The $(3, 4, \infty)$ family of complex six-spheres

V. Complex fourfolds from the family

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Abstract

We construct compact complex fourfolds from the torus-fibred complex six-spheres of the $(3, 4, \infty)$ family. The main construction is a family of complex three-tori over $\mathbb{P}^1$ with monodromy in $GL_6(\mathbb{Z})$: two multiple fibres, of multiplicities five and four, and one Kuhn cusp. Its periods come from the exterior square of the genus-two pencil $y^2 = x^5 + \lambda^3 x + \lambda^4$. Every member is a compact complex fourfold diffeomorphic, with its complex orientation, to $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$. The family is parametrized exactly by the half-plane $\operatorname{Im}\xi > c_*$, where $c_* \approx 1.09454$ is minus the minimum of an explicit real-analytic period function. We compute the Hodge numbers, the tangent cohomology (the Kuranishi germ is smooth of dimension one) and the automorphism group $\mathbb{C} \times \mathbb{Z}/2$, and we show that the family extends analytically across no boundary point of its parameter domain. We also construct a compact complex fourfold outside Fujiki's class $\mathcal{C}$ as a small crepant resolution of the fibre product of a complex six-sphere with an elliptic surface; give an effective classification of the integral monodromy data of this kind that yield three-connected eight-manifolds, all of which are diffeomorphic to $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$; and describe shifted fibre products of complex dimension five.

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1 Results and their proofs in outline

1.1 The series and this paper

This paper is the fifth of a series on the holomorphic families built from the $(3, 4, \infty)$ family of torus-fibred complex structures on S^6 (Claude and L. Alpöge, 2026; Engel, 2026). The table gives the main result of each paper of the series and the papers it uses.

Paper	Main result	Uses
I	Exact periods, moduli, fibre strata and the two ends	—
II	Torus-fibred threefolds with $b_1 = b_2 = 0$: the Campana–Demailly–Peternell argument and the sign theorem for all degenerations	I, IV
III	Maps from the six-spheres and the manifolds they give	I, II, IV, V
IV	Smooth monodromy in Θ_7	I, II, III, V
V	Complex fourfolds on $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$ and a resolved fibre product	I, III, IV

1.2 The main theorems

Theorem A concerns the fibre product of a complex six-sphere of the family with an elliptic surface (Part I). Theorems B–F concern a second construction (Part II): a family of complex three-tori over $\mathbb{P}^1$ whose monodromy in $GL_6(\mathbb{Z})$ has orders five and four at two points and an index-one Kuhn cusp at the third. Theorem B gives its underlying smooth manifold, Theorem C its periods, Theorem D the compact complex fourfold and its diffeomorphism type, Theorem E the exact family, and Theorem F the classification of all integral data of this kind. The parameter ξ is taken with respect to the algebraic splitting σ_b of theorem 8.4. Part III treats the shifted fibre products, which are compact complex fivefolds.

Theorem A (The resolved transverse elliptic fibre product)**.** [tag `03KR`] The explicit crepant small resolution of the transverse elliptic fibre product gives a compact complex fourfold with $K = \Pi^* L^{-5}$ and algebraic dimension two; its Todd and Pontryagin calculations give signature two and middle form $\langle 1\rangle \oplus \langle 1\rangle$. The computed values $b_1 = 0$ and $h^{0,1} = 1$, preserved by a Kähler modification, obstruct membership in Fujiki's class $\mathcal{C}$ through the Kähler equality $b_1 = 2h^{0,1}$ (V.2.1, V.2.5).

Mechanism. The explicit period comparison and a small crepant Kuhn resolution preserve the root canonical calculation and the integral characteristic pairings. *Relation to literature.* The threefold factor is the construction of (Claude and L. Alpöge, 2026, Theorems 7.17 and 7.22, pp. 57 and 60), and the elliptic equation is the one in (Engel, 2026, §2, initial construction). The local Kuhn quotient is the $n = 3$, $D = C_3$ specialization of (Yang, 2026, Theorem 8.7 and Corollary 8.8, arXiv v1, pp. 57–59). The additional global construction is the small crepant resolution with the integral period morphism $AZ_3B^{-1} = \mathrm{diag}(Z_2, s + c_E)$ in theorem 2.1; its invariants are calculated in theorem 2.5. The characteristic polynomials are the known identities (Esole and Kang, 2018, equations (1.2)–(1.4), pp. 3–4), and the class-$\mathcal{C}$ obstruction uses (Jia and Meng, 2022, Theorem 1.1 and Remark 1.2, p. 2). Outline of the proof; proofs: V.2.1, V.2.5.

Theorem B (The integral rank-six construction)**.** [tag `03KS`] The rank-six integral monodromy of theorem 5.1, with free affine fillings of orders five and four and the index-one Kuhn cusp, gives a closed smooth eight-manifold that is three-connected, has free integral homology, and has middle intersection form $3\langle 1\rangle \oplus \langle -1\rangle$.

Mechanism. In the integral Leray spectral sequence over $\mathbb{P}^1$, each order-two torsion class coming from the order-four filling is hit by a differential from the cusp, so the homology is free; Todd integrality and the signature bound then force the middle form. *Relation to literature.* The local toroidal filling and its Euler number are the marked $n = 3$ specialization of (Yang,

2026, Theorem 8.7 and Corollary 8.8, arXiv v1, pp. 57–59). The global integral specialization method extends the calculation in (Claude and L. Alpöge, 2026, §7.7). The rank-six data and the integral computation are new. Outline of the proof; proofs: V.5.1, V.5.5, V.7.1.

Theorem C (The genus-two period construction)**.** [tag 03KT] The pencil $y^2 = x^5 + \lambda^3 x + \lambda^4$ has one nodal fibre away from $0, \infty$, at $\lambda_* = -256/3125$. Its good root charts have orders eight and five, and on that root curve its Hodge bundle is

$$\mathcal{H} = T^3 \oplus T, \qquad T = \mathcal{O}((2/5)[\infty] - (3/8)[0]), \qquad \deg T = 1/40.$$

The meromorphic differential dx/y generates $\ell_0 = T^3$. In the full rank-four de Rham bundle, projection of $\ell_0 \wedge \mathcal{H}_{\mathrm{dR}}$ to $\bigwedge^2 \mathcal{H}_{\mathrm{dR}}/\mathbb{C}\omega$ is a rank-three bundle P. Its holomorphic lift to the rank-six extension is $(\sigma_b + \xi\alpha e)(P)$, with the normalization of e, α, σ_b in theorem 8.1. It is transverse to its conjugate on the completed base exactly for the parameters $\xi \in \mathcal{U}$ of Theorem E. The exterior-square and lift maps descend the finite root orders to five and four; the cusp has period $Z(q) = (\log q/(2\pi i))I_3 + C(q)$ with C holomorphic at zero (theorems 6.1 to 6.3, 6.6 and 8.4).

Mechanism. The wedge image of the genus-two Hodge line has rank three, and its split lift becomes transverse after the normalized imaginary shift. *Relation to literature.* The construction uses the explicit morphism $\ell_0 \otimes (\mathcal{H}_{\mathrm{dR}}/\ell_0) \to \bigwedge^2 \mathcal{H}_{\mathrm{dR}}/\mathbb{C}\omega$, $u \otimes [v] \mapsto [u \wedge v]$, and the index-two integral map in theorem 6.4. The pencil, the integral quotient and the split rank-three plane are new; the toroidal model at the cusp is (Yang, 2026, Theorem 8.7), applied in theorem 6.8. Outline of the proof; proofs: V.6.1, V.6.2, V.6.3, V.6.6.

Theorem D (The oriented diffeomorphism type)**.** [tag 03KU] The period planes extend over the two finite root charts and the marked cusp, giving a compact complex fourfold diffeomorphic to $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$. Its characteristic vector is carried explicitly to $(1, 1, 1, 1)$, after which the known smooth classification and homotopy-sphere absorption identify the oriented manifold.

Mechanism. The middle-lattice isometry transports both Euler and Pontryagin clutching data, after which the cited absorption theorem removes the homotopy sphere. *Relation to literature.* The oriented extended-form classification and Kramer–Stolz absorption theorem are the known results cited in theorem 7.2. The almost-complex compatibility is also known from (Yang, 2012, Theorems 1–2, pp. 2–3) and (Mandal, 2026, Theorem 1.3 and Corollary 1.4, pp. 3–4). Explicitly, transporting the constructed integrable structure by the oriented diffeomorphism gives a complex vector bundle (TN, J) whose realification is TN, with $c_2 = -\sum x_i$, $c_4 = 6u$, and $p_2 = 14u$. In Mandal's normalization $x_i^2 = \epsilon_i u$, $\epsilon = (1, 1, 1, -1)$, his two conditions are $14 \equiv 2 \pmod 4$ and $14 - 2 = 2 \cdot 6$. The compact complex realization is new. Outline of the proof; proofs: V.6.9, V.7.2.

Theorem E (The exact family and its two ends)**.** [tag 03KV] Put

$$c_* = -h_{b,\min} = \frac{96}{625\pi} \operatorname{Re}(X H_\infty)_{03}, \qquad \mathcal{U} = \{\xi \in \mathbb{C} : \Im \xi > c_*\},$$

where X is the normalized integral intertwiner of theorem 8.1 and H_∞ is its regular infinity-root fundamental-solution limit, normalized at $\lambda = 1$ as in theorem 8.4. Thus

$$c_* = 1.094540250311402780116268263054945703 7\ldots.$$

The marked family is parametrized exactly by $\mathcal{U}$: its members are pairwise nonisomorphic, have algebraic dimension one and automorphism group $\mathbb{C} \times \mathbb{Z}/2$, and admit no antiholomorphic isomorphism for this fan. The real-intersection function determines the parameter domain; the invariant line and one half-period determine the automorphisms, while the reflected cusp map fails to preserve a ray of the chosen fan. On the effective root curve put $L = \mathcal{O}((4/5)[\infty] - (3/4)[0])$. Then

$$R^q h_* \mathcal{O} = \bigwedge^q (\mathcal{O} \oplus L^{-1} \oplus L^{-2}), \qquad K_M = h^* L^{-6}.$$

With p indexing rows and q indexing columns, the Hodge table is

$h^{p,q}$	0	1	2	3	4
0	1	1	0	0	0
1	0	2	1	0	0
2	0	1	6	1	0
3	0	0	1	2	0
4	0	0	0	1	1

The tangent tuple is $h^q(T_M) = (1, 1, 1, 2, 0)$ and its Kuranishi germ is smooth of dimension one. A full parameter disc across the transversality boundary has the proper-filling obstruction of theorem 10.1. A proper smooth family on a punctured disc whose fibres belong to $\mathcal{U}$ has the smooth filling after the explicit degree-two torsor cover of theorem 10.2.

Mechanism. The normalized Frobenius period minimum fixes the exact domain, and the marked period and cap maps determine the family invariants and isomorphisms. *Relation to literature.* The root-base and direct-image method is the one in (Du et al., 2026, Lemmas 6.1–6.2, pp. 74–75, and Proposition 6.10, pp. 79–80); theorem 9.1 applies its cup-product and base-change maps to the displayed rank-three bundle on the $(5, 4)$ root curve. The constant c_* and the cohomology, deformation, isomorphism and boundary results are new. The torus-deformation input for theorem 10.1 is (Catanese, 2004, Theorem 2.1, p. 264); its application is to the smooth compact torus fibres produced inside that argument. The punctured-disc filling of theorem 10.2 uses its interior parameter limit and the explicit $\mathbb{C} \times \mathbb{Z}/2$ torsor reduction.

Outline of the proof; proofs: V.8.4, V.9.1, V.9.2, V.9.3, V.9.4, V.10.1, V.9.5, V.10.2.

Theorem F (The integral input classification)**.** [tag `03KW`] Let $k \in \{2, 3\}$ and $\Lambda = \mathbb{Z}^6$. Consider ordered integral tuples $(M_1, \dots, M_k, M_0)$ with each M_i of finite order and having a nonzero fixed vector, with $M_0 = (M_1 \cdots M_k)^{-1}$, and with $N = M_0 - I$ satisfying $N^2 = 0$, rank three, and $\operatorname{im} N = \ker N$ as integral sublattices of Λ. Every such tuple has a computable canonical representative and an explicit simultaneous unimodular conjugator; equality of the representatives is equivalent to simultaneous integral conjugacy. For input entry height H, the finite stopping bound is $(2H + 1)^{36(k+1)}$ candidates before the proved rejection filters. The ordered sector with both finite linear orders two has exactly 56 classes. Each has free affine lifts of affine order four, and every free affine filling of this sector has nontrivial fundamental group.

Specify the free affine actions $\tau_i(x) = M_i x + v_i/m_i$, with $v_i \in \Lambda$ and affine order m_i, an integral translation clutch $\kappa \in \Lambda$, and a primitive Kuhn cusp marking. The algorithms compute the full fundamental group and every integral homology group of the resulting marked gluing for every such input. Their constructive parametrization and finite tests identify exactly the inputs with $\pi_1 = 1$ and $H_2 = H_3 = 0$; there is no fixed bound on the orders, twists or lattice indices (theorem 11.33).

On $B^* = \mathbb{P}^1 \setminus \{p_0, \dots, p_k\}$ with $k \geq 1$, every representation $\rho : \pi_1(B^*) \to GL^+(6, \mathbb{R})$ whose p_0 monodromy satisfies $(M_0 - I)^2 = 0$ and $\operatorname{rank}(M_0 - I) = 3$ has a holomorphic period family transverse to its conjugate everywhere on B^*. In the marking $M_0 = \left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$, its cusp period is

$$Z(t) = \frac{\log t}{2\pi i} I_3 + C(t), \qquad C \text{ holomorphic at } 0,$$

with any prescribed finite jet of C at zero. An integral primitive cusp admits an integral marking. The other p_i remain punctures in this assertion (theorem 12.5).

For a polarized algebraic abelian-surface family with toric-rank-two cusp, the construction that takes a fibrewise degree-zero line bundle and quotients its punctured bundle by a lifted translation cannot have $b_1 = b_2 = 0$ with the stipulated rank-three square-zero Kuhn cusp; this holds for both proportional and nonproportional source sections. The filtration and Euler-parity proof is theorem 12.1.

Mechanism. Finite-order module restrictions and integral cusp transvections reduce the marked inputs to the stated finite list and its explicit analytic realizations. *Relation to literature.* The lattice-isomorphism algorithm of Bley et al. (2022) and the Oka-1 theory of Alarcón and Forstnerič (2025) are cited at their locators in theorems 11.33 and 12.5; the classification and the period theorem are new. Outline of the proof; proofs: V.11.33, V.12.5, V.12.1.

1.3 The proofs in outline

Theorem A: The resolved transverse elliptic fibre product

1. Pull back the Weierstrass family to the root curve of the threefold fibration. The integral matrices A, B give the period comparison $AZ_3B^{-1} = \operatorname{diag}(Z_2, s + c_E)$. See V.2.1.

2. The unimodular Kuhn fan resolves the local fibre product without adding rays. Its period action is proper and its marked section extends face by face. The resolution is therefore small and crepant. See V.2.1.

3. The canonical formula on the root curve gives $K = \Pi^* L^{-5}$. The split relative elliptic complex and Leray give $(h^{0,q}) = (1, 1, 0, 0, 0)$; the explicit meromorphic product comparison gives algebraic dimension two. See V.2.5.

4. A Kähler modification in class $\mathcal{C}$ preserves b_1 and $h^{0,1}$. Here $b_1 = 0$ and $h^{0,1} = 1$, contradicting the Kähler equality $b_1 = 2h^{0,1}$. This proves the stated exclusion through these invariants. See V.2.5.

5. The universal Todd and Pontryagin formulas give $c_2^2 = 2$, $c_4 = 6$ and signature two. The middle lattice is positive definite unimodular of rank two, hence has the displayed form. See V.2.5.

Theorem B: The integral rank-six construction

1. The matrices $M_1 = A^t$, $M_2 = B^t$ have orders five and four, and $N = (M_1M_2)^{-1} - I$ has square zero and primitive image of rank three. The displayed unimodular marking takes N to $\left(\begin{smallmatrix} 0 & I_3 \\ 0 & 0 \end{smallmatrix}\right)$; the invariant covector $\ell(x) = x_0$ proves freeness of every nonidentity affine power (theorem 5.1, theorem 5.2). See V.5.1, V.5.2.

2. The cusp and finite quotient charts glue to a closed smooth manifold. Killing the invariant normal closure of the cusp angular lattice leaves the central fibre generator h and the relations $5a = h$, $4b = -h$, $a + b = 0$; their integral relation matrix is unimodular, so the fundamental group vanishes (theorem 5.1, theorem 5.6). See V.5.1, V.5.6.

3. Unimodular coordinates identify the finite core monodromies with a five-cycle and a four-cycle plus a sign. Their covering maps induce invariant vertical cochains and norm sums on the prism cochains. Together with the primitive cusp collapse, these maps give the displayed integral sheaf complex; its signed orbit bases and determinant minors give the sheaf groups (theorem 5.5, items 1–4; theorem 5.7). See V.5.5, V.5.7.

4. For the order-four core and normal loops g, h, the fibre and transverse loops are $v = g^4h^{-1}$ and $t = g$; their integral basis matrix has determinant one. A torsion representative w pulls back to $(Nw, w) = (0, w)$ in these two directions. Thus its transgression is $\pm[w]$. The printed modulo-two rows detect all three nonzero global classes (theorem 5.5, item 5; lemma 5.4). See V.5.4.

5. The three torsion differentials are isomorphisms onto $\mathbb{Z}/2$. On the cusp term $\mathbb{Z} \oplus \mathbb{Z}/2$, the remaining kernel is generated by $(1, \epsilon)$, where ϵ is the image of its free generator. The Leray

filtration therefore gives $H^3 = 0$ and $H^4 = \mathbb{Z}^4$. Universal coefficients, duality and Hurewicz give the asserted free homology and three-connectivity (theorem 5.5, item 6; theorem 5.8). See V.5.5, V.5.8.

6. The compatible almost complex structure has $c_1 = c_3 = 0$. Todd integrality and the signature bound $|\sigma| \le 4$ force $c_2^2 = 2$, $p_1^2 = 8$, $p_2 = 14$ and $\sigma = 2$. Its unimodular middle lattice is consequently the odd form $I_{3,1}$ (theorem 7.1). See V.7.1.

Theorem C: The genus-two period construction

1. The pencil has one node at $\lambda_* = -256/3125$ and good reductions after root orders eight and five. Its Hodge bundle on that root curve is $T^3 \oplus T$, with $T = \mathcal{O}((2/5)[\infty] - (3/8)[0])$ and $\deg T = 1/40$ (theorem 6.1). See V.6.1.

2. In the exterior-square lattice, projection induces $j : \omega^\perp \to (\bigwedge^2 \mathbb{Z}^4)/\mathbb{Z}\omega$ with Smith factors $(1, 1, 1, 1, 2)$. This exact index-two map identifies the rational systems, and the quotient is the dual of the primitive integral lattice. The lower five-dimensional blocks of the given rank-six tuple are the quotient blocks under this map (theorem 6.4). See V.6.4.

3. Put $P = p(\ell_0 \wedge \mathcal{H}_{\mathrm{dR}})$, where $\ell_0 = \langle dx/y \rangle$ and p is projection modulo ω. The wedge map has rank three and its image meets $\mathbb{C}\omega$ trivially: a decomposable two-vector has square zero, whereas $\omega^2 \neq 0$. This proves the required rank before a lift is chosen (theorem 6.2, theorem 6.3). See V.6.2, V.6.3.

4. The orbifold extension over P splits because $H^1(P^*) = 0$, as computed from its two line-bundle extensions and their integral floor degrees. A chosen splitting σ and the canonical functional $\alpha : P \to \mathcal{O}$ give $F_\xi = (\sigma + \xi\alpha e)P$ (theorem 6.3, theorem 6.5). See V.6.3, V.6.5.

5. On the real line $P \cap \overline{P}$, choose n with $\alpha(n) = 1$ and write $\sigma(n) - \overline{\sigma(n)} = 2ihe$. Then $F_\xi \cap \overline{F_\xi} = 0$ exactly when $h + \operatorname{Im}\xi \neq 0$. At the cusp the period matrix is $Z = (\log q)/(2\pi i)I_3 + C(q)$, with C holomorphic and bounded; hence $h \to +\infty$ and a sufficiently large imaginary shift works on the whole completed base (theorem 6.6, theorem 6.7). See V.6.6, V.6.7.

Theorem D: The oriented diffeomorphism type

1. The lifted periods commute with the finite root actions and have the prescribed bounded cusp correction, so the same quotient and toroidal charts are holomorphic and proper. Their collar identifications are the marked maps used in the smooth construction (theorem 6.9, theorem 6.8). See V.6.9, V.6.8.

2. The preceding integral calculation gives the intersection lattice $I_{3,1}$. The spin characteristic vector $p_1/2$ has square two, by the Chern-number calculation (theorem 7.1). See V.7.1.

3. In an odd diagonal basis write its coordinates, after sign changes, as positive odd integers (a, b, c, d). A reflection in $(1, 1, 1, 1)$ replaces d by $2d - a - b - c$, whose absolute value is smaller when $d > 1$. Iteration terminates at $(1, 1, 1, 1)$, proving an isometry of the lattice together with its characteristic vector (theorem 7.2). See V.7.2.

4. In the quaternionic clutching coordinates, the map is $(a, b) \mapsto (e, p_1/2) = (a - b, a + b)$, with inverse $((e + p_1/2)/2, (p_1/2 - e)/2)$. Thus the pairing and characteristic vector determine the normal datum; the Poincaré-duality map transports it with the middle lattice isometry (theorem 7.2). See V.7.2.

5. Each quaternionic projective-plane summand has $p_1 = 2u$. The oriented connected-sum pairing and characteristic vector are therefore the same pair $(I_{3,1}, (1, 1, 1, 1))$. Wall's stated

extended-form classification gives the oriented diffeomorphism up to a homotopy eight-sphere (theorem 7.2). See V.7.2.

6. Kramer–Stolz's Theorem A absorbs that sphere in one positive $\mathbb{HP}^2$ summand. Its diffeomorphism has degree $+1$ because the square of a middle generator evaluates to 1 on both sides. Oriented connected-sum maps therefore give the required oriented diffeomorphism of the full fourfold (theorem 7.2). See V.7.2.

Theorem E: The exact family and its two ends

1. The real coordinate and marked deck maps preserve the normalized height. The infinity-root rotation fixes its differential at zero; an invertible real system and the positive Hessian prove a strict local minimum. Convergent root and cusp series and exact collar continuation enclose all 1600 leaves of the closed covers above that value. The cusp graph has a positive determinant on the remaining inner disc and tends to positive height without bound. These maps prove the unique global minimum and identify the transversality domain exactly with $\mathcal{U}$ (theorem 8.4, proof, items 1–7). See V.8.4.

2. On the effective root curve of orders five and four put $L = \mathcal{O}((4/5)[\infty] - (3/4)[0])$. The direct images are $R^q h_* \mathcal{O} = \bigwedge^q (\mathcal{O} \oplus L^{-1} \oplus L^{-2})$, and $K_M = h^* L^{-6}$. The toric normalization complex proves the exterior-algebra assertion at the cusp, and the root-chart invariant cohomology proves it at the finite fibres (theorem 9.1). See V.9.1.

3. The logarithmic cotangent sequence and its explicit residue matrices give the Hodge table below. Serre duality applied to the twisted logarithmic and residue sequences gives $h^q(T_M) = (1, 1, 1, 2, 0)$. The normal tangent map and the period lift give a nonzero Kodaira–Spencer class; the local analytic-ring comparison makes the Kuranishi germ smooth of dimension one (theorem 9.2, theorem 9.3). See V.9.2, V.9.3.

4. Restriction of a horizontal divisor to a regular torus fibre gives an invariant alternating class. The computed invariant space is zero, while positive pairing with a torus Kähler form rules out a nonzero effective divisor of zero class. Thus the meromorphic field is the base field $\mathbb{C}(\lambda)$. Every isomorphism therefore preserves the base and its three distinguished values. The simultaneous linear centralizer is $\mathbb{Q}I$; the order-five affine norm excludes its negative integral unit. The translation sheaf has identity component $\mathbb{C}$ and an integral parabolic quotient $\mathbb{Z}/2$, represented by the displayed half-period, proving the automorphism group and equality of parameters for isomorphic members (theorem 9.4). See V.9.4.

5. The two integral anti-intertwiners are $\pm R$. One is excluded by the finite cap norm. For the other, the based cusp correction acts on rays by $-P$, and takes e_2 to $(0, -1, 1)$, outside the component's rays $\pm(\{0, 1\}^3 \setminus \{0\})$. The corresponding monomial map contracts a divisor, contradicting a biholomorphism (theorem 9.5). See V.9.5.

6. A proposed proper Hausdorff filling across a boundary point of a full parameter disc recovers the marked base from the moving pencil of K_M^{-4}, whose two sections have ratio λ. Closed relative forms give torus fibres outside a proper analytic subset. Their marked periods continue as F_ξ, while the real hypersurface $h_b + \operatorname{Im} \xi = 0$ cannot be contained in that analytic subset. A torus fibre there violates real transversality (theorem 10.1). For a family on a punctured disc whose fibres belong to $\mathcal{U}$, the unique classifying map is holomorphic; its Cayley transform extends and the maximum principle keeps its limit inside $\mathcal{U}$. After the explicit degree-two removal of the finite torsor, that family has a smooth filling (theorem 10.2). See V.10.1, V.10.2.

Theorem F: The integral input classification

1. Primitive square-zero cusp matrices have the explicit integral normal form and centralizer $\left(\begin{smallmatrix} P & Q \\ 0 & P \end{smallmatrix}\right)$. Eigenvalue one is necessary for a free affine lift, leaving the finite cyclotomic list and its complex-compatibility filter (proposition 4.4, proposition 4.5). See V.4.4, V.4.5.

2. The rational intertwiner algebra, its radical and its semisimple quotient reduce the simultaneous integral-conjugacy problem to the cited lattice-isomorphism algorithm in its verified hypothesis. Searching by height and lexicographic order terminates by the input height, yielding a canonical representative and an explicit unimodular conjugator (theorem 11.30, theorem 11.31). See V.11.30, V.11.31.

3. For that conjugator R, the coordinate map $x \mapsto Rx$ transports (M_i, v_i, κ, P) to $(RM_iR^{-1}, Rv_i, R\kappa, RP)$. Since $(RP)^{-1}(Rx) = P^{-1}x$, it intertwines the cusp charts as well as the affine caps and clutch. The resulting finite cellular cone and group presentation compute all integral homology and the full fundamental group (theorem 11.25, theorem 11.29, theorem 11.33). See V.11.25, V.11.29, V.11.33.

4. The exact two-point tests and the three-point orbifold quotient give the complete constructive parametrization; the separate involution branch has 56 classes. The displayed unbounded trace family shows why the terminating bound depends on the input (corollary 11.24, theorem 11.32, theorem 11.33). See V.11.24, V.11.32, V.11.33.

5. Nine explicitly given complex-line directions and a uniform singular-value bound put a spanning tube in the unpolarized period domain. The local-spray approximation theorem, with the relative extension-point argument proved here, produces a holomorphic transverse period section and retains any prescribed bounded cusp jet (theorem 12.2, lemma 12.4, theorem 12.5). See V.12.2, V.12.4, V.12.5.

6. For a polarized abelian-surface source, semisimplicity and the maximal square-zero cusp force the middle rank-four system to have no invariant or coinvariant vectors. The filtration $0 \subset \mathbb{Q} \subset E \subset T$, with middle quotient that system and top quotient $\mathbb{Q}$, then forces both relevant invariant ranks to be one; the integral Euler-parity relation excludes $b_1 = b_2 = 0$ (theorem 12.1). See V.12.1.

1.4 Known and new

Theorem	Status and sources
A	New. The six-sphere is that of (Claude and L. Alpöge, 2026, Theorems 7.17 and 7.22), the elliptic surface that of (Engel, 2026, §2); the local cusp model is (Yang, 2026, Theorem 8.7 and Corollary 8.8); the characteristic-number identities are (Esole and Kang, 2018, (1.2)–(1.4)); the class-$\mathcal{C}$ criterion is (Jia and Meng, 2022, Theorem 1.1).
B	New. The local toroidal filling is (Yang, 2026, Theorem 8.7 and Corollary 8.8); the integral specialization method follows (Claude and L. Alpöge, 2026, §7.7).
C	New.
D	New. The smooth classification is Wall's, in the form of (Conway et al., 2023, Definition 3.6 and Theorem 3.7(1)), with (Kramer and Stolz, 2007, Theorem A); that $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$ admits an almost complex structure is (Mandal, 2026, Corollary 1.4).
E	New. The direct-image method is that of (Du et al., 2026, Lemmas 6.1–6.2 and Proposition 6.10); the torus-deformation input is (Catanese, 2004, Theorem 2.1).
F	New. The lattice-isomorphism algorithm is (Bley et al., 2022, Algorithm 8.3); the Oka-1 theory is Alarcón and Forstnerič (2025).

1.5 Methodology

This text is AI-produced; no part of it was written by a human. GPT-5.6 Sol (OpenAI), in Codex at reasoning effort ultra, wrote a first working edition (24 August–9 September 2026). GPT-6 Astra (OpenAI), in ChatGPT Pro mode, carried out the research runs (12–16 November 2026), an independent run that re-derived the stated results without their proofs (16–18 November), and the writing runs (16–20 November). Claude Fable 5.2 (Anthropic), at effort high and at effort max, and Claude Opus 5.5 (Anthropic), at effort max, read, checked and recomputed the results between the runs (12–20 November); what they found went into the next run. Claude Opus 5.5 at effort max revised the presentation of this version on 20 November 2026, without changing the mathematics of the results.

Each statement was checked by targeted proof checks (every step derived before the printed step was read, then compared), by independent re-proofs, and by independent computations; the numerical constants rest on interval-arithmetic certificates. For the statements of this paper the check record contains 369 targeted proof checks, 53 independent re-proofs and 157 computations or formal certificates. No human has checked the mathematics.

The programs named in the proofs, their outputs, the records of the runs and the check record of every statement are in the repository s6-complex-structure-workbench, commit `83d76022bc`, folder `pro_returns/S6_SERIES_GRIND_checkpoint_10_20261120T0211Z`; the programs are in its folder `code`, and the check record of each statement is in `S6_SERIES_GRIND_checks_V.md` under its tag. We dedicate this text to the public domain under CC0 1.0 Universal.

1.6 How the paper is organized

Part I (Section 2) constructs the fibre-product fourfold of Theorem A. Part II constructs the rank-six family: the cusp and the finite fillings and the integral eight-manifold (Sections 3–5), the periods (Section 6), the diffeomorphism type (Section 7), the parameter domain (Section 8), cohomology, deformations and isomorphisms (Section 9), the boundary of the family (Section 10), the classification of the integral data (Section 11) and the existence of periods for every such datum (Section 12). Part III treats the shifted fibre products. The figure shows which main theorems use which, and the table names the argument that carries each dependency.

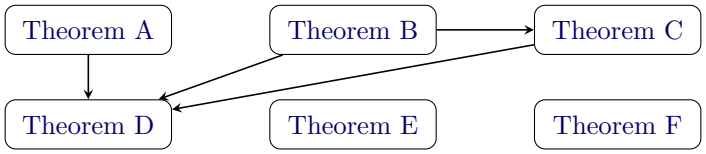


Figure 1: Dependencies among the main theorems; the table names the argument that carries each.

Arrow	Argument used
$B \to C$	V.5.1 $\longrightarrow$ V.6.3. The exterior-square identity (F.11) gives the explicit period representation for the stated integral matrices.
$A \to D$	V.2.1 $\longrightarrow$ V.6.9. The same marked Kuhn period-action proof supplies smoothness and properness for the nonsymmetric bounded cusp correction.
$B \to D$	V.5.5 $\longrightarrow$ V.7.2. The integral homology and middle form enter the oriented extended-form classification.
$C \to D$	V.6.6 $\longrightarrow$ V.6.9. The constructed preceding period family is conjugated on base and fibres to give precisely the homology monodromy.

Part I
The resolved fibre product

2 The resolved fibre-product fourfold

In this section $X = X_c$ has the canonical sphere twists $v_1 = (1, 2, -4, 0)^t$, $v_2 = (-1, -3, 3, 0)^t$, the untwisted cusp collar, and an arbitrary admissible parameter c. Orient all complex manifolds by their complex structures. Cohomology is integral unless a coefficient field is displayed. Put

$$L = \mathcal{O}_X(S_1 - S_2).$$

The divisor comparisons $3S_1 \sim 4S_2 \sim W$ give $L^4 = \mathcal{O}_X(S_1)$, $L^6 = \mathcal{O}_X(2S_2)$ and $L^{12} = \mathcal{O}_X(W)$. The comparisons use the meromorphic functions t, $t - 1$ and their quotient; thus these are isomorphisms of holomorphic line bundles.

Theorem 2.1 (A smooth compact model of the fibre product)**.** [tag 02M5]

There is a smooth compact complex fourfold Z with proper holomorphic maps $g : Z \to B$ and $\Pi : Z \to X$, with $g = f\Pi$, with the local models described below, agreeing with the marked fibre product over the punctured base. The map Π has a holomorphic section. The construction is a small crepant resolution of a Weierstrass model over X, and is bimeromorphic to $X \times_B S$.

Proof. ⟨1⟩1. Let a be a nonzero scalar multiple of the divisor section of L^4 and let b be a nonzero scalar multiple of the square of the divisor section of $\mathcal{O}_X(S_2) = L^3$. Choose the scalars so that $4a^3 + 27b^2$ is the pullback of the section of $\mathcal{O}_B(1)$ vanishing at p_0. This is possible because a^3 and b^2 have divisors $3S_1$ and $4S_2$, which correspond to a basis of $H^0(B, \mathcal{O}_B(1))$. The Weierstrass equation

$$y^2 z = x^3 + axz^2 + bz^3 \tag{E.1}$$

in the projective bundle associated to $\mathcal{O}_X \oplus L^2 \oplus L^3$ defines a proper complex space $\mathscr{W} \to X$. Its section at infinity is disjoint from its singular locus. Its j-function is $1728t$.

⟨1⟩2. Over B°, equation (E.1) is the product with Engel's elliptic curve: both have that marked Weierstrass family. At p_1, pass to the root atlas $t = s^3$. The coefficient a has a simple zero in s and b is a unit. At p_2, on $t - 1 = s^4$, a is a unit and b has order two. These are the smooth good-reduction families E. The deck action on $F \times E$ is diagonal, with the $A_j \oplus C_j$ linear part and translation $(v_j, 0)/m_j$. It is free because its projection to F is free. Its quotient is therefore smooth and is exactly the local model asserted in the theorem. The section at infinity is invariant. Notice that the full fibres here have multiplicities three and four; only their reductions are smooth.

⟨1⟩3. At the cusp, use the angular–deck marking of the toroidal model, and write the two-torus period as $Z_2 = sB_0 + C(t_c)$, where

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad Q = B_0^{-1} P, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here $PS = I_2$. Put $k = (1, 1, 1)^t$, $r = (0, 0, 1)$, and write the Tate period of the elliptic surface as $s + c_E(t_c)$, with c_E holomorphic and bounded. Define

$$Z_3 = sI_3 + C_3(t_c), \qquad C_3 = SCQ + kc_E r. \tag{E.2}$$

For $A = \binom{P}{r}$ and $B = \binom{Q}{r}$ we have

$$\det A = \det B = 1, \qquad AZ_3B^{-1} = \operatorname{diag}(Z_2, s + c_E). \tag{E.3}$$

This is an integral holomorphic product identification on the punctured overlap, including the bounded correction.

⟨1⟩4. Take the fan over the periodic Kuhn tetrahedra

$$[m, m+e_{\pi_1}, m+e_{\pi_1}+e_{\pi_2}, m+(1,1,1)], \quad m \in \mathbb{Z}^3, \quad \pi \in S_3. \tag{E.4}$$

Every cone is unimodular, since subtracting its first vertex reduces its determinant to that of a permuted triangular integral matrix with diagonal entries one. Every maximal chart is therefore $\mathbb{C}^4$, with $t_c = z_0 z_1 z_2 z_3$. The action

$$(x, t_c) \longmapsto (e^{2\pi i C_3(t_c)\lambda} t_c^{\lambda} x, t_c), \qquad \lambda \in \mathbb{Z}^3, \tag{E.5}$$

extends to the fan by translation of its cells. On the boundary a fixed point would require a bounded cell to equal its nonzero translate. On a punctured fibre, invertibility of $\operatorname{Im} Z_3$ excludes a fixed point. If $y = \operatorname{Im} s$ and $K = \sup \| \operatorname{Im} C_3 \|$, shrinking the disc gives

$$\tfrac{1}{2}\|\lambda\| \le \|(I + y^{-1} \operatorname{Im} C_3)\lambda\| \le \tfrac{3}{2}\|\lambda\|.$$

This proves proper discontinuity by tropical coordinates. Rounding the three coordinates of $(I + y^{-1} \operatorname{Im} C_3)^{-1} \operatorname{trop}(x)$ puts every orbit in finitely many toric charts with bounded coordinates. Over a smaller closed disc their closures are compact. Thus the quotient N_3^C is smooth and proper over that disc.

⟨1⟩5. The map (P, r) takes (E.4) to a subdivision of the product of an A_2 triangle and an interval. There are three tetrahedra in each triangular prism. No ray is added. Consequently the induced map is a small toric resolution of the local equation $xy = w_0 w_1 w_2$ in (E.1); over a double curve it resolves $xy = w_0 w_1$ times a smooth factor. All rays have height one, so the resolution is crepant. The formula $PZ_3 = Z_2 Q$ proves equivariance of its map to the cusp of X. The slice $x_3 = 1$ gives the zero section: its lattice map is S, and S takes each A_2 triangle to a face of a Kuhn tetrahedron. Thus the section extends over every double curve and both triple points, rather than only over the punctured cusp.

⟨1⟩6. We glue N_3^C and the two finite models to the punctured product by (E.3) and their root-atlas identifications. These overlap maps are holomorphic isomorphisms and satisfy the cocycle condition because they are identifications with the same marked product. Properness over the three closed discs and compactness over the remaining compact base give compactness of the glued Hausdorff space. It is smooth in all its charts. The maps to X and the zero section agree on their overlaps. Since both this model and the literal fibre product agree on a dense open set, they are bimeromorphic.

□

Theorem 2.2 (the integral cusp calculation)**.** [tag 02M6]
For the central fibre W_3 of (E.5),

$$H_q(W_3; \mathbb{Z}) = \mathbb{Z}^{b_q}, \qquad (b_0, \dots, b_6) = (1, 3, 9, 9, 10, 3, 1). \tag{E.6}$$

Integral specialization is onto the monodromy invariant lattice in every degree. Its only kernel is a copy of $\mathbb{Z}$ in degree four.

Proof. ⟨1⟩1. The normalization is the three-dimensional permutohedral toric variety. Its paired boundary surfaces consist of four copies of dP_6 and three copies of $\mathbb{P}^1 \times \mathbb{P}^1$. There are twelve normalized triple curves and six quadruple points. The normalization complex for the constant sheaf is the alternating branch complex. Local exactness follows from

the augmented simplex on the branches of $z_0z_1z_2z_3 = 0$. Its nonzero rows after taking cohomology are

q	$p=0$	$p=1$	$p=2$	$p=3$	$\operatorname{rank} d^0$	$\operatorname{rank} d^1$ or d^2
0	1	7	12	6	0	4, 5
2	11	22	12	0	5	9
4	11	7	0	0	4	0
6	1	0	0	0	0	0

(E.7)

Every nonzero Smith invariant in these matrices is one.

⟨1⟩2. The integral matrices are as follows. The fourteen rays are $\pm v$, with $v \in \{0,1\}^3 \setminus \{0\}$. The twenty-four maximal cones are the translated Kuhn tetrahedra containing zero. This enumeration requires exactly the eight cubes with lower vertex in $\{-1,0\}^3$; no omitted cube can contain zero. The Picard group of the normalization is the group of ray coefficients modulo the three character relations. We set the coefficients at e_1, e_2, e_3 to zero, giving an integral basis of rank eleven. On a boundary surface D_v, choose adjacent rays u_1, u_2 for which (v, u_1, u_2) is unimodular and set the two remaining character coefficients to zero. Restriction is evaluated by subtracting the unique integral linear function with the prescribed values at these three rays. The paired restriction is $\mathrm{res}_{D_v} - \mathrm{res}_{D_{-v}}$, with the second adjacent ray corresponding to $u - v$.

⟨1⟩3. For a triangle $(0, v, u)$ with $0 < v < u$ coordinatewise, the next map is the alternating sum of the degrees on the three boundary curves of D_{u-v}, D_u, D_v. In a surface fan, if the neighbours of a ray w are h_1, h_2 and $h_1 + h_2 = bw$ modulo v, that degree is $a_{h_1} + a_{h_2} - ba_w$. In degree four, the restriction map is dual to the seven divisor differences $[D_v] - [D_{-v}]$. In degree zero the matrices are the cellular incidence matrices of the one-vertex Kuhn triangulation of T^3. These rules define (E.7) over $\mathbb{Z}$, and both compositions are zero. Its exact Smith reductions give the ranks and primitivity just stated.

⟨1⟩4. For completeness, the possible higher differential of this normalization spectral sequence vanishes integrally. Replace C_3 by hC_3, $0 \le h \le 1$. The preceding uniform properness estimate applies simultaneously. Local parameter vector fields preserving t_c and each coordinate stratum patch by a partition of unity; their flow identifies the central fibres and their specialization maps. We may therefore use $C_3 = 0$. This toroidal degeneration is projective after passing to the finite cover with deck lattice $2\mathbb{Z}^3$. An explicit coherent polarization is given by

$$Q(n) = 4\sum n_i^2 - (\sum n_i)^2.$$

For the tetrahedron ordered $1, 2, 3$, its affine interpolant is $3n_1 + n_2 - n_3$, and their difference is

$$\sum_i n_i(n_i - 1) + \sum_{i<j}(n_i - n_j)(n_i - n_j - 1) \ge 0 \quad (n \in \mathbb{Z}^3).$$

Equality holds exactly at its four vertices. Translation and permutation give strict coherence on every face. The finite cover has simple normal crossings, and its descent complex is a complex of pure toric cohomology groups. Over $\mathbb{Q}$ the higher differentials vanish by their different weights (Deligne, 1974, Proposition 8.3.5, pp. 42–43). Taking finite deck invariants returns the original fibre. The integral E_2 groups of (E.7) are free, so a higher differential which is zero over $\mathbb{Q}$ is zero integrally. Summing its diagonals proves (E.6), with no extension torsion.

⟨1⟩5. The same projective comparison allows the local invariant cycle theorem, in the form of (Chen, 2021, Theorem B), followed by exact finite-group invariants over $\mathbb{Q}$. The cusp monodromy on H_1 is $\left(\begin{smallmatrix} I_3 & I_3 \\ 0 & I_3 \end{smallmatrix}\right)$. Its exterior-power invariant dimensions are $(1, 3, 9, 9, 9, 3, 1)$,

by a computation with exact exterior matrices. Comparison with (E.6) proves the last assertion. We now strengthen this to an integral statement. Write b_i for the three degree-one invariant deck classes and a_i for the angular classes, so the infinitesimal monodromy sends a_i to b_i. At $C_3 = 0$, every integral symmetric matrix S gives a line bundle on the toroidal family. Its descent data are as follows. Assign to a fan vertex n the integer

$$q_S(n) = \frac{n^t S n - \sum_i S_{ii} n_i}{2}.$$

Interpolate on each unimodular Kuhn tetrahedron. This is an integral piecewise linear support function. Translation by λ changes it by the integral affine function $n^t S\lambda + q_S(\lambda)$, giving the deck linearization $t_c^{-q_S(\lambda)} x^{-S\lambda}$. The identity $q_S(\lambda+\mu) = q_S(\lambda) + q_S(\mu) + \lambda^t S\mu$ is exactly the cocycle condition. On a smooth fibre its Chern class is $\sum S_{ij} a_i b_j$. Restriction of this line bundle to the central fibre therefore supplies every class

$$a_i b_i, \qquad a_i b_j + a_j b_i \quad (i < j)$$

in the specialization image, as well as the b_i and their products. These elements generate the full integral invariant exterior algebra for three pairs. To check saturation, one enumerates all monomials of weighted degree at most six in these nine generators and reduces their coordinates in an integral kernel basis of $\bigwedge^q U - I$. In every degree the nonzero Smith factors are all one. This finite check has only 2^3 possible products of the odd generators and at most three even factors. It can also be seen directly: in degree three the mixed kernel is the trace-zero rank-eight lattice, generated by six off-diagonal terms and two primitive diagonal differences; the complementary degrees are generated by the corresponding products of two or three $a_i b_i$. The parameter isotopy already constructed identifies the integral specialization maps for general C_3. Thus the images are primitive and surjective in every degree. Comparing their ranks with (E.6) gives exactly the asserted free degree-four kernel.

□

Theorem 2.3 (fundamental group and rational homology)**.** [tag 02M7]

The fourfold in Theorem 2.1 has

$$\pi_1(Z) = 1, \qquad \chi(Z) = 6, \qquad (b_0, \dots, b_8) = (1, 0, 2, 1, 2, 1, 2, 0, 1). \tag{E.8}$$

The first homology groups of its two finite reduced fibres are, respectively, $\mathbb{Z}^2 \oplus \mathbb{Z}/3$ *and* $\mathbb{Z}^2 \oplus \mathbb{Z}/2$.

Proof. ⟨1⟩1. The section of Π splits the fundamental-group sequence over its smooth locus $X \setminus W$. Every loop in Z can be moved off the discriminant inverse image and is a product of a loop in the elliptic fibre and a section loop. A section loop becomes trivial in Z because it bounds in the embedded simply connected X. Thus $\pi_1(Z)$ is a quotient of the coinvariants of $\mathbb{Z}^2$ under C_1, C_2. Loops realizing these two monodromies are the root-circle paths closed by the affine identifications in $X \setminus W$. The Smith form of $(C_1 - I \mid C_2 - I)$ is $(1, 1)$, so these coinvariants vanish.

⟨1⟩2. Euler characteristic localizes on the exceptional fibres: all regular torus fibres and the finite free torus quotients have Euler characteristic zero. The cusp has six zero-dimensional torus strata, and every positive-dimensional torus stratum has Euler characteristic zero. Hence $\chi(Z) = 6$.

⟨1⟩3. Write $V = R^1 g_* \mathbb{Q}$ on the punctured base, and j for its inclusion. Finite quotient specialization is an isomorphism onto invariants over $\mathbb{Q}$. Theorem 2.2 gives $R^q g_* \mathbb{Q} = j_* \bigwedge^q V$ except

for one kernel $\mathbb{Q}_{p_0}$ in degree four. For any local system L' on the punctured sphere,

$$H^2(B, j_*L') = (L')_{\pi_1}, \qquad \chi(B, j_*L') = -\operatorname{rank} L' + \sum_{j=0}^{2} \dim(L')^{M_j}. \tag{E.9}$$

The first equality is compactly supported duality and the second is Euler additivity on the open stratum and the three point stalks. The exact exterior-matrix computation gives the following rows of the rational Leray E_2 page:

q	0	1	2	3	4	5	6
$\dim E_2^{0,q}$	1	1	2	3	3	1	1
$\dim E_2^{1,q}$	0	1	0	1	0	1	0
$\dim E_2^{2,q}$	1	1	2	3	2	1	1.

(E.10)

The differential ranks $r_q = \operatorname{rank}(E_2^{0,q} \to E_2^{2,q-1})$ are

$$(r_1, \dots, r_6) = (1, 1, 2, 2, 1, 1). \tag{E.11}$$

Indeed Π^* transports the four nonzero transgressions of the sphere fibration to the corresponding direct summands here. These four transgressions are forced over $\mathbb{Q}$ by the one-dimensional rows and $H^*(X;\mathbb{Q}) = H^*(S^6;\mathbb{Q})$; this also checks the particular reduction of Engel's Lemma 7.6 which we use. The divisor class of the zero section gives a permanent class u restricting to the elliptic orientation class. Multiplication by u gives another independent image in degree three, and proves $r_3 = 2$, $r_5 = 1$, $r_6 = 1$, and $r_4 \geq 2$. The target summands are independent by the marked decomposition $\Lambda_X \oplus \Lambda_E$. The resulting formulas for b_3 and b_5 are $b_3 = 1$ and $b_5 = 3 - r_4$. Poincaré duality of the compact smooth fourfold gives $r_4 = 2$. No later differential exists because the base has real dimension two. Taking diagonal sums of (E.10) with (E.11) proves (E.8).

⟨1⟩4. Finally, for a finite quotient its affine fundamental group has generators $\Lambda \oplus \mathbb{Z}a$ and abelianized relations $(A_j \oplus C_j - I)\Lambda$ and $m_j a - (v_j, 0)$. The complete relation matrix is computed by the program `fourfolds_checks.py`. Its nonzero Smith factors are $(1,1,1,1,3)$ and $(1,1,1,1,2)$, respectively. This proves the integral assertions about the finite fibres.

□

Theorem 2.4 (integral homology of the fourfold)**.** [tag `02M8`]

The fourfold of Theorem 2.1 has torsion-free integral homology:

$$H_i(Z;\mathbb{Z}) = \mathbb{Z}^{b_i}, \qquad (b_0, \dots, b_8) = (1, 0, 2, 1, 2, 1, 2, 0, 1).$$

In particular the finite-fibre torsion in 2.3 does not survive as torsion in the homology of the total space.

Proof. ⟨1⟩1. We compute the integral covering cokernels. Write $\mathcal{V} = R^1 g_* \mathbb{Z}|_{B^\circ}$, and put $\mathcal{J}_q = j_* \bigwedge^q \mathcal{V}$. The cohomology monodromies are the inverse transposes of $A_1 \oplus C_1$, $A_2 \oplus C_2$, $M_0 \oplus C_0$. All lattices below are in this marking.

⟨1⟩2. First we describe a finite filling. Since the first coordinate of each twist v_j is ± 1, the integral change of basis with first column $v_j/(v_j)_\gamma$ and remaining columns the last five standard basis vectors changes its linear monodromy to $\operatorname{diag}(1, D_j)$ and its translation to $\pm e_0/m_j$. The reduced finite fibre is the mapping torus of the linear automorphism $D_j : T^5 \to T^5$.

The covering $T^6 \to B_j$ is the m_j-fold cover of its suspension circle. Set $T_j = D_j^{-t}$. The two vertical matrices on homology, in coordinates (u, w, δ, e_1, e_2), are

$$D_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The Wang sequences of the mapping torus and its cover give the following exact description of the cokernel of covering pullback:

$$C_j^q \simeq \frac{\ker(\bigwedge^{q-1} T_j - I)}{\operatorname{im} \mathcal{N}_j^{q-1}}, \qquad \mathcal{N}_j^{q-1} = \sum_{a=0}^{m_j-1} (\bigwedge^{q-1} T_j)^a. \tag{E.I1}$$

Here the target of pullback is the invariant lattice in $H^q(T^6; \mathbb{Z})$. To verify (E.I1), its restriction to the vertical torus surjects onto $\ker(\bigwedge^q T_j - I)$, and restriction of a mapping-torus class has exactly the same image, by Wang exactness. Thus every cokernel class has a representative containing the suspension covector. On this kernel the covering map is the displayed norm: cut the covered circle into its m_j consecutive intervals and transport the vertical cochain through them. This proves both surjectivity and the precise kernel of the horizontal description.

⟨1⟩3. For $q = 2$, the two quotients in (E.I1) are respectively zero and $\mathbb{Z}/2$. For $q = 3$, they are

$$C_1^3 = \mathbb{Z}/3, \qquad C_2^3 = (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4. \tag{E.I2}$$

The same Wang sequences show that the kernels of finite specialization in degree two are $\mathbb{Z}/3$ and $\mathbb{Z}/2$, and that the degree-three and degree-four kernels vanish. The cusp specialization is integrally onto, with the sole free degree-four kernel of 2.2.

⟨1⟩4. We give the integral global maps needed to use these local groups. The global invariant two-form $q_X = u^*w^* + 6\gamma\delta^*$ maps onto C_2^2. Indeed, if its pullback descended to the six-dimensional finite fibre, restriction to the elliptic zero section would make it descend to the original bielliptic surface S_2. Its square there would be $\pm 12/4 = \pm 3$, which is impossible in the even intersection form of S_2. That form is even because $H^2(S_2; \mathbb{Z})$ is torsion free and its canonical class is torsion, as checked in III.5.4 (p. 34) . Thus the global-to-local map in degree two is onto its only nonzero covering cokernel.

⟨1⟩5. For degree three all three global invariant classes contain γ, and remain purely horizontal in both adapted finite markings. They are

$$\gamma e_1^* e_2^*, \qquad \gamma(-u^* e_1^* + w^* e_2^*), \qquad \gamma u^* w^*. \tag{E.I3}$$

Juxtaposition denotes exterior product. Consequently their images in (E.I2) are exactly the vertical two-forms in (E.I3) modulo the norm lattices. In integral invariant bases, the two norm matrices and this common image matrix are

$$N_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & 1 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 2 & -1 & 0 & -1 & 0 \\ 3 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -2 & 0 & 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & -1 & -1 & 0 \\ 4 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every nonzero Smith factor of $(\mathrm{diag}(N_3, N_4) \mid (G, G)^t)$ is one; there are eight. Thus the global classes surject onto the direct sum of the two degree-three covering cokernels.

⟨1⟩6. We also compute the integral parabolic groups, with their torsion. The degree-two global coboundary matrix, obtained by stacking the two monodromy-minus-identity matrices, has a single nontrivial Smith factor six. Its torsion generator restricts with orders three, two and one at the three boundary circles. Hence no nonzero torsion class is parabolic. Its rational parabolic rank is zero, so $H^1(B, \mathcal{J}_2) = 0$. In degree three the global coboundary image is primitive and the rational parabolic rank is one; hence $H^1(B, \mathcal{J}_3) = \mathbb{Z}$. These assertions use the cocycle description as the kernel of restriction from the free-group cohomology to the three peripheral circle groups. The matrices and the restriction orders are computed exactly in `fourfolds_integral_audit.py`.

⟨1⟩7. Passing from $\mathcal{J}_q$ to $R^q g_* \mathbb{Z}$ adds the point-supported kernels just described and replaces the finite stalks by the pullback sublattices. A point-supported sheaf has no positive cohomology. The long exact sequences, together with the two surjections onto covering cokernels, therefore give

$$H^1(B, R^2 g_* \mathbb{Z}) = 0, \qquad H^1(B, R^3 g_* \mathbb{Z}) = \mathbb{Z}. \tag{E.I4}$$

They also identify degree-two base cohomology with the global coinvariants. In the relevant degrees these are torsion-free, with

$$H^2(B, R^1 g_* \mathbb{Z}) = \mathbb{Z}, \qquad H^2(B, R^2 g_* \mathbb{Z}) = \mathbb{Z}^2. \tag{E.I5}$$

The first is the corresponding group from X. The second is the direct sum of that from $\bigwedge^2 \mathcal{V}_X$ and the elliptic orientation line; the mixed coinvariants are zero. Smith forms show that these identifications are primitive.

⟨1⟩8. The integral differential onto the first group in (E.I5) is surjective by naturality from the sphere fibration: its own corresponding Leray quotient must vanish because $H^3(X) = 0$. The differential onto the second group is surjective as well. The X summand is killed by the differential transported from $H^4(X) = 0$; the elliptic orientation summand is killed by multiplying the primitive degree-one transgression of X by the permanent class of the elliptic zero-section divisor. That divisor has degree one on each elliptic fibre. Thus these are two primitive independent target generators.

⟨1⟩9. Now $H^0(B, R^3 g_* \mathbb{Z})$ and $H^0(B, R^4 g_* \mathbb{Z})$ are free: in the first degree specialization has no kernel, and in the second its only kernel is the free cusp class. There is no later Leray differential, since the base has cohomological dimension two. Equation (E.I4) and the first surjectivity in (E.I5) show that $H^3(Z; \mathbb{Z})$ is a free group, of rank one by 2.3. Equation (E.I4) and the second surjectivity show that $H^4(Z; \mathbb{Z})$ is an extension of two free groups of rank one, and hence is $\mathbb{Z}^2$. Simple connectivity makes $H^2(Z; \mathbb{Z})$ torsion free. The universal coefficient theorem now proves successively that $H_2(Z; \mathbb{Z}) = \mathbb{Z}^2$ and $H_3(Z; \mathbb{Z}) = \mathbb{Z}$. Poincaré duality determines the other degrees and gives exactly the displayed groups.

□

Theorem 2.5 (complex invariants and middle form)**.** [tag `02M9`]

For the fourfold of Theorem 2.1,

$$K_Z = \Pi^* L^{-5}, \qquad a(Z) = 2, \qquad (h^{0,0}, \dots, h^{0,4}) = (1, 1, 0, 0, 0). \tag{E.12}$$

It is not in Fujiki's class $\mathcal{C}$. Moreover $c_1(Z) = 0$ integrally, $\int_Z c_2^2 = 2$, $\sigma(Z) = 2$, and the intersection form on $H^4(Z; \mathbb{Z})/\mathrm{torsion}$ is $\langle 1 \rangle \oplus \langle 1 \rangle$.

Proof. ⟨1⟩1. We first compute the direct images of the structure sheaf. The orbifold base has roots of orders three and four. Write its line generator as $\mathcal{O}(1)$; it pulls back to L. For the coarse map ρ,

$$\rho_*\mathcal{O}(k) = \mathcal{O}_B(d(k)), \qquad d(k) = \frac{k}{12} - \frac{k \bmod 3}{3} - \frac{-k \bmod 4}{4}. \tag{E.13}$$

This follows by dividing a local invariant section by its necessary root-coordinate power. Finite-group invariants over $\mathbb{C}$ are exact. In particular $\mathcal{O}(-1)$ and $\mathcal{O}(-2)$ are acyclic, since their coarse degree is -1.

⟨1⟩2. For $\bar{f} : X \to \mathfrak{B}$, the direct images are

$$R^0\bar{f}_*\mathcal{O}_X = \mathcal{O}, \quad R^1\bar{f}_*\mathcal{O}_X = \mathcal{O} \oplus \mathcal{O}(-1), \quad R^2\bar{f}_*\mathcal{O}_X = \mathcal{O}(-1). \tag{E.14}$$

We verify (E.14). The cusp branch resolution of $\mathcal{O}_W$ is the cellular complex of its dual two-torus, and has cohomology dimensions $(1, 2, 1)$ with nonzero exterior cup pairing. Proper flat base change therefore gives vector bundles with those ranks. The relative logarithmic vertical bundle has a fixed line generated by $\widehat{\delta}$, and its determinant has twelfth power $\mathcal{O}(-p_0)$, by the modular discriminant. Thus the determinant is $\mathcal{O}(-1)$. Relative duality gives the last term of (E.14). The invariant covector γ gives a nowhere zero section of $R^1\bar{f}_*\mathcal{O}_X$: on a torus its class in $\Lambda^*_{\mathbb{C}}/\operatorname{rowspan}\Pi_c$ is nonzero by the last two columns of Π_c; at the cusp it is the nonzero deck cocycle on the dual two-torus. Its determinant and the nonzero cusp cup pairing then give the extension $0 \to \mathcal{O} \to R^1\bar{f}_*\mathcal{O}_X \to \mathcal{O}(-1) \to 0$. It splits because $H^1(\mathfrak{B}, \mathcal{O}(1)) = 0$, also from (E.13). The same logarithmic determinant gives $K_X = L^{-6}$. These computations use the root atlases and the reduced cusp.

⟨1⟩3. The Weierstrass adjunction calculation gives $K_{\mathscr{W}} = \Pi^*(K_X \otimes L)$, and the small toric resolution is crepant. This proves the canonical formula in (E.12). Every fibre of Π is a connected genus-one curve; over the deeper strata its toric resolution is a cycle of rational curves. Thus Π is flat, by equidimensionality between smooth spaces, and relative duality gives $R^1\Pi_*\mathcal{O}_Z = L^{-1}$ and $R^0\Pi_*\mathcal{O}_Z = \mathcal{O}_X$. Restriction to the section splits the unit map in the derived direct image, so

$$R\Pi_*\mathcal{O}_Z \simeq \mathcal{O}_X \oplus L^{-1}[-1].$$

Projection formula, (E.13), and (E.14) show that $H^*(X, L^{-1}) = 0$ and $h^*(X, \mathcal{O}_X) = (1, 1, 0, 0)$. This proves the asserted Hodge numbers.

⟨1⟩4. A meromorphic trivialization of L makes the coordinate x in (E.1) a meromorphic function over X, of degree two on each elliptic fibre. It gives a generically finite meromorphic map to $X \times \mathbb{P}^1$. Traces and elementary symmetric functions over its finite generic fibre show that the meromorphic function field is an algebraic extension of that of the target. Since $a(X) = 1$, this proves $a(Z) = 2$. If Z were in class $\mathcal{C}$, a sequence of blowups along smooth centres would give a compact Kähler model, by the form of the model theorem in (Jia and Meng, 2022, Theorem 1.1 and Remark 1.2). Both b_1 and $h^{0,1}$ are unchanged by such blowups: the exceptional fibres have no first homology and no positive-degree structure-sheaf cohomology. The Kähler identity $b_1 = 2h^{0,1}$ would give $0 = 2$, a contradiction.

⟨1⟩5. Since L is pulled from a line on S^6, its integral first Chern class vanishes, so $c_1(Z) = 0$. Riemann–Roch and the signature formula, in the explicit normalization of (Esole and Kang, 2018, (1.2)–(1.4)), yield

$$0 = \chi(\mathcal{O}_Z) = \frac{3c_2^2 - c_4}{720}, \qquad \sigma(Z) = \frac{3c_2^2 + 14c_4}{45}.$$

Here $c_4[Z] = 6$, hence $c_2^2[Z] = 2$ and $\sigma(Z) = 2$. The integral middle pairing modulo torsion is unimodular by Poincaré duality, has rank two by (E.8), and is positive definite by its signature. The rank-at-most-four lattice classification (Peters and Sterk, 2024, Theorem 2.2.1) therefore gives precisely $\langle 1\rangle \oplus \langle 1\rangle$.

