric quotient used for the cusp is the construction in Yang's theorem *Higher-dimensional periodic quotient*, and its Kuhn-grid Euler number is the formula in his corollary *The standard grid family* (Yang, 2026, Theorem 8.7, pp. 57–59, and Corollary 8.8, p. 59, arXiv v1). For the present primitive rank-three grid, this gives Euler number $3! = 6$. The full cap and boundary complexes, their integral inclusion, and the torsion groups are calculated in the proof above.

**Theorem 14.6** (Complete integral homology of a marked gluing)**.** [tag 02NQ]

*For every admissible datum there is a terminating exact integer algorithm computing every integral homology group of the resulting eight-manifold, including every torsion invariant factor. The input consists of the finite homology matrices $M_1, \dots, M_k$, free affine numerators $v_i/m_i$ of their affine orders, the integral clutch vector $\kappa$, and the chosen primitive cusp marking*

$$M_0 = (M_1 \cdots M_k)^{-1}, \qquad M_0 P = P \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}, \qquad P \in \mathrm{GL}(6, \mathbb{Z}). \tag{14.28}$$

*Here $k = 2$ or $3$ and the genuine orders are at least two; order-one sites are first eliminated as in 14.1. If no marking is given, the algorithm chooses one by Smith reduction. The complete chain prefix through degree nine has at most 4003 generators in any degree.*

*Proof.* 1. We construct both boundary maps for the global gluing. Let $F_k$ be the free group with ordered generators $s_1, \dots, s_k$, and let $M(w)$ be the product specified by a reduced word $w$. The torus-bundle complement over the punctured sphere retracts to the corresponding bundle over a graph. Its fundamental group and multiplication are

$$\Gamma = \mathbb{Z}^6 \rtimes F_k, \qquad (x, w)(y, u) = (x + M(w)y, wu). \tag{14.29}$$

The fibre and base are aspherical and the base has no two-cells. Hence a chosen section identifies the complement up to homotopy with $B\Gamma$. These section and peripheral choices are fixed by the input clutch convention below.

2. *A finite complement complex with its full bar projection.* Use reduced words for $F_k$. The tree resolution of the free group has $K_0 = \mathbb{Z}[F_k]$ and $K_1 = \bigoplus_i \mathbb{Z}[F_k]e_i$, with $de_i = s_i - 1$, and no higher terms. Its explicit augmented contraction sends a word to its signed path in the Cayley tree: for each positive letter $s_i$ add its preceding prefix times $e_i$, and for a negative letter $s_i^{-1}$ subtract the prefix including that letter times $e_i$. This is the Fox derivative, and its boundary is $w - 1$. Together with the bar insertion contraction it gives, by the recursion (14.23), a strong contraction of the free-group bar resolution to this tree resolution. Every recursion decreases bar degree and every word is finite.

3. For a bar simplex $[(x_1, w_1)|\cdots|(x_n, w_n)]$ of $\Gamma$, put $y_j = M(w_1 \cdots w_{j-1})x_j$. The exact Alexander–Whitney chain map to the torus/free-group double bar complex is

$$\sum_{p+q=n} (-1)^{pq}[w_1|\cdots|w_p] \otimes [y_{p+1}|\cdots|y_n]. \tag{14.30}$$

   Terms with an identity bar entry are zero. The horizontal first face acts on torus vectors by $\rho(w) = M(w)^{-1}$, and the horizontal boundary has sign $(-1)^q$. Notice that $\rho$ is a right action: $\rho(wu) = \rho(u)\rho(w)$. This order is essential when the matrices do not commute. Formula (14.30) follows from the diagonal bisimplicial bar construction; alternatively, its chain-map identity follows by pairing adjacent interior faces, with the first face using $M(w_1)^{-1}y_j = M(w_2 \cdots w_{j-1})x_j$. The usual shuffle inverse is finite in each degree, so the map is a chain equivalence.

4. Apply the torus contraction and (14.25), then the free-group tree contraction and the same finite perturbation formula. Higher terms decrease group bar degree at least two, whereas the tree complex only has degrees zero and one. Thus the resulting complement complex is

$$C_n(E) = \bigwedge^n \mathbb{Z}^6 \ \oplus \bigoplus_{i=1}^{k} \bigwedge^{n-1} \mathbb{Z}^6, \qquad d(e_i \otimes \xi) = (-1)^{n-1}(\bigwedge^{n-1} M_i^{-1} - 1)\xi. \tag{14.31}$$

   Its differential alone has this short form; its projection from group bars retains the finite higher perturbation terms. Those projection terms are used for every boundary map.

5. *Finite affine boundary maps.* For the $i$th finite cap use the orbit groupoid of 14.3. Its one-edge homotopy colimit for the affine generator is the boundary mapping torus. Vertically contracting it gives a bounded complex with $C_n(Y_i) = \bigwedge^n \mathbb{Z}^6 \oplus \bigwedge^{n-1} \mathbb{Z}^6$ and differential $(-1)^{n-1}(\bigwedge^{n-1} M_i - 1)$ on the second summand. Its inclusion into the cyclic bar construction, followed by the cap projection in (14.25), is the full map $Y_i \to B_i$.

6. Here is an explicit group-bar map for its other inclusion. Send a torus-groupoid arrow labelled $\lambda$ to the translation $x^\lambda \in \Gamma$. With the integral offsets $z_1(a)$ in (14.24), put $\eta_a = s_i^{-1}x^{z_1(a)}$. For an arrow $f : a \to b$, the exact identity

$$\eta_a \, \psi(gf) = \psi(f) \, \eta_b \tag{14.32}$$

   is the natural-transformation identity: it is just $s_i^{-1}x^{M_i\lambda + z_1(b)} = x^\lambda s_i^{-1}x^{z_1(b)}$. For a $q$-simplex, the edge map is $(-1)^q$ times the alternating prism which inserts $\eta$ at every position, uses the original arrows before the insertion and the transformed arrows afterwards. Its boundary is precisely the signed difference of the two end maps, by (14.32). Compose this prism with the finite vertical lift of the one-edge complex, then with (14.30) and the complement projection. This proves the full chain-map identity. The offsets record the integer affine lift: replacing $v_i$ by $v_i + m_i\lambda$ changes the offsets and the map, even if the affine map on the torus has not changed. An optional unimodular local basis simply applies its matrix to every translation in (14.32).

7. We make the distinction between a bar cell and the geometric positive root explicit. Put $a_i = z_1(0) = \lfloor v_i/m_i \rfloor$, coordinatewise, and let $\tau_i$ be the positive one-edge bar cell closed using the zero-labelled return arrow to object zero. With the first-face convention above its geometric loop is

$$\tau_i = g_i^{-1} x^{a_i}, \qquad F_E(\tau_i) = -e_i + M_i^{-1} a_i \in C_1(E), \qquad g_i = x^{a_i} \tau_i^{-1}. \tag{14.33}$$

Thus the bar cell has the opposite peripheral orientation, with the displayed integer correction when the numerator has a nonzero integer part. Formula (14.32) sends its group element to $s_i^{-1} x^{a_i}$; consequently the geometric positive root in the last expression of (14.33) maps to $s_i$ exactly. This is the stated conversion between the boundary and complement chain conventions. In a local basis $Q$, replace $a_i$ in the group and complement formulas by $Q\lfloor Q^{-1} v_i/m_i \rfloor$. Writing $\mathcal{N}_i = \sum_{j=0}^{m_i-1} M_i^j$, the cyclic degree-two cell has boundary $m_i\tau_i + w_i - \mathcal{N}_i a_i$. Indeed $\tau_i^{m_i} = x^{\mathcal{N}_i a_i - w_i}$ follows directly from $g_i^{m_i} = x^{w_i}$. This establishes the affine power relation and the degree-one prism sign for every integer numerator.

8. There is an optional exact geometric shortcut. If a verified local integral basis gives $M_i = \mathrm{diag}(1, D)$ and $v_i = \epsilon e_0$, $\epsilon \in \{1, -1\}$, eliminate the first fibre generator from the cap group using $g^{m_i} = u^{\epsilon}$. The cap is the mapping torus with group $\mathbb{Z}^5 \rtimes_D \mathbb{Z}$. The fibre vector $(u, a)$ maps to $x^a g^{\epsilon m_i u}$, and the root generator maps to $g$. Substituting this homomorphism for $\psi$ in the same prism gives the boundary map to this bounded geometric cap model. The shortcut is used only after checking the displayed integral normal form; the general periodic construction remains available for every other input.

9. To check the peripheral convention directly, the universal cover of $Y_i$ has the root generator $g_i$ with linear part $M_i$ and affine displacement $v_i/m_i$. Its integral translations and $g_i$ map to the same fibre translations and $s_i$ in (14.29). The full base-circle translation obeys $g_i^{m_i} = x^{w_i} t$, where $w_i$ is the power vector of 14.1. Filling the disc kills $t$, giving exactly $s_i^{m_i} = x^{w_i}$. The straight paths between the chosen orbit points give (14.32), so the prism is this marked map, including numerators with nonzero integer parts.

10. *The marked cusp boundary map.* Write $P = [A\ B]$ in (14.28), and set

$$w_0 = (s_1 \cdots s_k)^{-1}, \qquad c = w_0 x^{-\kappa} \in \Gamma. \tag{14.34}$$

Send the three angular generators to the columns of $A$, the three tropical generators to the columns of $B$, and the base-phase generator to $c$. Relation (14.28) gives $c b_i c^{-1} = b_i + a_i$ and all the other nilpotent boundary relations. Filling kills the angular generators and $c$, so (14.34) gives exactly $s_1 \cdots s_k x^{\kappa} = 1$, in agreement with 14.1.

11. This homomorphism has a direct chain implementation on the full Kuhn diagram. For an incidence with translation label $w$, assign the group element $x^{Bw}$. A vertical phase $(n, a) \in \mathbb{Z} \oplus \mathbb{Z}^3$ maps to $c^n x^{Aa}$. The incidence shear is $S_w(n, a) = (n, a + nw)$, and

$$(c^n x^{Aa}) x^{Bw} = x^{Bw} (c^n x^{A(a+nw)}). \tag{14.35}$$

For a flag of $p$ incidence arrows and a vertical bar simplex of length $q$, sum over all $(p, q)$ shuffles. Each incidence step contributes its assigned group element; each vertical step contributes its phase transformed by all preceding incidence shears. Give the shuffle the sign $(-1)^{\#\{\text{incidence steps before vertical steps}\}}$. Interior face cancellations use exactly (14.35), and adjacent incidence faces use addition of their translation labels. This is therefore a chain map from the double bar complex to $C_*(B\Gamma)$. Precompose with the finite Kuhn lift of 14.4 and postcompose with (14.30) and the complement projection. The other cusp map is the full logarithmic collapse already computed in 14.4. Both the cusp boundary and the complement are aspherical, so the displayed marked homomorphism also verifies the homotopy class of this geometric inclusion.

12. *The final finite integer calculation.* Take the mapping cone of the just constructed maps

$$\bigoplus_{i=0}^{k} C_*(Y_i) \longrightarrow C_*(E) \oplus \bigoplus_{i=0}^{k} C_*(B_i), \qquad y \longmapsto (f_E(y), -f_B(y)), \tag{14.36}$$

where $B_0 = W$ is the Kuhn cap. Gluing collars makes this the homotopy pushout model of the closed manifold. Consequently its cone computes integral homology.

13. The chain ranks give an explicit finite size bound. Put $t_n = \sum_{q=0}^{\min(n,6)} \binom{6}{q}$, with binomial coefficients outside their range interpreted as zero. Let $w_n$ and $y_n$ be the raw cap and boundary ranks of 14.4, zero outside their listed ranges. Then the cone rank in degree $n$ is

$$\binom{6}{n} + k\binom{6}{n-1} + kt_n + w_n + k\binom{7}{n-1} + y_{n-1}. \tag{14.37}$$

For $0 \le n \le 9$ its maximum is 3890 when $k = 2$ and 4003 when $k = 3$. Every bar comparison and perturbation series used to produce these matrices terminates by its degree filtration. Word reductions and affine orbit enumerations terminate. Computing integer kernels followed by Smith normal forms now gives $H_0, \dots, H_8$; higher groups vanish by dimension. Unit cancellations are optional exact integral chain equivalences which substantially reduce the matrices before Smith reduction. This proves the algorithm and the bound.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Remark 14.7** (An orientation-reversing gluing)**.** [tag PV20] Take the gluing with $A = I_3$, $B = \mathrm{diag}(0, 1, 0)$ in (15.8), affine orders $4, 4$, numerators $a_1, a_2$, and zero clutching. These two linear actions reverse orientation. Integral unit cancellation and Smith reduction of its gluing cone give free ranks

$$(1, 0, 4, 11, 10, 1, 6, 3, 0).$$

The nonzero torsion groups are

$$\begin{aligned} &\operatorname{Tor} H_1 = \operatorname{Tor} H_5 = (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/4, \quad &&\operatorname{Tor} H_2 = (\mathbb{Z}/2)^2, \\ &\operatorname{Tor} H_3 = \operatorname{Tor} H_7 = \mathbb{Z}/2, &&\operatorname{Tor} H_4 = (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^5. \end{aligned}$$

Abelianizing its van Kampen presentation gives the same $H_1$. Its Euler number is six, its mod-two Betti vector is $(1, 4, 10, 14, 20, 14, 10, 4, 1)$, and its integral top groups are those of a closed nonorientable eight-manifold.

# 15 Canonical representatives of integral monodromy

**Theorem 15.1** (Simultaneous integral conjugacy for every input)**.** [tag 02NR]

*There is an unconditional exact algorithm which decides simultaneous* $\mathrm{GL}_6(\mathbb{Z})$ *conjugacy of any two admissible linear tuples and returns a conjugating matrix when one exists. More generally it applies to any integral tuple containing a matrix* $1 + N$ *with* $N^2 = 0$ *and* $\operatorname{rank} N = 3$.

*Proof.* 1. We use column vectors and seek $P$ satisfying $PX_i = Y_iP$ for every marked index, including the cusp index. Put

$$\begin{aligned} H &= \{P \in \mathrm{Mat}_6(\mathbb{Z}) : PX_i = Y_iP \ (\forall i)\}, \\ \mathcal{O} &= \{C \in \mathrm{Mat}_6(\mathbb{Z}) : CX_i = X_iC \ (\forall i)\}, \qquad E = \mathbb{Q}\mathcal{O}. \end{aligned} \tag{15.1}$$

Both lattices are saturated integer kernels of explicitly displayed linear equations in thirty-six unknowns. Thus Hermite or Smith reduction computes bases of these saturated integer kernels.

2. We first decide rational isomorphism. For a basis $h_1, \dots, h_s$ of $H$, form the polynomial $f(c) = \det(\sum c_j h_j)$. If it is zero, there is no rational isomorphism. Otherwise successively substitute one of $0, 1, \dots, 6$ for each variable while keeping the remaining polynomial nonzero. At each stage one of these seven values works, because the degree in that variable is at most six. The final matrix $D \in H$ is invertible over $\mathbb{Q}$. This gives a terminating exact test for rational isomorphism.

3. The full lattice $L = D^{-1}H \subset E$ is a right $\mathcal{O}$-module. Let $J$ be the Jacobson radical of $E$, let $B = E/J$, and use bars for images under the quotient map. We next verify the splitting hypothesis needed for the principal-ideal algorithm. In a rational $N$-adapted basis,

$$N = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}, \qquad E \subseteq \left\{ \begin{pmatrix} A & T \\ 0 & A \end{pmatrix} : A, T \in \mathrm{Mat}_3(\mathbb{Q}) \right\}. \tag{15.2}$$

   Projection to $A$ has square-zero kernel and image an algebra $S \subseteq \mathrm{Mat}_3(\mathbb{Q})$. A nilpotent ideal is contained in the radical, so $E/J \cong S/J(S)$.

4. Choose a composition series of the faithful $S$-module $\mathbb{Q}^3$. The kernel of its action on the associated graded module lowers this filtration and is therefore nilpotent. Conversely the radical kills every simple factor. Thus this kernel is exactly $J(S)$ and every simple factor of $S/J(S)$ acts nontrivially on a composition factor. For a simple algebra $\mathrm{Mat}_n(D)$, where $D$ has centre $K$ and degree $d$ over that centre, every nonzero simple module has $\mathbb{Q}$-dimension $nd^2[K : \mathbb{Q}]$. If $D$ were noncommutative this would be at least four, which is impossible in $\mathbb{Q}^3$. Consequently

$$B \cong \prod_j \mathrm{Mat}_{n_j}(K_j), \qquad n_j[K_j : \mathbb{Q}] \le 3. \tag{15.3}$$

   Every noncommutative factor is therefore $\mathrm{Mat}_2(\mathbb{Q})$ or $\mathrm{Mat}_3(\mathbb{Q})$; the remaining factors are number fields of degree at most three. This proves hypothesis (H) of (Bley et al., 2022, hypothesis (H), p. 2; Algorithm 8.3, p. 21) for every tuple under consideration.

5. The decomposition can be computed as follows. Rational linear algebra computes $J$ as the radical of the regular trace pairing and computes the quotient multiplication table. Its simple factors are obtained from its centre. A commutative factor is a number field. In a noncommutative factor of dimension $n^2$, we enumerate integral combinations $e$ of a rational basis until the left ideal $Be$ has dimension $n$. Such an $e$ exists because the factor is $\mathrm{Mat}_n(\mathbb{Q})$, so its rank-one matrices are rational points; clearing denominators expresses one among the enumerated combinations. Left multiplication on $Be$ then supplies an explicit isomorphism to $\mathrm{Mat}_n(\mathbb{Q})$. The multiplication table verifies the algebra homomorphism on every pair of basis elements.

6. The images $\overline{\mathcal{O}}$ and $\overline{L}$ are a full order and a full right lattice in $B$. If the right order of $\overline{L}$ differs from $\overline{\mathcal{O}}$, this lattice cannot be principal over that order. Otherwise Algorithm 8.3 of (Bley et al., 2022, Algorithm 8.3 and its correctness proof, p. 21), in its right-ideal form, decides principality and returns a generator $\overline{\alpha}$ when one exists. The cited algorithm passes through a maximal order and its central conductor, tests local freeness and principality over the maximal order, and solves the remaining reduced-norm congruence in a finite quotient. Its correctness proof uses its Theorem 4.5(b) and Theorem 7.9; its Theorem 8.4 specifies the number-field, finite-field and integer factorization subproblems. These have unconditional exact algorithms. Use unconditional class-group and unit-group computations for every number-field component before applying the principal-ideal decision.

7. If $\overline{L}$ is not principal, $L$ cannot be principal. If it is principal, solve the integer linear equations expressing $\overline{\alpha}$ as a combination of the images of the chosen basis of $L$. This produces a lift $\alpha \in L$ and an integral intertwiner $P = D\alpha \in H$. We claim that a simultaneous integral conjugacy exists if and only if $|\det P| = 1$.

8. The sufficient direction is immediate. For the necessary direction, suppose $P_0 \in H$ is unimodular. Every $h \in H$ has $P_0^{-1}h \in \mathcal{O}$, so $L = \alpha_0\mathcal{O}$, where $\alpha_0 = D^{-1}P_0$. Write $\alpha = \alpha_0 u$ with $u \in \mathcal{O}$. Because both image elements generate $\overline{L}$, $\overline{u}$ is a unit of $\overline{\mathcal{O}}$. Lift its inverse to $v \in \mathcal{O}$. Then $uv = 1 + j$ with $j \in J \cap \mathcal{O}$, and the finite geometric series for $(1 + j)^{-1}$ shows that $u^{-1} = v(1 + j)^{-1} \in \mathcal{O}$. Thus $u$ is an integral unimodular matrix and $P = P_0 u$ is unimodular. This also proves that any chosen lift of the image generator gives the same yes-or-no answer. It is the right-module form of (Bley et al., 2022, Theorem 3.4, p. 6; Propositions 8.1–8.2, p. 21), with the needed determinant condition written out.

9. Every branch either proves nonexistence or returns a matrix whose integrality, determinant and all intertwining equations are verified exactly. This completes the decision procedure. □

**Theorem 15.2** (A complete canonical-representative algorithm)**.** [tag 02NS]

*Every admissible linear tuple, for either $k = 2$ or $k = 3$, has a computable canonical integral representative and a returned simultaneous unimodular conjugator. Two inputs have the same representative if and only if they are simultaneously $\mathrm{GL}_6(\mathbb{Z})$ conjugate. The algorithm uses a proved finite bound depending on the input. No fixed global entry bound can represent every class.*

*This is an effective classification. The ordered involution sector has the separate complete finite normal form 15.3, consisting of 56 classes.*

*Proof.* 1. We remove the existential free-lift condition by an explicit construction. Let $d > 1$ be the order of $M$ and choose a primitive $u \in \ker(M - I) \cap \mathbb{Z}^6$. Such a vector is obtained from a nonzero rational kernel vector by clearing denominators and dividing by the greatest common divisor of its integer coordinates; both operations retain the equation $(M-I)u = 0$. Put $g(x) = Mx + u/d^2$ and $S_d = \sum_{a=0}^{d-1} M^a$. Then $g^j(x) = M^j x + ju/d^2$, $S_d M^j = S_d$ and $S_d u = du$. A fixed point of $g^j$ gives $(M^j - I)x + ju/d^2 = \lambda \in \mathbb{Z}^6$. Applying $S_d$ gives $ju/d = S_d\lambda \in \mathbb{Z}^6$. A Bezout row pairing to one with $u$ therefore gives $d \mid j$. Writing $j = dr$, the original equation becomes $ru/d = \lambda$, so the same row gives $d \mid r$. Thus only multiples of $d^2$ can have fixed points. Since $g^{d^2}$ is translation by the integer vector $u$, its torus action is the identity. This proves both its affine order $d^2$ and freeness of every nonidentity power. For $M = I$, translation by $u/2$ for primitive $u$ has affine order two and is free. Conversely, when $I - M$ is invertible over $\mathbb{R}$, every affine map $Mx + a$ fixes the torus class of $(I - M)^{-1}a$. The free-lift test is consequently exactly $\ker(M - I) \neq 0$.

2. The finite linear orders here are the already proved list of proposition 10.4:

$$d \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}. \tag{15.4}$$

Taking these finitely many powers, checking the fixed kernel, the product relation and the cusp Smith factors therefore gives an exact admissibility test. The affine order is the specified gluing datum; the construction in item 1 gives its exact relation to the chosen finite lift.

3. For an admissible full tuple $T = (M_1, \ldots, M_k, M_0)$ let $H(T)$ be its maximum absolute entry. There is at most one identity finite slot in a three-point tuple: two such identities would make $M_0$ finite order, whereas $M_0 = I + N$, $N^2 = 0$ and $\operatorname{rank} N = 3$ give $(I + N)^d = I + dN \neq I$ for every positive integer $d$. At a fixed identity slot, delete that matrix and reinsert $I$ at the same slot after classifying the remaining ordered tuple. These two maps are inverse, and both commute with $T \mapsto (QM_1Q^{-1}, \ldots, QM_0Q^{-1})$ for the same $Q \in \mathrm{GL}(6, \mathbb{Z})$. They therefore retain the returned simultaneous conjugator. The remaining invertible integral matrices have height at least one, so inserting $I$ leaves height unchanged. Every entry in the inserted block is constant; the first unequal row-major entry of two tuples is therefore the

same before and after insertion. This proves preservation of the lexicographic comparison as well. The affine order and numerator at that slot are retained as gluing data and are transported by this same $Q$. In the two-involution sector use the canonical representative of 15.3, with its own specified convention. In all remaining sectors, order full integer tuples first by $H$, then lexicographically by their row-major entries in the displayed marked order, and take the first admissible member of the simultaneous conjugacy class.

4. To compute the representative, enumerate heights $1, \dots, H(T)$ and all integer entries in the stated order. There are at most

$$(2H(T)+1)^{36(k+1)} \tag{15.5}$$

integer lists in this box. Retain the invertible tuples satisfying the ordered product relation and the stated maximal square-zero primitive cusp condition, which are finite integer tests, and apply the decision-and-witness algorithm of theorem 15.1 to each candidate and the tuple. Its general rank-three square-zero input permits precisely this call. A positive answer transfers all finite-order, orientation and admissibility conditions through its integral conjugator. The tuple itself occurs, with the identity as a possible witness, so a positive answer is reached in finitely many tests. The first one is the required representative. Record the returned convention as $P_T^{-1} T_i P_T = \mathrm{Can}(T)_i$. The candidate bound depends on $H(T)$; theorem 15.1 constructs the integral witness $P_T$.

5. Every simultaneous orbit has a least element in this order. Its height is at most the height of each given member, so (15.5) includes that global least element. Equivalent inputs have exactly the same ordered orbit and therefore the same representative. Conversely, if $T, T'$ have the same representative, the matrix $Q = P_T P_{T'}^{-1}$ satisfies $Q^{-1} T_i Q = T'_i$ for every slot. The separate involution convention has the same property by theorem 15.3, with the product of its returned witnesses giving the corresponding map. Conjugation preserves the ordered finite orders, so the involution sector is preserved by those same maps. This proves the classification assertion and its finite stopping rule.

6. Finally, 12.1 already gives an unbounded admissible integral family. In its homology convention direct symbolic multiplication gives

$$\mathrm{tr}(M_1 M_2^3) = -v^4 + v^3 - v^2 + 2. \tag{15.6}$$

This is unbounded on the free affine subfamily $v \in 20\mathbb{Z}$ of theorem 12.1. It is invariant under simultaneous conjugacy. A fixed finite entry box contains only finitely many tuples and hence only finitely many such traces; it cannot contain a representative of every class. The input-dependent nature of (15.5) is therefore necessary for this particular exhaustive procedure.

□

**Theorem 15.3** (The complete ordered involution stratum)**.** [tag 02NT]

*There are exactly 56 simultaneous* $\mathrm{GL}_6(\mathbb{Z})$ *conjugacy classes of ordered tuples* $(M_1, M_2, M_0)$ *satisfying the primitive maximal-cusp conditions of definition 10.1 with both finite linear orders equal to two. An explicit integral normalizer returns a canonical representative and an explicit unimodular conjugating matrix for every such integral input. Every representative has free affine lifts of affine order four at both finite points. None can give a simply connected filling, for any choices of free affine orders or twists.*

*Proof.* 1. First choose an integral cusp basis, so that

$$M_0 = I_6 + N, \qquad N = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}. \tag{15.7}$$

This is effective by Smith reduction. In fact choose vectors $\beta_i$ whose images $\alpha_i = N\beta_i$ form a primitive basis of $\operatorname{im} N = \ker N$. The index-one hypothesis says that their classes form a basis of the quotient, so $[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3]$ is unimodular.

2. Since $M_2 = M_1(I - N)$ and $M_1^2 = M_2^2 = I$, we have $M_1 N M_1 = -N$. Solving this block equation gives precisely

$$M_1 = \begin{pmatrix} A & B \\ 0 & -A \end{pmatrix}, \qquad M_2 = \begin{pmatrix} A & B - A \\ 0 & -A \end{pmatrix}, \qquad A^2 = I_3, \quad AB = BA. \tag{15.8}$$

Every integral conjugator preserving (15.7) is $\begin{pmatrix} C & D \\ 0 & C \end{pmatrix}$, with $C \in \mathrm{GL}_3(\mathbb{Z})$ and $D$ arbitrary integral. Thus one first classifies $A$ integrally; after fixing it, $B$ is considered modulo $AD + DA$ and modulo conjugation by the integral centralizer of $A$.

3. We include the integral involution classification needed here. For the lattice $L = \mathbb{Z}^3$, let $L_+$ and $L_-$ be the primitive kernels of $A - I$ and $A + I$. They satisfy $2L \subseteq L_+ \oplus L_- \subseteq L$. The quotient $L/(L_+ \oplus L_-)$ is a vector space over $\mathbb{F}_2$. Its projections into $L_+/2L_+$ and $L_-/2L_-$ are injective: a glue vector with only one nonzero projection would give a new integral vector in the corresponding eigensublattice, contradicting its definition. It is consequently the graph of an isomorphism between two subspaces of equal dimension $r$. Unimodular changes of bases in $L_\pm$ put this graph in coordinate form. For each glued pair $e_+, e_-$ the vectors $(e_+ + e_-)/2$ and $(e_+ - e_-)/2$ form a swap block. The unglued vectors give trivial and sign blocks. Hence there is a unique type

$$A \cong I_p \oplus (-I_q) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus r}, \qquad p + q + 2r = 3. \tag{15.9}$$

The dimensions of the two rational eigenspaces and the index $[L : L_+ \oplus L_-] = 2^r$ prove uniqueness. Here $r$ is zero or one, so this normalization only requires integer kernels, one Smith reduction and, in the glued case, one explicit half-sum pair whose integrality is checked.

4. For fixed (15.9), compute the quotient of the commuting matrices by the subgroup $\{AD + DA\}$. On the trivial and sign diagonal blocks the norm is respectively $2D$ and $-2D$. The commuting off-diagonal blocks between the trivial and sign summands are zero. Every commuting block involving a swap summand is a norm. For example a trivial-to-swap block is a row $(a, a)$ and is obtained as $(d, e) + (e, d)$; a sign-to-swap block has the same calculation with a difference. For two swap summands, the commuting block is $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, and applying the norm to $\begin{pmatrix} d & e \\ f & g \end{pmatrix}$ gives $\begin{pmatrix} e+f & d+g \\ d+g & e+f \end{pmatrix}$. The reverse mixed blocks have the transposed calculations. Therefore the full integral quotient is exactly

$$\mathrm{End}_{\mathbb{Z}[A]}(L)/\{AD + DA\} \cong \mathrm{Mat}_p(\mathbb{F}_2) \oplus \mathrm{Mat}_q(\mathbb{F}_2). \tag{15.10}$$

The displayed block formulas are surjective over $\mathbb{Z}$, so they compute the full norm image.

5. The centralizer action on (15.10) is precisely independent similarity in the two matrix factors. Indeed, in a product of commuting block matrices, every contribution factoring through a swap block is a norm. The remaining diagonal blocks are just the reductions of the trivial and sign blocks of the centralizer matrix. If that matrix is an integral unit, its inverse shows that both reductions are invertible. Conversely block-diagonal unimodular changes on the trivial and sign summands induce every element of $\mathrm{GL}_p(\mathbb{F}_2) \times \mathrm{GL}_q(\mathbb{F}_2)$; elementary matrices give unimodular lifts. An intertwiner of a trivial and a sign lattice is zero over $\mathbb{Z}$, which proves independence of the two factors.

6. The remaining classification is over the finite field. Enumerate all $2^{n^2}$ matrices of size $n = 0, 1, 2, 3$ over $\mathbb{F}_2$ and conjugate by every invertible matrix of that size. Their respective numbers of similarity classes are $1, 2, 6, 14$. Taking the smallest bit-coded representative in each orbit gives a fixed canonical choice. The six integral involution types and their counts are

| $(p, q, r)$ | $(0,3,0)$ | $(1,2,0)$ | $(2,1,0)$ | $(3,0,0)$ | $(0,1,1)$ | $(1,0,1)$ |
|---|---|---|---|---|---|---|
| classes | 14 | 12 | 12 | 14 | 2 | 2 |

and their sum is 56. A representative is (15.8) with $B$ equal to the chosen 0–1 representatives in its trivial and sign blocks, and zero in every other block.