□

Part II
The marked rank-six construction and its family

3 Geometric and characteristic conventions

The index and signature polynomials used below are (Esole and Kang, 2018, equations (1.2)–(1.4), pp. 3–4); the rank-at-most-four integral classification is (Peters and Sterk, 2024, Theorem 2.2.1, p. 61). In the almost complex application, the Todd polynomial and the cotangent-twisted polynomial are the corresponding elliptic-symbol indices, so their integrality does not require an integrable complex structure. For $c_1 = c_3 = 0$ the substitution gives

$$\mathrm{Td} = \frac{3c_2^2 - c_4}{720}, \qquad \chi^1 = \frac{3c_2^2 - 31c_4}{180}, \qquad \sigma = \frac{3c_2^2 + 14c_4}{45}. \tag{1}$$

In particular $\chi^1 - 4\,\mathrm{Td} = -c_4/6$, and $\mathrm{Td} = (3\sigma - c_4)/48$. For the rank-four middle group and $c_4 = 6$ this forces $\sigma = 2$, since $\sigma \equiv 2 \pmod{16}$ and $|\sigma| \leq 4$. The cited lattice theorem then gives $I_{3,1}$ and the displayed polynomials give the complete numerical record

$$H^4(X;\mathbb{Z}) \cong I_{3,1}, \qquad c_4 = 6, \qquad c_2^2 = 2, \qquad \sigma = 2, \qquad \mathrm{Td} = 0. \tag{2}$$

These are the known inputs used in the construction.

Mandal's connected-sum criterion is used only in its stated orientation: $\ell\mathbb{HP}^2 \# m(-\mathbb{HP}^2)$ has an orientation-compatible almost complex structure exactly when $\ell = 2m + 1$ and m is odd (Mandal, 2026, Corollary 1.4, p. 4, and its proof in §5). Its normalized tangent classes are those of (Mandal, 2026, Lemma 2.1, p. 5, and Theorem 2.4, p. 7).

4 The marked cusp and preliminary lattice restrictions

Definition 4.1 (Admissible data)**.** Let $\Lambda = \mathbb{Z}^6$ and $k \in \{2, 3\}$. An *admissible linear tuple* is a tuple $(M_1, \ldots, M_k, M_0)$ of integral matrices with each M_i of finite order and with a nonzero fixed vector, $M_0 = (M_1 \cdots M_k)^{-1}$, and $N = M_0 - I$ satisfying $N^2 = 0$, $\operatorname{rank} N = 3$ and $\operatorname{im} N = \ker N$ as integral sublattices of Λ (the *index-one condition*). An *admissible datum* is an admissible linear tuple together with free affine lifts $\tau_i(x) = M_i x + v_i/m_i$ of affine orders m_i, an integral translation clutch $\kappa \in \Lambda$ and a primitive marking of the Kuhn cusp. Gluing the affine caps, the index-one Kuhn cap and the torus bundle over the punctured sphere gives a closed smooth eight-manifold, the *gluing* of the datum.

Theorem 4.2 (Euler number of a Kuhn cusp)**.** [tag 02MU]

Let $g : M^8 \to S^2$ be a proper triangulable map, a T^6-bundle off finitely many values. Suppose its unique non-quotient special fibre is the rank-three Kuhn toroidal fibre with period-lattice index d, and all its other special reduced fibres are free finite quotients of T^6. Then $\chi(M) = 6d$.

Proof. ⟨1⟩1. The rank-three Kuhn fibre is the case $n = 3$ of (Yang, 2026, Corollary 8.8, arXiv v1, p. 59): the identical height-one fan and the inclusion $\Gamma \subset \mathbb{Z}^3$ identify its index-d quotient, and give central Euler number $3!d = 6d$.

⟨1⟩2. The regular fibre has Euler number $\chi(T^6) = 0$. Each other reduced fibre is a free finite quotient of T^6, so its Euler number is also zero. Euler additivity for the stated proper triangulable map therefore gives $\chi(M) = 6d$.

□

Proposition 4.3 (the diagonal obstruction)**.** [tag 02MX]

If $A \in \mathrm{GL}_n(\mathbb{Z})$ has no eigenvalue one, every affine map $x \mapsto Ax + b$ of $\mathbb{R}^n/\mathbb{Z}^n$ has a fixed point. In particular, finite monodromies without eigenvalue one, such as the diagonal elliptic ones of orders four and six, admit no free affine lifts.

Proof. ⟨1⟩1. The real map $I - A$ is invertible. A real solution of $(I - A)x = b$ maps to a fixed point in the torus. The diagonal copies of the order-four and order-six elliptic rotations have no eigenvalue one, so the conclusion applies to every translation part, not only to integral or invariant choices.

□

Proposition 4.4 (the cusp normal form)**.** [tag 02MY]

Under the index-one condition of definition 4.1, the cusp matrix is conjugate in $\mathrm{GL}_6(\mathbb{Z})$ to

$$U = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}.$$

Its integral centralizer consists exactly of $\left(\begin{smallmatrix} P & Q \\ 0 & P \end{smallmatrix}\right)$ with $P \in \mathrm{GL}_3(\mathbb{Z})$ and $Q \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$.

Proof. ⟨1⟩1. Let $N = M_0 - I$. The index-one condition says that the image is the saturated kernel lattice and that $\Lambda/\ker N \to \operatorname{im} N$ is an isomorphism. Choose a basis a_1, a_2, a_3 of this image and lifts b_i with $Nb_i = a_i$. Saturation shows that the six vectors form an integral basis. It gives the displayed matrix. Commuting a block matrix with N forces its lower-left block to vanish and its two diagonal blocks to agree. Invertibility over $\mathbb{Z}$ is then equivalent to unimodularity of that block.

□

Proposition 4.5 (a complete finite-order filter)**.** [tag 02N9]

The linear order of a finite matrix in $\mathrm{GL}_6(\mathbb{Z})$ which admits a free affine lift belongs to

$$\{1, 2, 3, 4, 5, 6, 8, 10, 12\}. \tag{F.3}$$

There are 38 possible degree-six cyclotomic characteristic polynomials with an eigenvalue one, and 19 of them commute with some real complex structure. These are necessary types, not 38 or 19 conjugacy classes of admissible tuples.

Proof. ⟨1⟩1. Proposition 4.3 requires a nonzero one-eigenspace. The other cyclotomic factors have total degree at most five. The elementary identity

$$\frac{\phi(n)^2}{n} = \prod_{p^a \| n} p^{a-2}(p-1)^2 \geq \frac{1}{2}$$

implies $n \leq 50$ when $\phi(n) \leq 5$. The complete enumeration through 50 gives $n = 1, 2, 3, 4, 5, 6, 8, 10, 12$. Enumerating their multiplicities with total degree six and at least one factor $T - 1$ gives 38 polynomials. A commuting complex structure exists on a real finite-order module exactly when the multiplicities of the real eigenvalues 1 and -1 are even; nonreal pairs already carry such a structure. Imposing those two parity conditions leaves 19. Taking least common multiples of their root orders gives exactly (F.3), also for the 38-polynomial list. This is a bounded exhaustive enumeration for the stated filter. It does not bound affine multiplicities when the affine action has larger order than its linear part.

□

Theorem 4.6 (the self-dual parity obstruction)**.** [tag 0201]

Suppose a compact smooth eight-manifold has the torus fibration and the Kuhn cusp of 4.2 with index one, and its other reduced fibres are free quotients of complex three-tori by finite holomorphic affine actions. If its rational H^1 monodromy representation is self-dual, then it cannot satisfy $b_1 = b_2 = 0$. In particular no admissible datum with symplectic monodromy has a three-connected gluing, for either $k = 2$ or $k = 3$ (or any number k of finite points).

Proof. ⟨1⟩1. Let $\mathcal{V}$ be the rank-six cohomology local system on the punctured sphere and let j be its inclusion. On a finite quotient fibre, averaging gives $R^1 g_* \mathbb{Q} = (j_* \mathcal{V})$ at that point. At the Kuhn cusp, its normalization components are simply connected and the degree-one branch complex is the cochain complex of its dual T^3. Specialization takes its three deck classes to the covectors annihilating the angular lattice, exactly the three-dimensional monodromy invariant subspace. Hence the same equality holds at the cusp. It holds everywhere.

⟨1⟩2. Put $a = \dim H^0(B, j_* \mathcal{V})$ and $b = \dim H^2(B, j_* \mathcal{V})$. The only differential that can kill a class in $H^0(R^1)$ or in $H^2(R^0)$ is $d_2 : H^0(R^1) \to H^2(R^0) = \mathbb{Q}$. If $b_1 = b_2 = 0$, it must be an isomorphism, so $a = 1$. Moreover $H^1(B, j_* \mathcal{V})$ must vanish because its position $(1, 1)$ has no possible incoming or outgoing differential. Self-duality and compactly supported duality give $b = a = 1$.

⟨1⟩3. At the ith finite point, let r_i be the dimension of the fixed subspace in H^1. The derivative is complex linear, so its one-eigenspace has even real dimension. Each r_i is therefore even. The cusp fixed dimension is three. Euler additivity gives

$$\chi(B, j_* \mathcal{V}) = 6(1 - k) + 3 + \sum_{i=1}^{k} r_i,$$

an odd integer. Consequently

$$\dim H^1(B, j_* \mathcal{V}) = a + b - \chi(B, j_* \mathcal{V})$$

is odd and cannot vanish. This is the required contradiction. A preserved nondegenerate alternating form identifies a representation with its dual, so symplectic monodromy is included. The explicit form $(6\gamma \wedge \delta + u \wedge w) \oplus \omega_E$ checks this hypothesis for the fibre-product example of Part I.

□

Proposition 4.7 (the remaining rank patterns)**.** [tag 0204]

If a possibly non-self-dual holomorphic datum with the same local models had $b_1 = b_2 = 0$, put $b = \dim H^0(B, \mathcal{V}^*)$. For $k = 2$, the only possible unordered pairs $(r_1, r_2; b)$ are

$$(2, 2; 0), \qquad (2, 4; 2).$$

For $k = 3$, they are

$$(2, 2, 6; 0),\ (2, 4, 4; 0),\ (2, 4, 6; 2),\ (4, 4, 4; 2). \tag{F.4}$$

Proof. ⟨1⟩1. The preceding Leray argument still gives $a = 1$ and vanishing $H^1(B, j_* \mathcal{V})$. Euler additivity therefore gives $b = \sum r_i + 3 - 1 - 6(k - 1)$. Each r_i belongs to $\{2, 4, 6\}$ because a free affine lift requires an eigenvalue one and complex linearity makes its multiplicity even. Also $0 \le b \le \min(3, r_1, \dots, r_k)$, by inclusion in each local invariant subspace. Enumerating these three-value lists gives exactly (F.4). No existence of a tuple or of a period family with these ranks is asserted.

□

5 The integral eight-manifold

Theorem 5.1 (The integral eight-manifold)**.** [tag 02NW]

There is a closed smooth oriented eight-manifold with the prescribed index-one Kuhn cusp, two free affine fillings of multiplicities $5, 4$*, trivial fundamental group, and rational Betti numbers*

$$(1, 0, 0, 0, 4, 0, 0, 0, 1). \tag{F.5}$$

It admits an almost complex structure. Its signature is two and its middle intersection lattice modulo torsion is $3\langle 1\rangle \oplus \langle -1\rangle$*. Its integral homology is computed in theorem 5.5, and a complex structure on it is constructed in theorem 6.9.*

Proof. $\langle 1\rangle 1$. We use the following matrices on the marked dual lattice:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Put $M_1 = A^t$, $M_2 = B^t$, $M_0 = (M_1 M_2)^{-1}$ and

$$v_1 = (1, 3, -2, -1, 2, 0)^t, \qquad v_2 = (-1, 0, 0, 0, 0, 0)^t.$$

Theorem 5.2 gives their orders, the primitive cusp marking, the coinvariant homomorphism $\ell(x) = x_0$, and freeness of every nonidentity power of the affine actions. Consequently

$$(T^6 \times \Delta_u)/\langle (x, u) \mapsto (M_i x + v_i/m_i, \zeta_{m_i} u)\rangle, \qquad (m_1, m_2) = (5, 4),$$

are smooth oriented finite caps. On the universal angular cover $u = re^{2\pi i a}$ their marked collar maps are

$$\alpha_i(x, u) = x - a v_i, \qquad \alpha_i(M_i x + v_i/m_i, \zeta_{m_i} u) = M_i \alpha_i(x, u).$$

At the cusp we take the index-one Kuhn cap of Theorem 6.8, in the marking of Proposition 4.4. The three displayed boundary bundles therefore glue to the marked torus bundle over the intervening pair of pants, with zero additional clutch translation. Every boundary is paired. The result is a closed connected smooth eight-manifold, oriented by the cusp complex structure and the continued fibre/base orientation.

$\langle 1\rangle 2$. The small cusp cap retracts onto its central fibre C through the mapping-cylinder homeomorphism (Achinger and Ogus, 2020, Theorem 4.1.1 and Lemmas 4.1.2–4.1.3, pp. 480–482). Indeed its divisorial log chart is $\mathbb{N} \to \mathbb{N}^r$, $1 \mapsto (1, \dots, 1)$; it is log smooth, saturated and vertical, and the cap is proper by Theorem 6.8. In the resulting cylinder the homotopy $[x, r] \mapsto [x, (1 - t)r]$ fixes C and retracts onto it.

Before the period quotient, the components of C and their nonempty intersections are smooth complete toric strata. Each is simply connected: a loop moves into its dense torus by general position, and the one-parameter loops along the rays of a smooth maximal cone bound the coordinate discs; those rays are an integral basis. Van Kampen sends component-crossing paths to edges and triple intersections to triangles of the Kuhn nerve in $\mathbb{R}^3$. Thus the covering core has trivial fundamental group. Its period quotient has deck group $\mathbb{Z}^3$, so $\pi_1(C) = \mathbb{Z}^3$. The nearby-torus map is the quotient of Λ by the saturated image of $M_0 - I$, by the H_1 collapse in Theorem 5.7. It is onto, and the cusp meridian dies because its zero section extends.

We also record the invariant normal closure needed below. Let K be the smallest $\langle M_1, M_2\rangle$-invariant subgroup containing $(M_0 - I)\Lambda$, and let $J = (M_1 - I)\Lambda + (M_2 - I)\Lambda = \ker \ell$. The equality $\ell M_i = \ell$ gives $K \subseteq \ker \ell$. On the abelian quotient Λ/K, the induced maps a, b satisfy $ab = 1$, $a^5 = 1$ and $b^4 = 1$. Thus $a^4 = a^5 = 1$, so $a = b = 1$ and $J \subseteq K$. Consequently $K = J = \ker \ell$ as integral subgroups.

Let a_i be the two finite peripheral lifts. The universal covering deck transformation of the ith cap has m_ith power equal to translation by v_i, since $M_i v_i = v_i$. The collar map in Step 1 sends a_i to that transformation. Van Kampen therefore gives

$$a_i \lambda a_i^{-1} = M_i \lambda, \qquad a_1^5 = v_1, \quad a_2^4 = v_2, \quad a_1 a_2 = 1.$$

Write $a_1 = t$ and $a_2 = t^{-1}$. Then $t = t^5 t^{-4} = v_1 + v_2$ lies in the abelian fibre subgroup. Conjugation is consequently trivial, and that subgroup is a quotient of $\Lambda_{\langle M_1, M_2\rangle} = \mathbb{Z}$, with coordinate ℓ. Now $\ell(v_1 + v_2) = 0$, so $t = 0$, while $t^5 = v_1$ kills its generator $\ell(v_1) = 1$. Every generator vanishes, proving $\pi_1 = 1$. Equivalently, after this abelian reduction, the relations on (ℓ, a_1, a_2) have matrix

$$\begin{pmatrix} -1 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \det = -1.$$

⟨1⟩3. We calculate rational cohomology directly from the marked restriction maps. For $1 \le q \le 4$ put

$$L_q = \bigwedge^q \Lambda^\vee \otimes \mathbb{Q}, \quad A_q = \bigwedge^q A, \quad B_q = \bigwedge^q B, \quad C_q = (B_q A_q)^{-1}.$$

The finite restriction images are $J_{1,q} = \ker(A_q - I)$ and $J_{2,q} = \ker(B_q - I)$: transfer followed by restriction is multiplication by the finite covering degree, which is invertible over $\mathbb{Q}$. The marked collapse in Theorem 5.7 gives $J_{0,q} = \ker(C_q - I)$ at the cusp, with one additional rational kernel in degree four. Proper base change identifies these with the stalk maps (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1; chapter PDF p. 30).

Cutting the sphere along the two based peripheral arcs gives the following cellular complex for the sheaf of restriction images:

$$J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q,$$

$$d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0), \qquad \Delta_q(x, y) = (A_q - I)x + (B_q - I)y.$$

The cusp value is transported across the first arc by A_q before the second subtraction. The relative disc boundary is Δ_q; their composition is zero because $B_q A_q j_0 = j_0$. The exterior matrices are their minors in lexicographically ordered exterior bases. Rational elimination gives

q	$\dim L_q$	$\dim J_{1,q}$	$\dim J_{2,q}$	$\dim J_{0,q}$	$\operatorname{rank} d_0$	$\operatorname{rank} \Delta_q$
1	6	2	2	3	6	6
2	15	3	3	9	15	15
3	20	4	4	9	17	20
4	15	3	3	9	15	15

Here are explicit nonzero minors for the two rank calculations. For each $J_{i,q}$ use the rational nullspace basis obtained from reduced row echelon form: order the free columns increasingly, set one free coordinate to one and the others to zero, and solve the pivot coordinates. Number rows and columns from zero. For Δ_q use all rows and the following columns:

q	columns	det
1	$1, 2, 3, 5, 7, 8$	2
2	$0, 1, 2, 4, \dots, 12, 16, 20, 24$	-6
3	$0, \dots, 7, 10, \dots, 17, 20, 24, 30, 36$	-6
4	$0, \dots, 7, 10, 11, 12, 13, 15, 21, 25$	6

For d_0, use columns $(0,1,2,3,5,6)$ when $q=1$ and all columns otherwise, with the following row sets:

q	rows	det
1	$0,1,3,4,6,7$	1
2	$0,\dots,8,10,12,13,15,20,21$	$-2/3$
3	$0,\dots,10,13,16,20,21,30,31$	$-4/3$
4	$0,\dots,11,15,16,25$	$1/3$

All are determinants of the matrices specified above. They give the stated full row ranks for Δ_q and full column ranks for d_0 in degrees two through four. In degree one the common fixed covector e_0 gives the nonzero kernel vector (e_0,e_0,e_0) of d_0, so its rank is at most six; the displayed minor gives equality. The finite invariant dimensions count exterior products of the specified eigenvalues with product one; the cusp dimensions are those of Theorem 5.7.

These ranks determine the cohomology: H^0 of the image sheaf is $\mathbb{Q}$ in coefficient degree one, H^1 is $\mathbb{Q}^3$ in coefficient degree three, and every other entry in these four coefficient degrees is zero. The cusp skyscraper kernel adds $\mathbb{Q}$ to $H^0(R^4\pi_*\mathbb{Q})$ and no higher cohomology. Connected fibres give $R^0\pi_*\mathbb{Q}=\mathbb{Q}$.

In the Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4; chapter PDF p. 22), the only remaining low-degree differential is $d_2^{0,1}:\mathbb{Q}\to\mathbb{Q}$. Its kernel is $H^1(M;\mathbb{Q})=0$ by Step 2, so it is an isomorphism. Hence $b_1=b_2=b_3=0$ and $b_4=3+1=4$. Poincaré duality determines all remaining Betti numbers and gives (F.5). Also, the smooth and finite quotient fibres have Euler number zero, while the Kuhn cusp has Euler number six by Theorem 6.8. Additivity over the base strata therefore gives $\chi(M)=6$ directly.

⟨1⟩4. Each finite monodromy admits an invariant real linear complex structure: its 1 and -1 real eigenspaces have even dimension, and each remaining conjugate eigenpair gives a real two-plane. Choose a complex structure on each summand. Reversing it on one fixed two-plane selects the required orientation. These structures commute with M_i, so their direct sums with the root-disc structure descend to the affine caps. The cusp cap already has its complex structure. Their vertical complex structures on the boundary collars are translation-invariant in the marked torus fibres.

The space $\mathcal{J}=\mathrm{GL}_6^+(\mathbb{R})/\mathrm{GL}_3(\mathbb{C})$ of allowable vertical structures retracts to $\mathrm{SO}(6)/\mathrm{U}(3)$ by the explicit conjugation

$$J\longmapsto P_tJP_t^{-1},\qquad P_t=\bigl((1-t)I+t(I+J^tJ)\bigr)^{1/2}.$$

At $t=1$ the structure is orthogonal, since J preserves the metric $I+J^tJ$; an orthogonal J is fixed throughout this homotopy. The inclusion $\mathrm{U}(3)\to\mathrm{SO}(6)$ sends $\mathrm{diag}(e^{i\theta},1,1)$ to one-plane rotation and induces the surjection $\mathbb{Z}\to\mathbb{Z}/2$ on fundamental groups. Indeed the determinant fibration and $\mathrm{SU}(2)=S^3\to\mathrm{SU}(3)\to S^5$ give the first group; the quaternion cover of $\mathrm{SO}(3)$ and the sphere fibrations for $\mathrm{SO}(n)$ give the second, with a full rotation lifting from 1 to -1. The homogeneous-space exact sequence therefore gives $\pi_0\mathcal{J}=\pi_1\mathcal{J}=0$.

Over the two-dimensional pair of pants the prescribed boundary sections of the associated $\mathcal{J}$-bundle extend over relative one-cells by connectedness and over relative two-cells by simple connectivity. Smooth approximation relative to the collars gives a smooth vertical structure. Horizontal splittings form an affine space, so a partition of unity extends the prescribed complex splittings from those collars. Combining the vertical structure with the pulled-back base structure and gluing the caps gives the required almost complex structure on M.

⟨1⟩5. We have $c_1 = c_3 = 0$ rationally because $H^2(M;\mathbb{Q})$ and $H^6(M;\mathbb{Q})$ vanish, and $c_4[M] = \chi(M) = 6$. The characteristic number formulas of (Esole and Kang, 2018, equations (1.2)–(1.4), pp. 3–4) give

$$\mathrm{Td}[M] = \frac{3c_2^2[M] - 6}{720}, \qquad \sigma(M) = \frac{3c_2^2[M] + 84}{45} = 16\,\mathrm{Td}[M] + 2.$$

The Todd genus of this almost complex manifold is an integer (Hirzebruch, 1978, Theorem 24.5.4 and corollary, p. 184). Since $|\sigma(M)| \le b_4 = 4$, the displayed integer relation forces $\mathrm{Td}[M] = 0$ and $\sigma(M) = 2$. Thus the real middle pairing has positive and negative indices 3 and 1.

Integral Poincaré duality makes the middle pairing modulo torsion unimodular. An even unimodular lattice has signature divisible by eight, so this lattice is odd. The classification of indefinite odd unimodular lattices (Peters and Sterk, 2024, Theorem 2.2.1, p. 61) gives its integral isometry class $3\langle 1\rangle \oplus \langle -1\rangle$.

□

We use column vectors. The matrices A, B in the data act on cohomology; $M_1 = A^t$, $M_2 = B^t$ act on $\Lambda = \mathbb{Z}^6 = H_1(T^6;\mathbb{Z})$, and $M_0 = (M_1M_2)^{-1}$. Positive based meridians satisfy $g_1g_2g_0 = 1$. The exterior-square basis of $\bigwedge^2 \mathbb{Z}^4$ is $(01, 02, 03, 12, 13, 23)$. All assertions about an integral matrix below are checked over $\mathbb{Z}$ or $\mathbb{Q}$, without floating-point arithmetic.

Theorem 5.2 (The rank-six monodromy)**.** [tag `02QI`] *The finite homology monodromies have orders* $5, 4$*, respectively, and characteristic polynomials* $(T-1)^2\Phi_5(T)$ *and* $(T-1)^2(T+1)^2(T^2+1)$*. For* $N = M_0 - I$ *we have* $N^2 = 0$*,* $\operatorname{rank} N = 3$*, and nonzero Smith factors* $1, 1, 1$*. Moreover*

$$\Lambda^{\langle M_1, M_2\rangle} = 0, \qquad \Lambda_{\langle M_1, M_2\rangle} = \mathbb{Z},$$

where the quotient homomorphism is $\ell(x) = x_0$*. On the cohomology lattice the corresponding groups are* $\mathbb{Z}e_0$ *and* $\mathbb{Z}/2$*. Every nonidentity power of each specified affine action* τ_i *is free.*

Proof. ⟨1⟩1. Exact multiplication gives

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix S is unimodular and $S^{-1}M_0S = \left(\begin{smallmatrix} I_3 & I_3 \\ 0 & I_3 \end{smallmatrix}\right)$. The columns of S are henceforth denoted $(a_1, a_2, a_3, b_1, b_2, b_3)$, so that $Nb_j = a_j$. This is also an integral certificate of the Smith assertion. The Hermite basis of $(M_1 - I)\Lambda + (M_2 - I)\Lambda$ is $(e_1, \dots, e_5)$; the stacked matrix of the two fixed-vector equations has rank six. Transposing the matrices gives a fixed lattice $\mathbb{Z}e_0$ and coinvariant Smith factors $(1, 1, 1, 1, 1, 2)$. The pairing $\Lambda^\vee \times \Lambda \longrightarrow \mathbb{Z}$ identifies the fixed cohomology vector e_0 with the covector ℓ.

⟨1⟩2. We have $M_iv_i = v_i$, $\ell(v_1) = 1$, and $\ell(v_2) = -1$. Thus $\tau_i^k(x) = M_i^kx + kv_i/m_i$, and $\tau_i^{m_i} = 1$ on the torus. A fixed point of τ_i^k would give $k\ell(v_i)/m_i = 0$ in $\mathbb{R}/\mathbb{Z}$. For $0 < k < m_i$ this is impossible. The orders, characteristic polynomials, fixed-space dimensions, coinvariant Smith factors, primitive cusp image and every affine-power obstruction are recomputed exactly in `independent_V_connection_spectra_06.py`. The displayed matrix S and its standard-form equality are also verified in `independent_V_markings_06.py`.

□

Theorem 5.3 (Homology of the finite fillings)**.** [tag 02QU] *The two finite filling pieces have first integral homology* $\mathbb{Z}^2$ *and* $\mathbb{Z}^2 \oplus \mathbb{Z}/2$*.*

Proof. ⟨1⟩1. For $K_i = T^6/\langle \tau_i \rangle$, the universal covering deck group has commuting translations x^u, $u \in \Lambda$, and a generator a_i, with the complete relations

$$a_i x^u a_i^{-1} = x^{M_i u}, \qquad a_i^{m_i} = x^{v_i}.$$

Indeed the affine lifts satisfy these relations, and they reduce every word uniquely to $x^u a_i^r$, $0 \le r < m_i$. The abelianization relation matrix has columns $((M_i - I)e_j, 0)$ and $(-v_i, m_i)$. Smith reduction of these explicit seven-by-seven matrices gives the groups above. The disk quotient $(T^6 \times \Delta)/\langle (\tau_i, \zeta_{m_i}) \rangle$ retracts onto K_i by equivariant radial contraction. Freeness supplies ordinary smooth quotient charts by disjoint translates of small balls.

□

Lemma 5.4 (The torsion transgression at the order-four filling)**.** [tag 02QF]

Use the order-four filling and the marked matrices of theorem 5.1. Let $T = \bigwedge^{q-1} D^t$ *on the integral vertical cochains of the five-torus. If* w *represents a torsion element of* $\operatorname{coker}(T - I)$*, its point-supported degree-*q *class has Leray differential equal, up to the one boundary-orientation sign, to the image of* w *in the global degree-*$(q-1)$ *coinvariants. For each of* $q = 2, 3, 4$*, the unique nonzero local torsion class maps to the nonzero class in the global* $\mathbb{Z}/2$ *quotient.*

Proof. ⟨1⟩1. Use the explicit matrix C_2 in the proof of theorem 5.5. In the original vertical marking D is the bottom-right block of M_2; multiplication gives $DC_2 = C_2 D_2$, where D_2 is the four-cycle followed by the sign on its fifth coordinate. The change of cochain coordinates is $\bigwedge^k C_2^t$, and the inverse is $\bigwedge^k C_2^{-t}$. Thus the product cell structure of the signed-permutation torus, transported by this integral torus isomorphism, has zero cellular differential and the stated action $T = \bigwedge^{q-1} D^t$. The fibre cells and their prisms give the mapping-torus cochain complex. The degree-q suspension classes are consequently the coinvariants of T on vertical degree-$q-1$ cochains.

⟨1⟩2. Write $u = -x_0$ in the original marking. The affine action is $(u, y, z) \mapsto (u + 1/4, Dy, iz)$, and $W = e^{-2\pi i u} z$ is invariant. With core angle $a = 4u \bmod 1$ and normal angle $k = \arg W/(2\pi)$, the coarse angle is $a + 4k$. Let g, h denote the two corresponding loops. The integral basis

$$v = g^4 h^{-1}, \qquad t = g, \qquad [v, t]_{[g,h]} = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}$$

has determinant one. Its coarse degrees are zero and one; its vertical monodromies are $T^4 = I$ and T respectively. It specifies the generic six-torus fibre and the primitive base loop in the boundary product.

⟨1⟩3. A suspension cochain representing $[w]$ has values $(w, 0)$ on (g, h). Following the four consecutive core prisms transports their values by I, T, T^2, T^3, giving

$$(w, 0)_{(g,h)} \longmapsto (N_T w, w)_{(v,t)}, \qquad N_T = I + T + T^2 + T^3. \tag{FT.1}$$

For a torsion coinvariant there are an integer $n \neq 0$ and an integral cochain b with $nw = (T - I)b$. Therefore $nN_T w = N_T(T - I)b = 0$, and $N_T w = 0$ in the free cochain lattice. The fibre restriction vanishes; its base-circle component is $[w]$. Replacing w by $w + (T - I)b$ changes that component by its cellular coboundary. The formula is consequently a well-defined map of the stated integral coinvariant groups.

⟨1⟩4. The base cover by a disc at the finite point and its complement gives the relative Čech connecting map. On their annular intersection the degree-zero fibre restriction is zero and the next component is the one-cocycle just computed. The connecting map sends it to $\pm[w]$ in the global coinvariants; the two ordinary peripheral loops impose the relations $\bigwedge^{q-1} M_1^t - I$ and $\bigwedge^{q-1} M_2^t - I$. These are the two successive boundaries in the same base–fibre complex, so their composition is the Leray d_2 of (F.6). The one boundary-orientation choice accounts for the sign. Its coefficient is one, as witnessed by the degree-one loop t above. Point-supported kernels or cokernels in the comparison of restriction sheaves have no positive cohomology, so they preserve this degree-two base quotient.

⟨1⟩5. In signed-permutation coordinates the negative orbits in vertical degrees $1, 2, 3$ are represented respectively by e_4, $e_0 \wedge e_2$ and $e_0 \wedge e_2 \wedge e_4$. The orbit relations give exactly one local $\mathbb{Z}/2$ in each case. Transport by $\bigwedge^k C_2^{-t}$ and then include vertically in the six-lattice. This is the same map as the vertical inclusion followed by $\bigwedge^k Q_2$ used in theorem 5.5, because $P_2 = (-1) \oplus C_2$. The explicit w_k, ρ_k, E_k, z_k there satisfy

$$\rho_k \Delta_k = 0 \pmod 2, \qquad \rho_k w_k = 1 \pmod 2, \qquad E_k z_k = 2w_k, \qquad |\det E_k| = 2.$$

Thus the unique nonzero local class maps to the unique nonzero global class in each of the three degrees. Formula (FT.1) sends the zero representative to zero as well.

□

Theorem 5.5 (Integral homology and three-connectivity)**.** [tag 02NX]

The manifold of Theorem 5.1 has free integral homology with Betti numbers (F.5)*. In particular $H_2 = H_3 = 0$, and it is three-connected.*

Proof. ⟨1⟩1. We use the marked construction of theorem 5.1 and the integral cusp-collapse assertion of theorem 5.7. Explicitly, $M_1 = A^t$, $M_2 = B^t$ and the vectors v_1, v_2 are those printed in item 1 of the proof of theorem 5.1; S is the displayed integral marking in item 1 of the proof of theorem 5.2. Write $m_1 = 5$, $m_2 = 4$, and put

$$C_1 = \begin{pmatrix} -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the translation vectors v_i, the matrices $P_i = [v_i, (0; C_i)]$ are unimodular. Direct multiplication gives

$$P_i^{-1} M_i P_i = 1 \oplus D_i, \qquad Q_i = P_i^{-t},$$

where D_1 sends e_j to $e_{j+1 \bmod 5}$, and D_2 sends e_j to $e_{j+1 \bmod 4}$ for $0 \le j < 4$ and e_4 to $-e_4$. These equalities give the integral coordinate maps. In these coordinates the affine action is

$$(u, y, z) \longmapsto (u + 1/m_i, D_i y, e^{2\pi i/m_i} z).$$

Thus its core F_i is the mapping torus of $D_i : T^5 \to T^5$. The function $W = e^{-2\pi i u} z$ trivializes its normal disc bundle. With $\theta = m_i u \bmod 1$, the coarse-base function on that bundle is

$$q = W^{m_i} e^{2\pi i \theta}.$$

In particular the boundary is $F_i \times S^1_W$ with this specified map to the base circle.

⟨1⟩2. We compute the integral restriction images on this model. Let $V = \mathbb{Z}^5$ be the vertical cohomology lattice, let $T_{i,k} = \bigwedge^k D_i^t$, and let $N_{i,k} = \sum_{a=0}^{m_i-1} T_{i,k}^a$. The signed permutations preserve the product cell structure with one vertex and one edge on each circle. Its cellular differential is zero. Fibre cells and their prisms give the mapping-torus complex

$$C^q(F_i) = \bigwedge^q V \oplus \bigwedge^{q-1} V, \qquad d(a,b) = (0, (T_{i,q} - I)a).$$

The m_i-fold covering by the generic six-torus pulls back (a,b) to $(a, N_{i,q-1}b)$: the horizontal cell is the union of the m_i consecutive prisms, with their transported coefficients. Consequently its image in $L_q = \bigwedge^q \mathbb{Z}^6$ is

$$J_{i,q} = \bigwedge^q Q_i \left(\operatorname{Inv} T_{i,q} \ \oplus \ e_0 \wedge N_{i,q-1} \bigwedge^{q-1} V \right).$$

Here vertical coordinates occupy slots $1, \dots, 5$ in the six-lattice. This formula specifies both the lattice and its inclusion.

We fix its column basis as follows. Exterior coordinates are in increasing lexicographic order, with indices starting at zero. For each signed-permutation orbit, start at its least coordinate and sum its successive signed images until it returns there. A positive orbit of length l gives a primitive invariant column; the norm image has the column m_i/l times that column. A negative orbit has zero invariant lattice and one $\mathbb{Z}/2$ in its coinvariant group. Indeed the successive relations identify all its coordinates with signed copies of the first, and the last relation is twice that first coordinate. Order the positive orbits by their least coordinates, with vertical columns before horizontal columns. For D_1 there is no torsion in vertical degrees 1 through 4. For D_2 there is one negative orbit in each such degree; the positive-orbit counts are $1, 2, 2, 1$.

⟨1⟩3. At the cusp, theorem 5.7 supplies a primitive collapse image and free homology, with the only collapse cokernel equal to $\mathbb{Z}$ in degree four. Dualizing those integral maps therefore gives

$$J_{0,q} = \operatorname{Inv}(\bigwedge^q M_0^t),$$

with cohomological restriction kernel zero for $q = 1, 2, 3$ and $\mathbb{Z}$ for $q = 4$. Here are explicit primitive bases, which also fix the column order in the calculation below. In the cusp marking S, write the cohomology basis as $x_0, x_1, x_2, y_0, y_1, y_2$; its action sends x_i to $x_i + y_i$ and fixes y_i. In degree one take (y_0, y_1, y_2). In degree two take, in order,

$$y_i \wedge y_j \ (i < j), \qquad x_i \wedge y_i \ (0 \le i \le 2), \qquad x_i \wedge y_j + x_j \wedge y_i \ (i < j).$$

In degree three take $y_0 \wedge y_1 \wedge y_2$, then $x_i \wedge y_j \wedge y_k$ in lexicographic (j,k) order with $j < k$ and $i = j, k$, and then

$$x_0 \wedge y_1 \wedge y_2 + x_1 \wedge y_0 \wedge y_2,$$
$$x_0 \wedge y_1 \wedge y_2 - x_2 \wedge y_0 \wedge y_1.$$

Their coordinate equations show that these bases are primitive and have ranks $3, 9, 9$. For degree four set $J = \left(\begin{smallmatrix} 0 & I_3 \\ -I_3 & 0 \end{smallmatrix}\right)$, and let $H(e_I)$ be the complementary exterior basis vector, with the sign of the permutation (I, I^c). Apply $H \bigwedge^2 J$ to the displayed degree-two basis. The identity $JU = U^{-t}J$ gives the invariant equation, and the signed-permutation maps H and $\bigwedge^2 J$ preserve primitivity. Transport all these columns by $\bigwedge^q S^{-t}$ to obtain $J_{0,q}$ in the original marking. Thus its degree-four rank is nine.

⟨1⟩4. Let $\mathcal{R}_q = R^q f_* \mathbb{Z}$ on the coarse base S^2, let j include its thrice-punctured locus, and let $\mathcal{S}_q$ be the image of $\mathcal{R}_q \to j_* L_q$. Since M is compact and the base is Hausdorff, the proper-base-change map identifies its stalks with fibre cohomology (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1; chapter PDF p. 30). Its three stalk images are exactly the lattices above. The kernel $\mathcal{K}_q$ is supported at those points: for $q = 2, 3$ it is the order-four $\mathbb{Z}/2$, and for $q = 4$ it is that $\mathbb{Z}/2$ together with the cusp $\mathbb{Z}$. Point-supported sheaves have no positive cohomology, so $H^p(\mathcal{R}_q) = H^p(\mathcal{S}_q)$ for $p > 0$.

Put $A_q = \bigwedge^q M_1^t$ and $B_q = \bigwedge^q M_2^t$. The based marking gives $\bigwedge^q M_0^t = (B_q A_q)^{-1}$. Oriented cellular duality on the punctured sphere and its two-loop spine gives the compact-support complex $L_q^2 \to L_q$ in degrees one and two. Attaching the three stalk lattices gives the full complex

$$J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q,$$
$$d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0),$$
$$\Delta_q(a, b) = (A_q - I)a + (B_q - I)b.$$

For the cusp column, the two consecutive peripheral paths carry j_0 and $A_q j_0$; their boundary is $(B_q A_q - I)j_0 = 0$. The reverse cusp orientation gives their negatives. The two finite columns are the corresponding single peripheral loops. This is the relative cellular sequence for extension by zero and the three stalk inclusions, so it retains their integral indices. It also verifies $\Delta_q d_0 = 0$ with the displayed transports.

Here are finite determinant certificates for its cohomology. They use precisely the ordered bases just given, so every matrix is specified by the marked M_i, S, C_i and exterior minors. For each q select the following columns $\mathcal{P}_q$ of Δ_q and call their square matrix E_q:

$$\begin{aligned}
\mathcal{P}_1 &= (1, 2, 5, 8, 9, 10),\\
\mathcal{P}_2 &= (0, 1, 2, 8, 9, 11, 12, 13, 16, 17, 23, 24, 25, 26, 28),\\
\mathcal{P}_3 &= (1, 2, 3, 5, 6, 10, 11, 12, 13, 15, 16, 18, 19,\\
&\qquad 26, 29, 31, 32, 34, 35, 37),\\
\mathcal{P}_4 &= (0, 1, 5, 7, 9, 10, 11, 13, 16, 17, 20, 23, 26, 28, 29).
\end{aligned}$$

Direct integer multiplication gives

$$\det E_q = (2, 2, 2, -2)_q, \qquad E_q^{-1}\Delta_q \in \mathrm{Mat}(\mathbb{Z}).$$

Hence $E_q \mathbb{Z}^{\binom{6}{q}}$ is the entire image of Δ_q and its cokernel is $\mathbb{Z}/2$. Let $\mathcal{F}_q$ be the increasing complement of $\mathcal{P}_q$ in $\{0, \dots, 2\binom{6}{q} - 1\}$. These free-coordinate rows identify $\ker \Delta_q$ integrally: the pivot coordinates of a kernel vector are $-E_q^{-1}(\Delta_q)_{\mathcal{F}_q}$ times its free coordinates. The relation matrix is therefore $R_q = (d_0)_{\mathcal{F}_q}$. Its sizes, ranks and primitive minors are

q	size of R_q	rank R_q	maximal minors
1	6×7	6	$-20,\ -5,\ 4$
2	15×15	15	$\det R_2 = 1$
3	20×17	17	-1
4	15×15	15	$\det R_4 = -1$

For $q = 1$ the three column sets are respectively $(0, 1, 2, 3, 5, 6)$, $(0, 1, 2, 4, 5, 6)$, $(0, 2, 3, 4, 5, 6)$, using all rows. For $q = 3$ use all columns and rows $(0, 1, \dots, 13, 15, 16, 17)$. These determinants have greatest common divisor one, so the quotient by the relations is free. The stalk column counts are $(2, 2, 3)$, $(3, 3, 9)$, $(4, 4, 9)$ and $(3, 3, 9)$. Thus the kernel dimensions

and the quotient ranks give exactly

q	$H^0(\mathcal{S}_q)$	$H^1(\mathcal{S}_q)$	$H^2(\mathcal{S}_q)$
1	$\mathbb{Z}$	0	$\mathbb{Z}/2$
2	0	0	$\mathbb{Z}/2$
3	0	$\mathbb{Z}^3$	$\mathbb{Z}/2$
4	0	0	$\mathbb{Z}/2$.

⟨1⟩5. We compute the torsion differential on the boundary. Let g be the core-circle loop of F_2 and h the W-circle loop. Their coarse-base degrees are one and four. The basis

$$v = g^4h^{-1}, \qquad t = g, \qquad [v,t]_{[g,h]} = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}$$

has determinant one; v lies in the generic torus fibre and t projects once around the base. For $k = q-1$ let $T = \bigwedge^k D_2^t$ and $N = I + T + T^2 + T^3$. A torsion class $[w] \in \operatorname{coker}(T-I)$ has a mapping-torus prism representative with values w on g and zero on h. Subdivision into successive transported prisms gives values Nw on v and w on t. An integral equation $nw = (T-I)a$ with $n \neq 0$ implies $nNw = 0$, hence $Nw = 0$ in the free lattice. Its fibre restriction is therefore zero and its first boundary Wang component is exactly $[w]$. The relative-base connecting map takes that component to its global coinvariant class. In the cellular fibre and base Čech double complex these are precisely the two successive boundaries defining the Leray differential. Thus, up to the single boundary-orientation sign,

$$d_2[w] = \pm[w] \in L_{q-1}/\operatorname{im}\Delta_{q-1}. \tag{F.6}$$

Changing w by $(T-I)a$ changes a boundary representative by a coboundary. The coefficient is one because the loop t has degree one over the coarse-base circle.

For $k = 1, 2, 3$ the negative signed orbits have representatives e_4, $e_0 \wedge e_2$, $e_0 \wedge e_2 \wedge e_4$. Map them by the vertical inclusion followed by $\bigwedge^k Q_2$; call the resulting columns $w_k \in L_k$. The following rows ρ_k, read modulo two, annihilate Δ_k:

$$\begin{aligned}
\rho_1 &= (0,1,1,1,1,1),\\
\rho_2 &= (1,1,1,1,1,1,1,0,0,1,0,1,0,1,1),\\
\rho_3 &= (0,0,0,0,0,0,0,0,0,0,1,1,0,1,0,1,1,1,0,0).
\end{aligned}$$

The explicit representatives and integer witnesses are

$$\begin{aligned}
w_1 &= (0,0,1,1,1,0)^t,\\
w_2 &= (0,0,0,0,0,0,0,0,-1,0,0,-1,0,-1,-1)^t,\\
w_3 &= (0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0)^t,\\
z_1 &= (0,0,0,-1,-1,-1)^t,\\
z_2 &= (-1,0,1,-2,-2,-2,4,2,-1,-3,2,0,-2,0,2)^t,\\
z_3 &= (-2,-2,-3,0,0,-1,-1,-1,2,2,-1,3,-1,4,3,\\
&\qquad -3,-2,1,-4,1)^t.
\end{aligned}$$

They satisfy $E_k z_k = 2w_k$ and $\rho_k w_k = 1 \bmod 2$. The nonzero local torsion class therefore maps to the nonzero global class in each degree $q = 2, 3, 4$.

⟨1⟩6. Since $H^0(\mathcal{S}_q) = 0$ in these three degrees, the point-supported kernels give

$$H^0(\mathcal{R}_2) = \mathbb{Z}/2, \quad H^0(\mathcal{R}_3) = \mathbb{Z}/2, \quad H^0(\mathcal{R}_4) = \mathbb{Z} \oplus \mathbb{Z}/2.$$

The Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4; chapter PDF p. 22) has only base columns zero, one and two. The maps from $(0,2)$ to $(2,1)$ and from $(0,3)$ to $(2,2)$ are the isomorphisms just proved. The restriction of the map from $(0,4)$ to $(2,3)$ to its torsion summand is also an isomorphism. Writing the image of a free cusp generator as $\epsilon \in \mathbb{Z}/2$, its kernel is the free group generated by $(1,\epsilon)$. All total-degree-three terms consequently vanish. In total degree four the remaining terms are $\mathbb{Z}$ and $\mathbb{Z}^3$. Their extension is free, so

$$H^3(M;\mathbb{Z}) = 0, \qquad H^4(M;\mathbb{Z}) = \mathbb{Z}^4.$$

By theorem 5.1, $\pi_1(M) = 0$ and the rational homology in degrees two and three vanishes. Universal coefficients (Hatcher, 2002, Theorem 3.2, p. 195, and Corollary 3.3, p. 196) identify the torsion of H_2 and H_3 with the corresponding Ext subgroups of H^3 and H^4; both are zero. These finitely generated groups have rational rank zero, hence $H_2 = H_3 = 0$. Integral Poincaré duality (Hatcher, 2002, Theorem 3.30, p. 241) determines the remaining groups as $\mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}$ in degrees $0, 4, 8$ and zero otherwise. Finally, the Hurewicz homomorphisms (Hatcher, 2002, Theorem 4.32, p. 366) give successively $\pi_2 = H_2 = 0$ and $\pi_3 = H_3 = 0$.

□

Theorem 5.6 (The compact complex fourfolds M_c)**.** [tag 02QJ] *For every real parameter $c > -h_{\min}$ the period family F_c of theorem 6.7 gives a compact smooth complex fourfold M_c and a proper holomorphic map to $\mathbb{P}^1$ with the two multiple fibres of theorem 5.1 and the index-one Kuhn cusp. With its complex orientation, $\pi_1(M_c) = 1$ and $\chi(M_c) = 6$.*

Proof. ⟨1⟩1. Evaluation of Λ on F_c maps $\Lambda_{\mathbb{R}}$ isomorphically onto the underlying real bundle of $F_c^\vee$: the kernel is zero exactly by $F_c \cap \bar{F}_c = 0$. The image is consequently a full discrete lattice. Quotient by its holomorphic period sections gives a holomorphic torus bundle on the nonspecial base.

⟨1⟩2. On a finite root chart $u^{m_i} = t_i$, divide the pullback torus bundle by

$$(x,u) \longmapsto (M_i x + v_i/m_i, \zeta_{m_i} u).$$

Freeness proves smoothness, including the central fibre. The special divisor is m_i times the reduced free quotient. On the punctured root disk translation by $-(\log u/(2\pi i))v_i$, interpreted by its period section, identifies this affine quotient with the original Jacobian bundle. A change of logarithm gives an integral lattice period; replacing u by $\zeta_{m_i} u$ cancels exactly the affine term v_i/m_i. This proves the holomorphic gluing and its sign. At the cusp use theorem 6.8 and the exponential period map. The zero section $\zeta = 0$ extends to $z = 1$ in the dense torus of the component marked by the vertex zero. The resulting map is proper by a finite compact cover of the base by the three local pieces and a compact remainder with torus fibres. All total-space charts and their overlaps have already been specified, so this produces a smooth compact complex fourfold, and in particular an almost complex structure.

⟨1⟩3. For completeness we record every van Kampen relation. The main bundle has its zero section and fundamental group $\Lambda \rtimes \langle g_1, g_2\rangle$. The finite pieces impose $g_i^{m_i} = v_i$ and $g_i x^u g_i^{-1} = x^{M_i u}$; the logarithmic gluing above supplies these signs. The cusp zero section kills $g_0 = (g_1 g_2)^{-1}$ and the cusp collapse kills $\operatorname{im} N$. The latter assertion follows directly from the toric moment model in theorem 5.7: angular cycles a_j collapse and the base cycles b_j survive. Its fundamental group is $\mathbb{Z}^3$, since the simply connected complete toric components and their connected toric intersections have dual complex $\mathbb{R}^3/\mathbb{Z}^3$; alternatively, van Kampen on the moment cells kills the angular loops generated by the primitive facet normals and retains exactly the three base loops. Thus the full presentation is

$$\left\langle \Lambda, g_1, g_2 \;\middle|\; [\Lambda,\Lambda] = 1, \quad g_i x^u g_i^{-1} = x^{M_i u}, \quad g_1^5 = v_1,\ g_2^4 = v_2,\ g_1 g_2 = 1,\ \operatorname{im} N = 0 \right\rangle.$$

The last relation is redundant after the preceding ones but records the cusp map. Since $g_2 = g_1^{-1}$, the two induced actions are inverse; their orders divide both five and four and are therefore one. The remaining fibre group is $\mathbb{Z}$ through ℓ. With its generator z, the additive relations are $5g_1 - z = 0$, $4g_2 + z = 0$, $g_1 + g_2 = 0$. Their matrix

$$\begin{pmatrix} 5 & 0 & -1 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

has determinant -1, proving triviality. Euler additivity gives $\chi(M_c) = 6$: smooth tori and finite free torus quotients have Euler number zero, and theorem 6.8 gives the remaining contribution.

□

Theorem 5.7 (The central cusp fibre and its collapse map)**.** [tag 02QK] *The index-one central cusp fibre C has torsion-free integral homology with ranks $(1, 3, 9, 9, 10, 3, 1)$. The collapse $\kappa : T^6 \to C$ has primitive image in every degree; it is onto except in degree four, where its cokernel is $\mathbb{Z}$. Its image ranks are $(1, 3, 9, 9, 9, 3, 1)$. In every degree its kernel is the saturation of $(\bigwedge^k M_0 - I)\bigwedge^k \Lambda$.*

Proof. ⟨1⟩1. We first remove the unit matrix $C(q)$ topologically, by an explicit deformation, without changing the integral marking. Use the family $C_u(q) = (1-u)C(q) + uiI_3$, followed by the reverse of the same family starting at $C(q) = 0$. The two flows compose to a marked comparison with $C(q) = 0$. The action and properness estimates of theorem 6.8 are uniform on both parameter intervals. Local quotient charts form a smooth family of pairs, with the local equation $q = x_0x_1x_2x_3$ independent of u. Local lifts of $\partial/\partial u$, tangent to every coordinate stratum and annihilating q, can be patched by a partition of unity. Over a smaller compact disk their flow exists for the whole interval. This gives a stratified diffeomorphism of the central fibres and a diffeomorphism of the nearby fibres, preserving the period marking. The same construction on the real oriented blowup preserves the collapse map. We may therefore calculate at $C(q) = 0$.

⟨1⟩2. Here is a concrete topological model. Put $G = 4I_3 - \mathbf{1}\mathbf{1}^t$. Its positive eigenvalues are $1, 4, 4$. The Voronoi polytope in dual coordinates is cut out by

$$|v^t y| \le \tfrac{1}{2} v^t G v \quad (0 \ne v \in \{0,1\}^3).$$

The value on a binary vector with k nonzero entries is $k(4-k)$. These fourteen primitive facet normals give the star of a vertex in the Kuhn triangulation. One checks its Delaunay tetrahedra by the successive chains $0, e_i, e_i + e_j, e_1 + e_2 + e_3$ for the six permutations (i, j, k); solving their three perpendicular-bisector equations and the fourteen inequalities verifies precisely their common faces. The translates of this Voronoi polytope by $G\mathbb{Z}^3$ tile $\mathbb{R}^3$.

⟨1⟩3. For a simplex σ with vertices $v_0, \dots, v_r$, put $L_\sigma = \sum_i \mathbb{Z}(v_i - v_0)$ and $D_\sigma = \bigcap_{v\in\sigma} P_v$, where $P_v = P + Gv$. The component torus has lattice $A = \mathbb{Z}^3$; the common-stratum marking is the explicit map

$$(A \oplus \mathbb{Z})/\sum_i \mathbb{Z}(v_i, 1) \longrightarrow A/L_\sigma, \qquad [(a, k)] \longmapsto a - kv_0.$$

The Kuhn unimodularity makes L_σ primitive.

We choose compatible moment maps as follows. The polytope $2P_v$ is integral, since its vertex facet normals are unimodular and its facet constants are integral. Embed the component X_v using the characters χ^{-m} for $m \in \mathcal{A}_v = 2P_v \cap A^\vee$, and define

$$\mu_v([Z_m]) = \frac{\sum_{m\in\mathcal{A}_v} m|Z_m|^2}{2\sum_{m\in\mathcal{A}_v} |Z_m|^2}.$$

The sign is specified: the ray $w - v$ tends to the face on which $\langle w - v, -m\rangle$ is minimal, hence to the upper common Voronoi face of P_v and P_w. Thus the character polytope is $-2P_v$, and μ_v is minus one-half its usual moment map. The monomial moment theorem (Cox et al., 2011, equations (12.2.3)–(12.2.4) and Theorem 12.2.5, p. 571) identifies the nonnegative toric part with P_v. For the squared weights here, the intervening map on that part is $[a_m] \mapsto [a_m^2]$, with inverse $[b_m] \mapsto [\sqrt{b_m}]$: both preserve the homogeneous binomial relations and positive common projective rescaling. Write $s_v : P_v \to X_v$ for the resulting continuous positive section.

On a shared face put $\mathcal{A}_\sigma = 2D_\sigma \cap A^\vee$ and choose $m_0 \in \mathcal{A}_\sigma$. For $m \in \mathcal{A}_\sigma$, the difference $m - m_0$ annihilates L_σ. The ratio Z_m/Z_{m_0} is consequently the restriction of the same character

$$(-(m - m_0), \langle m - m_0, v\rangle) \in (A \oplus \mathbb{Z})^\vee$$

for every vertex v of σ. At $C = 0$ all deck unit multipliers are one, and the toric gluing identifies these character functions exactly. On either component the restricted moment map is therefore the identical expression

$$\frac{\sum_{m\in\mathcal{A}_\sigma} m|\chi^{-(m-m_0)}|^2}{2\sum_{m\in\mathcal{A}_\sigma} |\chi^{-(m-m_0)}|^2}.$$

The equality persists on every smaller face by restricting the index set. Positive character values also agree, so the sections s_v glue on the entire shared orbit closure. The angular maps are the displayed quotient homomorphisms $A \to A/L_\sigma$.

The period deck map is $(a, k) \mapsto (a + k\nu, k)$, sending v to $v + \nu$. Its component marking is unchanged because $a + k\nu - k(v + \nu) = a - kv$. Moreover $\mathcal{A}_{v+\nu} = \mathcal{A}_v + 2G\nu$; the common character factor cancels in projective ratios. Hence

$$\mu_{v+\nu}(T_\nu p) = \mu_v(p) + G\nu, \qquad s_{v+\nu}(y + G\nu) = T_\nu s_v(y).$$

These give an explicit well-defined map

$$\mathcal{Q} = ((\mathbb{R}^3/G\mathbb{Z}^3) \times (\mathbb{R}^3/\mathbb{Z}^3))/\sim \longrightarrow C,$$
$$[y, \theta] \longmapsto e^{2\pi i\theta}\, s_v(y) \quad (y \in P_v).$$

where over relint D_σ the relation collapses the subtorus $L_\sigma \otimes (\mathbb{R}/\mathbb{Z})$. On each stratum the inverse takes its moment value and its angular character class in T_A/T_{L_σ}; the latter is obtained by $z \mapsto z/|z|$ on its nonzero characters. The map is a continuous bijection, and its source is compact and C Hausdorff, so it is a homeomorphism. The angular identity $[y, 0]$ gives the continuous section, with the same face and deck compatibilities just proved.

⟨1⟩4. The nearby-fibre collapse is the quotient map in this model. Give the semistable family its divisorial logarithmic structure. The local equations $q = x_0x_1x_2x_3$ and their face restrictions make it fine, saturated, log smooth and vertical over the standard logarithmic disc. The family is proper by theorem 6.8. In its logarithmic central fibre, retain all arguments of the x_j with their sum fixed by $\arg q$. The integral angular lattice over a face with vertices $v_0, \dots, v_r$ is the kernel of the height map on $\mathbb{Z}^3 \oplus \mathbb{Z}$. Forgetting the arguments of the zero coordinates induces exactly

$$\mathbb{Z}^3 \longrightarrow (\mathbb{Z}^3 \oplus \mathbb{Z})/\sum_i \mathbb{Z}(v_i, 1) \;\cong\; \mathbb{Z}^3/L_\sigma.$$

The last isomorphism sends the class of (a, k) to $a - kv_0 \pmod{L_\sigma}$; its inverse sends a to $(a, 0)$.

The polar maps agree under the integral monomial overlaps. At $C(q) = 0$ the deck multipliers have positive real values, so the logarithmic fibre at angle one is the product of the base and angular three-tori in item 3, with their given markings.

The mapping-cylinder comparison for a proper vertical log-smooth family identifies this logarithmic forgetful map with the nearby-fibre collapse, compatibly with the central strata (Achinger and Ogus, 2020, formula (3-3-1), p. 478; Theorem 4.1.1 and Lemmas 4.1.2–4.1.3, pp. 480–482). Its angular cycles are a_j and its base cycles are b_j. In these column-basis conventions monodromy fixes each a_j and sends b_j to $b_j + a_j$, as specified by $Nb_j = a_j$. The stratified deformation in item 1 transports this map back to the original bounded period matrix.

⟨1⟩5. Filter $\mathcal{Q}$ by the dimension d of its dual base cells. Over the interior of the dual cell of a p-simplex, $d = 3 - p$ and the angular quotient has lattice $Q_\sigma = \mathbb{Z}^3/L_\sigma$ of rank d. Excision for the cell and its boundary gives the relative chain groups of the resulting homology spectral sequence:

$$E^1_{d,q} = \bigoplus_{\dim \sigma = 3-d} \bigwedge^q Q_\sigma.$$

Orient dual cells by the orientation of the base and the ordered simplex. The differential is the signed quotient map for each codimension-one coface. More explicitly, with integral projection matrices Q_σ and chosen right inverses S_σ,

$$D_{p,q}[\tau, \sigma] = \sum_{\sigma = \partial_j \tau \bmod \mathbb{Z}^3} (-1)^j \bigwedge^q (Q_\tau S_\sigma). \tag{1.11.1}$$

All L_σ are primitive because the tetrahedra are unimodular. Thus these projections and right inverses are integral. Formula (1.11.1) specifies every integer entry, including multiple incidences under translation identifications. The program `reconstruct_V_cusp_06.py` enumerates all 26 simplex classes and 74 codimension-one incidences and computes these matrices exactly.

⟨1⟩6. The nonzero $E^2_{d,q}$ ranks are

$q\backslash d$	0	1	2	3
0	1	3	3	1
1	0	6	8	3
2	0	0	7	3
3	0	0	0	1

and every group is free. The reconstructed incoming Smith matrices have only unit nonzero factors and reproduce every entry of this table. Support leaves only the possible higher differential $d_2 : E^2_{3,0} \to E^2_{1,1}$. The angular identity section splits the base projection as a map of filtered spaces, so the base fundamental class survives. This kills that differential. All subsequent differentials have zero source or target. The filtration extensions are extensions of free abelian groups and therefore split, proving the asserted integral homology.

⟨1⟩7. The product-to-quotient map induces on E^1 the exterior powers of the quotient projections. The program `reconstruct_V_cusp_06.py` constructs the induced maps on integral cycle and quotient bases from the face projections and the Poincaré-dual Alexander–Whitney formula in step 9. Their images are primitive, and the only nonzero graded cokernel is $\mathbb{Z}$ at $(d, q) = (2, 2)$. The summed image ranks are

$$r_k = (1, 3, 9, 9, 9, 3, 1).$$

The same exact exterior-power calculation gives these ranks for $\ker(\bigwedge^k M_0 - I)$. The collapse satisfies $\kappa_*(\bigwedge^k M_0 - I) = 0$, because a monodromy circuit has the same retraction

to the central fibre in the mapping-cylinder comparison. Therefore its rank is at most r_k. A filtered map has rank at least the sum of the ranks of its graded maps; hence the rank is exactly r_k.

Choose integral bases compatible with the source and target filtrations; their graded groups are free. In each graded block choose the unit Smith pivots of the computed map. The corresponding r_k rows and columns of the filtered map form a block triangular submatrix with unimodular diagonal pivot blocks. Its determinant is ± 1. Since the full map has rank r_k, these unit pivots prove that its image is primitive. Thus the map is onto in every degree except four, where its cokernel is the free group of rank $10 - 9 = 1$. This rank argument also rules out additional lower-filtration images of source classes in the kernels of the graded maps.

⟨1⟩8. We represent a base cycle b_J by Poincaré duality from the Alexander–Whitney cup cocycle on the complementary coordinates. On an ordered p-simplex its coefficient is

$$\operatorname{sgn}(J, J^c) \prod_{r=1}^{p} (v_r - v_{r-1})_{(J^c)_r}.$$

Tensoring with an angular exterior vector and applying (1.11.1) gives the marked matrices. Their source basis is the ordered exterior basis $(a_1, a_2, a_3, b_1, b_2, b_3)$; the output gives the integral target homology projections and all Smith basis maps. They annihilate $\bigwedge^k M_0 - I$ exactly. The collapse also annihilates that operator, since turning the smoothing parameter does not change its image in the central fibre. The computed ranks equal the ranks of the monodromy coinvariants; primitivity therefore identifies both kernels with the saturation of its image. Consequently these are the matrices of the collapse in suitable integral target bases.

□

Theorem 5.8 (Integral homology of M_c)**.** [tag `02QL`] *The fourfold M_c has torsion-free integral homology*

$$H_k(M_c; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, 8, \\ \mathbb{Z}^4, & k = 4, \\ 0, & k = 1, 2, 3, 5, 6, 7. \end{cases}$$

In particular it is three-connected.

Proof. The fourfold M_c is glued from the same affine caps, cusp fan and clutching as the manifold of theorem 5.1, and the proof of theorem 6.9 shows that the underlying smooth manifold does not depend on the bounded corrections or on the complex structures of the fibres. Hence theorem 5.5 applies. An independent computation by the Mayer–Vietoris sequence, with integral cellular models of the torus bundle over the punctured sphere, the two finite caps and the cusp cap and with all maps integral, gives the same groups (programs `reconstruct_V_global_06.py` and `verify_V_global_saved_07.py`). □

6 The genus-two pencil and the lifted period planes

Theorem 6.1 (The genus-two pencil and its Hodge bundle)**.** [tag `02QV`] *The genus-two pencil has one nodal fibre away from $0, \infty$, at $\lambda_* = -256/3125$. Good reduction at $0, \infty$ requires root orders $8, 5$. On this root orbifold its Hodge bundle is $\mathcal{H} = T^3 \oplus T$, where*

$$T = \mathcal{O}((2/5)[\infty] - (3/8)[0]), \qquad \deg T = 1/40,$$

and T^3 is the line generated meromorphically by dx/y.

Proof. ⟨1⟩1. The discriminant of $x^5 + \lambda^3 x + \lambda^4$ is $\lambda^{15}(3125\lambda + 256)$. At λ_* the double root is $64/625$; the second x-derivative and first λ-derivative there are nonzero. The remaining roots are simple. The odd-degree hyperelliptic compactification is smooth at its branch point at infinity. This gives one node, a smooth total surface at that node, and genus-one normalization.

⟨1⟩2. At zero set $\lambda = u^8$, $x = u^6 X$, $y = u^{15} Y$; at infinity set $\lambda = u^{-5}$, $x = u^{-4} X$, $y = u^{-10} Y$. The resulting smooth equations and differentials are

	equation	dx/y	$x\,dx/y$
0	$Y^2 = X^5 + X + u^2$	$u^{-9} dX/Y$	$u^{-3} X\,dX/Y$
∞	$Y^2 = X^5 + uX + 1$	$u^6 dX/Y$	$u^2 X\,dX/Y.$

Their central polynomials are square-free. The deck maps at zero and infinity are $(X, Y) \mapsto (iX, \zeta_8 Y)$ and $(X, Y) \mapsto (\zeta_5^{-1} X, Y)$, respectively. The eigenvalues on $H^{1,0}$ are (ζ_8, ζ_8^3) and $(\zeta_5^{-1}, \zeta_5^{-2})$. Including conjugates gives characteristic polynomials $T^4 + 1$ and Φ_5, proving minimality of the root orders. The fourth power at zero is the hyperelliptic involution, acting as $-I$ on H^1 and as I on its exterior square.

⟨1⟩3. The two meromorphic differential generators have orbifold divisors $(6/5)[\infty] - (9/8)[0]$ and $(2/5)[\infty] - (3/8)[0]$. At the node they are independent sections of the dualizing Hodge extension: one has opposite nonzero residues at the two points of the normalization, and the linear combination vanishing at the node is a regular nonzero elliptic differential. Thus there is no additional divisor at λ_*. These computations prove the asserted direct sum and its degrees. The discriminant, the nodal root and its transverse derivatives, both exact root substitutions, square-freeness of the central polynomials, and the line degree are recomputed in `independent_V_markings_06.py`, with its exact JSON output.

□

Theorem 6.2 (The rank-three bundle P)**.** [tag 02QW] *The wedge map from $\ell_0 \otimes (\mathcal{H}_{\mathrm{dR}}/\ell_0)$, followed by projection modulo the nondegenerate polarization bivector, is injective and has rank-three image. This image defines the bundle P used in the period construction.*

Proof. ⟨1⟩1. There is an injective holomorphic map

$$\ell_0 \otimes (\mathcal{H}_{\mathrm{dR}}/\ell_0) \longrightarrow \bigwedge^2 \mathcal{H}_{\mathrm{dR}}, \quad u \otimes [v] \longmapsto u \wedge v.$$

Every nonzero image vector is decomposable and has wedge square zero; $\omega^2 \neq 0$. Projection modulo $\mathcal{O}\omega$ is therefore injective on its rank-three image. This defines the bundle P used below.

□

Theorem 6.3 (the geometric rank-five quotient)**.** [tag 0207]

Put $\lambda_* = -256/3125$ and $U = \mathbb{P}^1 \setminus \{0, \infty, \lambda_*\}$. The smooth projective models of

$$C_\lambda : \quad y^2 = x^5 + \lambda^3 x + \lambda^4 \tag{F.7}$$

give a polarized rank-four rational cohomology system $\mathcal{H}$ on U. Its primitive exterior square is rationally isomorphic to the rank-five quotient of the system with matrices A, B in Theorem 5.1 by its invariant first coordinate line. On the orbifold $\mathcal{B}'$ obtained by taking roots of orders five at infinity and eight at zero, the extended Hodge bundle is

$$\mathcal{E} = \mathcal{T}^3 \oplus \mathcal{T}, \qquad \mathcal{T} = \mathcal{O}_{\mathcal{B}'}(2D_\infty - 3D_0), \qquad \deg \mathcal{T} = \frac{1}{40}. \tag{F.8}$$

Here D_∞, D_0 are the reduced root divisors; the extension at λ_* is the nilpotent canonical extension.

Proof. ⟨1⟩1. The discriminant of the degree-five polynomial in (F.7) is exactly $\lambda^{15}(3125\lambda + 256)$. At λ_* its unique double root is $r = 64/625$ and the coefficient of $(x-r)^2$ is $10r^3 \neq 0$. Its other roots are simple. Thus the fibre has one ordinary node and irreducible genus-one normalization. The local model is a smoothing of a nonseparating node; the vanishing circle is a primitive nonseparating homology class. Cutting and regluing an annular neighborhood proves that its monodromy is a single Dehn twist, and hence a primitive rank-one transvection on H^1. In particular its logarithm N_H has square zero and rank one.

⟨1⟩2. At zero, the substitutions $\lambda = s^8$, $x = s^6X$, $y = s^{15}Y$ give

$$Y^2 = X^5 + X + s^2.$$

This is smooth at $s = 0$. The generator $s \mapsto \zeta_8 s$ at fixed (x, y) acts by $(X, Y) \mapsto (iX, \zeta_8 Y)$, so its eigenvalues on the two holomorphic differentials are ζ_8, ζ_8^3. The full rational characteristic polynomial is $g(T) = T^4 + 1$. At infinity put $\lambda = s^{-5}$, $x = s^{-4}X$, $y = s^{-10}Y$. The resulting smooth family is

$$Y^2 = X^5 + sX + 1.$$

The deck generator acts by $(X, Y) \mapsto (\zeta_5^4 X, Y)$, with holomorphic eigenvalues ζ_5^4, ζ_5^3. Its full characteristic polynomial is $f(T) = T^4 + T^3 + T^2 + T + 1$. Inverses have the same two rational characteristic polynomials.

⟨1⟩3. We identify the representation. Choose positive peripheral generators T_1, T_2 at infinity and zero, in that order, with T_1T_2 inverse to nodal monodromy. Set $R = T_1^{-1}$ and $S = T_2$. Then $R - S$ has rank one and their characteristic polynomials are f, g. The companion presentation is Levelt's theorem in (Beukers and Heckman, 1989, Theorem 3.5, pp. 331–332; Corollary 3.6, p. 332). Its generators are related to the present pair by

$$h_\infty = R = T_1^{-1}, \qquad h_0 = S^{-1} = T_2^{-1}, \qquad h_1 = R^{-1}S = T_1T_2, \qquad h_\infty h_1 h_0 = I.$$

The following calculation supplies the rational change of basis for this presentation. If u spans $\operatorname{im}(R - S)$, it is cyclic for R: the span of its iterates is stable under both R and S, and the irreducibility of f makes that span all of $\mathbb{Q}^4$. In the rational basis u, Ru, R^2u, R^3u, the matrix of R is the companion matrix of f, and $S = R - u\varphi$ for a covector φ. The determinant identity

$$g(T) = f(T) + \varphi \operatorname{adj}(TI - R)u \tag{F.9}$$

determines φ uniquely: the four coordinates of $\operatorname{adj}(TI - R)u$ are a basis of the polynomials of degree at most three, as is seen from their successively distinct leading degrees in the cyclic basis. It follows that any two such pairs are simultaneously rationally conjugate. Apply the lemma to the explicit companion pair

$$C_f = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad C_g = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{F.10}$$

Their difference has rank one and their characteristic polynomials are f, g. More explicitly, put

$$\begin{aligned} \mathsf{B} &= (u, Ru, R^2u, R^3u), \\ d &= (C_f - C_g)e_4 = (0, -1, -1, -1)^t, \\ \mathsf{B}_c &= (d, C_f d, C_f^2 d, C_f^3 d). \end{aligned}$$

Here $\det \mathsf{B}_c = 1$. Uniqueness of the covector in (F.9) gives the intertwiner $\mathsf{P} = \mathsf{B}\mathsf{B}_c^{-1}$, with $R\mathsf{P} = \mathsf{P}C_f$ and $S\mathsf{P} = \mathsf{P}C_g$. Thus (T_1, T_2) is rationally conjugate to (C_f^{-1}, C_g), with the chosen peripheral conventions.

⟨1⟩4. Use the lexicographic basis in $\bigwedge^2 \mathbb{Q}^4$. Its common invariant vector is $(1,1,1,1,1,1)^t$. Let Q be the identity matrix of size six with first column replaced by this vector. It is unimodular. Direct multiplication gives the exact identities

$$\left(Q^{-1}\bigwedge^2(C_f^{-1})Q\right)_{2:6,2:6} = A_{2:6,2:6}, \qquad \left(Q^{-1}\bigwedge^2(C_g)Q\right)_{2:6,2:6} = B_{2:6,2:6}. \tag{F.11}$$

They are checked over $\mathbb{Z}$ in `fourfolds_periods.py`. The invariant line is the inverse of the polarization, up to a nonzero rational scalar. Projection along that line identifies the quotient with the primitive exterior square $\mathcal{W}$.

⟨1⟩5. The differentials $\omega_1 = dx/y$ and $\omega_2 = x\,dx/y$ span the extended Hodge bundle everywhere. At the node they are independent dualizing differentials: ω_1 has nonzero opposite residues on the normalization and $(x-r)\omega_1$ is a nonzero holomorphic differential there. The extension assertion can be checked on the nodal annulus. In coordinates $uv = t$ a relative dualizing differential has a convergent Laurent expansion $\sum_n a_n(t)u^n\,du/u$. Integration on the vanishing circle is $2\pi i a_0(t)$. Integration across the annulus has the single logarithmic term $a_0(t)\log t$; integration of every nonconstant Laurent monomial and of the complementary fixed arcs gives a holomorphic remainder. Thus the period vector has the form $\exp(sN_H)$ times a holomorphic vector, $s = \log t/(2\pi i)$. Untwisting by $\exp(-sN_H)$ is exactly the nilpotent canonical extension used here. The two independent dualizing limits remain a basis of its Hodge subbundle. Their isotropic pairing, zero away from zero, remains zero at the limit by holomorphic continuation, so that limit is Lagrangian. At zero their expressions in the root coordinates are $s^{-9}dX/Y$ and $s^{-3}X\,dX/Y$; at infinity they are $s^6 dX/Y$ and $s^2 X\,dX/Y$. Thus their divisor lines are $\mathcal{O}(6D_\infty - 9D_0)$ and $\mathcal{O}(2D_\infty - 3D_0)$, proving (F.8). The nonzero residue of ω_1 also shows

$$N_H\ell_\infty \neq 0, \qquad \ell = \mathcal{T}^3 \subset \mathcal{E}, \tag{F.12}$$

where ℓ_∞ denotes its limit at the nodal cusp. Indeed integration on the vanishing circle tends to $2\pi i$ times that residue; the transvection sends the cohomology class of a differential to that period times its vanishing cohomology vector.

□

Theorem 6.4 (The integral exterior-square quotient)**.** [tag `02QT`] *The lower blocks of A, B are precisely the quotient representations of $\bigwedge^2(C_f^{-1}), \bigwedge^2 C_g$ in the basis Q. The integral quotient is the dual of the primitive exterior-square lattice. The natural map from the primitive lattice into that quotient has cokernel $\mathbb{Z}/2$; these two integral representations are not isomorphic. Their rationalizations are isomorphic by that same map.*

Proof. ⟨1⟩1. The two matrices obtained from their two-by-two minors are

$$Q^{-1}\bigwedge^2(C_f^{-1})Q = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix},$$

$$Q^{-1}\bigwedge^2 C_g Q = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Their lower five-by-five blocks are the asserted blocks, entry by entry. Put $L = \bigwedge^2 \mathbb{Z}^4$, let $(\ ,\ )$ be its unimodular wedge pairing, and set $\omega = (1,1,1,1,1,1)^t$. Then $(\omega,\omega) = 2$ and the primitive sublattice $P_{\mathbb{Z}} = \omega^{\perp}$ is the kernel of $(1,-1,1,1,-1,1)$. In the basis with columns

$$\begin{pmatrix} 1 & -1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the inclusion followed by projection $j : P_{\mathbb{Z}} \longrightarrow L/\mathbb{Z}\omega$ has matrix

$$j = \begin{pmatrix} 0 & 1 & 1 & -1 & 1 \\ -1 & 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & -1 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & -1 & 2 \end{pmatrix}, \qquad \det j = 2.$$

Its Smith factors are $(1,1,1,1,2)$. The unimodular pairing on L identifies $L/\mathbb{Z}\omega$ with $P_{\mathbb{Z}}^{\vee}$; under this identification j is the pairing map.

$\langle 1\rangle 2$. We rule out every possible integral intertwiner. Denote the lower blocks above by A_5, B_5. The Krylov matrix $(e_3, A_5e_3, A_5^2e_3, A_5^3e_3, A_5^4e_3)$ has determinant -1, so A_5 is cyclic. Any rational endomorphism commuting with A_5 therefore has the form $u_0I + u_1A_5 + \cdots + u_4A_5^4$. The entries $(1,2),(1,4),(2,4),(3,1)$ of its commutator with B_5 give, on $(u_1,u_2,u_3,u_4)^t$, the coefficient matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \qquad \det = -1.$$

Thus $u_1 = u_2 = u_3 = u_4 = 0$, proving that the simultaneous rational centralizer is exactly $\mathbb{Q}I$. The natural map j above is equivariant and rationally invertible. For any rational equivariant map $f : P_{\mathbb{Q}} \to (L/\mathbb{Z}\omega)_{\mathbb{Q}}$, the composite fj^{-1} lies in this centralizer. Consequently $f = cj$ for some $c \in \mathbb{Q}$. An integral isomorphism would have determinant ± 1, whereas $\det(cj) = 2c^5$. Its 2-adic valuation would give $1 + 5v_2(c) = 0$, impossible because $v_2(c) \in \mathbb{Z}$. This proves the asserted failure of integral isomorphism and retains j itself as the explicit rational isomorphism.

□

Theorem 6.5 (Integral marking of the pencil)**.** [tag 02QY] *The integral H^1 monodromy of the pencil has a companion marking with finite pair (C_f^{-1}, C_g), and cusp product $C_fC_g^{-1}$. Its exterior-square quotient is consequently the integral quotient representation appearing in theorem 6.4.*

Proof. $\langle 1\rangle 1$. Let $L = H^1(C_\lambda, \mathbb{Z})$ with its integral cup pairing, and write T_1, T_2 for the finite generators of theorem 6.3. Put $A = C_f$ and $B = C_g$. The rational intertwiner proved there is a map

$$q : L_{\mathbb{Q}} \longrightarrow \mathbb{Q}^4, \qquad qT_1q^{-1} = A^{-1}, \quad qT_2q^{-1} = B.$$

Its relation to the Levelt presentation is $(h_\infty, h_1, h_0) = (T_1^{-1}, T_1T_2, T_2^{-1})$; the ordered product is one. The known rigidity input and its explicit rational change of basis are recorded in theorem 6.3. The nodal peripheral is $c = (T_1T_2)^{-1}$. Its primitive Picard–Lefschetz class $\delta \in L$ and coefficient one are the geometric node input proved there: the nonseparating vanishing circle has a transverse integral cycle meeting it once, and the

annulus $xy = t$ turns once. Thus T_1T_2 acts by $v \mapsto v + \epsilon\delta\langle\delta, v\rangle$, where $\epsilon \in \{1, -1\}$ records the fixed inverse and pairing conventions.

⟨1⟩2. A B-invariant alternating form has upper entries (u, v, u, u, v, u). Invariance under A, evaluated on the first and fourth basis vectors, gives $v = -u$. The common invariant forms are therefore exactly the multiples of

$$J = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The Pfaffian of J is 1, and direct multiplication gives

$$A^tJA = B^tJB = J, \qquad A^{-1}B - I = de_4^t, \qquad d^tJ = e_4^t.$$

Here e_4 is the fourth standard vector. Write the transported cup form as sJ and $q(\delta) = kd$, with $s, k \in \mathbb{Q}^*$. Comparison with the Picard–Lefschetz formula gives $\epsilon sk^2 = 1$.

⟨1⟩3. The integral classes $\delta_i = T_1^{-i}\delta$, $0 \le i \le 3$, have rational images equal to the columns of kD, where

$$D = (d, Ad, A^2d, A^3d) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \det D = 1.$$

Their pairing is ϵG, with

$$G = D^tJD = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \det G = 1.$$

These four classes are a basis of $L_{\mathbb{Q}}$. For any $v \in L$, write $v = \sum_i x_i\delta_i$ with $x_i \in \mathbb{Q}$. The vector $(\langle\delta_i, v\rangle)_i = \epsilon Gx$ is integral. Since G^{-1} is integral, so is x. Thus the four classes are an integral basis of L.

⟨1⟩4. The required integral marking is the precise map $\mu = k^{-1}q : L \longrightarrow \mathbb{Z}^4$. Indeed $q(L) = kD\mathbb{Z}^4 = k\mathbb{Z}^4$. Its inverse is specified on standard vectors by

$$\mu^{-1}(e_j) = \sum_{i=0}^{3} (D^{-1})_{i+1,j}\delta_i, \qquad D^{-1} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

This supplies both integral arrows and retains the cup form ϵJ. In this marking the finite pair is (A^{-1}, B).

⟨1⟩5. The exact simultaneous companion reversal is

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It satisfies $W^2 = I$, $\det W = 1$, $WAW = A^{-1}$, $WBW = B^{-1}$ and $W^tJW = -J$. These identities give the integral comparison of the two companion conventions, including the pairing sign. For the based cusp used in the statement take $c' = T_1^{-1}cT_1 = (T_2T_1)^{-1}$. Then

$$\mu c\mu^{-1} = B^{-1}A, \qquad \mu c'\mu^{-1} = A(B^{-1}A)A^{-1} = AB^{-1}.$$

The homology matrices are the inverse transposes under the evaluation pairing with L.

⟨1⟩6. Apply $\bigwedge^2 \mu$ to the integral wedge lattice. In the lexicographic basis of theorem 6.4, the invariant vector is $\omega = (1,1,1,1,1,1)^t$. Let Q have first column ω and remaining columns the last five standard vectors, as in that result. The induced quotient is the map

$$[u \wedge v] \longmapsto \mathrm{pr}_{2,\dots,6}\, Q^{-1}(\mu(u) \wedge \mu(v)).$$

Its kernel before passage to the quotient is precisely the invariant line. Its inverse is induced by $(\bigwedge^2 \mu^{-1})Q$ on the last five standard vectors. The finite quotient matrices are the lower five-by-five blocks of

$$Q^{-1}(\bigwedge^2 A^{-1})Q, \qquad Q^{-1}(\bigwedge^2 B)Q.$$

They are exactly the integral quotient matrices of theorem 6.4, and the cusp follows by the same based word $(T_2T_1)^{-1}$. The invariant form, Picard–Lefschetz rank-one map, unimodular orbit matrix, integral inverse, reversal, based peripheral word and exterior-quotient identities are recomputed exactly in `independent_V_markings_06.py`. The integral saturation argument in steps 2–4 supplies the map μ for the geometric lattice.

□

Theorem 6.6 (a global nonpolarized period family)**.** [tag 0208]

The integral rank-six system with cohomology monodromies A, B admits a holomorphic subbundle $\mathcal{F}$ of rank three over U satisfying $\mathcal{F} \cap \overline{\mathcal{F}} = 0$. It extends holomorphically on the finite root charts. At the cusp an integral marking puts its periods in the form

$$Z(t) = \frac{\log t}{2\pi i} I_3 + C(t), \tag{F.13}$$

where C is holomorphic at zero and bounded there.

Proof. ⟨1⟩1. Write the exact sequence given by (F.11) as

$$0 \longrightarrow \mathbb{Q}e \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0. \tag{F.14}$$

Here e is the first coordinate vector. The nilpotent canonical extension at the cusp and the ordinary extensions on finite root charts preserve this sequence. This follows directly by using the frames $\exp(-sN)$: each residue has only eigenvalue zero, and the same exponential construction respects the inclusion and the quotient. At finite roots the statement is ordinary exactness of bundles with trivialized local monodromy. Denote these extended bundles by $\mathcal{H}_D, \mathcal{V}_D, \mathcal{W}_D$.

⟨1⟩2. Let $\omega \in \bigwedge^2 \mathcal{H}_D$ be the real, flat nondegenerate bivector inverse to the polarization, and let $p : \bigwedge^2 \mathcal{H}_D \to \mathcal{W}_D$ be primitive projection. Define the holomorphic rank-three bundle

$$\mathcal{P} = p(\ell \wedge \mathcal{H}_D). \tag{F.15}$$

Projection is injective on $\ell \wedge \mathcal{H}_D$: every nonzero bivector there is decomposable whereas ω is not. This argument applies also at the cusp. The polarization gives $\mathcal{H}_D/\mathcal{E} \simeq \mathcal{E}^*$, including there, since $\mathcal{E}$ is Lagrangian. Hence

$$0 \longrightarrow \mathcal{T}^4 \longrightarrow \mathcal{P} \longrightarrow \mathcal{O} \oplus \mathcal{T}^2 \longrightarrow 0. \tag{F.16}$$

After dualizing, the end terms are $\mathcal{O} \oplus \mathcal{T}^{-2}$ and $\mathcal{T}^{-4}$.

⟨1⟩3. We compute the obstruction to lifting $\mathcal{P}$ in (F.14). For the coarse projection $\rho : \mathcal{B}' \to \mathbb{P}^1$, the elementary root-chart calculation of invariant sections gives

$$\rho_* \mathcal{T}^k = \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{2k}{5} \right\rfloor + \left\lfloor \frac{-3k}{8} \right\rfloor\right). \tag{F.17}$$

Higher direct images vanish, because taking invariants under a finite group in characteristic zero is exact. For $k = 0, -2, -4$ the displayed degrees are respectively $0, -1, -1$. Their first cohomology vanishes. The dual of (F.16) therefore gives $H^1(\mathcal{B}', \mathcal{P}^*) = 0$. The pullback of (F.14) to $\mathcal{P}$ consequently splits as holomorphic bundles: choose a holomorphic lift $\sigma : \mathcal{P} \to \mathcal{V}_D$ of its inclusion in $\mathcal{W}_D$. To specify the choice concretely, take a finite root-coordinate cover with local bundle lifts σ_i and write their differences as $\xi_{ij}e$, where ξ_{ij} is a Cech cocycle in $\mathcal{P}^*$. Choose any solution $b_i - b_j = \xi_{ij}$ of this explicitly vanishing cohomology class and glue the maps $\sigma_i - b_i e$. Every such solution is allowed in the construction that follows.

⟨1⟩4. There is also a canonical holomorphic functional $\alpha : \mathcal{P} \to \mathcal{O}$. For $v \in \mathcal{P}$, its unique preimage in $\ell \wedge \mathcal{H}_D$ is

$$j(v) = v + \alpha(v)\omega \tag{F.18}$$

under the primitive decomposition. At a smooth point, the positive Hodge metric implies that ℓ and $\bar{\ell}$ form a nondegenerate symplectic pair. Their common wedge line is $\ell \wedge \bar{\ell}$. The bivector ω does not belong to $\ell \wedge \mathcal{H} + \bar{\ell} \wedge \mathcal{H}$, since its component on the two-dimensional symplectic orthogonal complement is nonzero. Projection p therefore preserves the intersection, and

$$\mathcal{P} \cap \bar{\mathcal{P}} = p(\ell \wedge \bar{\ell}). \tag{F.19}$$

This is the complexification of a real line. If u spans ℓ, then $iu \wedge \bar{u}$ is real and its coefficient on ω is nonzero, by positivity. Thus α is real and nonzero on that real intersection line. There is a unique smooth real section n of it such that $\alpha(n) = 1$.

⟨1⟩5. Since the quotient of $\sigma(n)$ is real, there is a unique real smooth function h on U such that

$$\sigma(n) - \overline{\sigma(n)} = 2ihe. \tag{F.20}$$

For $c \in \mathbb{R}$ set

$$\mathcal{F}_c = (\sigma + ic\alpha e)(\mathcal{P}). \tag{F.21}$$

This is a holomorphic rank-three subbundle projecting isomorphically to $\mathcal{P}$. An intersection with its complex conjugate must project to (F.19). On its real generator the two lifts differ by $2i(h + c)e$. It follows, including arbitrary complex multiples of that generator, that

$$\mathcal{F}_c \cap \overline{\mathcal{F}_c} = 0 \quad \Longleftrightarrow \quad h + c \neq 0. \tag{F.22}$$

⟨1⟩6. We verify the required cusp property before choosing c. In a complex symplectic basis a, b, d_1, d_2 of the limiting $\mathcal{H}_D$, relation (F.12) permits us to arrange $N_H b = a$, $N_H a = N_H d_1 = N_H d_2 = 0$ and $\ell_\infty = \mathbb{C}b$. Then

$$\mathcal{P}_\infty = \langle k = p(b \wedge a),\ b \wedge d_1,\ b \wedge d_2 \rangle, \quad N_W(b \wedge d_i) = a \wedge d_i, \quad N_W k = 0. \tag{F.23}$$

The line generated by k is $\ker N_W / \operatorname{im} N_W$; all other kernel directions come from the displayed image. The exact matrices give $N_V^2 = 0$, $\operatorname{rank} N_V = 3$ and $\operatorname{rank} N_W = 2$. Every lift of an image vector $a \wedge d_i$ is killed by N_V, because that vector has a lift in $\operatorname{im} N_V$ and $N_V e = 0$. Rank three now forces N_V to take any lift of k to a nonzero multiple of e. Therefore $N_V : \mathcal{F}_{c,\infty} \to \operatorname{im} N_V$ is an isomorphism for every c. In particular its domain is a complement to $\ker N_V$.

⟨1⟩7. The primitive index-one cusp has a real integral basis a_i, b_i with $N_V b_i = a_i$, by Proposition 4.4, also for the dual or inverse convention. In this basis the untwisted subbundle near zero is the graph $b + C_c(t)a$, with C_c holomorphic and bounded. Restoring the flat frame by $\exp(sN_V)$, $s = \log t/(2\pi i)$, gives (F.13). Thus $\operatorname{Im} Z = \operatorname{Im} s\, I_3 + \operatorname{Im} C_c$ is nonsingular on a sufficiently small punctured disc. Apply this first with $c = 0$. Equation (F.22) shows that h is nonzero and has constant sign on that connected punctured disc.

⟨1⟩8. At the two finite root points, all ingredients in (F.19)–(F.20) are smooth and $\alpha(n) = 1$ remains valid, so h extends and is bounded. On the compact complement of the cusp disc it has a finite bound K. Choose c with the same sign as h near the cusp by the exact rule $c = \operatorname{sign}(h(t_0))(K + 2)$, where t_0 is any point of that punctured disc and K is the supremum of $|h|$ on the compact complement. Then $h + c$ is nonzero both on that disc and on its complement. Equations (F.21)–(F.22) prove global real transversality, and the preceding cusp calculation proves the bounded correction for this same c.

⟨1⟩9. Finally, the fourth power of the order-eight action on $\mathcal{H}$ is $-I$. It acts trivially on $\mathcal{W}, \mathcal{P}, \alpha$ and $\mathcal{V}$. Hence these bundles, the lift and the constructed periods descend from $\mathcal{B}'$ to the orbifold $\mathcal{B}$ with orders five and four. Indeed any equivariant bundle map between bundles with trivial action of the extra order-two stabilizer descends on its root chart. This gives exactly the finite orders required for the rank-six datum.

□

Theorem 6.7 (The exact parameter domain of the lifted family)**.** [tag 02QZ] *With the rank-three bundle P of theorem 6.2, the rank-six system admits a holomorphic lifted family $F_c = (\sigma + ic\alpha e)(P)$ on the punctured base. For a specified splitting σ there is a real analytic function h with finite attained minimum $h_{\min}$, and the exact domain of real parameters giving $F_c \cap \overline{F_c} = 0$ on the completed base, including both finite root points and excluding only the cusp before its toroidal filling, is $c > -h_{\min}$. At the cusp its periods have the form $Z(q) = (\log q/(2\pi i))I_3 + C(q)$ with C holomorphic and bounded. The construction extends to the two finite root charts and descends to orders $5, 4$ on the rank-six system.*

Proof. ⟨1⟩1. Work first on the root orbifold with orders $5, 8$, leaving the nodal point as a logarithmic point. The symplectic pairing and the Hodge subbundle give an exact sequence $0 \to \mathcal{H} \to \mathcal{H}_{\mathrm{dR}} \to \mathcal{H}^\vee \to 0$. This remains exact at the node by the residue pairing in its plumbing chart. With $\ell_0 = T^3$ the bundle P fits into

$$0 \longrightarrow T^4 \longrightarrow P \longrightarrow \mathcal{O} \oplus T^2 \longrightarrow 0.$$

Contraction in $\ell_0 \wedge \mathcal{H}_{\mathrm{dR}}$ defines $\alpha : P \to \mathcal{O}$, using the unique lift given by theorem 6.2; it is surjective.

⟨1⟩2. For $a = 2, 4, -2, -4$ the invariant coarse direct image of T^a has degree -1, by taking the floors of the two coefficients of its rational divisor. Finite-stabilizer invariants are exact in characteristic zero by averaging. Consequently $H^0(T^a) = H^1(T^a) = 0$ for these exponents. The two-chart Laurent-series computation for $\mathcal{O}(-1)$ proves this vanishing directly. The extension therefore splits, and

$$P \cong T^4 \oplus T^2 \oplus \mathcal{O}, \qquad H^1(P^\vee) = 0, \quad H^0(P^\vee) = \mathbb{C}\alpha, \quad H^0(P) = \mathbb{C}g,$$

where $\alpha(g) = 1$.

⟨1⟩3. Let V be the logarithmic extension of the rank-six flat bundle. At the cusp use its nilpotent exponential frame, and on each finite root chart use the trivial flat extension with the deck action. These charts give the exact bundle sequence $0 \to \mathcal{O}e \to V \to W \to 0$. Its pullback to P splits because $H^1(P^\vee) = 0$. All holomorphic splittings are $\sigma + z\alpha e$, $z \in \mathbb{C}$. To remove this scalar choice we require the e-coordinate of $\sigma(g)$ to be zero in the fixed flat frame at $z = 1/2$.

⟨1⟩4. Choose a local generator u of ℓ_0 and put $r = i\pi(u \wedge \bar{u}) \in W_{\mathbb{R}}$, $A = \alpha(r) > 0$; the sign is that of the polarization. The symplectic orthogonal decomposition into $\langle u, \bar{u}\rangle$ and its two-dimensional complement proves $P \cap \bar{P} = \mathbb{C}r$. Indeed $u \wedge v - \bar{u} \wedge w = t\omega$ has zero

component in the exterior square of that complement on the left and a nonzero such component on the right unless $t = 0$. With $t = 0$ its intersection is the line generated by $u \wedge \bar{u}$. Define h by the exact equality

$$\sigma(r) - \overline{\sigma(r)} = 2iAhe.$$

Both r and A scale by $|a|^2$ when u scales by a, so this is well defined. Comparing the unique lifts of a multiple of r proves

$$F_c \cap \overline{F_c} = 0 \quad \Longleftrightarrow \quad h + c \neq 0.$$

⟨1⟩5. The sign and growth at the cusp follow from the extension matrices. Put $D = C_f C_g^{-1} - I$ and $N_V = (BA)^{-1} - I$. Only the zeroth column of D is nonzero, and it is $(0, 1, 1, 1)^t$; the top row of N_V is $(0, 0, 0, -1, 1, -1)$. For arbitrary $a, b \in \mathbb{C}^4$, two-by-two minor expansion gives

$$(N_V)_{0,*}(0, \pi(a \wedge Db)) = \operatorname{contr}(a \wedge Db) = a_0 b_0, \qquad N_W \pi(a \wedge Db) = 0.$$

The full scalar and quotient-kernel identities for arbitrary symbolic entries of a, b are checked directly from the displayed companion and rank-six matrices in `independent_V_markings_06.py`. Write $u = \exp(sD)a(q)$, $s = \log q/(2\pi i)$ and $y = \operatorname{Im} s$. The vector a is holomorphic, and $a_0(0) \neq 0$ because dx/y has nonzero nodal residue. In the logarithmic frame we obtain

$$\exp(-sN_W)r = i\pi(a \wedge \bar{a}) + 2y\pi(a \wedge D\bar{a}), \quad A = i \operatorname{contr}(a \wedge \bar{a}) + 2y|a_0|^2.$$

The splitting is bounded in that frame. On returning by $\exp(sN_V)$, the only quadratic contribution to the imaginary e-coordinate is $2|a_0|^2 y^2$. All other terms are $O(y)$, uniformly for $\operatorname{Re} s$ in a fixed unit interval. Dividing by $A = 2|a_0|^2 y + O(1)$ proves

$$h(q) = y + O(1) = -\frac{\log|q|}{2\pi} + O(1).$$

The function extends continuously over both finite root charts and tends to $+\infty$ at the cusp. Compactness away from that cusp gives an attained minimum. Connectedness and the intermediate value theorem give $h(\widehat{B} \setminus \{\text{cusp}\}) = [h_{\min}, \infty)$, where both finite root points are included. Thus the domain is precisely the displayed half-line. Replacing σ by $\sigma + ib\alpha e$ replaces h by $h + b$ and sends F_c to F_{c+b}, proving the stated comparison between choices.

⟨1⟩6. At $q = 0$ the map $N_V : F_{\log} \to \ker N_V$ is an isomorphism. On the quotient its rank is two and its kernel is generated by $\pi(a \wedge Da)$; the displayed scalar identity sends a lift of this generator to $a_0^2 e \neq 0$. In the integral Jordan marking, the logarithmic three-plane is therefore a graph of a holomorphic matrix $C(q)$; applying $\exp(sN_V)$ adds sI_3. Evaluation on the homology lattice gives the period matrix $(I_3, Z(q))$. The fourth power of the order-eight deck transformation acts trivially on V and P, so the graph descends to the order-four chart. The other chart has order five. The affine translations are holomorphic translations on the resulting tori, and their linear parts preserve this graph by construction.

□

Theorem 6.8 (The Kuhn filling at the cusp)**.** [tag `02R0`] *The periodic Kuhn fan, in the marking S, gives a smooth proper holomorphic filling of the periods $Z(q) = sI_3 + C(q)$ for a sufficiently small disk. Its central fibre is reduced with normal crossings. Modulo the index-one period lattice it has respectively $1, 7, 12, 6$ strata of dimensions $3, 2, 1, 0$, each isomorphic to the corresponding complex torus $(\mathbb{C}^*)^d$, and Euler number 6. An index-d period sublattice gives d times these stratum counts and Euler number $6d$.*

Proof. ⟨1⟩1. Apply (Yang, 2026, Theorem 8.7 and Corollary 8.8, arXiv v1, pp. 57–59) with $n = 3$, $t = q$, $D(q) = C(q)$, $B = I_3$, and the displayed periodic Kuhn triangulation. These substitutions give exactly the action

$$(z, q) \longmapsto (q^{\lambda} \exp(2\pi i C(q)\lambda)z, q)$$

and the period order $[I_3 \mid sI_3 + C(q)]$, where $q = \exp(2\pi i s)$. Thus the identity on the dense torus, with the same integral monomial charts, identifies our quotient with the cited quotient. The source theorem supplies its smoothness, Hausdorffness, properness, and the reduced local equations $q = x_0 \cdots x_k$. Flatness follows since this nonconstant function is a nonzero divisor in each smooth chart.

⟨1⟩2. For an index-d period sublattice $\Gamma = B\mathbb{Z}^3$, the same substitutions give the source's quotient map induced by $\Gamma \subset \mathbb{Z}^3$. It has degree d, and Corollary 8.8 gives $\chi(W) = 3!d = 6d$ for this marked Kuhn fan.

□

Theorem 6.9 (compact holomorphic filling)**.** [tag 0209]

The smooth eight-manifold of Theorem 5.5 carries an integrable complex structure. There is a proper holomorphic map to $\mathbb{P}^1$ whose regular fibres are complex three-tori, whose two finite reduced fibres are the free affine quotients of orders five and four specified in Theorem 5.1, and whose remaining fibre is the index-one Kuhn degeneration.

Proof. ⟨1⟩1. We first fix the dual convention. The preceding construction has cohomology monodromies A, B; Theorem 5.1 uses homology monodromies A^t, B^t. All three special values are real. Choose a positive real basepoint and paths along the positive real circle to infinity and to zero. Complex conjugation fixes these paths and reverses the respective peripheral meridians. Pull back the preceding period family by conjugation of the base and conjugate its complex fibres. Its Hodge subbundle is holomorphic, with local expression $\overline{\mathcal{F}(\bar{\lambda})}$, and its cohomology monodromies are A^{-1}, B^{-1}. Its homology monodromies are therefore exactly A^t, B^t. Put $M_1 = A^t$, $M_2 = B^t$, $M_o = M_2M_1$, $M_0 = (M_1M_2)^{-1}$ and $J = \mathrm{diag}(I_3, -I_3)$. The based peripheral word is $\kappa(g_0) = g_1^{-1}g_0^{-1}g_1$, so $M_0 = M_1M_o^{-1}M_1^{-1}$. For the target marking S, direct integral multiplication gives

$$S^{-1}M_0S = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}, \qquad S_o = M_1^{-1}SJ, \qquad S_o^{-1}M_oS_o = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}.$$

Any two markings of this positive standard form differ by $G = \left(\begin{smallmatrix} P & Q \\ 0 & P \end{smallmatrix}\right)$, where $P \in \mathrm{GL}_3(\mathbb{Z})$ and Q is integral: this follows by multiplying G with the displayed unipotent matrix. The period change is

$$(I_3, Z)G = (P, Q + ZP), \qquad P^{-1}(P, Q + ZP) = (I_3, sI_3 + P^{-1}Q + P^{-1}CP).$$

Thus the bounded holomorphic logarithmic form holds in S_o. In the real-normalized cusp coordinate, $q_o = \bar{q}$ and $s_o = -\bar{s}$. Base and fibre conjugation therefore give $(I_3, -sI_3 + C^{\kappa}(q))$, with $C^{\kappa}(q) = \overline{C(\bar{q})}$. The period-cycle change J gives the target marking $M_1S_oJ = S$ and the positive periods $(I_3, sI_3 - C^{\kappa}(q))$.

⟨1⟩2. For a holomorphic Hodge subbundle $\mathcal{F}$ with $\mathcal{F} \cap \bar{\mathcal{F}} = 0$, restriction of integral homology functionals embeds the homology lattice discretely in $\mathcal{F}^*$. Real transversality makes its real span the whole real vector space. The quotient $\mathcal{F}^*/\mathcal{V}_{\mathbb{Z}}^*$ is consequently a holomorphic family of compact complex three-tori over U, with its zero section. This also proves local properness, by a smoothly varying fundamental parallelepiped.

⟨1⟩3. On the root disc at the finite point of order m_i, the family extends smoothly and its deck map has linear part M_i. The invariant vector v_i of Theorem 5.1 gives a holomorphic torsion section v_i/m_i. Use the diagonal affine deck action

$$(z, s) \longmapsto (M_i z + v_i/m_i, \zeta_{m_i} s). \tag{F.24}$$

Here $M_i z$ denotes the induced complex-linear map $\mathcal{F}_s^* \to \mathcal{F}_{\zeta_{m_i} s}^*$ from the marked lattice action, not multiplication of a complex three-vector by a six-by-six matrix. Likewise v_i/m_i means its evaluation as a point of the target torus. It has order m_i, since the m_ith translation is integral. Every nonidentity power is fixed-point-free on the central torus: the invariant integral covector ℓ takes v_i to 1 or -1, so a fixed point would require the noninteger $\pm j/m_i$ to vanish in $\mathbb{R}/\mathbb{Z}$. Away from the central fibre the base rotation is already free. The quotient is therefore smooth and proper over the coarse disc, with precisely the required multiple fibre.

⟨1⟩4. Its marked punctured family is holomorphically identified with the original torus family by logarithmic translation. On the root cover use $r(s) = (\log s/(2\pi i))v_i$. Then $r(\zeta_{m_i} s) - M_i r(s) = v_i/m_i$, modulo the integral lattice; the change of branch of $\log s$ is itself an integral vector. Thus translation by r conjugates the linear deck action to (F.24). This fixes the affine clutching, including its sign.

⟨1⟩5. At the cusp use the integral marking of (F.13), the fan (E.4) and the action (E.5) with $C_3 = C$. Its maximal charts are $\mathbb{C}^4$ with map $t = z_0 z_1 z_2 z_3$, and its transition maps are the toric changes of coordinates multiplied by the holomorphic units $\exp(2\pi i C(t)\lambda)$. The multiplicative comparison is

$$q = \bar{q}_o, \qquad x = (\bar{x}_o)^{-1}, \qquad \nu = -\nu_o, \qquad C_{\mathrm{new}}(q) = -C^{\kappa}(q).$$

It carries the old period action to $x \mapsto \exp(2\pi i C_{\mathrm{new}}(q)\nu) q^{\nu} x$. On cocharacters it is $\Phi = \mathrm{diag}(-I_3, 1)$ together with conjugation. The Kuhn simplex at v with ordering σ is carried to the simplex at $-v - (1,1,1)$ with reversed ordering. If N_o and N are their height-one generator matrices and R reverses four columns, then $N = \Phi N_o R$ and $N^{-1}\Phi = R N_o^{-1}$. Consequently the full affine-chart map is $z_j = \overline{z_{o,3-j}}$. These character identities also give equality on every monomial overlap; the product of the four coordinates is the conjugate base coordinate. The already displayed period-action equality makes the map descend to the analytic period quotient. Hence the fan and marking in this comparison are precisely the prescribed ones.

The proof of Theorem V.2.1 used only boundedness of C, integrality of the standard cusp, and unimodularity of that fan. It therefore proves smoothness, proper discontinuity and properness for this model without a symmetry condition on C. Glue by the displayed integral marking and use no additional cusp translation.

⟨1⟩6. The three punctured overlap maps are holomorphic isomorphisms with the same marked torus family, so their cocycle condition is literal. Proper local models and a compact complementary base give a Hausdorff compact total space. Every chart is smooth. The affine vectors, cusp fan and clutching are exactly those in Theorems 5.1–5.5. Varying their bounded corrections and varying the fibre complex structures along a smooth path does not change these marked smooth fillings: on the finite root models the underlying real torus family is the fixed local-system bundle; on the cusp use the stratum-preserving parameter vector fields and uniform properness estimate of Theorem V.2.2. Thus the resulting underlying smooth manifold is the one whose integral topology was computed there.

□

7 The oriented diffeomorphism type

Theorem 7.1 (Characteristic numbers and the middle lattice)**.** [tag 02QM] *With the complex orientation the fourfold has*

$$\sigma(M_c) = 2, \qquad H^4(M_c;\mathbb{Z}) \cong I_{3,1}, \qquad w_2(M_c) = 0,$$

and there is a diagonal integral basis in which $p_1/2 = (1,1,1,1)$*. Its Chern numbers are* $c_4 = 6$*,* $c_2^2 = 2$*, and* $c_1^4 = c_1^2 c_2 = c_1 c_3 = 0$*; its Pontryagin numbers are* $p_1^2 = 8$*,* $p_2 = 14$*.*

Proof. ⟨1⟩1. Theorem 5.8 implies $c_1 = c_3 = 0$ as integral cohomology classes; $c_4[M_c] = \chi(M_c) = 6$. The Chern-root identities give $p_1 = -2c_2$, $p_2 = c_2^2 + 2c_4$. The symbol-index interpretation in §3, with the polynomial (Esole and Kang, 2018, equation (1.2), p. 3), gives integrality of the Todd genus. Expanding $\prod x_j/(1-e^{-x_j})$ to degree four gives

$$\frac{3c_2^2 - 6}{720} \in \mathbb{Z}, \quad \text{hence} \quad c_2^2 = 2 + 240k \quad (k \in \mathbb{Z}).$$

The signature polynomial (Esole and Kang, 2018, equation (1.4), p. 4), applied to this closed oriented smooth eight-manifold gives

$$\sigma = \frac{7p_2 - p_1^2}{45} = \frac{3c_2^2 + 84}{45} = 2 + 16k.$$

The rank-four middle form has $|\sigma| \le 4$, so $k = 0$. This proves the displayed characteristic numbers.

⟨1⟩2. Duality makes the middle form unimodular. Its signature is two and its positive and negative indices are three and one. The rank-at-most-four integral classification (Peters and Sterk, 2024, Theorem 2.2.1, p. 61) applied to these positive and negative indices gives the odd form $I_{3,1}$. Since $H^2(M_c;\mathbb{Z}/2) = 0$, its w_2 is zero. For the complex tangent bundle, $w_4 = c_2 \bmod 2 = (p_1/2) \bmod 2$. The Wu identity in degree four, with $w_1 = w_2 = 0$, says that w_4 is characteristic for the middle pairing. Thus the vector $p_1/2$ is characteristic and has square 2.

⟨1⟩3. We give the required integral isometry of this pair explicitly as a terminating procedure. In a diagonal basis a characteristic vector has four odd coordinates. Change their signs to write it as (a,b,c,d) with all positive and $a^2+b^2+c^2 = d^2+2$. For $d > 1$,

$$d < a + b + c < 3d.$$

The first inequality follows by squaring and using the positive cross terms; the second follows from Cauchy–Schwarz and $d > 1$. Reflection in the integral norm-two vector $(1,1,1,1)$ sends this vector to itself minus $(a+b+c-d)(1,1,1,1)$. Its fourth coordinate is $2d-a-b-c$, whose absolute value is strictly smaller than d. The reflection preserves the integral lattice and characteristic parity. Repeating, with sign changes at each step, terminates at $d = 1$, when the norm equation forces $a = b = c = 1$. This supplies an integral isometry taking $p_1/2$ to $(1,1,1,1)$, and proves the claimed pair, not just its square. □

Theorem 7.2 (a complex structure on the stated connected sum)**.** [tag 020C]

The compact complex fourfold of Theorem 6.9 is three-connected, has Euler number six, and is orientation-preservingly diffeomorphic to

$$3\mathbb{HP}^2 \# (-\mathbb{HP}^2). \tag{F.25}$$

Its integral middle lattice is $I_{3,1}$ *and, in a diagonal integral basis,* $p_1/2 = (1,1,1,1)$*.*

Proof. ⟨1⟩1. The integral calculation in theorem 5.5 gives $\pi_1(M) = 0$, $H_2(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0$, and torsion-free homology of ranks $1, 4, 1$ in degrees $0, 4, 8$. Hurewicz applied successively gives $\pi_2(M) = \pi_3(M) = 0$, and $e(M) = 6$. The complex orientation is the one in theorem 6.9. The characteristic calculations of theorem 7.1 give

$$Q = I_{3,1}, \qquad c_2^2 = 2, \qquad p := p_1/2 = -c_2, \qquad p^2 = 2.$$

Here $c_1 = c_3 = 0$ because the corresponding integral groups vanish. The spin Wu identity makes p characteristic for Q. Evaluating its characteristic congruence on each diagonal basis vector shows that all four coordinates of p are odd.

⟨1⟩2. Put $v = (1, 1, 1, 1)^t$. Since $v^t Q v = 2$, the reflection $R = I - v(v^t Q)$ is an integral isometry of Q with $R^2 = I$. By the descent in the proof of theorem 7.1, a product U of such reflections and of coordinate sign changes satisfies $Up = v$.

⟨1⟩3. We compute the complete normal clutching datum in fixed coordinates. Orient $\mathbb{H}$ by $(1, i, j, k)$ and use

$$\mathrm{Spin}(4) = S^3_L \times S^3_R, \qquad (u, w)h = uhw^{-1}.$$

The covering induces an isomorphism on π_3. Let (a, b) be the degrees of the two clutching maps. Evaluation on 1 gives uw^{-1}, of degree $a - b$, so $e = a - b$, with the sphere orientation fixed by the left degree-one Euler generator. The left action is complex linear for right multiplication by i, whose real complex basis $(1, i, j, -k)$ has the opposite quaternion orientation. Its degree-one generator therefore has $c_2 = -1$, and hence $p_1 = 2$. The right inverse action is complex linear for left multiplication by i, with real basis $(1, i, j, k)$. Its Euler number and c_2 are both -1, so its p_1 is again 2. The determinant-one complex actions have $c_1 = 0$. Consequently the exact homomorphism and its inverse on its image are

$$(a, b) \longmapsto (e, p_1/2) = (a - b, a + b), \qquad (a, b) = \left(\frac{e + p_1/2}{2}, \frac{p_1/2 - e}{2}\right).$$

The image consists of pairs of the same parity, and the map is injective. The tangent clutching class of S^4 has $(e, p_1) = (2, 0)$, hence coordinates $(1, -1)$. The homotopy sequence of $SO(4) \to SO(5) \to S^4$ identifies its span with the kernel of stabilization and gives surjectivity on π_3. Subsequent stabilizations are isomorphisms there, since $\pi_3(S^n) = \pi_4(S^n) = 0$ for $n \geq 5$. The stable coordinate is therefore $a + b = p_1/2$, with the left generator sent to 1.

⟨1⟩4. For $x \in H_4(M;\mathbb{Z})$, use the embedded sphere representative in the setup of (Conway et al., 2023, §3.3, pp. 11–12). Orient its normal bundle ν_x by the ambient orientation. Self-intersection and the tangent splitting give

$$e(\nu_x) = x \cdot x, \qquad \langle p_1(\nu_x), [S^4] \rangle = \langle p_1(TM), x \rangle,$$

because TS^4 is stably trivial. Thus its two clutching coordinates are exactly

$$\left(\frac{x \cdot x + \langle p, x \rangle}{2}, \frac{\langle p, x \rangle - x \cdot x}{2}\right), \qquad S\alpha_M(x) = \langle p, x \rangle.$$

Their integrality is the characteristic congruence in step 1. This is the stable normal homomorphism of (Conway et al., 2023, Definition 3.6, p. 12) with its generator. For an oriented middle-cohomology isometry ψ from M to a comparison manifold N, the corresponding homology map is

$$\Phi = \mathrm{PD}_N \circ \psi \circ \mathrm{PD}_M^{-1}.$$

When $\psi(p_M) = p_N$, the cup-product evaluation identity gives $\langle p_N, \Phi(x) \rangle = \langle p_M, x \rangle$. The map Φ preserves the intersection pairing and both displayed normal coordinates, so it intertwines the complete extended forms.

⟨1⟩5. Take $N = 3\mathbb{HP}^2\#(-\mathbb{HP}^2)$. On each summand choose the generator u with $p_1 = 2u$, as in (Mandal, 2026, Lemma 2.1, p. 5). Reversing orientation changes the evaluation of u^2 and leaves the tangent Pontryagin class unchanged. The collapse maps of the oriented connected sum give

$$H^4(N;\mathbb{Z}) = \bigoplus_{i=1}^{4} \mathbb{Z}u_i, \qquad Q_N = I_{3,1}, \qquad p_N = u_1 + u_2 + u_3 + u_4;$$

the tangent formula is (Mandal, 2026, Theorem 2.4, p. 7). Van Kampen, the connected-sum homology sequence and Hurewicz show that N is three-connected. The isometry U of step 2 is an isometry $\psi : H^4(M;\mathbb{Z}) \to H^4(N;\mathbb{Z})$ carrying p_M to p_N. Step 4 supplies the isomorphism of their smooth extended forms. Wall's classification in (Conway et al., 2023, Theorem 3.7(1), pp. 13–14) now gives an orientation-preserving diffeomorphism $M \cong N\#\Sigma$ for an oriented homotopy eight-sphere Σ.

⟨1⟩6. Kramer–Stolz's (Kramer and Stolz, 2007, Theorem A, p. 178) applies to $\mathbb{HP}^2$: it is smooth, simply connected, has dimension eight, and has total integral homology $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. It gives $\mathbb{HP}^2\#\Sigma \cong \mathbb{HP}^2$. This diffeomorphism preserves orientation: its map on H^4 sends a generator to that generator or its negative, whose square still evaluates to 1, forcing its degree to be $+1$. The oriented associativity and permutation maps for connected sum place Σ beside one positive summand. Choose the punctured balls compatibly with that diffeomorphism and use the identity on the remaining punctured summands. These maps give

$$\begin{aligned} N\#\Sigma &\cong (\mathbb{HP}^2\#\Sigma)\#\mathbb{HP}^2\#\mathbb{HP}^2\#(-\mathbb{HP}^2) \\ &\cong N. \end{aligned}$$

Composing with step 5 supplies the required oriented smooth diffeomorphism. Transporting the integrable structure of theorem 6.9 along it gives the asserted complex structure.

□

8 The normalized connection and the exact parameter domain

Theorem 8.1 (The algebraic connection)**.** [tag 02QO] *The rank-six flat extension, as a complex flat bundle with its invariant line, has the following explicit algebraic connection model. Set $\omega_k = x^k dx/y$, $0 \le k \le 3$, and use the quotient frame*

$$(w_2, w_1, w_0, w_{-1}, w_{-2}) = (\omega_{01}, \omega_{02}, \omega_{03}, \omega_{13}, \omega_{23}), \qquad \omega_{12} = -3\omega_{03}.$$

These are meromorphic frames of $(L^2, L, \mathcal{O}, L^{-1}, L^{-2})$, where $L = T^2$ on the root orbifold of orders $5, 4$. If a frame is written as a row of vectors, its derivative is right multiplication by

$$\Gamma_6 = \begin{pmatrix} 0 & b \\ 0 & \Gamma_W \end{pmatrix}, \qquad b = \frac{1}{\lambda - \lambda_*}\left(-\frac{25}{6\lambda^2}, -\frac{125}{48\lambda}, 0, 1, 0\right), \tag{1.15.1}$$

where

$$\Gamma_W = \frac{1}{3125\lambda + 256}\begin{pmatrix} -8(625\lambda+48)/\lambda & -8 & 10\lambda & 75\lambda^2/2 & 0 \\ -125/(2\lambda) & -4(625\lambda+48)/\lambda & 8 & 0 & 75\lambda^2/2 \\ 300/\lambda^2 & -375/\lambda & 0 & -48 & 60\lambda \\ 120/\lambda^3 & 0 & 125/(2\lambda) & 4(625\lambda+48)/\lambda & -8 \\ 0 & 120/\lambda^3 & 50/\lambda^2 & -125/(2\lambda) & 8(625\lambda+48)/\lambda \end{pmatrix}. \tag{1.15.2}$$

This model specifies a canonical algebraic splitting, denoted σ_b. The compatible contraction used for the real parameter is $\alpha = (625\pi i/128)\alpha_{\mathrm{alg}}$, where $\alpha_{\mathrm{alg}}(w_0) = 4/3$ and it vanishes on w_1, w_2.

Proof. ⟨1⟩1. Reduction modulo exact differentials is completely explicit. To differentiate ω_k, solve the polynomial identity

$$-\tfrac{1}{2}x^k\partial_\lambda f = f\sum_{j=0}^{3} G_{kj}x^j + R_k' f - \tfrac{1}{2}R_k f', \qquad \deg R_k \le 4,$$

where $f = x^5 + \lambda^3 x + \lambda^4$. The last two terms are the numerator of $d(R_k/y)$. Matching the nine coefficients gives the four-by-four Gauss–Manin matrix exactly. At the branch point at infinity, a local coordinate $z = x^{-1/2}$ gives $\omega_k = -2z^{2-2k}(1 + O(z^8))\,dz$. Residues of a primitive times a differential give the alternating pairings $J_{03} = 4/3$, $J_{12} = 4$, with their negatives in reverse order and all other entries zero. This form is nondegenerate. The invariant bivector is therefore $3\omega_{03} + \omega_{12}$. Taking the exterior-square connection and this quotient gives (1.15.2). The polynomial reduction, all exact primitives, the horizontal polarization relation and all twenty-five quotient entries are reconstructed in `independent_V_connection_spectra_06.py` and its JSON output.

⟨1⟩2. The root-chart powers computed from $x^k dx/y$ are u^{6k-9} at zero and u^{6-4k} at infinity. Their nondegenerate pairing shows that the four resulting regular frames form the full de Rham bundle on those charts. At the node the computed connection has a simple pole and square-zero residue; hence these are the nilpotent logarithmic extension frames there. Thus the four line bundles are T^3, T, T^{-1}, T^{-3}, and their quotient exterior-square lines are the stated powers of L.

⟨1⟩3. All $H^1(L^j)$ for $j = 0, \pm1, \pm2$ vanish by the coarse-degree calculation already used in theorem 6.7. Consequently every holomorphic extension of this W by $\mathcal{O}$ splits as a bundle. In such a splitting its flat connection is an upper block extension by a row b. Regularity at the finite root charts and a logarithmic pole at the cusp require, entry by entry,

$$b = \frac{1}{\lambda - \lambda_*}(a\lambda^{-2}, b'\lambda^{-1}, 0, c, d\lambda).$$

For example the L^j entry, after $\lambda = u^4$, has exponent $3j + 3 + 4p$ for a term $\lambda^p/(\lambda - \lambda_*)$; the infinity chart requires $-4j - 1 - 5p \ge 0$. These two inequalities leave exactly the four displayed terms. The equation that the full cusp residue have square zero is $b_{\mathrm{res}}(\Gamma_W)_{\mathrm{res}} = 0$. Its solution space has basis

$$(-25/6, -125/48, 1, 0), \qquad (5/3, -25/3, 0, 1).$$

The holomorphic splitting changes are the constant multiples of the row $r_0 = (0, 0, 1, 0, 0)$ selecting w_0. Its connection coboundary is the third row $r_0\Gamma_W$, whose parameter vector is $(-48b_1 + 60b_2)/3125$. The quotient by this exact gauge direction is one-dimensional, with coordinate $s = c + 4d/5$. The first vector has $s = 1$ and rank-three square-zero cusp residue.

The extension is nontrivial, since its dual has no common invariant vector by theorem 5.2; consequently its coordinate s is nonzero. Set $t = 3125d/60$. Its full row is $\beta = sb + tr_0\Gamma_W$. The constant bundle maps

$$C = \begin{pmatrix} 1 & -tr_0 \\ 0 & sI_5 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & (t/s)r_0 \\ 0 & s^{-1}I_5 \end{pmatrix}$$

are holomorphic in the stated root frames and fix the invariant section e. Direct multiplication gives

$$C^{-1}\Gamma_6 C + C^{-1}dC = \begin{pmatrix} 0 & \beta \\ 0 & \Gamma_W \end{pmatrix}.$$

This proves the required equivalence of the complete flat bundles with their fixed invariant line. Evaluation at the marked basepoint gives a common intertwiner for both specified meridians. Thus its existence follows from the exact connection maps.

⟨1⟩4. For the last normalization, the kernel of the quotient cusp residue on (w_2, w_1, w_0) is generated by $v = (-4096/1171875, 64/1875, 1)^t$. It satisfies $b_{\mathrm{res}}(v) = 625/192$ and $\alpha_{\mathrm{alg}}(v) = 4/3$. The logarithmic monodromy is $2\pi i$ times this residue. The exact cusp identity of theorem 6.7 characterizes the contraction on this line by $N_V(\widetilde{v}) = \alpha(v)e$. Since the space of global contraction forms is one-dimensional, the ratio $(625/192)/(4/3) = 625/256$ fixes its scalar everywhere and gives the displayed $625\pi i/128$.

□

Definition 8.2 (The height function h_b)**.** Let $Y' = Y\Gamma_6$. Normalize $Y(1) = X$ by requiring that continuation around the meridians intertwine exactly with A, B, and that $Xe_0 = e_0$. Existence follows from the flat-extension identification in theorem 8.1. The exact simultaneous commutator equations for A, B have a one-dimensional solution space spanned by the identity; this is checked in `independent_V_extension_06.py`.

The condition $Xe_0 = e_0$ removes the scalar freedom and proves uniqueness. At each nonspecial point there is a unique pair $v_2, v_1 \in \mathbb{C}$ such that the lower five coordinates of $Y(0, v_2, v_1, -i, 0, 0)^t$ are real. The intersection-line proof in theorem 6.7 and the normalization of α prove existence and uniqueness. Define the real function h_b associated with σ_b by

$$h_b(\lambda) = \frac{96}{625\pi}\,\mathrm{Im}(Y_{01}v_2 + Y_{02}v_1 - iY_{03}). \tag{1.15.3}$$

Its periods for the real parameter c are the three columns $Y_{*,1}, Y_{*,2}, Y_{*,3} - (625\pi/96)c\,e_0$.