7. The construction also gives an effective normalizer for every input. Normalize its cusp and then its involution by the preceding Smith and glue constructions. Use the finite-field orbit lookup to normalize its two reductions. In dimensions at most three the binary lift of an invertible binary matrix is already unimodular: its determinant is odd and has absolute value at most two. The latter bound follows by adjoining a row and column of ones and applying the $4 \times 4$ Hadamard bound. Finally solve the full integral linear equation $AD + DA = B - B_{\mathrm{can}}$ by Smith reduction. The preceding norm calculation proves solvability. Conjugating by $\begin{pmatrix} I & D \\ 0 & I \end{pmatrix}$ subtracts this norm from the upper-right block. The product of these explicitly recorded conjugators is integral unimodular and carries all three marked matrices to their canonical representative.

8. Both matrices in (15.8) have a three-dimensional fixed space. Choose a primitive integral fixed vector $v_i$. Then $g_i(x) = M_i x + v_i/4$ has affine order four: $g_i^2$ is the nontrivial translation $v_i/2$. This order-two subgroup is free, and every nontrivial subgroup of a cyclic group of order four contains it, so the full action is free. For any other finite affine lift the affine order is still even, since its linear part has order two. The two-point orbifold quotient of 14.1 is therefore the nontrivial cyclic group of order $\gcd(m_1, m_2)$. Simple connectivity is impossible for every choice of twists and affine orders.

□

**Theorem 15.4** (The full effective classification and all marked gluings)**.** [tag 02NU]

*The admissible data and the topology of their gluings are classified as follows. For $k = 2$ and $k = 3$, every admissible ordered integral tuple has the canonical representative of 15.2, with an explicit simultaneous unimodular conjugator. Equality of these representatives is necessary and sufficient for simultaneous integral conjugacy. The finite stopping bound for an input of height $H$ is $(2H + 1)^{36(k+1)}$ candidates before the proved rejection filters. The separate involution normal form has exactly 56 classes and is computed completely in 15.3.*

*For every choice of free affine orders and integer numerators, integral clutch vector, and specified primitive Kuhn cusp marking, 14.1 and 14.6 compute the full fundamental group and every integral homology group of the gluing. The complete list of data with $\pi_1 = 1$ and $H_2 = H_3 = 0$ is the constructive parametrization of 13.6, with its exact lattice enumeration and necessary-and-sufficient finite tests. Each such filling has the homology and oriented smooth type stated there.*

*Proof.* 1. The input is finite, so the prime-subgroup test in 14.1 decides whether every claimed affine cyclic action is free and has the stated affine order. The integral cusp test and its adapted basis are 2.2. The complete linear admissibility filter and the canonical representative, including the proved finite stopping bound, are proposition 10.4 and theorems 15.1 to 15.3. The characteristic-polynomial filters used by that algorithm are conjugacy invariants, so none can remove a possible representative. The unbounded trace family (15.6) proves that no fixed finite box could replace the input-dependent bound.

2. We spell out how the linear classification is used with a specified gluing. If $R$ is its returned conjugator and $P$ is the cusp marking, change the data by

$$M_i' = RM_iR^{-1}, \qquad v_i' = Rv_i, \qquad \kappa' = R\kappa, \qquad P' = RP. \tag{15.11}$$

The affine orders are unchanged. The coordinate map $x \mapsto Rx$ on the torus intertwines the finite quotient charts, since it sends $M_ix + v_i/m_i$ to $M_i'x' + v_i'/m_i$. At the cusp, its coordinates in the marked Kuhn model are unchanged: $(P')^{-1}x' = P^{-1}x$. Hence it intertwines every toric chart, translated face identification and collapse map of that fixed filling. It also carries the clutch path to the path with vector $\kappa'$. These maps agree on the collars and give a diffeomorphism of the two fully marked gluings. Thus homology can be computed in the original coordinates, while the canonical data and their group presentation are also returned. The cusp marking in this comparison is $P' = RP$.

3. An affine order-one site is ordinary, even if its integer numerator has been recorded. Delete it by the explicit comparison of theorem 14.1: if it is the $i$th site, replace $\kappa$ by $\kappa+(M_{i+1}\cdots M_k)^{-1}v_i$ before deletion. The cusp matrix and its given marking are unchanged. There cannot be an order-one site in an admissible two-point tuple: its other finite matrix would be the inverse of a nontrivial unipotent matrix. Nor can a three-point tuple have two such sites. Thus this reduction leaves precisely the genuine two- or three-point inputs of 14.6 and loses no case from the input class.

4. For each remaining input, the exact unsaturated normal closure, cyclic or triangle-group extension and exceptional spherical Euler relation of 14.1 give the full fundamental group. The finite cap, complement and marked Kuhn chain maps of theorems 14.3, 14.4 and 14.6 give a finite integer Mayer–Vietoris cone through degree nine. Integer kernels and Smith reduction therefore give all $H_0, \dots, H_8$ with their complete torsion factors. Its degree-one homology agrees with the abelianization of the van Kampen presentation.

5. The results from theorem 11.1 through corollary 13.6 prove both directions of the classification. Genuine three-point orbifold quotients exclude simple connectivity. The two-point filters reduce every three-connected datum to the minimal order-$(5, 4)$ spectrum; the rational normal form, finite stable-lattice and extension algorithms include every integral representation above it. The universal affine normalization includes every twist, and the six exact integral tests decide three-connectivity. Conversely every datum that passes them satisfies all three topological conditions by those tests. 13.6 computes its common integral homology and applies the almost-complex and smooth-classification arguments to identify its oriented type. Applying the canonical-representative algorithm identifies precisely the simultaneously integrally conjugate inputs in this parametrization.

□

# 16 Source obstructions and the punctured-base period decision

**Theorem 16.1** (Exclusion of the abelian-surface source)**.** [tag 0206]

*Apply the degree-zero line-bundle and lifted-translation operations of (Engel, 2026, §2, Definitions 2.3 and 2.5, p. 5; Remark 8.5, p. 24) to a polarized algebraic family of abelian surfaces with a toric-rank-two cusp. The resulting quotient of the punctured line bundle cannot have $b_1 = b_2 = 0$ with the stipulated rank-three square-zero Kuhn cusp. The two sections need not be proportional.*

*Proof.* 1. Let $W$ be the rank-four rational homology local system of the abelian surfaces. It is semisimple by (Deligne, 1971, Definition 4.2.4 and Proposition 4.2.5, p. 44; Theorem

4.2.6, p. 45). Its toric-rank-two cusp has $N_W^2 = 0$ and $\ker N_W = \operatorname{im} N_W$ of dimension two. Semisimplicity splits $W = W^\Gamma \oplus W'$. The image of $N_W$ lies in $W'$ whereas $W^\Gamma$ lies in its kernel. The equality of image and kernel forces $W^\Gamma = 0$. The coinvariants vanish as well.

2. The rank-six rational homology system $T$ after the two Engel operations has a natural filtration
$$0 \subset L \subset E \subset T, \qquad L \simeq \mathbb{Q}, \quad E/L \simeq W, \quad T/E \simeq \mathbb{Q}.$$
The lower line is the circle in the punctured line bundle and the upper quotient is the lifted-translation generator. Here polarization is used through the semisimplicity of the middle term $W$. The triviality of $T/E$ gives $\operatorname{im} N_T \subset E$. Maximal square-zero cusp monodromy gives $\ker N_T = \operatorname{im} N_T$; thus every globally fixed vector lies in $E$. Since $W^\Gamma = 0$, the fixed vectors of $E$ are exactly $L$. Hence $\dim T^\Gamma = 1$. The dual filtration, whose middle term is $W^*$, gives $\dim(T^*)^\Gamma = 1$ by the identical argument.

3. Consequently the two ranks $a, b$ in Theorem 10.5 are both one. Its Euler-parity calculation applies without requiring self-duality of $T$, and contradicts $b_1 = b_2 = 0$. This excludes the algebraic abelian-surface construction and, more generally, every such filtration with semisimple middle term.

□

**Theorem 16.2** (The unpolarized period domain has LSAP)**.** [tag 020A]
*Let $D_3$ be the component of*
$$\{F \in \mathrm{Gr}(3, \mathbb{C}^6) : F \cap \overline{F} = 0\}$$
*containing the standard complex structure, so that $D_3 \cong GL^+(6,\mathbb{R})/GL(3,\mathbb{C})$. At every point, $D_3$ is dominated by a spanning tube of nine complex lines. In particular, it has the local spray approximation property (LSAP) and is Oka–1.*

*Proof.* 1. Use the graph chart $F_Z = \operatorname{im}\binom{Z}{I_3}$. The open set
$$\Omega = \{Z \in M_3(\mathbb{C}) : \det(\operatorname{Im} Z) > 0\} \tag{16.1}$$
lies in $D_3$ and contains $iI_3$. For a real column $w$, write $K_w x = w \times x$, put $q_w = w^t w$, and define
$$A_w = q_w I_3 - w w^t + K_w. \tag{16.2}$$
Take the six vectors
$$e_1,\ e_2,\ e_3,\ e_1+e_2,\ e_1+e_3,\ e_2+e_3$$
and the corresponding six matrices $A_w$, followed by the three matrices $K_{e_1}, K_{e_2}, K_{e_3}$. Call these nine matrices $A_1, \dots, A_9$. Their symmetric parts are a basis of the six-dimensional space of real symmetric matrices: the first three give all diagonal matrices and the last three add the three distinct off-diagonal entries. The final three skew matrices form a basis of the skew matrices. Thus the nine directions are linearly independent over $\mathbb{R}$ and also over $\mathbb{C}$. For reference, the determinant of their nine row-major coordinate columns is $-16$.

2. Each $A_w$ is real normal, is zero on $\mathbb{R}w$, and has eigenvalues $q_w \pm i\sqrt{q_w}$ on $w^\perp \otimes \mathbb{C}$. For every real $s$ its two transverse squared singular values are
$$(1+q_w s)^2 + q_w s^2 = q_w(q_w+1)\left(s + \frac{1}{q_w+1}\right)^2 + \frac{1}{q_w+1} \geq \frac{1}{3}, \tag{16.3}$$
since $q_w \in \{1, 2\}$; the remaining squared singular value is one. For a pure skew direction $K_{e_j}$ all singular values of $I_3 + sK_{e_j}$ are at least one. Their determinants are positive at $s = 0$ and never vanish, so they are positive for all real $s$.

3. Let $\Lambda = \bigcup_{j=1}^{9}(iI_3 + \mathbb{C}A_j)$ and let $B_{1/4}$ be the open Frobenius ball of radius $1/4$ in $M_3(\mathbb{C})$. Then
$$T = \Lambda + B_{1/4} \subset \Omega. \tag{16.4}$$
Indeed, the imaginary part of $iI_3 + tA_j + E$ is $I_3 + (\operatorname{Im} t)A_j + \operatorname{Im} E$. Its unperturbed least singular value is at least $1/\sqrt{3}$, whereas $\|\operatorname{Im} E\|_{\mathrm{op}} < 1/4$. It therefore stays invertible throughout the straight perturbation from $E = 0$, preserving the positive determinant. The set $T$ is open and connected and is a union of affine translates of the spanning tree $\Lambda$. Thus it is a spanning tube in the sense of (Alarcón and Forstnerič, 2025, Definition 2.1, p. 4). Its inclusion in $D_3$ is a local biholomorphism at $iI_3$.

4. The real group $GL^+(6,\mathbb{R})$ acts holomorphically and transitively on $D_3$. Transporting this tube gives the same domination at every point. The observation preceding (Alarcón and Forstnerič, 2025, arXiv v5, Proposition 7.13 and preceding paragraph, pp. 23–24; Theorem 2.2, p. 5) gives LSAP from this domination, and its part (b), or Theorem 2.2 of that paper, gives Oka–1.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 16.3** (Holomorphic periods on every open base)**.** [tag 020B]

*Let $B$ be an open connected Riemann surface and $\rho : \pi_1(B) \to GL^+(6,\mathbb{R})$ any representation. Its associated flat $D_3$-bundle admits a holomorphic section. More generally, every continuous section which is holomorphic near a compact Runge set can be deformed through sections to a holomorphic section, with arbitrary uniform approximation there and any prescribed finite-order interpolation at finitely many points of that set. For an integral representation, this gives a holomorphic family of complex three-tori with the prescribed lattice local system and a zero section over $B$.*

*Proof.* 1. The flat bundle $\mathscr{D} = \widetilde{B} \times_\rho D_3 \to B$ is a holomorphic fibre bundle: its transition maps are constant real linear transformations, acting holomorphically on the Grassmannian. Its fibre has LSAP by Theorem 16.2, so the bundle map is an Oka–1 map by (Alarcón and Forstnerič, 2025, arXiv v5, Proposition 7.13(d), p. 24). The lifting property in (Alarcón and Forstnerič, 2025, arXiv v5, Definition 7.7, p. 22), applied to the identity map of $B$, gives precisely the stated section deformation and approximation, including the finite jets.

2. There is a continuous section to start with. An open Riemann surface has the homotopy type of a one-dimensional CW complex; a bundle with connected fibre admits a section over its vertices and edges, and the homotopy lifting property transports this section to $B$. Connectedness holds for the chosen component $D_3$. Taking the Runge set empty therefore proves existence.

3. For an integral representation let $\mathbb{V}_{\mathbb{Z}}$ be its lattice local system and let $\mathcal{F} \subset \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{O}_B$ be the resulting holomorphic rank-three subbundle. Real transversality says that projection identifies the real vector bundle with the underlying real bundle of $\mathcal{Q} = (\mathbb{V}_{\mathbb{Z}} \otimes \mathcal{O}_B)/\mathcal{F}$. Hence the image of $\mathbb{V}_{\mathbb{Z}}$ is a full lattice in every fibre of $\mathcal{Q}$. Locally it is uniformly discrete and acts by holomorphic translations. The quotient $\mathcal{Q}/\mathbb{V}_{\mathbb{Z}} \to B$ is consequently a smooth proper holomorphic family of compact complex three-tori, with its evident zero section and the required marking.

□

**Lemma 16.4** (Isolated extension points)**.** [tag 0202]

*Let $R$ be an open Riemann surface, $V \to R$ a holomorphic vector bundle, and $\mathscr{U} \subset \mathrm{Gr}_r(V)$ an open subset whose projection onto $R$ is surjective. Let $S \subset R$ be finite. Suppose that over $R \setminus S$ this projection is a holomorphic fibre bundle with an LSAP fibre. Let $K \subset R$ be a compact Runge*

*set containing $S$ in its interior. Every continuous section of $\mathscr{U}$ which is holomorphic near $K$ can be deformed through sections to a holomorphic section, with arbitrary uniform approximation on $K$ and interpolation of any given finite jets at finitely many points of its interior.*

*Proof.* 1. We give the relative version of the LSAP argument in (Alarcón and Forstnerič, 2025, arXiv v5, Lemma 5.1, p. 14; Theorem 6.1, p. 17; Proposition 7.13(d), p. 24). Enlarge $K$ slightly inside the domain of holomorphicity, retaining the Runge property. Choose a strongly subharmonic Morse exhaustion of $R$ starting with this compact set. All subsequent boundary arcs, attached discs and critical one-handles can and will be chosen disjoint from $S$. On them $\mathscr{U}$ is an ordinary holomorphic bundle with LSAP fibre.

2. The global spray on the old compact set can be taken to be vertical even though the fibres at $S$ differ. Indeed, let $f$ be the current holomorphic section on a Stein neighborhood $W$ of that set. Put $A$ equal to the finite interpolation set, including $S$, and choose an ideal sheaf $\mathcal{I}$ vanishing to one more than the prescribed jet orders at its points. Cartan's theorem A supplies finitely many holomorphic sections $T_1, \dots, T_N$ of $\mathrm{End}(V)|_W \otimes \mathcal{I}$ which generate $\mathrm{End}(V)$ on the compact attaching arcs, after shrinking $W$ if needed. The formula

$$F(z,a) = \exp\left(\sum_{j=1}^{N} a_j T_j(z)\right) f(z) \tag{16.5}$$

defines a holomorphic spray of sections of the Grassmannian bundle. Its differential in $a$ surjects onto the vertical tangent space at every attaching point, since $\mathrm{End}(V_z) \to \mathrm{Hom}(f(z), V_z/f(z))$ is onto. For $a$ sufficiently small it has range in $\mathscr{U}$ on a neighborhood of the old compact set, by openness and compactness. Every parameter value preserves all the chosen jets.

3. For a noncritical extension, subdivide the new annular regions into small discs lying in bundle trivializations over $R \setminus S$. Restrict (16.5) to a thin attaching disc. LSAP of the fibre extends this spray across the attached disc, approximating the old spray as closely as desired on the intersection. The spray gluing used in the proof of (Alarcón and Forstnerič, 2025, arXiv v5, Lemma 5.1, equations (5.6)–(5.8), pp. 16–17) now applies verbatim to vertical sprays: the inverse-function step is needed only on the attaching intersection, where (16.5) is dominating. It produces reparametrizations of the two sprays and a glued section. The old part remains in $\mathscr{U}$ because it is evaluated in (16.5); the new part remains in $\mathscr{U}$ because the LSAP spray takes values in the bundle fibre. The fixed jets survive the reparametrizations, being independent of the spray parameters on the old side.

4. For a critical extension, first extend over the attached real arc using the given continuous section and the homotopy already constructed on the old compact set. Apply Mergelyan approximation with the same finite jets to a neighborhood of the union of the old set and this arc, and then apply the noncritical step. To see explicitly that this approximation is by sections, trivialize $V$ holomorphically on the open Riemann surface: its underlying complex bundle is trivial over a graph spine, and the frame bundle therefore has a continuous section. Its fibre $GL(\mathrm{rank}\, V, \mathbb{C})$ has the dominating spray $(g, A) \mapsto g \exp A$, so (Forstnerič and Prezelj, 2002, Theorem 1.2, preprint p. 3) gives a holomorphic frame. The section is then a map to the fixed Grassmannian, to which (Alarcón and Forstnerič, 2025, Theorem 2.13, p. 9) applies. The base coordinate is kept unchanged. Close approximation on the compact arc and old set, followed by shrinking its neighborhood, preserves membership in the open subset $\mathscr{U}$. This is the critical case of Theorem 6.1 of that paper. The projection is a submersion everywhere because $\mathscr{U}$ is open in a Grassmannian bundle.

5. Choose successive errors summably small on the preceding compact sets and smaller than their distances to the complement of $\mathscr{U}$. The sections converge uniformly on every compact

set to a holomorphic section with range in $\mathscr{U}$. The finite jets persist. The close homotopies on successive stages concatenate in the usual exhaustion construction. All new attachment regions lie outside $S$; the old section remains holomorphic on a full neighborhood of each point of $S$ at every stage. This proves the lemma.

□

**Theorem 16.5** (The bounded-cusp period decision)**.** [tag 0203]

*Let $B^* = \mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_k\}$, where $k \geq 1$, and let $\rho : \pi_1(B^*) \to GL^+(6, \mathbb{R})$ be a representation. Assume that the monodromy $M_0$ at $p_0$ satisfies $(M_0 - I)^2 = 0$ and $\operatorname{rank}(M_0 - I) = 3$. Then there is a holomorphic period family with this monodromy and nondegenerate real transversality everywhere on $B^*$, equivalently a nondegenerate imaginary part in each local normalized period chart. In a marking with $M_0 = \left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$ its cusp period is*

$$Z(t) = \frac{\log t}{2\pi i} I_3 + C(t), \qquad C(t) \text{ holomorphic at } t = 0. \tag{16.6}$$

*Any prescribed finite jet of $C$ at zero may be imposed. For an integral primitive cusp the marking can be integral. Consequently every admissible datum with three-connected gluing admits such a period family on $B^*$. The points $p_1, \dots, p_k$ remain punctures of its base.*

*Proof.* 1. Write $N = M_0 - I$ and choose a real basis with $N = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$. Its integral version is proposition 2.2. Choose the component of the period domain selected by the positive logarithmic model in (16.6). This choice matters only when an orientation has been fixed independently of the marking; every element of $GL^+(6, \mathbb{R})$ preserves either component.

2. Adjoin the cusp to obtain the open surface

$$B = \mathbb{P}^1 \setminus \{p_1, \dots, p_k\}.$$

The flat complex vector bundle on $B^*$ has a holomorphic extension $V_D$ across $p_0$: in a local coordinate $t$ use the single-valued frame obtained by the change $\exp(-sN)$, where $s = (\log t)/(2\pi i)$. Changing the logarithm by $2\pi i$ is cancelled by $M_0$. This specifies the extension directly.

3. Over $B^*$ let $\mathscr{U}$ be the flat bundle of real-transverse planes in the chosen component. At $p_0$ adjoin only the planes transverse to $\ker N_{\mathbb{C}}$, an affine space of matrices $C \in M_3(\mathbb{C})$. This gives an *open* subset $\mathscr{U} \subset \operatorname{Gr}_3(V_D)$. Indeed, in the chart $F_C = \operatorname{im}\left(\begin{smallmatrix} C \\ I_3 \end{smallmatrix}\right)$ the flat period is $\exp(sN)F_C = F_{sI_3 + C}$. For $t \neq 0$ membership is exactly

$$\det(y(t) I_3 + \operatorname{Im} C) > 0, \qquad y(t) = -\frac{\log |t|}{2\pi}. \tag{16.7}$$

For every fixed $C_0$, and uniformly for $C$ in a sufficiently small bounded neighborhood of $C_0$, the matrix in (16.7) is invertible with positive determinant when $|t|$ is sufficiently small. This follows, for example, from $y(t) > \| \operatorname{Im} C \|_{\mathrm{op}}$ and a straight-line homotopy to $y(t) I_3$. Thus a full neighborhood of $(0, C_0)$ lies in the adjoined set, proving openness. The projection is consequently a holomorphic submersion. Its fibre at $p_0$ is the affine chart $\mathbb{C}^9$ just described; over $B^*$ its fibre is $D_3$.

4. Start with a polynomial $C_0(t)$ realizing the desired finite jet. Formula (16.7) makes it a holomorphic section on a small closed cusp disc $K$, which is Runge in $B$. This section extends continuously over $B$. To justify the relative extension, $D_3$ is connected and simply connected: it retracts onto $SO(6)/U(3)$, and $\pi_1(U(3)) \to \pi_1(SO(6))$ is onto, since a unitary rotation in one complex coordinate is a generator of the real two-plane rotation loop. A cellular decomposition of the surface outside the cusp disc now extends its boundary section over edges by connectedness and over two-cells by simple connectivity. The flat bundle is trivial over each cell.

5. Over $B \setminus \{p_0\}$ the fibre is the LSAP manifold $D_3$ from Theorem 16.2. Lemma 16.4, with $S = \{p_0\}$, produces a holomorphic section of $\mathscr{U}$ over all of $B$, with the chosen jet at $p_0$. Since its value at $p_0$ lies in the affine chart just adjoined, it has a holomorphic matrix $C(t)$ there. Undoing the canonical frame gives exactly (16.6), with a bounded holomorphic correction. Everywhere else membership in $\mathscr{U}$ is precisely nondegeneracy of the complex-torus periods. More explicitly, choose three members of a real lattice basis whose projected periods are complex-linearly independent, and normalize them to $I_3$. The remaining three periods form a matrix $Z$, and real transversality is exactly $\det \operatorname{Im} Z \neq 0$. This normalization persists locally.

6. For admissible data with three-connected gluing, Theorem 11.1 excludes $k = 3$ and Proposition 11.2 proves orientation preservation of the finite matrices for $k = 2$. The cusp determinant is also one. Thus all three-connected data satisfy the hypotheses.

□

# Part III
# The resolved fibre product

## 17 The resolved fibre-product fourfold

**Definition 17.1** (The resolved-product datum)**.** [tag PV21] Let $f : X \to B = \mathbb{P}^1_t$ be a canonical marked member of the $(3, 4, \infty)$ family, with $p_1 = 0$, $p_2 = 1$, $p_0 = \infty$, sphere twists $v_1 = (1, 2, -4, 0)^t$, $v_2 = (-1, -3, 3, 0)^t$, and the untwisted cusp collar. Put $B^\circ = B \setminus \{p_0, p_1, p_2\}$. Write $S_1, S_2$ for the reduced multiple fibres, of multiplicities three and four, and $W$ for the cusp divisor. Let $S \to B$ be the smooth relatively minimal elliptic surface whose Weierstrass model is

$$y^2 = x^3 - 3t^3(t-1)x + 2t^4(t-1)^2.$$

It has $j = 1728t$ and singular fibres of types $IV^*$, $III$, and $I_1$ at $p_1, p_2, p_0$, respectively (Engel, 2026, §2, p. 3).

Use the homology bases $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$ for $X$ and $(e_1, e_2)$ for $S$, with dual basis $(\gamma, u^*, w^*, \delta^*, e_1^*, e_2^*)$ for their direct sum. The marked homology monodromies are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}, \qquad A_0 = (A_1 A_2)^{-1},$$

$$C_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C_0 = (C_1 C_2)^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Write $\Pi_X$ for the period matrix of $X$ in this canonical marking; its last two columns are $I_2$, as in I.2 (p. 21) . Orient complex manifolds by their complex structures and use integral cohomology unless a coefficient field is specified. Put $L = \mathcal{O}_X(S_1 - S_2)$. The meromorphic functions $t$, $t-1$ and their quotient give $3S_1 \sim 4S_2 \sim W$. Tensoring these divisor isomorphisms gives

$$L^3 \cong \mathcal{O}_X(S_2), \qquad L^4 \cong \mathcal{O}_X(S_1), \qquad L^6 \cong \mathcal{O}_X(2S_2), \qquad L^{12} \cong \mathcal{O}_X(W).$$

**Theorem 17.2** (A smooth compact model of the fibre product)**.** [tag 02M5]

*There is a smooth compact complex fourfold $Z$ with proper holomorphic maps $g : Z \to B$ and $\Pi : Z \to X$, with $g = f\Pi$, with the local models described below, agreeing with the marked fibre product over the punctured base. The map $\Pi$ has a holomorphic section. The construction is a small crepant resolution of a Weierstrass model over $X$, and is bimeromorphic to $X \times_B S$.*

*Proof.* 1. Let $a$ be a nonzero scalar multiple of the divisor section of $L^4$ and let $b$ be a nonzero scalar multiple of the square of the divisor section of $\mathcal{O}_X(S_2) = L^3$. Choose the scalars so that $4a^3 + 27b^2$ is the pullback of the section of $\mathcal{O}_B(1)$ vanishing at $p_0$. This is possible because $a^3$ and $b^2$ have divisors $3S_1$ and $4S_2$, which correspond to a basis of $H^0(B, \mathcal{O}_B(1))$. The Weierstrass equation

$$y^2 z = x^3 + axz^2 + bz^3 \tag{17.1}$$

in the projective bundle associated to $\mathcal{O}_X \oplus L^2 \oplus L^3$ defines a proper complex space $\mathscr{W} \to X$. Its section at infinity is disjoint from its singular locus. Its $j$-function is $1728t$.

2. Over $B^\circ$, equation (17.1) is the product with Engel's elliptic curve: both have that marked Weierstrass family. At $p_1$, pass to the root atlas $t = s^3$. The coefficient $a$ has a simple zero in $s$ and $b$ is a unit. At $p_2$, on $t - 1 = s^4$, $a$ is a unit and $b$ has order two. These are the smooth good-reduction families $E$. The deck action on $F \times E$ is diagonal, with the $A_j \oplus C_j$ linear

part and translation $(v_j, 0)/m_j$. It is free because its projection to $F$ is free. Its quotient is therefore smooth and is exactly the local model asserted in the theorem. The section at infinity is invariant. Notice that the full fibres here have multiplicities three and four; only their reductions are smooth.

3. At the cusp, use the angular–deck marking of the toroidal model, and write the two-torus period as $Z_2 = sB_0 + C(t_c)$, where

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad Q = B_0^{-1}P, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here $PS = I_2$. Put $k = (1, 1, 1)^t$, $r = (0, 0, 1)$, and write the Tate period of the elliptic surface as $s + c_E(t_c)$, with $c_E$ holomorphic and bounded. Define

$$Z_3 = sI_3 + C_3(t_c), \qquad C_3 = SCQ + kc_E r. \tag{17.2}$$

For $A = \binom{P}{r}$ and $B = \binom{Q}{r}$ we have

$$\det A = \det B = 1, \qquad AZ_3B^{-1} = \operatorname{diag}(Z_2, s + c_E). \tag{17.3}$$

This is an integral holomorphic product identification on the punctured overlap, including the bounded correction.

4. Apply theorem 2.4 to the bounded holomorphic correction $C_3$ in (17.2). The fan is (2.1); its action is (2.2) with $(q, C) = (t_c, C_3)$. This identifies the quotient $N_3^C$ by the identity on its dense torus and gives a smooth proper filling.

5. The map $(P, r)$ takes (2.1) to a subdivision of the product of an $A_2$ triangle and an interval. There are three tetrahedra in each triangular prism. No ray is added. Consequently the induced map is a small toric resolution of the local equation $xy = w_0w_1w_2$ in (17.1); over a double curve it resolves $xy = w_0w_1$ times a smooth factor. All rays have height one, so the resolution is crepant. The formula $PZ_3 = Z_2Q$ proves equivariance of its map to the cusp of $X$. The slice $x_3 = 1$ gives the zero section: its lattice map is $S$, and $S$ takes each $A_2$ triangle to a face of a Kuhn tetrahedron. Thus the section extends over every double curve and both triple points.

6. We glue $N_3^C$ and the two finite models to the punctured product by (17.3) and their root-atlas identifications. These overlap maps are holomorphic isomorphisms and satisfy the cocycle condition because they are identifications with the same marked product. Properness over the three closed discs and compactness over the remaining compact base give compactness of the glued Hausdorff space. It is smooth in all its charts. The maps to $X$ and the zero section agree on their overlaps. Since both this model and the literal fibre product agree on a dense open set, they are bimeromorphic.

□

**Theorem 17.3** (Fundamental group and rational homology)**.** [tag 02M7]
*The fourfold in Theorem 17.2 has*

$$\pi_1(Z) = 1, \qquad \chi(Z) = 6, \qquad (b_0, \dots, b_8) = (1, 0, 2, 1, 2, 1, 2, 0, 1). \tag{17.4}$$

*The first homology groups of its two finite reduced fibres are, respectively,* $\mathbb{Z}^2 \oplus \mathbb{Z}/3$ *and* $\mathbb{Z}^2 \oplus \mathbb{Z}/2$.

*Proof.* 1. The section of $\Pi$ splits the fundamental-group sequence over its smooth locus $X \setminus W$. Every loop in $Z$ can be moved off the discriminant inverse image and is a product of a loop in the elliptic fibre and a section loop. A section loop becomes trivial in $Z$ because it bounds in the embedded simply connected $X$. Thus $\pi_1(Z)$ is a quotient of the coinvariants of $\mathbb{Z}^2$ under $C_1, C_2$. Loops realizing these two monodromies are the root-circle paths closed by the affine identifications in $X \setminus W$. The Smith form of $(C_1 - I \mid C_2 - I)$ is $(1, 1)$, so these coinvariants vanish.

2. Euler characteristic localizes on the exceptional fibres: all regular torus fibres and the finite free torus quotients have Euler characteristic zero. The cusp has six zero-dimensional torus strata, and every positive-dimensional torus stratum has Euler characteristic zero. Hence $\chi(Z) = 6$.