This is a different splitting normalization from the one chosen in theorem 6.7. To compare them exactly, parallel-transport the fixed marking from $\lambda = 1$ to $\lambda = \lambda_*/2$ along a specified path in the upper half-plane, and put $a = (\sigma_b(g))_e$ at that point, where $g = (96/(625\pi i))w_0$ and $\alpha(g) = 1$. Then the splitting of theorem 6.7 is $\sigma = \sigma_b - a\alpha e$, so $h = h_b - \mathrm{Im}\,a$. For the full complex splitting parameter ξ, the comparison is $\xi_b = \xi - a$. This gives the translation between the two families, including its real part.

Theorem 8.3 (Certified values of h_b)**.** [tag `02QP`] *In the canonical splitting σ_b of theorem 8.1,*

$$-1.034276622416321799494248845378346 1 < h_b(1),$$
$$h_b(1) < -1.034276622416321799494248845378346 0.$$

In particular $h_b(1) < -1$. The parameter $c = 5/(3\pi)$, the exceptional value of theorem 9.2, lies outside the global transversality domain.

Proof. ⟨1⟩1. **The rational connection and the marked paths.** Put $a = 256/3125 = -\lambda_*$ and

$$D(\lambda) = \mathrm{diag}(1, \lambda^2, \lambda, 1, \lambda^{-1}, \lambda^{-2}).$$

For the row-frame equation $Y' = Y\Gamma_6$ of theorem 8.1, the exact change $\widetilde{Y} = YD$ gives

$$\widetilde{\Gamma} = D^{-1}\Gamma_6 D + D^{-1}D' = \frac{R_0}{\lambda} + \frac{R_*}{\lambda + a}, \qquad R_0 = \lim_{\lambda\to 0}\lambda\widetilde{\Gamma}, \quad R_* = \lim_{\lambda\to -a}(\lambda + a)\widetilde{\Gamma}.$$

These formulas specify both rational residue matrices entry by entry from (1.15.1)–(1.15.2). Direct rational multiplication verifies the identity. Their maximum absolute row-sum norms are

$$\|R_0\|_\infty = \frac{3125}{256}, \qquad \|R_*\|_\infty = \frac{14575}{768}.$$

Let $r = 1/32$. The zero meridian is the polygon with successive vertices

$$1,\ r,\ r+ir,\ -r+ir,\ -r-ir,\ r-ir,\ r,\ 1.$$

The infinity meridian has successive vertices

$$1,\ 1-i,\ -1-i,\ -1+i,\ 1+i,\ 1.$$

The first runs counterclockwise about 0 and excludes $-a$; the second runs clockwise about both finite poles, hence positively about infinity. The zero square has radial distance between r and $\sqrt{2}r < a$, and the infinity square between 1 and $\sqrt{2}$, with $1 > a$. Linear interpolation of these positive radial distances to r and 1, respectively, gives based homotopies through the corresponding punctured regions to their circular meridians; the zero stem remains the positive segment. Thus these are the based meridians of theorem 8.1, with matrices B and A, respectively. Since $D(1) = I$, the gauge keeps their based matrices unchanged. The open comparison path from 1 to 2 is the positive real segment.

⟨1⟩2. **The Taylor recurrence and its complete error bound.** Write $M_1 = R_0 + R_*$ and $M_0 = aR_0$. At a rational complex center z, set

$$Q_0 = z(z+a), \quad Q_1 = 2z+a, \quad P_0 = zM_1 + M_0.$$

For the fundamental solution $T(t) = \sum_{n\ge 0} T_n t^n$ normalized by $T_0 = I$, multiplication of $T' = T(M_1(z+t)+M_0)/((z+t)(z+a+t))$ gives, with $T_{-1} = 0$,

$$(n+1)Q_0T_{n+1} = T_nP_0 + T_{n-1}M_1 - nQ_1T_n - (n-1)T_{n-1}.$$

This recurrence uses only rational connection coefficients and outward-rounded complex balls.

For each step put

$$d_0 = \max(|\operatorname{Re} z|, |\operatorname{Im} z|), \quad d_* = \max(|\operatorname{Re}(z+a)|, |\operatorname{Im} z|), \quad R = \tfrac{1}{2}\min(d_0, d_*).$$

The segment is subdivided with rational increments h satisfying $|h|_1 = |\operatorname{Re} h| + |\operatorname{Im} h| \le R/4$. The whole complex disc $|t| \le R$ avoids both poles, and on it

$$\|\widetilde{\Gamma}(z+t)\|_\infty \le M = \frac{\|R_0\|_\infty}{d_0 - R} + \frac{\|R_*\|_\infty}{d_* - R}.$$

The integral equation and Gronwall's estimate (Teschl, 2012, Lemma 2.7, pp. 42–43), followed by Cauchy's coefficient formula (National Institute of Standards and Technology, 2026, §1.9(iii), (1.9.30)–(1.9.31)), give the entrywise degree-N tail bound

$$E_N = \exp(MR)\frac{(|h|_1/R)^{N+1}}{1 - |h|_1/R}. \tag{1.16.1}$$

Adding the real and imaginary error balls of radius E_N to each entry therefore encloses the exact transition. All subsequent matrix products and inverses retain those radii. The order starts at 32 and is increased by 16 until $E_N < 10^{-100}$. There are 120 steps for the zero meridian, 66 for the infinity meridian and 6 for the open path, hence 192 in total; the largest order is 192.

⟨1⟩3. **The normalized intertwiner and the height.** Let H_0, H_∞ denote the two computed based transition matrices. The exact equations are

$$XH_0 = BX, \qquad XH_\infty = AX, \qquad Xe_0 = e_0.$$

Theorem 8.1 proves their simultaneous existence and uniqueness. The fixed first column leaves thirty unknowns. We select thirty of the sixty scalar equations in the other five columns and certify their coefficient matrix inverse in complex-ball arithmetic. The exact common solution therefore belongs to the resulting matrix ball; the remaining equations are also checked to contain zero. Their consistency is given by theorem 8.1.

Likewise, the real intersection in (1.15.3) is computed from an invertible four-equation subsystem in $(\operatorname{Re} v_2, \operatorname{Im} v_2, \operatorname{Re} v_1, \operatorname{Im} v_1)$. Theorem 6.7 proves the existence and uniqueness of the full intersection; the unused equation is checked with its interval radius retained. At $\lambda = 1$ the resulting height ball has radius less than $4 \cdot 10^{-91}$. It is contained in the following wider displayed interval:

$$h_b(1) \in \big[-1.03427662241632179949424884537834605259496477790218821132 \\ \pm 8.11 \cdot 10^{-57} \big].$$

It lies strictly between the two decimal endpoints in the statement. Since $5/(3\pi) < 1$, we have $h_b(1) + 5/(3\pi) < 0$. The cusp asymptotic in theorem 6.7 gives $h_b + c \to +\infty$ along a path to the cusp. Continuity on a path from 1 to that region therefore supplies a zero for $c = 5/(3\pi)$, proving its inadmissibility.

⟨1⟩4. **The regular-root evaluation and the exact comparison.** The second method uses the regular frames with powers $(0, 6, 3, 0, -3, -6)$ for $\lambda = u^4$ at zero and $(0, -8, -4, 0, 4, 8)$ for $\lambda = u^{-5}$ at infinity. Their transformed connections have denominators $3125u^4 + 256$ and $3125 + 256u^5$. The regular-root recurrences, finite deck characters and normalized intertwiner construction are displayed in the proof of theorem 8.4. They specify X_{root} and the constant infinity frame F_∞ without integrating a closed loop.

Let T_{12} be the Fuchsian transition on the positive segment. The first method gives

$$Y_2 = X T_{12} D(2)^{-1}.$$

For the second method put $u = 2^{-1/5}$ and $D_\infty(u) = \operatorname{diag}(1, u^{-8}, u^{-4}, 1, u^4, u^8)$. Let U_∞ solve $U_\infty' = U_\infty \Gamma_\infty$ with $U_\infty(0) = I$. The second evaluation is

$$Y_{2,\mathrm{root}} = F_\infty U_\infty(2^{-1/5}) D_\infty(2^{-1/5})^{-1}.$$

The explicit polynomial numerator of Γ_∞ is obtained by the stated gauge and chain rule from Γ_6; its coefficient matrices are saved. On $|u| \le 5/4$, its denominator has modulus at least 2343.75, and the row-sum connection bound is 65.6375. The degree-880 Taylor polynomial and (1.16.1) consequently give an error less than $5.39 \cdot 10^{-103}$ at $2^{-1/5}$. The complete ball comparisons give

$$\max_{i,j} |(X - X_{\mathrm{root}})_{ij}| < 1.37 \cdot 10^{-73}, \qquad \max_{i,j} |(Y_2 - Y_{2,\mathrm{root}})_{ij}| < 4.71 \cdot 10^{-74}.$$

Thus both period matrices agree to more than thirty decimal digits under the explicit frame maps above.

⟨1⟩5. **The period planes.** For the real parameter c the period matrix is

$$W_c = (Y_{*,1},\ Y_{*,2},\ Y_{*,3} - (625\pi/96) c\, e_0).$$

Conjugation of $\det(W_c, \overline{W_c})$ interchanges two blocks of three columns, so it multiplies the determinant by $(-1)^9 = -1$. Thus the determinant is purely imaginary. At the two points above and $c = 10$, its positive imaginary parts are

$$88279007523.216745184606660708829970947 9845\ldots,$$
$$3178499608.363760851758323601075286113244 1\ldots.$$

The complete stored balls have radii less than $4 \cdot 10^{-79}$ and $2 \cdot 10^{-80}$, respectively. These are evaluations at the two specified points. The global admissibility of $c = 10$ follows from the complete minimum argument in theorem 8.4. The regular-root calculation there also gives

$$h_b(\infty) = -1.094540250311402780116268263054945703725533\ldots,$$
$$h_b(0) = -0.931793376356523505430795105796011114990828\ldots.$$

The computation is done by the program `period_loop_comparison.py` in interval arithmetic (python-flint, Arb) with 140 decimal digits; a second program, `validate_saved_period_comparison.py`, rechecks the saved enclosures and every displayed bound.

□

Theorem 8.4 (The minimum c_*)**.** [tag `02QR`] *In the algebraic splitting σ_b, the function h_b has its unique global minimum at the finite-root point $\lambda = \infty$. The minimum is the exact Frobenius period constant*

$$h_{b,\min} = -\frac{96}{625\pi}\,\mathrm{Re}(XH_\infty)_{03}, \qquad h_{b,\min} = -1.094540250311402780116268263054945703 7\ldots, \tag{1.18.1}$$

where X is the normalized integral intertwiner of theorem 8.1 and H_∞ is the regular infinity-root limit of its fundamental solution, normalized to be the identity at $\lambda = 1$ before multiplication by X. Formula (1.18.1), the rational connection, and those two normalizations specify the constant exactly. A certified enclosure has radius less than $4 \cdot 10^{-76}$. Consequently the exact domain of completed transverse periods is

$$c > -h_{b,\min}, \qquad \mathrm{Im}\,\xi > -h_{b,\min}$$

for the real and complex parameters, respectively. The member $c = 10$ used in the numerical period checks is therefore admissible on the whole completed base.

For the point-normalized splitting of theorem 6.7, continued from 1 to $\lambda_*/2$ by the polygon $1, 1+i, \lambda_*/2+i, \lambda_*/2$, the exact comparison constant a_{split} of (1.15.4) has certified value

$$\mathrm{Re}\,a_{\mathrm{split}} = -0.709988567912642777172347345501230 8149\ldots,$$
$$\mathrm{Im}\,a_{\mathrm{split}} = -0.913401705936415336214322922382837 3269\ldots.$$

Thus the minimum in that splitting is $-0.181138544374987443901945340672108376 8\ldots$, attained at the same base point. Its real parameter interval is

$$c > 0.181138544374987443901945340672108376 8\ldots.$$

The full parameter comparison is $\xi_b = \xi_{\mathrm{original}} - a_{\mathrm{split}}$; both its real and imaginary parts are necessary to identify the two normalized families.

Proof.

⟨1⟩1. **The height, parameter, and real coordinate maps.** Put $k = 625\pi/96$, $c_0 = k^{-1}$, and use the right connection $H' = H\Gamma_6$, $H(1) = I$, with $Y = XH$. For the uniquely normalized real quotient-intersection vector write

$$q = (0, v_2, v_1, -i, 0, 0)^t, \qquad z = Yq, \qquad h_b = c_0\,\mathrm{Im}\,z_0.$$

The lower five entries of z are real. Every marked monodromy matrix T is real and fixes e_0. Its top entry on z is z_0 plus a real linear combination of those lower entries, so $\mathrm{Im}(Tz)_0 =$

$\operatorname{Im} z_0$. This proves branch independence in the given marking. The normalization gives $\alpha(q) = k$, since the coefficient of w_0 is $-i$ and $\alpha(w_0) = ki$. The lifted vector for parameter ξ therefore has top coordinate $z_0 + k\xi$ and the same lower coordinates. The exact obstruction in the real quotient-intersection line is

$$\operatorname{Im}(z_0 + k\xi) = k(h_b + \operatorname{Im}\xi).$$

The transversality criterion of theorem 6.7 consequently uses this same function for complex and real parameters.

The numerical preconditioner has an explicit real coordinate map. For a constant limiting root frame H_c, let B have lower-five columns

$$\operatorname{Im}(H_c)_1, \quad \operatorname{Re}(H_c)_1, \quad \operatorname{Im}(H_c)_2, \quad \operatorname{Re}(H_c)_2, \quad \operatorname{Im}(H_c)_3.$$

The certified inverse of B defines

$$r = (\operatorname{Im}(H_c)_{01}, \operatorname{Re}(H_c)_{01}, \operatorname{Im}(H_c)_{02}, \operatorname{Re}(H_c)_{02}, 0)B^{-1}, \qquad T_c = \begin{pmatrix} 1 & -r \\ 0 & B^{-1} \end{pmatrix}.$$

This is a real invertible matrix fixing e_0. Realness of $B^{-1}z_{\mathrm{low}}$ is equivalent to realness of z_{low}, and the change of top entry is real on that vector. Thus it preserves both the normalized pair and h_b exactly. Writing $v_2 = a + ib$, $v_1 = c + id$ gives the real equations with row $(\operatorname{Im} Y_{i1}, \operatorname{Re} Y_{i1}, \operatorname{Im} Y_{i2}, \operatorname{Re} Y_{i2})$ and right-hand entry $\operatorname{Re} Y_{i3}$. Approximate row selection only chooses four rows; the inverse and every ensuing enclosure are certified in ball arithmetic.

⟨1⟩2. **The marked conjugation map.** The connection has real rational coefficients. Conjugating the specified ordered polygonal meridians gives their inverse paths. The matrix $K_0 = X\overline{X}^{-1}$ therefore satisfies $K_0A = A^{-1}K_0$, $K_0B = B^{-1}K_0$, and $K_0e_0 = e_0$. The normalized intertwiner uniqueness in theorem 8.1 applies: the quotient of two invertible solutions is in the common centralizer fixing e_0. Direct exact solution gives

$$K = \begin{pmatrix} 1 & 2 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplication verifies the two intertwiner equations, $Ke_0 = e_0$, $K^2 = I$, and $\det K = -1$, hence $K = K_0$. On conjugate continuation branches, $Y(\overline{\lambda}) = K\overline{Y(\lambda)}$. The vector $q' = -\overline{q}$ has the required $-i$ normalization, and its image is $-K\overline{z}$, with real lower coordinates. Uniqueness therefore identifies the normalized intersection vector and gives $h_b(\overline{\lambda}) = h_b(\lambda)$. This is the coordinate map used in reflecting the covered regions.

⟨1⟩3. **Rigorous analytic enclosures.** The following formulas specify the complete-domain computation. All matrix coefficients come from the displayed rational connection and the gauges of theorem 8.1, theorem 8.3. A regular root frame is the pullback followed by its stated diagonal gauge D; its right connection is exactly $D^{-1}\lambda'\Gamma_6 D + D^{-1}D'$. At a positive anchor the constant limiting frame is obtained by the map $H_c = Y_{\mathrm{root}}(z_c)U(z_c)^{-1}$, using the normalized continuation and a certified matrix inverse.

⟨2⟩1. **Normalization from the two finite roots.** The computation uses the rational matrix Γ_6 of theorem 8.3, the integral matrices A, B of theorem 8.1, and the normalization $H(1) = I$, $Xe_0 = e_0$. It derives the two gauged connections by exact rational substitution:

$$\lambda = u^4, \qquad D_0 = \operatorname{diag}(1, u^6, u^3, 1, u^{-3}, u^{-6}),$$
$$\lambda = u^{-5}, \qquad D_\infty = \operatorname{diag}(1, u^{-8}, u^{-4}, 1, u^4, u^8).$$

Write $U_0(0) = U_\infty(0) = I$ for their regular fundamental solutions. For a positive local root rotation ζ with gauge exponents d_i, the character is $T = \mathrm{diag}(\zeta^{d_i})$. Uniqueness of the regular right ODE gives $U(\zeta u) = T^{-1}U(u)T$ and $(UD^{-1})(\zeta u) = T^{-1}(UD^{-1})(u)$. Since $D(1) = I$, the two based matrices are exactly

$$H_0 = U_0(1)^{-1}T_0^{-1}U_0(1), \qquad H_\infty^{\mathrm{mon}} = U_\infty(1)^{-1}T_\infty^{-1}U_\infty(1).$$

The equations solved are $XH_0 = BX$, $XH_\infty^{\mathrm{mon}} = AX$, $Xe_0 = e_0$. The exact geometric existence and normalized uniqueness are given by theorem 8.1; a certified invertible square subsystem encloses that exact solution. Every omitted equation is also checked against its enclosure. Residual containment is used as a consistency check; existence is the cited geometric input. The limiting frames are $F_0 = XU_0(1)^{-1}$ and $F_\infty = XU_\infty(1)^{-1}$.

The zero-root transport starts at $u = 1/4$, with a series radius $2/5$, and then uses the exact positive real steps $\delta = \min(u/32, 1-u)$, of radius $u/8$, until $u = 1$. The infinity root is evaluated directly at $u = 1$ with radius $5/4$. For any shifted rational connection $P(t)/Q(t)$, the certificate first verifies

$$d = |Q_0| - \sum_{j\geq 1} |Q_j|R^j > 0, \qquad M = \frac{\sum_j \|P_j\|_w R^j}{d}.$$

The degree-N transition error is then bounded by $e^{MR}(r/R)^{N+1}/(1-r/R)$ for $r < R$, by the integral equation and the coefficient integral below. Each step records its rational centre, radius, order, positive denominator bound and full propagated ball error. The Taylor coefficients and subdivision leaves are generated by the stated recurrences and rational-domain algorithm.

⟨2⟩2. **The analytic enclosures and their matrix maps.** For positive weights w_i let $W = \mathrm{diag}(w_i)$. The specified norm is the maximum absolute row-sum norm of the conjugate $W^{-1}MW$:

$$\|M\|_w = \max_i \sum_j |M_{ij}| \frac{w_j}{w_i}.$$

The equality $W^{-1}MNW = (W^{-1}MW)(W^{-1}NW)$ and the rowwise triangle inequality prove $\|MN\|_w \leq \|M\|_w\|N\|_w$ and $|M_{ij}| \leq \|M\|_w w_i/w_j$. For matrices with centres C, D and entrywise error bounds E, F, multiplication has entrywise error at most

$$|C|F + E|D| + EF,$$

with the computed rounding error also retained. In particular, a left factor H_c acts on an entrywise error matrix by the sum $|H_c|_+E$, where $|H_c|_+$ bounds its entries including their ball radii.

⟨2⟩3. **Root series.** The listed regular-root connection matrices already include the specified diagonal gauges. Their frame map is $E_{\mathrm{root}} = E_{\mathrm{original}}D_{\mathrm{root}}$, and the numerical coordinate change is exactly $G_z(z) = \alpha G_{\mathrm{root}}(\alpha z)$, where $\alpha^4 = 256/3125$ at zero and $\alpha^5 = 3125/256$ at infinity. Multiplying $G_{\mathrm{root}}(t)$ by $1 + t^m/\alpha^m$ gives the exact rational polynomial $\sum_{j=0}^{m-1} C_j t^j$, so $B_j = \alpha^{j+1}C_j$. Thus

$$U' = U\frac{\sum_{j=0}^{m-1} B_j z^j}{1+z^m}, \qquad U(0) = I. \tag{1.18.2}$$

Coefficient comparison in $(1+z^m)U' = UB$ gives

$$(n+1)U_{n+1} = \sum_{j=0}^{\min(m-1,n)} U_{n-j}B_j - (n-m+1)U_{n-m+1},$$

where the last term is absent for $n < m$. The order of every matrix product is as displayed. Picard iteration on any smaller disc converges by the bound $(Lr)^k/k!$ on its k-fold integral, so these are the coefficients of the normalized solution.
Use $w = (1, 1/128, 1/128, 1/32, 1/64, 1/32)$ and

$$\mathfrak{b} = \max_i \sum_{j,k} |(B_k)_{ij}| \frac{w_j}{w_i}.$$

On a ray of radius $t < 1$ the connection norm is at most $\mathfrak{b}/(1-t^m)$. The scalar Gronwall inequality (Teschl, 2012, Lemma 2.7, pp. 42–43) applies to the matrix integral equation. The elementary estimate

$$\frac{1}{1-t^m} \le \frac{1}{m(1-t)} + \frac{m-1}{m}$$

holds because its numerator difference is $\left(\sum_{j=1}^{m-1} t^j - (m-1)t^m\right)/m \ge 0$. Consequently

$$\|U(z)\|_w \le K(R) := \exp\left\{\mathfrak{b}\left[-\frac{\log(1-R)}{m} + \frac{m-1}{m}R\right]\right\} \quad (|z| \le R < 1).$$

Cauchy's coefficient integral, with the positively oriented circle of radius R (National Institute of Standards and Technology, 2026, §1.9(iii), (1.9.30)–(1.9.31)), gives $\|U_n\|_w \le K(R)R^{-n}$. Hence the error after degree N on the whole disc $|z| \le r < R$ is

$$\tau_N(r; R) = K(R)\frac{(r/R)^{N+1}}{1 - r/R}. \tag{1.18.3}$$

Its ij entry is bounded by $\tau_N w_i/w_j$; after left multiplication by H_c the bound is $\tau_N \sum_l |(H_c)_{il}|_+ w_l/w_j$. The computation stores coefficients through $N = 3000$ and uses $R = 999/1000$. A shorter prefix may be evaluated at a centre; its own degree and modulus are used in the same tail formula. On the central squares the whole-square modulus is used. Every coefficient and evaluation radius is retained. Polynomial evaluation is grouped into blocks of 32 coefficients; this is the identity $p(z) = \sum_j z^{32j} p_j(z)$, with no change of polynomial or of its error bound.

⟨2⟩4. **Enclosing an entire rectangle.** Let z_c be its centre, ρ its Euclidean half-diagonal, and $\mathcal{A}$ a nonnegative entrywise bound for $|G(z)|$ on the whole rectangle. For any point z the segment $z_c + t(z - z_c)$ remains in it. Iteration of the integral equation on that segment bounds the k-th ordered integral by $|Y(z_c)|_+ |z - z_c|^k \mathcal{A}^k/k!$. Subtracting its zeroth term and summing proves the entrywise simultaneous enclosure

$$|Y(z) - Y(z_c)| \le |Y(z_c)|_+ (\exp(\rho\mathcal{A}) - I).$$

The centre enclosure and matrix-exponential rounding radii are retained, so this bounds every point of the rectangle.
The midpoint-polynomial bound has the same explicit domain control. For $p(z) = \sum_{j=0}^N c_j z^j$, coefficient discs with midpoints $\widehat{c}_j$ and radii ϵ_j, and midpoint evaluation rounding radius η, the centre error is $\eta + \sum_j \epsilon_j |z_c|^j$. Integrating p' on each segment bounds the remaining variation by

$$\rho \sum_{j=1}^{N} j(|\widehat{c}_j| + \epsilon_j) r^{j-1},$$

where r is the rectangle's maximum modulus. Add the analytic tail at that same r. This formula includes coefficient uncertainty in the derivative bound as well as at the centre.

⟨2⟩5. **The cusp holomorphic factor.** The exact connection here is $R/q + A_0/(1+q)$, with $R^2 = 0$. In $Y = H(I + R\log q)P(q)$, $P(0) = I$, the inverse of the logarithmic factor is $I - R\log q$. Multiplication by that inverse gives

$$P' = \frac{\mathrm{ad}(P)}{q} + \frac{PA_0}{1+q}, \qquad \mathrm{ad}(Q) = QR - RQ.$$

Thus

$$(nI - \mathrm{ad})P_n = \sum_{j=0}^{n-1} (-1)^{n-1-j} P_j A_0.$$

Equivalently its right side is $\mathrm{ad}(P_{n-1}) + P_{n-1}A_0 - (n-1)P_{n-1}$. Direct expansion gives $\mathrm{ad}^2(Q) = -2RQR$ and $\mathrm{ad}^3(Q) = 0$, so the inverse is

$$(nI - \mathrm{ad})^{-1} = \frac{1}{n}\left(I + \frac{\mathrm{ad}}{n} + \frac{\mathrm{ad}^2}{n^2}\right).$$

This fixes both the commutator sign and the side of the A_0 multiplication.
Use the cusp weights $(100, 1, 1, 1, 1, 1)$ and put $r_0 = \|R\|_w$, $a_0 = \|A_0\|_w$, $C_0 = 1 + 2r_0 + 4r_0^2$, $\gamma = C_0 a_0$. For the exact matrices,

$$r_0 = \frac{5103}{8000}, \quad a_0 = \frac{367}{512}, \quad C_0 = \frac{62452609}{16000000}, \quad \gamma = \frac{22920107503}{8192000000}.$$

The norm of the inverse operator is at most C_0/n. Therefore $\|P_n\|_w \le (\gamma/n)\sum_{j<n}\|P_j\|_w$. The nonnegative scalar majorant $c_0 = 1$, $c_n = (\gamma/n)\sum_{j<n} c_j$ satisfies $c_n = (\gamma + n - 1)c_{n-1}/n$ and has generating function $(1-s)^{-\gamma}$. Its radius is at least one by the coefficient ratio. Thus the matrix series converges normally on $|q| < 1$, solves the equation term by term, and obeys

$$\|P(q)\|_w \le K_0(s) := (1-s)^{-\gamma} \quad (|q| \le s < 1).$$

The coefficient integral and geometric sum give its degree-N tail on $|q| \le r < s$ as

$$K_0(s)\frac{(r/s)^{N+1}}{1 - r/s},$$

with the exact entry factor w_i/w_j. The current cusp series uses $N = 600$ and $s = 0.4$ for $r \le 0.25$. On the inner disc $r = 0.01$ the enclosure retains all coefficient absolute sums through degree 600 and this tail. It bounds P itself; the subsequent logarithmic factor retains its specified branch and its own matrix entries.

⟨2⟩6. **Collar continuation.** At a centre z_c put $t = z - z_c$ and expand

$$1 + (z_c + t)^5 = \sum_{j=0}^{5} q_j t^j, \qquad \widetilde{B}_j = \sum_{k=j}^{4} \binom{k}{j} B_k z_c^{k-j}.$$

The right transition $T(0) = I$ has the coefficients

$$T_{n+1} = \frac{\displaystyle\sum_{j=0}^{\min(4,n)} T_{n-j}\widetilde{B}_j - \sum_{j=1}^{\min(5,n)} q_j(n-j+1)T_{n-j+1}}{(n+1)q_0}.$$

Starting with $R = 1/8$, the program halves R until $d = |q_0| - \sum_{j=1}^{5}|q_j|R^j > 0$. It sets $M = \sum_j \|\widetilde{B}_j\|_w R^j/d$, takes a step $|\delta| \le R/2$, and increases N from 32 in increments of 16 until the certified error

$$e^{MR}\frac{(|\delta|/R)^{N+1}}{1 - |\delta|/R} < 10^{-45}.$$

Every coefficient, centre and step is an enclosed complex number; the strict denominator and tail tests are ball comparisons. The map is exactly $Y(z_c+\delta) = Y(z_c)T_{z_c}(\delta)$. At a collar centre of modulus at least 0.99, continuation starts at $(49/50)z_c/r_c$, where r_c is the outward upper endpoint of $|z_c|$. This point has modulus at most $49/50$, its value is enclosed by the root series, and the continuation follows the radial segment to the centre. The resulting whole-rectangle enclosure uses the matrix-exponential estimate above. The completed cover uses 412 complex Taylor transitions. Their individual tails are propagated before every right matrix multiplication.

⟨2⟩7. **The real implicit system and Hessian.** Use the fixed four rows on the central square and write the system as $Mv = b$, with

$$M_{i,*} = (\operatorname{Im} Y_{i1}, \operatorname{Re} Y_{i1}, \operatorname{Im} Y_{i2}, \operatorname{Re} Y_{i2}), \qquad b_i = \operatorname{Re} Y_{i3}.$$

For the top row define c by the same four coordinate projections, $d = \operatorname{Re} Y_{03}$ and $h = c_0(cv - d)$, $c_0 = 96/(625\pi)$. On each of the four closed central subsquares the certified matrix inverse exists throughout the square, so it is the inverse there. Their union is the whole central square. For $a, b \in \{x, y\}$, direct differentiation gives

$$\begin{aligned} v_a &= M^{-1}(b_a - M_a v), \\ v_{ab} &= M^{-1}(b_{ab} - M_{ab}v - M_a v_b - M_b v_a), \\ h_a &= c_0(c_a v + c v_a - d_a), \\ h_{ab} &= c_0(c_{ab}v + c_a v_b + c_b v_a + c v_{ab} - d_{ab}). \end{aligned}$$

The repeated-index cases retain both cross terms. The complex derivatives supplying these real projections are

$$\begin{aligned} Y_x &= YG, & Y_y &= iYG, \\ Y_{xx} &= Y(G^2 + G'), & Y_{xy} &= iY(G^2 + G'), \\ Y_{yy} &= -Y(G^2 + G'). \end{aligned}$$

Taking the stated real and imaginary parts and carrying out these products with rigorous full-square balls consequently encloses all Hessian entries throughout the domain. Approximate row selection merely chooses a minor; it supplies neither its inverse nor its error bound.

⟨1⟩4. **The infinity value and its strict local minimum.** Set $a = 256/3125$ and use the exact scaled root coordinates

$$w = a^{1/5}u, \quad \lambda = aw^{-5}, \qquad v = a^{-1/4}u_0, \quad \lambda = av^4.$$

The infinity regular frame is $\widetilde{Y}(u) = XH(u^{-5})D_\infty(u)$. Its third indexed column, the w_0 column, is unchanged by the root gauge. The real four-row system and its certified inverse extend the uniquely normalized coefficient pair to zero; its fifth reality equation extends there by continuity. For $\zeta = e^{2\pi i/5}$ the exact deck formula is

$$\widetilde{Y}(\zeta u) = A\widetilde{Y}(u)T_\zeta, \qquad T_\zeta = \operatorname{diag}(1, \zeta^{-8}, \zeta^{-4}, 1, \zeta^4, \zeta^8).$$

At zero this changes the unique coefficient pair to $(\zeta^8 v_2, \zeta^4 v_1)$ and preserves the $-i$ coefficient. Thus both variables vanish and

$$h_b(\infty) = -c_0 \operatorname{Re}(F_\infty)_{03} = -\frac{96}{625\pi} \operatorname{Re}(XH_\infty)_{03}.$$

The same deck map gives $h_b(\zeta u) = h_b(u)$, so the real gradient at zero vanishes. The four closed quadrants of $Q = [-1/10, 1/10]^2$ are the central certificate. On each, the implicit derivative formulas give

$$\min(h_{xx}, h_{yy}) - |h_{xy}| > \frac{1}{4}.$$

Every real vector therefore satisfies $t^t D^2 h_b\, t \geq |t|^2/4$ throughout Q. Integration along the segment from zero, which remains in the convex square, gives

$$h_b(aw^{-5}) - h_b(\infty) \geq \frac{1}{8}|w|^2. \tag{1.18.5}$$

The centre is the unique minimum on this square.

The normalized root calculation gives the outward enclosure

$$h_b(\infty) = -1.094540250311402780116268263054945703725532693940533089777705\ldots,$$
$$\text{stored midpoint radius} < 4 \cdot 10^{-76}.$$

In particular the following exact terminating-decimal bounds hold:

$$\begin{aligned} -1.094540250311402780116268263054945703726 &< h_b(\infty), \\ h_b(\infty) &< -1.094540250311402780116268263054945703725. \end{aligned} \tag{1.18.4}$$

⟨1⟩5. **The closed covers and their strict lower bounds.** Put $R = 49/50$ and $S = 103/100$. The first-quadrant w disc $|w| \leq R$ contains the sector $0 \leq \arg w \leq 2\pi/5$; under $\lambda = aw^{-5}$ this covers $|\lambda| \geq aR^{-5}$. Its intersection with $[0, 1/10]^2$ is covered by the central Hessian certificate. The first-quadrant v disc $|v| \leq R$ maps by $\lambda = av^4$ onto $|\lambda| \leq aR^4$. The respective finite deck maps are the real integral matrices A, B, which fix e_0 and preserve the height.

The collar uses $R \leq |w| \leq S$ in the first quadrant. Its image contains every argument and every radius between aS^{-5} and aR^{-5}. The exact rational inequality $R^4 \geq S^{-5}$ proves overlap with the zero-root disc. The exact cusp coordinate is $q = \lambda/\lambda_* - 1 = -w^{-5} - 1$. Boxes contained in $|q| \leq 1/4$ are covered by the cusp certificate. For that disc the seed rectangle is $[-1/4, 1/4] \times [0, 1/4]$ in the q plane; the exact marked conjugation map in item 2 gives its lower-half counterpart. The inner subdisc $|q| \leq 1/100$ is treated in item 6. All boundaries are retained. These maps and the rational overlap prove coverage of the full punctured base, including its two finite-root points.

The three first-quadrant seed rectangles are $[0, R]^2$ for each root and $[0, S]^2$ for the collar. Each subdivision is into four closed rational rectangles with disjoint interiors. For each leaf the program either proves the strict inequality on the whole rectangle or records a checked geometric reason: outside the required radius, inside the central Hessian square, inside the inner cusp disc, or inside the cusp disc already covered in its own coordinate. A new real preconditioner of item 1 may be applied at each centre. Its exact formula fixes e_0 and preserves the height; the whole-box ODE bound is then applied to that same constant real change of coordinates.

All strict comparisons use the exact rational decimal

$$U = -1.094540250311402780116268263054945703725 > h_b(\infty).$$

The complete partition has the following counts. The last column is an exact terminating decimal strictly below every accepted lower endpoint in that row, and strictly above U:

region	strict	geometric	total	uniform lower bound
infinity root	335	89	424	−1.094216
zero root	200	77	277	−1.083056
cusp annulus	113	8	121	−1.083777
collar	440	338	778	−1.030556

Thus there are 1088 strict and 512 geometric leaves, 1600 in total, in addition to the four central Hessian squares. The root normalization uses 120-decimal-digit Arb arithmetic; the complete outer cover uses 80 digits, with all input radii retained. The maximum subdivision depth is 8.

A second program reconstructs every rectangle from its quadtree address, checks that the leaves partition the four seed regions (of areas 2401/2500, 2401/2500, 1/8 and 10609/10000) and rechecks every exclusion and every strict comparison.

⟨1⟩6. **The cusp graph and the inner disc.** The exact cusp substitution and gauge are

$$\lambda = \lambda_*(1+q), \qquad D_\lambda = \operatorname{diag}(1, \lambda^2, \lambda, 1, \lambda^{-1}, \lambda^{-2}), \qquad YD_\lambda = H_c(I + R_c \log q)P(q).$$

The chain rule gives the exact q connection

$$\Gamma_q = D_\lambda^{-1}\lambda_*\Gamma_6(\lambda(q))D_\lambda + D_\lambda^{-1}\frac{dD_\lambda}{dq} = \frac{R_c}{q} + \frac{A_0}{1+q}.$$

The rational matrices R_c, A_0 are obtained by this substitution. For the anchor take

$$q_a = 1/4, \quad \lambda_a = -\tfrac{5}{4}a, \quad w_a = (4/5)^{1/5}e^{\pi i/5}, \quad u_a = a^{-1/5}w_a.$$

The right connection map is

$$H_c = Y_{\mathrm{root},\infty}(w_a)D_\infty(u_a)^{-1}D_{\lambda_a}P(1/4)^{-1}(I - R_c\log(1/4)).$$

It fixes the continuation branch and every gauge factor. The resulting logarithmic monodromy and an explicit unimodular marking are

$$N = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The enclosure of $2\pi i H_c R_c H_c^{-1}$ contains this unique integer matrix. Exact multiplication gives

$$N = B^{-1}A^{-1} - I, \qquad N^2 = 0, \qquad \det J = \pm 1,$$
$$J^{-1}NJ = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}, \qquad e_J = J^{-1}e_0 = (1,0,0,0,0,0)^t.$$

This anchor's marking is related explicitly to the cusp marking $N' = A^{-1}B^{-1} - I$ by $N = AN'A^{-1}$. The left map is $z \mapsto Az$; its top coordinate changes by $z_3 - z_4 + z_5$, which is real on the normalized intersection vector. It fixes e_0 and preserves h_b. For the marking

$$J' = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

the exact matrix

$$J^{-1}AJ' = \begin{pmatrix} -1 & 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & -1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

has block form $\left(\begin{smallmatrix} D & E \\ 0 & D \end{smallmatrix}\right)$. It changes a graph Z to $DZD^{-1} + ED^{-1}$, so its imaginary part changes by the real similarity $B \mapsto DBD^{-1}$. This supplies the full graph-coordinate comparison.

Set $c_* = -h_b(\infty)$, $s = kc_*$ and $V = J^{-1}H_cP(q)$. The shifted columns are $V_1, V_2, V_3 - se_J$. Their upper and lower blocks are U_0, L; the right scaling from the algebraic columns is $\mathrm{diag}(\lambda^2, \lambda, 1)$, preserving their span and the third-column shift. The coefficient absolute sums through degree 600 and the proved cusp tail enclose $P(q)$ on the entire closed disc $|q| \le 1/100$. The certified inverse of L exists throughout this disc. Since $Ne_0 = 0$, the logarithmic factor gives exactly

$$Z(q) = \frac{\log q}{2\pi i} I_3 + C(q), \qquad C = U_0 L^{-1}.$$

Put $B = \operatorname{Im} C$ and $y = -\log|q|/(2\pi)$. Then

$$\det \operatorname{Im} Z = y^3 + \operatorname{tr}(B) y^2 + \sum_{i<j} (B_{ii}B_{jj} - B_{ij}B_{ji}) y + \det B. \tag{1.18.6}$$

Set $y_0 = \log 100/(2\pi)$ and $t = y - y_0 \ge 0$. The coefficients of $\det(tI_3 + y_0 I_3 + B(q))$, in increasing degree, have the uniform certified lower bounds

$$\frac{3}{2}, \qquad \frac{9}{2}, \qquad \frac{15}{4}, \qquad 1,$$

with the last coefficient exactly one. Indeed the stored lower endpoints are strictly greater than $1.7115, 4.6510, 3.8702$ in the first three positions; the last is exactly one. Thus the determinant is positive throughout the punctured disc. A graph vector (Zv, v) is real exactly when v is real and $(\operatorname{Im} Z)v = 0$, proving transversality of the c_* plane. Item1 therefore gives $h_b - h_b(\infty) \neq 0$ there. The punctured disc is connected; the anchor $q = 1/100$ has $h_b = -0.450442200907412868179061298423334508 4\ldots > h_b(\infty)$. Hence $h_b > h_b(\infty)$ on the whole punctured inner disc.

These maps also prove the cusp limit. The lower block of e_J is zero, so L is independent of a real parameter c, and C_c is holomorphic at zero and affine in c. For every fixed real M, it is uniformly bounded on a smaller closed disc and the compact interval between c_* and $-M$. As $y \to \infty$, the Neumann inverse makes $yI + \operatorname{Im} C_c$ invertible uniformly on that interval. The real quantity $h_b + c$ cannot cross zero along it; its sign at c_* is positive, so $h_b - M > 0$ at the other endpoint. Thus $h_b \to +\infty$ at the cusp. The closed covers and the central strict estimate prove the global minimum and uniqueness.

⟨1⟩7. **The parameter domain and the full splitting translation.** Item1 and theorem 6.7 give $c > -h_b(\infty)$ and $\operatorname{Im}\xi > -h_b(\infty)$ on the completed base. The enclosure gives $-h_b(\infty) < 1.095 < 10$. For the upper-half-plane polygon specified in the statement, $g = (96/(625\pi i))w_0$ has $\alpha(g) = 1$, and

$$a_{\mathrm{split}} = \frac{96}{625\pi i} Y_{03}(\lambda_*/2).$$

The evaluation is at $v = 2^{-1/4}e^{\pi i/4}$ in the zero-root coordinate $\lambda = av^4$. Its continuation from the positive real axis lies in the upper half-plane and is homotopic there, relative endpoints, to the stated polygon. The root gauge has third indexed diagonal entry one, so the displayed top coordinate is unchanged. This proves the precise path and gauge comparison. Both components of a_{split} have outward radius less than $8 \cdot 10^{-76}$.

The defining equality $\sigma_{\mathrm{original}} = \sigma_b - a_{\mathrm{split}}\alpha e$ gives the exact plane equality

$$(\sigma_{\mathrm{original}} + \xi_{\mathrm{original}}\alpha e)(P) = (\sigma_b + (\xi_{\mathrm{original}} - a_{\mathrm{split}})\alpha e)(P).$$

Hence $\xi_b = \xi_{\text{original}} - a_{\text{split}}$ and $h_{\text{original}} = h_b - \operatorname{Im} a_{\text{split}}$ pointwise. The resulting minimum is

$$-0.18113854437498744390194534067210837680361520441866877988 93\ldots,$$

$$\text{stored midpoint radius} < 8 \cdot 10^{-76},$$

attained at the same base point. The original real line $\xi_{\text{original}} = ic$ maps exactly to $\xi_b = -\operatorname{Re} a_{\text{split}} + i(c - \operatorname{Im} a_{\text{split}})$; its criterion is $c > -h_{\text{original,min}}$.

The programs implementing these bounds, their outputs and the commands that run them are in the folder `code/paper_v` of the repository (§1.5).

□

9 Cohomology, deformations and isomorphisms

We keep the complex orientation and the integral marking of theorem 5.2. Let $\widehat{B}$ be the root orbifold of orders $5, 4$, let $h : M \to \widehat{B}$ be the map given by the two multiple divisors, and let $\pi : \widehat{B} \to \mathbb{P}^1$ be its coarse map. Write $L = T^2$; in the pencil coordinate it is

$$L = \mathcal{O}_{\widehat{B}}(\tfrac{4}{5}[\infty] - \tfrac{3}{4}[0]), \qquad \deg L = \tfrac{1}{20}.$$

The cusp point has line bundle L^{20}, and $T_{\widehat{B}} = L^9$. These identities include the stabilizer characters: their quotients have integral divisors of degree zero on $\mathbb{P}^1$. Denote the cusp by W, its normalization by Y, and its reduced singular locus by D.

Theorem 9.1 (Direct images and the canonical bundle)**.** [tag `02R1`] *For every admissible member, its vertical translation bundle on $\widehat{B}$ and its first coherent direct image are*

$$E_0 = F^\vee = \mathcal{O} \oplus L^{-1} \oplus L^{-2}, \qquad G = R^1 h_* \mathcal{O}_M = \mathcal{O} \oplus L^{-1} \oplus L^{-2}.$$

Moreover $R^q h_* \mathcal{O}_M = \bigwedge^q G$ for $0 \le q \le 3$, $K_M = h^* L^{-6}$, and

$$(h^0(\mathcal{O}_M), \ldots, h^4(\mathcal{O}_M)) = (1, 1, 0, 0, 0).$$

Proof. ⟨1⟩1. The evaluation of the marked lattice on the lifted plane F identifies the Lie bundle of the smooth tori with $F^\vee$. Theorem 6.7 gives $F = \mathcal{O} \oplus L \oplus L^2$; its lift as a bundle has the same splitting. The logarithmic toric vector fields extend this Lie bundle over the cusp, so the identification holds on the whole orbifold, including both finite root charts.

⟨1⟩2. On the smooth locus the constant-cocycle description of torus cohomology gives $G = V/F$. In the rank-five orthogonal quotient, $K = \ker(\alpha : P \to \mathcal{O}) = L \oplus L^2$ is the radical of the restricted pairing on P. Indeed every wedge containing the fixed line pairs to zero with another such wedge except through their contraction components; removing that scalar component leaves exactly K. The nondegenerate pairing consequently identifies W/P with $K^\vee$. Thus

$$0 \longrightarrow \mathcal{O}e \longrightarrow V/F \longrightarrow L^{-1} \oplus L^{-2} \longrightarrow 0.$$

For every integer j, descent by invariant local Laurent series gives

$$\pi_* L^j = \mathcal{O}_{\mathbb{P}^1}(d(j)), \qquad d(j) = \lfloor 4j/5 \rfloor + \lfloor -3j/4 \rfloor. \tag{3}$$

The extension group $H^1(L \oplus L^2)$ therefore vanishes, so this sequence splits as a holomorphic vector-bundle sequence.

⟨1⟩3. We verify that this quotient is the coherent extension at the cusp. The normalization resolution of $\mathcal{O}_W$ has one toric threefold, seven toric surfaces, twelve toric curves and six points. All positive coherent cohomology of those smooth complete toric strata vanishes (Cox et al., 2011, Theorem 9.2.3, p. 410). Their constants form the cellular cochain complex of the periodic Kuhn triangulation modulo $\mathbb{Z}^3$: its simplices are the faces of $(0, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, (1,1,1))$ for $\pi \in S_3$, identified by integral translations. The quotient map $x \mapsto x \bmod \mathbb{Z}^3$ identifies its realization with the dual three-torus and respects every face map. Its cohomology is $\bigwedge^q \mathbb{C}^3$, with the exterior cup product. The face incidences, unit Smith factors and the Alexander–Whitney products of the three coordinate cocycles are reconstructed in `independent_V_analytic_06.py` and its JSON output. The invariant flat cohomology $\ker N$ maps to these classes by constant Cech cocycles on the toroidal covering. A component translation in that cover evaluates the corresponding lattice coordinate. In the nilpotent logarithmic frame, $\ker N$ maps isomorphically to V/F, since $N : F \to \ker N$ is an isomorphism. Thus these are the extending quotient classes.

⟨1⟩4. Flatness over each smooth root disc follows because its base is a DVR and the smooth total space is torsion free over it. Properness was proved in theorems 5.6 and 6.8. The coherent fibre dimensions are the constant binomial dimensions on smooth tori, finite root fibres, and the cusp. (Grauert, 1960, §7, Satz 5), with these hypotheses, gives local freeness and base change. The cup-product maps are isomorphisms on every fibre, including the preceding central cellular model, so they identify the remaining direct images with $\bigwedge^q G$.

⟨1⟩5. There is an exact sequence of locally free logarithmic bundles

$$0 \longrightarrow h^* L^{11} \longrightarrow \Omega^1_M(\log W) \longrightarrow h^* F \longrightarrow 0. \tag{4}$$

In a chart $q = x_0 \cdots x_r$, its first arrow sends dq/q to $\sum dx_i/x_i$ and its quotient is the height-zero logarithmic cotangent bundle of the torus. These formulas also prove exactness on every stratum. Taking determinants and cancelling $\mathcal{O}_M(W) = h^* L^{20}$ yields $K_M = h^* L^{11+3-20} = h^* L^{-6}$.

⟨1⟩6. For any bundle which is a sum of powers of L, projection and the exact finite averaging projector compute its direct images by tensoring with $\bigwedge^q G$ and applying (3). All exponents $-1, \dots, -5$ have coarse degree -1, while exponent zero has degree zero. The Leray sequence on the curve then gives the asserted structure-sheaf cohomology. This includes the nonreduced coarse fibres, since the computation was made before exact finite descent.

□

Theorem 9.2 (The Hodge numbers)**.** [tag `02R2`] *The complete Hodge table of every admissible fourfold is the following, with rows indexed by p and columns by q:*

$h^{p,q}$	0	1	2	3	4
0	1	1	0	0	0
1	0	2	1	0	0
2	0	1	6	1	0
3	0	0	1	2	0
4	0	0	0	1	1

Proof. ⟨1⟩1. All maps needed for the two nontrivial rows can be evaluated at the cusp. We give their matrices and the reason evaluation determines their global ranks. In the algebraic splitting of theorem 8.1 write the three generators of F as (f_0, f_1, f_2) of weights $(0, 1, 2)$, and those of G as (g_0, g_1, g_2) of weights $(0, -1, -2)$. The residue of the quotient connection is

$$R : F_0 \longrightarrow G_0, \qquad R = \begin{pmatrix} 0 & A & B \\ C & 0 & D_0 \\ E & D_0 & 0 \end{pmatrix},$$

where

$$A = -125(192z - 3125)/12288, \qquad B = -78125(96z + 3125)/393216,$$
$$C = -125/512, \qquad E = 78125/32768,$$
$$D_0 = -146484375/2097152.$$

Here the lifted first vector is $w_0 + (4/3)ze$. These entries result from the exact Gauss–Manin reduction of theorem 8.1, with the quotient projection taken after lifting F. The full projected residue and both connecting determinants are recomputed in `independent_V_analytic_06.py`. Its JSON output also records the root-line cohomology sums used below.

⟨1⟩2. The connecting map of (4), on relative cohomology, sends $f \otimes \eta$ to $R(f) \wedge \eta$ at the cusp. To see the extension class, trivialize logarithmic forms on the toric covering. A deck translation changes a relative logarithmic form by its period derivative times dq/q. The constant Cech class of this derivative is exactly the residue of the quotient connection. This proves the formula with the indicated frames. It also proves its exterior power version, with the usual alternating signs.

⟨1⟩3. The floor formula gives the following absolute cohomology:

	h^0	h^1	h^2	h^3	h^4
h^*F	1	3	2	0	0
h^*L^{11}	0	2	3	1	0
$h^*(\bigwedge^2 F)$	0	2	3	1	0
$h^*(L^{11} \otimes F)$	2	6	7	3	0.

Every group in this display comes from global sections of the indicated direct image; none comes from H^1 of a line on the coarse base. Every contributing summand has coarse degree zero, so evaluation at the cusp is injective on its space of sections. For the equal-dimensional source and target spaces used below it is an isomorphism onto the displayed weight subspaces. Thus fibrewise evaluation computes the ranks of the absolute connecting maps.

⟨1⟩4. For one-forms, the three middle source classes are $f_i \otimes g_i$. In the ordered target (g_0g_1, g_0g_2, g_1g_2), the matrix is

$$C_1 = \begin{pmatrix} -C & A & 0 \\ -E & 0 & B \\ 0 & -D_0 & D_0 \end{pmatrix}, \qquad \det C_1 = \frac{476837158203125}{281474976710656}(384z + 3125).$$

The degree-zero connecting map is nonzero because $R(f_0)$ has nonzero g_1, g_2 coefficients. The degree-two connecting map, from the classes $f_1g_0g_1, f_2g_0g_2$ to $g_0g_1g_2$, has rank one because $D_0 \neq 0$.

⟨1⟩5. We must exclude the sole possible zero of $\det C_1$ on the whole admissible parameter domain. Theorem 8.1 identifies $z = (625\pi i/128)\xi$ on the full complex parameter plane, and hence $z = -(625\pi/128)c$ when $\xi = ic$. The unique exceptional parameter is therefore $\xi = 5i/(3\pi)$. By theorem 8.4, the admissible half-plane is $\operatorname{Im}\xi > -h_{b,\min} > 1$, whereas $5/(3\pi) < 1$. Thus this zero lies outside the whole admissible parameter domain. The bound used here is theorem 8.4. Consequently the three connecting ranks are $1, 3, 1$. The long exact sequence of (4) gives

$$h^q(\Omega^1_M(\log W)) = (0, 1, 1, 0, 0).$$

⟨1⟩6. The residue sequence

$$0 \to \Omega^1_M \to \Omega^1_M(\log W) \to \nu_*\mathcal{O}_Y \to 0$$

is exact even at self-intersections: in a local branch chart it records independently the coefficient of each dx_i/x_i on $x_i = 0$. The normalization Y is smooth complete toric, so its only coherent structure cohomology is $H^0 = \mathbb{C}$ (Cox et al., 2011, Theorem 9.2.3, p. 410). The one-form row follows immediately.

⟨1⟩7. For two-forms take the second exterior power of (4). Its short exact sequence has left term $h^*(L^{11} \otimes F)$ and right term $h^*(\bigwedge^2 F)$. In degree two the three source classes are $f_i f_j \otimes g_i g_j$, $i < j$. The target is $F \otimes g_0 g_1 g_2$. Its connecting matrix is

$$C_2 = \begin{pmatrix} -D_0 & D_0 & 0 \\ E & 0 & -B \\ 0 & -C & A \end{pmatrix}, \qquad \det C_2 = -\det C_1.$$

Thus it is an isomorphism, and $h^3(\Omega^2_M(\log W)) = 1$.

⟨1⟩8. For completeness the ordinary/logarithmic comparison in this degree is the exact residue resolution

$$0 \to \Omega^2_M \to \Omega^2_M(\log W) \to \nu_*\Omega^1_Y(\log \partial Y) \to \nu^{[2]}_*\mathcal{O}_{W^{[2]}} \to 0.$$

The last space is the disjoint normalization of the seven double surfaces. Local exactness is verified by writing a logarithmic two-form as a sum of $a_{ij}\, dx_i/x_i \wedge dx_j/x_j$ and terms with ordinary differentials. The two residues of an ij term have opposite sign; their cancellation is the sole compatibility. Every local double-surface coefficient lifts to one branch, which proves surjectivity at the last term. Both final sheaves have zero positive cohomology: the first is the pushforward of the trivial rank-three toric logarithmic bundle, and the second has complete toric surface components. Writing $Q = \Omega^2_M(\log W)/\Omega^2_M$, we obtain $H^q(Q) = 0$ for $q \geq 2$. Hence $h^3(\Omega^2_M) = 1$. In degree zero the last residue map is zero: the fourteen toric boundary divisors of Y occur in seven opposite pairs; their primitive normal vectors have opposite signs and their gluing is a torus translation. This gives the zero 7-by-3 residue matrix on global logarithmic one-forms. Thus $H^0(Q) = \mathbb{C}^3$. The two global logarithmic two-forms come from the L^{12}, L^{13} summands of $L^{11} \otimes F$; their residues are the independent f_1, f_2 directions at the cusp. Therefore $h^0(\Omega^2_M) = 0$.

⟨1⟩9. Serre duality (Serre, 1955, Theorem 4, p. 22) on this compact smooth complex fourfold gives $h^{2,1} = h^{2,3} = 1$. Finally the analytic Riemann–Roch theorem (Hirzebruch, 1978, Theorem 25.3.1, p. 187; §25.4(a), pp. 188–190) applies to the Dolbeault elliptic operator of Ω^p_M. No Kähler hypothesis occurs there. Using $c_1 = c_3 = 0$, the Chern-root product gives

$$\chi_y(M) = \frac{(1+y)^4}{240}\, c_2^2 + \left(\frac{(-1+10y-y^2)^2}{144} - \frac{(1+y)^4}{120} \right) c_4 = -y + 4y^2 - y^3.$$

The exact Chern-number substitution and the resulting polynomial are verified in `independent_V_analytic_06.py`. It gives $\chi(\Omega^2_M) = 4$, hence $h^{2,2} = 6$. Theorem 9.1 and Serre duality supply all other entries of the table.

□

Theorem 9.3 (Tangent cohomology and the Kuranishi germ)**.** [tag `02R3`] *For every admissible fourfold,*

$$(h^0(T_M), h^1(T_M), h^2(T_M), h^3(T_M), h^4(T_M)) = (1, 1, 1, 2, 0).$$

The complex parameter family has a nonzero Kodaira–Spencer map at every member and supplies a smooth one-dimensional miniversal Kuranishi germ.

Proof. ⟨1⟩1. Put $a = -256/3125$, and denote the whole reduced cusp fibre by W. The logarithmic sequence established in the proof of theorem 9.2, tensored with $K_M = h^*L^{-6}$ from theorem 9.1, is

$$0 \longrightarrow h^*L^5 \longrightarrow \Omega^1_M(\log W) \otimes K_M \longrightarrow h^*(F \otimes L^{-6}) \longrightarrow 0.$$

Here $K_{\widehat{B}}(a) = L^{11}$. More precisely, in the meromorphic L-frames of theorem 8.1, $\kappa = d\lambda/(\lambda - a)$ is λ^9 times the L^{11}-frame: both sides have divisor $3D_0 - D_\infty$. The logarithmic sequence is exact in each chart $q = z_0 \cdots z_r$, where its first arrow sends dq/q to $\sum_{j=0}^{r} dz_j/z_j$; the quotient is the relative translation cotangent bundle h^*F. These arrows agree on the monomial and deck transitions and descend through the free finite root-chart actions.

⟨1⟩2. For a summand L^n, the coarse degree is $d(n) = \lfloor 4n/5 \rfloor + \lfloor -3n/4 \rfloor$. Projection formula and $R^j h_* \mathcal{O}_M = \bigwedge^j G$ compute the two end terms. For h^*L^5 the weights in relative degrees $j = 0, 1, 2, 3$ are

$$\{5\}, \quad \{5, 4, 3\}, \quad \{4, 3, 2\}, \quad \{2\}.$$

Weights $4, 5$ have degree zero and weights $2, 3$ have degree -1. Thus $h^q(h^*L^5) = (1, 2, 1, 0, 0)$. The three weights of $F \otimes L^{-6}$ are $-4, -5, -6$. After tensoring with $\bigwedge^j G$, every weight is between -9 and -4. We have $d(-9) = -2$ and $d(n) = -1$ for $-8 \le n \le -4$. Weight -9 occurs once in each of relative degrees 2 and 3, so

$$h^q\big(h^*(F \otimes L^{-6})\big) = (0, 0, 0, 1, 1).$$

The base has cohomological dimension one, so the two-column Leray spectral sequences have no further differential. Every connecting map for the displayed logarithmic sequence has zero target. Consequently

$$h^q(\Omega^1_M(\log W) \otimes K_M) = (1, 2, 1, 1, 1).$$

⟨1⟩3. Let $\nu : Y \to W$ be the branch normalization. The residue sequence, with its branchwise maps, is

$$0 \longrightarrow \Omega^1_M \otimes K_M \longrightarrow \Omega^1_M(\log W) \otimes K_M \longrightarrow \nu_* \mathcal{O}_Y \otimes K_M|_W \longrightarrow 0.$$

Since K_M is pulled back from the base, its restriction to W is a constant line. The complete smooth toric variety Y has $H^0(\mathcal{O}_Y) = \mathbb{C}$ and $H^{>0}(\mathcal{O}_Y) = 0$ (Cox et al., 2011, Theorem 9.2.3, p. 410, with divisor zero). The sole global section of the logarithmic middle term comes from h^*L^5. Its meromorphic expression is $\lambda^{-5}\kappa$ tensored with the L^{-6}-frame: the unique L^5 section is λ^4 times its frame, and κ contributes λ^9. Its residue at a is $a^{-5} \neq 0$. Hence the map on global sections is an isomorphism onto the residue line, and

$$h^q(\Omega^1_M \otimes K_M) = (0, 2, 1, 1, 1).$$

The pointwise perfect evaluation pairing $T_M \otimes (\Omega^1_M \otimes K_M) \to K_M$ induces analytic Serre duality (Serre, 1955, §13, Theorem 4, p. 22; §15, p. 23). It identifies $H^q(T_M)^\vee$ with $H^{4-q}(\Omega^1_M \otimes K_M)$ and gives the stated tangent cohomology dimensions.

⟨1⟩4. We check the class of the parameter family on the smooth fibre $A = h^{-1}(1)$. In the fixed frame its period plane has basis

$$w_2, \quad w_1, \quad w_0 + \frac{625\pi i}{96}\xi e.$$

Differentiation in ξ, followed by projection to $G = H/F$, gives the rank-one map $P : F \to G$ with sole nonzero matrix entry $P_{13} = 625\pi i/96$ in the ordered bases above and (e, w_{-1}, w_{-2})

of the quotient. Differentiation in the base coordinate gives, by projecting the connection in theorem 8.1, the matrix

$$Q = \frac{1}{3125(1-a)} \begin{pmatrix} -78125/6 + 300r & -390625/48 - 375r & 0 \\ 120 & 0 & 125/2 \\ 0 & 120 & 50 \end{pmatrix}, \qquad r = \frac{625\pi}{96}(-i\xi).$$

Its determinant is $3/[625(1-a)^3] \neq 0$. The translation-valued Čech cocycles obtained by differentiating the lattice gluing identify $H^1(T_A) = \mathrm{Hom}(F, G)$; contraction with a relative one-form sends the two cocycles to the displayed projected period derivatives, in the same convention.

⟨1⟩5. The normal tangent sequence on this fibre is

$$0 \longrightarrow T_A \longrightarrow T_M|_A \longrightarrow \mathcal{O}_A \otimes T_{\widehat{B},1} \longrightarrow 0.$$

Its connecting map sends a base tangent vector to its Kodaira–Spencer class Q. As $H^0(\mathcal{O}_A) = \mathbb{C}$, the kernel of $H^1(T_A) \to H^1(T_M|_A)$ is exactly $\mathbb{C}Q$. The nonzero matrix P has determinant zero; every nonzero scalar multiple of Q has nonzero determinant. Therefore the image of P in $H^1(T_M|_A)$ is nonzero. The global parameter Kodaira–Spencer cocycle restricts to this image, so its class in $H^1(T_M)$ is nonzero. The finite affine actions are fixed, and the cusp multipliers $\exp(2\pi i C(q,\xi)\nu)$ vary holomorphically in ξ. The same finite fan charts, with C bounded on a small parameter neighbourhood, give the proper local family of theorem 6.8, theorem 5.6. Thus the cocycle just computed is the derivative of this compact family.