3. Write $V = R^1 g_* \mathbb{Q}$ on the punctured base, and $j$ for its inclusion. Finite quotient specialization is an isomorphism onto invariants over $\mathbb{Q}$. Theorem 2.6 gives $R^q g_* \mathbb{Q} = j_* \bigwedge^q V$ except for one kernel $\mathbb{Q}_{p_0}$ in degree four. For any local system $L'$ on the punctured sphere,

$$H^2(B, j_* L') = (L')_{\pi_1}, \qquad \chi(B, j_* L') = -\operatorname{rank} L' + \sum_{j=0}^{2} \dim (L')^{M_j}. \tag{17.5}$$

The first equality is compactly supported duality and the second is Euler additivity on the open stratum and the three point stalks. The exact exterior-matrix computation gives the following rows of the rational Leray $E_2$ page:

| $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\dim E_2^{0,q}$ | 1 | 1 | 2 | 3 | 3 | 1 | 1 |
| $\dim E_2^{1,q}$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\dim E_2^{2,q}$ | 1 | 1 | 2 | 3 | 2 | 1 | 1. |

(17.6)

The differential ranks $r_q = \operatorname{rank}(E_2^{0,q} \to E_2^{2,q-1})$ are

$$(r_1, \dots, r_6) = (1, 1, 2, 2, 1, 1). \tag{17.7}$$

Indeed $\Pi^*$ transports the four nonzero transgressions of the sphere fibration to the corresponding direct summands here. These four transgressions are forced over $\mathbb{Q}$ by the one-dimensional rows and $H^*(X; \mathbb{Q}) = H^*(S^6; \mathbb{Q})$; this also checks the particular reduction of (Engel, 2026, Lemma 7.6, p. 21) which we use. The divisor class of the zero section gives a permanent class $u$ restricting to the elliptic orientation class. Multiplication by $u$ gives another independent image in degree three, and proves $r_3 = 2$, $r_5 = 1$, $r_6 = 1$, and $r_4 \geq 2$. The target summands are independent by the marked decomposition $\Lambda_X \oplus \Lambda_E$. The resulting formulas for $b_3$ and $b_5$ are $b_3 = 1$ and $b_5 = 3 - r_4$. Poincaré duality of the compact smooth fourfold gives $r_4 = 2$. No later differential exists because the base has real dimension two. Taking diagonal sums of (17.6) with (17.7) proves (17.4).

4. Finally, for a finite quotient its affine fundamental group has generators $\Lambda \oplus \mathbb{Z}a$ and abelianized relations $(A_j \oplus C_j - I)\Lambda$ and $m_j a - (v_j, 0)$. The complete relation matrix is obtained by stacking these integral relations; see Table 1. Its nonzero Smith factors are $(1, 1, 1, 1, 3)$ and $(1, 1, 1, 1, 2)$, respectively. This proves the integral assertions about the finite fibres.

□

**Theorem 17.4** (Integral homology of the fourfold)**.** [tag 02M8]

*The fourfold of Theorem 17.2 has torsion-free integral homology:*

$$H_i(Z; \mathbb{Z}) = \mathbb{Z}^{b_i}, \qquad (b_0, \dots, b_8) = (1, 0, 2, 1, 2, 1, 2, 0, 1).$$

*In particular the finite-fibre torsion in 17.3 does not survive as torsion in the homology of the total space.*

*Proof.* 1. We compute the integral covering cokernels. Write $\mathcal{V} = R^1 g_* \mathbb{Z}|_{B^\circ}$, and put $\mathcal{J}_q = j_* \bigwedge^q \mathcal{V}$. The cohomology monodromies are the inverse transposes of $A_1 \oplus C_1, A_2 \oplus C_2, A_0 \oplus C_0$. All lattices below are in this marking.

2. First we describe a finite filling. Since the first coordinate of each twist $v_j$ is $\pm 1$, the integral change of basis with first column $v_j/(v_j)_\gamma$ and remaining columns the last five standard basis vectors changes its linear monodromy to $\operatorname{diag}(1, D_j)$ and its translation to $\pm e_0/m_j$. The reduced finite fibre is the mapping torus of the linear automorphism $D_j : T^5 \to T^5$. The covering $T^6 \to B_j$ is the $m_j$-fold cover of its suspension circle. Set $T_j = D_j^{-t}$. The two vertical matrices on homology, in coordinates $(u, w, \delta, e_1, e_2)$, are

$$D_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The Wang sequences of the mapping torus and its cover give the following exact description of the cokernel of covering pullback:

$$C_j^q \simeq \frac{\ker(\bigwedge^{q-1} T_j - I)}{\operatorname{im} \mathcal{N}_j^{q-1}}, \qquad \mathcal{N}_j^{q-1} = \sum_{a=0}^{m_j-1} (\bigwedge^{q-1} T_j)^a. \tag{17.8}$$

Here the target of pullback is the invariant lattice in $H^q(T^6; \mathbb{Z})$. To verify (17.8), its restriction to the vertical torus surjects onto $\ker(\bigwedge^q T_j - I)$, and restriction of a mapping-torus class has exactly the same image, by Wang exactness. Thus every cokernel class has a representative containing the suspension covector. On this kernel the covering map is the displayed norm: cut the covered circle into its $m_j$ consecutive intervals and transport the vertical cochain through them. This proves both surjectivity and the precise kernel of the horizontal description.

3. For $q = 2$, the two quotients in (17.8) are respectively zero and $\mathbb{Z}/2$. For $q = 3$, they are

$$C_1^3 = \mathbb{Z}/3, \qquad C_2^3 = (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4. \tag{17.9}$$

The same Wang sequences show that the kernels of finite specialization in degree two are $\mathbb{Z}/3$ and $\mathbb{Z}/2$, and that the degree-three and degree-four kernels vanish. The cusp specialization is integrally onto, with the sole free degree-four kernel of 2.6.

4. We give the integral global maps needed to use these local groups. The global invariant two-form $q_X = u^* w^* + 6\gamma\delta^*$ maps onto $C_2^2$. Indeed, if its pullback descended to the six-dimensional finite fibre, restriction to the elliptic zero section would make it descend to the original bielliptic surface $S_2$. Its square there would be $\pm 12/4 = \pm 3$, which is impossible in the even intersection form of $S_2$. That form is even because $H^2(S_2; \mathbb{Z})$ is torsion free and its canonical class is torsion, as checked in III.5.4 (p. 34) . Thus the global-to-local map in degree two is onto its only nonzero covering cokernel.

5. For degree three all three global invariant classes contain $\gamma$, and remain purely horizontal in both adapted finite markings. They are

$$\gamma e_1^* e_2^*, \qquad \gamma(-u^* e_1^* + w^* e_2^*), \qquad \gamma u^* w^*. \tag{17.10}$$

Juxtaposition denotes exterior product. Consequently their images in (17.9) are exactly the vertical two-forms in (17.10) modulo the norm lattices. In integral invariant bases, the two norm matrices and this common image matrix are

$$N_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & 1 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 2 & -1 & 0 & -1 & 0 \\ 3 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -2 & 0 & 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & -1 & -1 & 0 \\ 4 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every nonzero Smith factor of $\left(\operatorname{diag}(N_3, N_4) \mid (G, G)^t\right)$ is one; there are eight. Thus the global classes surject onto the direct sum of the two degree-three covering cokernels.

6. We also compute the integral parabolic groups, with their torsion. The degree-two global coboundary matrix, obtained by stacking the two monodromy-minus-identity matrices, has a single nontrivial Smith factor six. Its torsion generator restricts with orders three, two and one at the three boundary circles. Hence no nonzero torsion class is parabolic. Its rational parabolic rank is zero, so $H^1(B, \mathcal{J}_2) = 0$. In degree three the global coboundary image is primitive and the rational parabolic rank is one; hence $H^1(B, \mathcal{J}_3) = \mathbb{Z}$. These assertions use the cocycle description as the kernel of restriction from the free-group cohomology to the three peripheral circle groups. The matrices and restriction orders are obtained by integer kernels and Smith reduction; see Table 1.

7. Passing from $\mathcal{J}_q$ to $R^q g_* \mathbb{Z}$ adds the point-supported kernels just described and replaces the finite stalks by the pullback sublattices. A point-supported sheaf has no positive cohomology. The long exact sequences, together with the two surjections onto covering cokernels, therefore give

$$H^1(B, R^2 g_* \mathbb{Z}) = 0, \qquad H^1(B, R^3 g_* \mathbb{Z}) = \mathbb{Z}. \tag{17.11}$$

They also identify degree-two base cohomology with the global coinvariants. In the relevant degrees these are torsion-free, with

$$H^2(B, R^1 g_* \mathbb{Z}) = \mathbb{Z}, \qquad H^2(B, R^2 g_* \mathbb{Z}) = \mathbb{Z}^2. \tag{17.12}$$

The first is the corresponding group from $X$. The second is the direct sum of that from $\bigwedge^2 \mathcal{V}_X$ and the elliptic orientation line; the mixed coinvariants are zero. Smith forms show that these identifications are primitive.

8. The integral differential onto the first group in (17.12) is surjective by naturality from the sphere fibration: its own corresponding Leray quotient must vanish because $H^3(X) = 0$. The differential onto the second group is surjective as well. The $X$ summand is killed by the differential transported from $H^4(X) = 0$; the elliptic orientation summand is killed by multiplying the primitive degree-one transgression of $X$ by the permanent class of the elliptic zero-section divisor. That divisor has degree one on each elliptic fibre. Thus these are two primitive independent target generators.

9. Now $H^0(B, R^3 g_* \mathbb{Z})$ and $H^0(B, R^4 g_* \mathbb{Z})$ are free: in the first degree specialization has no kernel, and in the second its only kernel is the free cusp class. There is no later Leray differential, since the base has cohomological dimension two. Equation (17.11) and the first surjectivity in (17.12) show that $H^3(Z; \mathbb{Z})$ is a free group, of rank one by 17.3. Equation (17.11) and the second surjectivity show that $H^4(Z; \mathbb{Z})$ is an extension of two free groups of rank one,

and hence is $\mathbb{Z}^2$. Simple connectivity makes $H^2(Z;\mathbb{Z})$ torsion free. The universal coefficient theorem now proves successively that $H_2(Z;\mathbb{Z}) = \mathbb{Z}^2$ and $H_3(Z;\mathbb{Z}) = \mathbb{Z}$. Poincaré duality determines the other degrees and gives exactly the displayed groups.

□

**Theorem 17.5** (Complex invariants and middle form)**.** [tag 02M9]
*For the fourfold of Theorem 17.2,*

$$K_Z = \Pi^* L^{-5}, \qquad a(Z) = 2, \qquad (h^{0,0}, \dots, h^{0,4}) = (1,1,0,0,0). \tag{17.13}$$

*It is not in Fujiki's class $\mathcal{C}$. Moreover $c_1(Z) = 0$ integrally, $\int_Z c_2^2 = 2$, $\sigma(Z) = 2$, and the intersection form on $H^4(Z;\mathbb{Z})/\text{torsion}$ is $\langle 1\rangle \oplus \langle 1\rangle$.*

*Proof.* 1. We first compute the direct images of the structure sheaf. The orbifold base has roots of orders three and four. Write its line generator as $\mathcal{O}(1)$; it pulls back to $L$. For the coarse map $\rho$,

$$\rho_* \mathcal{O}(k) = \mathcal{O}_B(d(k)), \qquad d(k) = \frac{k}{12} - \frac{k \bmod 3}{3} - \frac{-k \bmod 4}{4}. \tag{17.14}$$

This follows by dividing a local invariant section by its necessary root-coordinate power. Finite-group invariants over $\mathbb{C}$ are exact. In particular $\mathcal{O}(-1)$ and $\mathcal{O}(-2)$ are acyclic, since their coarse degree is $-1$.

2. For $\bar{f} : X \to \mathfrak{B}$, the direct images are

$$R^0 \bar{f}_* \mathcal{O}_X = \mathcal{O}, \quad R^1 \bar{f}_* \mathcal{O}_X = \mathcal{O} \oplus \mathcal{O}(-1), \quad R^2 \bar{f}_* \mathcal{O}_X = \mathcal{O}(-1). \tag{17.15}$$

We verify (17.15). The cusp branch resolution of $\mathcal{O}_W$ is the cellular complex of its dual two-torus, and has cohomology dimensions $(1,2,1)$ with nonzero exterior cup pairing. Proper flat base change therefore gives vector bundles with those ranks. The relative logarithmic vertical bundle has a fixed line generated by $\widehat{\delta}$, and its determinant has twelfth power $\mathcal{O}(-p_0)$, by the modular discriminant. Thus the determinant is $\mathcal{O}(-1)$. Relative duality gives the last term of (17.15). The invariant covector $\gamma$ gives a nowhere zero section of $R^1 \bar{f}_* \mathcal{O}_X$: on a torus its class in $\Lambda^*_{\mathbb{C}}/\operatorname{rowspan} \Pi_X$ is nonzero by the last two columns of $\Pi_X$; at the cusp it is the nonzero deck cocycle on the dual two-torus. Its determinant and the nonzero cusp cup pairing then give the extension $0 \to \mathcal{O} \to R^1 \bar{f}_* \mathcal{O}_X \to \mathcal{O}(-1) \to 0$. It splits because $H^1(\mathfrak{B}, \mathcal{O}(1)) = 0$, also from (17.14). The same logarithmic determinant gives $K_X = L^{-6}$. These computations use the root atlases and the reduced cusp.

3. The Weierstrass adjunction calculation gives $K_{\mathscr{W}} = \Pi^*(K_X \otimes L)$, and the small toric resolution is crepant. This proves the canonical formula in (17.13). Every fibre of $\Pi$ is a connected genus-one curve; over the deeper strata its toric resolution is a cycle of rational curves. Thus $\Pi$ is flat, by equidimensionality between smooth spaces, and relative duality gives $R^1\Pi_* \mathcal{O}_Z = L^{-1}$ and $R^0\Pi_* \mathcal{O}_Z = \mathcal{O}_X$. Restriction to the section splits the unit map in the derived direct image, so

$$R\Pi_* \mathcal{O}_Z \simeq \mathcal{O}_X \oplus L^{-1}[-1].$$

Projection formula, (17.14), and (17.15) show that $H^*(X, L^{-1}) = 0$ and $h^*(X, \mathcal{O}_X) = (1,1,0,0)$. This proves the asserted Hodge numbers.

4. A meromorphic trivialization of $L$ makes the coordinate $x$ in (17.1) a meromorphic function over $X$, of degree two on each elliptic fibre. It gives a generically finite meromorphic map to $X \times \mathbb{P}^1$. Traces and elementary symmetric functions over its finite generic fibre show that the meromorphic function field is an algebraic extension of that of the target. Since $a(X) = 1$, this proves $a(Z) = 2$. If $Z$ were in class $\mathcal{C}$, a sequence of blowups along smooth centres would

give a compact Kähler model, by the form of the model theorem in (Jia and Meng, 2024, Theorem 1.1 and Remark 1.2, p. 2). Both $b_1$ and $h^{0,1}$ are unchanged by such blowups: the exceptional fibres have no first homology and no positive-degree structure-sheaf cohomology. The Kähler identity $b_1 = 2h^{0,1}$ would give $0 = 2$, a contradiction.

5. Since $H^2(X;\mathbb{Z}) = 0$, we have $c_1(L) = 0$. The canonical formula gives $c_1(Z) = 5\Pi^* c_1(L) = 0$. Riemann–Roch and the signature formula, in the explicit normalization of (Esole and Kang, 2018, (1.2)–(1.4), pp. 3–4), yield

$$0 = \chi(\mathcal{O}_Z) = \frac{3c_2^2 - c_4}{720}, \qquad \sigma(Z) = \frac{3c_2^2 + 14c_4}{45}.$$

Here $c_4[Z] = 6$, hence $c_2^2[Z] = 2$ and $\sigma(Z) = 2$. The integral middle pairing modulo torsion is unimodular by Poincaré duality, has rank two by (17.4), and is positive definite by its signature. The rank-at-most-four lattice classification (Peters and Sterk, 2024, Theorem 2.2.1, p. 61) therefore gives precisely $\langle 1\rangle \oplus \langle 1\rangle$.

□

# Part IV
# Shifted fibre products

## 18 The shifted product and its bordism

The same sphere fibrations give compact complex fivefolds by placing their exceptional fibres over disjoint subsets of the base. Their topology is governed by a torus-knot peripheral group and the integral cusp maps.

**Definition 18.1** (Shifted fibre product)**.** [tag PV22] Let $X$ and $X'$ be canonical marked members of the $(3,4,\infty)$ family, and choose $\varphi \in \mathrm{PGL}_2(\mathbb{C})$ so that the exceptional values of $X$ and $\varphi^* X'$ are disjoint. Their *shifted fibre product* is $P = X \times_B \varphi^* X'$.

**Theorem 18.2** (A shifted second sphere)**.** [tag 02MN]

*Let $X'$ be a second canonical member and choose $\varphi \in \mathrm{PGL}_2(\mathbb{C})$ so that its three exceptional values are disjoint from those of $X$. The smooth fivefold $P = X \times_B \varphi^* X'$ has Euler characteristic zero and an infinite fundamental group with presentation*

$$\begin{aligned} G = \langle a_1, a_2, b_1, b_2, h_1, h_2 \mid\ & h_1, h_2 \text{ central}, \quad a_1^3 = h_1,\ a_2^4 = h_1^{-1}, \\ & b_1^3 = h_2,\ b_2^4 = h_2^{-1}, \quad a_1 a_2 b_1 b_2 = 1 \rangle. \end{aligned} \tag{18.1}$$

*Its abelianization is $\mathbb{Z}$, and its rational Betti numbers are*

$$(1, 1, 1, 20, 63, 88, 63, 20, 1, 1, 1). \tag{18.2}$$

*Proof.* 1. Let $f : X \to B$ and $g : \varphi^* X' \to B$ be the two maps. At a point $(x, x')$ of the fibre product, their disjoint critical-value sets imply that at least one of $df_x$ and $dg_{x'}$ is surjective. Consequently the difference differential $(v, w) \mapsto df_x(v) - dg_{x'}(w)$ maps $T_x X \oplus T_{x'} \varphi^* X'$ onto $T_{f(x)} B$. In a local coordinate on $B$ it is the differential of the defining function $f - g$. The holomorphic implicit function theorem therefore makes $P$ smooth of complex dimension $3 + 3 - 1 = 5$. It is compact because it is closed in the compact product $X \times \varphi^* X'$. At each special value the other factor is a smooth four-torus. The Euler characteristic of every fibre is therefore zero, and Euler additivity gives $\chi(P) = 0$. We apply van Kampen with the cusp relations. Over the six-punctured sphere the fibre group is $\Lambda_X \oplus \Lambda_{X'}$ and the base is free on five peripheral loops. Each monodromy acts on its own direct summand. Filling the first cusp kills its angular generators $\widehat{w}, \widehat{\delta}$ and its cusp meridian. Their normal closure under $A_1$ also kills $\widehat{u}$, since $A_1 \widehat{w} = \widehat{u} - \widehat{w}$. The same calculation in the other summand leaves only $h_1 = \widehat{\gamma}$ and $h_2 = \widehat{\gamma}'$. The finite monodromies now act trivially on them. The four finite filling relations are precisely those in (18.1), since $\gamma(v_1) = 1$ and $\gamma(v_2) = -1$. Both cusp collars are untwisted; their meridians are killed. The product of the remaining four meridians gives the last relation. These are all relations introduced by the six fillings, so van Kampen gives exactly (18.1).

2. Quotienting by $h_1, h_2$ gives the orbifold group of the sphere with cone orders $3, 4, 3, 4$. Its orbifold Euler characteristic is $-5/6$, so its universal orbifold cover is the disc and the group is infinite. Equivalently one can build the Seifert $T^2$-bundle with these four primitive translation vectors; over the disc it is a trivial torus bundle. Its exact sequence shows that the central subgroup in (18.1) is $\mathbb{Z}^2$. The abelianized five-by-six relation matrix has Smith factors $(1, 1, 1, 1, 1)$ by integral row and column operations, proving $G_{\mathrm{ab}} = \mathbb{Z}$.

3. For the rational homology there is no extra cusp kernel: each cusp is the rank-two cusp of the sphere fibration times a smooth torus, and its specialization is an isomorphism onto invariants

over $\mathbb{Q}$. The two monodromy groups act on independent summands. In exterior degrees $q = 0, \dots, 8$, the $H^0$ and $H^2$ rows are both

$$(1, 2, 3, 4, 5, 4, 3, 2, 1),$$

and the $H^1$ row is

$$(0, 0, 18, 60, 86, 60, 18, 0, 0).$$

The local invariant polynomials of the first factor are $A(z) = 1 + 2z + 2z^2 + 2z^3 + z^4$ at a finite point and $C(z) = A(z) + 2z^2$ at its cusp. Hence the Euler polynomial of the direct-image rows is

$$-4(1+z)^8 + (4A(z) + 2C(z))(1+z)^4.$$

Formula (17.5), with six point strata, gives the displayed rows.

4. Each invariant algebra of a factor has one generator in degrees zero through four. Its invariant-to-coinvariant map is nonzero in even degrees and zero in odd degrees. In the marking used above, this follows from the invariant form $6\gamma \wedge \delta + u \wedge w$ and its exterior products: the odd invariant classes pair to zero with every invariant complementary class, and the even ones have nonzero square or top pairing. Its four transgressions are all nonzero, as the factor is a sphere. The derivation rule therefore sends a tensor of bidegree $(i, j)$ into bidegrees $(i-1, j)$ and $(i, j-1)$, with the first term nonzero exactly when $i > 0$ and $j$ is even, and the second exactly when $j > 0$ and $i$ is even. The nonzero column supports are disjoint except for pairs in a single row, so their scalar normalizations do not affect ranks. This gives

$$(r_1, \dots, r_8) = (1, 2, 2, 3, 3, 2, 2, 1).$$

Row reduction of these exterior-product matrices gives the displayed ranks. There are no later differentials. Their diagonal sums give (18.2), which also satisfies Poincaré duality and Euler zero.

□

**Proposition 18.3** (The first integral torsion of the shifted product)**.** [tag 02MP]

*Let $Z_\varphi = X \times_B \varphi^* X$ be the transverse fivefold in 18.2, with the two sets of three critical values disjoint. Its second integral homology is*

$$H_2(Z_\varphi; \mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/12)^7.$$

*Proof.* 1. Use subscripts one and two for the two rank-four cohomology local systems. At a critical value of one factor the other factor is a smooth torus bundle over a disc, hence a product there. Each local specialization is therefore the tensor product of the original specialization with the identity on the exterior algebra of $\mathbb{Z}^4$. The original finite reduced surfaces and the original cusp have free cohomology. Their specialization maps are injective; at the cusp they are integral isomorphisms onto invariants. For the latter, apply lemma 2.8 in dimension two. It supplies the invariant deck classes $b_1, b_2$, the three classes $a_1b_1, a_2b_2, a_1b_2 + a_2b_1$, and their products. Together with $b_1b_2$ these give an integral kernel basis in degree two; their products give integral bases in degrees three and four. Both the central-fibre and invariant lattices have ranks $(1, 2, 4, 2, 1)$, as follows also from the normalization matrix in III.5.2 (p. 32) at $d = 1$. Thus the specialization is onto between free groups of equal ranks in every degree, proving the claimed isomorphism. Thus the only difference between $R^q g_* \mathbb{Z}$ and $\mathcal{J}_q = j_* \bigwedge^q (\mathcal{V}_1 \oplus \mathcal{V}_2)$ is their finite covering cokernel, with no point-supported kernel.

2. The finite Wang calculation (17.8), now with the original three-dimensional vertical lattice, gives

$$C_3^1 = \mathbb{Z}/3, \quad C_4^1 = \mathbb{Z}/4, \qquad C_3^2 = 0, \quad C_4^2 = \mathbb{Z}/2.$$

The invariant covector $\gamma$ generates the first two cokernels, and the invariant two-form $q_X$ generates the last, by the same even-intersection argument as in 17.4. Consequently the degree-two covering cokernel at all four finite values, combined, is

$$C^2 = \Big((\mathbb{Z}/3)^4 \oplus (\mathbb{Z}/4)^4 \oplus \mathbb{Z}/2\Big)^2 .$$

The two monodromy groups act independently on the two factors. The three global invariant two-forms have integral basis $q_1, \gamma_1\gamma_2, q_2$. The first and third map onto the respective $\mathbb{Z}/2$ summands and vanish in the other summands. The middle form maps to the primitive $\gamma$ coordinate of each of the four order-three or order-four blocks. After the two order-two summands have been removed, it therefore generates a direct cyclic summand of order twelve in $(\mathbb{Z}/12)^8$. Hence

$$\operatorname{coker}\{H^0(B, \mathcal{J}_2) \to C^2\} = (\mathbb{Z}/12)^7. \tag{18.3}$$

The eighteen-row integer presentation of these covering cokernels has Smith form with eleven unit factors and seven factors twelve.

3. In degree two the global coboundary image is primitive. Thus the parabolic group, a subgroup of the free-group cohomology on the punctured base, is torsion free. Its rank is eighteen: the six local invariant ranks are $(16, 16, 18, 16, 16, 18)$, and its global invariant and coinvariant ranks are both three. The Euler formula on the six-punctured sphere gives $3 + 3 + 4 \cdot 28 - 100 = 18$. These ranks and the primitivity are computed with exact exterior matrices. The sheaf sequence $0 \to R^2 g_* \mathbb{Z} \to \mathcal{J}_2 \to C^2 \to 0$ and (18.3) now give

$$H^1(B, R^2 g_* \mathbb{Z}) = \mathbb{Z}^{18} \oplus (\mathbb{Z}/12)^7.$$

Here the quotient by its finite subgroup is free, so this abelian-group extension splits.

4. The group $H^2(B, R^1 g_* \mathbb{Z})$ is the free rank-two group of coinvariants, one primitive summand from each factor. Pulling back the integral degree-two transgression of the original sphere fibration through each projection kills these two summands separately and surjectively. Thus $E_\infty^{2,1} = 0$ in the Leray sequence of $g$. Its other middle group $E_\infty^{0,3}$ is free, since $R^3 g_* \mathbb{Z}$ injects into $\mathcal{J}_3$. There are no higher differentials. Consequently $\operatorname{Tor} H^3(Z_\varphi; \mathbb{Z}) = (\mathbb{Z}/12)^7$. The universal coefficient theorem identifies this with $\operatorname{Tor} H_2(Z_\varphi; \mathbb{Z})$. Finally 18.2 gives $b_2 = 1$, proving the assertion.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 18.4** (Full integral homology of the shifted product)**.** [tag `02MQ`]

*For the transverse shifted product $P$ of 18.2, write*

$$A = (\mathbb{Z}/12)^7, \qquad B = (\mathbb{Z}/12)^{10} \oplus (\mathbb{Z}/2)^7, \qquad C = (\mathbb{Z}/12)^6 \oplus \mathbb{Z}/24 \oplus (\mathbb{Z}/2)^{17}.$$

*Its complete integral homology is*

| $i$ | $H_i(P; \mathbb{Z})$ |
|---|---|
| $0, 1, 8, 9, 10$ | $\mathbb{Z}$ |
| $2$ | $\mathbb{Z} \oplus A$ |
| $3$ | $\mathbb{Z}^{20} \oplus B$ |
| $4$ | $\mathbb{Z}^{63} \oplus C$ |
| $5$ | $\mathbb{Z}^{88} \oplus C$ |
| $6$ | $\mathbb{Z}^{63} \oplus B$ |
| $7$ | $\mathbb{Z}^{20} \oplus A.$ |

*In particular the higher integral filtration extensions are nontrivial: the middle torsion includes a cyclic factor of order* 24.

*Proof.* 1. *A decomposition with free groups on every piece.* Choose a separating circle in $B = \mathbb{P}^1$ so that one closed disc contains the three critical values of $X$ and the other contains the three critical values of $X'$. The smooth torus family of the other factor is trivial over each disc. Set

$$F = T^4, \qquad E = X \setminus \operatorname{int}(F \times D^2), \qquad E' = X' \setminus \operatorname{int}(F' \times D^2).$$

The product trivializations give the decomposition

$$P = (E \times F') \cup_{F \times F' \times S^1} (F \times E'). \tag{18.4}$$

The boundary identification preserves the two marked torus factors and reverses the normal circle, because the two base discs have opposite boundary orientations. The same regular-disc trivializations define both the boundary identifications of $E, E'$ and (18.4).

2. Since $X, X'$ are six-spheres, the Mayer–Vietoris sequence of $S^6 = E \cup (F \times D^2)$ gives, integrally,

$$H_0(E) = \mathbb{Z}, \qquad H_k(E) = H_{k-1}(F) \quad (1 \le k \le 4), \qquad H_k(E) = 0 \quad (k \ge 5). \tag{18.5}$$

The isomorphism is the normal-circle sweep $\alpha \mapsto [S^1] \times \alpha$. Indeed, in degrees one through four the boundary map to $H_k(E) \oplus H_k(F)$ is an isomorphism, and its second component is the identity on the unswept fibre summand and zero on the swept summand. In degree five the boundary fundamental class is the image of the sphere fundamental class, so $H_5(E) = 0$. Thus all homology groups of the two pieces and their common boundary in (18.4) are free abelian. The same statements hold for $E'$.

3. *Determining every boundary map integrally.* Let $V = H^1(F; \mathbb{Z})$, in the basis $(\gamma, u, w, \delta)$, and use the positively ordered volume $v = \gamma u w \delta$. The primitive global invariant classes of the original torus fibration are

$$I_0 = 1, \quad I_1 = \gamma, \quad I_2 = q = uw + 6\gamma\delta, \quad I_3 = \gamma uw, \quad I_4 = v.$$

The images of $H^k(E) \to H^k(F)$ are exactly

$$n_k \mathbb{Z} I_k, \qquad (n_1, n_2, n_3, n_4) = (12, 2, 2, 12). \tag{18.6}$$

We verify this integrally. On the disc the only local restrictions on a global invariant class are the finite-cover pullback lattices; the cusp specialization is an integral isomorphism onto its invariant lattice by 18.3. In degree one the finite images contain $3\gamma$ and $4\gamma$, giving the intersection $12\mathbb{Z}\gamma$. In degree two the order-three image contains $q$, while the order-four image requires an even coefficient of $q$, by the finite integral lattices computed in the proof of theorem 17.4. In degree three use the perfect integral pairing downstairs: an invariant class is a pullback precisely when its pairings with every pulled-back degree-one class are divisible by the covering degree. The other degree-one generators are $2u + w + 3\delta$ and $u + w + 2\delta$. Their pairings with $t\gamma uw$ are, up to signs, $3t$ and $2t$. Thus the first cover imposes no further condition, and the second requires $t$ even. In degree four the pullback degrees are three and four, giving $12\mathbb{Z}v$.

4. These local restrictions are also sufficient for the image from $E$. For a local system on a disc with punctures, the restriction from its free-group $H^1$ to the direct sum of the local coinvariants is onto: each puncture has its own independent free generator. Consequently $H^2(D, j_*\mathcal{L}) = 0$. Replacing $j_*\mathcal{L}$ by the direct-image sheaf changes it only by point-supported cokernels, and so does not change this vanishing. The Leray sequence over the disc therefore makes $H^k(E) \to H^0(D, R^k f_* \mathbb{Z})$ onto. This proves (18.6)

5. Write $\theta$ for the normal-circle covector. The dual of (18.5) identifies $H^k(E)$ with $\bigwedge^{k-1} V$, and its boundary restriction has the graph form

$$r_k(\alpha) = \phi_k(\alpha) + \theta\alpha, \qquad \phi_k : \bigwedge^{k-1} V \longrightarrow \bigwedge^{k} V. \tag{18.7}$$

The image of $\phi_k$ is exactly $n_k\mathbb{Z}I_k$ by (18.6). If $k + l = 5$, classes extending to $E$ have zero product evaluated on $\partial E$. Therefore

$$(-1)^k\phi_k(\alpha)\beta + \alpha\phi_l(\beta) = 0 \quad \text{in } \bigwedge^{4} V. \tag{18.8}$$

Each $I_k$ is primitive, and wedge pairing between the two complementary exterior powers is unimodular. Writing $\phi_k = f_kI_k$, equation (18.8), tested against a class pairing to one with $I_k$, proves that $f_k$ is an integral multiple of $\alpha \mapsto \langle \alpha I_{5-k}, [F]\rangle$. The exact image (18.6) fixes the absolute multiple. Thus

$$\begin{aligned} \phi_k(\alpha) &= c_k\langle \alpha I_{5-k}, [F]\rangle I_k, \\ (c_1, c_2, c_3, c_4) &= (12\varepsilon, 2\eta, -2\eta, -12\varepsilon), \qquad \varepsilon, \eta \in \{1, -1\}. \end{aligned} \tag{18.9}$$

This proves every boundary map up to two signs. We allow those signs independently for $E$ and $E'$ below; the answer is the same for all sixteen possibilities.

6. *The full integral Mayer–Vietoris matrices.* Let $i_p : H_p(F) \to H_p(E)$ be the transpose of $\phi_p$ in the sweep bases, with $i_0 = 1$; let $j_{p+1}$ denote the sweep isomorphism for $0 \le p \le 3$, and zero for $p = 4$. The analogous maps of the second copy are $i'_q, j'_{q+1}$. For a fibre tensor $\alpha \otimes \beta$, with degrees $p, q$, the matrix $D_k$ of (18.4) is

$$\alpha \otimes \beta \longmapsto (i_p\alpha \otimes \beta, -\alpha \otimes i'_q\beta), \qquad p + q = k. \tag{18.10}$$

For a swept tensor, place the normal circle first. Then

$$\theta \times \alpha \times \beta \longmapsto \big(j_{p+1}\alpha \otimes \beta, (-1)^p\alpha \otimes j'_{q+1}\beta\big), \qquad p + q = k - 1. \tag{18.11}$$

The second sign contains the Koszul sign from moving the circle past $\alpha$, the reversal of the two boundary-circle orientations, and the second minus in Mayer–Vietoris. The last two cancel. This specifies every integer entry of $D_k$.

7. In the exterior bases, exact Smith reduction of the four adjoint maps (18.9) and the full matrices (18.10)–(18.11) gives the following table. The source ranks are $\binom{9}{k}$; a zero binomial is understood outside $0 \le k \le 9$.

| $k$ | target rank | source rank | rank $D_k$ | unit factors | nonunit factors |
|---|---|---|---|---|---|
| 0 | 2 | 1 | 1 | 1 | $--$ |
| 1 | 10 | 9 | 9 | 9 | $--$ |
| 2 | 28 | 36 | 27 | 20 | $12^{[7]}$ |
| 3 | 64 | 84 | 53 | 36 | $2^{[7]}, 12^{[10]}$ |
| 4 | 114 | 126 | 82 | 58 | $2^{[17]}, 12^{[6]}, 24$ |
| 5 | 138 | 126 | 94 | 70 | $2^{[17]}, 12^{[6]}, 24$ |
| 6 | 104 | 84 | 73 | 56 | $2^{[7]}, 12^{[10]}$ |
| 7 | 44 | 36 | 35 | 28 | $12^{[7]}$ |
| 8 | 8 | 9 | 8 | 8 | $--$ |
| 9 | 0 | 1 | 0 | 0 | $--$ |
| 10 | 0 | 0 | 0 | 0 | $--$ |

The notation $a^{[r]}$ denotes $r$ entries equal to $a$. All sixteen choices of the two signs in each copy give precisely this table.

8. Finally the exact homology sequence gives

$$0 \longrightarrow \operatorname{coker} D_k \longrightarrow H_k(P) \longrightarrow \ker D_{k-1} \longrightarrow 0.$$

The last group is free, since the boundary homology is free. This sequence therefore splits as abelian groups, so the Smith factors in the table are the torsion of $H_k(P)$. Adding the free kernel ranks gives exactly the rational Betti vector in 18.2 and the integral groups asserted here. The resulting torsion also satisfies the required ten-dimensional duality $\operatorname{Tor} H_k(P) \cong \operatorname{Tor} H_{9-k}(P)$.

The corresponding exact calculations are implemented in the programs in Table 1. □

**Proposition 18.5** (A unique transverse family type and its stable tangent bundle)**.** [tag 02MR]

*For fixed marked sphere fibrations, all Möbius transformations whose critical triples are disjoint give orientation-preservingly diffeomorphic shifted products. The same conclusion holds while either sphere varies in its proper smooth marked period family. For every such fivefold $P$ there is an isomorphism of real vector bundles*

$$TP \oplus \mathbb{R}^4 \simeq \mathbb{R}^{14}. \tag{18.12}$$

*Thus all its positive-degree Stiefel–Whitney and Pontryagin classes vanish. For its given complex tangent bundle, $c_1(P) = c_2(P) = 0$ integrally.*

*Proof.* 1. The allowed transformations form the complement in $\mathrm{PGL}_2(\mathbb{C})$ of the nine analytic hypersurfaces imposing an equality of one critical value from each triple. This complement is path connected: a piecewise smooth path between two points can be perturbed relative to its endpoints to avoid these sets, which have real codimension two. Equivalently, use local complex coordinate balls and perturb each segment there. Choose a smooth such path.

2. The corresponding fibre products form a compact smooth family over its parameter interval. Indeed the vertical derivative of $f_1(x) - f_2(y)$ is surjective at every point, since its two factors cannot be critical at the same base value. The same surjectivity solves the equation for a lift of the interval coordinate vector field. A partition of unity supplies a global smooth lift; compactness makes its flow exist for the whole interval. This flow gives the claimed diffeomorphisms. The complex orientations vary continuously and hence are preserved. The identical argument applies to smooth proper period families of either factor, keeping the critical triples disjoint throughout.

3. View $P$ as the transverse inverse image of the diagonal in $B \times B$ under $f_1 \times f_2$. Its real oriented normal two-plane bundle is the underlying bundle of $g^*T_B$, where $g : P \to B$ is the common base map. It is trivial: its first Chern class is

$$c_1(g^*T_B) = 2g^*[B]^\vee = 2\,\mathrm{pr}_1^*\, f_1^*[B]^\vee = 0,$$

because $H^2(X;\mathbb{Z}) = 0$, and complex line bundles are classified by their first Chern class. The tangent bundles of the two six-spheres become trivial after adding one real line each. Restriction of their product tangent bundle gives

$$(TP \oplus \mathbb{R}^2) \oplus \mathbb{R}^2 \simeq \mathbb{R}^{14},$$

proving (18.12) and the characteristic-class assertions for the real tangent bundle. Finally the holomorphic tangent sequence is

$$0 \longrightarrow TP \longrightarrow \mathrm{pr}_1^* TX \oplus \mathrm{pr}_2^* TX' \longrightarrow g^*T_B \longrightarrow 0.$$

Both sphere factors have integral $c_1 = c_2 = 0$, since their cohomology in degrees two and four vanishes. The last line has zero first Chern class, as just proved. Multiplying total Chern classes in this exact sequence gives $c_1(P) = c_2(P) = 0$ integrally.

□

**Proposition 18.6** (The shifted product bounds with its natural stable framing)**.** [tag 02MS]

*Equip the two oriented six-spheres with their standard bounding stable tangent framings. Trivialize the oriented two-plane bundle $f_1^*T\mathbb{P}^1$ on the first sphere, and equip the transverse shifted product $P$ with the induced stable framing in (18.12). Then there is a compact smooth oriented eleven-manifold $V$ with*

$$\partial V = P, \qquad TV \oplus \mathbb{R}^3 \simeq \mathbb{R}^{14}, \qquad [P] = 0 \quad \text{in } \Omega^{\mathrm{fr}}_{10}. \tag{18.13}$$

*The boundary framing is the stated one. In particular its induced spin structure bounds.*

*Proof.* 1. Write $f_2 : S^6 \to \mathbb{P}^1$ for the shifted map of the second factor. By III.3.4 (p. 15) the first map $f_1 : S^6 \to \mathbb{P}^1$ is null-homotopic. It therefore extends to a smooth map $F : D^7 \to \mathbb{P}^1$, which can be chosen to equal $f_1$ on each sphere of a collar of the boundary. To see this with the boundary fixed, first use the null-homotopy to obtain a continuous extension constant in the collar direction. Smooth its interior in coordinate charts, with cutoffs supported away from the collar; on the overlap the extension is already smooth.

2. Let $C_2$ be the three critical values of $f_2$. The restriction $f_1$ is transverse to each point of $C_2$, since the two critical triples are disjoint. Perturb $F$ only in the interior to be transverse to all these points. The applicable finite simultaneous transversality statement is (Hirsch, 1976, Chapter 3, §2, Theorem 2.1, p. 74; Theorem 2.5, p. 78; cutoff globalization, pp. 75–76). For the relative assertion, the proof of that theorem uses local perturbations multiplied by cutoff functions. Their supports may be restricted to the complement of the collar, where transversality has not already been secured; the open condition on the collar persists. Compactness permits finitely many such perturbations.

3. Define the smooth fibre product

$$V = \{(x, y) \in D^7 \times S^6 : F(x) = f_2(y)\}. \tag{18.14}$$

At a point whose common value is outside $C_2$, the differential $df_2$ is onto. At a point over $C_2$, the differential $dF$ is onto. Consequently $dF - df_2$ is onto everywhere on (18.14). The same argument holds on its boundary, using $df_1$ there. The implicit-function theorem, including its boundary version, makes $V$ a smooth eleven-manifold with boundary exactly $P$. It is compact as a closed subset of $D^7 \times S^6$. The inverse-image orientation using the complex orientation of $T\mathbb{P}^1$ induces the complex orientation of $P$ at the boundary, because the collar keeps the first map fixed.

4. Let $g : V \to \mathbb{P}^1$ be the common map. The real normal two-plane bundle of (18.14) is $g^*T\mathbb{P}^1$, identified by $dF - df_2$. This is the pullback from $D^7$ of $F^*T\mathbb{P}^1$, which is trivial because the disc is contractible. A choice of its oriented framing restricts on the boundary to the chosen framing of $f_1^*T\mathbb{P}^1$ up to homotopy: two oriented framings of this two-plane bundle differ by a map $S^6 \to SO(2)$, and $H^1(S^6; \mathbb{Z}) = 0$. The tangent exact sequence and the bounding stable framing of the second sphere now give

$$\begin{aligned} TV \oplus \mathbb{R}^2 &\simeq T(D^7 \times S^6)|_V, \\ TV \oplus \mathbb{R}^3 &\simeq \mathbb{R}^7 \oplus (TS^6 \oplus \mathbb{R})|_V \simeq \mathbb{R}^{14}. \end{aligned}$$

At the boundary, the outward normal of $D^7$ supplies the additional real line in $TV|_P = TP \oplus \mathbb{R}$. The resulting framing is precisely the stable tangent framing obtained from the two sphere framings and the normal framing in 18.5. This proves (18.13) by the definition of framed bordism. It also proves the assertion about the induced spin structure, by forgetting the framing.

□

**Theorem 18.7** (Framed null-bordism over the fundamental group)**.** [tag 02MT]

*Let $G$ be the group in (18.1). There is a closed smooth aspherical four-manifold $N$ with*

$$\pi_1(N) = G, \qquad \widetilde{N} \simeq \mathbb{H}^2 \times \mathbb{R}^2, \qquad H_*(N;\mathbb{Z}) \cong H_*(S^1 \times S^3;\mathbb{Z}), \tag{18.15}$$

*and a smooth classifying map $c : P \to N$ inducing the displayed identification of fundamental groups. With the natural stable framing of 18.6, one has*

$$[P, c] = 0 \quad \text{in } \Omega^{\mathrm{fr}}_{10}(N). \tag{18.16}$$

*Thus $P$ has a compact stably framed eleven-dimensional filling over $BG$. This is stronger than the filling of 18.6.*

*Proof.* 1. *The aspherical target.* The quotient maps of I.22.4 (p. 298) are smooth maps $e_i : S^6 \to S^3$ over the base, and III.3.4 (p. 15) gives the exact factorizations $f_i = h_i e_i$. Here $h_i : S^3 \to \mathbb{P}^1$ is the weighted Hopf map of orders $3, 4$, with the second map shifted by the chosen Möbius transformation. Define

$$N = \{(u, v) \in S^3 \times S^3 : h_1(u) = h_2(v)\}, \qquad c(x, y) = (e_1(x), e_2(y)). \tag{18.17}$$

The critical values of each $h_i$ are only its two exceptional circle values. They are disjoint, so the same transversality argument as in III.9.3 (p. 317) makes $N$ a compact smooth four-manifold. The two circle actions on the weighted Hopf spheres give a principal $T^2$ bundle in the orbifold sense over

$$\mathcal{B} = S^2(3, 4, 3, 4).$$

At a cone of order $m$, its uniformizing chart is $(D^2 \times T^2)/C_m$, with the generator rotating the disc and translating one torus coordinate by a primitive fraction of denominator $m$. This local description is smooth and is the same finite-filling description as I.22.3 (p. 297) , with the unused circle kept.

2. The orbifold has Euler characteristic $2 - 2(1 - 1/3) - 2(1 - 1/4) = -5/6$. It is good and has hyperbolic orbifold universal cover, by (Scott, 1983, Theorems 2.3–2.4, p. 425; Euler-characteristic comparison, pp. 427–428). Pulling the principal torus bundle to $\mathbb{H}^2$ gives an ordinary principal $T^2$ bundle over a contractible base, hence the trivial bundle. Its universal cover is $\mathbb{H}^2 \times \mathbb{R}^2$. The local chart above shows that these orbifold pullbacks are ordinary covering maps of total spaces. This proves asphericity and injectivity of the fibre subgroup $\mathbb{Z}^2$.

3. Using the marked fibre generators $h_1, h_2$ and the four positive peripheral generators gives exactly

$$\pi_1(N) = \langle a_1, a_2, b_1, b_2, h_1, h_2 \mid h_1, h_2 \text{ central}, \quad a_1^3 = h_1, \quad a_2^4 = h_1^{-1}, \\ b_1^3 = h_2, \quad b_2^4 = h_2^{-1}, \quad a_1 a_2 b_1 b_2 = 1 \rangle.$$

The map $c$ sends the surviving regular-fibre generators $\widehat{\gamma}, \widehat{\gamma}'$ to $h_1, h_2$, respectively, and each peripheral generator to the one with the same name. This is because $e_i$ is the literal circle quotient on these fibres and finite collars. Its cusp replacement extends over a solid torus and kills precisely the cusp meridian in this presentation. Comparing with the full van Kampen calculation in 18.2 therefore shows that $c_*$ is the identity on the displayed presentation, and in particular is an isomorphism. As $N$ is aspherical, $c$ is a classifying map.

4. For completeness the abelian relation matrix has Smith diagonal $(1, 1, 1, 1, 1)$; a primitive integer kernel vector in the order $(a_1, a_2, b_1, b_2, h_1, h_2)$ is $(4, -3, -4, 3, 12, -12)$. Thus $H_1(N) = \mathbb{Z}$. Every stratum of its base has a torus fibre, so Euler additivity gives $\chi(N) = 0$. Integral Poincaré duality and the universal coefficient theorem give $H_3(N) = \mathbb{Z}$ and no torsion in $H_2(N)$. Euler characteristic then forces $H_2(N) = 0$, proving the integral homology assertion in (18.15).

5. *The bordism computation before taking the inverse image.* For a based sphere, the stable splitting $(S^3)_+ \simeq S^0 \vee S^3$ identifies

$$\Omega^{\mathrm{fr}}_6(S^3) = \pi^{\mathrm{S}}_6 \oplus \pi^{\mathrm{S}}_3. \tag{18.18}$$

We use Pontryagin–Thom in its form for maps to a space, (Davis and Kirk, 2001, Definition 8.16, p. 209; Theorem 8.17 and proof, pp. 209–210). The first component of $[S^6, e_i]$ is obtained by forgetting the map, so it is zero for the bounding sphere framing. The second component is the stable class of $e_i$, equivalently the framed inverse image of a regular point. It is $\eta^3$ by III.3.3 (p. 14) . One can also check the splitting directly: the Pontryagin–Thom collapse of the framed sphere into its disjoint-basepoint suspension has degree one on the sphere summand; composing with $e_i$ gives its stable homotopy class. Thus

$$[S^6, e_i] = (0, \eta^3).$$

Smashing the two sphere splittings gives

$$(S^3 \times S^3)_+ \simeq S^0 \vee S^3 \vee S^3 \vee S^6.$$

Under this splitting the external product $[S^6 \times S^6, e_1 \times e_2]$ has four components. The first three each contain a factor $[S^6] = 0$; the last is $\eta^3\eta^3 = \eta^6 = 0$. Indeed $\eta^4$ belongs to $\pi^{\mathrm{S}}_4 = 0$, as recorded in (Isaksen et al., 2023, Corollary 1.3 and Table 1, pp. 112–113). Therefore

$$[S^6 \times S^6, e_1 \times e_2] = 0 \quad \text{in } \Omega^{\mathrm{fr}}_{12}(S^3 \times S^3). \tag{18.19}$$

6. *The comparison with the classifying map.* By (18.19) there is a compact stably framed thirteen-manifold $Z$ with boundary $S^6 \times S^6$ and a map $E : Z \to S^3 \times S^3$ extending $e_1 \times e_2$. Smooth it relative to the boundary and make it transverse to $N$ there. Transversality on the boundary already holds: the derivative of its defining equation is $df_1 - df_2$, which is onto. Relative transversality therefore supplies the interior perturbation without changing that boundary.

7. The normal bundle of $N \subset S^3 \times S^3$ is the restriction of $h_1^* T\mathbb{P}^1$ pulled back from the first sphere. This oriented two-plane bundle is trivial because $H^2(S^3; \mathbb{Z}) = 0$. Choose a framing on it before restriction to $N$. Its pullback under $e_1 \times e_2$ is a framing of $f_1^* T\mathbb{P}^1$, agreeing with that in 18.6 up to homotopy since $H^1(S^6; \mathbb{Z}) = 0$. It follows that

$$V = E^{-1}(N), \qquad E|_V : V \longrightarrow N$$

is a compact stably framed eleven-manifold with boundary $(P, c)$ and the required boundary framing. This proves (18.16).

The finite algebra and combinatorial data are implemented in the programs in Table 1. □

## 19 The fibre complement and its homotopy coinvariants

**Definition 19.1** (Ordinary closed spin)**.** [tag `PV23`] For a closed submanifold $L \subset \operatorname{int} D^n$, its ordinary closed spin is $L \times S^1$ in $D^n \times S^1 \subset \partial(D^n \times D^2) = S^{n+1}$.

**Proposition 19.2** (The fibre complement and a spinning obstruction)**.** [tag `02MA`]

*Let $F = T^4$ be a regular fibre of the original sphere fibration and $E = S^6 \setminus \operatorname{int}(F \times D^2)$, with the marking of 18.4. Its group and peripheral homomorphism are*

$$\begin{aligned} K = \pi_1(E) = \langle a, b, h \mid h \text{ central},\ a^3 = h,\ b^4 = h^{-1} \rangle, \\ g \longmapsto h, \qquad u, w, d \longmapsto 1, \qquad \mu \longmapsto ab. \end{aligned} \tag{19.1}$$

*Here $\mu$ is the normal meridian, with the peripheral orientation chosen so that $[a] = 4[\mu]$, $[b] = -3[\mu]$ and $[h] = 12[\mu]$. The image of the boundary group is the embedded copy $\langle h, \mu\rangle \simeq \mathbb{Z}^2$. In the unique homotopy class of normal framings for which every pushed-off loop of $F$ has linking number zero, the image of $g$ is*

$$\lambda = h\mu^{-12}, \qquad \ker\{\pi_1(F) \longrightarrow K\} = \mathbb{Z}\langle u, w, d\rangle. \tag{19.2}$$

*In particular the peripheral structure is that of the $(3, 4)$ torus knot: the marked $g$ coordinate is its regular Seifert fibre, and (19.2) gives the preferred longitude. The embedded four-torus $F \subset S^6$ is not the ordinary threefold closed spin of that classical knot, even allowing ambient orientation reversal and arbitrary changes of its normal framing. More strongly, it is not a single ordinary closed spin of any embedded $T^3 \subset S^5$.*

*Proof.* 1. Apply the van Kampen presentation in III.3.10 (p. 23) with the regular base disc removed. The cusp kills $w, d$ and its peripheral loop. Conjugating $w$ by the order-three generator also kills $u$. The two finite caps impose $a^3 = h$ and $b^4 = h^{-1}$; there is now no relation $ab = 1$, since the outer boundary has not been filled. The outer meridian is $ab$ in the stated orientation. This proves (19.1), including the peripheral map. Eliminating $h$ gives $\langle a, b \mid a^3b^4 = 1\rangle$. Its quotient by $h$ is $C_3 * C_4$. The word $ab$ has infinite order in that free product, so a relation $h^r\mu^s = 1$ forces $s = 0$. Abelianization then forces $r = 0$. This proves peripheral injectivity. It also shows that $\lambda$ has infinite order: its image in $C_3 * C_4$ is $(ab)^{-12}$.

2. Write $\theta \in H^1(\partial E; \mathbb{Z})$ for the normal-circle covector and use $(\gamma, u, w, \delta)$ for $H^1(F; \mathbb{Z})$. The generator $\xi \in H^1(E; \mathbb{Z})$ restricts to $\theta + 12\gamma$. Homotopy classes of normal framings differ by $H^1(F; \mathbb{Z})$. Consequently there is a unique framing for which the associated section $s : F \to \partial E$ satisfies $s^*\xi = 0$: it has $s^*\theta = -12\gamma$. Its induced peripheral map is exactly (19.2). Simultaneously reversing the meridian conventions changes the displayed signs and preserves all the following assertions.

3. For this section set

$$\mathcal{R} = \operatorname{im}\{H^2(E; \mathbb{Z}) \longrightarrow H^2(F; \mathbb{Z})\}, \qquad \operatorname{rad}(\mathcal{R}) = \{v \in H_1(F; \mathbb{Z}) : \iota_v\omega = 0 \text{ for all } \omega \in \mathcal{R}\}. \tag{19.3}$$

This uses the canonical identification of the cohomology of a torus with the exterior algebra of its first cohomology. Equations (18.7)–(18.9) give classes whose boundary restrictions are $\theta u$ and $\theta w$: their unswept terms are zero because $u\gamma uw = w\gamma uw = 0$. Restricting them along $s$ proves

$$-12\gamma u, \ -12\gamma w \ \in \mathcal{R}. \tag{19.4}$$

The simultaneous contraction kernel of these two forms is $\mathbb{Z}d$. Indeed their contractions force successively $\gamma(v) = u(v) = w(v) = 0$. Thus

$$\operatorname{rad}(\mathcal{R}) \subseteq \ker\gamma = \ker\{\pi_1(F) \longrightarrow K\}. \tag{19.5}$$

Both the zero-linking framing and this inclusion are invariants of the embedded pair, independently of the torus coordinates.

4. Use the ordinary closed spin of Definition 19.1. Start with the closed classical knot $K_0 \subset \operatorname{int} D^3$ and repeat this operation three times, each time placing the previous compact pair in the interior of a disc. This is the ordinary torus spinning construction reviewed in (Winter, 2009, arXiv v3, §3, paragraph before Theorem 3.4, p. 4). If $\Sigma$ is a Seifert surface for $K_0$, the same three spins give the embedded Seifert hypersurface $\Sigma \times T^3$ with boundary $K_0 \times T^3$. Pushing the latter off using the normal of this hypersurface gives its zero-linking framing.

5. Let $v$ be the original knot-circle class and let $v_1, v_2, v_3$ be the spinning circles. Each pushed-off two-cycle $v \wedge v_i$ bounds the corresponding copy of $\Sigma \times S^1$ in a parallel copy of that hypersurface, disjoint from the knot. Hence every restriction of a complement two-class vanishes on $v \wedge H_1(F)$: $v$ belongs to $\mathrm{rad}(\mathcal{R})$. On the other hand its pushed-off loop is the original preferred longitude and is nontrivial for $K_0 = T(3,4)$, whereas the spinning circles are trivial in the complement group. These peripheral claims also follow directly at each spin: its complement is

$$((D^n \setminus \operatorname{int} \nu L) \times S^1) \ \cup_{S^{n-1} \times S^1} \ (S^{n-1} \times D^2),$$

so van Kampen kills precisely the new spinning circle and preserves the old complement group. They agree with the primary peripheral calculation (Winter, 2009, arXiv v3, Theorem 4.5 and following discussion, pp. 9–10) in the first spin. Thus the ordinary spun torus violates (19.5). An identification of embedded pairs would preserve its two sides, giving a contradiction.

6. For the stronger assertion, let $L = T^3 \subset S^5$ be arbitrary and suppose that its ordinary closed spin were our four-torus. Write $E_L$ for its exterior. Deleting a ball disjoint from $L$ changes cohomology only in degrees at least four. In the displayed spun-complement decomposition, with $n = 5$, the cap and intersection are $S^4 \times D^2$ and $S^4 \times S^1$. Mayer–Vietoris and the surjectivity onto the latter first cohomology therefore give, in degrees two and three, precisely the cohomology of $(E_L \setminus D^5) \times S^1$. Let $s$ be the spinning-circle covector and $v_s$ its primitive homology generator. Put $\mathcal{R}_L^j = \mathrm{im}\{H^j(E_L) \to H^j(L)\}$, using the zero-linking framing. One has $\mathcal{R}_L^1 = 0$. Also $\mathcal{R}_L^3 = 0$: the pushed-off $L$ bounds a parallel Seifert hypersurface in its exterior, so every top-degree restriction evaluates to zero. The product restrictions consequently satisfy

$$\mathcal{R}^2 = \mathcal{R}_L^2, \qquad \mathcal{R}^3 = \mathcal{R}_L^2 \wedge s. \tag{19.6}$$

In particular $v_s \in \mathrm{rad}(\mathcal{R}^2)$, and contraction by $v_s$ is injective on $\mathcal{R}^3$. For our four-torus, (19.4) forces $v_s = \pm d$. But (18.7)–(18.9), now applied to $\alpha = \gamma\delta$ in degree three, show that $\mathcal{R}^3$ contains $\pm 2\gamma uw$. Its swept term vanishes in the zero-linking framing, and this is a nonzero class annihilated by $d$. This contradicts that injectivity. No ordinary closed spin is possible. □

**Theorem 19.3** (Exact second-homotopy coinvariants)**.** [tag 02MB]
*For the classifying map $c : P \to N = K(G,1)$ of 18.7, one has*

$$c_* : H_3(P;\mathbb{Z}) \longrightarrow H_3(N;\mathbb{Z}) = \mathbb{Z} \quad \text{equal to zero}, \qquad (\pi_2(P))_G \simeq \mathbb{Z}^2 \oplus (\mathbb{Z}/12)^6. \tag{19.7}$$

*The pullback of a generator of $H^3(N;\mathbb{Z})$ has exact order twelve in $H^3(P;\mathbb{Z})$. In particular $\pi_2(P)$ is not a projective $\mathbb{Z}G$-module. The fivefold cannot be diffeomorphic, or homotopy equivalent, to $N \times S^6$ connected-summed with any finite collection of $S^r \times S^{10-r}$, $2 \le r \le 8$, and any homotopy ten-sphere.*

*Proof.* 1. Use the separating base circle in 18.4 also in the target of 18.7. There $M$ denotes the exterior of a regular orbit of the weighted Hopf map, namely the torus-knot exterior of I.22.3 (p. 297) . The two decompositions, and the restriction of $c$, are

$$\begin{array}{ccc} (E \times F') \cup (F \times E') & \longrightarrow & (M \times S^1) \cup (S^1 \times M') = N, \\ F \times F' \times S^1 & \longrightarrow & S^1 \times S^1 \times S^1. \end{array} \tag{19.8}$$

On the common boundary the map retains exactly the covectors $\gamma_1, \gamma_2, \theta$. It is the literal quotient there. Each knot exterior has the integral homology of a circle, so the third homology and cohomology of each target piece in (19.8) vanish.

2. We first prove the homological statement. In the degree-two source of (18.10)–(18.11) write $t_1, t_2, t_{12}$ for the coefficients of $\theta \times g_1$, $\theta \times g_2$, and $g_1 \times g_2$. The image of $i_2 : H_2(F) \to H_2(E)$ lies entirely in the $d$-sweep coordinate, by (18.9). Consequently the $g_1$-sweep coordinate in the first $H_2(E)$ summand of the target is just $t_1$, and the corresponding second coordinate is just $t_2$. Every vector in the kernel of the boundary matrix therefore has $t_1 = t_2 = 0$. Its first $H_1(E) \otimes \mathbb{Z}g_2$ coordinate then forces $12t_{12} = 0$, up to an immaterial orientation sign. These three coordinates are the only ones retained in $H_2(T^9) \to H_2(T^3)$. The target Mayer–Vietoris sequence injects $H_3(N)$ into $H_2(T^3)$; its source-piece third groups are zero. Naturality now proves $c_* H_3(P) = 0$.

3. The cohomological calculation determines the extension. On the target boundary write $v = \gamma_1\gamma_2$. The generators of the two degree-two piece cohomology groups restrict, with consistent meridian orientations, to

$$\theta\gamma_2 + 12v, \qquad \theta\gamma_1 + 12v. \tag{19.9}$$

Their image is a primitive rank-two subgroup of $H^2(T^3)$: the minor in the last two coordinates is a unit. Its quotient is freely generated by $v$. Since the degree-three piece groups are zero, the connecting class $\partial v$ is a generator of $H^3(N; \mathbb{Z})$.

4. In the source decomposition both $\theta\gamma_1$ and $\theta\gamma_2$ already belong to the image of the degree-two piece restrictions. This is (18.7) with $\alpha = \gamma$, since $\phi_2(\gamma) = 0$; the second normal circle is reversed, which only changes its sign. Subtracting one of these classes from the corresponding product of a degree-one class of $E$ with $\gamma_2$ proves that $12v$ is a source boundary image. Conversely the coefficient of $v$ in every such boundary image is divisible by twelve. The pure degree-two terms from a single factor have no $v$ coefficient, and the mixed terms use the restriction $\theta + 12\varepsilon\gamma$ of the degree-one generator, with $\varepsilon = \pm 1$. It follows that

$$c^*(\partial v) = \partial v \quad \text{has exact order 12.} \tag{19.10}$$

This uses the Mayer–Vietoris image.