⟨1⟩6. Douady's versality and semiuniversality theorems (Douady, 1974, §VII.8, pp. 598–599; §VIII.1, pp. 599–600) give a miniversal germ with tangent space $H^1(T_M) = \mathbb{C}$. Its local ring has a minimal presentation $\mathbb{C}\{u\}/I$, where $I \subset (u)^2$. The parameter disc classifies by $u \mapsto g(s)$ with $g'(0) \neq 0$, by the preceding step. Substitution $\mathbb{C}\{u\} \to \mathbb{C}\{s\}$ is therefore an isomorphism of convergent power-series rings. Since this map factors through $\mathbb{C}\{u\}/I$, it annihilates I; its injectivity gives $I = 0$. The miniversal germ is reduced and smooth of dimension one, and the classifying map is a local isomorphism. This proves miniversality of the given family at each member.

□

Theorem 9.4 (Algebraic dimension and automorphisms)**.** [tag 02R4] *The fourfold has algebraic dimension one and*

$$\mathrm{Aut}^0(M) = \mathbb{C}, \qquad \mathrm{Aut}(M) = \mathbb{C} \times \mathbb{Z}/2.$$

In a fixed holomorphic splitting, two members of the full admissible complex parameter half-plane are biholomorphic exactly when their parameters are equal.

Proof. ⟨1⟩1. An effective divisor dominating the base would restrict to a nonempty effective divisor on a general complex torus. Pairing its Chern class with the square of a Kähler form on that torus is strictly positive. The restriction class is an alternating integral form J on Λ fixed by both monodromies:

$$M_1^t J M_1 = J, \qquad M_2^t J M_2 = J.$$

The exact linear system in its fifteen alternating entries has rank fifteen, so $J = 0$. This contradicts the positive pairing, and every divisor is vertical. The pole divisor of a meromorphic function is vertical; its restriction to a general compact fibre is holomorphic and therefore constant. The compact reduced graph of the function maps properly to

$\mathbb{P}^1 \times \mathbb{P}^1$ by h on its first factor. Remmert's proper-image theorem, in (Grauert, 1960, §7, Satz 1, p. 60), makes its image analytic. It is generically a single graph and therefore defines the descended meromorphic function on the normal base curve, including the three special values. Conversely pullbacks from the base occur. Hence the meromorphic function field is $\mathbb{C}(t)$ and $a(M) = 1$. Every automorphism consequently induces a Möbius map on the base. The two distinct multiplicities and the cusp identify all three values separately, so this map is the identity.

⟨1⟩2. On smooth fibres an automorphism is affine. Its integral linear part is locally constant and commutes with M_1, M_2. The two simultaneous commutator equations have rational solution space $\mathbb{Q}I$, as recomputed in `independent_V_extension_06.py`; their integral invertible elements are therefore $\pm I$. At the order-five filling an affine lift with linear part ϵI would have to conjugate the same finite affine generator. For its translation b this requires

$$(I - M_1)b = (1 - \epsilon)v_1/5 \pmod{\Lambda}.$$

Applying the invariant integral covector ℓ makes the left side zero. For $\epsilon = -1$ the right side is $2/5 \pmod{}Z$, which is nonzero. Thus every automorphism is a translation.

⟨1⟩3. Let $\mathcal{A}$ be the sheaf of translations extending to the cusp model and finite root charts. There is an exact analytic sheaf sequence

$$0 \to j_*\Lambda \to E_0 \to \mathcal{A} \to 0$$

on the root orbifold. Here j omits the cusp and the stalk of $j_*\Lambda$ there is $\ker N$. On a finite root cover the torus evaluation sequence is exact and equivariant for the prescribed affine action; its translation part cancels under conjugation. We descend this sequence on the orbifold, with its finite stabilizer cohomology.

To check the last surjectivity at the cusp, lift a translation to the toroidal cover. It preserves the fan precisely when its valuation is an integral translation of the periodic fan. Subtract the corresponding period; the remaining characters are nonvanishing holomorphic functions on a disc and possess logarithms. These give a local section of E_0. The kernel consists exactly of the invariant integral periods. This also proves that the sequence accounts for all extendable translations, including possible component permutations.

⟨1⟩4. Write $A = M_1$, $B = M_2$ and $N_1 = I + A + \cdots + A^4$, $N_2 = I + B + B^2 + B^3$. The orbifold fundamental group away from the cusp has generators g_1, g_2 with relations $g_1^5 = g_2^4 = 1$. A lattice cocycle is therefore a pair (x_1, x_2) with $N_1 x_1 = N_2 x_2 = 0$. The degree-one exact sequence for j_* says that it extends across the cusp precisely when its value on $g_1 g_2$ is a coboundary:

$$x_1 + Ax_2 = (AB - I)u, \qquad u \in \Lambda.$$

Subtract the coboundary of u. The resulting pair is $(-Ay, y)$, where $N_1 y = N_2 y = 0$. The ambiguity in u is $t \in \ker(AB - I)$, which changes y by $(B - I)t$. Thus the cohomology group, with the finite-root torsion, is

$$H^1(\widehat{B}, j_*\Lambda) = \frac{\ker N_1 \cap \ker N_2}{(B - I)\ker(AB - I)}.$$

Primitive integral column bases for its numerator and for $\ker(AB - I)$ are respectively

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Direct multiplication gives

$$(B-I)T = KZ, \qquad Z = \begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The determinant of Z has absolute value two; an entry and a 2-by-2 minor have absolute value one. Its Smith factors are $1, 1, 2$, proving $H^1(\widehat{B}, j_*\Lambda) = \mathbb{Z}/2$. The norm-kernel ranks, primitive column bases, full identity $(B-I)T = KZ$ and Smith factors are recomputed exactly in `independent_V_analytic_06.py`.

⟨1⟩5. Now $H^0(j_*\Lambda) = 0$ by theorem 5.2, $H^0(E_0) = \mathbb{C}$, and $H^1(E_0) = 0$ by the floor calculation. The exact sequence gives

$$0 \to \mathbb{C} \to H^0(\mathcal{A}) \to \mathbb{Z}/2 \to 0.$$

The half-period $\tau = (0, 1, 1, 1, 1, 1)^t/2$ is fixed modulo Λ by every monodromy matrix. It extends over the finite quotients because the translations commute there. At the cusp $N\tau$ is integral; primitiveness of N provides $\lambda \in \Lambda$ with $N\lambda = N\tau$. The representative $\tau - \lambda \in \ker N$ is a bounded angular half-period and extends on every toric chart. It is a nonzero order-two global section. Since the additive group $\mathbb{C}$ has no nonzero torsion, it splits the displayed exact sequence. Translations commute, giving the asserted full group.

⟨1⟩6. For an isomorphism between two members, the same intrinsic base argument applies. Its integral linear part still lies in the common centralizer. Either scalar $\pm I$ preserves a complex period plane exactly by equality of the two planes. The graphs $(\sigma + \xi\alpha e)(P)$ are equal precisely when their parameters ξ agree, since $e \notin F$ and α is nonzero. Translation does not change this condition. This proves the parameter assertion.

□

Theorem 9.5 (No antiholomorphic isomorphisms)**.** [tag `02QQ`] *For the cusp marking S fixed in theorem 5.2 and the corresponding Kuhn fan, none of the completed fourfolds in this family admits an anti-holomorphic automorphism. In particular none admits a real structure. More strongly, two members with this same cusp marking are never anti-biholomorphic.*

Proof. ⟨1⟩1. We first recover the base intrinsically. An effective divisor on M_c cannot be horizontal: its first Chern class vanishes by $H^2(M_c; \mathbb{Z}) = 0$, whereas its nonzero effective restriction to a general torus fibre would have positive pairing with the square of a flat Kähler class. The pole divisor of any meromorphic function is therefore vertical. On every regular fibre outside that divisor the function is holomorphic and hence constant. It factors through a meromorphic function on the base, which is rational on $\mathbb{P}^1$. Thus the function field is exactly $\mathbb{C}(\lambda)$. An anti-biholomorphism consequently induces an anti-Möbius map of the base. The two distinct multiplicities $5, 4$ and the singular cusp support distinguish the three special values, so this base map fixes $\infty, 0, \lambda_*$. As those points are real, the map is precisely $\lambda \mapsto \bar{\lambda}$.

⟨1⟩2. A holomorphic map between compact complex tori is affine: after lifting to their vector spaces, its derivative is periodic and therefore constant, since it descends to holomorphic functions on a compact torus. Applying this to an anti-holomorphic map after conjugating its source gives the same assertion for its anti-linear lift. Its induced integral map on H_1 is constant along the regular base. Complex conjugation reverses the two finite meridians, so this matrix must simultaneously conjugate M_i to M_i^{-1}. Solving these exact integral linear

equations gives only R and $-R$ among unimodular matrices, where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad R^2 = I.$$

Indeed the simultaneous intertwiner space over $\mathbb{Q}$ has dimension one and this primitive matrix is unimodular; determinant one in absolute value permits only the scalar factors ± 1. The exact anti-intertwiner equations, their one-dimensional rational solution space, unimodularity of R, the based cusp block and the specific ray obstruction below are recomputed in `independent_V_markings_06.py` and its JSON output.

⟨1⟩3. The finite affine cap excludes $G = R$. Normalize the pullback by $t = u^{m_i}$; any base-preserving isomorphism lifts to this canonical root cover and conjugates its deck generator to the inverse. At its central torus the translation part t_i would satisfy

$$(I - M_i^{-1})t_i = -(v_i + Gv_i)/m_i \pmod{\Lambda}.$$

Applying ℓ, and using $\ell R = \ell$, gives $0 = -2\ell(v_i)/m_i$ in $\mathbb{R}/\mathbb{Z}$, impossible already for $m_i = 5$. Hence $G = -R$ is necessary.

⟨1⟩4. We now use the cusp marking, keeping track of the conjugation of based meridians. Conjugation sends $g_0 = (g_1 g_2)^{-1}$ to $g_2 g_1 = g_2 g_0^{-1} g_2^{-1}$. Thus the matrix in a fixed local cusp frame is $H = M_2^{-1} G$, and it anti-conjugates M_0. Exact multiplication in the marking S gives

$$S^{-1}HS = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} P & Q \\ 0 & -P \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}. \quad (1.17.1)$$

Changing the based cusp path by a cusp loop multiplies by a power of M_0 and does not change P.

⟨1⟩5. We explain why an arbitrary translation cannot remove the obstruction in (1.17.1). In the additive torus coordinates whose first periods are the a_j, the anti-linear map must have the form $\zeta' = P\bar{\zeta} + t(q)$, because its values on these three real periods already determine its linear part. In multiplicative coordinates it is $z' = u(q)\bar{z}^{-P}$. The source zero section limits to a smooth point of the central fibre. An isomorphism takes it to another smooth point. The central fibre is irreducible, and its smooth locus is exactly the dense torus of its toric normalization: every toric boundary divisor is a normal-crossings self-identification. In a chart about this image, the translation therefore has the form q^v times nonvanishing holomorphic units, with $v \in \mathbb{Z}^3$. Translating by the period v changes the selected component chart; the units act by torus translation. Neither changes the linear action $-P$ on the fan.

⟨1⟩6. The primitive rays of the normal fan of that component are

$$\{\pm v : 0 \neq v \in \{0,1\}^3\}.$$

An extending torus monomial isomorphism must permute these rays. This can be seen directly on a one-parameter arc $z = t^v$ meeting the generic point of its boundary divisor:

if the image vector lies in the interior of a cone of dimension at least two, that divisor is contracted, which is impossible for an isomorphism. Complex conjugation itself preserves the toric charts, so the same criterion applies to the anti-monomial map above. But $-Pe_2 = (0,-1,1)^t$ is not one of the displayed rays. This is a contradiction. We have exhausted both possible integral linear parts, so no anti-biholomorphism exists.

□

10 The boundary of the family

Fix one splitting and write the complex parameter as ξ, so that $F_\xi = (\sigma + \xi\alpha e)(P)$. Put $c_* = -\min h$ and $\mathcal{H} = \{\operatorname{Im}\xi > c_*\}$. The real family is $\xi = ic$, $c \in \mathbb{R}$. By an analytic extension across a boundary point we mean a proper Hausdorff complex-analytic family on a full complex disc whose restriction to the nonempty open part lying in $\mathcal{H}$ is the prescribed family. At infinity we use the coordinate $u = 1/\xi$. This definition permits singular central spaces, multiplicities, and additional central components.

The stability of complex tori under deformation in the large is established in (Catanese, 2004, definition, p. 261; Theorem 2.1, p. 264). The relative-form calculation below writes the torus quotient on the regular analytic open set used in the boundary argument.

Theorem 10.1 (No analytic extension across the boundary)**.** [tag 02R5] *There is no such analytic extension across any finite boundary point of $\mathcal{H}$, or across infinity. The same conclusion holds after any finite ramified change of the local boundary coordinate. In particular neither end of the real interval $c_* < c < +\infty$ has a proper Hausdorff analytic extension in this sense.*

Proof. ⟨1⟩1. We first identify the moving pencil on each known member. By theorem 9.1, $K_M^{-4} = h^*L^{24}$ and the fibres of h are connected. For the coarse map $\rho : \widehat{B} \to \mathbb{P}^1$, the root-divisor formula gives

$$\rho_* L^{24} = \mathcal{O}_{\mathbb{P}^1}\big(\lfloor 96/5 \rfloor\, \infty + \lfloor -72/4 \rfloor\, 0\big) = \mathcal{O}_{\mathbb{P}^1}(19\infty - 18\,0).$$

Its sections are spanned by $\lambda^{18}, \lambda^{19}$. On a root chart the invariant sections have a common local monomial factor, and their ratio cancels that factor. Thus $h^0(M, K_M^{-4}) = 2$, and the moving part of this complete system is exactly the marked base map, up to a projective coordinate change. In particular its base ideal is invertible on every member of the prescribed family.

⟨1⟩2. Suppose that a proper Hausdorff analytic extension over a full parameter disc exists. Take the reduced component containing the given nonempty open family and resolve it. Analytic resolution applies on a relatively compact neighbourhood of the compact central fibre and can be kept unchanged over the given smooth locus (Bierstone and Milman, 1997, Theorem 1.6 and its following analytic-neighbourhood paragraph, preprint p. 5). Write $p : Z \to \Delta$ for the resulting proper map from a smooth irreducible space. The proper images of the exceptional and critical loci are analytic (Grauert, 1960, §7, Satz 1, p. 60); they omit the given open parameter set. After shrinking the disc they are supported at its centre. Hence p is a proper holomorphic submersion over Δ^*, with compact fourfold fibres.

Put $\mathcal{J} = \omega_{Z/\Delta}^{-4}$. The sheaves $R^j p_* \mathcal{J}$ are coherent (Grauert, 1960, §6, Hauptsatz I, p. 59). Multiplication by a nonzero parameter germ is injective on $\mathcal{J}$ and on $p_*\mathcal{J}$, so $p_*\mathcal{J}$ is locally free over the disc and free after shrinking. The exact sequence for a fibre over t gives

$$0 \longrightarrow (p_*\mathcal{J})_t/\mathfrak{m}_t(p_*\mathcal{J})_t \longrightarrow H^0(Z_t, \mathcal{J}_t) \longrightarrow \ker\big((t'-t) : (R^1p_*\mathcal{J})_t \longrightarrow (R^1p_*\mathcal{J})_t\big) \longrightarrow 0.$$

The last term is supported in the torsion support of the coherent sheaf $R^1p_*\mathcal{J}$, a closed discrete analytic set. At a point of the given open family outside this support, the displayed

comparison shows that $p_*\mathcal{J}$ has rank two. Shrinking makes the support empty away from the centre. Thus a frame of $p_*\mathcal{J}$ gives the complete two-dimensional linear system on each punctured fibre.

⟨1⟩3. The two evaluation sections define a coherent base ideal and a meromorphic map $Z \dashrightarrow \mathbb{P}^1$ over the full disc. The ideal is locally invertible on the given open family by the first step. Its non-invertibility locus is analytic; its proper image in Δ is therefore discrete and can be removed away from the centre. Blowing up that ideal and resolving the graph consequently changes only the central fibre. It gives a proper holomorphic map to $\mathbb{P}^1 \times \Delta$. The construction uses the graph of the coherent evaluation map, so its coefficients have finite meromorphic order in the parameter.

Its critical image is proper analytic. Every component dominating the parameter disc meets the given open family, where there are precisely three critical values. Their degrees over the parameter are one. The multiplicities five and four distinguish two of them, and the third is the cusp. After shrinking, their normalizations give three distinct labelled meromorphic sections, and all vertical exceptional components lie over the centre. The projective transformation taking these sections to $\infty, 0, \lambda_*$ has meromorphic coefficients and is invertible on Δ^*. Resolving its graph gives the marked map H over the disc, still unchanged on punctured fibres. Its regular restriction is a proper holomorphic submersion

$$q : W \longrightarrow B = \left(\mathbb{P}^1 \setminus \{0, \infty, \lambda_*\}\right) \times \Delta^*.$$

Ehresmann transport makes the number of components of a fibre locally constant. The base is connected and the known fibres are connected, so every fibre of q is connected.

⟨1⟩4. We need the torus structure on an analytically open dense part of this regular family. The coherent sheaf $\mathcal{E} = q_*\Omega^1_{W/B}$ has generic rank three. Generic cohomology and base change for this flat bundle gives a proper analytic exceptional set outside which $\mathcal{E}$ is locally free of rank three and its fibre is the space of holomorphic one-forms. The constant-dimension comparison is (Grauert, 1960, §7, Satz 5, pp. 63–64). Use the global coherent evaluation map

$$q^*\mathcal{E} \longrightarrow \Omega^1_{W/B}.$$

Its cokernel has analytic support. The proper image of that support is analytic in B and omits the nonempty part of the known torus open set outside the generic exceptional set. Enlarge the exceptional set by this image. On its complement evaluation is an isomorphism of rank-three bundles.

Relative exterior differentiation defines an $\mathcal{O}_B$-linear map $\mathcal{E} \to q_*\Omega^2_{W/B}$. It vanishes on the known torus open set. The target is torsion-free, since it is the proper direct image of a vector bundle under a submersion. The identity theorem therefore makes this map zero. Each remaining compact connected fibre has three everywhere independent closed holomorphic one-forms. Their dual vector fields commute and are complete by compactness. Their flows give a transitive holomorphic $\mathbb{C}^3$-action: every orbit is open and the fibre is connected. Its stabilizer Γ is discrete, and compactness of $\mathbb{C}^3/\Gamma$ makes Γ a full real lattice. This is the complex-torus structure on each such fibre.

⟨1⟩5. Let $E \subset B$ be the proper analytic exceptional set just removed. On the universal cover of $B \setminus E$, use Gauss–Manin transport from the given open family to mark integral H^1. Integration of the closed relative forms gives a holomorphic map to $\mathrm{Gr}(3, 6)$. On the known open set it is the explicit graph $F_\xi = (\sigma + \xi\alpha e)(P)$. The identity theorem on the connected cover identifies these two maps everywhere. The explicit graph is a rank-three holomorphic plane for every finite ξ, including parameters for which it is not real-transverse. At infinity the same assertion is used with $\xi = 1/u$ on the punctured parameter disc. All changes of continuation marking are real integral and preserve the condition $F \cap \bar{F} = 0$.

⟨1⟩6. Let ξ_0 be a finite boundary point, so $\operatorname{Im}\xi_0 = c_* = -\min h$. The minimum may occur at a finite root point. Continuity there nevertheless gives regular λ with $h(\lambda)$ arbitrarily close to $\min h$. Every sufficiently small full parameter disc about ξ_0 therefore contains noncentral points of

$$\operatorname{Im}\xi + h(\lambda) = 0.$$

This equation defines nonempty smooth real hypersurface pieces in the complex surface B, since its derivative in $\operatorname{Im}\xi$ is one. A proper complex analytic subset of B has real dimension at most two; it cannot contain those three-dimensional pieces. Choose such a point outside E. Its fibre is a complex torus by the fourth step, so its period plane is real-transverse. The fifth step identifies that plane with F_ξ, whereas the exact criterion of theorem 6.7 gives $F_\xi \cap \overline{F_\xi} \neq 0$ at the chosen point. This is a contradiction.

At infinity fix a regular λ and set

$$\xi = R - ih(\lambda), \qquad u = \frac{1}{R - ih(\lambda)}, \qquad R \in \mathbb{R}.$$

These are noncentral points of the same failure locus and tend to $u = 0$ as $|R| \to \infty$. Varying λ and R again gives real hypersurface pieces in every punctured u-disc, so one can choose a point outside E and obtain the same contradiction. Under a finite ramified change of the boundary coordinate, the parameter map has nonzero derivative off the centre after shrinking. The inverse images of these hypersurface pieces remain nonempty real hypersurfaces there, and the argument is unchanged.

All contradictions occur on a noncentral compact smooth torus fibre of the recovered marked map. The reductions and modifications are proper and preserve the prescribed open family; they impose no smoothness, multiplicity or component condition on the original central fibre. The permitted central spaces therefore do not avoid the contradiction. Both nonextension assertions follow.

□

Theorem 10.2 (Families over a punctured disc)**.** [tag `02R6`] *Let $p : \mathcal{M} \to \Delta^*$ be a proper holomorphic submersion whose fibres are members of this fourfold family. Its classifying parameter has an interior limit. After a base change of degree at most two, it is isomorphic to the pullback of the constructed family by a holomorphic map of the full disc to $\mathcal{H}$, and therefore has a proper smooth filling. Such a family cannot approach either parameter end.*

Proof. ⟨1⟩1. At every punctured parameter, theorem 9.3 supplies local miniversality and a local holomorphic classifying map to $\mathcal{H}$. Theorem 9.4 makes this map unique on each geometric point. Thus these local maps agree on overlaps and define one holomorphic function $\xi : \Delta^* \to \mathcal{H}$. There is no quotient by parameter translations, by theorem 9.4.

⟨1⟩2. The Cayley coordinate

$$z = \frac{\xi - i(c_* + 1)}{\xi - i(c_* - 1)}$$

has $|z| < 1$. It extends holomorphically across the puncture by the removable-singularity theorem. Its value there has modulus strictly less than one, since a boundary value would force z to be constant by the maximum-modulus principle. The inverse Cayley map therefore extends ξ with value in $\mathcal{H}$.

⟨1⟩3. Local identifications with the pulled-back constructed family differ by its automorphisms. These form the constant holomorphic group $\mathbb{C} \times \mathbb{Z}/2$ over the parameter half-plane: the translation field and the half-period in theorem 9.4 extend over the entire family. The finite component torsor on Δ^* is killed by the double cover. For a coordinate z on Δ^*,

the functions z and z^{-1} confine the holomorphic hull of each compact set to a compact annulus; z also separates points and supplies a local coordinate. Thus Δ^* is Stein. Cartan's Theorem B (Cartan, 1952, Exposé 19, §5, pp. 19-14–19-15), applied to its coherent structure sheaf, gives $H^1(\Delta^*, \mathcal{O}) = 0$, so the remaining additive holomorphic torsor is a coboundary. Thus after that cover the local identifications glue to a global isomorphism. Pulling back the already constructed proper smooth family along the extended classifying map supplies the stated filling.

□

11 The effective integral input classification

Theorem 11.1 (the orbifold quotient)**.** [tag 02NY]

For the torus fillings of an admissible datum, with affine multiplicities $m_1, \dots, m_k$, the fundamental group of the total space has a canonical surjection onto

$$\langle x_1, \dots, x_k \mid x_i^{m_i} = 1,\ x_1 \cdots x_k = 1 \rangle. \tag{F.27}$$

Consequently a simply connected example with $k = 2$ requires $\gcd(m_1, m_2) = 1$. There is no simply connected example with $k = 3$ and $m_1, m_2, m_3 \geq 2$. An affine multiplicity-one point is an ordinary smooth point and reduces to the problem with one fewer multiple fibre.

Proof. ⟨1⟩1. Over the complement of small discs at the special values, the torus bundle projects surjectively on fundamental groups to the punctured base. Quotient its fundamental group by the normal closure of all regular-fibre loops. At a finite filling its root cover is a smooth torus bundle over a disc, and its free deck group is cyclic of order m_i. The covering exact sequence shows that the quotient of the filling group by its fibre subgroup is precisely this cyclic group. The boundary base meridian maps to its generator. Thus adjoining that filling imposes exactly $x_i^{m_i} = 1$ after the fibre loops have been killed. At the index-one Kuhn cusp the fundamental group is generated by the three deck loops of its dual torus; these are regular fibre loops. Its base meridian contracts. Hence its filling imposes the ordinary meridian relation and leaves no extra generator in this quotient. The base relation gives the product relation in (F.27). Van Kampen proves that (F.27) is the resulting quotient, in particular the asserted surjection.

⟨1⟩2. For two points, substitute $x_2 = x_1^{-1}$; the quotient is $\mathbb{Z}/\gcd(m_1, m_2)$. For three points of orders at least two it is a nontrivial triangle group. Nontriviality can be seen directly. Take a geodesic triangle with angles $\pi/m_1, \pi/m_2, \pi/m_3$ in the sphere, Euclidean plane or hyperbolic plane according as their sum is greater than, equal to or less than π. In the spherical case the possible orders are, after sorting, $(2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$, and the corresponding spherical triangles exist. The Euclidean and hyperbolic angle conditions give the other triangles directly. If r_1, r_2, r_3 are reflections in the opposite sides, the rotations $r_2 r_3, r_3 r_1, r_1 r_2$ have the stated orders and product one. They therefore give a nontrivial representation of (F.27) into the orientation-preserving isometry group. Its quotient cannot be trivial. Finally a free cyclic action of order one is the identity construction on a root disc of order one, hence an ordinary torus bundle filling and not a multiple fibre.

□

Proposition 11.2 (exclusion of an involutory finite part)**.** [tag 02NZ]

A simply connected datum with $b_2 = 0$, $k = 2$ and the stated finite affine fillings has neither finite linear monodromy of order one or two. Its unordered pair of finite linear orders must therefore belong to

$$\{(3, 4), (3, 5), (3, 8), (3, 10), (4, 5), (5, 6), (5, 8), (5, 12)\}. \tag{F.28}$$

The affine multiplicities may still be larger than these linear orders.

Proof. ⟨1⟩1. The linear orders divide the affine multiplicities and are therefore coprime by 11.1. The fixed-rank calculation of 11.3, which uses only 4.6 and the evenness in 11.9, gives fixed rank two at each finite generator. An identity generator would make the product finite and unipotent, hence the identity, contrary to the rank-three cusp.

⟨1⟩2. Suppose the cohomology generators are A, B with $A^2 = I$. The order of B is odd. The (-1)-eigenspace of A has dimension four, whereas $\ker(AB - I)$ has dimension three. Their intersection in the six-dimensional space contains a nonzero vector v. Then $ABv = v$ and $Av = -v$ imply

$$Bv = Av = -v,$$

which contradicts the odd order of B. Interchanging the two generators proves the same assertion for either finite site.

⟨1⟩3. Delete one and two from the complete finite-order list (F.3) and enumerate its coprime unordered pairs. They are exactly the eight pairs in (F.28). Only the divisibility of the linear orders into the affine multiplicities has been used.

□

Proposition 11.3 (the only first-cohomology pattern for two points)**.** [tag 0200]

For a simply connected two-multiple-fibre datum with $b_2 = 0$, the rational cohomology system V has one invariant, no coinvariants, and fixed rank two at each finite generator. Its quotient $W = V/\mathbb{Q}e$ by the invariant line has neither invariants nor coinvariants. If one finite characteristic polynomial on W is $(T-1)\Phi_d(T)$ with $\deg \Phi_d = 4$, then W is irreducible over $\mathbb{Q}$.

Proof. ⟨1⟩1. Write $a = \dim V^\Gamma$, $b = \dim(V^*)^\Gamma$ and f_i for the finite fixed ranks. Theorem 4.6 gives $a = 1$, vanishing first parabolic cohomology, and $f_1 + f_2 = 4 + b$. By 11.9 each f_i is positive and even, so b is even. The maximal square-zero cusp has $\ker N = \operatorname{im} N$. Therefore every globally fixed vector belongs to $\operatorname{im} N$, whereas every globally fixed covector kills $\operatorname{im} N$. The global invariant and coinvariant spaces pair to zero.

⟨1⟩2. Restriction of a globally invariant covector to the fixed space of a finite generator is injective: averaging that generator is the projection onto its fixed space, and the covector is unchanged by averaging. Its restriction kills the a global invariants in that space. Hence $a + b \le f_i$ for each i. If $b \ge 2$ is even, the even ranks satisfy $f_i \ge b + 2$, so $4 + b = f_1 + f_2 \ge 2b + 4$, a contradiction. Thus $b = 0$ and $f_1 = f_2 = 2$.

⟨1⟩3. The quotient W has no invariant covectors, since its dual embeds into V^*. If it had a fixed line, its preimage in V would be an extension of the trivial line by the trivial line. Every finite generator on this two-dimensional space would be unipotent of finite order, hence identity. This would give two invariants in V, a contradiction. Thus W also has no invariant vectors.

⟨1⟩4. If one finite characteristic polynomial is $(T-1)\Phi_d(T)$ with the second factor irreducible of degree four, every proper nonzero rational submodule has dimension one or four. In dimension one that generator acts as the identity. Since the product of the two finite generators is unipotent, their induced one-dimensional characters are inverse; hence the other generator also acts trivially. This would be an invariant line. In dimension four the one-dimensional quotient is trivial by the same argument, giving a coinvariant. Both have just been excluded.

□

The exterior-square representation of $\mathrm{Sp}(4)$**.** We recall the classical five-dimensional orthogonal representation and the comparison of unipotent classes used below. Let $E = \mathbb{C}^4$ have a symplectic basis e_1, e_2, f_1, f_2. The invariant bivector $\omega = e_1 \wedge f_1 + e_2 \wedge f_2$ has nonzero square. The wedge pairing on $\bigwedge^2 E$ is symmetric and nondegenerate. Its orthogonal complement $E_0 = \omega^\perp$ therefore has dimension five and is nondegenerate. Preservation of ω^2 gives determinant one on E; $\det(\bigwedge^2 g) = (\det g)^3$ and the fixed line $\mathbb{C}\omega$ therefore give determinant one on E_0. The action gives

$$\rho : \mathrm{Sp}(E) \longrightarrow \mathrm{SO}(E_0).$$

Its differential is injective. Indeed, a symplectic infinitesimal transformation acting trivially on E_0 also acts trivially on $\mathbb{C}\omega$, hence on all of $\bigwedge^2 E$. In a basis, the equations $Xe_i \wedge e_j + e_i \wedge Xe_j = 0$ for every $i < j$ first make every off-diagonal coefficient of X zero, and then give $X_{ii} + X_{jj} = 0$ for every distinct pair. Since there are four indices, these equations make every diagonal coefficient zero. Both Lie algebras have dimension ten, so the image contains a neighbourhood of the identity in $\mathrm{SO}(E_0)$. That complex special orthogonal group is connected. For completeness, orthogonal elimination expresses an orthogonal matrix as a product of reflections in vectors of nonzero norm: to fix a nonisotropic vector, reflection in the difference of its image and itself works when that difference is nonisotropic; otherwise reflection in their sum followed by reflection in the vector works. At least one of sum and difference is nonisotropic. Continue on its nondegenerate orthogonal complement. Determinant one makes the number of reflections even. The nonisotropic vectors of a complex vector space of dimension at least two form a path-connected set, since two generic complex line segments can avoid the zero set of the norm polynomial except at finitely many parameter values, around which the paths can be detoured. Varying one vector in a pair of reflections to the other connects that pair to the identity. The one-dimensional final complement contributes only the already accounted-for sign. This proves connectedness. Thus the open image of ρ is the whole special orthogonal group.

The kernel is $\{I, -I\}$. If an element acts trivially on E_0, it also fixes ω and hence every bivector. It preserves every two-plane, as follows by applying it to a nonzero decomposable bivector of that plane. Intersections of two-planes show that it preserves every line. A linear map preserving every line is scalar: apply it to the coordinate lines and their pairwise sums. Its scalar has square one. In particular every finite orthogonal element of order m has lifts whose mth powers belong to $\{I, -I\}$; the lifts are finite and semisimple.

The symplectic transvection fixing e_1, e_2, f_2 and sending f_1 to $f_1 + e_1$ acts on the following primitive basis:

$$e_1 \wedge e_2, \quad e_1 \wedge f_2, \quad e_2 \wedge f_1, \quad f_1 \wedge f_2, \quad e_1 \wedge f_1 - e_2 \wedge f_2.$$

Its difference from the identity sends the third vector to the negative of the first and the fourth to the second, and annihilates the other three. Its orthogonal Jordan type is therefore $(2, 2, 1)$. Conversely, for an orthogonal unipotent $I + N$ of that type, $N^2 = 0$ implies $N^* = -N$, because $(I + N)^* = (I + N)^{-1} = I - N$. Its image is an isotropic plane. After choosing a basis of that plane and a dual isotropic plane, use the isometry on their residual nondegenerate line to make the identification of two such decompositions have determinant one. Changing its sign changes that determinant and commutes with N, which kills the residual line. In these decompositions, N is determined by a nonzero alternating form on the dual two-dimensional space. All such forms are equivalent under its general linear group. Extending that change by the inverse transpose on the first plane gives an orthogonal change of basis; its determinant is one. Thus these unipotents form one special-orthogonal conjugacy class. Surjectivity of ρ lifts the conjugacy. The two lifts of $I + N$ are a symplectic transvection and its negative.

Finally a symplectic diagonal element with eigenvalues x, x^{-1}, y, y^{-1} acts on E_0 with eigenvalues

$$1, \quad xy, \quad x/y, \quad y/x, \quad (xy)^{-1}.$$

This gives the stated eigenvalue comparison through the primitive exterior-square representation.

Theorem 11.4 (exclusion of three further order pairs)**.** [tag 02MZ]

No simply connected datum with $k = 2$, $b_2 = 0$, and finite linear orders $(3,5)$, $(3,8)$ or $(3,10)$ has the required maximal square-zero cusp. Among such data, the remaining necessary unordered linear order pairs are

$$\{(3,4),(4,5),(5,6),(5,8),(5,12)\}. \tag{F.30}$$

When one order is five, the other finite characteristic polynomial cannot be $(T-1)^2\Phi_4(T)^2$ or $(T-1)^2\Phi_6(T)^2$.

Proof. ⟨1⟩1. For each proposed pair, 11.3 gives an irreducible rational rank-five quotient W. Its cusp has Jordan type $(2,2,1)$. One finite polynomial is $(T-1)\Phi_3(T)^2$ and the other is $(T-1)\Phi_d(T)$ for $d = 5, 8, 10$. For the two additional assertions, replace Φ_3^2 by Φ_4^2 or Φ_6^2, with $d = 5$. In all these cases a commuting endomorphism of W restricts to a rational scalar on the one-dimensional fixed space of the second generator. This restriction is injective: a nonzero endomorphism of an irreducible representation is invertible and cannot vanish on that line. Consequently

$$\mathrm{End}_\Gamma(W) = \mathbb{Q}.$$

⟨1⟩2. Consider $\mathcal{H} = \mathrm{Hom}(W, W^*)$. The local fixed dimensions are $9, 5, 13$. At the repeated finite site, pairing reciprocal eigenspaces of dimensions $1, 2, 2$ gives $1 + 4 + 4 = 9$. At the distinct finite site it gives five. For the cusp, the ordered pairs of blocks of sizes $2, 2, 1$ contribute the sum of their minimum sizes, namely thirteen. The invariant-sheaf Euler formula therefore gives

$$\chi(\mathbb{P}^1, j_*\mathcal{H}) = -25 + 9 + 5 + 13 = 2. \tag{F.31}$$

Here degree zero is $\mathrm{Hom}_\Gamma(W, W^*)$, and degree two is dual to $\mathrm{Hom}_\Gamma(W^*, W)$. At least one is nonzero, so irreducibility gives a nondegenerate invariant bilinear form $J : W \longrightarrow W^*$, inverting the map if necessary. Its alternating part is equivariant; if nonzero it would be invertible, which is impossible in odd dimension. Thus J is symmetric.

⟨1⟩3. An invariant bivector is an alternating equivariant map $K : W^* \longrightarrow W$. The comparison map

$$K \longmapsto KJ \in \mathrm{End}_\Gamma(W)$$

makes it a scalar, so K is a scalar multiple of the symmetric map J^{-1}. Hence $K = 0$. Applying the same argument to an alternating map $W \longrightarrow W^*$ gives

$$(\Lambda^2 W)^\Gamma = (\Lambda^2 W^*)^\Gamma = 0.$$

Both degree zero and degree two of $j_*\Lambda^2 W$ vanish. Its Euler characteristic must therefore be nonpositive.

⟨1⟩4. The repeated finite spectrum has four fixed bivectors and the distinct spectrum has two. At the cusp, write the unipotent module as two blocks of size two and one block of size one. Their exterior square has two trivial blocks from the individual wedges, blocks of sizes three and one from the product of the two size-two blocks, and two further size-two blocks. Its fixed rank is six. Consequently

$$\chi(\mathbb{P}^1, j_*\Lambda^2 W) = -10 + 4 + 2 + 6 = 2,$$

contradicting the preceding step. This proves all five stated exclusions. Removing the three excluded order pairs from (F.28) leaves exactly (F.30).

□

Proposition 11.5 (rational uniqueness of the extension)**.** [tag 02N0]

Let W be a rational rank-five representation of a twice-punctured affine line, with two finite generators having fixed rank one and product with unipotent Jordan type $(2,2,1)$. Suppose W has neither invariants nor coinvariants. Up to rescaling the invariant line, there is exactly one rational extension $0 \to \mathbb{Q}e \to V \to W \to 0$ in which the two finite generators retain their finite orders and the cusp has square-zero logarithm of rank three. Thus, for the two-point data of 11.3, the remaining rational representation problem is exactly the problem of the rank-five quotients.

Proof. ⟨1⟩1. Write the finite generators of W as S_1, S_2 and the extension generators as

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & S_i \end{pmatrix}.$$

If S_i has order m_i, retaining that finite order is equivalent to $a_i(I + S_i + \cdots + S_i^{m_i-1}) = 0$. Averaging identifies this condition with $a_i \in \operatorname{row}(S_i - I)$, a space of dimension four. A change of splitting by $b \in W^*$ sends the two rows to $a_i + b(S_i - I)$. This coboundary map is injective because W has no invariant covectors. Thus the extension space subject to the two finite conditions has dimension $8 - 5 = 3$.

⟨1⟩2. Put $U = S_1S_2 = I + N$, so $N^2 = 0$ and $\operatorname{rank} N = 2$. The product A_1A_2 has upper row $c = a_1S_2 + a_2$. Changing the splitting sends c to $c + bN$. Hence cusp restriction is the explicit map

$$[(a_1, a_2)] \longmapsto [c] \in W^*/\operatorname{row} N.$$

It is an isomorphism. Indeed, a class in its kernel can be represented with $c = 0$, so $a_2 = -a_1S_2$. The finite conditions then say

$$a_1 \in \operatorname{row}(S_1 - I) \cap \operatorname{row}(S_2 - I).$$

This intersection has dimension three: the two four-dimensional row spaces span W^* because W has no common fixed vector. The rows b satisfying $bN = 0$ also form a three-dimensional space. Their map $b \mapsto b(S_1 - I)$ lands in the displayed intersection, since $bU = b$ gives $b(S_1 - I)S_2 = b - bS_2$. It is injective, because $bS_1 = b$ and $bU = b$ together give $bS_2 = b$, forcing $b = 0$. It is therefore onto. Every pair with $c = 0$ is a coboundary, proving injectivity of cusp restriction. Both its source and target have dimension three, so it is an isomorphism.

⟨1⟩3. The square of the lifted product minus identity vanishes exactly when $cN = 0$. Under this restriction isomorphism the square-zero extensions form

$$\frac{\operatorname{ann}(\operatorname{im} N)}{\operatorname{row} N}, \tag{F.32}$$

of dimension $3 - 2 = 1$. The lifted nilpotent has the rows of N together with the row c, so its rank is three exactly when $c \notin \operatorname{row} N$. These are precisely the nonzero classes in (F.32). Rescaling the invariant line rescales this one-dimensional rational space and is transitive on its nonzero classes. This proves the asserted existence and uniqueness with the stated finite orders and maximal cusp.

□

Theorem 11.6 (an integral obstruction for a rational $(5,12)$ family)**.** [tag 02N1]

Let a rational rank-six cohomology representation have generators

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A_5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & c \\ 0 & B_5 \end{pmatrix}, \qquad c = (0,0,0,1,0),$$

where

$$A_5 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad B_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}. \tag{F.33}$$

For every invariant integral lattice with an index-one maximal square-zero cusp, and every free affine filling of multiplicities m_1, m_2 divisible respectively by five and twelve, a simply connected total space has

$$\operatorname{Tor}_3 H_2 \simeq \mathbb{Z}/3^{v_3(m_2)}. \tag{F.34}$$

In particular none of these data is three-connected. The standard lattice nevertheless has a primitive cusp, free affine fillings of orders five and twelve, and a simply connected total space with rational Betti vector $(1,0,0,0,4,0,0,0,1)$.

Proof. ⟨1⟩1. Put $R = \mathbb{Z}_{(3)}$ and use the rational coordinates of (F.33). Let $\mathcal{P}_R$ be the algebra of blocks $\left(\begin{smallmatrix} a & b \\ 0 & D \end{smallmatrix}\right)$, where $a \in R$, $b \in R^{1\times 5}$ and $D \in \operatorname{Mat}_5(R)$. We first determine the integral algebra generated by A, B. The following ordered words have 31 coordinate columns in the order $(0,j)$, $0 \le j \le 5$, followed by (i,j), $1 \le i,j \le 5$:

$$I, A, B, A^2, AB, BA, B^2, A^3, A^2B, ABA, AB^2, BA^2,$$
$$BAB, B^2A, B^3, A^3B, A^2BA, A^2B^2, ABA^2, AB^2A,$$
$$AB^3, BA^3, BA^2B, BAB^2, B^2A^2, B^3A, A^3BA,$$
$$A^3B^2, A^2BA^2, BA^3B, B^2A^3.$$

Their coordinate determinant is -32. Its adjugate divided by -32 therefore expresses every elementary matrix of $\mathcal{P}_R$ as an R-linear combination of these words. Since $A, B \in \mathcal{P}_R$, this proves $R[A,B] = \mathcal{P}_R$ with the coordinate map.

For a stable lattice L_R, the elementary idempotents split its six coordinate fractional ideals; the matrices E_{ij} for $i,j \ge 1$ make the last five equal. Thus

$$L_R = 3^a R e_0 \oplus 3^b R^5, \qquad b \ge a,$$

where the inequality follows from the matrices E_{0j}. The matrix $N = AB - I$ has square zero and nonzero Smith factors $1,1,1$, while $A_5B_5 - I$ has factors $1,1$. In this basis of L_R, its upper-right row is multiplied by 3^{b-a}. For $b > a$ its reduction modulo three has rank two, contradicting the primitive rank-three cusp. Hence $b = a$. The scalar map $3^{-a}I$ identifies L_R with R^6 and intertwines every exterior power of the two generators. The given cusp operator has the same elementary divisors: $BA - I = A^{-1}(AB - I)A$, and $(BA)^{-1} - I = -(BA)^{-1}(BA - I)$. This proves the asserted local marking for every stable lattice.

All arithmetic in this item is reproduced by a separate program (§1.5). The argument from that determinant treats all lattices, without a bound on their indices.

⟨1⟩2. Write $v_i = g_i^{m_i}$ for the finite power vectors. They are fixed by the homology matrices M_i. The base-orbifold quotient in theorem 11.1 shows that simple connectivity forces $\gcd(m_1, m_2) = 1$. After the cusp meridian relation the two base generators are inverse up to a fibre translation. Their actions on the abelian fibre lattice are therefore inverse. Their linear orders five and twelve force both actions to be the identity in this quotient. The exact Smith factors of $[A^t - I \ B^t - I]$ are five units, with first row zero, so its fibre coinvariant module over R is R through the primitive character $\ell = x_0$.

Set $a_i = \ell(v_i)$. An arbitrary cusp clutch contributes a scalar δ to $g_1 + g_2 = \delta z$. The remaining relation matrix is

$$\begin{pmatrix} m_1 & 0 & -a_1 \\ 0 & m_2 & -a_2 \\ 1 & 1 & -\delta \end{pmatrix}.$$

Its determinant is a unit for a simply connected total space. Since $3 \mid m_2$ and $3 \nmid m_1$, reduction modulo three makes a_2 a unit in R. This conclusion includes arbitrary clutch translations and twists.

⟨1⟩3. We determine the second finite-cap restriction over R. Choose a primitive integral invariant character α whose R-span is ℓ. The stable sublattice

$$\Lambda' = \mathbb{Z}v_2 \oplus \ker \alpha$$

has index $|\alpha(v_2)|$, prime to three. It gives a finite torus cover; the affine action lifts because it preserves v_2, preserves $\ker \alpha$, and translates by v_2/m_2. The corresponding reduced cap F_2' covers F_2 with the same degree. In its basis beginning with v_2, the action is

$$(u, y) \longmapsto (u + 1/m_2, Dy).$$

Thus F_2' is the mapping torus of $D : T^5 \to T^5$, and its regular six-torus is the m_2-fold cover in the circle direction. The coefficient change has determinant a unit in R; under the scalar marking of item 1, the vertical cohomology action D^t is the displayed B_5.

The matrices $D^t - I$ and $\bigwedge^2 D^t - I$ have no nonunit three-primary Smith factors. Their Wang sequences therefore make $H^q(F_2'; R)$ free for $q \le 3$. Pullback to the generic torus is injective in these degrees, because transfer kills its kernel by m_2 and the source is free. Deck transformations of $F_2' \to F_2$ act by translations on the generic torus cover. Their action there is trivial on cohomology, so the injectivity just proved makes their action on $H^q(F_2'; R)$ trivial as well. Transfer for this prime-to-three cover now identifies $H^q(F_2; R)$ with $H^q(F_2'; R)$, compatibly with their restriction maps. In particular the degree-two and degree-three restrictions have zero kernel.

Put $T = D^t$. Its polynomial is $(x - 1)\Phi_{12}(x)$, and

$$E = \Phi_{12}(T) = T^4 - T^2 + I, \qquad E^2 = E, \qquad \sum_{j=0}^{11} T^j = 12E.$$

The image of this integral rank-one idempotent is the primitive invariant line. Hence the norm for every stated m_2 is $m_2 E$. The Wang sequence in degree two has quotient $\mathrm{Inv}(\bigwedge^2 T)$ and kernel $\mathrm{coker}(T - I)$. Restriction to the m_2-fold cover projects identically onto the former group. On the latter it is the horizontal wedge with the norm: a horizontal cell lifts to m_2 successive transported prisms. Consequently the restriction image J_2 projects onto the vertical invariants, and its intersection with the horizontal invariant summand is $e_0 \wedge m_2 \,\mathrm{Inv}(T)$. The horizontal inclusion induces the isomorphism

$$\frac{\mathrm{Inv}(T)}{m_2 \,\mathrm{Inv}(T)} \xrightarrow{\sim} \frac{\mathrm{Inv}(\bigwedge^2 V_R)}{J_2} \cong R/3^r R.$$

No choice of lifts of the vertical Wang classes changes this quotient or its kernel. At the first finite cap, m_1 is a unit in R, so transfer and averaging give the full invariant image and zero kernel. At the cusp, theorem 5.7 gives injective restrictions onto the invariant lattices in degrees one, two and three, by dualizing its saturated-kernel collapse.

⟨1⟩4. The integral specialization-sheaf approach is used in (Claude and L. Alpöge, 2026, Lemma 7.25 and Propositions 7.26–7.27, pp. 61–63) and (Engel, 2026, section "The integral homology"). Here are the coefficient maps and calculations for the present lattice. Put $L_q = \bigwedge^q V_R$, $A_q = \bigwedge^q A$, $B_q = \bigwedge^q B$, and let j include the thrice-punctured base. For the full invariant-extension sheaf j_*L_q the oriented cellular complex is

$$\begin{gathered} J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q, \\ J_{1,q} = \ker(A_q - I), \quad J_{2,q} = \ker(B_q - I), \quad J_{0,q} = \ker(B_q A_q - I), \\ d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0), \\ \Delta_q(a, b) = (A_q - I)a + (B_q - I)b. \end{gathered}$$

These are the base attachment maps: the two cusp paths carry j_0 and $A_q j_0$, whose combined boundary is $(B_q A_q - I)j_0 = 0$. Primitive invariant bases and a unimodular kernel basis of Δ_q give the following certificates:

q	$\operatorname{rank} H^0$	$\operatorname{rank} H^1$	$\operatorname{SNF}(d_0 \text{ in } \ker \Delta_q)$	$\operatorname{SNF}(\Delta_q)$
1	1	0	1^6	$1^5, 2$
2	0	0	1^{15}	1^{15}
3	0	3	$1^{16}, 2$	$1^{19}, 2.$

Here 1^s denotes s entries equal to one. The row-coordinate matrices and their unimodular HNF transformations are specified by the preceding formulas. In particular $H^0(j_*L_2) = H^1(j_*L_2) = 0$, $H^2(j_*L_1) = 0$ over R, and $H^0(j_*L_3) = 0$.

Write $\mathcal{R}_q = R^q f_* R$. The stalk/fibre identification is proper base change (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1). The local maps of item 3 give

$$0 \longrightarrow \mathcal{R}_2 \longrightarrow j_*L_2 \longrightarrow Q \longrightarrow 0, \qquad Q_{p_2} = R/3^r R,$$

with no other support. Hence its connecting homomorphism gives $H^1(\mathcal{R}_2) = R/3^r R$. The degree-one quotient is also point supported, so $H^2(\mathcal{R}_1) = H^2(j_*L_1) = 0$. The degree-three sheaf map is injective, so $H^0(\mathcal{R}_3) = 0$.

⟨1⟩5. The only total-degree-three term in the Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4) is therefore $E_2^{1,2} = R/3^r R$. It has no possible differential because the base has cohomological dimension two. Thus $H^3(M; R) = R/3^r R$. Universal coefficients (Hatcher, 2002, Theorem 3.2, p. 195) identify this torsion module with $\operatorname{Ext}_R(H_2(M; R), R)$: the quotient $\operatorname{Hom}_R(H_3(M; R), R)$ is free and must vanish. The elementary divisor decomposition over the PID R proves (F.34).

⟨1⟩6. For the standard lattice take

$$v_1 = (3, 0, 0, 0, 0, 0)^t, \qquad v_2 = (-7, 7, -7, 7, 0, 7)^t, \qquad (m_1, m_2) = (5, 12).$$

They are fixed by A^t, B^t respectively. Their ℓ-values $3, -7$ are units modulo the respective multiplicities; applying ℓ to a fixed-point equation proves that every nonidentity affine power is free. The primitive cusp in item 1 has the integral marking of proposition 4.4. Glue the two finite quotient caps and the index-one Kuhn cap to the marked torus bundle over a pair of pants, with zero additional clutch translation. The finite boundary identification is the prescribed logarithmic fibre translation; the cusp is attached in that integral marking. These local models give a closed smooth oriented manifold. Its remaining van Kampen matrix is

$$\begin{pmatrix} 5 & 0 & -3 \\ 0 & 12 & 7 \\ 1 & 1 & 0 \end{pmatrix},$$

with determinant of absolute value one. It is simply connected.

Over $\mathbb{Q}$ the finite images are full invariants. Item 4 gives vanishing $H^1(j_*V_{\mathbb{Q}})$, $H^0(j_*\bigwedge^2 V_{\mathbb{Q}})$, $H^1(j_*\bigwedge^2 V_{\mathbb{Q}})$ and $H^2(j_*V_{\mathbb{Q}})$, and the degree-three invariant group is zero. The five-term Leray sequence and $H^1(M;\mathbb{Q}) = 0$ make the map from the one-dimensional $H^0(\mathcal{R}_1)$ to $H^2(\mathbb{P}^1;\mathbb{Q})$ an isomorphism. Thus $b_2 = b_3 = 0$. Euler additivity gives $\chi(M) = 6$ from the Kuhn cusp; all other fibres have Euler number zero. Poincaré duality therefore gives $(1,0,0,0,4,0,0,0,1)$, as claimed.

□

Theorem 11.7 (two further rational families excluded integrally)**.** [tag 02N2]
Let C_f, C_g denote the degree-four companion matrices. Take $f = \Phi_5$ and either

$$\textit{case A: } g = \Phi_{12}, \qquad \textit{case B: } g = \Phi_3\Phi_6. \tag{F.36}$$

Let W be the quotient of $\bigwedge^2(C_f^{-1}, C_g)$ by its invariant line, and let V be its unique rational extension of 11.5. For every invariant integral lattice in V with primitive maximal square-zero cusp, every simply connected free affine filling, with multiplicities m_1, m_2 divisible by five and six respectively, fails three-connectivity. More precisely, put $r = v_3(m_2) \geq 1$. In case A, $\operatorname{Tor}_3 H_3$ has order 3^{2r-1}. In case B, $\operatorname{Tor}_3 H_2 \simeq \mathbb{Z}/3^r$. For $m_2 = 6$ the respective obstructing group is $\mathbb{Z}/3$.

Proof. ⟨1⟩1. We first give rational models, including the quotient maps. Use the exterior basis $(01, 02, 03, 12, 13, 23)$ for the companion pair. Its common primitive invariant vectors are

$$\omega_{\mathrm{A}} = (1,2,1,2,2,1)^t, \qquad \omega_{\mathrm{B}} = (1,0,1,0,0,1)^t.$$

Let T have columns $\omega, e_1, \dots, e_5$; its determinant is one. The last five rows of T^{-1} give the quotient map. In that quotient the two matrices a, b, and an extension row c, are

	a	b
A	$\begin{pmatrix} 2 & -1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 \\ 3 & 0 & -2 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix}$
B	$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix}$

$$c_{\mathrm{A}} = (-2,-3,1,2,0), \qquad c_{\mathrm{B}} = (0,-1,1,0,0), \qquad A = \operatorname{diag}(1,a), \quad B = \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix}.$$

Exterior eigenvalue products give characteristic polynomial $(x-1)\Phi_5$ for a, and $(x-1)(x+1)^2\Phi_3$ or $(x-1)(x+1)^2\Phi_6$ for b. Each has fixed rank one. Direct multiplication gives $A^5 = B^6 = I$, $(AB-I)^2 = 0$, and nonzero Smith factors $1,1,1$ for $AB-I$ and $1,1$ for $ab-I$. The row conditions are $c\sum_{j=0}^{5} b^j = 0$ and $c(ab-I) = 0$. The full quotient matrix algebra verified in the next item implies that this rank-five quotient is irreducible, with no invariant vectors or covectors. Thus proposition 11.5 identifies these models with the rational extensions, up to rescaling their invariant line.

⟨1⟩2. Put $R = \mathbb{Z}_{(3)}$ and let $\mathcal{P}_R$ be the algebra of blocks $\left(\begin{smallmatrix} u & v \\ 0 & D \end{smallmatrix}\right)$, with $u \in R$, $v \in R^{1\times 5}$ and $D \in \operatorname{Mat}_5(R)$. In the coordinate order $(0,j)$, $0 \leq j \leq 5$, followed by (i,j), $1 \leq i,j \leq 5$, the

following 31 words have coordinate determinants 512 and -512 in cases A and B:

$$I, A, B, A^2, AB, BA, B^2, A^3, A^2B, ABA, AB^2, BA^2, BAB, B^2A,$$
$$A^3B, A^2BA, A^2B^2, ABA^2, AB^2A, BA^3, BA^2B, BAB^2, B^2A^2, B^2AB,$$
$$A^3BA, A^3B^2, A^2BA^2, ABA^3, ABA^2B, BA^3B, A^3BA^2.$$

Their adjugates divided by the displayed determinants express every elementary matrix of $\mathcal{P}_R$ as an R-linear combination of these words. Consequently $R[A, B] = \mathcal{P}_R$. Its lower block also proves the irreducibility used in item 1.

For an invariant lattice, the elementary projectors and lower-block matrix units give

$$L_R = 3^\alpha Re_0 \oplus 3^\beta R^5, \qquad \beta \geq \alpha;$$

the inequality is forced by E_{0j}. In this basis the upper nilpotent row is multiplied by $3^{\beta-\alpha}$. If $\beta > \alpha$, reduction modulo three has rank two, contradicting the primitive rank-three cusp. Thus $\beta = \alpha$ and the map $3^{-\alpha}I$ identifies L_R with R^6, intertwining every exterior power. The other product conventions are related by

$$BA - I = A^{-1}(AB - I)A, \qquad (BA)^{-1} - I = -(BA)^{-1}(BA - I).$$

This proves the assertion for all invariant lattices, with no bound on their indices.

⟨1⟩3. We record the consequence of simple connectivity for the affine translations. Write a finite affine generator, after a translation of the torus, as $x \mapsto M_i x + v_i/m_i$, where $M_i v_i = v_i$ and v_i is its integral power vector. This change has an explicit map. If t is a lift of the original translation and d is the linear order, put $t_0 = d^{-1} \sum_{j=0}^{d-1} M_i^j t$ and $b_0 = t - t_0$. Then

$$h = \frac{1}{d} \sum_{j=0}^{d-1} (d - 1 - j) M_i^j b_0, \qquad (I - M_i)h = b_0.$$

For $T_h(x) = x + h$, the conjugate $T_h^{-1} g T_h$ has translation t_0, and $v_i = m_i t_0$ is integral because the given affine action has period m_i.

The base-orbifold quotient in theorem 11.1 shows that $\gcd(m_1, m_2) = 1$. In particular m_1 is a unit in R. The homology augmentation matrix $[A^t - I \mid B^t - I]$ has first row zero and rank-five minors -4 and -1 in its last five rows, respectively. Its coinvariant quotient over R is therefore R through $\ell(x) = x_0$. With $a_i = \ell(v_i)$, the finite power relations and cusp clutch give the quotient relation matrix

$$\begin{pmatrix} m_1 & 0 & -a_1 \\ 0 & m_2 & -a_2 \\ 1 & 1 & -\delta \end{pmatrix}.$$

Here δ is the character value of the cusp clutch. The character annihilates the collapsed angular cycles, so these maps on the finite caps and cusp agree on the boundary and define a quotient of the abelianized fundamental group. Simple connectivity makes its determinant a unit. Reduction modulo three forces a_2 to be a unit. Arbitrary clutch translations are included in δ.

⟨1⟩4. Let α be the primitive integral invariant character whose R-span is ℓ, and let $K = \ker \alpha$. The invariant sublattice $\Lambda' = \mathbb{Z}v_2 \oplus K$ has index $|\alpha(v_2)|$, prime to three. It gives a torus cover and a cover $F_2' \to F_2$ of the second reduced finite fibre. In coordinates beginning with v_2, its affine map is

$$(\theta, y) \longmapsto (\theta + 1/m_2, Dy).$$

Thus F_2' is a five-torus mapping torus, and its generic six-torus is the m_2-fold cover in the circle direction. Restriction to K is the dual quotient $(x_0, x') \mapsto x'$ from V_R to W_R, intertwining B with b. Reversing the mapping-torus circle replaces b by b^{-1}; the identities $b^{-1} - I = -b^{-1}(b - I)$ and $\sum_{j=0}^{5} b^{-j} = \sum_{j=0}^{5} b^{j}$ give the exact comparison.

Put $b_k = \bigwedge^k b$. The nonzero Smith factors of $b_k - I$ for $0 \le k \le 4$ are units in R. The Wang sequences make $H^q(F_2'; R)$ free for $q \le 4$. Pullback to its generic torus is injective, since transfer kills its kernel by m_2 and the source is free. Deck transformations of $F_2' \to F_2$ act by translations on that generic cover. Their action on its cohomology is trivial, and injectivity makes it trivial on $H^q(F_2'; R)$ as well. Transfer for this prime-to-three cover therefore identifies the restriction maps with those for F_2.

The covering map on the horizontal term of the Wang sequence is the norm

$$\mathcal{N}_{m_2,k} = \sum_{j=0}^{m_2-1} b_k^j : \operatorname{coker}(b_k - I) \longrightarrow \operatorname{Inv}(b_k).$$

Indeed, the lift is the union of the m_2 successive prisms with their transported coefficients. On the vertical invariant quotient pullback is the identity. Consequently horizontal inclusion induces the isomorphism

$$\frac{\operatorname{Inv}(b_{q-1})}{\mathcal{N}_{m_2,q-1} \bigwedge^{q-1} R^5} \xrightarrow{\sim} \frac{\operatorname{Inv}(\bigwedge^q V_R)}{J_{2,q}}.$$

Surjectivity follows by subtracting a restriction with the same vertical component; its kernel is exactly the horizontal norm image. Thus no choice of a Wang splitting changes this quotient.

⟨1⟩5. Write $\lambda = m_2/6$. Since $b^6 = I$, $\mathcal{N}_{m_2,k} = \lambda \mathcal{N}_{6,k}$. In case A a primitive basis of $\operatorname{Inv}(b_2)$, in lexicographic exterior order, is the column matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 1 & 4 \\ 2 & 4 \\ 0 & 2 \\ 0 & 3 \\ 1 & 4 \\ 0 & 3 \\ 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

The norm coordinates in this basis are

$$\begin{pmatrix} 0 & 3 & 0 & -3 & 0 & 0 & -4 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \end{pmatrix}.$$

Their image has basis $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 6 \end{smallmatrix}\right)$, and multiplication by $\left(\begin{smallmatrix} 1 & 0 \\ -4 & 1 \end{smallmatrix}\right)$ gives $\operatorname{diag}(1, 6)$. This gives the explicit quotient map $(x, y) \mapsto (x, y - 4x)$ modulo $(\lambda, 6\lambda)$. In case B the primitive invariant vector of b is $(1, 0, 0, 1, 0)^t$, and the norm coordinates are $(3, 0, 0, 3, 0)$. After multiplication by the remaining units of R, these give

$$Q_{2,3}^{\mathrm{A}} \cong R/3^{r-1}R \oplus R/3^r R, \qquad Q_{2,2}^{\mathrm{B}} \cong R/3^r R.$$

⟨1⟩6. We use the integral specialization and base-sheaf method of (Claude and L. Alpöge, 2026, Lemma 7.25 and Propositions 7.26–7.27, pp. 61–63) and (Engel, 2026, section "The integral homology"), with the following explicitly computed coefficients. Put $L_q = \bigwedge^q R^6$, $A_q = \bigwedge^q A$, $B_q = \bigwedge^q B$. The based-sphere complex for j_*L_q is

$$J_{1,q} \oplus J_{2,q} \oplus J_{0,q} \xrightarrow{d_0} L_q^2 \xrightarrow{\Delta_q} L_q,$$
$$J_{1,q} = \ker(A_q - I), \quad J_{2,q} = \ker(B_q - I), \quad J_{0,q} = \ker(B_q A_q - I),$$
$$d_0(j_1, j_2, j_0) = (j_1 - j_0, j_2 - A_q j_0),$$
$$\Delta_q(x, y) = (A_q - I)x + (B_q - I)y.$$

The cusp paths carry j_0 and $A_q j_0$, so their combined boundary is zero. Inverse generator conventions give a chain isomorphism: use cusp basis $A_q J_{0,q}$ for the inverse pair, and maps $-I$, $\mathrm{diag}(-A_q, -B_q)$, I in degrees zero, one and two. Both chain identities follow by multiplication.

Unimodular HNF transformations give primitive invariant bases and kernel coordinates for d_0 in $\ker \Delta_q$. In both cases the exact Smith records are

q	$\operatorname{rank} H^0$	$\operatorname{rank} H^1$	$\operatorname{rank} H^2$	SNF(d_0 in $\ker \Delta_q$)	SNF(Δ_q)
1	1	0	0	1^6	$1^5, 2$
2	0	0	0	1^{15}	$1^{14}, 2$
3	0	3	0	1^{17}	$1^{19}, 2$
4	0	0	0	1^{15}	$1^{14}, 2$

All displayed groups are therefore free over R. These computations are done by three separate programs (§1.5). Their computations use the printed models.

⟨1⟩7. Write $\mathcal{R}_q = R^q f_* R$. Proper base change (The Stacks Project Authors, 2026, Tag 09V5, Lemma 20.18.1) identifies the stalks with the fibres. At the first finite cap, m_1 is a unit in R, so transfer gives the full invariant restriction image and zero kernel. At the second finite cap, item 4 gives injectivity and the norm quotient. At the prescribed cusp, dualizing the primitive-image collapse of theorem 5.7 gives the invariant restriction image, with zero kernel in degrees one through three and a free kernel R in degree four. The latter is dual to the free degree-four cokernel of the collapse and must be retained.

For $q = 2, 3$ there is consequently an exact sequence

$$0 \longrightarrow \mathcal{R}_q \longrightarrow j_*L_q \longrightarrow Q_q \longrightarrow 0,$$

where Q_q is supported at the second finite point. Since $H^0(j_*L_q) = 0$, its connecting map embeds Q_q in $H^1(\mathcal{R}_q)$, with the free quotient $H^1(j_*L_q)$ computed above. Its torsion is exactly Q_q. Also $H^2(\mathcal{R}_1) = H^2(\mathcal{R}_2) = H^2(\mathcal{R}_3) = 0$. Injectivity gives $H^0(\mathcal{R}_3) = 0$. The degree-four free cusp kernel, together with $H^0(j_*L_4) = 0$, gives $H^0(\mathcal{R}_4) = R$.

⟨1⟩8. In the Leray sequence (The Stacks Project Authors, 2026, Tag 01F2, Lemma 20.13.4), total degree three has only $H^1(\mathcal{R}_2)$. Its middle base column has no incoming or outgoing differential. Thus case B gives $H^3(M; R) = R/3^r R$. In total degree four, the surviving terms are $H^1(\mathcal{R}_3)$ and $H^0(\mathcal{R}_4) = R$; the degree-two base term is zero and the possible differential from the latter term lands in $H^2(\mathcal{R}_3) = 0$. The extension by the free module R splits. Hence in case A

$$\operatorname{Tor} H^4(M; R) \cong R/3^{r-1}R \oplus R/3^r R.$$

Universal coefficients (Hatcher, 2002, Theorem 3.2, p. 195) identify this torsion with the torsion of $H_3(M; R)$, and similarly identify the case-B torsion with that of $H_2(M; R)$.

Localization retains the finite three-primary subgroups by the identity on each $\mathbb{Z}/3^s$ summand. This proves the stated orders and the case-B isomorphism. At $m_2 = 6$ the first case-A summand vanishes and the second is $\mathbb{Z}/3$; case B is also $\mathbb{Z}/3$. These groups obstruct three-connectivity.

⟨1⟩9. The two displayed rational models have explicit smooth marked realisations with the required free caps and simply connected filling. Use their displayed standard cohomology lattice $\mathbb{Z}^6$ and its dual homology basis, so $M_1 = A^t$, $M_2 = B^t$. The independent exact calculation gives $A^5 = B^6 = I$, exact orders five and six, and determinant one. Put $K = BA - I$. Then $K^2 = 0$, and the third minor on rows $(0, 3, 4)$ and columns $(1, 2, 3)$ is one in both models. Hence K has rank three and all its nonzero Smith factors are one. The marked cusp is

$$M_0 - I = ((M_1M_2)^{-1} - I) = -K^t.$$

Set $b_1 = e_0, b_2 = e_3, b_3 = e_4$, $a_j = (M_0 - I)b_j$ and $S = (a_1, a_2, a_3, b_1, b_2, b_3)$. Direct multiplication gives $S \in \mathrm{GL}_6(\mathbb{Z})$ and

$$(M_0 - I)S = S \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}.$$

This is the exact integral marking used to transport the index-one Kuhn cap and its collapse maps.

The affine power vectors in these same homology coordinates are

	v_1	v_2
A	$(1, 0, 0, 0, 0, 0)$	$(-1, 2, 0, -1, 0, 0)$
B	$(1, 0, 0, 0, 0, 0)$	$(-1, 0, 0, 0, 0, 1)$.

They satisfy $M_iv_i = v_i$. The first-coordinate covector ℓ is fixed by both M_i and has values $1, -1$ on v_1, v_2. Consequently every nonidentity power of $x \mapsto M_1x + v_1/5$ or $x \mapsto M_2x + v_2/6$ changes the ℓ-circle by $j/5$ or $-j/6$, respectively. These are free cyclic actions of the exact stated orders. Their diagonal actions on the product with the rotating disc therefore give the smooth finite quotient caps.

Take the affine torus bundle over the three-holed sphere with these two boundary holonomies and their inverse product on the third boundary. Glue the two finite caps and the proper index-one Kuhn cap of theorem 6.8, theorem 5.7 by the specified marked mapping-torus collars and zero cusp clutch. For an affine cusp monodromy $x \mapsto M_0x + c$, a path of translations $d(s)$ with $d(0) = 0$, $d(1) = c$ gives the collar map $(x, s) \mapsto (x + d(s), s)$ from the linear mapping torus. The prescribed zero clutch fixes its integral lift. The lattice map on this collar is the displayed S.

The cap fundamental-group maps retain these choices. On the finite cap the universal covering $\mathbb{R}^6 \times \Delta$ has the deck transformation $(x, t) \mapsto (M_ix + v_i/m_i, e^{2\pi i/m_i}t)$; its m_i-th power is the lattice translation v_i. Thus the boundary generators a, b satisfy $a^5 = v_1$, $b^6 = v_2$. On the Kuhn cap the polar projection onto its base three-torus and angular-identity section induce inverse maps on fundamental groups. In its dual-cell model the one-skeleton consists of the base edges, and each angular loop contracts after transport to a dual vertex where that angular fibre is collapsed. The fibre map is therefore exactly $\Lambda \to \Lambda/\operatorname{im}(M_0 - I)$, expressed by the last three rows of S^{-1}. The meridian bounds the toric coordinate disc at a smooth point of a central component; the zero clutch identifies it with $(ab)^{-1}$. Hence the cusp imposes $ab = 1$.

For $J = [M_1 - I \mid M_2 - I]$, the first row is zero. Its lower five rows have a unit minor on columns $(1, 2, 3, 6, 10)$ in case A and $(1, 2, 3, 4, 6)$ in case B, with determinants -1 and

1. The inverse of that minor supplies integral preimages of $e_1, \dots, e_5$. Thus $\operatorname{im} J = \ker \ell$ integrally. Van Kampen now gives simple connectivity at the group level: from $b = a^{-1}$ and the two power relations,

$$a^{-1} = a^5 a^{-6} = v_1 v_2 \in \Lambda.$$

So a, b lie in the abelian image of the fibre lattice, their conjugation actions become trivial, and J is killed. The exact remaining fibre quotient is $\mathbb{Z}$ via ℓ, with generator h and images $v_1 = h$, $v_2 = -h$. The displayed identity forces $a = 1$, and $a^5 = v_1$ then kills h. Every generator is trivial. Equivalently, after this nonabelian reduction the residual relation matrix on (h, a, b) is

$$\begin{pmatrix} -1 & 5 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \det = 1.$$

Finally use the checked rational invariant-extension table for these exact models. The finite covering restrictions identify rational cap cohomology with invariants. The primitive cusp collapse of 5.7 gives the same identification through degree three and a one-dimensional kernel in degree four. Hence $R^q p_* \mathbb{Q} = j_* \Lambda^q V$ for $1 \le q \le 3$, and

$$0 \longrightarrow \mathbb{Q}_{\mathrm{cusp}} \longrightarrow R^4 p_* \mathbb{Q} \longrightarrow j_* \Lambda^4 V \longrightarrow 0.$$

The table therefore leaves, in total degree at most four, only $E_2^{0,0} = E_2^{2,0} = E_2^{0,1} = \mathbb{Q}$, $E_2^{1,3} = \mathbb{Q}^3$ and $E_2^{0,4} = \mathbb{Q}$. Simple connectivity forces the $d_2 : E_2^{0,1} \to E_2^{2,0}$ to be an isomorphism. There are no other possible nonzero differentials on these terms. Thus $(b_0, \dots, b_4) = (1, 0, 0, 0, 4)$; oriented duality on the closed smooth eight-manifold gives the full tuple $(1, 0, 0, 0, 4, 0, 0, 0, 1)$.

□

Theorem 11.8 (four order pairs and eight local spectral pairs)**.** [tag 02N3]

Every simply connected two-finite-point datum with $b_2 = 0$ has one finite linear order equal to five and the other equal to four, six, eight or twelve. After suppressing the common factor $(T-1)^2$, its unordered pair of finite characteristic polynomials is exactly one of the following necessary possibilities:

orders	*order-five factor*	*other factor*
$(5, 4)$	Φ_5	$\Phi_2^2\Phi_4$
$(5, 6)$	Φ_5	$\Phi_2^2\Phi_3,\ \Phi_2^2\Phi_6,\ \Phi_3\Phi_6$
$(5, 8)$	Φ_5	Φ_8
$(5, 12)$	Φ_5	$\Phi_{12},\ \Phi_3\Phi_4,\ \Phi_4\Phi_6.$

(F.39)

Its rank-five quotient is rationally irreducible. In particular all order-three pairs, including $(3, 4)$, are impossible. The table gives necessary conditions only.

Proof. ⟨1⟩1. By proposition 11.3, proposition 11.9, the cohomology system V has a one-dimensional invariant space I, no invariant covectors, and two-dimensional finite fixed spaces F_1, F_2. The finite orders are among the coprime pairs of proposition 11.2. At a finite quotient, averaging identifies the degree-two specialization with the fixed part of $\Lambda^2 V$. At the cusp, theorem 5.7 gives the surjective homology collapse with kernel the saturation of $(\Lambda^2 M_0 - I)\Lambda^2\Lambda$. Its dual has image $\ker(\Lambda^2 M_0^t - I)$ under the evaluation pairing. Consequently the natural specialization morphism identifies

$$R^2 g_* \mathbb{Q} = j_* \Lambda^2 V.$$

The term $H^0(j_* \Lambda^2 V)$ has no incoming Leray differential, and its only possible outgoing differential targets $H^2(j_* V) = H^0(V^*)^* = 0$. It survives to $H^2(M; \mathbb{Q}) = 0$. Thus $a_2' := \dim(\Lambda^2 V)^\Gamma = 0$.

⟨1⟩2. Let $\beta : V \to V^*$ be an invariant alternating form. Since $\beta(I) \subset (V^*)^\Gamma = 0$, its radical contains I. For the finite averaging projection $P_i : V \to F_i$, invariance gives $\beta(u, x) = \beta(u, P_i x)$ when $u \in F_i$. The alternating restriction to the two-dimensional space F_i has I in its radical and is therefore zero. Hence $F_1 + F_2 \subset \operatorname{rad}\beta$. This sum has dimension three because $F_1 \cap F_2 = I$. The alternating rank is even, so every such form has rank at most two.

Suppose β, γ were independent invariant forms. They and their sum have rank two, so $\beta \wedge \gamma = 0$. Their image planes in V^* are distinct invariant two-planes meeting in a rational line. Each finite generator acts on that line by a sign. At least one finite order is odd, and the unipotent product acts trivially on the line; both signs are therefore $+1$. This contradicts $(V^*)^\Gamma = 0$. It follows that $b_2' := \dim(\Lambda^2 V^*)^\Gamma \le 1$.

⟨1⟩3. At the cusp, the exterior square of three size-two unipotent blocks has fixed rank nine: three fixed vectors come from their individual exterior squares and two from each of the three pairwise tensor products. If d_i denotes a finite fixed rank on $\Lambda^2 V$, the invariant-sheaf Euler identity gives

$$-15 + d_1 + d_2 + 9 = a_2' - h^1(j_*\Lambda^2 V) + b_2' \le 1. \tag{F.40}$$

Here the degree-two group is dual to $(\Lambda^2 V^*)^\Gamma$, by compact-support duality and the extension-by-zero sequence.

⟨1⟩4. The two-dimensional unit eigenspace leaves four dimensions for the nonunit cyclotomic factors. Local complex compatibility requires even multiplicity of -1. Partitioning four into the cyclotomic degrees gives the following complete list:

3	Φ_3^2
4	$\Phi_4^2,\ \Phi_2^2\Phi_4$
5	Φ_5
6	$\Phi_6^2,\ \Phi_2^2\Phi_3,\ \Phi_2^2\Phi_6,\ \Phi_3\Phi_6$
8	Φ_8
10	Φ_{10}
12	$\Phi_{12},\ \Phi_3\Phi_4,\ \Phi_4\Phi_6$.

The unit eigenspace contributes one fixed bivector. A repeated nonreal quadratic pair contributes four further fixed bivectors, so $d_i = 5$ for $\Phi_3^2, \Phi_4^2, \Phi_6^2$. Every other listed factor contributes two, so $d_i = 3$. Inequality (F.40) excludes every occurrence of $d_i = 5$. In particular all order-three pairs disappear. The coprime order list of proposition 11.2 therefore reduces to $(5,4), (5,6), (5,8), (5,12)$; their remaining factors are exactly the eight entries of (F.39). Each pair contains the order-five factor, so proposition 11.3 proves irreducibility of V/I.

□

Proposition 11.9 (automatic local complex compatibility)**.** [tag 02N4]

Every admissible datum with simply connected gluing and two finite affine fillings automatically has finite linear actions commuting with real complex structures. This assertion concerns the individual finite actions and does not produce simultaneous global periods.

Proof. ⟨1⟩1. Let m_i be the order of the free cyclic affine action defining the ith filling. Its linear order divides m_i. By 11.1 the integers m_1, m_2 are coprime, so at least one of them is odd. The corresponding linear determinant is one, because a finite real matrix of odd order cannot have determinant minus one. The relation $M_1 M_2 M_0 = I$ and the unipotence of M_0 imply $\det M_1 \det M_2 = 1$, so both determinants are one.

⟨1⟩2. A finite-order real matrix is semisimple. Its nonreal eigenvalues occur in conjugate pairs, and its only possible real eigenvalues are 1 and -1. Determinant one says that the -1 eigenspace has even real dimension. Since the total dimension is six, the 1 eigenspace also has even real dimension. Choose a complex structure on each of these real eigenspaces by pairing a real basis. On each nonreal isotypical real summand, multiplication by i in its complex eigenspace supplies a commuting complex structure. Their direct sum commutes with M_i. The fixed eigenspace is nonzero because the affine action is free, as in 4.3, so its even dimension is at least two.

⟨1⟩3. Each chosen structure makes $\mathbb{R}^6/\mathbb{Z}^6$ a complex torus and the given affine map holomorphic; thus it supplies precisely the local complex-compatibility hypothesis used in the subsequent reductions. For three genuine finite fillings 11.1 has already excluded simple connectivity without this argument.

□

Theorem 11.10 (an unbounded integral family)**.** [tag 02N5]

There are infinitely many simultaneous rational conjugacy classes of integral rank-six tuples of the linear type $(5,4)$ *satisfying definition 4.1, with primitive maximal square-zero cusp. An explicit infinite subfamily has free affine lifts of orders five and four. Apart from the parameter zero, the rank-five irreducible quotient in the family is not self-dual. Thus no finite bound on rational conjugacy classes is valid for the unrestricted input tuples.*

Proof. ⟨1⟩1. Let $v \in \mathbb{Z}$, put $q = v^2 - v + 1$, and let $Ae_j = e_{j+1}$ on $\mathbb{Q}^5$, with indices modulo five. Define

$$X = (e_0,\ e_2 + ve_4), \qquad Y = \begin{pmatrix} 0 & -q & -v^2 & -1 & v \\ 0 & 1-vq & -vq & v^2-v & q \end{pmatrix}, \qquad N = XY. \tag{F.U1}$$

Multiplying Y by the two columns of X gives $YX = 0$, so $N^2 = X(YX)Y = 0$. Consequently

$$C = (I+N)A, \qquad B = A^4(I-N) = C^{-1}.$$

The cycle gives $A^5 = I$. Exact multiplication and determinant expansion give

$$N^2 = 0, \qquad A^5 = C^4 = I_5, \qquad \chi_C(x) = (x-1)(x+1)^2(x^2+1). \tag{F.U2}$$

For the determinant calculation one may substitute

$$(xI - A)^{-1} = \frac{x^4 I + x^3 A + x^2 A^2 + xA^3 + A^4}{x^5 - 1}$$

in the rank-two determinant identity

$$\det(xI - C) = \det(xI - A)\det(I_2 - YA(xI - A)^{-1}X).$$

The identity $C^4 = I$ is checked separately; it is not inferred from the repeated factor $x + 1$ of the characteristic polynomial. The factor $x^2 + 1$ shows that B and C have exact order four.

⟨1⟩2. Let P have columns $\mathbf{1}/q, e_1, e_2, e_3, e_4$, and put $A' = P^{-1}AP$, $B' = P^{-1}BP$. Since $q > 0$ and $\det P = 1/q$, these are rational changes of coordinates. Define

$$s = v^2 + v - 1, \qquad r = -\frac{(v+1)(v^2-2)}{2}, \qquad \beta = (s,\ s - vr,\ v^2 s + vr,\ r,\ -vs - r) \tag{F.U3}$$

and

$$\widehat{A} = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} 1 & \beta \\ 0 & B' \end{pmatrix}. \tag{F.U4}$$

The lattice $P\mathbb{Z}^5$ has basis $f_0 = \mathbf{1}/q, f_1 = e_1, \ldots, f_4 = e_4$. Its images under A are f_0, f_2, f_3, f_4 and $qf_0 - f_1 - f_2 - f_3 - f_4$, respectively; $A^{-1} = A^4$ therefore also preserves this lattice. The columns of X are $qf_0 - f_1 - f_2 - f_3 - f_4$ and $f_2 + vf_4$, while

$$Y' := YP = \begin{pmatrix} -2 & -q & -v^2 & -1 & v \\ 2(1-v) & 1-vq & -vq & v^2-v & q \end{pmatrix}.$$

Thus X and Y are integral maps between the lattices. The already proved $YX = 0$ shows that $I + N$ and its inverse $I - N$ preserve $P\mathbb{Z}^5$. Since $B = A^{-1}(I - N)$, both A', B' belong to $\mathrm{GL}_5(\mathbb{Z})$. For r, odd v makes $v + 1$ even, and even v makes $v^2 - 2$ even; hence r and all entries of β are integers. The hatted matrices are therefore integral automorphisms.

The exact unit needed in the hatted lattice is

$$z = (-v^3 + v - 1, 0, 0, 2, 2(1 - v^2))^t.$$

Writing $a = -v^3 + v - 1$, the first coordinate of $Y'z$ is $-2a - 2 + 2v(1 - v^2) = 0$. Half its second coordinate is $(1 - v)a + (v^2 - v) + q(1 - v^2) = 0$, since the first two terms sum to $v^4 - v^3 + v - 1$ and the last is its negative. Furthermore

$$\beta z = s(v^3 - v - 1) + 2v^2 r = 1. \tag{F.U5}$$

Indeed the two summands expand to $v^5 + v^4 - 2v^3 - 2v^2 + 1$ and $-v^5 - v^4 + 2v^3 + 2v^2$. Since $AB = I - N$ and $NPz = X(Y'z) = 0$, the six-dimensional map satisfies

$$(\widehat{A}\widehat{B} - I)(0, z)^t = e_0.$$

This establishes the unit in the displayed hatted coordinates.

The row identity $\beta(I + B' + B'^2 + B'^3) = 0$ proves $\widehat{B}^4 = I$; also $\widehat{A}^5 = I$. Their quotient actions show that these orders are exact.

Use the lattice change

$$L = \frac{1}{4}\begin{pmatrix} -2 & 3 & -5 & -1 & 5 & -3 \\ 0 & 2 & -2 & 2 & -2 & 2 \\ 0 & -4 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -4 & 4 & -4 \\ 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & -4 & 4 & 0 \end{pmatrix}, \qquad A_v = L^{-1}\widehat{A}L, \quad B_v = L^{-1}\widehat{B}L. \tag{F.U6}$$

Here $\det L = -1/4$. Expanding these products gives entries in $\mathbb{Z}[v]$ for both matrices. The complete expanded entries and the exact identities $LA_v = \widehat{A}L$, $LB_v = \widehat{B}L$ are given by the polynomial certificate described in the last item. Thus the defined matrices act on the stated $\mathbb{Z}^6$ for every integer parameter. Their determinants are one and their inverses are the integer powers A_v^4, B_v^3. Their characteristic polynomials are

$$(x-1)^2\Phi_5(x), \qquad (x-1)^2(x+1)^2(x^2+1),$$

respectively.

⟨1⟩3. Put $K = B_vA_v - I$. Multiplication gives $K^2 = 0$. The following are third minors of K, in zero-based indices:

rows	columns	determinant
$(0,1,4)$	$(1,2,3)$	$v(v+1)$
$(0,1,4)$	$(1,3,4)$	v^2
$(0,3,4)$	$(1,2,3)$	$(v+1)^2$.

Their integer combination $(v+1)^2 - 2v(v+1) + v^2$ equals one. Hence K has rank at least three, while $K^2 = 0$ in dimension six gives rank at most three. The gcd of its third minors is one, so its three nonzero Smith factors are all one. This proves primitivity directly in the lattice, for every v. For $M_1 = A_v^t$ and $M_2 = B_v^t$ the cusp map is

$$M_0 = (M_1 M_2)^{-1} = (I + K^t)^{-1} = I - K^t.$$

Thus $M_0 - I = -K^t$, which has the same integral Smith factors and the required maximal square-zero type.

⟨1⟩4. Both A_v, B_v fix e_0. The quotient morphism to the original rank-five pair is

$$R = P \operatorname{pr}_5 L : \mathbb{Q}^6 \longrightarrow \mathbb{Q}^5, \qquad RA_v = AR, \quad RB_v = BR.$$

It is surjective and has kernel $\mathbb{Q}e_0$, since $Le_0 = -e_0/2$. We prove that this quotient is irreducible over $\mathbb{Q}$. The coprime factors $x - 1$ and $\Phi_5(x)$ of χ_A give the projector $(I + A + A^2 + A^3 + A^4)/5$. Its image is $\mathbb{Q}\mathbf{1}$ and its kernel is $S = \{\sum x_i = 0\}$. Irreducibility of Φ_5 shows that these are the only nonzero proper rational A-invariant subspaces.

Neither is C-invariant. Indeed,

$$N\mathbf{1} = -2qX \begin{pmatrix} 1 \\ v-1 \end{pmatrix}$$

has nonzero first entry and zero entries in positions one and three, so $C\mathbf{1} = \mathbf{1} + N\mathbf{1}$ is not a multiple of $\mathbf{1}$. Write $\epsilon = (1,1,1,1,1)$. If S were C-invariant, then ϵN would be a multiple of ϵ; its first entry is zero, so that multiple would vanish. But its entries in columns three and four are $v^3 - v - 1$ and $v^3 + v + 1$. Their simultaneous vanishing would give $v = -1$, at which the first is -1. This is impossible. Since $B = C^{-1}$, the pair (A, B) is irreducible, as claimed.

⟨1⟩5. The rank-two update gives exact traces

$$\operatorname{tr}(AC) = \operatorname{tr}(YA^2X) = 1 - v^2 q, \qquad \operatorname{tr}((AC)^{-1}) = -\operatorname{tr}(YA^3X) = 1 + v^2 q.$$

Here $\operatorname{tr}(A^2) = \operatorname{tr}(A^3) = 0$ and $B = A^4(I - N)$ give the displayed equalities. An equivariant isomorphism F to the contragredient representation would satisfy $FA = A^{-t}F$ and $FB = B^{-t}F$, and therefore identify the character of the word $AC = AB^{-1}$ with the character of its contragredient matrix. The two displayed traces would be equal. Their difference $-2v^2q$ is nonzero when $v \neq 0$, proving the required nonselfduality.

For completeness, at $v = 0$ the symmetric matrix $Q_{ij} = 1$ for $j - i \equiv 1, 4 \pmod 5$ and $Q_{ij} = 0$ otherwise has determinant two and satisfies $A^tQA = Q$, $B_0^tQB_0 = Q$. It gives the equivariant isomorphism in that case.

For the rank-six matrices, the same word has trace

$$h(v) := \operatorname{tr}(A_v B_v^{-1}) = 2 - v^2 q, \qquad h(v+1) - h(v) = -4v^3 - 3v^2 - 3v - 1.$$

Thus h is strictly decreasing on the nonnegative integers, in particular on the nonnegative multiples of twenty. A simultaneous rational conjugacy of the ordered matrices would preserve this trace. There are consequently infinitely many such rational conjugacy classes already in that subfamily.

⟨1⟩6. For $v \in 20\mathbb{Z}$, put $a = (1, 3, -2, -1, 2, 0)^t$, $e = (1, 0, 0, 0, 0, 0)^t$, and define

$$w_1 = \frac{1}{5} \sum_{j=0}^{4} (A_v^t)^j a, \qquad w_2 = -\frac{1}{4} \sum_{j=0}^{3} (B_v^t)^j e. \tag{F.U7}$$

The polynomial matrices reduce to their parameter-zero values modulo five and four, respectively. At zero, $M_1 a = a$ and $M_2 e = e$. The two sums are therefore divisible by their indicated denominators, so w_1, w_2 are integral. Their defining averages give $M_i w_i = w_i$. The common first-coordinate covector ℓ is fixed by both M_i and satisfies $\ell(w_1) = 1$, $\ell(w_2) = -1$.

For $(m_1, m_2) = (5, 4)$ let $g_i(x) = M_i x + w_i/m_i$ on $\mathbb{R}^6/\mathbb{Z}^6$. Then $g_i^j(x) = M_i^j x + j w_i/m_i$, so $g_i^{m_i}$ is the identity. Every power with $0 < j < m_i$ changes the ℓ-circle coordinate by $j/5$ or $-j/4$, respectively. It has no fixed point. These are free actions of the exact orders five and four, with the affine lifts.

⟨1⟩7. Substitution of $v = 0$ in the displayed construction gives exactly the marked pair in Theorem 5.1, and the averages give $w_1(0) = a$, $w_2(0) = -e$. This identifies both the matrices and the affine numerators without a further change of basis.

A separate program computes the expanded polynomial matrices and checks every matrix, quotient, trace and affine identity above over $\mathbb{Q}[v]$, with coefficient integrality in $\mathbb{Z}[v]$. It checks all four hundred third minors as well as the displayed integer combination. Its separate Smith computations at ten stated integer parameters are additional finite checks; the universal primitivity assertion follows from the displayed identity equal to one. The conclusions concern the unrestricted input tuples in the statement.

□

Proposition 11.11 (an odd torsion obstruction in this lattice)**.** [tag 02N6]

For the integral tuple (A_v, B_v) in (F.U6), any simply connected filling with free finite affine caps and the fixed primitive Kuhn cusp has a subgroup

$$\mathbb{Z}/(v^2 - v + 1) \ \subset \operatorname{Tor} H_2(M; \mathbb{Z}). \tag{F.U9}$$

In particular, in this specified family of lattices only $v = 0$ and $v = 1$ can be three-connected, regardless of the affine multiplicities and twists. The stronger exclusion for every stable lattice in each rational class is proved in Theorem 11.13 below.

Proof. ⟨1⟩1. First compute the global coinvariants in the hatted lattice. In the rank-five quotient, the A' relations identify $f_1 = f_2 = f_3 = f_4 = t$ and $q f_0 = 5t$. The number $q = v^2 - v + 1$ is relatively prime to five, as its five residue values show. Thus these coinvariants are infinite cyclic, with $f_0 = 5h$ and $t = qh$. The B' relations become $q\kappa_j h$, where

$$(\kappa_0, \dots, \kappa_4) = (2v^2,\ v(v^3 + v - 1),\ v(v^3 + v + 1), -v^3 + v + 1,\ -v^3 - v - 1). \tag{F.U10}$$

Their gcd is one: it divides $2v$ by subtracting the second and third entries, and it divides $v^3 - v - 1$ by the fourth; the latter is odd and is -1 modulo v. Hence the rank-five global coinvariants are $\mathbb{Z}/q$. The extension line is killed in rank-six coinvariants by (F.U5), since $(\widehat{A}\widehat{B} - I)(0, z)^t = e$. Consequently the hatted rank-six coinvariants are also $\mathbb{Z}/q$. The lattice change (F.U6) is an isomorphism over every $\mathbb{Z}_{(p)}$ with p odd, so the odd part of the rank-six global coinvariants is precisely $\mathbb{Z}/q$.

⟨1⟩2. Let V be the degree-one fibre cohomology local system. The only difference between $R^1 g_* \mathbb{Z}$ and $j_* V$ is a finite cokernel at finite caps; the cusp degree-one specialization is onto. Thus

$$H^2(\mathbb{P}^1, R^1 g_* \mathbb{Z}) = H^2(\mathbb{P}^1, j_* V) = V_\Gamma,$$

including its q-primary part just computed. The space of global degree-two invariants is zero: the irreducible odd rank-five quotient has no invariant alternating tensor, and the extension filtration $0 \to W \to \bigwedge^2 V \to \bigwedge^2 W \to 0$ gives $(\bigwedge^2 V)^\Gamma = 0$.

⟨1⟩3. It remains to check every possible source of the differential $d_2 : H^0(\mathbb{P}^1, R^2 g_* \mathbb{Z}) \to H^2(\mathbb{P}^1, R^1 g_* \mathbb{Z})$. The cusp has no kernel in degree two. At a finite cap the kernel is torsion in H^2 of that cap, namely torsion in its first homology. Simple connectivity of the total filling forces the distinguished affine numerator to be prime to the affine multiplicity, by the determinant argument of 11.6. The cap is therefore a mapping torus of a torus automorphism whose order divides the linear order, five or four. Its first-homology torsion is supported only at the primes five or two, respectively: averaging over the finite order kills the torsion of the vertical coinvariants. Hence the source of this d_2 has no torsion at any prime dividing q, since q is odd and prime to five.

⟨1⟩4. The subgroup $\mathbb{Z}/q$ in bidegree $(2,1)$ survives. There are no longer differentials on a two-dimensional base, and this terminal filtration term is a subgroup of $H^3(M;\mathbb{Z})$. Universal coefficients identify its torsion with a subgroup of $\operatorname{Tor} H_2(M;\mathbb{Z})$. This proves (F.U9). Finally, $v^2 - v + 1 = 1$ for an integer v exactly when $v = 0$ or $v = 1$.

□

Theorem 11.12 (A three-connected datum with non-self-dual quotient)**.** [tag 02N7]
The parameter $v = 1$ in (F.U6), with affine orders five and four and twists

$$w_1 = (1, 3, -2, -1, 2, 0)^t, \qquad w_2 = (-1, 0, 0, 0, 0, 0)^t,$$

gives a three-connected smooth eight-manifold with $H_4 = \mathbb{Z}^4$ and all other reduced homology zero except $H_8 = \mathbb{Z}$. It is almost complex and is oriented diffeomorphic to $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$. Its irreducible rank-five quotient has no invariant bilinear form, so integral three-connectivity does not force that quotient to be orthogonal. By theorem 12.5 it has holomorphic periods with bounded cusp correction on the punctured base; a compact complex structure on it is not constructed here.

Proof. ⟨1⟩1. The first cohomological matrix is the same A as in Theorem 5.1. The second is

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 1 & -3 \\ 0 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}. \tag{F.U11}$$

Take $M_1 = A^t$, $M_2 = B_1^t$ and $M_0 = (M_1 M_2)^{-1}$. The displayed twists are fixed integral vectors with $\ell(w_1) = 1$, $\ell(w_2) = -1$. They therefore give free affine actions. The cusp is primitive by 11.10. Exact Hermite reduction gives

$$\operatorname{col}_{\mathbb{Z}}[\, M_0 - I \mid M_1(M_0 - I) \mid M_2(M_0 - I)\,] = \langle e_1, \dots, e_5 \rangle.$$

Van Kampen leaves only $5a = h$, $4b = -h$, $a + b = 0$, whose relation matrix is unimodular. This proves $\pi_1 = 1$ for the gluing, including zero cusp clutching.

⟨1⟩2. We give the integral data needed to reuse the differential calculation of 5.5. The rational exterior invariant and parabolic ranks are exactly the table there. The finite cap first homology groups are $\mathbb{Z}^2$ and $\mathbb{Z}^2 \oplus \mathbb{Z}/2$. On integral invariant two-form bases their suspension periods divided by the covering orders are, respectively, $(-1, 0, -1)$ and $(0, 0, 1)$. The determinants of the invariant degree-three skew pairings are 5^4 and 4^4, while the degree-two/degree-four pairings have absolute determinants 5^3 and 4^3. Thus all the finite covering indices used in degrees two, three and four are one.

⟨1⟩3. For exterior coefficient degrees two, three and four, the global coboundary Smith form has one factor two and all other nonzero factors one. Its torsion class has peripheral triviality flags (true, false, true), so the parabolic groups are torsion free, of ranks $0, 3, 0$. The degree-one coboundary is primitive and the degree-one finite covering cokernels are killed by the common invariant covector and the Chinese remainder theorem, exactly as in 5.5. The global coinvariants in degrees one through four are each $\mathbb{Z}/2$.

⟨1⟩4. Write $M_2 = \operatorname{diag}(1, D)$. This time use the unimodular marking

$$P_1 = \begin{pmatrix} -1 & 1 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 \end{pmatrix}, \qquad \det P_1 = -1.$$

It satisfies $DP_1 = P_1 \operatorname{diag}(\text{four-cycle}, -1)$. The cap therefore has the same cellular mapping-torus model used to prove (F.6). With the unchanged twist $w_2 = -e_0$, the boundary coordinates remain $\theta = a + 4k$, $h = k$; the suspension representative restricts to w in the base direction and to $-\operatorname{Norm}_4(w) = 0$ in the auxiliary fibre direction. Consequently the coefficient in the torsion differential is again one.

⟨1⟩5. The new vertical representatives in lexicographic exterior bases are

$$\begin{aligned} q = 2 &: (0, -1, -1, -1, 0), \\ q = 3 &: (0, 0, 1, 0, 0, 0, 0, 0, 0, 1), \\ q = 4 &: (0, 0, 0, 0, 0, 0, 0, -1, -1, 0). \end{aligned}$$

Each has residue one in its global coinvariant $\mathbb{Z}/2$. Thus all three torsion differentials in (F.6) are isomorphisms. The integral Leray table and its free degree-one differential are therefore exactly those of 5.5. They give $H^2 = H^3 = 0$ and $H^4 = \mathbb{Z}^4$. Universal coefficients, Hurewicz and Poincaré duality give the stated integral homology and three-connectivity.

⟨1⟩6. The relative almost-complex extension over the pair of pants in 5.1 applies: the finite matrices have the same even real eigenspace dimensions and the cusp is the same marked Kuhn type. The purely smooth characteristic-vector and normal-invariant argument in 7.2 then gives the stated oriented diffeomorphism type. No integrability at the finite punctures is used for this smooth classification.

⟨1⟩7. Finally, Theorem 11.10 gives nonselfduality of the irreducible rank-five quotient at $v = 1$. The exact program `fourfolds_v1_homology.py` runs every matrix, Smith, Hermite, suspension-period, pairing and differential certificate just displayed. The proof uses the explicit monomial marking displayed above.

□

Theorem 11.13 (a residual obstruction for every lattice)**.** [tag `02N8`]

Let a rational rank-six datum have finite linear orders five and four, a primitive maximal cusp on a stable integral lattice, and invariant line L, with irreducible rank-five quotient W and no rational global coinvariants. Suppose that at a prime $p \neq 2, 5$ the semisimplified reduction of a stable lattice in W is the sum of the trivial module and an absolutely irreducible module of dimension four. Then no stable integral lattice in that rational datum, with primitive maximal cusp, and no free finite affine twists can give $\pi_1 = 1$ and $H_2 = 0$ simultaneously.

For every $v \notin \{0, 1\}$, the rational family in 11.10 satisfies this hypothesis at each prime dividing $v^2 - v + 1$. Consequently the exclusion of those parameters holds for all *stable lattices and all twists, not just the lattice in* (F.U6)*. The two remaining parameters give the three-connected manifolds of theorems 5.5 and 11.12.*

Proof. ⟨1⟩1. We first give the lattice-independent mechanism. For any stable lattice $V_{\mathbb{Z}}$, use its primitive invariant line $L_{\mathbb{Z}}$ and quotient lattice $W_{\mathbb{Z}}$. The semisimplified reduction of a rational representation is independent of its stable lattice. One elementary way to see this is to compare two commensurable lattices by their finite quotient T. The exact sequence obtained by reduction modulo p has end terms $T[p]$ and T/pT, which have the same composition factors by the filtration of T by powers of p. Hence the two lattice reductions have equal classes in the Grothendieck group of finite-length modules. Chaining such comparisons proves the assertion for any two lattices.

⟨1⟩2. Thus $W_{\mathbb{Z}}/pW_{\mathbb{Z}}$ has length two, with one trivial constituent and one irreducible four-dimensional constituent. It has either a trivial submodule or a trivial quotient (and possibly both).

⟨1⟩3. Suppose first that it has a trivial submodule. Its inverse image in $V_{\mathbb{Z}}/pV_{\mathbb{Z}}$ is an extension of a trivial module by a trivial module. Each finite generator acts on this two-dimensional space by a unipotent matrix. Since its order is five or four, prime to p, it acts as the identity. There are therefore at least two global invariant vectors in degree-one fibre cohomology modulo p. Write their dimension as $a_p \geq 2$; this is also the dimension of the global coinvariants of fibre homology over $\mathbb{F}_p$, by the perfect pairing between fibre homology and cohomology.

⟨1⟩4. The abelianized van Kampen presentation now shows

$$b_1(M;\mathbb{F}_p) \geq a_p - 1. \tag{F.U12}$$

Indeed, after the monodromy relations the fibre contributes a_p dimensions, and the two finite meridians contribute two more. The two finite power relations and the single base relation impose at most three linear relations. The cusp introduces no further fibre relation in this abelianization. On the dual homology lattice its nilpotent operator is still primitive and maximal, so its angular lattice is $\ker(M_0 - I) = \operatorname{im}(M_0 - I)$ by primitivity, and the latter image already vanishes among the global monodromy relations. Its base meridian is the base relation just counted. Changing affine twists changes coefficients of these three relations, not their number. Thus (F.U12) contradicts simple connectivity.

⟨1⟩5. Suppose instead that the reduction of W has a trivial quotient. Its integral global coinvariants have nonzero p-primary torsion, since they have rational rank zero. The natural map from the global coinvariants of V to those of W is an isomorphism. To check its kernel, a primitive generator e of $L_{\mathbb{Z}}$ lies in $\ker(M_0 - I) = \operatorname{im}(M_0 - I)$, and hence is already zero among global coinvariants. Therefore $H^2(R^1)$ has nonzero p-primary torsion. If the total space were simply connected, the finite-cap argument of 11.11 would make every possible torsion source in $H^0(R^2)$ supported only at two and five. The invariant part of $\bigwedge^2 V$ is zero by irreducibility of the odd-dimensional quotient. Consequently the nonzero p-primary subgroup survives to $H^3(M;\mathbb{Z})$ and then to $\operatorname{Tor} H_2(M;\mathbb{Z})$, again a contradiction. This proves the general criterion.

⟨1⟩6. For its application to 11.10, let $p \mid q = v^2 - v + 1$. We have $p \neq 2, 5$. In the original cycle lattice of (F.U1), $Y\mathbf{1} = 2q(-1, 1-v)^t$, so $\mathbf{1}$ gives a trivial submodule after reduction. On the four-dimensional quotient the first finite generator has characteristic polynomial Φ_5, and $C = B^{-1}$ has polynomial $(x+1)^2(x^2+1)$. This quotient is absolutely irreducible. Here is a characteristic-independent proof over $\overline{\mathbb{F}}_p$.

⟨1⟩7. A one-dimensional composition factor would make the eigenvalues of A and C equal, because CA^{-1} is unipotent. An eigenvalue of order dividing both five and four must be one, which is absent from Φ_5. If the quotient were reducible, its only remaining possible

composition-factor dimensions would be two and two. On each factor, the determinants of A and C must agree and have orders dividing five and four, so both determinants are one. The two C eigenvalue pairs must therefore be $(-1,-1)$ and $(i,-i)$ in some order. On the former factor $C=-I$, since its order is prime to p. Then the unipotence of $CA^{-1}=-A^{-1}$ would force the eigenvalues of A to be -1, impossible for Φ_5. This proves absolute irreducibility and establishes the required residual composition factors at every $p \mid q$. When $v \notin \{0,1\}$, the integer q is greater than one and has such a prime divisor. The criterion applies.

□

Theorem 11.14 (the complete rational $(5,4)$ normal form)**.** [tag 02NA]

Consider ordered rational rank-five pairs (A,C) with

$$\chi_A = (x-1)\Phi_5, \qquad \chi_C = (x-1)(x+1)^2(x^2+1).$$

Put

$$N = CA^{-1} - I, \qquad N^2 = 0, \quad \operatorname{rank} N = 2,$$

where $A^5 = C^4 = I$ and the pair is irreducible over $\mathbb{Q}$. Their simultaneous rational conjugacy classes are parametrized bijectively by

$$\{[r:s:t] \in \mathbb{P}^2(\mathbb{Q}) : D(r,s,t) \neq 0\} \;\sqcup\; \mathbb{A}^1(\mathbb{Q}), \tag{F.R1}$$

where

$$\begin{aligned} D = {} & r^3 - 4r^2s - r^2t + 7rs^2 + rst - 3rt^2 \\ & - 7s^3 + 11s^2t - 11st^2 + 7t^3. \end{aligned} \tag{F.R2}$$

The second summand lies over the single exceptional point $[4:1:3]$ of $D=0$. The construction below is explicit and uses only a ten-variable linear system. By 11.5 it also parametrizes the rational rank-six maximal-cusp extensions of all these pairs, with $B=C^{-1}$. This is a rational classification; the stable integral lattices, the affine twists and the homology of the gluings are treated below.

Proof. ⟨1⟩1. Write $K=\mathbb{Q}(\zeta)$, $\zeta^4+\zeta^3+\zeta^2+\zeta+1=0$, and identify the rational space with $\mathbb{Q}\oplus K$. The matrix of A is multiplication by $(1,\zeta)$:

$$A = \operatorname{diag}(1,J), \qquad J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad R = (A+I)^{-1}. \tag{F.R3}$$

Its rational centralizer is $\mathbb{Q}^* \times K^*$. Let $I_N = \operatorname{im} N$. If X is any basis matrix for this plane and $N=XY$, then $YX=0$. The identity

$$C + I = (A+I) + XYA$$

shows that its two-dimensional kernel has the same dimension as the kernel of $I_2 + YARX = I_2 - YRX$. Hence

$$YRX = I_2, \qquad I_N \cap RI_N = 0. \tag{F.R4}$$

Conversely the two matrix equations in (F.R4), together with $YX=0$, give precisely that two-dimensional -1 eigenspace.

⟨1⟩2. The plane I_N projects nontrivially to $\mathbb{Q}$: otherwise the first-coordinate covector is invariant under both A and C, contrary to irreducibility. It contains no nonzero vector of $\mathbb{Q} \oplus 0$, because R acts as $1/2$ on that line and (F.R4) would fail. Write a basis of the plane as $(0, u), (1, v)$, where $u \in K^*$. Multiplication on K by u^{-1} puts it into

$$I_N = \langle (0,1), (1,\beta) \rangle_{\mathbb{Q}}.$$

Changing the second basis vector by a rational multiple of the first changes β by a rational constant. Two such planes are in the same centralizer orbit exactly when $\beta' = a\beta + b$, with $a \in \mathbb{Q}^*$ and $b \in \mathbb{Q}$. Indeed a centralizer element carrying one to the other must preserve the kernel of the projection to $\mathbb{Q}$, so its K^* component is itself rational; the displayed affine change follows, and every such change is realized. We may therefore take

$$\beta = r\zeta + s\zeta^2 + t\zeta^3, \qquad [r:s:t] \in \mathbb{P}^2(\mathbb{Q}), \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & r \\ 0 & s \\ 0 & t \end{pmatrix}. \tag{F.R5}$$

The triple is nonzero; a rational β would make the plane contain the forbidden first-coordinate line. The stabilizer of such a plane in the centralizer consists only of common scalar matrices. Thus it introduces no further identifications of the resulting nilpotent operators.

⟨1⟩3. Write Y as an unknown two-by-five matrix. It is determined by the following ten linear equations:

$$YX = 0, \qquad YRX = I_2, \qquad \operatorname{tr}(YAX) = -1, \qquad \operatorname{tr}(YA^{-1}X) = 1. \tag{F.R6}$$

The first eight are (F.R4). The last two are exactly $\operatorname{tr} C = \operatorname{tr} C^{-1} = -1$, since $\operatorname{tr} A = \operatorname{tr} A^{-1} = 0$ and $C^{-1} = A^{-1}(I - XY)$. They suffice to obtain the required characteristic polynomial: the two -1 roots and determinant one leave a monic cubic of constant term -1, and the two traces fix that cubic to $(x-1)(x^2+1)$. These remaining three roots are distinct, and the -1 eigenspace already has dimension two, so C is semisimple and $C^4 = I$. Also $YRX = I_2$ ensures that N has rank two. Thus every solution of (F.R6) gives all the asserted local linear conditions.

⟨1⟩4. In row-major order on the ten entries of Y, the determinant of the coefficient matrix in (F.R6) is $D/4$. For $D \neq 0$ this proves uniqueness and gives the explicit formula $Y = \operatorname{reshape}(E^{-1}b)$, where E is the coefficient matrix of the displayed linear equations and b their displayed right side. The exact expanded matrix and all ten rational functions are printed by `fourfolds_relative_normalform.py`. There is no search or degree cutoff in this calculation.

⟨1⟩5. We now treat the exceptional locus. Set

$$Q_1 = r^2 - rs - rt + s^2 + t^2, \qquad Q_2 = r^2 - 2rs - 3rt + 4s^2 - 4st + 4t^2.$$

The first column of the unique generic solution is

$$y_{10} = \frac{2Q_1Q_2}{D}, \qquad y_{20} = \frac{2(r-s-t)Q_1}{D}. \tag{F.R7}$$

Here y_{10}, y_{20} are the first column of Y. The two equalities after multiplication by D are polynomial linear combinations of (F.R6), by its adjugate matrix. The program prints the

corresponding two polynomial rows and verifies those identities, so they remain necessary also when $D = 0$. Moreover

$$2Q_1 = (r-s)^2 + (r-t)^2 + s^2 + t^2 > 0$$

for every nonzero rational triple. Consequently consistency on $D = 0$ forces $r = s + t$ and $Q_2 = 0$. Substitution gives

$$D(s+t, s, t) = (t - 3s)(2t - s)^2, \qquad Q_2(s+t, s, t) = (3s - t)(s - 2t). \tag{F.R8}$$

There are precisely two resulting projective points, $[3 : 2 : 1]$ and $[4 : 1 : 3]$. At the former, $[X \mid RX]$ has rank less than four, so (F.R4) is impossible. At the latter the coefficient and augmented matrices of (F.R6) both have rank nine, and every solution is

$$Y_z = \begin{pmatrix} 4z + 16/5 & 0 & -2z - 1 & 4/5 - 2z & 2z \\ 2z - 2/5 & 0 & -z & 2/5 - z & z \end{pmatrix}, \qquad z \in \mathbb{Q}. \tag{F.R9}$$

Direct substitution proves both necessity and sufficiency. This establishes the exceptional affine line in (F.R1).

⟨1⟩6. Every pair just constructed is irreducible over $\mathbb{Q}$. An A-stable proper rational subspace has dimension one or four. A one-dimensional subrepresentation or quotient must have trivial character, since its finite eigenvalues of orders dividing five and four must agree under the unipotent product. The only A-invariant covector is the first-coordinate covector. Its product with N is the second row of Y, which is nonzero by $YRX = I_2$. The only A-invariant vector is the first-coordinate vector; its image under N vanishes exactly if both entries in (F.R7) vanish. For $D \neq 0$, positivity of Q_1 and (F.R8) rule this out. On the exceptional line the two entries $4z + 16/5$ and $2z - 2/5$ never vanish simultaneously. There are therefore neither invariant vectors nor covectors, which excludes both possible dimensions.

⟨1⟩7. Finally, rescaling (r, s, t) is exactly the centralizer change already described; it gives a conjugate pair. Conversely any conjugacy of two normalized pairs centralizes A and hence identifies their projective parameters as proved above. At the exceptional point its stabilizer is scalar, so distinct z give distinct pairs. This proves the stated bijection, including completeness and uniqueness. The saved program checks all determinant, adjugate, characteristic-polynomial and finite-order identities over the rational function field, and checks the exceptional line symbolically for every z.

□

Corollary 11.15 (all good-prime lattices are forced)**.** [tag 02NB]

For a $(5, 4)$ datum with $\pi_1 = 1$ and $H_2 = 0$, the rank-five quotient has absolutely irreducible reduction at every prime $p \neq 2, 5$. At each such prime its stable $\mathbb{Z}_p$-lattice is unique up to homothety. The same is true of the rank-six extension among lattices with primitive maximal cusp. Thus, once a rational pair in 11.14 admits one three-connected datum, all remaining integral-lattice ambiguity is supported at the primes two and five.

Proof. ⟨1⟩1. Work first over an algebraic closure of $\mathbb{F}_p$. Both finite generators are semisimple because $p \neq 2, 5$. A one-dimensional composition factor must be trivial: its A and C eigenvalues agree by the unipotent product, and their orders divide five and four. There can be at most one such factor, since the 1 eigenvalue of A occurs once. If it occurs, the remaining four-dimensional module is absolutely irreducible by the spectral argument in 11.13. The only other possible reducible composition pattern is $2 + 3$. On each factor, determinants of A and C agree and have orders dividing five and four, hence are one.

The two-dimensional C eigenvalue pair is therefore $(-1,-1)$ or $(i,-i)$. The first would give $C=-I$ on that factor, contradicting the unipotence of CA^{-1} and the absence of -1 among the eigenvalues of A. In the second case the three-dimensional factor has C eigenvalues $1,-1,-1$, so $C^2=I$ there; it has an A-fixed vector because the other factor has determinant one and cannot contain the simple 1 eigenvalue of A.

⟨1⟩2. Here is the required elementary involutory identity in the present characteristic. Put $B=A^{-1}$ on this factor. From $(CB-I)^2=0$ and $C^2=I$ one obtains

$$BC+CB^{-1}=2I.$$

For $Bv=v$ this says $(B+I)Cv=2v$. The operator $B+I$ is invertible, since $B^5=I$ and $p\neq 2$, so $Cv=v$. The resulting common fixed line contradicts simplicity of the three-dimensional factor. We have proved that any reducible residual representation has exactly the constituents 1 and an absolutely irreducible four-dimensional module. Theorem 11.13 excludes that possibility for a three-connected datum, proving absolute irreducibility at every indicated prime.

⟨1⟩3. Let $R=\mathbb{Z}_p$ and $K_p=\mathbb{Q}_p$. For two stable quotient lattices, multiply one by a power of p until it is contained in the other but not in its p-multiple. Its nonzero residual image must then be the whole quotient by absolute irreducibility. Nakayama's lemma, or the Smith form of the inclusion, makes the lattices equal. This proves quotient-lattice uniqueness.

⟨1⟩4. For completeness the extension-lattice comparison is also integral. Fix a primitive-cusp lattice V_R, with invariant line Re and quotient W_R. The coboundary map

$$W_R^{*}\longrightarrow W_R^{*}\oplus W_R^{*}$$

is primitive, because its reduction has trivial kernel. Thus its cokernel, the degree-one cohomology of the free group on the two generators, is torsion free, and an integral coboundary over K_p is already integral over R. The extension class α is primitive in this cokernel. Indeed, divisibility by p would make the extra cusp map

$$\ker N_W/\operatorname{im} N_W\longrightarrow Re$$

divisible by p, contrary to the three unit Smith factors of the full cusp. Here N_W has primitive image: it is the quotient of the primitive image of N_V by $Re\subset\operatorname{im} N_V$.

⟨1⟩5. Normalize any other stable lattice by the already proved quotient homothety so that its quotient is W_R. Its invariant line is p^rRe for some integer r. The extension class in that basis is $p^{-r}\alpha$; primitivity makes it integral only when $r\leq 0$. Primitivity of the coboundary map eliminates any additional rational change of splitting. The extra cusp map is then multiplied by p^{-r}, so its Smith factor is a unit exactly when $r=0$. The lattices coincide after the common homothety. This proves the assertion.

□

Theorem 11.16 (a terminating lattice algorithm for each rational pair)**.** [tag 02NC]

Let (A,C) be any irreducible rational rank-five pair in one of the eight spectral cases of 11.8, with $A^5=I$, $B=C^{-1}$, and $(AB-I)^2=0$ of rank two. There is an explicit terminating algorithm which decides whether the pair preserves an integral lattice, and, if so, lists all such lattices up to simultaneous integral conjugacy. It then lists all rank-six extensions with the same finite orders and primitive maximal square-zero cusp, up to simultaneous integral conjugacy. The bounds below depend on the input rational pair and lose no lattice.

Proof. ⟨1⟩1. We give the algorithm and its termination proof. The fixed line of A has dimension one. Choose a nonzero vector e on it and a nonzero fixed covector φ. Irreducibility implies that the vectors ge, with g a word in A, C, span the whole rational space; the analogous statement holds for the covectors φh. In fact words of length at most four suffice. The spans of words of lengths at most j increase from dimension one; if one step does not increase the span, it is invariant under both generators and hence already the whole space. There are only four possible increases.

⟨1⟩2. Choose words $g_1, \dots, g_5$ and $h_1, \dots, h_5$ giving bases in this way, and put

$$P = I + A + \cdots + A^4, \qquad T_{ij} = g_i P h_j. \tag{F.L1}$$

The rank-one operator P is a nonzero scalar multiple of $e\varphi$. Hence the 25 matrices T_{ij} are a rational basis of $\mathrm{Mat}_5(\mathbb{Q})$. Each belongs to $\mathcal{R} = \mathbb{Z}[A, C]$. Let $\mathcal{R}_0$ be their integral span and set

$$\mathcal{R}_0^{\#} = \{X : \mathrm{tr}(XT) \in \mathbb{Z} \text{ for all } T \in \mathcal{R}_0\}. \tag{F.L2}$$

The trace pairing on the full matrix algebra is nondegenerate, as is seen from the matrix units, so this is another full rational lattice, computed by inverting the 25-by-25 Gram matrix.

⟨1⟩3. If any stable lattice exists, every element of $\mathcal{R}$ acts integrally on it. Consequently $\mathrm{tr}(XY) \in \mathbb{Z}$ for all $X, Y \in \mathcal{R}$, and necessarily

$$\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_0^{\#}. \tag{F.L3}$$

A nonintegral Gram entry therefore proves nonexistence. Otherwise let $d = [\mathcal{R}_0^{\#} : \mathcal{R}_0]$, the absolute Gram determinant. Start with the lattice generated by $\mathcal{R}_0$ and I, and repeatedly replace a current lattice L by $L + AL + CL$, using exact Hermite reduction. If it leaves $\mathcal{R}_0^{\#}$, (F.L3) proves that there is no stable lattice. While it remains inside, every strict enlargement reduces its index by a factor at least two. There can be at most $\lfloor \log_2 d \rfloor$ such enlargements starting from $\mathcal{R}_0$, so a bound of $1 + \lfloor \log_2 d \rfloor$ tests after adjoining I is sufficient. Stabilization gives exactly $\mathcal{R}$: the stable lattice contains I and all words in A, C, and every element introduced belongs to their ring. It is then an order, and $\Lambda_0 = \mathcal{R}\mathbb{Z}^5$ is an explicitly computed stable lattice. This proves the decision step, including termination in the nonexistence case.

⟨1⟩4. Use a basis of Λ_0 from now on, so $\mathcal{R} \subset \mathrm{Mat}_5(\mathbb{Z})$. Smith reduction gives an integer $c \geq 1$ with

$$c\,\mathrm{Mat}_5(\mathbb{Z}) \subset \mathcal{R}; \tag{F.L4}$$

one may take the largest Smith factor of the inclusion of the order in the full matrix lattice. Normalize any stable lattice by a rational homothety so that it is contained in $\mathbb{Z}^5$ and the gcd of all coordinates of a lattice basis is one. The operators cE_{ij} preserve it. The coordinate values of its vectors generate $\mathbb{Z}$, so Bezout's identity, applied to cE_{ij}, shows that it contains every ce_i. Therefore every normalized lattice lies in the finite interval

$$c\mathbb{Z}^5 \subset \Lambda \subset \mathbb{Z}^5. \tag{F.L5}$$

Conversely every lattice in this interval stable under A, C is retained by the algorithm. The only rational operators commuting with the pair are scalars: they act by a scalar on the one-dimensional fixed line of A, and then by the same scalar on the spanning vectors $g_i e$. Thus rational equivalence of two stable lattices is precisely homothety. The coordinate-gcd normalization is unique, so (F.L5) gives the integral conjugacy classification without further identifications.

⟨1⟩5. Here is a finite enumeration bound which is also implemented. For each $p^a \| c$, enumerate all subspaces of $\mathbb{F}_p^5$ in reduced echelon form. Beginning with $\mathbb{Z}^5$, pass through a levels of stable lattices between pH and H, retaining only those containing $p^a\mathbb{Z}^5$ and identifying equal Hermite forms. Every desired lattice occurs: for any target Λ, the chain $\Lambda + p^j\mathbb{Z}^5$, $0 \le j \le a$, is such a chain and ends at Λ. If s_p denotes the number of all subspaces of $\mathbb{F}_p^5$, there are at most s_p^a branches before deduplication. Intersect the resulting lattices for the different primes and discard those of coordinate gcd greater than one. The Chinese remainder theorem proves that this enumerates (F.L5) completely.

⟨1⟩6. For each quotient lattice $W_\mathbb{Z}$ so obtained, put $N_W = AB - I$. Its image must be primitive: for a primitive full cusp, $\operatorname{im} N_V = \ker N_V$ contains the primitive invariant line, and quotienting that image by the line gives a primitive image in $W_\mathbb{Z}$. Reject a quotient whose two nonzero Smith factors are not one. For the remaining quotients use extension rows $c_1, c_2 \in \mathbb{Z}^{1\times 5}$ and impose

$$\begin{gathered} c_1(I + A + \cdots + A^4) = 0, \qquad c_2(I + B + \cdots + B^{m-1}) = 0, \\ (c_2 + c_1B)N_W = 0, \qquad m = \operatorname{ord}(B). \end{gathered} \tag{F.L6}$$

These are integral linear equations. Their integral kernel modulo the coboundaries

$$z \longmapsto \big(z(A - I), z(B - I)\big), \qquad z \in \mathbb{Z}^{1\times 5}, \tag{F.L7}$$

is a finitely generated group G of rank one by 11.5. Compute its free and torsion factors by Smith reduction. Choose an integral generator w of the free rank-one quotient $\ker N_W / \operatorname{im} N_W$. The homomorphism

$$\kappa : G \longrightarrow \mathbb{Z}, \qquad [(c_1, c_2)] \longmapsto (c_2 + c_1B)w \tag{F.L8}$$

is well defined and nonzero: (F.L6) kills the image, and (F.L7) kills w; rational nonvanishing is 11.5. The full cusp has unit Smith factors exactly when $\kappa = \pm 1$. Indeed the quotient rank-two image is already primitive, and the remaining invariant-line image is generated by that one integer. If $\operatorname{im}\kappa \ne \mathbb{Z}$, reject the lattice. Otherwise the inverse image of 1 is a finite coset of $\operatorname{Tor} G$, and lists all extensions. Changing the sign of the invariant-line generator identifies $\kappa = -1$ with $\kappa = 1$. There are no further identifications: an integral isomorphism on the quotient is $\pm I$, on the invariant line is ± 1, and, after $\kappa = 1$ is fixed, their ratio is one. The remaining row is exactly an integral change of splitting (F.L7). This proves completeness and termination of the extension step as well.

⟨1⟩7. The order, trace-dual bound, conductor and finite-interval steps are implemented in `fourfolds_lattice_algorithm.py`; the extension-kernel and Smith steps are implemented in `fourfolds_two_classes_lattices.py` with the finite order as an argument. Their applications and exact exhaustive bounds are recorded next.