5. Put $A = (\mathbb{Z}/12)^7$ and $L = (\pi_2(P))_G$. The low-degree homology sequence of the universal-cover fibration, together with 18.4 and 18.7, gives

$$0 \longrightarrow \mathbb{Z}z \longrightarrow L \longrightarrow \mathbb{Z} \oplus A \longrightarrow 0. \tag{19.11}$$

The zero on the left uses $c_* H_3(P) = 0$, just proved. To determine this extension, replace $c$ by its mapping cylinder and use relative groups for $(N, P)$. The induced map on fundamental groups is an isomorphism and the target has zero second homotopy group. On universal covers the relative pair is two-connected; relative Hurewicz gives $H_3(\widetilde{N}, \widetilde{P}) = \pi_2(P)$ and all lower relative groups vanish. The relative covering homology sequence therefore identifies $H_3(N, P) = L$ and $H_2(N, P) = 0$. The universal coefficient theorem and the relative cohomology sequence identify the map

$$H^3(N, P) = \operatorname{Hom}(L, \mathbb{Z}) \longrightarrow H^3(N) = \mathbb{Z}$$

with evaluation on $z$. Its image is $12\mathbb{Z}$ by (19.10).

6. The free summand of the quotient in (19.11) splits. Lift its seven order-twelve generators to $x_1, \dots, x_7$ in the remaining extension. There are integers $e_i$, defined modulo twelve, with the complete abelian presentation

$$\langle z, x_1, \dots, x_7 \mid 12x_i = e_i z \ (1 \le i \le 7) \rangle. \tag{19.12}$$

Evaluation of homomorphisms to $\mathbb{Z}$ on $z$ has image $(12/\gcd(12, e_1, \dots, e_7))\mathbb{Z}$. Thus that gcd is one. The relation matrix in (19.12) has Smith factors $1, 12, 12, 12, 12, 12, 12$, giving $\mathbb{Z} \oplus (\mathbb{Z}/12)^6$. Adding the split free summand proves (19.7).

7. If $\pi_2(P)$ were projective over $\mathbb{Z}G$, it would be a direct summand of a free module. Taking coinvariants preserves that splitting, so $L$ would be a projective, hence free, abelian group. The torsion in (19.7) contradicts this. Finally $N \times S^6$ has zero second homotopy group. Connected sum in dimension ten with a simply connected sphere product adds its second homotopy as a free induced $\mathbb{Z}G$-module, as seen by lifting the connected-sum necks and using their simply connected boundary spheres. Only $r = 2, 8$ contribute, with one free generator each; a homotopy sphere contributes none. Their second homotopy modules are therefore free, so none is homotopy equivalent to $P$.

The finite algebra and combinatorial data are implemented in the programs in Table 1. □

**Theorem 19.4** (Central localizations of the homotopy module)**.** [tag 02MC]

*Take the central generators $h_1, h_2$ of $G$ from 18.7 and put*

$$R = \mathbb{Z}G, \qquad R_i = R[(h_i - 1)^{-1}], \qquad R_{12} = R[(h_1 - 1)^{-1}, (h_2 - 1)^{-1}]. \tag{19.13}$$

*These are flat central localizations. As modules over the indicated localized group rings, the second homotopy group satisfies*

$$\begin{aligned} R_1 \otimes_R \pi_2(P) &\simeq R_1/(h_2 - 1), \\ R_2 \otimes_R \pi_2(P) &\simeq R_2/(h_1 - 1), \\ R_{12} \otimes_R \pi_2(P) &= 0. \end{aligned} \tag{19.14}$$

*More generally, $H_j(P; R_1)$ is zero outside $2 \le j \le 7$; in those six degrees it is respectively a direct sum of*

$$1,\ 3,\ 4,\ 4,\ 3,\ 1 \tag{19.15}$$

*copies of $R_1/(h_2 - 1)$. The analogous statement holds with the indices interchanged. All homology with coefficients in $R_{12}$ vanishes. For the fibre complement of 19.2, with $S = \mathbb{Z}K[(h-1)^{-1}]$, one likewise has*

$$S \otimes_{\mathbb{Z}K} \pi_2(E) \simeq S, \qquad H_4(E; S) \simeq S, \tag{19.16}$$

*and its other localized homology groups vanish. Moreover, if $q = (h_1 - 1)(h_2 - 1)$, then*

$$q^2 \pi_2(P) = 0, \qquad \dim_{\mathbb{Q}}(\pi_2(P) \otimes \mathbb{Q}) = \infty. \tag{19.17}$$

*Consequently the universal cover has no finite homotopy type. There is no fibration of $P$ over $N$ with compact manifold fibre and inducing an isomorphism on fundamental groups.*

*Proof.* 1. *The cusp module with its marked deck transformation.* In the original cusp, the angular generators $w, d$ vanish and $\pi_1(N_0) = \mathbb{Z}\langle g, u\rangle$. In $E$ their images are respectively $h, 1$; in the first factor of $P$ they are $h_1, 1$. Thus the connected cover of this cusp occurring inside the universal cover is the cover $\widehat{W}$ of its central fibre in which the $g$ deck transformation is $z$, whereas $u$ remains a loop. Write $A = \mathbb{Z}[z, z^{-1}]$. Its ordinary homology modules are

$$H_0(\widehat{W}) = H_1(\widehat{W}) = A/(z-1), \quad H_2(\widehat{W}) = A \oplus (A/(z-1))^2, \quad H_3(\widehat{W}) = 0, \quad H_4(\widehat{W}) = A. \tag{19.18}$$

Here is an integral normalization calculation of this input. The dual complex of the cover is an infinite cylinder, whose ordinary homology is $A/(z-1)$ in degrees zero and one and zero in degree two. The normalization degree-four row is a free copy of $A$. In degree two, the homology map from the three double curves to the four component classes is the transpose, up to the unimodular intersection marking and possible inversion of $z$, of

$$B(z) = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & z \\ -z & -z & -1 & -z \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z-1 & 0 & 0 \\ 0 & 0 & z-1 & 0 \end{pmatrix}. \tag{19.19}$$

For example multiply on the left by $\left(\begin{smallmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -z & 0 & 1 \end{smallmatrix}\right)$, then use as new columns $e_1, e_4-e_1, e_3-e_1, e_2-e_1$. Both operations are unimodular over $A$. The transpose is injective and its cokernel is the displayed degree-two module. All strata have only even homology, and the normalization direction has length two, so no higher differential is possible. There is no degree-two extension from the dual complex, whose second homology is zero. This proves (19.18). Ordinary chains in this locally finite normalization have compact support and use direct sums.

2. *Localization and the vanishing of every other piece.* The central subgroup $\langle h_1, h_2\rangle$ is an injected $\mathbb{Z}^2$ by 18.7. The group ring is free over its central Laurent subring, by choosing coset representatives. In particular $h_i - 1$ and their products are non-zero-divisors. Inverting them is central localization and is flat; it commutes with homology of the universal-cover chain complex. The analogous assertions hold for $h$ in $K$.

3. Use the regular torus bundle over a graph and the six closed cap neighborhoods in 18.2. For $R_1$ coefficients, the first regular torus has its $g_1$ loop acting by $h_1$. The torus chain complex is the Koszul complex of its commuting loop actions. Its $h_1 - 1$ term is invertible, providing a chain contraction. Thus every regular fibre has zero homology with these coefficients. The homology sequence of a bundle over a graph, and the same calculation over each boundary circle, show that the whole regular piece and every cap boundary are acyclic. Each cap of the second factor is a product with a first-factor regular torus over a disc, so it too is acyclic.

4. At a finite cap of the first factor, the affine generator $a_j$ has $a_j^{m_j} = h_1^{\pm 1}$ in its image in $G$. The connected cover of the reduced flat surface corresponding to its map onto $\langle a_j\rangle$ is $T^3 \times \mathbb{R}$. The $m_j$-th power acts by an integral translation with linear part $A_j^{m_j} = I$, and therefore acts identically on all of its homology. Thus $h_1 - 1$ annihilates every such homology module before localization. Keeping the other regular torus does not change that assertion, and induction to $G$ commutes with the central action. Both first-factor finite caps are therefore acyclic after localization as well.

5. *The one surviving cap and induction.* The first cusp is $N_{0,1} \times F_2$, since the two critical triples are disjoint. Its image in $G$ is precisely $H = \langle h_1, h_2\rangle$. A connected component of its preimage in the universal cover is
$$\widehat{W} \times \mathbb{R} \times T^3$$
up to deformation retraction: $g_1$ acts on $\widehat{W}$ by $z = h_1$, and $g_2 = h_2$ translates the real factor. It acts trivially on the homology of that three-torus. All components together give induction from $\mathbb{Z}H$ to $R$. By (19.18), inverting $h_1 - 1$ leaves one free $\mathbb{Z}[h_1^{\pm 1}, (h_1-1)^{-1}]$ summand in cusp degrees two and four and nothing else. The integral product homology with $T^3$ has no torsion correction. Inducing the result gives
$$H_j(N_{0,1} \times F_2; R_1) \simeq (R_1/(h_2-1))^{\binom{3}{j-2}+\binom{3}{j-4}}, \tag{19.20}$$
where a binomial coefficient is zero outside its usual range. All other pieces and their intersections are acyclic, as proved above, so Mayer–Vietoris identifies this with $H_j(P; R_1)$. This proves (19.15) and (19.20), and interchanging the factors proves the second version. Inverting $h_2 - 1$ also kills (19.20), proving the double-localization assertion. Since $\widetilde{P}$ is simply connected, Hurewicz identifies its second homology with $\pi_2(P)$; flat localization now gives (19.14).

6. For $E$ the same decomposition has only one cusp and two finite caps. Every other piece is acyclic after $h - 1$ is inverted by the identical argument. Its cusp image is $\langle h\rangle$, so inducing (19.18) after localization gives one free $S$ summand in degrees two and four. This proves (19.16).

7. *A uniform annihilator without localization.* Put $I = (h_1 - 1, h_2 - 1) \subset R$. Before localization, the homology of a regular eight-torus with $R$ coefficients is

$$(R/I) \otimes_{\mathbb{Z}} \bigwedge^{j} \mathbb{Z}^6 \quad (0 \le j \le 6). \tag{19.21}$$

Indeed the two nontrivial loop images are the independent central generators $h_1, h_2$. Their differences from one form a regular sequence in the central Laurent ring, and $R$ is free over that ring. The two-variable Koszul complex therefore has only its degree-zero quotient as homology; the other six loop differentials are zero. This proves (19.21). The homology sequence of the regular bundle over a graph has only base degrees zero and one. Both filtration quotients are annihilated by $I$, by (19.21), so $I^2$ annihilates the homology of the regular piece. The same two-stage Wang argument applies to every boundary mapping torus.

8. At a finite cap its connected lift is a product of six torus directions and two real directions up to deformation retraction. Each central deck generator acts trivially on homology: the one belonging to that cap is the finite affine power already checked above, and the other translates the regular real direction. Hence $I$ annihilates its induced homology. At the first cusp, $h_2$ translates the real factor of the other regular torus, so $h_2 - 1$ annihilates all of its homology; the second cusp is annihilated by $h_1 - 1$. Thus $q$ annihilates the homology of every piece and every intersection, since $q \in I^2$. The decomposition has a regular piece and disjoint caps with disjoint boundary collars, so its Mayer–Vietoris sequence gives in every degree

$$0 \longrightarrow \operatorname{coker} D_j \longrightarrow H_j(P; R) \longrightarrow \ker D_{j-1} \longrightarrow 0.$$

Both end groups are killed by $q$, and therefore the middle group is killed by $q^2$. Since $H_2(P; R) = \pi_2(P)$, this proves the first assertion of (19.17) integrally.

9. *Infinite rational rank.* If $\pi_2(P) \otimes \mathbb{Q}$ were finite dimensional, localizing its linear automorphism $h_1$ at $h_1 - 1$ would remain finite dimensional. Explicitly, this removes the generalized eigenspace with eigenvalue one and inverts an automorphism on the remaining finite-dimensional quotient. But (19.14), tensored with $\mathbb{Q}$, contains a copy of $\mathbb{Q}[h_1^{\pm 1}, (h_1 - 1)^{-1}]$, as follows by taking the identity coset in the free module over the central Laurent subring and setting $h_2 = 1$. That vector space is infinite dimensional, a contradiction. This proves the second assertion and rules out a finite homotopy type of the universal cover. A fibration over the aspherical $N$ inducing a fundamental-group isomorphism would have simply connected fibre, and pulling it to $\widetilde{N}$ would identify that fibre up to homotopy with $\widetilde{P}$. A compact manifold fibre has finite homotopy type, so such a fibration is impossible.

The corresponding exact calculations are implemented in the programs in Table 1. □

## 20 The circle fibration and its torus twist

**Theorem 20.1** (A circle fibration and infinite-order return)**.** [tag 02MD]
*The shifted fivefold admits a smooth fibre bundle*

$$Y^9 \longrightarrow P \xrightarrow{\rho} S^1 \tag{20.1}$$

*with connected closed fibre, inducing the primitive character*

$$\chi(a_1, a_2, b_1, b_2, h_1, h_2) = (4, -3, -4, 3, 12, -12) \tag{20.2}$$

*on the presentation of 18.7. Its fibre group is the fundamental group of the oriented principal circle bundle of Euler number $-1$ over a surface of genus six. Here the base has its complex*

*orientation and the positive circle is $h_1h_2$; reversing that circle changes the Euler number to $+1$. In particular*

$$\pi_1(Y) = \ker \chi, \qquad H_1(Y;\mathbb{Z}) = \mathbb{Z}^{12}. \tag{20.3}$$

*The return diffeomorphism of (20.1) has infinite order even among unbased homotopy classes of self-equivalences of $Y$, although its induced outer automorphism of $\pi_1(Y)$ has exact order twelve. No connected finite cover of $P$ is homotopy equivalent to a product $S^1 \times Z$.*

*Proof.* 1. *A closed one-form with a nonzero derivative at every point.* Use the smooth map $c = (e_1, e_2) : P \to N$ constructed in 18.7. The manifold $N$ has the locally free $T^2$ action induced by the two weighted Hopf actions. Take a closed one-form representing the primitive integral class $\chi$ on $N$ and average it over this compact torus. Its average $\alpha$ is closed, represents the same integral class, and is invariant. If $V_i$ is the fundamental vector field with time-one orbit $h_i$, Cartan's formula gives $d(\alpha(V_i)) = 0$. Integrating along an ordinary orbit therefore gives

$$\alpha(V_1) = 12, \qquad \alpha(V_2) = -12. \tag{20.4}$$

The integrality of the periods integrates $\alpha$ to a smooth map $\rho_N : N \to \mathbb{R}/\mathbb{Z}$.

2. Choose the two cusp replacements defining $e_i$ in disjoint base discs, themselves disjoint from the other factor's critical values. At every point of $P$, at least one factor is a regular torus at which its $e_i$ is the literal character quotient of I.22.1 (p. 295) . In a local flat marking of that factor, the constant vertical vector $g_i$ maps to $V_i$. Holding the other factor fixed gives a tangent vector to $P$, and

$$(c^*\alpha)(g_i) = 12 \quad \text{or} \quad -12.$$

Thus $c^*\alpha$ is nowhere zero. The map $\rho = \rho_N c$ is a proper submersion. Equivalently, choose a vector field $W$ with $(c^*\alpha)(W) = 1$; its complete flow supplies local trivializations and a return diffeomorphism. The fibre is connected because $\chi$ is primitive: the image in $\pi_1(S^1)$ of a connected component of a bundle's total space has index equal to the number of fibre components. The homotopy sequence now proves $\pi_1(Y) = \ker \chi$.

3. *The exact surface cover and its circle Euler class.* Write $\Gamma = G/\langle h_1, h_2\rangle$ for the oriented orbifold group of $S^2(3,4,3,4)$. The character modulo twelve descends to the surjection

$$\overline{\chi} : \Gamma \longrightarrow \mathbb{Z}/12, \qquad (a_1, a_2, b_1, b_2) \longmapsto (4, -3, -4, 3). \tag{20.5}$$

Its restriction to every cone group is injective. Hence its kernel $\Gamma_0$ is torsion free and defines a degree-twelve ordinary surface cover of the orbifold. Its Euler characteristic is $12(-5/6) = -10$, so that surface is $\Sigma_6$. The image of $\ker \chi$ in $\Gamma$ is exactly $\Gamma_0$: a lift of an element of $\Gamma_0$ has character a multiple of twelve, which can be cancelled by a power of $h_1$. Its central kernel is the primitive subgroup

$$\ker \chi \cap \langle h_1, h_2\rangle = \mathbb{Z}\langle h_1 h_2\rangle. \tag{20.6}$$

4. We compute the circle Euler number in the following orientation convention. On either weighted Hopf sphere put $z_1 = \cos r\, e^{i\varphi_1}$ and $z_2 = \sin r\, e^{i\varphi_2}$. For the positive action with weights $(4,3)$ the smooth invariant connection is

$$\vartheta = \tfrac{1}{4}\cos^2 r\, d\varphi_1 + \tfrac{1}{3}\sin^2 r\, d\varphi_2.$$

It evaluates to one on the generator with period $2\pi$. With the boundary orientation of the unit ball in $\mathbb{C}^2$,

$$\frac{1}{(2\pi)^2}\int_{S^3} \vartheta \wedge d\vartheta = \frac{1}{12}. \tag{20.7}$$

For the associated complex line whose unit circle is this positive circle bundle, the convention agreeing with the tautological Hopf line is $c_1 = -[d\vartheta/(2\pi)]$. Thus each orbifold circle Euler

number is $-1/12$. Both shifted base identifications preserve the complex orientation. Pullback to $\Sigma_6$ multiplies the two Euler numbers by twelve, giving $(-1,-1)$; this is also the covering formula of (Scott, 1983, Theorem 3.6, p. 436). In the integral fibre basis

$$v = (1,1), \qquad w = (0,1)$$

the determinant is one, the Euler vector is $-v$, and $\chi(v) = 0$, $\chi(w) = -12$. Consequently the circle kernel has Euler number $-1$ with generator $h_1h_2$.

5. One can see this bundle directly in $N$. Invariance of $\alpha$ implies $\rho_N((s_1,s_2)x) = \rho_N(x) + 12(s_1 - s_2)$, where $s_i \in \mathbb{R}/\mathbb{Z}$. A level contains twelve diagonal circles over an ordinary base point. The peripheral permutations (20.5) give exactly the connected cover $\Sigma_6$; they have no stabilizers at the cone points. The diagonal circle acts freely, since at every cone only one of the two Hopf actions has a nontrivial stabilizer. The level is therefore the circle bundle just computed. This identifies its fundamental group with $\ker\chi$ and gives a three-dimensional aspherical model. The homology sequence of a circle bundle of Euler number $\pm1$ gives its first homology $\mathbb{Z}^{12}$, proving the second assertion in (20.3).

6. *Faithful central action and the return obstruction.* The centre of $G$ is exactly $\langle h_1, h_2\rangle$: the quotient is a nonelementary oriented hyperbolic orbifold group with trivial centre. This centre acts faithfully on $\pi_2(P)$. Indeed if $h_1^a h_2^b$ acts as the identity, localize first at $h_1 - 1$ and apply 19.4. On $R_1/(h_2-1)$ its action is multiplication by $h_1^a$. This module is free over the central ring $\mathbb{Z}[h_1^{\pm1}, (h_1-1)^{-1}]$ after choosing coset representatives, so this is the identity only if $a = 0$. The symmetric localization then gives $b = 0$.

7. If $H < G$ has finite index, its image in $\Gamma$ is again nonelementary. Its centralizer in $\Gamma$ is trivial: commuting with two hyperbolic elements with distinct axes fixes at least three boundary points and hence is the identity. It follows that $Z(H) = H \cap Z(G)$. A homotopy product $S^1 \times Z$ for the corresponding finite cover would supply an infinite central element acting trivially on $\pi_2$, namely the circle factor; this contradicts the faithful action just proved. If a positive power of the return of (20.1) were homotopic to the identity, its mapping torus would be homotopy equivalent to $S^1 \times Y$. That mapping torus is the corresponding finite cyclic cover of $P$, so this too is impossible.

8. Finally take $t = a_1a_2$, for which $\chi(t) = 1$, as a stable letter. The image of $\ker\chi$ in $\Gamma$ is $\Gamma_0$, so the same centralizer argument gives $C_G(\ker\chi) = Z(G)$. The automorphism $\mathrm{Ad}(t^n)|_{\ker\chi}$ is inner precisely when $t^n$ differs from an element of $\ker\chi$ by a central element. Since $\chi(Z(G)) = 12\mathbb{Z}$, this holds precisely for $12 \mid n$. Thus the outer action has order twelve, while the return has infinite homotopy order.

   The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 20.2** (The cyclic-cover splice and its separating torus twist)**.** [tag 02ME]

*Let $E_{12} \to E$ be the connected cyclic cover defined by reduction modulo twelve of the positive meridian character. The circle fibration of 20.1 may be chosen with fibre*

$$Y = (E_{12} \times T^3) \cup_{T^8} (T^3 \times E_{12}), \tag{20.8}$$

*with the following exact boundary marking. Write $\theta_i'$ for the lifted normal circle and use the two regular-torus markings. Then*

$$\theta_1' = \gamma_2 - \gamma_1, \qquad \theta_2' = \gamma_1 - \gamma_2. \tag{20.9}$$

*There is a return representative $\psi$ whose twelfth power is the separating torus twist*

$$\begin{gathered} T(x,s) = (x + \eta(s)(g_1+g_2), s) \quad \text{on } T^8 \times [-1,1], \\ T = \mathrm{id} \ \text{ elsewhere.} \end{gathered} \tag{20.10}$$

*Here $\eta$ is zero near $-1$ and one near $1$, and circle coordinates have period one. In particular (20.10) is the identity near both ends of its collar. Its rational cohomological variation satisfies*

$$(T^* - I)^2 = 0, \qquad \operatorname{rank}(T^* - I)|_{H^j(Y;\mathbb{Q})} = \begin{cases} 1 & j = 4, 5, \\ 0 & j \neq 4, 5. \end{cases} \tag{20.11}$$

*The rational Betti numbers of the fibre are*

$$(b_0, \dots, b_9)(Y) = (1, 12, 121, 407, 693, 693, 407, 121, 12, 1). \tag{20.12}$$

*Every eigenvalue of $\psi^*$ is a twelfth root of unity. At eigenvalue one there is exactly one Jordan block of size two in each of degrees four and five; all its other blocks, and all blocks at other eigenvalues, have size one. Thus the infinite order is already visible on ordinary rational cohomology.*

*Proof.* 1. *Adapting the fibration to the splice.* Cut the base along the circle used in 18.4, with the two critical triples on opposite sides. Over the first disc the second factor is a regular marked $T^4$, and over the second disc the first factor is regular. Let $\xi_E : E \to \mathbb{R}/\mathbb{Z}$ represent the positive meridian class. Its restriction to the boundary can be chosen exactly

$$\xi_E = \theta + 12\gamma. \tag{20.13}$$

Indeed this is its integral restriction by 19.2; homotopy extension makes the representative equal to that linear map on a collar. The two maps

$$\rho_A = \xi_{E_1} - 12\gamma_2, \qquad \rho_B = 12\gamma_1 - \xi_{E_2} \tag{20.14}$$

agree exactly on the common boundary, where $\theta_2 = -\theta_1$. Smooth them with this common product collar. The regular-factor circle derivative is $-12$ on the first piece and 12 on the second, so they give a proper submersion throughout, including both cusp pieces. These are also permissible choices in theorem 20.1: on the quotient $N$, an invariant closed one-form differs from these representatives only by the differential of a single-valued function on each of the two pieces. Absorb that function by translation in the unused regular circle. Thus this construction uses the same quotient and primitive character.

2. A level of $\xi_E - 12\gamma_2$ is the pullback of the degree-twelve circle cover along $\xi_E$, times the three remaining regular circles. The pullback is $E_{12}$. Its boundary is connected because the normal meridian has character one. The covering of that boundary has $\theta = 12\theta'$ and unchanged $\gamma, u, w, \delta$. Equation (20.13) in the level condition gives precisely (20.9). This proves (20.8) with its smooth boundary maps.

3. Choose the vector field integrating (20.14) to be $-g_2/12$ on the first piece off its collar and $g_1/12$ on the second. On the common regular collar interpolate by

$$W_s = \frac{\eta(s)g_1 - (1 - \eta(s))g_2}{12}. \tag{20.15}$$

It has $d\rho(W_s) = 1$, and it preserves the collar coordinate $s$ and the original base point. Its flow therefore defines a return $\psi$. At time twelve it is the identity off the collar, since the two indicated regular circles have period one. On the collar it translates by $\eta(s)g_1 - (1 - \eta(s))g_2$; removing the integral translation $-g_2$ gives (20.10). This comparison uses a globally regular factor on each piece.

4. *The variation map and the invariant boundary calculation.* Put $D = T^8$ and let $r : H^j(Y;\mathbb{Q}) \to H^j(D;\mathbb{Q})$ be restriction. Denote the Mayer–Vietoris connecting map by $\partial : H^{j-1}(D;\mathbb{Q}) \to H^j(Y;\mathbb{Q})$. Writing $v = g_1 + g_2$, the collar pullback of a translation invariant form is

$$T^*\omega = \omega + d\eta \wedge \iota_v\omega.$$

Average its restriction to the torus and change the global representative by an exact form to make this equality applicable on the collar. The difference represents the connecting class, and hence, with the connecting sign chosen by the increasing collar coordinate,

$$T^* - I = \partial\, \iota_v\, r. \tag{20.16}$$

As $r\partial = 0$, its square is zero.

5. The cyclic deck action on $E_{12}$ is a translation on the boundary torus, hence acts trivially on boundary cohomology. Every noninvariant deck eigenspace therefore restricts to zero. Rational transfer identifies the invariant part with $H^*(E)$, whose marked restriction is 18.4. Its pullback to (20.9) is

$$R_{k,1}(a) = \phi_k(a) + 12(\gamma_2 - \gamma_1)a, \qquad R_{k,2}(b) = \phi_k(b) + 12(\gamma_1 - \gamma_2)b. \tag{20.17}$$

Use the other factor's three covectors $u, w, \delta$, and take exterior products to obtain the two subspaces $U_j^k \subset \bigwedge^k \mathbb{Q}^8$. The image of $r$ is exactly $U_1^k \cap U_2^k$, and the kernel of $\partial$ is $U_1^{k-1} + U_2^{k-1}$. Consequently the rank in (20.11) is exactly the rank of

$$U_1^k \cap U_2^k \xrightarrow{\iota_v} \bigwedge^{k-1} \mathbb{Q}^8/(U_1^{k-1} + U_2^{k-1}). \tag{20.18}$$

6. Here are the full dimensions for this small rational calculation. Let $d_k$ be twice the degree-$k$ dimension of $H^*(E) \otimes H^*(T^3)$ and let $r_k = \dim(U_1^k + U_2^k)$:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $d_k$ | 2 | 8 | 20 | 44 | 70 | 68 | 36 | 8 | 0 |
| $r_k$ | 1 | 8 | 19 | 34 | 47 | 46 | 27 | 8 | 0 |

(20.19)

Contraction in (20.18) has rank one only for $k = 4, 5$ and is zero in every other degree. These entries result from exterior products of (20.17) and row reduction of the induced contraction maps. One can normalize the remaining sign in $\phi_2$ by the ring structure: with $r_1(1) = \theta + 12\gamma$, the equality $r_1(1)r_2(\delta) = r_3(a)$ forces the coefficient $c_2 = -2$, and then $c_3 = 2$, in the notation of 18.4.

7. *The finite-cover cohomology needed for the full Betti numbers.* The full rational cohomology of $E_{12}$ is computed as follows. Let $\mathcal{L}_\lambda$ be the rank-one system on $E$ whose meridian acts by $\lambda$, with $\lambda^{12} = 1$. On a regular torus it is trivial. Its three base transports in vertical degree $q$ are

$$\lambda^4 \bigwedge^q A_1^{-t}, \qquad \lambda^{-3} \bigwedge^q A_2^{-t}, \qquad \bigwedge^q M_0^{-t}. \tag{20.20}$$

The exponents carry the affine meridian character.

8. At a finite cap, averaging on its finite root cover identifies the cap cohomology with the invariant part of (20.20). At the unchanged cusp the system is trivial, because $g$ has character twelve and $u$ has character zero. Its rational specialization is an isomorphism onto the monodromy invariants in every degree. The cusp Betti numbers are $(1, 2, 4, 2, 1)$ by its normalization calculation. In degree two, the integral surjectivity of the specialization on homology (proof of proposition 18.3) gives injectivity on cohomology; the monodromy-invariant subspace has the same dimension four. Its image therefore includes $uw$ and $\gamma\delta$. In degree one the image is $\langle \gamma, u \rangle$; their products with those two degree-two classes give the two independent invariant classes in degree three and the top class in degree four. Thus all the specialization maps have the asserted ranks, with no additional skyscraper summand. The proof of proposition 18.3 gives these specialization assertions integrally in all degrees.

9. It follows that $R^q f_* \mathcal{L}_\lambda = j_* L_q$ over the disc containing the three critical values. On this disc $H^p(j_* L_q) = 0$ for $p \geq 2$. To check this directly, retract the punctured disc to a free graph on three loops. Restriction of a graph cocycle to the three local coinvariant spaces is onto: its values on the three free generators can be chosen independently. The exact restriction sequence therefore kills $H^2(j_* L_q)$. If $d_q = \binom{4}{q}$, $g_q$ is the common fixed dimension of (20.20), and $k_{i,q}$ are their three individual fixed dimensions, this same graph calculation gives

$$\dim H^0(j_* L_q) = g_q, \qquad \dim H^1(j_* L_q) = 2d_q + g_q - \sum_i k_{i,q}. \tag{20.21}$$

The Leray sequence has only these two base degrees and degenerates. Exact exterior-matrix ranks give the following table; a row applies to each character of the indicated order:

| ord($\lambda$) | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 4 | 6 | 4 | 0 |
| 2 | 0 | 0 | 4 | 6 | 4 | 0 |
| 3 | 0 | 0 | 3 | 4 | 3 | 0 |
| 4 | 0 | 0 | 3 | 4 | 3 | 0 |
| 6 | 0 | 1 | 5 | 6 | 5 | 1 |
| 12 | 0 | 1 | 4 | 4 | 4 | 1 |

(20.22)

Here $b_j = \dim H^j(E; \mathcal{L}_\lambda)$ and $b_6 = 0$. For the nontrivial characters all $g_q$ are zero; for the trivial one they are all one. For (20.20)–(20.22), represent the cyclotomic fields by rational companion matrices and compute each local fixed space by exact row reduction. Summing the rows with multiplicities $1, 1, 2, 2, 2, 4$ gives

$$(b_0, \ldots, b_6)(E_{12}) = (1, 7, 46, 56, 46, 6, 0). \tag{20.23}$$

Both sides have Euler characteristic 24, as required by the degree-twelve covering of $E$.

10. *The Betti and Jordan calculations on the fibre.* The return on each of the two pieces of (20.8), up to isotopy allowing its boundary to move, is its cyclic deck transformation times the identity of $T^3$. This follows also by moving the interpolation in (20.15) entirely into the other piece's collar. It acts trivially on the cohomology of $D$. The generalized eigenvalue-one part of the Mayer–Vietoris sequence therefore uses just the invariant part $H^*(E)$ in (20.19). Its dimensions are

$$(1, 0, 1, 19, 45, 45, 19, 1, 0, 1). \tag{20.24}$$

Every other eigenspace is the direct sum of the two piece eigenspaces, because the boundary has none. In particular those eigenspaces are semisimple. Their combined Poincaré polynomial is

$$2(1 + t)^3(6t + 42t^2 + 50t^3 + 42t^4 + 6t^5), \tag{20.25}$$

by (20.22)–(20.23). Adding (20.24) proves (20.12).

11. Finally the Wang sequence for $P = M_\psi$ and the rational Betti numbers in theorem 18.2 give the numbers of Jordan blocks at eigenvalue one as

$$(1, 0, 1, 19, 44, 44, 19, 1, 0, 1). \tag{20.26}$$

The only possible block sizes are one and two: the twelfth power is (20.10), and (20.16) squares to zero. Subtracting (20.26) from (20.24) gives exactly one block of size two in each of degrees four and five. This agrees with the direct variation ranks (20.18). All other eigenvalues and blocks are already described by (20.22) and (20.25).

The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 20.3** (Integral fibre groups and the coefficient of the torus twist)**.** [tag 02MF]

*The groups $H^*(E_{12};\mathbb{Z})$ are torsion free, with the ranks in (20.23). For the fibre $Y$, the only torsion in cohomology is*

$$H^3(Y;\mathbb{Z})_{\mathrm{tors}} = H^7(Y;\mathbb{Z})_{\mathrm{tors}} = (\mathbb{Z}/6)^2. \tag{20.27}$$

*Equivalently, its integral homology has the ranks (20.12) and torsion $(\mathbb{Z}/6)^2$ exactly in degrees two and six. The twelfth return $T = \psi^{12}$ acts as the identity on all integral cohomology except in degrees four and five. In each of those two free groups it has, in an integral basis, the form*

$$\begin{pmatrix} 1 & 72 \\ 0 & 1 \end{pmatrix} \oplus I_{691}. \tag{20.28}$$

*For $n \geq 1$, let $P_{12n}$ be the connected cyclic cover of $P$ of degree $12n$ defined by $\chi$. Its Betti numbers, independent of $n$, are*

$$(1, 13, 133, 528, 1099, 1384, 1099, 528, 133, 13, 1). \tag{20.29}$$

*Its integral homology torsion is exactly*

$$H_j(P_{12n};\mathbb{Z})_{\mathrm{tors}} = \begin{cases} (\mathbb{Z}/6)^2, & j = 2, 3, 6, 7, \\ \mathbb{Z}/(72n), & j = 4, 5, \\ 0, & \textit{otherwise.} \end{cases} \tag{20.30}$$

*In particular these covers are pairwise nonhomotopy-equivalent.*

*Proof.* 1. *The integral cohomology of the cover.* Apply the proof of (20.21) with the coefficient lattice $\mathbb{Z}[C_{12}]$, using its regular permutation matrix $P_{12}$. In vertical degree $q$ set

$$\begin{gathered} T_{1,q} = P_{12}^4 \otimes \bigwedge^q A_1^{-t}, \quad T_{2,q} = P_{12}^{-3} \otimes \bigwedge^q A_2^{-t}, \\ T_{0,q} = I_{12} \otimes \bigwedge^q M_0^{-t}, V_q = \mathbb{Z}^{12\binom{4}{q}}. \end{gathered} \tag{20.31}$$

This coefficient system computes $H^*(E_{12};\mathbb{Z})$. At the first finite cap the cover has four components, each the ordinary degree-three torus root cover; at the second there are three ordinary degree-four torus root covers. An invariant vector is determined by its value on one sheet of such an orbit, with the other values its integral monodromy translates. This identifies cap cohomology with the full primitive invariant lattice. The cusp has twelve unchanged components and the integral specialization of the proof of proposition 18.3. Consequently its direct-image sheaf is exactly the integral $j_*L_q$, including at every critical point.

2. The three graph-cocycle values again surject onto the three local coinvariant groups, integrally. Thus $H^2(j_*L_q) = 0$, and its remaining cohomology is the kernel and cokernel of the exact lattice map

$$\begin{gathered} V_q \longrightarrow \bigoplus_{i=0}^{2} \mathrm{im}(T_{i,q} - I), \\ v \longmapsto ((T_{i,q} - I)v)_i. \end{gathered} \tag{20.32}$$

Use Hermite column bases for the three image lattices. The source and target ranks, and the rank of (20.32), are respectively

| $q$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\mathrm{rank}\, V_q$ | 12 | 48 | 72 | 48 | 12 |
| $\mathrm{rank} \bigoplus \mathrm{im}(T_{i,q} - I)$ | 17 | 92 | 126 | 92 | 17 |
| rank (20.32) | 11 | 47 | 71 | 47 | 11 |

(20.33)

Every nonzero Smith entry is one. Hence $H^0(j_*L_q)$ has rank one and $H^1(j_*L_q)$ is free with ranks $(6, 45, 55, 45, 6)$. The two-column Leray filtration has free end groups and therefore splits as abelian groups. This proves the torsion-free assertion and (20.23) integrally. The integer matrices are determined by the Hermite image bases in (20.32); their Smith reductions give the displayed invariants.

3. *The integral images.* Each piece $A = E_{12} \times T^3$ in (20.8) has free integral homology. By Poincaré–Lefschetz duality, $H^{k+1}(A, \partial A; \mathbb{Z})$ is free. The cokernel of $H^k(A; \mathbb{Z}) \to H^k(\partial A; \mathbb{Z})$ embeds in that free group, so the boundary image is a primitive lattice. Its rational span has already been computed in (20.17). Therefore the integral image is precisely the saturation of that displayed image in $\bigwedge^k \mathbb{Z}^8$.

4. Choose saturated column bases $\overline{A}_k, \overline{B}_k$ for the two images. The combined matrix $[\overline{A}_k \ \overline{B}_k]$ has the ranks in the second row of (20.19). All its nonzero Smith entries are one except that, in each of degrees two and six, exactly two entries are six. The Mayer–Vietoris exact sequence has a free kernel term on its right, since all piece cohomology is free. Thus its abelian extension splits, and these Smith entries give exactly (20.27). The free ranks remain (20.12).

5. *The integral variation, including possible torsion components.* Formula (20.16) holds integrally. One can use the singular prism of the loop of translations instead of differential forms: the action map $S^1 \times T^8 \to T^8$ has cohomology formula

$$a \longmapsto 1 \otimes a + [S^1]^* \otimes \iota_v a.$$

   The difference between the two collar homotopies is exactly this loop, so its connecting class is $\partial \iota_v r(a)$. This proves the integral formula also on torsion and still gives $(T^* - I)^2 = 0$.

6. For complete lattice control, take an integral kernel basis of $[\overline{A}_k \ -\overline{B}_k]$. Its first block, multiplied by $\overline{A}_k$, is a basis of the common boundary lattice. Contract it by $v$ and reduce modulo $[\overline{A}_{k-1} \ \overline{B}_{k-1}]$ using the integral Smith transforms. In degrees four and five the result has rank one and unique nonzero Smith entry 72. In degree three its two possible components modulo six are both zero; in degree seven the common boundary image is zero. All the remaining reductions are zero. The reductions use the integral unimodular Smith transforms. In degrees four and five the connecting cokernel is free and embeds as a primitive summand of $H^k(Y)$, because the following Mayer–Vietoris kernel is free. Thus the divisibility in the whole cohomology group is 72.

7. A rank-one integral map of Smith entry 72 has the form $72u\ell$, where $u$ and $\ell$ are primitive. Its square is zero precisely when $\ell(u) = 0$. Extend $u$ to a basis of $\ker \ell$ and choose a vector on which $\ell$ is one. This gives (20.28). The same calculation proves identity on every other integral cohomology group, including its torsion.

8. *The finite-cover groups and the two potentially nontrivial extensions.* The cover $P_{12n}$ is the mapping torus of $T^n$. Since $(T^* - I)^2 = 0$, its variation is $n(T^* - I)$, with Smith entry $72n$ in degrees four and five. For every $k$ the integral Wang sequence gives

$$0 \longrightarrow \operatorname{coker}(T^{n*} - I)_{k-1} \longrightarrow H^k(P_{12n}; \mathbb{Z}) \longrightarrow \ker(T^{n*} - I)_k \longrightarrow 0. \tag{20.34}$$

   The ranks immediately give (20.29). Whenever the right group is free, (20.34) splits as an abelian sequence. In particular degrees four and eight have torsion $(\mathbb{Z}/6)^2$, contributed by $H^3(Y)$ and $H^7(Y)$ respectively, while degrees five and six have torsion $\mathbb{Z}/(72n)$. Every other such degree has no torsion.

9. The only right groups in (20.34) that are not free occur in degrees three and seven. Instead use the nondegenerate integral torsion linking pairing of the closed oriented ten-manifold $P_{12n}$:

cohomological torsion in degree $k$ is Pontryagin dual to that in degree $11-k$. The already fixed torsion in degree eight therefore forces the degree-three torsion to be $(\mathbb{Z}/6)^2$, and the fixed degree-four torsion forces the same answer in degree seven. The free ranks are already known. This resolves precisely the two possibly absorbing extensions. The universal coefficient theorem now translates these cohomology groups to (20.30). The order $72n$ in the middle groups distinguishes the covers for distinct $n$.

The corresponding exact calculations are implemented in the programs in Table 1. □

# 21 The homotopy module and the quadratic cusp attachment

**Theorem 21.1** (The full complement module and its peripheral map)**.** [tag 02MG]

*Let $K=\pi_1(E)$ have the marking (19.1), and put*

$$R=\mathbb{Z}K,\qquad Q=K/\langle h\rangle=C_3*C_4,\qquad \overline{R}=\mathbb{Z}Q.$$

*Use right deck modules; conversion to left modules applies the involution $g\mapsto g^{-1}$ to every coefficient. Then*

$$\pi_2(E)\simeq R\oplus\overline{R}^{\,2}\oplus L,\qquad L=\langle x,t\mid t(a-1)=t(b-1)=0,\ x(1+b)+3t=0\rangle_{\overline{R}}. \tag{21.1}$$

*Equivalently, replacing $x$ by $x+t$ and eliminating $t$ gives*

$$L\simeq \overline{R}/((b^2-1)\overline{R}+(1+b)(a-1)\overline{R}). \tag{21.2}$$

*There is a nonsplit exact sequence*

$$0\longrightarrow \mathbb{Z}\langle t\rangle\longrightarrow L\longrightarrow \mathbb{Z}_-\otimes_{\mathbb{Z}C_4}\overline{R}\longrightarrow 0, \tag{21.3}$$

*where $b$ acts as $-1$ on $\mathbb{Z}_-$ and the first term has trivial $Q$ action. Its extension class is the nonzero element of an order-two extension group. In the connected lift of the peripheral $T^5$ to the universal cover of $E$, the degree-two classes $uw, ud, wd$ map respectively to*

$$t,\quad 0,\quad 0. \tag{21.4}$$

*Thus the central torsion summand of the module is annihilated by $h-1$ itself. Formula (21.1) is a full integral module calculation, including the central value $h=1$. There is also a chain-level peripheral comparison. For the splitting constructed below, let $e$ generate the free $R$ summand and let $y,z$ generate the two copies of $\overline{R}$. The projection of the peripheral degree-two chain map to (21.1) is*

| *cell* | $gu$ | $gw$ | $gd$ | $uw$ | $ud$ | $wd$ |
|---|---|---|---|---|---|---|
| *value* | $0$ | $0$ | $6t$ | $t$ | $0$ | $0$ |

(21.5)

*and, writing $\Theta$ for the outer base circle with group image $\mu=ab$, use the cone convention: the notation $v\Theta$ denotes the suspended cell $\Theta\times v$, with the interval factor first. In this convention,*

| | |
|---|---|
| $g\Theta$ | $eh-2x-6yb+z(5-2b)$ |
| $u\Theta$ | $x-yb+zb$ |
| $w\Theta$ | $-y(1+b)+z$ |
| $d\Theta$ | $x+2t+z(b-1).$ |

(21.6)

*The tables give the chain map; its induced map on lifted peripheral homology is (21.4). In particular the free coefficient $h$ on $g\Theta$ is nonzero even though the free summand receives zero from the lifted peripheral homology in (21.4).*

*Proof.* 1. We give the local chain comparison underlying (21.1).

2. *The cusp boundary and its derived attachment.* Write $A = \mathbb{Z}[h, h^{-1}]$ and $q = h - 1$. Let $\widehat{W}$ be the gamma cover used in 19.4, and let $\widehat{M}$ be the corresponding boundary cover. Ordinary chains with the deck action give finite free $A$ chain models. The homology of $\widehat{W}$ is

$$H_0 = H_1 = A/(q), \quad H_2 = A \oplus (A/(q))^2, \quad H_3 = 0, \quad H_4 = A. \tag{21.7}$$

For the boundary, start with the torus chain complex $C = A \otimes \bigwedge\langle g, u, w, d\rangle$, whose differential contracts $g$ and multiplies by $q$. The cellular monodromy has degree-one matrix

$$F(M_0)g = g - hd, \quad F(M_0)u = u + w, \quad F(M_0)w = w, \quad F(M_0)d = d. \tag{21.8}$$

Its higher matrices are exterior powers. The mapping-torus complex is the cone of $I - F(M_0)$, with differential

$$d(c, e) = \big(dc + (I - F(M_0))e, -de\big). \tag{21.9}$$

For example $d(g\theta) = hd - q\theta$ and $d(u\theta) = -w$. Cancelling these two unit pairs leaves the degree-one classes $u$ and $\theta$, with resolutions $q$ and a unit times $q^2$. In degree two the four surviving classes are $uw, ud, w\theta, d\theta$, each with resolution $q$. In particular

$$H_0(\widehat{M}) = A/(q), \quad H_1(\widehat{M}) = A/(q^2) \oplus A/(q), \quad H_2(\widehat{M}) = (A/(q))^4. \tag{21.10}$$

Geometrically this is the Heisenberg three-manifold times a circle: the deck transformation shears the meridian by the $d$ circle. The homology map to (21.7) is the identity in degree zero, projection to $u$ in degree one, and the primitive projection $(uw, ud, w\theta, d\theta) \mapsto (U, D, 0, 0)$ in degree two. The last assertion uses the integral specialization of the proof of proposition 18.3 and the collapsed meridian tori.

3. The extension is determined by a relative intersection calculation. Let $S_i$ denote the normalized hexagon component with gamma index $i$. In its standard Picard basis $H, E_1, E_2, E_3$, the six boundary curves are

$$E_1,\ H - E_1 - E_2,\ E_2,\ H - E_2 - E_3,\ E_3,\ H - E_1 - E_3. \tag{21.11}$$

The first opposite pair is self-glued in the surviving $u$ direction; the other two pairs join $S_i$ to its two gamma neighbours. Put $C_0 = -H + E_1$, the primitive generator of the free kernel of the normalization matrix $B(h)$ in 19.4. The divisor $S_i$ restricts on its own normalization to $2C_0$: the total central divisor is principal, and the two self-gluing preimages cancel their terms in the normal restriction. It restricts to $-C_0$ on each neighbouring component and to zero on every other component. Thus the free part of the equivariant intersection map is

$$H_4(\widehat{W}) = A \longrightarrow H_4(\widehat{W}, \widehat{M}) \longrightarrow A, \qquad 2 - h - h^{-1} = -h^{-1}q^2. \tag{21.12}$$

Poincaré–Lefschetz duality over $A$, with its usual involution, gives the same coefficient, up to a unit, for the free part of $H_2(\widehat{W}) \to H_2(\widehat{W}, \widehat{M})$. The latter relative group is $A$: the universal-coefficient sequence in degree four has $\mathrm{Hom}_A(H_4, A) = A$, zero $H_3$, and no higher extension terms. Indeed every module in (21.7) has a projective resolution of length at most one. The relative sequence therefore includes

$$0 \longrightarrow A \xrightarrow{\text{unit } q^2} A \longrightarrow A/(q^2) \longrightarrow 0. \tag{21.13}$$

Consequently the boundary map has a unit component in $\mathrm{Ext}^1_A(A/(q^2), A) = A/(q^2)$. Every unit of $A/(q^2) = \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ lifts to a unit $\pm h^n$ of $A$.

4. This determines the low-degree chain map, as follows. All the modules in (21.7) and (21.10) have projective dimension at most one. Their complexes split in the derived category into their two-term projective resolutions: the obstructions to splitting consecutive truncations are

extension groups in degree at least two and vanish. In degrees relevant to $H_2$ of a mapping cone, a map between these split complexes consists of the displayed homology maps and extension components from $H_i$ to $H_{i+1}$. Normalize the component in (21.13) by a unit on the free $H_2$ summand of the target. Its components to the two torsion $H_2$ summands can then be removed by target maps $A \to (A/(q))^2$: reduction of the unit modulo $q$ is $\pm 1$. The remaining components from the $u$ summand are removed by target degree-one extension maps from $H_1(\widehat{W})$ to $H_2(\widehat{W})$. Finally the degree-zero component is removed using the isomorphism on $H_0$ and a target extension map from $H_0$ to $H_1$.

5. Each change just described is a unit on the diagonal of the split complex and is triangular with extension entries off the diagonal. It is therefore invertible. Replacing the split resolutions by the original finite free complexes represents it by a chain homotopy equivalence. The change is applied to the cusp target and its entire incoming boundary map simultaneously. The other vertices and all maps into the regular core are unchanged. It thus gives a chain homotopy equivalence of the global mapping-cone complex that is the identity on the external peripheral complex. Tensoring with $R$ preserves these comparisons, since $R$ is free over $A$.

6. One resulting cusp model in degrees zero through three has ranks $(1, 2, 4, 2)$, generators

$$(1); \quad (g, u); \quad (gu, E, U, D); \quad (R_U, R_D),$$

and differentials $dg = q$, $d(gu) = qu$, $dR_U = qU$, $dR_D = qD$. The map (21.9) to this model is, through degree two,

$$\begin{array}{c|ccccc} & g & u & w & d & \theta \\ \hline i_1 & g & u & 0 & 0 & 0 \end{array}, \qquad \begin{array}{c|cccccc} & gu & gw & gd & uw & ud & wd \\ \hline i_2 & gu & -hD & E & U & D & 0 \end{array}, \qquad i_2(v\theta) = 0. \tag{21.14}$$

The signs and the factor $h$ satisfy the chain equation directly. At $h = 1$, choose $E = -\overline{C}_1 - \overline{C}_2$, $U = \overline{C}_2 + \overline{C}_3$, and $D = \overline{C}_2$.

7. *Finite caps and the global complex.* For any marked monodromy $A = \left(\begin{smallmatrix} 1 & 0 \\ t & D \end{smallmatrix}\right)$ use the chain lift $F(A) = \left(\begin{smallmatrix} 1 & 0 \\ ht & D \end{smallmatrix}\right)$ and its exterior powers. These lifts multiply exactly and commute with contraction by $qg^*$.

8. This exterior lift represents the equivariant torus map. The gamma cover is $\mathbb{R}_g \times T^3$, and the marked linear map is $(g, v) \mapsto (g, Dv + gt)$. At every integer value of $g$ the translation is integral, so the vertex torus map is exactly $D$. In a single unit strip, concentrate the loop of translations $v \mapsto v + st$ at its terminal endpoint by an equivariant cellular approximation relative to the two vertex tori. The translation prism on the minimal cellular complex of $T^3$ is exterior multiplication by $t$: its value on each product torus is the homology class of the literal translation sweep, and all differentials in that minimal torus complex are zero. The terminal vertex has deck coefficient $h$. Consequently

$$v \longmapsto \bigwedge D(v), \qquad g \wedge v \longmapsto g \wedge \bigwedge D(v) + ht \wedge \bigwedge D(v), \tag{21.15}$$

which is exactly the exterior lift above. There is only this one strip parameter. Composing two such maps adds the translation loops by $t_A + D_A t_B$, the same strict multiplication law as $F(A)F(B) = F(AB)$. A regular boundary with deck image $s$ is the cone of $sI - F(A)$. This convention gives the group relation $s\lambda s^{-1} = A\lambda$.

9. At finite cap $j$, take its adapted first fibre vector $n_j v_j$, where

$$(m_1, n_1, v_1) = (3, 1, (1, 2, -4, 0)), \qquad (m_2, n_2, v_2) = (4, -1, (-1, -3, 3, 0)). \tag{21.16}$$

It has gamma coordinate one and is fixed by $A_j$. The other three coordinates form $L_0 = \langle u, w, d \rangle$, with monodromy $D_j = A_j|_{L_0}$. The cap is the mapping torus of this three-torus,

and its complex has differential $sI - \bigwedge^r D_j$ on the swept cells. The fibre-to-cap map is the identity on $\bigwedge L_0$; on the adapted long-circle factor it is the integral norm matrix $N_{m_j n_j}(s, D_j)$ determined by

$$(sI - D_j)N_{m_j n_j}(s, D_j) = (h-1)I, \qquad s = a \text{ or } b. \tag{21.17}$$

For positive $n_j$ the norm is $\sum_{r=0}^{m_j-1} s^r D_j^{m_j-1-r}$, and for $n_j = -1$ it is minus $s^{-m_j}$ times that sum. In phase degree $r$, the map is $N_{m_j n_j}(s, \bigwedge^r D_j)$. The boundary base-circle generator maps to the cap circle, while the cell containing both that meridian and the adapted long circle maps to zero. These are winding maps in the mapping-torus coordinates. The linear three-torus return fixes its origin.

10. Form the core over a bouquet of three circles with deck images $a, b, 1$ and monodromies $A_1, A_2, M_0$. Attach the two finite complexes by (21.17) and the cusp by (21.14), using their double mapping cylinders. Through degree three the resulting free $R$ complex has ranks $(4, 20, 49, 62)$. Write its group-ring entries in the integral normal form $h^r$ times an alternating word in $a, a^2$ and $b, b^2, b^3$; exact unit cancellations use multiplication in that normal form.

11. The final reduction is as follows. Cancelling degree-zero and degree-one unit pairs leaves ranks $(1, 2, 34, 62)$, with $d_1 = (a-1, b-1)$. The only nonzero columns of $d_2$ are $F$ and $F(b-1)$, where

$$F = \begin{pmatrix} 1 + a + a^2 \\ h(1 + b + b^2 + b^3) \end{pmatrix}. \tag{21.18}$$

The first component of $F$ is injective on $R$: multiplication by $a-1$ converts it to $h-1$, and $R$ is free over the Laurent ring $A$. Thus removing the $F$ column after subtracting its multiple gives a free basis of $\ker d_2$ of rank 33. The corresponding row of $d_3$ is zero by the chain equation. Unit reductions of the remaining presentation leave five generators $(x, t, y, z, e)$. One generator $e$ occurs in no relation. Three pure $t$ relations are

$$tq = 0, \qquad t(1 - a^{-1}) = 0, \qquad t(b - a^{-1}) = 0. \tag{21.19}$$

They generate exactly the augmentation relations on $t$: multiply the second by $a$ to get $t(a-1) = 0$, and subtract it from the third to get $t(b-1) = 0$. Every other pure $t$ coefficient has augmentation zero and is consequently redundant. Reducing the remaining $t$ coefficients by augmentation, three relations give successively $yq = 0, zq = 0, xq = 0$. The final three relations are

$$x(1 + b) + 3t = 0, \quad x(b + b^2) + 3t = 0, \quad x(1 + b + b^2 + b^3) + 6t = 0. \tag{21.20}$$

The last two are respectively the first multiplied by $b$, and the first multiplied by $1 + b^2$. These identities prove (21.1).

12. The invariant subgroup generated by $t$ injects in $L$. One way to see this is to start with the exact sequence of right $\mathbb{Z}C_4$ modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[C_4/C_2] \longrightarrow \mathbb{Z}_- \longrightarrow 0,$$

where the first map is the sum of the two cosets. Induce to $Q$ and push out its first term by the augmentation to the trivial module $\mathbb{Z}$. The resulting presentation is (21.1), after the change $x \mapsto x + t$ which replaces the coefficient 3 by 1. Pushout preserves the displayed injection. Restriction to $C_4$ shows nonsplitting: an invariant correction of the lift $x$ changes this odd coefficient by an even number. Equivalently the standard cyclic resolution computes $\mathrm{Ext}^1_{\mathbb{Z}C_4}(\mathbb{Z}_-, \mathbb{Z}) = \mathbb{Z}/2$. Induction–restriction gives the same extension group for (21.3).

13. Finally insert the fibre cycles $uw, ud, wd$ in the core before any cancellation and carry their coordinate columns through the same basis changes. The five-generator presentation gives respectively
$$t, \qquad t(a^{-1}-a), \qquad t(a^{-1}-1). \tag{21.21}$$
Relations (21.19) reduce these to (21.4). The external boundary cover has homotopy type $T^3$ and these are its three degree-two generators. Taking $K$ coinvariants gives $2x+3t=0$, so $t$ is twice a primitive class, exactly the coefficient 2 in 18.4.

14. For the remaining peripheral data, use the boundary word $abc$, where the cusp generator $c$ has deck image 1. On swept fibre chains its map to the three core edges has blocks
$$bI, \qquad F(A_1), \qquad F(A_1)F(A_2). \tag{21.22}$$
The identity
$$(aI-F(A_1))b + F(A_1)(bI-F(A_2)) + F(A_1)F(A_2)(I-F(M_0)) = (ab-1)I$$
checks its chain equation. This is the usual prism subdivision of the peripheral word, with its given ribbon marking. Tracking all ten degree-two columns through the same reductions, and using only $t(a-1)=t(b-1)=0$ and $xq=yq=zq=0$, gives (21.5)–(21.6).
The corresponding exact calculations are implemented in the programs in Table 1. □

**Theorem 21.2** (The full shifted homotopy module)**.** [tag 02MH]
*Use the right-module convention of 21.1. Write*
$$\begin{gathered} R=\mathbb{Z}G, \qquad q_i=h_i-1, \qquad S=\mathbb{Z}\Gamma, \\ \Gamma = G/\langle h_1,h_2\rangle = \langle a,b,c,d \mid a^3=b^4=c^3=d^4=abcd=1\rangle. \end{gathered} \tag{21.23}$$
*where $a,b$ belong to the first complement, $c,d$ to the second, and $\mu = ab = (cd)^{-1}$. The full integral second-homotopy module is*
$$\pi_2(P) \simeq R/q_2R \;\oplus\; R/q_1R \;\oplus\; T_L \oplus T_R. \tag{21.24}$$
*The last two modules are inflated from the following explicit six-generator presentations over $S$. Put $N_a = 1+a+a^2$, $N_b = 1+b+b^2+b^3$, and define $N_c, N_d$ similarly.*
*For $T_L$ use generators $p_i, r_i$ for $0 \le i \le 2$, put $v_i = p_i - r_i$, and define*
$$\begin{gathered} \Omega_L = v_0(1+d-d^2-d^3) + v_1(d^3-d) + v_2(d^2+d^3), \\ p_iN_a = 0, \qquad r_iN_b = 0 \quad (0\le i\le 2), \\ \Omega_L(a-1) = \Omega_L(b-1) = \Omega_L(c-1) = \Omega_L(d-1) = 0. \end{gathered} \tag{21.25}$$
*For $T_R$ use generators $B_i, C_i$, put $w_i = B_id + C_i$, and define*
$$\begin{gathered} \Omega_R = w_0a(1-b-b^2+b^3) + w_1a(b^2-1) - w_2a(1+b^3), \\ B_iN_c = 0, \qquad C_iN_d = 0 \quad (0\le i\le 2), \\ \Omega_R(a-1) = \Omega_R(b-1) = \Omega_R(c-1) = \Omega_R(d-1) = 0. \end{gathered} \tag{21.26}$$
*These are complete presentations: six generators and ten relations in each block. In particular*
$$q_1q_2\pi_2(P) = 0, \qquad \ker(q_1:\pi_2(P)\to\pi_2(P)) \cap \ker(q_2:\pi_2(P)\to\pi_2(P)) = T_L\oplus T_R. \tag{21.27}$$
*This sharpens the annihilator of theorem 19.4 and determines the central-intersection summand. Its coinvariants are $(\mathbb{Z}/12)^6$, so (21.24) recovers $\pi_2(P)_G = \mathbb{Z}^2 \oplus (\mathbb{Z}/12)^6$.*
*Let $\Pi = \pi_1(Y)$ and let $\Sigma \simeq \pi_1(\Sigma_6)$ be its quotient by $z = h_1h_2$, as in 20.1. Then*
$$\pi_2(Y) \simeq (\mathbb{Z}\Pi)^{24} \oplus \mathrm{Res}^{\Gamma}_{\Sigma}(T_L\oplus T_R), \qquad \pi_2(Y)_\Pi \simeq \mathbb{Z}^{110} \oplus (\mathbb{Z}/6)^2. \tag{21.28}$$
*The surface-group summand is inflated to $\Pi$, so $z$ acts trivially on it.*

*Proof.* 1. *A small chain presentation with the peripheral map.* The calculation in 21.1 permits the following model through degree three for the first complement. Its bases have ranks $(1,2,6,6)$, with degree-one basis $(a,b)$ and degree-two basis $(r,x,t,y,z,e)$. The first two differentials are

$$d_1 = (a-1,\, b-1), \qquad d_2(r) = \begin{pmatrix} N_a \\ h_1 N_b \end{pmatrix}, \quad d_2(x) = d_2(t) = d_2(y) = d_2(z) = d_2(e) = 0. \tag{21.29}$$

The six degree-three columns are the six relations

$$t(a-1), \quad t(b-1), \quad xq_1, \quad yq_1, \quad zq_1, \quad x(1+b) + 3t. \tag{21.30}$$

The degree-two norm $N_a$ in (21.29) is the polynomial $1+a+a^2$ of (21.25), evaluated in $R$ with $a^3 = h_1$. The second complement is the identical model with $(a,b,h_1)$ replaced by $(c,d,h_2)$.

2. The ordinary one-relator complex $C_K$ is the subcomplex on degrees zero and one and the cell $r$. It is an aspherical model: it is the double mapping cylinder of the degree-three and degree-four maps of circles, and its universal cover is a tree of contractible lines and strips. Equivalently it is the standard Seifert spine of the torus-knot exterior. Projection of (21.29)–(21.30) onto $C_K$ retains the peripheral map. In degree one this map is

$$g \longmapsto (N_a, 0), \qquad u, w, d \longmapsto 0, \qquad \Theta \longmapsto (b, 1). \tag{21.31}$$

Its degree-two $r$ coefficient is $b-1$ on the swept cell $g\Theta$ and zero on the other nine cells. Its remaining components are exactly (21.5)–(21.6).

3. Here and below the small complex is only a presentation through the degree needed for $H_2$. Replacing redundant degree-three columns by the six generators of their image in (21.30) changes no degree-two cycle or boundary. Reducing a peripheral degree-two column modulo that image changes it by a chain homotopy with values in degree three. Products with a torus and the mapping cones below therefore retain the same degree-two homology. For the comparison with an aspherical target one may first keep the original, full complex: its projection to $C_K$ extends by zero in degrees at least three, since the Fox column in (21.29) is injective and hence the $r$ row of $d_3$ is zero. Only after taking the kernel is its image in degree two shortened as in (21.30).

4. *The exact kernel complex for the splice.* Let $T(s)$ be the four-torus chain complex whose first circle has deck image $s$ and whose other three circles have trivial image. Its differential contracts that first circle and multiplies by $s-1$. The two vertex complexes for the splice are

$$V_1 = C_{E_1} \otimes T(h_2), \qquad V_2 = C_{E_2} \otimes T(h_1). \tag{21.32}$$

We have reversed the two product factors in the second vertex only for notation; the Koszul signs below account for that reversal. The boundary complex $B$ is the nine-torus complex with ordered coordinates

$$(g_1, u_1, w_1, d_1, g_2, u_2, w_2, d_2, \Theta), \quad \text{deck images } (h_1, 1, 1, 1, h_2, 1, 1, 1, \mu). \tag{21.33}$$

Its differential is the usual exterior contraction by those deck images minus one.