□

Theorem 11.17 (Lattice classes of the two three-connected members)**.** [tag `02ND`]

For each of the two rational representations $v = 0$ and $v = 1$ of 11.10, there are exactly two homothety classes of stable rank-five quotient lattices, and exactly three integral conjugacy classes of rank-six extensions with primitive maximal cusp. Exactly one of those three can have simply connected finite affine fillings. It is the integral class already realized in 5.5, respectively 11.12. Thus arbitrary stable-lattice changes produce no further linear classes with three-connected gluing anywhere in the family 11.10. The affine twists and multiplicities are treated in theorem 11.18.

Proof. ⟨1⟩1. Apply the preceding algorithm in the original five-cycle lattice (F.U1). For both parameters the order is obtained after four strict enlargements. Its Smith factors in the full

matrix lattice are 21 factors one and four factors two; thus the conductor $c = 2$ in (F.L4) is exact. All 374 subspaces of $\mathbb{F}_2^5$, including zero and the whole space, are enumerated, with no height restriction or neighbor-depth assumption. After imposing stability and coordinate gcd one, the two lattice representatives are

$$\mathbb{Z}^5, \qquad \Lambda_* = 2\mathbb{Z}^5 + \mathbb{Z}(1,1,1,1,1)^t. \tag{F.L9}$$

The latter has index 16. The generic prime-by-prime implementation of (F.L5) and an independent bit-vector enumeration give exactly the same two Hermite forms.

⟨1⟩2. For $\mathbb{Z}^5$, the extension group in (F.L6)–(F.L7) is $\mathbb{Z}$ and κ is a unit on its generator. For Λ_*, that group is $\mathbb{Z} \oplus \mathbb{Z}/2$, again with primitive κ. Hence there are exactly one and two primitive extensions, respectively. For each extension the program computes the image of the primitive global invariant homology covector ℓ on the two finite fixed lattices. The resulting positive divisibilities are

quotient lattice	extension class	$(\mathrm{div}_1, \mathrm{div}_2)$
$\mathbb{Z}^5$	unique	$(1,2)$
Λ_*	first torsion coset	$(1,1)$
Λ_*	second torsion coset	$(1,2)$.

(F.L10)

The table is identical for $v = 0$ and $v = 1$. All six full matrices, the integral kernel bases, the Smith factors and the finite divisibilities are computed exactly by `fourfolds_two_classes_lattices.py` .

⟨1⟩3. For a simply connected filling the determinant quotient of 11.6 requires $\gcd(\ell(v_i), m_i) = 1$ at each finite cap. The second affine multiplicity is divisible by four. A divisibility of two in (F.L10) therefore excludes that class for every choice of twists and larger multiplicities. The sole remaining class has an explicit integral intertwiner of determinant ± 1 with the corresponding pair in (F.U6), as printed and checked by the program. Those pairs already have the verified simply connected, three-connected fillings. 11.13 excludes every other rational parameter for all lattices. Together these facts prove all the claims, including the absence of any additional linear class with three-connected gluing in the unbounded family. □

Theorem 11.18 (All three-connected twists in the unbounded family)**.** [tag `02NE`]

Use the fixed Kuhn cusp and the affine translation clutching convention of 11.6. In the entire rational family 11.10, allow every stable lattice, every free affine finite lift, and every pair of finite multiplicities. The data with $\pi_1 = 1$ and $H_2 = H_3 = 0$ are precisely the following, up to the integral linear classification already proved and the translation identifications described below:

$$\begin{aligned} v \in \{0,1\}, \qquad & \text{the unique eligible integral class of 11.17,} \\ (m_1, m_2) = (5,4), \qquad & 20\kappa + 4r_1 + 5r_2 = \pm 1. \end{aligned} \tag{F.T1}$$

Here $r_i = \ell(v_i)$ is the invariant affine numerator, and $\kappa \in \mathbb{Z}$ is the invariant coordinate of the cusp translation clutching, with signs fixed by the displayed van Kampen relation below. The remaining fixed-direction components of the twists impose no condition. Every manifold in (F.T1) is oriented diffeomorphic to its corresponding known example and hence to $3\mathbb{HP}^2 \# (-\mathbb{HP}^2)$. This completes the classification for this rational family.

Proof. ⟨1⟩1. 11.13 excludes all other rational parameters, and 11.17 excludes all other integral linear classes. We therefore work with either known integral pair, in the marking of (F.U6). Its primitive invariant homology covector is $\ell(x) = x_0$. Each finite fixed lattice has an integral basis u_i, λ_i, where $\ell(u_i) = 1$, $\ell(\lambda_i) = 0$, and λ_i is primitive in $\ker \ell$. We may write

the affine numerator as $v_i = r_i u_i + s_i \lambda_i$. This covers arbitrary finite affine lifts: translating the origin makes their translation vector invariant by averaging, and its m_ith iterate is then the integral vector v_i. The resulting boundary translation change is included in the specified affine clutching.

⟨1⟩2. For both linear pairs, the normal closure of the cusp angular lattice in the fibre lattice is exactly $\ker \ell$, by 5.1 and 11.12. Van Kampen therefore leaves the abelian presentation

$$m_1 a = r_1 h, \qquad m_2 b = r_2 h, \qquad a + b + \kappa h = 0. \tag{F.T2}$$

This also defines the sign of κ. Its determinant is, up to sign,

$$p = m_1 m_2 \kappa + m_2 r_1 + m_1 r_2. \tag{F.T3}$$

Thus simple connectivity is exactly $p = \pm 1$. It forces $\gcd(m_1, m_2) = 1$ and $\gcd(r_i, m_i) = 1$ by reduction modulo each multiplicity.

⟨1⟩3. We first prove that a three-connected datum has the minimal multiplicities. Set $n_1 = 5$, $n_2 = 4$ and $m_i = n_i k_i$. In the splitting $\Lambda = \mathbb{Z} u_i \oplus \ker \ell$, write the finite linear map as $1 \oplus D_i$. The reduced finite fibre is a mapping torus of T^5 over the circle $m_i \ell$; its fibre is connected because r_i is prime to m_i. Its vertical monodromy is a power of D_i prime to n_i, possibly followed by a translation, which changes no integral Wang group.

⟨1⟩4. The primitive fixed vertical loop λ_i gives the only additional free two-cycle beyond vertical cycles. Consequently an invariant integral two-form ω on the covering torus descends integrally exactly when

$$\frac{\omega(v_i, \lambda_i)}{m_i} = \frac{r_i \omega(u_i, \lambda_i)}{m_i} \quad \text{is an integer.} \tag{F.T4}$$

For precision, the torus made from the affine generator and λ_i projects with degree r_i to the mapping- torus base, whereas the ordinary u_i suspension projects with degree m_i. A Bezout combination is a primitive suspension. Its integrality condition is equivalent to (F.T4), because the ordinary covering periods are already integral and $\gcd(r_i, m_i) = 1$. The Wang sequence shows that these vertical cycles and one primitive suspension generate the free second homology; there is no omitted period condition.

⟨1⟩5. The integral invariant two-form computations in 5.5 and 11.12 give

$$\gcd_{\omega} \omega(u_i, \lambda_i) = n_i. \tag{F.T5}$$

They are also independently printed for both pairs by `fourfolds_family_twists.py`. Hence the covering-image cokernel in degree two at the ith finite point is exactly $\mathbb{Z}/k_i$. The global invariant group of $\bigwedge^2 \mathcal{V}$ and its integral first parabolic group both vanish. The skyscraper cokernel sequence therefore gives

$$H^1(B, R^2 g_* \mathbb{Z}) = \mathbb{Z}/k_1 \oplus \mathbb{Z}/k_2. \tag{F.T6}$$

Point-supported torsion kernels do not change this degree-one base cohomology, and the cusp specialization is onto. The term (F.T6), in bidegree $(1, 2)$, survives every Leray differential on the two-dimensional base. All rational terms of total degree three vanish, independently of the multiplicities and twists, by the same exterior local-system table as 5.5. Thus H^3 is finite. If either $k_i > 1$, its surviving nonzero filtration quotient makes H^3 nonzero; universal coefficients then give nonzero torsion in H_2. This proves $m_1 = 5$, $m_2 = 4$ for every three-connected datum.

⟨1⟩6. We now identify all twists with these multiplicities. The following integral fact permits removal of each s_i:

$$\mathrm{Norm}_{n_i} : \ker \ell \longrightarrow \mathbb{Z}\lambda_i \quad \text{is onto.} \tag{F.T7}$$

Here the norm is $I + D_i + \cdots + D_i^{n_i-1}$. One can read (F.T7) directly from (F.T5). Let φ_i generate the primitive invariant covector lattice on $\ker \ell$. The splitting by u_i shows that the gcd in (F.T5) is $|\varphi_i(\lambda_i)|$: its forms are $\ell \wedge \varphi_i$ plus invariant vertical forms. Thus that value is n_i. The norm has the form $x \mapsto d_i\varphi_i(x)\lambda_i$ for an integer d_i; evaluating φ_i gives $|d_i|n_i = n_i$, hence $|d_i| = 1$. This proves surjectivity. The saved program additionally gives integral preimages $q_i \in \ker \ell$ with $\mathrm{Norm}_{n_i} q_i = \lambda_i$.

⟨1⟩7. Put

$$w_i = -\frac{s_i}{n_i} \sum_{j=0}^{n_i-1} j M_i^j q_i.$$

Then

$$(I - M_i)w_i = -s_i\lambda_i/n_i + s_i q_i. \tag{F.T8}$$

Translation by w_i conjugates the finite affine action with numerator $r_iu_i + s_i\lambda_i$ to that with numerator r_iu_i: the remaining s_iq_i is integral. It gives a diffeomorphism of the entire finite cap. In the standard linear boundary marking, its translation section is $w_i + \theta s_i\lambda_i/n_i$. Its endpoint mismatch with the linear deck map is exactly s_iq_i by (F.T8), so its integral boundary residue has ℓ-coordinate zero.

⟨1⟩8. We explain why these local changes extend simultaneously, including the prescribed cusp marking. Let P be the complementary pair of pants and $\mathcal{T} \to P$ its flat torus group bundle. A section on the boundary extends over the relative one-skeleton because the torus is connected. The only remaining obstruction is the two-cell obstruction with values in the fibre fundamental-group local system:

$$H^2(P, \partial P; \Lambda) \simeq H_0(P; \Lambda) = \Lambda_\Gamma. \tag{F.T9}$$

The equality follows from the oriented relative cellular complex, equivalently integral Poincaré–Lefschetz duality with local coefficients. Its boundary residues map to the indicated global coinvariants. For both linear pairs the horizontal monodromy relation matrix has five unit Smith factors, so $\Lambda_\Gamma = \mathbb{Z}$, detected exactly by ℓ. This is independently checked in the saved program. Thus the residues s_iq_i, with the identity section at the cusp, have zero obstruction and extend over P. A section may be smoothed relative to its boundary collars; translation by it is a fibre-preserving diffeomorphism. Together with the finite-cap maps it removes all s_i without changing the cusp. The same argument removes all components of translation clutching whose ℓ-coordinate is zero.

⟨1⟩9. It remains to handle the integer numerators and the remaining cusp coordinate. If $p = -1$, the congruences from (F.T3) give

$$r_1 = 1 + 5a, \qquad r_2 = -1 + 4b, \qquad a + b + \kappa = 0. \tag{F.T10}$$

Adding $5au_1$ or $4bu_2$ changes no finite affine action; in its linear boundary marking it changes the translation residue by au_1 or bu_2. The difference of the cusp marking has residue of ℓ-coordinate κ, by (F.T2). Their total obstruction in (F.T9) is therefore $a + b + \kappa = 0$. The same extension argument identifies the glued manifold with the original twists $(u_1, -u_2)$ and zero cusp clutching. If $p = 1$, the identical calculation starts from $(-u_1, u_2)$.

⟨1⟩10. Finally simultaneous fibre inversion identifies these two sign choices by an orientation-preserving diffeomorphism. It conjugates every affine translation to its negative. It extends

over the Kuhn cusp: the lattice involution $(x, h) \mapsto (-x, h)$ preserves its toric fan, since a simplex given by a chain in a unit cube is taken to the chain with reversed coordinate order in the opposite cube. It normalizes the cusp lattice action and therefore descends to its quotient, while fixing the base parameter. The resulting toric map is holomorphic, and in the regular real six-torus inversion has determinant $(-1)^6 = 1$. All translation maps also preserve orientation. Thus every datum in (F.T1) gives the stated oriented smooth type and has $H_2 = H_3 = 0$, proving sufficiency as well as necessity.

□

Theorem 11.19 (the universal degree-two suspension obstruction)**.** [tag `02NF`]

Every admissible datum whose gluing has $\pi_1 = 1$ and $H_2 = H_3 = 0$ has exactly two genuine finite fillings and has minimal affine multiplicities. After ordering the order-five point first, its finite characteristic polynomials are one of precisely the following two remaining necessary possibilities:

$$\begin{array}{c|c|c} (m_1, m_2) & \chi_{M_1} & \chi_{M_2} \\ \hline (5,4) & (x-1)^2\Phi_5 & (x-1)^2\Phi_2^2\Phi_4 \\ (5,6) & (x-1)^2\Phi_5 & (x-1)^2\Phi_2^2\Phi_3. \end{array} \tag{F.S1}$$

In particular all order-eight and order-twelve spectral loci, and the other two order-six spectral loci of 11.8, are excluded for every rational relative position, every stable lattice and every free affine twist. There is no multiplicity search bound in this statement: the only possible affine multiplicities are the two displayed pairs.

Proof. ⟨1⟩1. Theorem 11.1 excludes three genuine finite fillings. For two fillings, 11.3 and 11.8 give a primitive global invariant homology covector ℓ, finite fixed rank two, and an irreducible rank-five cohomology quotient. In particular

$$H^0(B, j_* \bigwedge^2 \mathcal{V}) = 0. \tag{F.S2}$$

This is also immediate from the exterior filtration and irreducibility in odd dimension, as in 11.11. Simple connectivity gives the determinant quotient of 11.6, so each invariant affine numerator $r_i = \ell(v_i)$ is prime to the multiplicity m_i, and the two multiplicities are coprime. As before one may conjugate any finite affine lift by a translation so that its numerator v_i is integral and fixed by its linear part.

⟨1⟩2. Consider one finite point. Let M have linear order n, so $n \mid m$, and let its complementary degree-four factor be $f(x)$, with $\chi_M = (x-1)^2 f(x)$. The image of ℓ on the finite fixed lattice is $g\mathbb{Z}$ for some $g \mid n$: averaging $I + M + \cdots + M^{n-1}$ on any vector of ℓ-value one proves this divisibility. Since $g \mid r$ and $\gcd(r, m) = 1$, we have $g = 1$. Choose a fixed integral vector u with $\ell(u) = 1$. It gives an integral invariant splitting

$$\Lambda = \mathbb{Z}u \oplus H, \qquad H = \ker \ell, \qquad M = 1 \oplus D, \qquad \chi_D = (x-1)f(x). \tag{F.S3}$$

The order of D is n. Let λ generate its primitive fixed lattice and let φ generate its primitive invariant covector lattice. Choose their signs so that

$$h = \varphi(\lambda) > 0. \tag{F.S4}$$

Their rational pairing is nonzero, since the finite map is semisimple and its averaging projection has rank one.

⟨1⟩3. Both $\mathrm{Norm}_n(D)$ and $f(D)$ are integral operators, have image on the line $\mathbb{Q}\lambda$, and have invariant defining covector proportional to φ. Primitivity of λ and φ therefore implies

$$\mathrm{Norm}_n(D) = \frac{n}{h}\lambda\varphi, \qquad f(D) = \frac{f(1)}{h}\lambda\varphi, \qquad h \mid n, \quad h \mid f(1). \tag{F.S5}$$

For example, the integer coefficient in the second formula is seen by applying the integral operator to a vector on which φ is one; evaluating on λ determines that coefficient. No lattice splitting over a localization or semisimplification modulo a prime is used here.

⟨1⟩4. Write $v = ru + s\lambda$. The quotient finite fibre is a connected T^5 mapping torus over the circle $m\ell$, because r is prime to m. Its vertical cycles already have integral periods for an invariant integral two-form on the covering torus. The Wang sequence leaves one additional primitive free suspension cycle, from λ. Exactly as in (F.T4), descent of an invariant form ω is equivalent to $m \mid r\omega(u, \lambda)$, and hence to $m \mid \omega(u, \lambda)$. In the integral splitting (F.S3), every invariant two-form is an integral multiple of $\ell \wedge \varphi$ plus an invariant vertical two-form. Its possible values on (u, λ) are therefore exactly $h\mathbb{Z}$. It follows that the degree-two specialization image at this point has cokernel

$$C_p^2 \simeq \mathbb{Z}/(m/h). \tag{F.S6}$$

The integer m/h is defined because $h \mid n \mid m$. Torsion in the finite fibre cohomology is a specialization kernel, not an additional restriction on these free periods. The primitive suspension and vertical cycles generate the free second homology, so (F.S6) is the full cokernel.

⟨1⟩5. Let I^2 be the image sheaf of $R^2 g_* \mathbb{Z} \to j_* \bigwedge^2 \mathcal{V}$. The cusp specialization is onto by V.2.2, and all kernels and cokernels at finite points are skyscraper sheaves. Thus $H^1(R^2 g_* \mathbb{Z}) = H^1(I^2)$. Equation (F.S2) and the cokernel exact sequence give an injection

$$\bigoplus_{p \text{ finite}} C_p^2 \hookrightarrow H^1(B, R^2 g_* \mathbb{Z}). \tag{F.S7}$$

The Leray term of bidegree $(1, 2)$ has no incoming or outgoing differential on a real two-dimensional base. If $H_2 = H_3 = 0$, the universal coefficient theorem gives $H^3 = 0$, so that term must vanish. Every cyclic cokernel in (F.S7) is therefore zero. Combining (F.S5)–(F.S6) yields the universal necessary condition

$$m = h = n, \qquad n \mid f(1). \tag{F.S8}$$

This both fixes the multiplicity and tests the spectrum.

⟨1⟩6. For completeness, apply (F.S8) to the entire eight-case list of 11.8. The order-five factor gives $\Phi_5(1) = 5$. At the other point the exact values are

n	f	$f(1)$	$n/\gcd(n, f(1))$
4	$\Phi_2^2\Phi_4$	8	1
6	$\Phi_2^2\Phi_3$	12	1
6	$\Phi_2^2\Phi_6$	4	3
6	$\Phi_3\Phi_6$	3	2
8	Φ_8	2	4
12	Φ_{12}	1	12
12	$\Phi_3\Phi_4$	6	2
12	$\Phi_4\Phi_6$	2	6.

(F.S9)

The last column is a lower bound for the nonzero local covering-cokernel order already at the smallest allowed multiplicity. Only the first two rows pass. The complete table is checked by `fourfolds_suspension_filter.py`; its input is exactly these eight spectra, with no matrix-entry or relative-position cutoff. This proves (F.S1) and all claimed universal exclusions.

□

Theorem 11.20 (the complete rational remaining $(5,6)$ locus)**.** [tag 02NG]

The irreducible rational rank-five pairs in the second row of (F.S1)*, with square-zero rank-two cusp, are classified up to simultaneous rational conjugacy by*

$$\{[r:s:t]\in\mathbb{P}^2(\mathbb{Q}) : D(r,s,t)\neq 0,\ Q_-(r,s,t)\neq 0\}\ \sqcup\ \mathbb{A}^1(\mathbb{Q}),\qquad Q_- = r^2-rs+st, \tag{F.R10}$$

where D is the same cubic as in (F.R2)*. The exceptional line again lies over $[4:1:3]$. Thus the two rational spectral loci left by 11.19 both have complete explicit normal forms, including their entire rational exceptional loci.*

Proof. ⟨1⟩1. Use precisely the space $\mathbb{Q}\oplus\mathbb{Q}(\zeta_5)$, matrices A, R, X and projective plane normalization in (F.R3)–(F.R5). The centralizer and image-plane arguments in 11.14 require only irreducibility and the two-dimensional -1 eigenspace of C; both hold here. Consequently every pair has a unique projective image-plane parameter, with only scalar stabilizer. The unknown Y now satisfies

$$YX=0,\qquad YRX=I_2,\qquad \operatorname{tr}(YAX)=-2,\qquad \operatorname{tr}(YA^{-1}X)=2. \tag{F.R11}$$

Its coefficient matrix is unchanged from (F.R6), so has determinant $D/4$. The two traces and the -1 eigenspace imply

$$\chi_C=(x-1)(x+1)^2(x^2+x+1).$$

Its other three roots are distinct, and the -1 eigenspace has its full multiplicity, so $C^6=I$. The cusp XY still has square zero and rank two. For $D\neq 0$, the ten linear equations give the unique explicit solution.

⟨1⟩2. The two first-column adjugate identities become

$$Dy_{10}=2Q_-Q_2,\qquad Dy_{20}=2(r-s-t)Q_-, \tag{F.R12}$$

with the same Q_2 as in (F.R7). For $D\neq 0$, these entries vanish simultaneously exactly when $Q_-=0$. Indeed, if $Q_-\neq 0$ they would instead force $r=s+t$ and $Q_2=0$, and (F.R8) would give $D=0$. The invariant-covector argument of 11.14 is unchanged. Thus the pair is rationally irreducible exactly off the conic $Q_-=0$ in the generic locus.

⟨1⟩3. No nonzero rational triple satisfies $D=Q_-=0$. If $s=0$, the conic forces $r=0$, and then $D=7t^3$. If $s\neq 0$, rescale to $s=1$ and put $r=u$, $t=u-u^2$. Direct substitution gives

$$D(u,1,u-u^2)=-(7u^2-11u+7)(u^4-u^3+u^2-u+1). \tag{F.R13}$$

The first factor has discriminant -75 and positive leading coefficient. The second is strictly positive, as the identity

$$u^4-u^3+u^2-u+1=\left(u^2-\frac{u}{2}\right)^2+\frac{3}{4}\left(u-\frac{2}{3}\right)^2+\frac{2}{3}$$

shows. In particular neither has a rational zero. On the exceptional cubic the necessary identities (F.R12) therefore force the same two points $[3:2:1]$ and $[4:1:3]$ as before. The first is again impossible because $[X\mid RX]$ has rank less than four. At the second, the complete solution is

$$Y_z=\begin{pmatrix} 4z+26/5 & 0 & -2z-2 & -2z-1/5 & 2z+1\\ 2z-2/5 & 0 & -z & 2/5-z & z\end{pmatrix},\qquad z\in\mathbb{Q}. \tag{F.R14}$$

Its first-column entries cannot vanish simultaneously, so each pair on this line is irreducible. There are no further identifications, by the scalar stabilizer argument already proved in 11.14. This establishes the claimed bijection.

⟨1⟩4. The exact program is `fourfolds_relative_six_probe.py`. It checks the modified characteristic polynomials, finite-order identities, first-column adjugate expressions, conic substitution and entire exceptional line symbolically. Its auxiliary trace-parameter computations for the other repeated-negative spectrum are recorded as well, although that spectrum is now excluded integrally by 11.19. No finite sample is used to infer a rational classification. The finite lattice and extension algorithm of 11.16 applies to every point of (F.R10); it does not by itself decide which rational parameters survive the remaining integral homology tests.

□

Theorem 11.21 (the entire remaining order-six locus is excluded)**.** [tag 02NH]

Every simply connected datum in the second row of (F.S1)*, at every rational relative position and every stable integral lattice with primitive maximal cusp, has*

$$\operatorname{Tor}_3 H_3(M;\mathbb{Z}) \neq 0$$

if its degree-two suspension cokernels vanish. Consequently no such datum is three-connected. Together with 11.19 this leaves only the minimal order pair $(5,4)$ *as a possible three-connected admissible datum.*

Proof. ⟨1⟩1. We use cohomology with coefficients in the discrete valuation ring $R=\mathbb{Z}_3$. Suppose the degree-two suspension cokernels vanish; otherwise 11.19 already gives nonzero H_2 or H_3. That theorem forces $m=n=h=6$ at the order-six point, in the notation (F.S3)–(F.S5). We first determine its local integral representation over R.

⟨1⟩2. The splitting by the fixed vector u is integral: $\Lambda_R = Ru\oplus H_R$. On H_R, the finite map D has characteristic polynomial $(x-1)(x+1)^2\Phi_3(x)$. Since two is a unit in R, the idempotents $(I\pm D^3)/2$ give an integral direct sum

$$H_R = L_+\oplus L_-, \qquad \operatorname{rank} L_+ = 3, \quad \operatorname{rank} L_- = 2.$$

On L_- the map is exactly $-I$, by its rational spectrum. On L_+ it has order three. The primitive fixed vector λ lies in L_+, and the primitive invariant covector φ vanishes on L_-, so their restrictions remain primitive. Equation (F.S5) with $h=6$ gives

$$2(I+D+D^2)|_{L_+} = \lambda\varphi|_{L_+}. \tag{F.X1}$$

This operator is nonzero modulo three. But modulo three $I+D+D^2=(D-I)^2$ and $(D-I)^3=0$. The reduction of D on the rank-three space therefore has one Jordan block of length three. Choose $q\in L_+$ whose image has $(D-I)^2q\neq 0$ modulo three. Then q, Dq, D^2q are a basis modulo three and hence an R-basis. In this basis D is the three-cycle permutation matrix, since $D^3=I$. We have proved, without a global lattice classification, that the full finite representation over R is

$$1\oplus \operatorname{Perm}_3\oplus(-I_2). \tag{F.X2}$$

Its dual has the same form, by reversing the three-cycle.

⟨1⟩3. Let u,a,b,c,d,e be a cohomology basis for (F.X2), with a,b,c cycled and d,e negated. A full integral basis of the invariant degree-three lattice is

$$u(ab+bc+ca), \qquad abc, \qquad ude, \qquad (a+b+c)de. \tag{F.X3}$$

Here juxtaposition denotes exterior multiplication. To check completeness, terms with an odd number of d,e are negated by the cube of the finite map and have no invariants.

The terms with zero or two such factors give exactly the four orbit sums in (F.X3), each primitive. With volume $uabcde$, their skew pairing has one block with coefficient three and one block with coefficient a unit. Thus its determinant has three-adic valuation two. The program `fourfolds_universal_56_obstruction.py` checks the full invariant kernel, its primitivity, and the pairing independently by exterior multiplication and six-column determinants; its Smith factors are $(1, 1, 3, 3)$.

⟨1⟩4. Let B_6 be the reduced finite fibre. Its torus cover has degree six. Its free middle cohomology has rank four and unimodular skew pairing, by integral Poincaré duality. If I denotes the index of its pullback lattice in the invariant lattice, multiplication of pairings by the covering degree gives

$$2v_3(I) + 2 = 4v_3(6) = 4, \qquad v_3(I) = 1. \tag{F.X4}$$

Consequently the three-primary degree-three covering cokernel is $R/3R$. Translations of the affine action do not enter this calculation: the cover degree, linear invariant lattice and oriented pairings are those of the free quotient for every allowed affine numerator.

⟨1⟩5. We explain why this local group must appear in total homology even when the global integral lattice varies. Let $\mathcal{V}$ be the rational rank-six cohomology local system. It is an extension of the irreducible rank-five system W by a trivial line. The determinant of W is trivial. Irreducibility in odd dimension implies that neither $\bigwedge^2 W$ nor its dual has a nonzero invariant: an invariant alternating form would have an invariant radical and would have to be nondegenerate, impossible in odd dimension. Also $\bigwedge^3 W \simeq \bigwedge^2 W^*$ by its trivial determinant. The exterior filtration therefore gives

$$(\bigwedge^3 \mathcal{V})^\Gamma = 0, \qquad ((\bigwedge^2 \mathcal{V})^*)^\Gamma = 0. \tag{F.X5}$$

These rational vanishing statements remain true over $\mathbb{Q}_3$, since they are kernels of rational matrices.

⟨1⟩6. As in (F.S7), take the image of $R^3 g_* R \to j_* \bigwedge^3 \mathcal{V}_R$. Its kernel and cokernel are point-supported sheaves. The first vanishing in (F.X5) makes the local group $R/3R$ inject into $H^1(B, R^3 g_* R)$. Bidegree $(1, 3)$ has no incoming or outgoing Leray differential on the two-dimensional base, so this injection survives to $E_\infty^{1,3}$.

⟨1⟩7. There is one extension issue, and it cannot remove this torsion. The second vanishing in (F.X5) says that $H^2(B, R^2 g_* R)$ has rational rank zero. Point-supported changes do not alter this degree-two base cohomology; it is the coinvariant module, as in (E.9). It is a finitely generated torsion R-module and hence a finite group. Therefore $E_\infty^{2,2}$ is finite as well. In the filtration of $H^4(M; R)$, take the inverse image of the injected $R/3R \subset E_\infty^{1,3}$. It is an extension of $R/3R$ by the finite group $E_\infty^{2,2}$. It is thus a nonzero finite three-primary subgroup of $H^4(M; R)$; in particular it cannot be absorbed as a finite quotient of a free filtration kernel. Universal coefficients identify this torsion with the three-primary torsion of $H_3(M; \mathbb{Z})$. This proves the claim at every rational position and every integral lattice, and completes the exclusion.

□

Theorem 11.22 (universal fundamental group and twist normalization)**.** [tag `02NI`]

In the remaining $(5, 4)$ locus, fix the marked standard Kuhn cap and allow affine translation clutching. Let $\Lambda = \mathbb{Z}^6$ be fibre homology and let Λ_Γ be its global monodromy coinvariants. Write $\Lambda_\Gamma = \mathbb{Z}h \oplus T$, with T finite, and let ℓ be the primitive invariant covector detecting the free factor. The following are complete reductions for any datum that could satisfy $\pi_1 = 1$ and $H_2 = H_3 = 0$.

The multiplicities must be $(5,4)$*, and the local vertical fixed pairings must be* $h_1 = 5, h_2 = 4$*, as in 11.19. For any affine twists and clutching the fundamental group is abelian. If its free presentation determinant is nonzero, its order is*

$$|\pi_1| = |20\kappa + 4r_1 + 5r_2|\,|T|, \qquad r_i = \ell(v_i). \tag{F.T11}$$

If that determinant is zero the group is infinite. In particular simple connectivity is equivalent to

$$T = 0, \qquad 20\kappa + 4r_1 + 5r_2 = \pm 1. \tag{F.T12}$$

Whenever these conditions and $h_1 = 5, h_2 = 4$ *hold, every such twist choice is oriented diffeomorphic to the normalized filling with twists* $(u_1, -u_2)$ *and zero cusp clutching, where* u_i *is any integral fixed vector with* $\ell(u_i) = 1$*. Thus each eligible integral linear datum has exactly one normalized smooth filling to test for the remaining homology conditions; no affine-twist enumeration remains.*

Proof. ⟨1⟩1. The image of the fibre fundamental group is a normal abelian subgroup in the van Kampen presentation. After quotienting by it, the two finite meridians satisfy $a^5 = b^4 = ab = 1$. That quotient is trivial, because five and four are coprime. The entire fundamental group is therefore equal to the image of the fibre group and is abelian. This argument does not assume that the normal closure of cusp vanishing cycles already exhausts $\ker \ell$.

⟨1⟩2. In abelianization the monodromy relations give Λ_Γ. The primitive cusp angular lattice is the image of its nilpotent operator, and this image is already zero among those monodromy relations. The remaining presentation is

$$5a = v_1, \qquad 4b = v_2, \qquad a + b + v_0 = 0 \quad \text{in } \Lambda_\Gamma \oplus \mathbb{Z}a \oplus \mathbb{Z}b, \tag{F.T13}$$

where $\ell(v_0) = \kappa$ fixes the sign convention. Use any Smith presentation of T and split the free part $\mathbb{Z}h$. The full relation matrix is block triangular, with the presentation of T as one diagonal block and the three-by-three free matrix of determinant $\pm(20\kappa + 4r_1 + 5r_2)$ as the other. Its determinant gives (F.T11); when the latter free matrix has rank less than three the group has positive rank and is infinite. For determinant ± 1, elimination of the three free variables leaves precisely T, even if the twists have nonzero torsion components. This proves the exact simple-connectivity criterion.

⟨1⟩3. We give the global twist comparison to make its scope precise. Under (F.T12), the finite fixed covector image is one at each point, so choose integral fixed u_i with $\ell(u_i) = 1$. The primitive vertical fixed vector λ_i and invariant covector φ_i satisfy $\varphi_i(\lambda_i) = m_i$, by the assumed maximal local pairings. Formula (F.S5) makes the finite norm onto $\mathbb{Z}\lambda_i$. Consequently every vertical numerator component $s_i\lambda_i$ is removed by the explicit cap translation (F.T8), whose boundary residue has ℓ-coordinate zero.

⟨1⟩4. For the complementary pair of pants the full relative obstruction group for extending boundary translation sections is $H^2(P, \partial P; \Lambda) = \Lambda_\Gamma$. Now $T = 0$, so it is exactly $\mathbb{Z}$ detected by ℓ, not just a rational or free quotient. Prescribing the identity translation at the cusp therefore allows all the vertical finite changes to extend over P. The same argument removes the zero-ℓ component of any cusp translation clutching. For determinant -1, write $r_1 = 1 + 5a$, $r_2 = -1 + 4b$; then (F.T12) says $a + b + \kappa = 0$. The two integer changes of finite boundary marking and the remaining cusp change thus also have zero total relative obstruction and extend. This gives the normalized twists $(u_1, -u_2)$. For determinant 1 the same argument gives their negatives. The orientation-preserving fibre inversion extending over the Kuhn fan, proved in 11.18, identifies the two choices. All maps used in the comparison are cap translations or translations by smooth torus-bundle sections

over P; the prescribed cusp is retained throughout. This proves the claimed complete twist normalization.

□

Theorem 11.23 (A finite integral test for three-connectivity)**.** [tag 02NJ]

Take any rational pair in 11.14 and any of its integral, primitive-cusp rank-six extensions given by 11.16. Write M_1, M_2 for geometric transport on fibre homology $\Lambda = \mathbb{Z}^6$, in the peripheral convention $M_0M_1M_2 = I$. First require the preliminary conditions

$$\Lambda_\Gamma = \mathbb{Z}, \qquad \ell(\Lambda^{M_i}) = \mathbb{Z}, \qquad h_1 = 5, \quad h_2 = 4. \tag{F.HC1}$$

Here ℓ is the primitive global invariant covector, and h_i is the integral fixed-vector/fixed-covector pairing on $\ker \ell$ from 11.19. Failure of any condition excludes the datum. If they hold, use the unique normalized affine filling of 11.22, with twists $(u_1, -u_2)$ and zero cusp clutching. The following six finite integer-matrix tests are necessary and sufficient for this filling to have $H_2 = H_3 = 0$.

Let

$$T_i = M_i^{-t}, \quad T_0 = (T_1T_2)^{-1}, \qquad G_q = \operatorname{coker}[\textstyle\bigwedge^q T_1 - I \mid \bigwedge^q T_2 - I\,] \quad (1 \le q \le 4). \tag{F.HC2}$$

All G_q are finite. Let P_q be the integral parabolic cohomology group of the exterior local system $\bigwedge^q \mathcal{V}$, for $q = 2, 4$. Let $t_q \in G_{q-1}$, $2 \le q \le 5$, be the natural global coinvariant image of the unique nonzero torsion suspension class at the order-four cap, as specified below. Let $c_4 \in G_3$ be the natural image of the unique nonzero torsion class in $\operatorname{coker}(\bigwedge^3 T_0 - I)$. Then the six tests are

	first test	*second test*	*third test*
$H^3 = 0$	$G_1 = \langle t_2 \rangle$	$P_2 = 0$	$t_3 \neq 0$
$H^5 = 0$	$G_3 = \langle t_4, c_4 \rangle$	$P_4 = 0$	$t_5 \neq 0$.

(F.HC3)

Every group, class, and equality here is computed by finitely many integer kernels, Smith forms, and exterior minors. This is a decision criterion for every rational parameter and every integral lattice in the normal form.

Proof. ⟨1⟩1. We first establish the uniform local calculations needed for the criterion, so the test does not assume that a general finite matrix is globally conjugate to the example in 5.5. The fixed vectors u_i split the homology lattice as $\mathbb{Z}u_i \oplus H_i$, where $H_i = \ker \ell$. Write $D_i = M_i|_{H_i}$. The maximal pairing $h_i = m_i$ gives

$$\operatorname{Norm}_{m_i}(D_i) = \lambda_i \varphi_i, \tag{F.HC4}$$

with both factors primitive.

⟨1⟩2. At five, work over $\mathbb{Z}_5$. Modulo five the norm is $(D_1 - I)^4$ and is nonzero. Choose q with $\operatorname{Norm}_5 q$ primitive. The five vectors $q, D_1q, \dots, D_1^4q$ are independent modulo five: otherwise the nilpotent Jordan block generated by q would have length less than five. They are therefore a $\mathbb{Z}_5$ basis. Thus D_1 is the regular five-cycle permutation locally.

⟨1⟩3. At four, work over $\mathbb{Z}_2$. The norm is $(D_2 - I)^3$ modulo two and is nonzero. The vectors q, D_2q, D_2^2q, D_2^3q similarly generate a primitive regular four-cycle submodule Q of rank four. The quotient is rank one with $D_2 = -1$, by the characteristic polynomial. For a lift z of its generator, put $a = (D_2 + I)z \in Q$. Since $D_2^4 = I$,

$$(D_2^3 - D_2^2 + D_2 - I)a = 0.$$

On the regular four-cycle module the kernel of the displayed alternating norm is exactly $(D_2 + I)Q$, integrally. One can see this by writing its four coefficients; equivalently $D_2 + I$ has three nonzero Smith factors, all one, and the indicated rational image. Subtracting a suitable element of Q from z makes $D_2 z = -z$. Consequently

$$M_1 \sim 1 \oplus \mathrm{perm}_5 \quad \text{over } \mathbb{Z}_5, \qquad M_2 \sim 1 \oplus \mathrm{perm}_4 \oplus(-1) \quad \text{over } \mathbb{Z}_2. \tag{F.HC5}$$

These local comparisons apply to cap cohomology, not just to abstract representation lattices. Indeed the integral lattice of intertwiners is the kernel of an integer matrix and its tensor with $\mathbb{Z}_p$ is the lattice of local intertwiners. Approximate a local invertible intertwiner by an integral one whose determinant is a p-adic unit. Keeping the u_i summand fixed produces an equivariant torus isogeny of degree prime to p, preserving the normalized translation $\pm u_i/m_i$. The resulting cap and boundary coverings induce isomorphisms on $\mathbb{Z}_p$ cohomology: the vertical torus maps are cohomology isomorphisms over $\mathbb{Z}_p$, so the mapping-torus Wang sequences give the assertion. At primes not dividing m_i, transfer already makes finite-cover pullback an isomorphism onto invariants.

⟨1⟩4. Here are the full required integral results for these two monomial models. The ranks of their invariant exterior lattices in degrees zero through six are $(1, 2, 3, 4, 3, 2, 1)$. The order-five cap has no cohomology torsion in degrees two through five. The order-four cap has exactly $\mathbb{Z}/2$ in each of those four degrees. This follows from the Wang sequence for the mapping torus of the vertical D_i: its torsion in degree q is

$$\mathrm{Tor}\,\mathrm{coker}(\textstyle\bigwedge^{q-1} D_i^t - I). \tag{F.HC6}$$

In particular it introduces no additional torsion primes.

⟨1⟩5. Integral pullback from either finite cap is onto the invariant lattice in degrees two through five. For degree two this is precisely the $m_i/h_i = 1$ suspension calculation of 11.19. For completeness, the determinants of the pairings between the invariant lattices in complementary degrees $(1, 5), (2, 4), (3, 3)$ are respectively m_i, m_i^3, m_i^4 in absolute value. The degree-one cover image has index m_i. The free downstairs complementary pairings are unimodular, and pullback multiplies each pairing by the covering degree m_i. The middle pairing first gives degree-three index one. The degree-two index one then gives degree-four index one; the degree-one index m_i gives degree-five index one. These determinant identities and all the Wang Smith forms are verified by the fixed, exhaustive computation in `fourfolds_local_caps.py`, for both orders and all exterior degrees zero through six. Thus no unexamined finite matrix is being inferred from an example.

⟨1⟩6. We next identify the boundary maps. Choose any unimodular matrix $P_2 = [u_2, H_2]$ realizing the splitting at four, and put $J = P_2^{-t}|_{\mathbb{Z}^5}$ for the inclusion of vertical cohomology. Choose the unique nonzero torsion representative

$$w_q \in \mathrm{coker}(\textstyle\bigwedge^{q-1} D_2^t - I), \qquad t_q = [(\textstyle\bigwedge^{q-1} J)w_q] \in G_{q-1}. \tag{F.HC7}$$

A Smith transformation supplies an integral w_q. Its norm is exactly zero, since it lies rationally in the image of $\bigwedge^{q-1} D_2^t - I$. The finite boundary coordinates of 5.5 satisfy $\theta = a + 4k$, $h = k$, hence $a = \theta - 4h$. The pullback of the suspension cocycle has θ component w_q and h component $-\mathrm{Norm}_4(w_q) = 0$. This holds in every exterior degree, including $q = 5$. It is an integral cellular assertion for the monomial model, where the torus cellular differentials vanish; the prime-to-two isogeny comparison transports it to every eligible cap. The base connecting map therefore gives exactly the class t_q in (F.HC7). Different representative choices change it by a global monodromy relation. Any sign change is immaterial for an order-two class.

⟨1⟩7. There is one further boundary map, from the free degree-four cusp kernel. The primitive cusp is integrally conjugate to $\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$. Its degree-three exterior difference has ten Smith factors one and one Smith factor two; its cokernel is $\mathbb{Z}^9 \oplus \mathbb{Z}/2$. The other exterior differences in degrees one, two, four and five have torsion-free cokernels. The exact matrices and a torsion representative are saved by `fourfolds_cusp_boundary.py`. Let C be the smooth cusp cap and $Y = \partial C$. The boundary Wang sequence gives $\operatorname{Tor} H^4(Y) = \mathbb{Z}/2$. In the Poincaré–Lefschetz exact sequence its image in

$$H^5(C, Y) \cong H_3(C) = \mathbb{Z}^9$$

is zero, so this torsion class lifts to $H^4(C)$. Its restriction to an ordinary fibre is zero. By V.2.2 the kernel of integral specialization is a primitive copy $K = \mathbb{Z}$ in degree four. The lift is in K, and the image of this cyclic group in $H^4(Y)$ contains the nonzero torsion class. A cyclic image containing nonzero torsion cannot also have a nonzero free part. Its primitive generator therefore restricts to the unique nonzero torsion class of $H^4(Y)$. Consequently the cusp contribution to d_2 is exactly c_4 as defined above. In particular this column must not be discarded merely because it vanishes in the first examples.

⟨1⟩8. It remains to prove that these are all the tests. For the irreducible odd-dimensional quotient W of $\mathcal{V}$, invariant alternating forms and bivectors vanish: a nonzero one would give an isomorphism $W \cong W^*$ with an odd-dimensional nondegenerate alternating pairing. The exact exterior sequences for $0 \to \mathbb{Q} \to \mathcal{V}_{\mathbb{Q}} \to W \to 0$, together with $\det W = 1$, give

$$H^0(B, j_* \textstyle\bigwedge^q \mathcal{V}) = 0 \quad (2 \le q \le 5), \qquad G_q \otimes \mathbb{Q} = 0 \quad (1 \le q \le 4). \tag{F.HC8}$$

For the endpoint assertions, the extension cannot split: on $\mathbb{Q} \oplus W$ a square-zero nilpotent has rank at most two, whereas the stipulated rank is three. An invariant covector of $\mathcal{V}_{\mathbb{Q}}$ would split this extension, since W has no trivial quotient. Thus $(\mathcal{V}_{\mathbb{Q}}^*)^{\Gamma} = 0$, which gives both $H^0(\bigwedge^5 \mathcal{V}_{\mathbb{Q}}) = 0$ and $G_1 \otimes \mathbb{Q} = 0$. Here and below the first assertion is integral as well, since the coefficient lattices are torsion free. The local invariant ranks in exterior degrees two and four are $3, 3, 9$ at the three punctures. The Euler formula for the rank-fifteen local system therefore gives $P_2 \otimes \mathbb{Q} = P_4 \otimes \mathbb{Q} = 0$. In degree one it similarly gives rational parabolic cohomology zero. The only global degree-one section is the invariant line. Simple connectivity and the universal coefficient theorem make its free transgression onto $H^2(B, \mathbb{Z})$; equivalently the rational second Betti number is zero, as in 5.5. Thus the normalized filling has rational $b_2 = b_3 = 0$.

⟨1⟩9. All specializations in exterior degrees two through five are onto, by the local calculations and V.2.2. Their only point-supported kernels are

q	2	3	4	5
kernel	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2.$

(F.HC9)

The free summand in degree four is at the cusp; the others are at four. Consequently $H^1(B, R^q g_* \mathbb{Z}) = P_q$ in the degrees used here, while $H^2(B, R^q g_* \mathbb{Z}) = G_q$. The latter equality also holds in degree one: changing a sheaf at isolated points changes no degree-two cohomology. The equality with coinvariants follows from compactly supported top cohomology on the punctured sphere and the sequence $j_! \to j_*$.

⟨1⟩10. The base has real dimension two. Thus the only possible nonzero differential in the integral Leray spectral sequence is $d_2 : E_2^{0,q} \to E_2^{2,q-1}$. The complete associated graded groups in total degrees three and five are therefore

$$\begin{aligned} &\operatorname{gr} H^3 : G_1/\langle t_2 \rangle, \quad P_2, \quad \ker(\mathbb{Z}/2 \xrightarrow{t_3} G_2), \\ &\operatorname{gr} H^5 : G_3/\langle t_4, c_4 \rangle, \quad P_4, \quad \ker(\mathbb{Z}/2 \xrightarrow{t_5} G_4). \end{aligned} \tag{F.HC10}$$

These are all finite. A finite filtered group vanishes exactly when every associated graded group vanishes, regardless of extension classes. Since $\pi_1 = 1$ and rational $b_2 = b_3 = 0$, the universal coefficient theorem gives $H^3 = \mathrm{Ext}(H_2, \mathbb{Z})$, so $H_2 = 0$ exactly when $H^3 = 0$. Integral Poincaré duality gives $H^5 \cong H_3$. This proves (F.HC3).

⟨1⟩11. For reproducibility, no sheaf calculation is left as a black box in the integer test. Given $A = \bigwedge^q T_1$, $B = \bigwedge^q T_2$ on a free lattice L, form the saturated integer kernel

$$K_q = \ker[\, A - I \mid A(B - I) \mid -(AB - I)\,] \subset L^3. \tag{F.HC11}$$

Its image under $(x, y, z) \mapsto ((A - I)x, (B - I)y)$ is precisely the integral parabolic cocycle lattice. Quotient by the columns of $\left(\begin{smallmatrix} A-I \\ B-I \end{smallmatrix}\right)$ to obtain P_q. This uses the integral images of peripheral differences, not their saturations. Smith transformations also supply every w_q, the cusp order-two representative, and the tests for their images and generated subgroups in G_q.

⟨1⟩12. The program `fourfolds_survivor_test.py` implements exactly these operations. Its executed test set is precisely $v = 0, 1, 2$ from 11.10. For $v = 0, 1$ all six tests pass; $G_1, \ldots, G_4$ are each $\mathbb{Z}/2$, $P_2 = P_4 = 0$, all four finite-cap images are nonzero, and the cusp image is zero. For $v = 2$ they are each $\mathbb{Z}/6$, $P_2 = P_4 = \mathbb{Z}/3$, and the first, second, fourth and fifth tests fail. The computation thus checks both accepted and rejected data.

⟨1⟩13. We finish by making the convention dictionary explicit. The imported family matrices called A, B in that program are geometric cohomology pullbacks, so the geometric homology inputs are (A^t, B^t). Their genuine left cohomology transport in (F.HC2) is (A^{-1}, B^{-1}), which is what the code uses. For comparison with computations written using (A, B), simultaneous reversal of the two peripheral generators sends

$$g_1 \mapsto g_1^{-1}, \qquad g_2 \mapsto g_2^{-1}, \qquad g_0 = (g_1 g_2)^{-1} \mapsto g_2 g_1 = g_1^{-1}(g_1 g_2) g_1. \tag{F.HC12}$$

This is an orientation-reversing marking of the pair of pants fixing its three ends: each peripheral conjugacy class is inverted. Thus the new cusp is conjugate to the inverse old cusp, not an independently chosen third matrix. Inversion preserves invariant and coinvariant lattices, and the indicated conjugating generator acts trivially on global coinvariants. All boundary classes in (F.HC3) have order two, so reversal signs do not change the tests. The implementation uses (F.HC2) directly and hence does not depend on this comparison for its validity.

□

Corollary 11.24 (Classification of the three-connected data)**.** [tag `02NL`]

For the marked Kuhn cap and affine translation clutchings used throughout this section, the following is a complete constructive classification of the data with $\pi_1 = 1$ and $H_2 = H_3 = 0$, up to integral conjugacy of their linear tuples and oriented diffeomorphism of their fillings.

There are no data with three genuine finite fillings. With two fillings their affine orders are five and four. Order the order-five point first. The complete, unbounded parameter set is the rational projective normal form of 11.14 together with its exceptional rational affine line. For each parameter, run the terminating order, conductor, stable-lattice, and primitive-extension algorithm of 11.16. Discard precisely the integral classes failing (F.HC1) *or one of* (F.HC3)*. The classes that pass these tests are exactly all integral linear tuples with three-connected gluing. For each such class all simply connected affine twists are exactly the choices satisfying* (F.T12)*; their fillings are all oriented diffeomorphic to its normalized filling. If ordered labels with the order-four point first are desired, interchange the finite labels and recompute $M_0 = (M_1 M_2)^{-1}$.*

Every such datum has

$$\pi_1 = 1, \qquad H_j = \begin{cases} \mathbb{Z}, & j = 0, 8, \\ \mathbb{Z}^4, & j = 4, \\ 0, & \textit{otherwise}, \end{cases} \tag{F.HC13}$$

and its oriented smooth manifold is $3\mathbb{HP}^2\#(-\mathbb{HP}^2)$*. It has bounded-cusp holomorphic periods in the sense of 12.5.*

The fundamental group and the integral homology of every other gluing are computed by theorems 11.25 and 11.29.

Proof. ⟨1⟩1. The exclusions 11.1, 11.8, 11.19, and 11.21 leave exactly the stated minimal orders and rational quotient problem. The rational normal form is injective on conjugacy classes, including the exceptional line. The finite interval and extension-coset enumeration in 11.16 gives every integral class over each rational point, with all bounds proved for that input. No distinct rational points can become integrally conjugate, since an integral conjugacy induces a rational quotient conjugacy. Theorem 11.22 and the necessary-and-sufficient criterion 11.23 prove both directions of the retention rule and account for every affine twist. This is also an exhaustive enumeration if desired: enumerate primitive projective integer triples with first nonzero coordinate positive by increasing height, and enumerate reduced rational exceptional parameters by increasing numerator and denominator height. Every finite stage has finitely many inputs, each terminates by 11.16, and every rational input occurs. No finite cutoff is imposed or needed.

⟨1⟩2. Hurewicz makes these simply connected manifolds three-connected. The only singular contribution to Euler number is the Kuhn cusp, whose value is six by V.2.2; torus bundles and the free finite caps have Euler number zero. Integral Poincaré duality and the universal coefficient theorem then give (F.HC13), including torsion freeness.

⟨1⟩3. For the smooth classification, 11.9 supplies a compatible complex structure at each finite cap, and the cusp already has its complex model. Choose their common fibre orientation to be the one induced by the cusp. Commuting complex structures at the finite caps can be chosen in that component: their two-dimensional fixed real summands allow the orientation to be reversed when necessary. The fibre of the bundle of orientation-compatible complex structures retracts to $\mathbb{CP}^3$. Its connectedness and simple connectivity extend the prescribed boundary structures across the two-dimensional pair of pants, exactly as in 5.1. A compatible horizontal distribution gives an almost complex structure on the total manifold. Theorem §3 therefore gives signature two, intersection lattice $I_{3,1}$, and $c_2^2 = 2$. Since $H^2 = 0$, $p_1/2 = -c_2$ is primitive, characteristic, and has square two. The complete integral reflection, Wall-form and Kramer–Stolz argument in 7.2 depends only on these almost complex characteristic numbers and three-connectivity. It applies without alteration and gives the claimed exact oriented diffeomorphism. Finally 12.5 applies to the resulting determinant-one monodromy and primitive cusp. It gives the punctured-base periods with bounded cusp correction.

□

Theorem 11.25 (complete fundamental-group data for every input)**.** [tag `02NM`]

Let $k = 2$ *or* 3*, and let an integral rank-six tuple satisfy the finite-order and primitive maximal-cusp conditions of definition 4.1. No irreducibility, determinant-one, or three-connectivity hypothesis is imposed. Specify each finite affine action as*

$$\tau_i(x) = M_i x + v_i/m_i, \qquad v_i \in \Lambda = \mathbb{Z}^6, \quad \tau_i^{m_i} = 1 \textit{ on } \Lambda_{\mathbb{R}}/\Lambda, \tag{F.G1}$$

and require its cyclic action to be free. The numerator need not be fixed by M_i. Define its integral power vector and the cusp lattice by

$$w_i = \frac{1}{m_i} \sum_{j=0}^{m_i-1} M_i^j v_i \in \Lambda^{M_i}, \qquad U = \operatorname{im}\big((M_1 \cdots M_k)^{-1} - I\big). \tag{F.G2}$$

Use left conjugation transport for the peripheral generators s_i, and record the integral translation clutching by $\kappa \in \Lambda$ in $s_1 \cdots s_k x^\kappa = 1$. Then the full fundamental group is

$$\left\langle x^\lambda, s_1, \dots, s_k \;\middle|\; \begin{array}{l} x^\lambda x^\mu = x^{\lambda+\mu}, \quad x^u = 1 \ (u \in U), \\ s_i x^\lambda s_i^{-1} = x^{M_i\lambda}, \quad s_i^{m_i} = x^{w_i}, \quad s_1 \cdots s_k x^\kappa = 1 \end{array} \right\rangle. \tag{F.G3}$$

Only a basis of Λ and of U is needed in this finite presentation. The program below also determines exactly which quotient of the apparent fibre subgroup injects.

Put

$$L = \sum_{g \in \langle M_1, \dots, M_k \rangle} gU, \qquad A = \Lambda / L. \tag{F.G4}$$

This is an integral sum, without saturation. For two points, put $d = \gcd(m_1, m_2)$, $M = M_1$, and $S_j(M) = I + M + \cdots + M^{j-1}$. Choose integers a, b with $am_1 + bm_2 = d$, and set, in A,

$$\begin{aligned} w_2' &= -w_2 - S_{m_2}(M)\kappa, \\ e &= (m_2/d)w_1 - (m_1/d)w_2', \qquad z = aw_1 + bw_2', \qquad \overline{A} = A/\langle e \rangle. \end{aligned} \tag{F.G5}$$

There is an exact sequence and a complete marked extension description

$$1 \longrightarrow \overline{A} \longrightarrow \pi_1 \longrightarrow C_d \longrightarrow 1, \qquad txt^{-1} = M(x), \quad t^d = z. \tag{F.G6}$$

Thus its order is $d|\overline{A}|$ if $\overline{A}$ is finite; otherwise it is infinite. In particular $d = 1$ gives precisely the abelian group $\overline{A}$.

For three genuine points, so $m_i \geq 2$, let

$$\Gamma = \langle q_1, q_2, q_3 \mid q_i^{m_i} = 1, \ q_1 q_2 q_3 = 1 \rangle, \qquad \epsilon = \sum_i 1/m_i - 1. \tag{F.G7}$$

The action $q_i \mapsto M_i$ is well defined on A. If $\epsilon \leq 0$, the exact sequence is

$$1 \longrightarrow A \longrightarrow \pi_1 \longrightarrow \Gamma \longrightarrow 1. \tag{F.G8}$$

If $\epsilon > 0$, Γ is the finite spherical triangle group. Using its action on A, form the exact transfer vector

$$e = \sum_{g \in \Gamma} g\kappa + \sum_{i=1}^{3} \sum_{g \in \Gamma/\langle q_i \rangle} gw_i. \tag{F.G9}$$

The second sums use left coset representatives and are independent of their choice. In this case the exact sequence has kernel $A/\langle e \rangle$, with the same lifted relations (F.G3). Consequently (F.G3), (F.G6), (F.G8), and (F.G9) give every such fundamental group by finite operations, including the precise injected abelian kernel and all extension data. These statements do not claim a conjugacy classification of the original tuples.

For every input, the full first integral homology is the Smith cokernel of the integer matrix

$$\begin{pmatrix} M_1 - I & \cdots & M_k - I & -w_1 & \cdots & -w_k & \kappa \\ 0 & \cdots & 0 & \multicolumn{3}{c}{\operatorname{diag}(m_1, \dots, m_k)} & \mathbf{1} \end{pmatrix}. \tag{F.G10}$$

Here the bottom left zero block has k rows. The code records the full matrix, not only its invariant factors.

Proof. ⟨1⟩1. *Actual finite and cusp caps.* The affine lift on $\mathbb{R}^6$ has m_ith power translation by w_i, as the displayed sum shows. It commutes with that power, so $M_i w_i = w_i$. The group generated by lattice translations and this lift is the fundamental group of the reduced flat six-manifold. Its presentation consists of the lattice, its s_i conjugation relations, and $s_i^{m_i} = w_i$. The finite cap deformation retracts onto that reduced manifold by the equivariant radial contraction of its root disc. Thus these are the cap relations, even for a numerator which is not an invariant lattice vector.

⟨1⟩2. For the primitive Kuhn cap, the compact toric strata in its normalization and all their intersection components are simply connected. Applying van Kampen successively to their tubular neighborhoods identifies the fundamental group of the central fibre with that of its dual three-torus. The nearby fibre map kills the angular lattice U and sends the long-period lattice Λ/U isomorphically onto this dual-torus group. This is also the explicit toric collapse used in V.2.2. The primitive index-one condition gives $U = \ker(M_0 - I)$ and makes Λ/U free of rank three. A section through a smooth central stratum bounds the base meridian by its parameter disc. The smooth cap retracts onto the central fibre, so it adds exactly these relations, with no extra meridian relation left unspecified.

⟨1⟩3. Over the punctured sphere the bundle has a section up to homotopy, because the base retracts to a graph. Its group is the appropriate semidirect product. Choose the cap sections and retain their relative translation clutching as κ. Van Kampen with the boundary maps just described gives exactly (F.G3). Changing the transport convention causes no group ambiguity: reverse the ordered list and replace each affine map by its inverse, and replace κ by $-\kappa$. The new linear matrices are $M_k^{-1}, \dots, M_1^{-1}$, the power vectors are $-w_k, \dots, -w_1$, and the cusp image lattice is unchanged. The substitutions $s_i \mapsto s_{k+1-i}^{-1}$ give the explicit isomorphism between these two presentations. This supplies the dictionary when geometric transport is recorded with opposite path-composition order.

⟨1⟩4. *Integral closure and two-point elimination.* In a normal abelian fibre subgroup, the normal closure of U is precisely the integer sum (F.G4). Start with its column Hermite basis and repeatedly append its images under every M_i and M_i^{-1}, then take Hermite normal form again. If the rational rank increases, it can do so at most three times. At the first nonincrease the rational span is invariant under every generator and its inverse, hence never increases again. If the current lattice has saturation index D in that span, every subsequent strict enlargement divides this index by an integer at least two. There can therefore be at most $\lfloor \log_2 D \rfloor$ such enlargements. This proves termination and gives the computable bound used by the code; it never replaces a finite-index lattice by its saturation.

⟨1⟩5. On A one has $M_2 = M^{-1}$ in the two-point case. The product relation eliminates $s_2 = t^{-1}x^{-\kappa}$, where $t = s_1$. Multiplying this word m_2 times gives

$$(t^{-1}x^{-\kappa})^{m_2} = t^{-m_2}x^{-S_{m_2}(M)\kappa}.$$

Thus the two power relations become $t^{m_1} = w_1$ and $t^{m_2} = w_2'$. Both right sides are fixed by M on A. Also $M^{m_1} = M^{m_2} = 1$, hence $M^d = 1$ there. Bezout elimination gives exactly $e = 0$ and $t^d = z$ in (F.G5). Replacing the Bezout pair changes z by a multiple of e and therefore changes no extension datum.

⟨1⟩6. To prove the asserted injection, construct $\overline{A} \rtimes_M \mathbb{Z}$ and quotient by its central element $t^d z^{-1}$. This cyclic subgroup has nonzero projection $d\mathbb{Z}$ to the infinite cyclic factor, so it intersects $\overline{A}$ trivially. The quotient has precisely the presentation obtained by elimination. This proves (F.G6), including the absence of any further hidden fibre relation.

⟨1⟩7. *The three-point kernel and the finite transfer.* For completeness this step computes the fibre injection; it does not infer it from the appearance of a normal subgroup in a presentation.

Let $\mathcal{B}$ be the orbifold classifying space obtained from the three-holed sphere by attaching, at its three boundary circles, the mapping cylinders of $S^1 \to BC_{m_i}$. Over the punctured sphere take the bundle with fibre $BA = K(A,1)$ and the action. For each boundary, the cyclic extension

$$E_i = \langle A, s_i \mid s_i a s_i^{-1} = M_i(a),\ s_i^{m_i} = w_i \rangle$$

contains A injectively: the same semidirect-product construction as above applies because $M_i^{m_i} = 1$ and $M_i w_i = w_i$. The associated fibration $BA \to BE_i \to BC_{m_i}$ restricts on S^1 to the required mapping-torus fibration. Replace the maps by their mapping-path fibrations before homotopy gluing, so that the fibre is preserved on every attaching cylinder. Glue these three fibrations, using the relative clutching κ on the punctured sphere. Van Kampen gives (F.G3) for the resulting total space. Thus the homotopy sequence of this fibration over $\mathcal{B}$ calculates precisely the image of A in the group under discussion.