5. The source signs are as follows. Write a boundary product cell as $\alpha \wedge \beta$ with an optional final $\Theta$, where $\alpha$ uses the first four coordinates and $\beta$ the second four. Put $\epsilon = 1$ when $\Theta$ occurs and 0 otherwise. Its first vertex map applies the 21.1 peripheral map to $(\alpha, \Theta)$, keeps $\beta$, and has sign $(-1)^{\epsilon(|\alpha|+|\beta|)}$. The second map applies that peripheral map to $(\beta, \Theta_2)$, keeps $\alpha$, and has sign $(-1)^{|\alpha|(|\beta|+\epsilon)+\epsilon|\beta|}$. If $\epsilon = 1$, multiply its coefficient on the right by $-\mu$. This is the exact cellular map of $\Theta_1 \mapsto \Theta_2^{-1}$, since $\mu_2 = cd = \mu^{-1}$. The suspended cells in (21.5)–(21.6) have the interval first; these signs convert from the ordered exterior cells in (21.33). Denote the two maps by $j_1, j_2$.

6. The chain complex of the splice is the cone of $(j_1, -j_2) : B \to V_1 \oplus V_2$. Project each $C_E$ to $C_K$ and each $T(s)$ to its first circle, and project $B$ to its three coordinates $g_1, g_2, \Theta$. The result is the complex of the aspherical graph of spaces

$$(K(K_1, 1) \times S^1) \ \cup_{T^3} \ (K(K_2, 1) \times S^1). \tag{21.34}$$

It is the model $N$ of 18.7. Both peripheral maps on groups are injective by 19.2; their common group is $\langle h_1, h_2, \mu \rangle = \mathbb{Z}^3$. Consequently its universal cover is a Bass–Serre tree of contractible vertex and edge spaces and is contractible. Its chain complex has ranks $(2, 7, 9, 5, 1)$, including its degree-four cell.

7. The indicated projection is a degreewise split surjection of the full chain complexes. The peripheral Fox coefficient in (21.31) makes the boundary square commute; higher degrees commute as well by the product construction and the zero higher groups of $C_K$. Let $D$ be its kernel. The target has zero positive-degree homology, so the long exact sequence gives $H_2(D) \simeq H_2(\widetilde{P}) = \pi_2(P)$. Using the small models for the relevant images, the ranks of $D$ through degree three are

$$(0, 6, 40, 123). \tag{21.35}$$

Its six degree-one generators are the unused regular-factor circles at the two vertices. The six shifted boundary circles $u_i, w_i, d_i$ map to them by unit entries, because each circle maps to zero in its own complement and identically to the opposite regular torus. Cancel these six unit pairs. This leaves a presentation of $H_2(D)$ with 34 generators and 123 relations.

8. Here is also an explicit block definition of that matrix. In each $V_i$ keep the cells killed by projection to $C_{K_i} \otimes C(S^1)$; their ranks in degrees two and three are 17 and 45. In $B_2$ keep the 33 cells containing at least one of $u_1, w_1, d_1, u_2, w_2, d_2$. In those ordered bases the matrix is

$$\mathcal{M} = \begin{pmatrix} (d_{V_1})_3 & 0 & (j_1)_2 \\ 0 & (d_{V_2})_3 & -(j_2)_2 \end{pmatrix} : R^{45} \oplus R^{45} \oplus R^{33} \longrightarrow R^{17} \oplus R^{17}. \tag{21.36}$$

Every entry is specified by (21.29)–(21.33) and (21.5)–(21.6); (21.36) itself is a complete finite presentation of the desired module. The formula includes the derived peripheral comparison.

9. *Reduction to the compact presentation.* The peripheral chain equations and $d_2 d_3 = 0$ hold by multiplication of their group-ring matrices. Cancelling the indicated unit pairs reduces (21.36) to 16 generators and 52 nonzero columns. The calculation takes place in the amalgam

$$G = (K_1 \times \mathbb{Z}) \underset{\mathbb{Z}^3}{*} (\mathbb{Z} \times K_2), \tag{21.37}$$

whose normal form is specified as follows: the torus-knot factor uses $h^r$ times an alternating word in $a, a^2$ and $b, b^2, b^3$. Modulo the peripheral subgroup, strip initial $ab$ and $b^3 a^2 = (ab)^{-1}$, allowing their last letter to be a partial power; choose $b^3$ in the sole length-one tie with $a$. Inverting these left coset representatives gives right coset representatives. The usual alternating amalgam normal form, followed by a final element of $\mathbb{Z}^3$, is unique. These word operations track both central exponents.

10. In the resulting 16-generator presentation two generators occur only in their respective single relations $q_2 e_1 = 0$ and $q_1 e_2 = 0$. They give the first two direct summands of (21.24). For the other 14 generators, two generators $t_0, t_1$ have trivial action of $a, b, c, d$. There are three generators initially killed by $q_2$ whose pure relations include $h_1 N_b$; the identity $h_1 N_b (b - 1) = 1 - h_1$ kills them by $q_1$ as well. There are three initially killed by $q_1$ with pure $N_c$ relations, and three more with pure $h_2 N_d$ relations. They are killed by $q_2$. The remaining generators already have both central relations. Thus the whole 14-generator summand is a module over $S$. The pure norm columns are exact differences of the recorded columns and are checked before setting either central element equal to 1.

11. The 14 generators separate into two seven-generator blocks. The first has generators $U_0, U_1, U_2, X, Y, Z, t_1$, with $t_1$ invariant, $U_i N_b = 0$, and

$$\begin{aligned} (U_0 + X - Yd + Zd)N_a &= 0, \\ (U_1 - Y(1+d) + Z)N_a &= 0, \\ (U_2 + X + 2t_1 + Z(d-1))N_a &= 0, \\ X(1+d) + 3t_1 &= 0. \end{aligned} \tag{21.38}$$

The second has generators $B_i, C_i, t_0$, with $B_i N_c = C_i N_d = 0$, $t_0$ invariant, and the single additional relation $t_0 = \Omega_R$. These formulas follow by collecting the displayed 16-generator columns.

12. In the first block put

$$\begin{aligned} v_0 &= X - Yd + Zd, \quad & v_1 &= -Y(1+d) + Z, \\ v_2 &= X + 2t_1 + Z(d-1), & r_i &= U_i, \quad p_i = U_i + v_i. \end{aligned} \tag{21.39}$$

The linear part of this change on $(X, Y, Z)$ is

$$\begin{pmatrix} 1 & 0 & 1 \\ -d & -1-d & 0 \\ d & 1 & d-1 \end{pmatrix}, \quad \det = 1,$$

over $\mathbb{Z}[d, d^{-1}]$. It is therefore an integral group-ring basis change. Its inverse gives

$$X = v_0(1-d^2) + v_1(d^2-d) + (v_2 - 2t_1)d^2.$$

Substitute in the last relation of (21.38), using the invariance of $t_1$. The result is $t_1 = \Omega_L$. Eliminate $t_1$ with this unit coefficient. Its four invariance relations become the four relations in (21.25). Eliminating $t_0$ in the second block gives (21.26). This proves the compact presentation in (21.24)–(21.26), including the integral central-intersection information.

13. For the two cyclic axis summands, the other central element is a non-zero-divisor: their group rings are free over its Laurent polynomial ring. Thus neither contributes to the common kernel of $q_1, q_2$. This proves (21.27). On coinvariants, the four invariance relations in each block vanish, while the norm relations become three copies of multiplication by 3 and three copies of multiplication by 4. Each block therefore has coinvariants $(\mathbb{Z}/12)^3$. Taking coinvariants therefore gives the groups in theorem 19.3.

14. *Restriction to the circle fibre.* The universal covers of $P$ and $Y \times \mathbb{R}$ are $\Pi$-equivariantly homotopy equivalent by 20.1. The map $\Pi \to G/\langle h_i \rangle$ is injective, since $\chi(h_i) = \pm 12$, and has index 12. Hence each of the first two group-ring summands in (21.24) is free of rank 12 over $\mathbb{Z}\Pi$. The last two factor through the index 12 surface subgroup $\Sigma < \Gamma$ and have trivial $z$ action. This proves the module formula in (21.28).

15. For its exact coinvariants use the regular 12-sheet permutation representation with exponents $(4, -3, -4, 3)$ on $(a, b, c, d)$. Evaluating the compact 12-by-20 presentation gives an integral 144-by-240 matrix. Its Smith form has 56 unit entries and two entries 6; its free cokernel rank is 86. Adding the 24 free coinvariants proves the last formula in (21.28).

The corresponding exact calculations are implemented in the programs in Table 1. □

**Definition 21.3** (Whitehead quadratic group)**.** [tag PV24] For an abelian group $A$, let $\Gamma(A)$ be Whitehead's universal quadratic group, with quadratic generator $q(a)$ and symmetric bilinear cross effect $[a, b] = q(a+b) - q(a) - q(b)$. In particular $[a, a] = 2q(a)$.

**Proposition 21.4** (The cusp quadratic attachment)**.** [tag 02MI]

*Keep the right-module presentation of 21.1 and write*

$$R = \mathbb{Z}K, \qquad M = \pi_2(E) = eR \oplus T, \qquad T = \overline{R}^2 \oplus L.$$

*There is a lift of the normalized cusp component* $S = \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ *to the universal cover* $\widetilde{E}$ *whose fundamental class has image*

$$\pm([e(h-1), t] - 12q(t)) \quad \textit{under} \quad H_4(\widetilde{E}; \mathbb{Z}) \longrightarrow \Gamma(M). \tag{21.40}$$

*The sign is solely the orientation chosen on the normalized component. Its mixed component in*

$$\Gamma(M) = \Gamma(eR) \oplus (eR \otimes_{\mathbb{Z}} T) \oplus \Gamma(T) \tag{21.41}$$

*is nonzero. Consequently* $E$ *is not homotopy equivalent to* $E_0 \#(S^2 \times S^4)$ *for any connected compact six-manifold* $E_0$*. Thus the free* $R$ *summand of its second homotopy group does not come from a sphere-product connected summand. Likewise the nine-dimensional fibre* $Y$ *is not homotopy equivalent to* $Y_0 \#(S^2 \times S^7)$ *for any connected closed nine-manifold* $Y_0$*.*

*Proof.* 1. We use the natural, deck-equivariant part of Whitehead's exact sequence

$$H_4(\widetilde{E}) \xrightarrow{\omega_E} \Gamma(\pi_2(E)) \longrightarrow \pi_3(E) \longrightarrow H_3(\widetilde{E}) \longrightarrow 0. \tag{21.42}$$

The exactness, identification of the quadratic term, and direct-sum formula (21.41) are given in (Whitehead, 1950, Theorem 1, p. 55; Section 2, p. 57; Theorem 5, p. 58). The article explicitly allows infinite complexes on p. 57, so these results apply to the simply connected universal cover. Naturality specifies the element in (21.40).

2. For the normalization calculation take the negatives of the complex-oriented Picard homology generators and denote them by $(H, E_1, E_2, E_3)$. Their intersection matrix is still $\mathrm{diag}(1, -1, -1, -1)$. This common reversal matches the phase-cell orientations of the cusp basis: the toric line with class $-(uw + 6\gamma\delta)$ has complex curve degrees $(-6, 0, 1)$, whereas its values on the corresponding phase classes are $(6, 0, -1)$; the support function $rs$ similarly fixes the middle sign. The exact wall-degree calculation in III.5.9 (p. 39) gives this comparison. With these orientations the six conductor classes are

$$E_1,\ H - E_1 - E_2,\ E_2,\ H - E_2 - E_3, E_3,\ H - E_1 - E_3.$$

The first opposite pair is self-glued in the surviving $u$ direction. In the gamma cover its relation is therefore $E_1 = H - E_2 - E_3$. The other two pairs connect adjacent gamma translates. Their relations, after eliminating $H$, are $(h-1)E_2 = (h-1)E_3 = 0$. Thus the image of normalization homology is, with these curve generators,

$$AE_1 \oplus (A/(h-1))E_2 \oplus (A/(h-1))E_3, \qquad A = \mathbb{Z}[h, h^{-1}], \qquad H = E_1 + E_2 + E_3. \tag{21.43}$$

This also follows directly from the transpose of the matrix $B(h)$ of 19.4, with the intersection signs of the Picard basis. It is the whole $H_2$ of the gamma cover: the degree-zero normalization row is the cellular complex of an infinite cylinder, whose second homology is zero.

3. We fix the comparison with the cusp basis of (21.14). In the ordinary central fibre the first conductor class $\overline{C}_1$ is the self-$u$ pair; the other two are the images of $E_2, E_3$, in either order. The chosen cusp basis is

$$E = -\overline{C}_1 - \overline{C}_2, \qquad U = \overline{C}_2 + \overline{C}_3, \qquad D = \overline{C}_2.$$

The two torsion coordinates in (21.43) are determined by this ordinary comparison, because each is annihilated by $h-1$. The normalization map is onto the whole group in (21.43), and $E_2, E_3$ generate its entire $A$-torsion submodule. Consequently the coefficient of $E_1$ in the free quotient generates $A$ and hence is a unit $\pm h^n$. The ordinary comparison fixes its sign. Translating the entire chosen normalization component by $h^{-n}$ makes this coefficient $-1$ and leaves both torsion coordinates fixed. The common Picard sign convention above changes every generator simultaneously. The resulting normalization map in the basis used for the full peripheral comparison is

$$E_1 \longmapsto -E - D, \qquad E_2 \longmapsto D, \qquad E_3 \longmapsto U - D, \qquad H \longmapsto -E + U - D. \tag{21.44}$$

Exchanging $E_2, E_3$ has no effect below.

4. The intersection form of $S$ is $\operatorname{diag}(1,-1,-1,-1)$. Its usual four-cell description, or the Hopf invariant of its two-dimensional classes, therefore gives

$$\omega_S[S] = q(H) - q(E_1) - q(E_2) - q(E_3).$$

Substitute $H = E_1 + E_2 + E_3$ and then (21.44). Bilinearity of the cross effect gives

$$\omega_S[S] \longmapsto [E_1, E_2] + [E_1, E_3] + [E_2, E_3] = -[E, U] - 2q(D). \tag{21.45}$$

The normalization map $S \to E$ lifts to $\widetilde{E}$ because $S$ is simply connected. Naturality of (21.42) shows that the last expression is the image of its four-dimensional fundamental class.

5. The global cancellation also tracks the three cusp two-cycles in addition to the peripheral chains. It gives, in the generators $(x, t, y, z, e)$ of 21.1,

$$E \longmapsto (1-h)e + 6t, \qquad U \longmapsto t, \qquad D \longmapsto 0. \tag{21.46}$$

For clarity, before the last relation reductions its $t$ coefficient for $E$ is $-3h + 4ha + 4ha^2 + h^2$, whose augmentation is six; its free coefficient is $1-h$. Its coefficient for $D$ is $2h^{-1}a^2 - 1 - a$, which has augmentation zero and hence annihilates the invariant generator $t$. The other coordinates vanish. These identities follow by applying the integral group-ring cancellations of 21.1 to the displayed cusp-cycle columns. Applying $\Gamma$ to (21.46) in (21.45) gives

$$-[(1-h)e + 6t, t] = [e(h-1), t] - 12q(t),$$

as asserted. In ordinary coinvariants $t = 2v$ and $E = 12v$ for a primitive class $v$. The quadratic image is then $-48q(v)$, equal to the degree-two cup square evaluated on the normalization.

6. We finish with the connected-sum obstruction, allowing every possible free summand. Multiplication by $h-1$ is injective on $R$: a finitely supported group-ring element constant along an infinite $h$ orbit is zero. It annihilates $T$, so

$$\operatorname{Ann}_M(h-1) = T, \qquad \operatorname{Hom}_R(T, R) = 0. \tag{21.47}$$

Suppose that a copy $fR$ is a direct summand of $M$. Write $f = eu + t_0$ and let a retraction send $e$ to $v$. It vanishes on $T$, and hence $vu = 1$.

7. The ring $\mathbb{Z}K$ is directly finite. One elementary verification uses the index-twelve subgroup of $K$: its quotient by the central $\mathbb{Z}$ is the free group of the punctured genus-three surface, so it is $F_6 \times \mathbb{Z}$. This subgroup, and hence $K$, is residually finite. In every finite-group quotient the regular representation is finite dimensional over $\mathbb{Q}$, so $vu = 1$ implies $uv = 1$. Residual finiteness separates the finite support of $uv - 1$ and proves the same equality in $R$. Thus $u$ is a unit, the kernel of the retraction is exactly $T$, and every free splitting is obtained by replacing $e$ by $eu + t_0$.

8. Under such a change the mixed component of (21.40) is $fu^{-1}(h-1)\otimes t$. It is nonzero: $t$ has infinite order by (21.3), and $u^{-1}(h-1)$ is nonzero. The central generator may also be reversed by a group automorphism, which multiplies $h-1$ by a unit and does not affect the argument.

9. If $E \simeq E_0\#(S^2\times S^4)$, its universal cover has, through dimension four, the homotopy type of $\widetilde{E}_0$ wedged with one copy of $S^2\vee S^4$ for each deck element. This follows by lifting the connected-sum balls; deleting or regluing their boundary five-spheres does not alter the four-skeleton. The new two-spheres generate a free $R$ summand. The four-spheres have trivial attaching maps, and every remaining four-dimensional Whitehead image lies in $\Gamma(\pi_2(E_0))$. There can therefore be no mixed component between this free summand and its complement. This contradicts (21.40)–(21.47) for every choice of the summand, and proves the stated obstruction.

10. For the assertion about $Y$, first take the maps from the two complements to the splice. Carry the four cycles $(t_0,e_0,t_1,e_1)$ through the unit cancellations of 21.2. In the resulting sixteen-generator presentation they are four unit vectors. The seven-generator block changes therefore give
$$t_0=\Omega_R,\qquad t_1=\Omega_L,\qquad e_0\in R/q_2R,\quad e_1\in R/q_1R \tag{21.48}$$
with exactly the axes in (21.24). Both $t_i$ are invariant under $G$: the four displayed relations for each $\Omega$ in (21.25)–(21.26) annihilate it by $a-1,b-1,c-1,d-1$. On $\Gamma$-coinvariants their respective images are
$$-2(B_2+C_2),\qquad 2(p_2-r_2), \tag{21.49}$$
and each has order exactly six: the relevant two coordinates have independent orders three and four in (21.25)–(21.26). In particular an integer annihilating either $t_i$ in the full module must be divisible by six.

11. Put $A_\Pi=\mathbb{Z}\Pi$ and $z=h_1h_2$. Choose the twelve coset representatives giving the free restriction of each axis in (21.28), and denote the resulting twenty-four generators by $f_j$. The normalized cusp surfaces and all their chosen deck translates are maps to $\widetilde{P}$. Transfer them to $\widetilde{Y}$ using the $\Pi$-equivariant homotopy equivalence $\widetilde{P}\simeq\widetilde{Y}\times\mathbb{R}$. On the first axis $h_2$ is the identity, so multiplication by $h_1-1$ equals multiplication by $z-1$; the analogous statement holds on the second axis. Naturality of (21.42) therefore supplies, for every $j$, an element in the image of the Whitehead map of $\widetilde{Y}$ of the form
$$\pm\bigl([f_j(z-1),t_{i(j)}]-12q(t_{i(j)})\bigr), \tag{21.50}$$
where $i(j)=0$ on the first twelve axes and $1$ on the remaining twelve. Invariance of $t_i$ under $G$ removes any extra coset action in this formula.

12. Write $\pi_2(Y)=F\oplus T_Y$, with $F=A_\Pi^{24}$ and $(z-1)T_Y=0$ as in (21.28). Suppose a free copy $fA_\Pi$ is a direct summand, with a retraction $r:\pi_2(Y)\to A_\Pi$ satisfying $r(f)=1$. Multiplication by $z-1$ is injective on $A_\Pi$, so $r(T_Y)=0$. Write $r(f_j)=v_j$ and $f=\sum_j f_ju_j+t$; then
$$\sum_j v_ju_j=1. \tag{21.51}$$
In the splitting $fA_\Pi\oplus\ker r$, the mixed component of (21.50) is $fv_j(z-1)\otimes t_{i(j)}$. If all these mixed components were zero, every integer coefficient of $v_j(z-1)$ would be divisible by six, by (21.49). The group-ring basis makes this implication immediate even if $t_i$ has larger or infinite order. Multiplication by $z-1$ is also injective on $(\mathbb{Z}/6)[\Pi]$: along an infinite $z$ orbit a finitely supported coefficient function with zero difference is zero. Hence every $v_j$ belongs to $6A_\Pi$, contradicting (21.51).

13. Finally, a connected sum with $S^2 \times S^7$ has, through dimension four in its universal cover, the original space wedged with a free deck orbit of two-spheres. All images of its four-dimensional Whitehead map consequently have zero mixed component with that free summand. The preceding argument rules this out for every free summand of the module. This proves the assertion for $Y$.

The corresponding exact calculations are implemented in the programs in Table 1. □

## 22 Smooth plumbings and framed handles

**Proposition 22.1** (The relative smooth cusp model)**.** [tag 02MJ]

*For the marked cusp, let*

$$Z(q) = \frac{\log q}{2\pi i} B_0 + C(q), \qquad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

*where $C(q)$ is holomorphic and bounded on a disc about zero. For a sufficiently small closed disc the six-dimensional cusp tube is orientation-preservingly diffeomorphic, relative to its full marked torus-bundle boundary, to the unitless toric quotient defined by $C = 0$. The latter has the following finite normal-crossings plumbing description.*

*Start with the canonical complex line bundle of $S = \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ and its six toric curves in cyclic order*

$$\begin{aligned} &C_0 = E_1, && C_1 = H - E_1 - E_2, && C_2 = E_2, \\ &C_3 = H - E_2 - E_3, && C_4 = E_3, && C_5 = H - E_1 - E_3. \end{aligned} \tag{22.1}$$

*Pair $C_i$ with $C_{i+3}$ for $i = 0, 1, 2$. Both the surface-normal line and the canonical line restricted to every $C_i$ have degree $-1$. In the resulting $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ coordinates $(z, a, b)$, with $z = 0$ at the endpoint meeting $C_{i+1}$ and $b$ the surface-normal coordinate, the pairing is*

$$(z, a, b) \longmapsto (z^{-1}, zb, za). \tag{22.2}$$

*The function giving the cusp parameter is $q = zab$ in this coordinate chart. The full coordinate map (22.2) specifies the normal framing of each plumbing. The two triple points are the endpoint orbits $\{0, 2, 4\}$ and $\{1, 3, 5\}$, where endpoint $i$ means $C_i \cap C_{i+1}$. Ordered local complex triples undergo a cyclic permutation around each orbit, with threefold composition the identity. The plumbing is formed with collars and rounded corners; its distinguished outer boundary is $|q| = \eta$.*

*Proof.* 1. We first construct the comparison by a smooth family. Set $C_s(q) = (1 - s)C(q)$ for $0 \le s \le 1$. The affine toric cover is fixed. On its dense torus the deck translation for $\lambda \in \mathbb{Z}^2$ becomes

$$(x, q, s) \longmapsto \left(q^{B_0 \lambda} \exp(2\pi i C_s(q)\lambda)x,\ q,\ s\right). \tag{22.3}$$

The exponential is componentwise. The powers of $q$ permute the toric charts by translation of the $A_2$ triangulation, while the exponential factors are nowhere-zero analytic units on every chart. Thus (22.3) defines a smooth family of holomorphic deck actions on a fixed smooth toric covering space. On a central stratum the lattice translation permutes its bounded fan cells without fixed points. Off the central fibre, freeness and properness are exactly the nondegeneracy of the real period lattice.

2. These assertions hold uniformly after shrinking $\eta$. Indeed put $T = -\log|q|/(2\pi)$, and let $K$ bound $\| \operatorname{Im} C_s(q) \|$ on the compact parameter interval and closed disc. For $T > K + 1$ the matrix $TB_0 + \operatorname{Im} C_s(q)$ is invertible, since $B_0$ is orthogonal. A fundamental parallelogram has logarithmic position $B_0 a + T^{-1} \operatorname{Im} C_s(q) a$ with $a \in [0, 1]^2$; it meets a uniformly bounded number of toric charts. More explicitly choose a triangle containing the unperturbed point $B_0 a$

and let $P$ be its unimodular vertex matrix. Its barycentric coordinates $t_j$ are nonnegative, and the corresponding chart coordinates satisfy $\log|z_j| = t_j \log|q| + O(2\pi\|P^{-1}\|K\sqrt{2})$. There are only finitely many such triangles, so with $c = 2\pi\sqrt{2}\max\|P^{-1}\|$ every chosen chart coordinate has modulus at most $\exp(cK)$, uniformly in $q, s$ and all compact phases. Their closed polydiscs have compact closure in the affine charts. These finitely many compact chart images prove compactness of the quotient over $[0,1]$ and $|q| \le \eta$. The central charts and their incidence relations are fixed. Projection of this quotient to $s$ is consequently a proper smooth submersion, including on every normal-crossings stratum. Its only boundary is $|q| = \eta$ together with the parameter end faces, and its boundary restriction is also a submersion.

3. On $|q| = \eta$ prescribe the marked real coordinates $x = \exp(2\pi i(Z_s(q)a + b))$, $(a,b) \in T^4$. They give a smooth product trivialization in $s$ and precisely the prescribed mapping-torus marking. Extend its horizontal vector field over a collar, and then over the compact family. Local toric product charts provide extensions tangent to every central stratum; a partition of unity preserves that property and the lift of $\partial_s$. Integrating this field proves the asserted relative diffeomorphism. Its orientation is the complex orientation on each fibre. This is relative Ehresmann transport in a smooth family.

4. It remains to identify the endpoint $C = 0$ with the framed plumbing data. In the lattice $\mathbb{Z}^2$ use cyclic rays
$$(1,0),\ (0,1),\ (-1,1),\ (-1,0),\ (0,-1),\ (1,-1). \tag{22.4}$$
The star of the vertex zero in the $A_2$ triangulation consists of the six triangles spanned by zero and consecutive rays. The corresponding three-dimensional cones have primitive rays $(v,1)$ and determinant one. Their common divisor at the central ray is the toric surface $S$. A principal central fibre has normal line
$$\mathcal{O}_S(-C_0 - \cdots - C_5) = K_S. \tag{22.5}$$
The cyclic surface fan in (22.4) also proves this identification directly: it is the fan of the three torus-fixed blow-ups of $\mathbb{CP}^2$, and each neighboring-ray relation gives $C_i^2 = -1$. In the Picard basis (22.1), $K_S = -3H + E_1 + E_2 + E_3$, so $K_S \cdot C_i = -1$.

5. Near a double curve the two adjacent unimodular charts give the total space of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Its two base charts have coordinates
$$(z,a,b), \qquad (w,A,B) = (z^{-1}, za, zb), \qquad q = zab = wAB. \tag{22.6}$$
The period translation taking a neighboring component back to the chosen component exchanges the two normal directions. It reverses the chosen endpoint coordinate on the opposite toric curve. In the first chart this is exactly (22.2); in the target infinity chart it is simply $(z,b,a)$, so it extends smoothly over the endpoint. The map is involutive, preserves $q$, and its complex Jacobian determinant is 1. These equalities specify the normal-disk attachment, including its sign and angular twisting.

6. The endpoint $C_i \cap C_{i+1}$ is sent to $C_{i+2} \cap C_{i+3}$. At the first endpoint write the ordered ambient coordinates as the surface-normal coordinate, the $C_i$-normal coordinate and the $C_{i+1}$-normal coordinate. The next endpoint has their order cyclically permuted. The threefold composition is the identity on the ordered coordinates. Thus the two endpoint orbits give ordinary triple charts $q = z_0 z_1 z_2$ with trivial clutching around each triple point.

7. Equivalently, for a triangle with vertex matrix $P$ whose columns are $(v_i, 1)$, translation by $n \in \mathbb{Z}^2$ replaces it by $L_n P$, where
$$L_n = \begin{pmatrix} 1 & 0 & n_1 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{22.7}$$

It acts as the identity in the corresponding translated unimodular coordinates, since $(L_nP)^{-1}L_nP = I$. This verifies the preceding plumbing and triple identifications in the toric charts. More explicitly, order the star matrix as $P_i = ((0,1),(r_i,1),(r_{i+1},1))$ and reverse its columns when using the coordinate order $(z,a,b)$. For every $i$ the matrix of translation by $-r_i$ from this chart to the opposite following-endpoint chart is

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{22.8}$$

Its monomials are exactly (22.2). In normal-first order, $P_{i+2}^{-1}L_{-r_i}P_i$ is the cyclic permutation matrix. Thus the framing and triple-cycle assertions hold for all six fan identifications. Taking collars of the component core, three double-curve annuli and two triple charts, and rounding their outer corners, gives the asserted finite plumbing. These are subcharts of the smooth toric quotient, so the construction is Hausdorff and the collar identifications are the indicated smooth ones.

8. Finally the period translations for $g,u$ are $-e_2,e_1$. On $q = \eta e^{-2\pi i\theta}$ the marked expression is $x = \exp(2\pi i((-\theta+iT)B_0a+b))$. At $\theta = 1$ it agrees with $\theta = 0$ and $b$ replaced by $b - B_0a$. This is exactly the inverse endpoint convention for

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{22.9}$$

The comparison therefore preserves the full outer marking.

The finite algebra and combinatorial data are implemented in the programs in Table 1. □

**Proposition 22.2** (The explicit smooth plumbing presentation)**.** [tag 02ML]

*The following finite smooth plumbing presentation uses only standard pieces and specified attaching maps. Its conversion into ordinary framed handles and the relative smooth identification are given in 22.3.*

*Let $F = \mathbb{R}^4/\mathbb{Z}^4$, with basis $(g,u,w,d)$, and use the two matrices $A_j$ and fixed vectors $v_j$ of definition 17.1, in the ordered basis $(g,u,w,d) = (\widehat{\gamma},\widehat{u},\widehat{w},\widehat{\delta})$. Set*

$$(m_1,m_2) = (3,4), \quad (\kappa_1,\kappa_2) = (1,-1), \quad w_j = \kappa_j v_j, \qquad L = \ker\gamma,$$

*so that $\Lambda = \mathbb{Z}w_j \oplus L$ and $A_jw_j = w_j$. On $L$ the two matrices are*

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{22.10}$$

*Define the standard smooth four-manifolds*

$$\mathcal{S}_j = (T^3\times\mathbb{R})/((l,\sigma+1)\sim(B_j^{-1}l,\sigma)), \qquad \mathcal{N}_j = \mathcal{S}_j\times D_r^2. \tag{22.11}$$

*The cap boundary is identified with $M_{A_j}$ by the following explicit map. Write $x = \phi w_j + l$ and use the interval marking $(x,1)\sim(A_j^{-1}x,0)$ on $M_{A_j}$. Then*

$$[x,\theta] \longmapsto \left([l,\sigma = m_j\kappa_j\phi+\theta],\ r = e^{2\pi i\kappa_j\phi}\right). \tag{22.12}$$

*All angles on the right are taken in their stated mapping-torus quotients. In the ordered angular variables $(\phi,\theta)$ and $(\sigma,\alpha)$, where $r=e^{2\pi i\alpha}$, the map is*

$$\begin{pmatrix}\sigma\\ \alpha\end{pmatrix}=\begin{pmatrix}m_j\kappa_j & 1\\ \kappa_j & 0\end{pmatrix}\begin{pmatrix}\phi\\ \theta\end{pmatrix},\qquad \begin{pmatrix}\phi\\ \theta\end{pmatrix}=\begin{pmatrix}0 & \kappa_j\\ 1 & -m_j\end{pmatrix}\begin{pmatrix}\sigma\\ \alpha\end{pmatrix}. \tag{22.13}$$

*These are integral unimodular matrices.*

*To define a compact six-manifold $\mathcal{E}_{\mathrm{pl}}$, take the flat $F$-bundle over an oriented disc with three open interior discs removed. Its three marked inner loop monodromies are $A_1,A_2,M_0$, with $A_1A_2M_0=I$. Equivalently cut the planar surface along three disjoint paths to its outer boundary, take the product with $F$, and reglue the two banks by these three integral linear diffeomorphisms. This specifies smooth band attachments, including their product normal coordinates. Fill the first two boundary components by (22.11)–(22.13), and the third by the finite canonical-bundle plumbing of 22.1, using its complete marked $M_0$ boundary map. The remaining boundary is marked as $F\times S^1_\mu$, with the outer meridian orientation of (19.1). The core uses its zero section.*

*For the regular-fibre complement there is an orientation-preserving diffeomorphism*

$$E\ \cong\ \mathcal{E}_{\mathrm{pl}}\quad \textit{relative to } F\times S^1_\mu. \tag{22.14}$$

*Consequently the shifted fivefold has the explicit smooth presentation*

$$P\ \cong\ (\mathcal{E}_{\mathrm{pl}}\times T^4)\ \cup_{(x_1,x_2,\theta)\mapsto(x_1,x_2,-\theta)}\ (T^4\times\mathcal{E}_{\mathrm{pl}}). \tag{22.15}$$

*Every piece in this presentation is a standard torus product, a linear three-torus mapping torus times an oriented disk, or the canonical disk bundle of the explicitly specified toric blow-up with the normal-disk plumbings (22.2). All attachment maps have been specified on their normal coordinates.*

*Proof.* 1. We verify first that (22.11)–(22.13) are the finite caps, including the normal framing. The original cap is

$$(F\times D^2_s)/\big((x,s)\sim(A_jx+v_j/m_j,\,e^{-2\pi i/m_j}s)\big). \tag{22.16}$$

The good-reduction real flat marking is smooth on the entire root disc, so these are smooth coordinates even at $s=0$. Because $\gamma A_j=\gamma$ and $\gamma(v_j)=\kappa_j=\pm1$,

$$r=s\,e^{2\pi i\kappa_j\gamma(x)} \tag{22.17}$$

is invariant under the displayed action. It is single-valued on $F$, has the same absolute value as $s$, and is complex linear on every normal disk. The induced map $[x,s]\mapsto([x],r)$ identifies (22.16) smoothly with $S_j\times D^2$. An inverse is obtained by choosing any representative $x$ and setting $s=re^{-2\pi i\kappa_j\gamma(x)}$; invariance makes this independent of the representative in the quotient.

2. In the integral basis $(w_j,u,w,d)$ the affine generator is

$$(\phi,l)\longmapsto(\phi+\kappa_j/m_j,\,B_jl). \tag{22.18}$$

Thus $\sigma=m_j\kappa_j\phi$ gives exactly the mapping torus in (22.11). This also supplies a direct inverse to the product description: choose a representative $(l,\sigma)$, put $x=(\kappa_j\sigma/m_j)w_j+l$ and $s=re^{-2\pi i\sigma/m_j}$. Replacing that representative by the defining mapping-torus relation changes $(x,s)$ by the finite affine generator and hence not its class.

3. The boundary parametrization of the finite cap is

$$[x, \theta] \longmapsto [x + \theta v_j/m_j, e^{-2\pi i\theta/m_j}]. \tag{22.19}$$

Substitute (22.19) into (22.17)–(22.18). The coefficient of $\theta$ in the normal angle is $-1/m_j + \kappa_j^2/m_j = 0$, while the base angle is $m_j\kappa_j\phi + \theta$. This proves (22.12) and (22.13). At $\theta = 1$ that base angle advances by one, precisely replacing $l$ by $B_j^{-1}l$; this checks the mapping-torus seam. A lattice change of $\phi$ changes the base angle by $m_j\kappa_j$; the equality $B_j^{m_j} = I$ makes the map descend under this change.

4. The function of the cap that gives its original base parameter is, in these smooth coordinates,

$$q = r^{m_j} e^{-2\pi i\sigma}. \tag{22.20}$$

On the marked boundary it has angle $m_j\alpha - \sigma = -\theta$, the original clockwise base convention. A positive normal meridian at fixed $(l, \sigma)$ has $(\Delta\phi, \Delta\theta) = (\kappa_j, -m_j)$. Since $\kappa_j w_j = v_j$, it is exactly $t_{v_j}\rho_j^{-m_j}$, recovering the filling relation $\rho_j^{m_j} = t_{v_j}$.

5. Over the punctured planar core, the family has the smooth real flat marking with linear monodromies $A_j$ and its zero section. The logarithmic finite gluing has exactly the drift in (22.19); after subtracting it, the remaining translation has a single-valued real lift and is removed by the collar isotopy already proved in III.3.9 (p. 20) . At the cusp use the relative smooth diffeomorphism of 22.1. Each comparison has precisely the prescribed boundary value and can be made product on a smaller collar. They therefore glue to the relative diffeomorphism (22.14). In particular the normal framing of the unfilled ordinary fibre is unchanged.

6. Finally use the two hemispheres in the shifted fibre-product decomposition of 18.4. On each hemisphere the other factor is a regular marked torus bundle over a disc, hence a product preserving its zero section and flat marking. Their common boundary has unchanged torus coordinates and opposite normal angles. Taking the product of (22.14) with those regular factors and gluing gives (22.15), with its induced complex orientation.

The finite algebra and combinatorial data are implemented in the programs in Table 1. □

**Theorem 22.3** (Ordinary framed handles and the smooth type)**.** [tag 02MM]

*The following finite ordinary framed-handle prescription defines a compact oriented six-manifold $\mathcal{H}_6$ with marked boundary $T^4 \times S^1$, and a closed oriented ten-manifold $\mathcal{H}_{10}$. There are orientation-preserving diffeomorphisms*

$$E \cong \mathcal{H}_6 \quad \textit{relative to the full marked boundary,} \qquad P \cong \mathcal{H}_{10}. \tag{22.21}$$

*The respective numbers of ordinary handles, in increasing index, are*

$$(2, 5, 14, 22, 21, 10, 2), \qquad (4, 31, 119, 290, 485, 574, 485, 290, 119, 31, 4).$$

*The attaching embeddings and their normal coordinates are given below. The balls and complex normal charts carry their complex orientations; orient the product blocks by their displayed coordinate embeddings. For example the dense block below has orientation $-d\arg X \wedge d\arg Y \wedge d\mu_1 \wedge d\mu_2 \wedge d\operatorname{Re} q \wedge d\operatorname{Im} q$. This fixes the orientation convention even when the chosen order of angle and moment coordinates differs from the complex coordinate order.*

1. The two triple-point balls. *Use two complex coordinate balls, denoted $V_+$ and $V_-$, with ordered coordinates $Z = (Z_0, Z_1, Z_2)$ and $W = (W_0, W_1, W_2)$. The boundary faces are rounded coordinate faces of*

$$|Z_i| \le \epsilon_v, \quad |Z_0Z_1Z_2| \le \eta, \qquad |W_i| \le \epsilon_v, \quad |W_0W_1W_2| \le \eta. \tag{22.22}$$

*They are smooth six-balls. Throughout the prescription, rounding is in the nonnegative radii, preserves the angular coordinates, and is fixed on the attaching collars. Positive changes of the radii only give the corresponding collar isotopy. The complex products in (22.22) are the same function $q$. The coordinate formulas may first be read on manifolds with corners; apply the same radial collar reparametrization on the two sides of each attaching face before smoothing their union. This convention also specifies the maps at a corner meeting $|q| = \eta$.*

2. The three annular blocks, with their endpoint framings. *For $i = 0, 1, 2$ use a block $S^1 \times D^1 \times D^4$ with coordinates*

$$z = \exp(s + 2\pi i\theta), \quad -R \le s \le R, \qquad (a, b) \in D^4, \qquad q = zab. \tag{22.23}$$

*The normal four-ball here is a rounded radial domain $|a|, |b| \le \epsilon_e$, $|ab| \le \eta/|z|$; its positive radial trivialization identifies it with $D^4$. The ends attach to framed tubes about the following coordinate circles. A table entry $(Z_j; Z_k, Z_l)$ means that the curve coordinate is $Z_j$ and the two ordered complex normal coordinates are $Z_k, Z_l$; the same convention applies to $W$.*

| $i$ | $z = 0$ *end:* $(z; a, b)$ | $z = \infty$ *end:* $(w; A, B)$ |
|---|---|---|
| 0 | $(Z_2; Z_1, Z_0)$ | $(W_0; W_1, W_2)$ |
| 1 | $(W_2; W_1, W_0)$ | $(Z_1; Z_2, Z_0)$ |
| 2 | $(Z_0; Z_2, Z_1)$ | $(W_1; W_2, W_0)$ |

(22.24)

*The full-dimensional endpoint map between the two end charts is*

$$(w, A, B) = (z^{-1}, za, zb). \tag{22.25}$$

*In particular the second attaching circle has angle $-\theta$, and its two complex normal vectors each rotate by $+2\pi\theta$. The factors $|z|$ are the positive radial rescalings. Every coordinate circle of the two balls occurs exactly once in (22.24). The maps are embeddings of $S^1 \times \partial D^1 \times D^4$.*

3. The dense torus block and its full sector maps. *Attach $T^2 \times D^2 \times D^2_q$ along $T^2 \times \partial D^2 \times D^2_q$. Here is a completely explicit choice of its base disk and the attaching map. Use the rays $r_i$ of (22.4) and the hexagon*

$$\mathcal{P} = \{v \in \mathbb{R}^2 : \langle v, r_i\rangle \ge -1\}, \quad \mathrm{Vert}(\mathcal{P}) = \{(-1,-1), (0,-1), (1,0), (1,1), (0,1), (-1,0)\}. \tag{22.26}$$

*On the positive real part of $(\mathbb{C}^*)^2$, use the coordinates given by the toric moment map*

$$\mu(X, Y) = \frac{\displaystyle\sum_{v \in \mathrm{Vert}(\mathcal{P}) \cup \{0\}} |X^{v_1} Y^{v_2}|^2 v}{\displaystyle\sum_{v \in \mathrm{Vert}(\mathcal{P}) \cup \{0\}} |X^{v_1} Y^{v_2}|^2}. \tag{22.27}$$

*Choose a small inward parallel hexagon, with its corners rounded symmetrically, $\mathcal{P}_\epsilon \subset \operatorname{int} \mathcal{P}$. The inverse image is $T^2 \times \mathcal{P}_\epsilon$; its two circle coordinates are $\arg X, \arg Y$. On the boundary disk circle put $(X, Y)$ equal to the positive inverse of $\mu$, multiplied by these two phases. On the six sectors of that boundary, the attaching map into the preceding blocks is the following monomial map, with the whole complex normal coordinate $q$ included:*

| $i$ | $n_i$ | $a_i$ | $a_{i+1}$ |
|---|---|---|---|
| 0 | $q/(XY)$ | $X$ | $Y$ |
| 1 | $q/Y$ | $XY$ | $X^{-1}$ |
| 2 | $qX$ | $Y$ | $(XY)^{-1}$ |
| 3 | $qXY$ | $X^{-1}$ | $Y^{-1}$ |
| 4 | $qY$ | $(XY)^{-1}$ | $X$ |
| 5 | $q/X$ | $Y^{-1}$ | $XY$ |

$q = n_i a_i a_{i+1}$. (22.28)

*Use the cyclic identifications* $(n_2, a_2, a_3) = (Z_1, Z_2, Z_0)$, $(n_4, a_4, a_5) = (Z_2, Z_0, Z_1)$ *and the analogous two cyclic permutations of* $W$ *for vertices* $3, 5$. *Vertices* $0, 1$ *are* $Z, W$. *The chart at an edge is* $(z, a, b) = (a_{i+1}, a_i, n_i)$, *with* (22.25) *at its other end. These rules specify* (22.28) *also where a sector meets an annular block. The transition on a common sector is the monomial map* $P_{i+1}^{-1} P_i$, *where* $P_i = ((0,1), (r_i, 1), (r_{i+1}, 1))$; *the sum of its exponent rows is* $(1,1,1)$, *so it preserves* $n_i a_i a_{i+1}$. *Thus the sector maps specify the normal framing as well as the core map.*

4. Expansion into ordinary handles. *We use the following fixed elementary rule on every parameterized block in this prescription:*

$$T^r \times D^k \times D^{n-r-k} \quad \text{attached on} \quad T^r \times \partial D^k \times D^{n-r-k}. \tag{22.29}$$

*Cut each ordered parameter circle into its two closed semicircles, with small endpoint collars. The first semicircle is its zero handle and the second is its one handle. For a bit vector* $\varepsilon \in \{0,1\}^r$, *the resulting ordinary handle is of index* $k + |\varepsilon|$. *Attach by increasing* $|\varepsilon|$, *then lexicographically. On the original attaching face use the displayed full-dimensional attaching embedding of the block; on a newly exposed parameter face use the identity endpoint map in that circle. The normal coordinates are the remaining ordered semicircle coordinates followed by the original normal coordinates. Corner rounding and the permutation putting negative coordinates first complete an ordinary framed attaching embedding* $S^{k+|\varepsilon|-1} \times D^{n-k-|\varepsilon|}$. *For one parameter it is the literal two-piece decomposition in (Vogel, 2009, Section 4.1, equation (16), p. 103).*

*Apply* (22.29) *first to* (22.23), *with* $(r, k, n) = (1, 1, 6)$, *and then to the core, with* $(r, k, n) = (2, 2, 6)$. *Together with the two balls this gives cusp handle counts* $(2, 3, 4, 2, 1)$.

5. The torus bands and the finite caps. *Take the marked flat torus bundle of 22.2 over the disc with three holes, regarding the cusp boundary as the incoming boundary. Cut along the two marked disjoint arcs from the two finite boundary circles to the outer circle. The cut surface is an annulus; its torus bundle is the collar of the already constructed cusp boundary. Restoring the two strips attaches two blocks*

$$T^4 \times D^1 \times D^1. \tag{22.30}$$

*To fix their maps, use the oriented ribbon polygon whose successive based inner boundary words are* $A_1, A_2, M_0$ *and whose outer word is their product. The negative bank of the* $j$*th cut has coordinate* $(x, t)$, *the positive bank has* $(A_j x, t)$, *and the strip uses the product normal coordinate. Reversing a bank orientation replaces that matrix by its inverse. This is the cut-polygon definition of the flat bundle in 22.2; the identity* $A_1 A_2 M_0 = I$ *specifies the unfilled outer boundary. These gluing maps are linear. Thus* (22.30) *has its full attaching embedding, including the transverse interval, fixed by these linear maps. Expand each by* (22.29), *with* $(r, k, n) = (4, 1, 6)$. *The unused boundary circles are precisely the two finite boundaries and the marked ordinary boundary. For clarity, the initial bank coordinates on the already attached cusp boundary are the explicit coordinates of proposition 22.1:*

$$\begin{aligned} q = \eta e^{-2\pi i \theta}, \qquad T = -\log \eta/(2\pi), \\ (X, Y) = \exp\big(2\pi i((-\theta + iT) B_0 a + b)\big), \qquad a = (g, u),\ b = (w, d). \end{aligned} \tag{22.31}$$

*The exponential is componentwise; at an interval seam use* $b \mapsto b - B_0 a$. *Equations* (22.28) *and* (22.31) *therefore give the bank embeddings into the existing handle boundary, including their derivatives.*

*For the* $j$*th finite cap use its already explicit* $\mathcal{S}_j \times D_r^2$ *and the entire boundary embedding* (22.12). *The mapping torus* $\mathcal{S}_j$ *has the following product block construction: a block* $T^3 \times D^1$ *over the first semicircle of* $\sigma$, *followed by a block* $T^3 \times D^1$ *over the second semicircle, with endpoint maps* $I$ *and* $B_j^{-1}$. *The first is a three-parameter zero handle, the second a three-parameter one handle. As a filling relative to* $\mathcal{S}_j \times S_r^1$, *the normal disk supplies two negative directions.*

*Consequently the relative cap consists of*

| *base-circle block* | $(r,k,n)$ | *full attaching data* |
|---|---|---|
| 0 | $(3,2,6)$ | (22.12), *the first semicircle, and the* $r$*-disk* |
| 1 | $(3,3,6)$ | (22.12), *endpoint maps* $I, B_j^{-1}$, *and the* $r$*-disk.* |

(22.32)

*More explicitly an attaching face in the normal direction is sent to* (22.12) *at* $r = e^{2\pi i\alpha}$, *and an attaching face in a base direction is the identical face of the preceding base block, crossed with* $D_r^2$. *At the base seam use* $B_j^{-1}$ *on* $l$ *and the identity on* $r$. *The inverse angular formula* (22.13) *supplies the full map back to the boundary being filled. In particular the normal meridian is* $v_j - m_j\rho_j$. *Apply* (22.29) *to the two blocks in* (22.32). *Each cap adds the ordinary relative counts* $(1,4,6,4,1)$ *in indices* $2,\dots,6$. *This completes the definition of* $\mathcal{H}_6$.

6. The closed ten-dimensional handlebody. *For each handle of* $\mathcal{H}_6$, *take its product with the four ordered torus circles and apply* (22.29). *This gives* $\mathcal{H}_6 \times T^4$ *with its full boundary framing. On the other side take the dual relative handle decomposition of* $T^4 \times \mathcal{H}_6$*: reverse the handle order and exchange the negative and positive disks of every handle. A handle of index* $j$ *becomes one of index* $10-j$, *and its attaching map is the belt embedding of the original handle. Glue its initial boundary by the explicit map*

$$(x_1, x_2, \theta) \longmapsto (x_1, x_2, -\theta). \tag{22.33}$$

*All subsequent dual attaching maps are obtained by this fixed exchange of the two coordinate disks, in reverse order, on the displayed attaching embeddings. Use the factor order* $E, T^4$ *on the first half and* $T^4, E$ *on the second; their interchange has sign* $(-1)^{24} = 1$. *The boundary-angle reversal in* (22.33) *is the orientation-reversing boundary gluing. This defines* $\mathcal{H}_{10}$ *entirely by ordinary handles. If*

$$c(t) = 2 + 5t + 14t^2 + 22t^3 + 21t^4 + 10t^5 + 2t^6, \qquad c_{10}(t) = (c(t) + t^6 c(t^{-1}))(1+t)^4, \tag{22.34}$$

*the coefficients are the counts in the statement. Expanding these polynomials gives* 76 *handles in dimension six and* 2432 *in dimension ten.*

*Proof.* 1. The main geometric point is that the first three steps give *smooth blocks of the cusp tube*. We give the collar argument explicitly. Work in the smooth toric quotient of 22.1. Choose disjoint small coordinate neighborhoods of the two triple points. The three incident coordinate axes in each neighborhood are exactly the six ends in (22.24). Remove smaller endpoint disks from each normalized double curve. Its remaining annulus has the two complex normal lines $\mathcal{O}(-1)|_{\text{annulus}}$, which are trivialized by $(a,b)$ in (22.6). The resulting normal domain in the tube is exactly the domain in (22.23). Every ray in $\mathbb{C}^2$ meets its rounded boundary once and transversely. Indeed the three defining radius functions have positive radial derivative wherever the corresponding boundary face is active. Positive radial rescaling therefore gives a smooth family of four-balls as $z$ ranges over this compact annulus. Choose the radial change to be constant linear scaling on a common sufficiently small ball about the origin and interpolate monotonically only away from that ball; it is then smooth at the origin as well as at the boundary and preserves the stated normal frames. The same argument for the coordinate and product radius functions in (22.22) gives six-balls at the triple points. Rounding can be chosen constant on the end attaching collars.

2. At vertex $i$ the normal-first coordinates are $(n_i, a_i, a_{i+1})$. At vertex $i+2$ the deck identification is their cyclic permutation, and the threefold composition is the identity by 22.1. At the two ends of $C_i$ the coordinates are $(z,a,b) = (a_{i+1}, a_i, n_i)$ and $(w,A,B) = (a_{i-1}, a_i, n_{i-1})$. The ordinary $\mathcal{O}(-1)^2$ transition is (22.25). Substituting the two cyclic permutations gives exactly (22.24). This checks all six ends.

3. We next choose the stratum collars so that their remaining interface is the entire attaching face of the dense block. On the normalized toric surface take a small collar of its hexagonal toric boundary, choosing the corner neighborhoods first and the edge strips second. Choose the strips narrower than the corner collars and invariant under the opposite-edge inversion and the two cyclic triple identifications. The uncovered moment region is the rounded interior disk $\mathcal{P}_\epsilon$. This can be done directly with six inward parallel affine inequalities in (22.26), smoothing each corner inside its already chosen corner neighborhood. Thus the common frontier in the smooth part of the normalization is exactly $T^2 \times \partial\mathcal{P}_\epsilon$.

4. The map (22.27) gives the claimed coordinates: in logarithmic positive coordinates it is the gradient of a log-sum-exponential function. Its Hessian is twice the covariance matrix of the seven affinely spanning exponent vectors, hence positive definite. Strict convexity and the growth in each supporting direction give a diffeomorphism onto $\operatorname{int}\mathcal{P}$. The inverse on the compact disk and its frontier is smooth. The canonical line on this free torus locus is trivial; more specifically the toric function $q$ is a nonzero toric factor times its normal coordinate. Hence $q$ *itself*, including its phase, trivializes that normal line. The portion of the truncated tube over this disk is therefore exactly

$$(T^2 \times \mathcal{P}_\epsilon) \times D_q^2, \quad \text{with interface} \quad (T^2 \times \partial\mathcal{P}_\epsilon) \times D_q^2. \tag{22.35}$$

   In extending the preceding moment cuts off the zero fibre, hold their equations constant in this $q$ coordinate on an open neighborhood of the compact frontier. On their ends use the product cuts of the annular and triple charts. A collar interpolation between these choices is an interpolation of positive radial cut functions, fixed on each already chosen end collar. Their radial derivatives remain positive, so its flow is a collar isotopy. The cuts can consequently be chosen to agree with (22.35) exactly. The $q$ coordinate therefore gives a common normal trivialization across the six sector seams.

5. These are blocks in a sufficiently small truncated neighborhood of the compact central fibre. Their central parts cover that fibre: two triple points, the three open double annuli and the remaining normalized surface core. Compactness from 22.1 permits one common sufficiently small $\eta$: the union of their open chart neighborhoods contains the compact zero fibre, so properness implies that it contains the entire inverse image of a sufficiently small closed $q$-disc. The nested cuts partition this entire inverse image. If a larger originally chosen tube is used, the difference is an exterior collar: on the compact annular region without critical points choose a vector field lifting the radial direction of $q$, and integrate it. On the prescribed outer collar choose the lift in the marked real torus coordinates of 22.1. It is then the identity on all those marked coordinates during radial collar transport.

6. The embedding in (22.35) is (22.28). To check this, the exponent matrix for a unimodular chart is $P_i^{-1}$, applied to $(X, Y, q)$. This gives each of the six displayed rows. Consecutive changes are $P_{i+1}^{-1}P_i$ and telescope to the identity. The translations identifying the two triple orbits are exactly those already used in (22.24). All six normal coordinate multipliers in (22.28) are thereby fixed, and the sector seams are equal maps on overlaps. Positive rescaling of the chosen radial domains leaves these monomials unchanged before the collar reparametrization. This completes the smooth, fully framed identification of the cusp with steps 1–3.

7. The handle expansion rule follows from the product decomposition of the parameter circles. For one circle, write $S^1 = I_0 \cup I_1$. First attach $I_0 \times D^k \times D^{n-k-1}$; after rounding it is a $k$-handle. The remaining block is $(I_1 \times D^k) \times D^{n-k-1}$, a $(k+1)$-handle. Its attaching face consists of the original negative face and its two endpoint faces in the first handle. The stated maps on these three faces agree on their corners because they are restrictions of the same product coordinates. Iterate this argument on the ordered circle factors. It proves (22.29), including all newly created attaching faces and their normal framings.

8. The planar core relative to its incoming circle has two ordinary surface one-handles. Its flat bundle is a product over each strip, so restoring the two cut strips is exactly (22.30). The zero section and the full lattice marking fix the normal coordinates on these attachments. Each cap in (22.32) is the smooth product of 22.2. Its relative handle claim can also be checked by the function $-|r|^2 + \varepsilon f(l, \sigma)$, with a cutoff near the incoming boundary: choose $f$ by a two-critical-point function of $\sigma$, perturbed on its two critical tori by the product Morse function on $T^3$. On the remaining arcs its base derivative dominates the perturbation, so there are no further critical points. The indices are two plus the base indices, as in (22.32). Summing the blocks gives

$$(2, 3, 4, 2, 1, 0, 0) + 2(0, 1, 4, 6, 4, 1, 0) + 2(0, 0, 1, 4, 6, 4, 1) = (2, 5, 14, 22, 21, 10, 2).$$

   Every comparison has its prescribed boundary value, so the relative diffeomorphism of 22.2 gives the first assertion of (22.21). Taking the products, the dual relative handles and the boundary flip of 22.2 gives the second. The handle polynomials evaluated at $-1$ give Euler numbers 2 and 0.

9. The following relative uniqueness theorems determine the smooth type from the full attaching embeddings. The relevant relative uniqueness results are Milnor's Theorem 3.13, which identifies an elementary cobordism from its full characteristic embedding, Theorem 1.4, which makes smooth composition unique with the marked boundary fixed, and the isotopy-extension Theorem 5.8 (Milnor, 1965, Theorem 3.13, pp. 31–32; Theorem 1.4, p. 2 and pp. 25–26; Theorem 5.8, pp. 63–64). In the present notation the characteristic embedding is the whole map $S^{j-1} \times D^{n-j} \longrightarrow \partial W$. Its normal coordinates are part of (22.24)–(22.33). The elementary consequence needed here can also be seen directly. If $\phi_0, \phi_1$ are joined by the isotopy of full attaching embeddings, extend the velocity field on their images to a slightly larger boundary collar and multiply it by a cutoff which is zero on the protected marked boundary. Its flow extends the attaching isotopy to an ambient boundary isotopy. Extending this flow across a product collar of $W$, and using the identity on the new coordinate handle, identifies $W \cup_{\phi_0} (D^j \times D^{n-j})$ with $W \cup_{\phi_1} (D^j \times D^{n-j})$. The same cutoff argument applies to the small radial corner changes above. This derives the relative, framed version from the stated isotopy-extension theorem.

10. Apply this comparison successively in the displayed block and sector order. The initial ball comparisons are their coordinate identifications. At each next stage the attaching map and the prescribed attaching map agree in the computed normal coordinates, or differ by the explicitly described positive collar isotopy. The comparison of that stage is consequently the identity on the new coordinate handle and the product-collar extension on the previous stages. Theorems 3.13 and 1.4 give an inductive relative diffeomorphism of the whole six-manifold. The final ordinary boundary coordinates are unchanged. The same argument, with the belt embeddings and the boundary flip, applies to every handle on the dual ten-dimensional side. Each top handle is specified by its entire attaching map. The computed ordered full framed attaching data, under this relative uniqueness theorem, are the classifying data used for (22.21).

The quadratic attachment of the regular-fibre complement is described in proposition 21.4. The finite algebra and combinatorial data are implemented in the programs in Table 1. □

## Programs

The programs below are available in the repository at the immutable commit

`83d76022bcd442278b760db9ff438afceed6c6ea`

Each filename links to its source at this commit.

Table 1: Programs for the exact computations and interval enclosures.

| Computation and statements | Programs |
|---|---|
| Endpoint and global height enclosures<br>theorem 7.1 | `fresh_frobenius.py`<br>`fresh_local_domains.py`<br>`fresh_global_cover.py`<br>`verify_saved_certificate.py` |
| Cohomology of the resolved fourfold<br>theorems 17.3 and 17.4 | `fourfolds_checks.py`<br>`fourfolds_integral_audit.py` |
| Integral calculation for the second rational pair<br>theorem 12.3 | `fourfolds_v1_homology.py` |
| Relative rational normal forms<br>theorems 12.5 and 13.2 | `fourfolds_relative_normalform.py`<br>`fourfolds_relative_six_probe.py` |
| Stable lattices and primitive extensions<br>theorems 12.7 and 12.8 | `fourfolds_lattice_algorithm.py`<br>`fourfolds_two_classes_lattices.py` |
| Affine twists of the two rational classes<br>theorem 12.9 | `fourfolds_family_twists.py` |
| Finite spectral suspension calculation<br>theorem 13.1 | `fourfolds_suspension_filter.py` |
| Order-six middle-pairing obstruction<br>theorem 13.3 | `fourfolds_universal_56_obstruction.py` |
| Finite-cap and cusp tests for third and fifth cohomology<br>theorem 13.5 | `fourfolds_local_caps.py`<br>`fourfolds_cusp_boundary.py`<br>`fourfolds_survivor_test.py` |
| Marked fundamental-group presentation<br>theorem 14.1 | `fourfolds_vankampen.py` |
| Three-finite-point family examples<br>theorem 14.2 | `fourfolds_three_point_family.py` |
| Finite cap, complement and gluing chains<br>theorems 14.3, 14.4 and 14.6 | `fourfolds_bar_koszul.py`<br>`fourfolds_affine_cap_chains.py`<br>`fourfolds_kuhn_cap_chains.py`<br>`fourfolds_global_chains.py`<br>`fourfolds_total_homology.py`<br>`fourfolds_marked_gluing.py` |
| Algebra of the period construction<br>theorem 3.2 | `fourfolds_periods.py` |
| Spanning tube directions<br>theorem 16.2 | `fourfolds_oka_tubes.py` |

| Computation and statements | Programs |
|---|---|
| Integral homology of shifted products<br>proposition 18.3 and theorem 18.4 | `fourfolds_shifted_integral.py`<br>`manifolds_shifted_mv.py` |
| Bordism and low-degree boundary calculations<br>theorems 18.7, 19.3 and 19.4 | `shifted_group_bordism.py`<br>`maps_shifted_pi2.py`<br>`manifolds_cusp_alexander.py` |
| Circle fibration, cyclic cover and integral fibre<br>theorems 20.1 to 20.3 | `maps_shifted_circle.py`<br>`maps_shifted_return.py`<br>`maps_shifted_cover.py`<br>`maps_shifted_cover_integral.py`<br>`maps_shifted_fibre_integral.py` |
| Complement and splice modules and their quadratic maps<br>theorems 21.1 and 21.2 and proposition 21.4 | `maps_complement_module.py`<br>`maps_splice_module.py`<br>`maps_cusp_quadratic.py`<br>`maps_splice_quadratic.py` |
| Plumbing, finite-cap framings and handle inventory<br>propositions 22.1 and 22.2 and theorem 22.3 | `maps_cusp_smooth_plumbing.py`<br>`maps_finite_cap_framings.py`<br>`maps_ordinary_handles.py` |

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