⟨1⟩8. When $\epsilon \le 0$, the universal orbifold cover is the plane or hyperbolic plane. Its contractibility identifies $\mathcal{B}$ with $B\Gamma$: indeed its Borel construction has the just-described boundary-cylinder model and contractible universal fibre. The Euclidean and hyperbolic alternatives are the oriented two-orbifold classification in (Scott, 1983, Theorems 2.3–2.4, p. 425; Euler-characteristic comparison, pp. 427–428). Since $\pi_2(B\Gamma) = 0$, the homotopy sequence proves (F.G8).

⟨1⟩9. When $\epsilon > 0$, the universal orbifold cover is S^2. Pulling the constructed BA fibration to this finite cover trivializes its local system. Its transgression $\pi_2(S^2) \to A$ is its integral clutching vector. There is one lifted ordinary base cell for each $g \in \Gamma$, contributing $g\kappa$, and one lifted cone cap for each left coset of $\langle q_i \rangle$, contributing gw_i. To check the sign, the base relation has fibre term $+\kappa$ and each cone relation has term $-w_i$; the oriented sphere chain subtracts the cone cells from the base cell. The result is therefore the plus sign in both sums of (F.G9). Changing a coset representative by q_i^j does not change its contribution because w_i is fixed. The result is Γ invariant, as also checked by the code. The long homotopy sequence now says that the only additional kernel is exactly $\langle e \rangle$. This proves the spherical assertion, with the orientation fixed by this cell calculation.

⟨1⟩10. There is no implicit infinite search for a spherical group. After sorting its orders the possibilities are $(2,2,n), (2,3,3), (2,3,4), (2,3,5)$, with groups of orders $2n, 12, 24, 60$, respectively. These follow directly from the positive spherical-triangle angle condition. The code constructs the dihedral, tetrahedral, octahedral, or icosahedral permutation group and finds the marked peripheral generators in that finite group. Breadth-first words supply their actions on A; every generator edge is checked modulo the lattice L. It then computes the two finite sums in (F.G9) and their integer Smith form.

⟨1⟩11. An order-one site is not a genuine cone. Its matrix is the identity and its relation is $s_i = x^{w_i}$. Eliminate it from (F.G3); moving this vector to the right of the remaining peripheral product replaces the clutching by

$$\kappa + (M_{i+1} \cdots M_k)^{-1} w_i. \tag{F.G11}$$

This reduces the three-point case with such a site to the two-point computation. A primitive rank-three cusp cannot remain with only one finite linear monodromy: a finite-order unipotent matrix is the identity. Thus no smaller case has been omitted within the hypotheses of this theorem.

⟨1⟩12. *Freeness and abelianization.* Freeness of (F.G1) is itself decidable exactly. A nontrivial stabilizer in a cyclic group contains a subgroup of prime order, so it is enough to check $\tau_i^{m_i/p}$ for every prime $p \mid m_i$. For an affine map $Rx + b$ of a torus, a fixed point exists

precisely when every integral covector in $\ker(R^t - I)$ takes an integral value on b. Smith form proves this criterion: the image of $R - I$ on the torus is the connected subtorus annihilated by those primitive covectors. The code computes the powers in the rational affine matrix group and performs these finite tests.

⟨1⟩13. Abelianizing (F.G3) yields (F.G10). No cusp column was lost:

$$\operatorname{im}(M_0 - I) \subset \sum_i \operatorname{im}(M_i - I)$$

follows by telescoping the product on the right; inversion preserves the image of its difference from the identity. The Smith form therefore computes every free and torsion summand of H_1 exactly.

⟨1⟩14. The program `fourfolds_vankampen.py` saves all input matrices, power vectors, integral closure traces, group extension data, and homology presentation matrices. It checks the nine explicitly listed three-point parameters of its original test set, four two-point order/clutching choices, a genuinely nonfixed affine numerator, and a finite-index closure step. It also checks a nonabelian rank-six two-point tuple, a spherical $(2, 4, 2)$ example with injected kernel $\mathbb{Z}^2 \oplus \mathbb{Z}/4$, and the infinite perfect hyperbolic example with orders $(5, 4, 19)$. The nonabelian test deliberately belongs to the full GL_6 scope and has orientation- reversing finite matrices; it is not claimed to admit a complex compactification. All bounds above depend on the input.

□

Theorem 11.26 (a complete three-point logarithmic family)**.** [tag `02NN`]

Start with the compact complex fourfold of 6.9 and its normalized order-five and order-four affine fillings. Let ℓ be the primitive invariant covector on fibre homology, with the sign chosen in 11.22. At any additional ordinary fibre choose $v \in \mathbb{Z}^6$ and an integer $m \geq 2$ such that

$$r = \ell(v), \qquad \gcd(r, m) = 1. \tag{F.LT1}$$

Perform the marked logarithmic transformation with translation v/m described below. This gives a smooth compact complex fourfold $X_{m,v}$ with Euler number six and three genuine finite affine fillings, whose linear orders are five, four and one and whose affine orders are five, four and m. Its fundamental group is

$$\pi_1(X_{m,v}) = \langle a, b, c, h \mid h \text{ central},\ a^5 = h,\ b^4 = h^{-1},\ c^m = h^r,\ abc = 1 \rangle. \tag{F.LT2}$$

This is the group of the oriented Seifert manifold with invariants $(0; (5, -1), (4, 1), (m, -r))$, in the convention $x_i^{\alpha_i} h^{\beta_i} = 1$ and product $x_i = 1$. Its base orbifold is hyperbolic, and its central fibre h has infinite order.

Put $C_{m,r} = \mathbb{Z}/(20r - m)\mathbb{Z}$, with $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$, and let $\varepsilon = 1$ when $20r = m$ and zero otherwise. All the integral homology groups are

j	$H_j(X_{m,v}; \mathbb{Z})$
$0, 8$	$\mathbb{Z}$
$1, 6$	$C_{m,r}$
2	$\mathbb{Z}^{\varepsilon} \oplus (\mathbb{Z}/m)^5$
3	$(\mathbb{Z}/m)^{10}$
4	$\mathbb{Z}^4 \oplus (\mathbb{Z}/m)^{10}$
5	$(\mathbb{Z}/m)^5$
7	$\mathbb{Z}^{\varepsilon}$.

(F.LT3)

Under (F.LT1) the exceptional free-rank case is exactly $(m, r) = (20, 1)$. In particular $(m, r) = (19, 1)$ and $(21, 1)$ have $H_1 = 0$ but nontrivial, infinite fundamental groups. More generally this occurs whenever $|20r - m| = 1$. The result holds for every vertical component of v in $\ker \ell$.

Proof. ⟨1⟩1. We first specify the complex construction and its marking. Over a small disc at the ordinary value use a holomorphic period marking $T_t = \mathbb{C}^3/\Lambda_t$. Let $\omega_v(t)$ be the holomorphic period vector corresponding to v. On the root disc $t = s^m$ quotient the pulled-back family by

$$(s, Z) \longmapsto (\zeta_m s,\ Z + \omega_v(s^m)/m). \tag{F.LT4}$$

Condition (F.LT1) makes v have exact order m in $\mathbb{Z}^6/m\mathbb{Z}^6$. Every nontrivial power therefore acts freely, including over $s = 0$. The quotient is smooth, has a proper map to the disc, and has central fibre of multiplicity m. On the punctured disc its holomorphic identification with the original family is

$$(s, Z) \longmapsto \left(s^m,\ Z - \frac{\log s}{2\pi i}\omega_v(s^m)\right). \tag{F.LT5}$$

Changing the branch of $\log s$ changes the second coordinate by a lattice period. Changing s by $\zeta_m s$ changes its logarithm by $2\pi i/m$ and cancels the translation in (F.LT4). Thus (F.LT5) is a well-defined holomorphic isomorphism of the quotient with the original torus family off zero. Gluing on a fixed annulus supplies the asserted compact complex manifold: its charts are the original charts and the free finite quotient charts (F.LT4), and it is the union of compact pieces over closed base regions. The gluing is an isomorphism on collars, so the total space is Hausdorff and smooth. The Euler number is unchanged, since the removed ordinary torus cap, the new cap, and their boundary all have Euler number zero. This marking fixes the new relation $c^m = v$ and zero extra cusp clutching.

⟨1⟩2. In the original datum the invariant normal closure of the cusp angular lattice is exactly $\ker \ell$, as proved in 5.1. Adding the identity linear generator does not change that lattice. The fibre image in van Kampen therefore reduces to the central generator h, while the original two finite relations remain $a^5 = h$, $b^4 = h^{-1}$. The new relation is $c^m = h^{\ell(v)}$ and the base relation is $abc = 1$. This proves (F.LT2), including the absence of any dependence on the vertical part of v.

⟨1⟩3. One may verify the Seifert identification directly: take a product circle bundle on a three-holed sphere and attach its three solid tori with primitive meridian slopes $(5, -1), (4, 1), (m, -r)$. They are primitive by (F.LT1), and van Kampen gives exactly (F.LT2). Its orbifold base has $1/5 + 1/4 + 1/m < 1$ for every $m \geq 2$. Pulling back the circle fibration to its contractible hyperbolic orbifold cover gives a circle bundle on the plane, hence the trivial circle bundle. The subsequent universal cover is the plane times $\mathbb{R}$, and the fibre subgroup is infinite cyclic. This also proves directly that none of these groups is trivial. The rational Euler number in these conventions is $r/m - 1/20$.

⟨1⟩4. In abelianization the relations $5a = h$, $4b = -h$ give $a = 4x$, $b = -5x$, $h = 20x$ because five and four are coprime. The product relation gives $c = x$. The remaining relation is $(m - 20r)x = 0$. Thus $H_1 = C_{m,r}$ exactly, including its integral generator and the exceptional infinite-cyclic case.

⟨1⟩5. We now compute the higher integral groups, including all extensions. Write $\mathcal{R}^q$ for the higher direct-image sheaf of the original fourfold and $\mathcal{R}^q_m$ for that of $X_{m,v}$, after identifying the bases and their ordinary local systems. At the new point the reduced finite quotient is a torus with lattice $\mathbb{Z}^6 + \mathbb{Z}(v/m)$. Its covering by the original torus is cyclic of degree m. The degree-one pullback has Smith factors $1, 1, 1, 1, 1, m$, so its degree-q cokernel is

$$C^q = (\mathbb{Z}/m)^{\binom{5}{q-1}}, \qquad 1 \leq q \leq 6. \tag{F.LT6}$$

There is no cohomology torsion in this torus cap. The sheaf comparison, which does not assert a map between the total manifolds, is therefore

$$0 \longrightarrow \mathcal{R}^q_m \longrightarrow \mathcal{R}^q \longrightarrow C^q_{\{p\}} \longrightarrow 0. \tag{F.LT7}$$

It is the identity away from the new point and the covering image at that point.

⟨1⟩6. For $2 \le q \le 5$, all original global sections of $\mathcal{R}^q$ are supported at the original finite point or the cusp, by (F.HC8)–(F.HC9). Their map to $C^q_{\{p\}}$ is zero. Consequently their H^0 groups are unchanged, their H^2 groups are unchanged, and

$$0 \longrightarrow C^q \longrightarrow H^1(\mathcal{R}^q_m) \longrightarrow H^1(\mathcal{R}^q) \longrightarrow 0. \tag{F.LT8}$$

The original degree-one base cohomology of these sheaves is zero for $q = 2, 4, 5$ and is $\mathbb{Z}^3$ for $q = 3$. The assertion for $q = 5$ also follows immediately from the original $H^6 = 0$: its bidegree $(1, 5)$ has no incoming or outgoing differential. Thus (F.LT8) gives C^q in degrees two, four and five, and $\mathbb{Z}^3 \oplus C^3$ in degree three.

⟨1⟩7. The original four order-two torsion transgressions remain isomorphisms. Indeed their boundary suspension representatives and their images in global coinvariants are the ones in 11.23; adding the identity monodromy changes none of those coinvariant quotients. These point-supported boundary maps do not depend on the free degree-one transgression, which changes under the new logarithmic gluing. The free degree-four cusp column likewise remains zero, since its boundary torsion representative has zero image in that same unchanged global quotient. It follows that the full filtrations in degrees three and five have just one nonzero term:

$$H^3 = C^2 = (\mathbb{Z}/m)^5, \qquad H^5 = C^4 = (\mathbb{Z}/m)^{10}. \tag{F.LT9}$$

In degree four there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \oplus (\mathbb{Z}/m)^{10} \longrightarrow H^4 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

It splits as an abelian-group extension because its quotient is free, giving the stated middle group. Thus no torsion extension has been inferred from rational ranks.

⟨1⟩8. For the remaining free ranks, $H^1(X_{m,v}) = \mathrm{Hom}(C_{m,r}, \mathbb{Z})$ has rank ε. The degree-one global section group has rank one, and the only rational differential involving it is $d_2 : H^0(\mathcal{R}^1_m) \to H^2(\mathcal{R}^0_m)$. Its kernel has rank ε, hence its cokernel also has rank ε. The degree-one parabolic group is rationally zero, and there are no rational degree-two global sections. Therefore $b_2 = \varepsilon$. For an integral check of this degree-one row, its finite sheaf cokernel is $\mathbb{Z}/\gcd(20, m)$: the original invariant section restricts as 20ℓ, and its value at the new covering cokernel is $20r$ modulo m. This finite group changes no rational rank.

⟨1⟩9. Poincaré duality gives $H_3 = H^5$, $H_4 = H^4$, and $H_5 = H^3$. The universal coefficient theorem now identifies the torsion of H_2 with H^3 and its free rank with ε. It also gives $H^2 = C_{m,r}$: when $20r \neq m$ this is $\mathrm{Ext}(H_1, \mathbb{Z})$; when $20r = m$ it is the free rank-one group. Duality gives $H_6 = C_{m,r}$ and $H_7 = \mathbb{Z}^\varepsilon$. This proves the complete table (F.LT3).

⟨1⟩10. The program `fourfolds_three_point_family.py` checks the original four torsion images and zero cusp image once, then computes the cyclic covering Smith forms in all exterior degrees one through six and the exact van Kampen abelianization for the thirty specified pairs $m \in \{2, 3, 4, 5, 6, 8, 10, 19, 20, 21\}$, $r \in \{-1, 1, m + 1\}$. It includes a nonzero vertical component in every test vector. The unbounded assertion follows from (F.LT6)–(F.LT9), not from extrapolation of those finite tests.

□

Theorem 11.27 (arbitrary finite affine caps)**.** [tag `02NO`]

Let $g(x) = Mx + v/m$ be a free cyclic affine action of affine order $m \geq 2$ on T^r, with $M \in \mathrm{GL}(r, \mathbb{Z})$ and $v \in \mathbb{Z}^r$. For every specified degree bound d, there is an explicit terminating integer algorithm producing chains and boundary maps through degree $d+1$ for the pair

$$B = (D^2 \times T^r)/\langle (s, x) \mapsto (e^{2\pi i/m} s, g(x)) \rangle, \qquad Y = \partial B. \tag{F.CA1}$$

The cap complex has at most 2^r generators in each degree and the boundary complex at most 2^{r+1}. These are complexes of finite rank in each degree; only their indicated finite prefixes are required to compute $H_j(B)$, $H_j(Y)$ and $H_j(B, Y)$ for $j \leq d$. Their maps include the affine translations and all integral higher coherence terms.

Proof. ⟨1⟩1. We give the complete recursive algorithm. All chains below are finite integer dictionaries, so the displayed operations are effective over $\mathbb{Z}$.

⟨1⟩2. First consider the normalized bar resolution of $\mathbb{Z}^r$ over $R = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ and its Koszul resolution $K = R \otimes \bigwedge^* \mathbb{Z}^r$. The latter differential is contraction with $(t_i - 1)_i$. Write $t^a = t_1^{a_1} \cdots t_r^{a_r}$ and put $S_n(t) = (t^n - 1)/(t - 1)$, interpreted as its finite integral Laurent sum for positive, zero or negative n. The following $\mathbb{Z}$-linear contraction of the augmented Koszul complex fixes a completely explicit choice:

$$s_K(t^a e_I) = \sum_{j < \min I} t_{j+1}^{a_{j+1}} \cdots t_r^{a_r} S_{a_j}(t_j) e_j \wedge e_I, \tag{F.CA2}$$

where $\min \varnothing = r + 1$ and indices in I are increasing. The bar contraction is $s_B(t^a[g_1|\cdots|g_q]) = [a|g_1|\cdots|g_q]$; terms with a zero bar entry are zero. Set $p_0 = 1$ and define the R-linear projection and homotopy on coefficient-one bar generators recursively by

$$p_q = s_K p_{q-1} d_B, \qquad h_q = s_B(1 - ip_q - h_{q-1} d_B). \tag{F.CA3}$$

Here $i(e_I)$ is the alternating sum of the bars of the ordered unit vectors in I. The recursion decreases degree. The identities for the two contractions give, by induction, $pi = 1$, $dh + hd = 1 - ip$, $ph = hi = h^2 = 0$. For example $s_K^2 = s_B^2 = 0$, $ps_B = s_K p$, and the alternating unit-vector formula gives $pi = 1$; substituting these into (F.CA3) proves the three side identities in the same induction. Augmenting $R \to \mathbb{Z}$ gives a strong chain contraction of the ordinary torus bar complex to $\bigwedge^* \mathbb{Z}^r$.

⟨1⟩3. Translations must enter before this contraction. Let $b_j = \sum_{a=0}^{j-1} M^a v/m$ and let p_j be its coordinatewise fractional part. The m points p_j are distinct because the action is free. Use the full fundamental groupoid of T^r on these points: an arrow from object a to object b is an integer vector λ, represented by the affine path from p_a to $p_b + \lambda$. Composition adds the integer vectors. The generator acts strictly by shifting objects and, for its sth power, by

$$\lambda \longmapsto M^s \lambda + z_s(b) - z_s(a), \qquad z_s(a) = M^s p_a + b_s - p_{a+s} \in \mathbb{Z}^r. \tag{F.CA4}$$

The subscripts on objects are taken modulo m. The nerve of this groupoid is equivariantly homotopy equivalent to the torus: the simplex with successive arrow labels λ_i maps affinely to the torus with lifted vertices $p_{a_j} + \sum_{i \leq j} \lambda_i$. Forgetting the objects contracts its chains to the ordinary torus bar complex. An explicit homotopy is the negative alternating prism which, at each possible position, inserts the zero-labelled arrow to object zero and replaces all subsequent objects by zero. Combining this with (F.CA3) gives a contraction (i, p, h) of the groupoid chains to $\bigwedge^* \mathbb{Z}^r$ with the same side identities.

⟨1⟩4. Use the normalized cyclic bar construction on this strict groupoid action. Its differential is $d_v + \delta$, where δ is the ordinary group-bar boundary, multiplied by $(-1)^q$ on vertical degree

q; its first face applies (F.CA4). The vertical contraction transfers this to cyclic bar chains with exterior coefficients by the finite formulas

$$I = (1 + h\delta)^{-1} i, \qquad P = p(1 + \delta h)^{-1}, \qquad D = p\delta I. \tag{F.CA5}$$

Inverses mean their alternating geometric series. On a chain of cyclic bar degree p each series has at most $p + 1$ terms, since δ lowers that degree. Multiplication and the contraction identities directly give $(d_v + \delta)I = ID$ and $P(d_v + \delta) = DP$, and a chain homotopy inverse. Equivalently these are the finite homological perturbation identities, here derived by expanding the finite series. The first term of D is the cyclic bar differential for the exterior action of M. Every remaining term lowers cyclic bar degree by at least two.

⟨1⟩5. For completeness, the second contraction used in the algorithm is also explicit. The cyclic periodic resolution has one $\mathbb{Z}[C_m]$ generator in each degree, with differential $g - 1$ in odd degree and $1 + g + \cdots + g^{m-1}$ in even degree. On g^a, $0 \le a < m$, its augmented contraction is $\sum_{j=0}^{a-1} g^j$ in even degree and is 1 precisely when $a = m - 1$ in odd degree, zero otherwise. Apply (F.CA3) with this contraction to compare it with the cyclic bar resolution. Tensor its coefficients with the exterior action of M. Transferring the higher terms of D once more by (F.CA5) terminates: each higher term followed by the horizontal homotopy decreases cyclic degree by at least one. The resulting basis in total degree n is $\{(p, I) : p + |I| = n,\ I \subseteq \{1, \dots, r\}\}$. This proves the claimed 2^r bound and termination in every specified degree.

⟨1⟩6. **Explicit comparison with the periodic cyclic resolution.** Write $C_m = \langle g \rangle$, $R_m = \mathbb{Z}[C_m]$, and let e_n generate the periodic resolution in degree n. Let B_* be the normalized R_m-bar resolution, with the insertion contraction s_B. Besides the periodic contraction s_P just specified, fix the inclusion recursively by

$$i_0(e_0) = [], \qquad i_n(e_n) = s_B(i_{n-1}(d_P e_n)), \quad n > 0,$$

and extend it R_m-linearly. This is a finite formula. More explicitly, with every summation index between 1 and $m - 1$,

$$i_{2j}(e_{2j}) = \sum_{a_1, \dots, a_j} [g^{a_1} | g | \cdots | g^{a_j} | g],$$
$$i_{2j+1}(e_{2j+1}) = \sum_{a_1, \dots, a_j} [g | g^{a_1} | g | \cdots | g^{a_j} | g].$$

The empty sum of indices gives $i_0(e_0) = []$ and $i_1(e_1) = [g]$. The contraction identity for s_B proves $d_B i = i d_P$ by induction: the chain to which s_B is applied is a cycle, since the preceding comparison commutes with differentials and $d_P^2 = 0$.

⟨1⟩7. Define p and h on coefficient-one bar generators by the recursion already used for the torus comparison, substituting s_P for s_K, and extend them R_m-linearly. Their values can be checked without an unspecified comparison theorem. For $1 \le a_i < m$, put $\epsilon(a, b) = \lfloor (a + b)/m \rfloor$. The projection is

$$p_{2j}[g^{a_1} | \cdots | g^{a_{2j}}] = \prod_{\nu=1}^{j} \epsilon(a_{2\nu-1}, a_{2\nu})\, e_{2j},$$
$$p_{2j+1}[g^{a_1} | \cdots | g^{a_{2j+1}}] = \left(\sum_{b=0}^{a_1 - 1} g^b \right) \prod_{\nu=1}^{j} \epsilon(a_{2\nu}, a_{2\nu+1})\, e_{2j+1}.$$

These expressions follow by induction from $p_n = s_P p_{n-1} d_B$: the terms other than the first bar face are annihilated after the preceding contraction identity is used. In $pi(e_{2j})$ only the summand with every $a_\nu = m - 1$ survives. The same observation, with initial factor 1, proves $pi(e_{2j+1}) = e_{2j+1}$. Thus $pi = 1$ in every degree.

⟨1⟩8. The recursion $h = s_B(1 - ip - hd_B)$ gives $d_B h + hd_B = 1 - ip$ by induction. Moreover $ps_B = s_P p$ follows from the displayed projection formulas, while $s_P^2 = s_B^2 = 0$ and $s_B i = 0$ follow directly from their definitions. Substitution in the same recursion gives $ph = hi = h^2 = 0$. We therefore have the strong comparison required for the second finite perturbation series, including its inclusion, projection, homotopy and side identities. After tensoring with the exterior coefficient module, the same identities hold. Every higher perturbation followed by the horizontal homotopy lowers cyclic degree, so this added description changes neither the termination proof nor the 2^r generator bound.

⟨1⟩9. The free action makes the Borel construction equivalent to T^r/C_m, and radial contraction identifies this with B. For Y use exactly the same algorithm on T^{r+1} with matrix $\mathrm{diag}(1, M)$ and numerator $(1, v)$. The projection forgetting its first circle is equivariant and represents $Y \to B$ under the radial contraction. More generally an equivariant integral matrix L acts on groupoid arrows as $\lambda \mapsto L\lambda + k_b - k_a$, where $k_a = Lp_a - p'_a \in \mathbb{Z}^{r'}$. Transferring this strict bar map by $P'LI$ gives the required integer chain map. Its mapping cone computes the pair. Every series is finite with the proved degree bound, and integer kernels and Smith normal forms finish the calculation.

□

The programs `fourfolds_bar_koszul.py` and `fourfolds_affine_cap_chains.py` implement these formulas, including the prime-subgroup freeness check of 11.25. The first checks all 281 abelian and 498 cyclic bar simplices in its explicitly stated alphabets and degree ranges. The second checks translated circles of orders two and three, two affine presentations of the Klein bottle (one with a nonfixed numerator), and the three-dimensional half-turn manifold. The separate rank-six program checks every degree zero through seven for the order-five and order-four caps, against the independent integral Wang sequence of their five-torus mapping-torus descriptions. The boundary-map program checks both translated circles and the Klein bottle, including the full chain-map identities and the integral Thom-shift homology of each cone. These are finite verification sets, not an asserted exhaustive test over all input matrices; the terminating formulas prove generality.

Theorem 11.28 (the marked Kuhn cusp pair)**.** [tag `02NP`]

The fixed primitive rank-three Kuhn cusp admits an explicit finite integer chain complex for its cap W, its boundary Y, and the inclusion $Y \to W$. It is obtained from a category with 26 objects, 170 nonidentity morphisms, and respectively $26, 170, 288, 144$ strict flags of lengths $0, 1, 2, 3$. The construction records the full integral collapse map. In particular it gives

j	0	1	2	3	4	5	6	7	8
$\operatorname{rank} H_j(W)$	1	3	9	9	10	3	1	0	0
$\operatorname{rank} H_j(Y)$	1	4	12	18	18	12	4	1	0
$\operatorname{rank} H_j(W, Y)$	0	0	1	3	10	9	9	3	1

(F.CK1)

The only torsion in these three rows is $\operatorname{Tor} H_3(Y) = \mathbb{Z}/2$.

Proof. ⟨1⟩1. Let $\mathcal{K}$ be the periodic Kuhn triangulation of $\mathbb{R}^3$. Normalize every simplex by translating its lexicographically first vertex to zero. The objects are its simplex orbits; a morphism $\tau \to \sigma$ is every strict translated face inclusion $\tau + w \subset \sigma$. Keep distinct translated inclusions as distinct morphisms. Composition adds the translation vectors. Enumerating all nonempty faces of the six chains in the unit cube gives $(1, 7, 12, 6)$ simplex orbits in dimensions zero through three, and the displayed morphism and flag counts. The category nerve is the barycentric cell model of the dual decomposition of $\mathbb{R}^3/\mathbb{Z}^3$; the direction of the face order is reversed when interpreted as dual cells.

⟨1⟩2. For an object τ let $L_\tau \subset \mathbb{Z}^3$ be the primitive span of its vertex differences. Its primitivity follows from unimodularity of every Kuhn simplex. Choose by Smith reduction a quotient map $Q_\tau : \mathbb{Z}^3 \to \mathbb{Z}^{3-\dim\tau}$ with kernel L_τ, and an integral section R_τ. Define two strict diagrams of tori on this category:

	object at τ	map for $\tau + w \subset \sigma$
$\mathcal{W}$	$T^{3-\dim\tau}$	$Q_\sigma R_\tau$
$\mathcal{Y}$	T^4	$S_w = \begin{pmatrix} 1 & 0 \\ w & I_3 \end{pmatrix}$.

(F.CK2)

The first maps compose because Q_σ vanishes on L_τ, and the second compose because translations add. The maps $[\,0\ Q_\tau\,]$ form a natural transformation $\mathcal{Y} \to \mathcal{W}$: the only additional term is $Q_\sigma w = 0$, since w is a vertex of σ.

⟨1⟩3. Here these diagrams describe the local topology. The toric moment quotient on the dual cell to τ has angular torus $T^3/(L_\tau \otimes \mathbb{R}/L_\tau)$; passing to a face collapses exactly the next angular subtorus. Barycentrically subdividing the dual cells gives the homotopy colimit of $\mathcal{W}$ for the central fibre. For its real oriented logarithmic boundary, retain the argument θ_0 of the base parameter and the three angular coordinates θ of the toric chart. Translating the chart by w changes them to $(\theta_0, \theta + w\theta_0)$, which is precisely S_w. Thus the logarithmic boundary is the homotopy colimit of $\mathcal{Y}$, and forgetting the logarithmic phases on each stratum is exactly the natural transformation just given. This can also be read directly in the unimodular charts $x_0 \cdots x_k = t$: the moment coordinates form the dual simplex, the four logarithmic phases remain, and the ordinary stratum retains their quotient by the vertex differences. The finite face identifications give (F.CK2), so there are no further gluing constants in this fixed marked cap.

⟨1⟩4. The Kuhn map with its divisor log structures is proper, log smooth and vertical after restricting to a small disc. Achinger–Ogus, Theorem 4.1.1, identifies its topological germ with the mapping cylinder of this logarithmic collapse (Achinger and Ogus, 2020, Theorem 4.1.1 and Lemmas 4.1.2–4.1.3, pp. 480–482). Consequently the preceding homotopy-colimit diagrams and natural transformation model the pair (W, Y) itself. Fix the marking by letting b_i be the category loop with translation label e_i. A homotopy-colimit edge with map S_{e_i} has the relation $b_i^{-1}\theta_0 b_i = \theta_0 + a_i$; equivalently $\theta_0 b_i \theta_0^{-1} = b_i + a_i$. This specifies the required primitive cusp $U = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}$.

⟨1⟩5. **Hypotheses and marking of the logarithmic comparison.** Give the small disc its standard divisorial log structure and the Kuhn total space the divisorial structure of the reduced central fibre. In a unimodular chart the characteristic monoid map is

$$\mathbb{N} \longrightarrow \mathbb{N}^{k+1}, \qquad 1 \longmapsto (1, \ldots, 1).$$

Both monoids are fine and saturated. The group cokernel is free, and the underlying chart is the monomial semistable map $x_0 \cdots x_k = t$ with additional smooth coordinates; this is log smooth. The quotient monoid is a group, since the negative of each coordinate generator equals the sum of the other generators after the diagonal generator is killed. The morphism is therefore vertical. The periodic toric construction is proper over the sufficiently small disc.

⟨1⟩6. These are precisely the hypotheses of (Achinger and Ogus, 2020, Theorem 4.1.1, pp. 480–482). Its proof also notes that exactness is automatic over the standard log disc. It identifies the germ with the mapping cylinder of the logarithmic collapse. This homeomorphism

depends on a trivialization; Lemma 4.1.3 fixes that trivialization on the logarithmic central fibre. We keep the resulting radial boundary identification and its stated angular and tropical generators. This suffices to model the pair and its inclusion up to marked homotopy. It does not replace the cusp marking by another integral basis.

⟨1⟩7. We specify finite integer chains. Replace every torus by its ordinary normalized bar complex. The affine simplex map from the bar construction to the torus is natural for all the integer homomorphisms in (F.CK2), so this replacement does not change either homotopy colimit or its collapse map. Take the simplicial replacement over all strict flags of the finite category. Its vertical differential is the torus bar boundary; its horizontal differential composes adjacent arrows, with the first face applying the diagram map, and has the sign $(-1)^q$ in vertical degree q. Apply the first contraction and the finite series (F.CA5) object by object. There are no flags longer than three, so every series has at most four terms. The resulting finite basis consists of a strict flag beginning at τ and an exterior basis vector of its torus. Transfer the natural transformation by $P_{\mathcal{W}} F I_{\mathcal{Y}}$. This proves both the chain-map identity and completeness of the collapse map, including all integral coherence corrections.

⟨1⟩8. The raw cap chain ranks in degrees zero through six are

$$(26, 199, 688, 1231, 1226, 648, 144),$$

and the raw boundary ranks in degrees zero through seven are

$$(26, 274, 1124, 2420, 3010, 2186, 864, 144).$$

The program enumerates every object, every translated incidence and every flag, verifies composition, checks $d^2 = 0$ on every generator, and checks the collapse-chain identity on every generator. Integral cancellation of a unit pivot is a chain homotopy equivalence: clear its row by integer column operations, then remove the resulting contractible two-generator summand. The adjacent differential updates are forced by $d^2 = 0$. After respectively 2063, 4988 and 7087 such cancellations for the cap, boundary and collapse cone, Smith reduction gives (F.CK1). All matrices and the remaining differentials are saved by `fourfolds_kuhn_cap_chains.py`; the reduced cap and relative differentials vanish, and the boundary has one nonzero Smith factor, equal to two in degree four to three. This computes the integral groups without a filtration extension problem.

□

The finite-pair and cusp-pair chain models enter the global gluing cone of theorem 11.29. Its maps to the common torus-bundle complement retain the peripheral word and clutching translation.

The periodic toric quotient used for the cusp is the construction in Yang's theorem *Higher-dimensional periodic quotient*, and its Kuhn-grid Euler number is the formula in his corollary *The standard grid family* (Yang, 2026, Theorem 8.7, pp. 57–59, and Corollary 8.8, p. 59, arXiv v1). For the present primitive rank-three grid, this gives Euler number $3! = 6$. The full cap and boundary complexes, their integral inclusion, and the torsion groups are calculated in the proof above.

Theorem 11.29 (complete integral homology of a marked gluing)**.** [tag `02NQ`]

For every admissible datum there is a terminating exact integer algorithm computing every integral homology group of the resulting eight-manifold, including every torsion invariant factor. The input consists of the finite homology matrices $M_1, \dots, M_k$, free affine numerators v_i/m_i of their affine orders, the integral clutch vector κ, and the chosen primitive cusp marking

$$M_0 = (M_1 \cdots M_k)^{-1}, \qquad M_0 P = P \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix}, \qquad P \in \mathrm{GL}(6, \mathbb{Z}). \tag{F.GC1}$$

Here $k = 2$ or 3 and the genuine orders are at least two; order-one sites are first eliminated as in 11.25. If no marking is given, the algorithm chooses one by Smith reduction. The complete chain prefix through degree nine has at most 4003 generators in any degree.

Proof. ⟨1⟩1. We supply both global boundary maps which were left outstanding after 11.28. Let F_k be the free group with ordered generators $s_1, \dots, s_k$, and let $M(w)$ be the product specified by a reduced word w. The torus-bundle complement over the punctured sphere retracts to the corresponding bundle over a graph. Its fundamental group and multiplication are

$$\Gamma = \mathbb{Z}^6 \rtimes F_k, \qquad (x, w)(y, u) = (x + M(w)y, wu). \tag{F.GC2}$$

The fibre and base are aspherical and the base has no two-cells. Hence a chosen section identifies the complement up to homotopy with $B\Gamma$. These section and peripheral choices are fixed by the input clutch convention below.

⟨1⟩2. *A finite complement complex with its full bar projection.* Use reduced words for F_k. The tree resolution of the free group has $K_0 = \mathbb{Z}[F_k]$ and $K_1 = \bigoplus_i \mathbb{Z}[F_k]e_i$, with $de_i = s_i - 1$, and no higher terms. Its explicit augmented contraction sends a word to its signed path in the Cayley tree: for each positive letter s_i add its preceding prefix times e_i, and for a negative letter s_i^{-1} subtract the prefix including that letter times e_i. This is the Fox derivative, and its boundary is $w - 1$. Together with the bar insertion contraction it gives, by the recursion (F.CA3), a strong contraction of the free-group bar resolution to this tree resolution. Every recursion decreases bar degree and every word is finite.

⟨1⟩3. For a bar simplex $[(x_1, w_1)|\cdots|(x_n, w_n)]$ of Γ, put $y_j = M(w_1 \cdots w_{j-1})x_j$. The exact Alexander–Whitney chain map to the torus/free-group double bar complex is

$$\sum_{p+q=n} (-1)^{pq}[w_1|\cdots|w_p] \otimes [y_{p+1}|\cdots|y_n]. \tag{F.GC3}$$

Terms with an identity bar entry are zero. The horizontal first face acts on torus vectors by $\rho(w) = M(w)^{-1}$, and the horizontal boundary has sign $(-1)^q$. Notice that ρ is a right action: $\rho(wu) = \rho(u)\rho(w)$. This order is essential when the matrices do not commute. Formula (F.GC3) follows from the diagonal bisimplicial bar construction; alternatively, its chain-map identity follows by pairing adjacent interior faces, with the first face using $M(w_1)^{-1}y_j = M(w_2 \cdots w_{j-1})x_j$. The usual shuffle inverse is finite in each degree, so the map is a chain equivalence.

⟨1⟩4. Apply the torus contraction and (F.CA5), then the free-group tree contraction and the same finite perturbation formula. Higher terms decrease group bar degree at least two, whereas the tree complex only has degrees zero and one. Thus the resulting complement complex is

$$C_n(E) = \bigwedge^n \mathbb{Z}^6 \ \oplus \ \bigoplus_{i=1}^{k} \bigwedge^{n-1} \mathbb{Z}^6, \qquad d(e_i \otimes \xi) = (-1)^{n-1}\Big(\bigwedge^{n-1} M_i^{-1} - 1\Big)\xi. \tag{F.GC4}$$

Its differential alone has this short form; its projection from group bars retains the finite higher perturbation terms. Those projection terms are used for every boundary map.

⟨1⟩5. *Finite affine boundary maps.* For the ith finite cap use the orbit groupoid of 11.27. Its one-edge homotopy colimit for the affine generator is the boundary mapping torus. Vertically contracting it gives a bounded complex with $C_n(Y_i) = \bigwedge^n \mathbb{Z}^6 \oplus \bigwedge^{n-1} \mathbb{Z}^6$ and differential $(-1)^{n-1}(\bigwedge^{n-1} M_i - 1)$ on the second summand. Its inclusion into the cyclic bar construction, followed by the cap projection in (F.CA5), is the full map $Y_i \to B_i$. In particular this one-edge model avoids an unnecessary periodic resolution of the seven-dimensional boundary.

⟨1⟩6. Here is an explicit group-bar map for its other inclusion. Send a torus-groupoid arrow labelled λ to the translation $x^\lambda \in \Gamma$. With the integral offsets $z_1(a)$ in (F.CA4), put $\eta_a = s_i^{-1} x^{z_1(a)}$. For an arrow $f : a \to b$, the exact identity

$$\eta_a \, \psi(gf) = \psi(f) \, \eta_b \tag{F.GC5}$$

is the natural-transformation identity: it is just $s_i^{-1} x^{M_i\lambda + z_1(b)} = x^\lambda s_i^{-1} x^{z_1(b)}$. For a q-simplex, the edge map is $(-1)^q$ times the alternating prism which inserts η at every position, uses the original arrows before the insertion and the transformed arrows afterwards. Its boundary is precisely the signed difference of the two end maps, by (F.GC5). Compose this prism with the finite vertical lift of the one-edge complex, then with (F.GC3) and the complement projection. This proves the full chain-map identity. The offsets record the integer affine lift: replacing v_i by $v_i + m_i\lambda$ changes the offsets and the map, even if the affine map on the torus has not changed. An optional unimodular local basis simply applies its matrix to every translation in (F.GC5).

⟨1⟩7. We make the distinction between a bar cell and the geometric positive root explicit. Put $a_i = z_1(0) = \lfloor v_i/m_i \rfloor$, coordinatewise, and let τ_i be the positive one-edge bar cell closed using the zero-labelled return arrow to object zero. With the first-face convention above its geometric loop is

$$\tau_i = g_i^{-1} x^{a_i}, \qquad F_E(\tau_i) = -e_i + M_i^{-1} a_i \in C_1(E), \qquad g_i = x^{a_i} \tau_i^{-1}. \tag{F.GC5a}$$

Thus the bar cell has the opposite peripheral orientation, with the displayed integer correction when the numerator has a nonzero integer part. Formula (F.GC5) sends its group element to $s_i^{-1} x^{a_i}$; consequently the geometric positive root in the last expression of (F.GC5a) maps to s_i exactly. This is the stated conversion between the boundary and complement chain conventions. In a local basis Q, replace a_i in the group and complement formulas by $Q\lfloor Q^{-1} v_i/m_i \rfloor$. Writing $\mathcal{N}_i = \sum_{j=0}^{m_i-1} M_i^j$, the cyclic degree-two cell has boundary $m_i\tau_i + w_i - \mathcal{N}_i a_i$. Indeed $\tau_i^{m_i} = x^{\mathcal{N}_i a_i - w_i}$ follows directly from $g_i^{m_i} = x^{w_i}$. This independently checks the affine power relation and the degree-one prism sign, including negative numerators.

⟨1⟩8. There is an optional exact geometric shortcut. If a verified local integral basis gives $M_i = \mathrm{diag}(1, D)$ and $v_i = \epsilon e_0$, $\epsilon \in \{1, -1\}$, eliminate the first fibre generator from the cap group using $g^{m_i} = u^\epsilon$. The cap is the mapping torus with group $\mathbb{Z}^5 \rtimes_D \mathbb{Z}$. The fibre vector (u, a) maps to $x^a g^{\epsilon m_i u}$, and the root generator maps to g. Substituting this homomorphism for ψ in the same prism gives the boundary map to this bounded geometric cap model. The shortcut is used only after checking the displayed integral normal form; the general periodic construction remains available for every other input.

⟨1⟩9. To check the peripheral convention directly, the universal cover of Y_i has the root generator g_i with linear part M_i and affine displacement v_i/m_i. Its integral translations and g_i map to the same fibre translations and s_i in (F.GC2). The full base-circle translation obeys $g_i^{m_i} = x^{w_i} t$, where w_i is the power vector of 11.25. Filling the disc kills t, giving exactly $s_i^{m_i} = x^{w_i}$. The straight paths between the chosen orbit points give (F.GC5), so the prism is this marked map, including numerators with nonzero integer parts.

⟨1⟩10. *The marked cusp boundary map.* Write $P = [A \; B]$ in (F.GC1), and set

$$w_0 = (s_1 \cdots s_k)^{-1}, \qquad c = w_0 x^{-\kappa} \in \Gamma. \tag{F.GC6}$$

Send the three angular generators to the columns of A, the three tropical generators to the columns of B, and the base-phase generator to c. Relation (F.GC1) gives $c b_i c^{-1} = b_i + a_i$ and all the other nilpotent boundary relations. Filling kills the angular generators and c, so (F.GC6) gives exactly $s_1 \cdots s_k x^\kappa = 1$, in agreement with 11.25.

⟨1⟩11. This homomorphism has a direct chain implementation on the full Kuhn diagram. For an incidence with translation label w, assign the group element x^{Bw}. A vertical phase $(n, a) \in \mathbb{Z} \oplus \mathbb{Z}^3$ maps to $c^n x^{Aa}$. The incidence shear is $S_w(n, a) = (n, a + nw)$, and

$$(c^n x^{Aa}) x^{Bw} = x^{Bw} (c^n x^{A(a+nw)}). \tag{F.GC7}$$

For a flag of p incidence arrows and a vertical bar simplex of length q, sum over all (p, q) shuffles. Each incidence step contributes its assigned group element; each vertical step contributes its phase transformed by all preceding incidence shears. Give the shuffle the sign $(-1)^{\#\{\text{incidence steps before vertical steps}\}}$. Interior face cancellations use exactly (F.GC7), and adjacent incidence faces use addition of their translation labels. This is therefore a chain map from the double bar complex to $C_*(B\Gamma)$. Precompose with the finite Kuhn lift of 11.28 and postcompose with (F.GC3) and the complement projection. The other cusp map is the full logarithmic collapse already computed in 11.28. Both the cusp boundary and the complement are aspherical, so the displayed marked homomorphism also verifies the homotopy class of this geometric inclusion.

⟨1⟩12. *The final finite integer calculation.* Take the mapping cone of the just constructed maps

$$\bigoplus_{i=0}^{k} C_*(Y_i) \longrightarrow C_*(E) \oplus \bigoplus_{i=0}^{k} C_*(B_i), \qquad y \longmapsto (f_E(y), -f_B(y)), \tag{F.GC8}$$

where $B_0 = W$ is the Kuhn cap. Gluing collars makes this the homotopy pushout model of the closed manifold. Consequently its cone computes integral homology directly; there is no remaining filtration extension to choose.

⟨1⟩13. For clarity the exact finite size bound can be read off before running any matrix calculation. Put $t_n = \sum_{q=0}^{\min(n,6)} \binom{6}{q}$, with binomial coefficients outside their range interpreted as zero. Let w_n and y_n be the raw cap and boundary ranks of 11.28, zero outside their listed ranges. Then the cone rank in degree n is

$$\binom{6}{n} + k\binom{6}{n-1} + kt_n + w_n + k\binom{7}{n-1} + y_{n-1}. \tag{F.GC9}$$

For $0 \le n \le 9$ its maximum is 3890 when $k = 2$ and 4003 when $k = 3$. Every bar comparison and perturbation series used to produce these matrices terminates by its degree filtration. Word reductions and affine orbit enumerations are finite, with no imposed size cutoff. Computing integer kernels followed by Smith normal forms now gives $H_0, \dots, H_8$; higher groups vanish by dimension. Unit cancellations are optional exact integral chain equivalences which substantially reduce the matrices before Smith reduction. This proves the algorithm and the bound.

□

The programs `fourfolds_global_chains.py`, `fourfolds_total_homology.py` and `fourfolds_marked_gluing.py` implement this algorithm (§1.5).

A third full integration uses $A = I_3$, $B = \mathrm{diag}(0, 1, 0)$ in (F.IV2), affine orders $4, 4$, numerators a_1, a_2, and zero clutching. These two linear actions reverse orientation. Its 7345 integral unit cancellations give free ranks

$$(1, 0, 4, 11, 10, 1, 6, 3, 0).$$

The nonzero torsion groups are

$$\begin{aligned} &\operatorname{Tor} H_1 = \operatorname{Tor} H_5 = (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/4, && \operatorname{Tor} H_2 = (\mathbb{Z}/2)^2, \\ &\operatorname{Tor} H_3 = \operatorname{Tor} H_7 = \mathbb{Z}/2, && \operatorname{Tor} H_4 = (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^5. \end{aligned}$$

The independent van Kampen H_1 agrees. Its Euler number is six, its mod-two Betti vector is $(1, 4, 10, 14, 20, 14, 10, 4, 1)$, and its integral top groups have the required nonorientable form. Thus this integration also tests the orientation-reversing part of the stated scope.

The free-bar and semidirect tests, all four periodic-boundary comparisons, all six one-edge comparison/Thom tests, all seven peripheral-sign checks, the complete 10048-generator cusp comparison, and the cached generic three-connected datum computation have also been independently rerun successfully. The reduced-chain, van Kampen H_1, Euler and mod-two duality checks have additionally been independently replayed on all three complete integration outputs. The general algorithm is proved by the explicit chain equivalences and finite operations above; these finite execution sets are reported with their exact scope.

Theorem 11.30 (simultaneous integral conjugacy for every input)**.** [tag `02NR`]

There is an unconditional exact algorithm which decides simultaneous $\mathrm{GL}_6(\mathbb{Z})$ *conjugacy of any two admissible linear tuples and returns a conjugating matrix when one exists. The algorithm does not impose a bound on their entries, on the affine multiplicities, or on the integral conjugator. More generally it applies to any integral tuple containing a matrix* $1 + N$ *with* $N^2 = 0$ *and* $\operatorname{rank} N = 3$*; the primitive-image assumption is unnecessary for this statement.*

Proof. ⟨1⟩1. We use column vectors and seek P satisfying $PX_i = Y_iP$ for every marked index, including the cusp index. Put

$$\begin{aligned} H &= \{P \in \mathrm{Mat}_6(\mathbb{Z}) : PX_i = Y_iP \ (\forall i)\}, \\ \mathcal{O} &= \{C \in \mathrm{Mat}_6(\mathbb{Z}) : CX_i = X_iC \ (\forall i)\}, \qquad E = \mathbb{Q}\mathcal{O}. \end{aligned} \tag{F.IC1}$$

Both lattices are saturated integer kernels of explicitly displayed linear equations in thirty-six unknowns. Thus Hermite or Smith reduction computes full bases, rather than sublattices of finite index.

⟨1⟩2. We first decide rational isomorphism. For a basis $h_1, \dots, h_s$ of H, form the polynomial $f(c) = \det(\sum c_j h_j)$. If it is zero, there is no rational isomorphism. Otherwise successively substitute one of $0, 1, \dots, 6$ for each variable while keeping the remaining polynomial nonzero. At each stage one of these seven values works, because the degree in that variable is at most six. The final matrix $D \in H$ is invertible over $\mathbb{Q}$. This is a terminating exact test, including the negative case, and not a search for a lucky invertible intertwiner.

⟨1⟩3. The full lattice $L = D^{-1}H \subset E$ is a right $\mathcal{O}$-module. Let J be the Jacobson radical of E, let $B = E/J$, and use bars for images under the quotient map. We next verify the splitting hypothesis needed for the principal-ideal algorithm. In a rational N-adapted basis,

$$N = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}, \qquad E \subseteq \left\{ \begin{pmatrix} A & T \\ 0 & A \end{pmatrix} : A, T \in \mathrm{Mat}_3(\mathbb{Q}) \right\}. \tag{F.IC2}$$

Projection to A has square-zero kernel and image an algebra $S \subseteq \mathrm{Mat}_3(\mathbb{Q})$. A nilpotent ideal is contained in the radical, so $E/J \cong S/J(S)$.

⟨1⟩4. Choose a composition series of the faithful S-module $\mathbb{Q}^3$. The kernel of its action on the associated graded module lowers this filtration and is therefore nilpotent. Conversely the radical kills every simple factor. Thus this kernel is exactly $J(S)$ and every simple factor of $S/J(S)$ acts nontrivially on a composition factor. For a simple algebra $\mathrm{Mat}_n(D)$, where D has centre K and degree d over that centre, every nonzero simple module has $\mathbb{Q}$-dimension $nd^2[K : \mathbb{Q}]$. If D were noncommutative this would be at least four, which is impossible in $\mathbb{Q}^3$. Consequently

$$B \cong \prod_j \mathrm{Mat}_{n_j}(K_j), \qquad n_j[K_j : \mathbb{Q}] \le 3. \tag{F.IC3}$$

Every noncommutative factor is therefore $\mathrm{Mat}_2(\mathbb{Q})$ or $\mathrm{Mat}_3(\mathbb{Q})$; the remaining factors are number fields of degree at most three. This proves hypothesis (H) of (Bley et al., 2022, Algorithm 8.3, p. 21) for every tuple under consideration.

⟨1⟩5. For clarity this decomposition is effective without a split-algebra oracle. Rational linear algebra computes J as the radical of the regular trace pairing and computes the quotient multiplication table. Its simple factors are obtained from its centre. A commutative factor is a number field. In a noncommutative factor of dimension n^2, we enumerate integral combinations e of a rational basis until the left ideal Be has dimension n. Such an e exists because the factor is $\mathrm{Mat}_n(\mathbb{Q})$, so its rank-one matrices are rational points; clearing denominators expresses one among the enumerated combinations. Left multiplication on Be then supplies an explicit isomorphism to $\mathrm{Mat}_n(\mathbb{Q})$. The program verifies multiplication on every pair of basis elements.

⟨1⟩6. The images $\overline{\mathcal{O}}$ and $\overline{L}$ are a full order and a full right lattice in B. If the right order of $\overline{L}$ differs from $\overline{\mathcal{O}}$, this lattice cannot be principal over that order. Otherwise Algorithm 8.3 of (Bley et al., 2022, Algorithm 8.3 and its correctness proof, p. 21), in its right-ideal form, decides principality and returns a generator $\overline{\alpha}$ when one exists. The cited algorithm passes through a maximal order and its central conductor, tests local freeness and principality over the maximal order, and solves the remaining reduced-norm congruence in a finite quotient. Its correctness proof uses its Theorem 4.5(b) and Theorem 7.9; its Theorem 8.4 specifies the number-field, finite-field and integer factorization subproblems. These have unconditional exact algorithms. In the recorded implementation the class and unit groups of every number-field component are certified with `GRH=false` before the principal-ideal decision is used. The negative decision thus does not presume the generalized Riemann hypothesis.

⟨1⟩7. If $\overline{L}$ is not principal, L cannot be principal. If it is principal, solve the integer linear equations expressing $\overline{\alpha}$ as a combination of the images of the chosen basis of L. This produces a lift $\alpha \in L$ and an integral intertwiner $P = D\alpha \in H$. We claim that a simultaneous integral conjugacy exists if and only if $|\det P| = 1$.

⟨1⟩8. The sufficient direction is immediate. For the necessary direction, suppose $P_0 \in H$ is unimodular. Every $h \in H$ has $P_0^{-1}h \in \mathcal{O}$, so $L = \alpha_0\mathcal{O}$, where $\alpha_0 = D^{-1}P_0$. Write $\alpha = \alpha_0 u$ with $u \in \mathcal{O}$. Because both image elements generate $\overline{L}$, $\overline{u}$ is a unit of $\overline{\mathcal{O}}$. Lift its inverse to $v \in \mathcal{O}$. Then $uv = 1 + j$ with $j \in J \cap \mathcal{O}$, and the finite geometric series for $(1 + j)^{-1}$ shows that $u^{-1} = v(1 + j)^{-1} \in \mathcal{O}$. Thus u is an integral unimodular matrix and $P = P_0 u$ is unimodular. This also proves that any chosen lift of the image generator gives the same yes-or-no answer. It is the right-module form of (Bley et al., 2022, Theorem 3.4, p. 6; Propositions 8.1–8.2, p. 21), with the needed determinant condition written out.

⟨1⟩9. Every branch either proves nonexistence or returns a matrix whose integrality, determinant and all intertwining equations are verified exactly. This completes the decision procedure. □

Theorem 11.31 (a complete canonical-representative algorithm)**.** [tag `02NS`]

Every admissible linear tuple, for either $k = 2$ or $k = 3$, has a computable canonical integral representative and a returned simultaneous unimodular conjugator. Two inputs have the same representative if and only if they are simultaneously $\mathrm{GL}_6(\mathbb{Z})$ *conjugate. The algorithm uses a proved finite bound depending on the input. No fixed global entry bound can represent every class.*

This is an effective classification. The ordered involution sector has the separate complete finite normal form 11.32, and all of its 56 *classes have been computed.*

Proof. ⟨1⟩1. We remove the existential free-lift condition by an explicit construction. Let $d > 1$ be the order of M and choose a primitive $u \in \ker(M - I) \cap \mathbb{Z}^6$. Such a vector is obtained from a nonzero rational kernel vector by clearing denominators and dividing by the greatest common divisor of its integer coordinates; both operations retain the equation $(M-I)u = 0$. Put $g(x) = Mx + u/d^2$ and $S_d = \sum_{a=0}^{d-1} M^a$. Then $g^j(x) = M^j x + ju/d^2$, $S_d M^j = S_d$ and $S_d u = du$. A fixed point of g^j gives $(M^j - I)x + ju/d^2 = \lambda \in \mathbb{Z}^6$. Applying S_d gives $ju/d = S_d\lambda \in \mathbb{Z}^6$. A Bezout row pairing to one with u therefore gives $d \mid j$. Writing $j = dr$, the original equation becomes $ru/d = \lambda$, so the same row gives $d \mid r$. Thus only multiples of d^2 can have fixed points. Since g^{d^2} is translation by the integer vector u, its torus action is the identity. This proves both its affine order d^2 and freeness of every nonidentity power. For $M = I$, translation by $u/2$ for primitive u has affine order two and is free. Conversely, when $I - M$ is invertible over $\mathbb{R}$, every affine map $Mx + a$ fixes the torus class of $(I - M)^{-1}a$. The free-lift test is consequently exactly $\ker(M - I) \neq 0$.

⟨1⟩2. The finite linear orders here are the already proved list of proposition 4.5:

$$d \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}. \tag{F.CN1}$$

Taking these finitely many powers, checking the fixed kernel, the product relation and the cusp Smith factors therefore gives an exact admissibility test. The affine order is the separate retained gluing datum; the construction in item1 gives its exact relation to the chosen finite lift.

⟨1⟩3. For an admissible full tuple $T = (M_1, \dots, M_k, M_0)$ let $H(T)$ be its maximum absolute entry. There is at most one identity finite slot in a three-point tuple: two such identities would make M_0 finite order, whereas $M_0 = I + N$, $N^2 = 0$ and $\operatorname{rank} N = 3$ give $(I + N)^d = I + dN \neq I$ for every positive integer d. At a fixed identity slot, delete that matrix and reinsert I at the same slot after classifying the remaining ordered tuple. These two maps are inverse, and both commute with $T \mapsto (QM_1Q^{-1}, \dots, QM_0Q^{-1})$ for the same $Q \in \mathrm{GL}(6, \mathbb{Z})$. They therefore retain the returned simultaneous conjugator. The remaining invertible integral matrices have height at least one, so inserting I leaves height unchanged. Every entry in the inserted block is constant; the first unequal row-major entry of two tuples is therefore the same before and after insertion. This proves preservation of the lexicographic comparison as well. The affine order and numerator at that slot are retained as gluing data and are transported by this same Q. In the two-involution sector use the canonical representative of 11.32, with its own specified convention. In all remaining sectors, order full integer tuples first by H, then lexicographically by their row-major entries in the displayed marked order, and take the first admissible member of the simultaneous conjugacy class.

⟨1⟩4. To compute the representative, enumerate heights $1, \dots, H(T)$ and all integer entries in the stated order. There are at most

$$(2H(T) + 1)^{36(k+1)} \tag{F.CN2}$$

integer lists in this box. Retain the invertible tuples satisfying the ordered product relation and the stated maximal square-zero primitive cusp condition, which are finite integer tests, and apply the decision-and-witness algorithm of theorem 11.30 to each candidate and the tuple. Its general rank-three square-zero input permits precisely this call. A positive answer transfers all finite-order, orientation and admissibility conditions through its integral conjugator. The tuple itself occurs, with the identity as a possible witness, so a positive answer is reached in finitely many tests. The first one is the required representative. Record the returned convention as $P_T^{-1} T_i P_T = \mathrm{Can}(T)_i$. The displayed candidate bound makes no assumption about entries of P_T; the terminating witness construction is the stated input from theorem 11.30.

⟨1⟩5. Every simultaneous orbit has a least element in this order. Its height is at most the height of each given member, so (F.CN2) includes that global least element. Equivalent inputs have exactly the same ordered orbit and therefore the same representative. Conversely, if T, T' have the same representative, the matrix $Q = P_T P_{T'}^{-1}$ satisfies $Q^{-1} T_i Q = T'_i$ for every slot. The separate involution convention has the same property by theorem 11.32, with the product of its returned witnesses giving the corresponding map. Conjugation preserves the ordered finite orders, so the involution sector is preserved by those same maps. This proves the classification assertion and its finite stopping rule.

⟨1⟩6. Finally, 11.10 already gives an unbounded admissible integral family. In its homology convention direct symbolic multiplication gives

$$\operatorname{tr}(M_1 M_2^3) = -v^4 + v^3 - v^2 + 2. \tag{F.CN3}$$

This is unbounded even on the previously constructed free affine subfamily $v \in 20\mathbb{Z}$. It is invariant under simultaneous conjugacy. A fixed finite entry box contains only finitely many tuples and hence only finitely many such traces; it cannot contain a representative of every class. The input-dependent nature of (F.CN2) is therefore necessary for this particular exhaustive procedure.

□

Theorem 11.32 (the complete ordered involution stratum)**.** [tag 02NT]

There are exactly 56 simultaneous $\mathrm{GL}_6(\mathbb{Z})$ *conjugacy classes of ordered tuples* (M_1, M_2, M_0) *satisfying the primitive maximal-cusp conditions of definition 4.1 with both finite linear orders equal to two. An explicit integral normalizer returns a canonical representative and a checked unimodular conjugating matrix for every such input, without any bound on its entries. Every representative has free affine lifts of affine order four at both finite points. None can give a simply connected filling, for any choices of free affine orders or twists.*

Proof. ⟨1⟩1. First choose an integral cusp basis, so that

$$M_0 = I_6 + N, \qquad N = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}. \tag{F.IV1}$$

This is effective by Smith reduction. In fact choose vectors β_i whose images $\alpha_i = N\beta_i$ form a primitive basis of $\operatorname{im} N = \ker N$. The index-one hypothesis says that their classes form a basis of the quotient, so $[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3]$ is unimodular.

⟨1⟩2. Since $M_2 = M_1(I - N)$ and $M_1^2 = M_2^2 = I$, we have $M_1 N M_1 = -N$. Solving this block equation gives precisely

$$M_1 = \begin{pmatrix} A & B \\ 0 & -A \end{pmatrix}, \qquad M_2 = \begin{pmatrix} A & B - A \\ 0 & -A \end{pmatrix}, \qquad A^2 = I_3, \quad AB = BA. \tag{F.IV2}$$

Every integral conjugator preserving (F.IV1) is $\begin{pmatrix} C & D \\ 0 & C \end{pmatrix}$, with $C \in \mathrm{GL}_3(\mathbb{Z})$ and D arbitrary integral. Thus one first classifies A integrally; after fixing it, B is considered modulo $AD + DA$ and modulo conjugation by the integral centralizer of A.

⟨1⟩3. We include the integral involution classification needed here. For the lattice $L = \mathbb{Z}^3$, let L_+ and L_- be the primitive kernels of $A - I$ and $A + I$. They satisfy $2L \subseteq L_+ \oplus L_- \subseteq L$. The quotient $L/(L_+ \oplus L_-)$ is a vector space over $\mathbb{F}_2$. Its projections into $L_+/2L_+$ and $L_-/2L_-$ are injective: a glue vector with only one nonzero projection would give a new integral vector in the corresponding eigensublattice, contradicting its definition. It is consequently

the graph of an isomorphism between two subspaces of equal dimension r. Unimodular changes of bases in $L_\pm$ put this graph in coordinate form. For each glued pair e_+, e_- the vectors $(e_+ + e_-)/2$ and $(e_+ - e_-)/2$ form a swap block. The unglued vectors give trivial and sign blocks. Hence there is a unique type

$$A \cong I_p \oplus (-I_q) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus r}, \qquad p + q + 2r = 3. \tag{F.IV3}$$

The dimensions of the two rational eigenspaces and the index $[L : L_+ \oplus L_-] = 2^r$ prove uniqueness. Here r is zero or one, so this normalization only requires integer kernels, one Smith reduction and, in the glued case, one explicit half-sum pair whose integrality is checked.

⟨1⟩4. For fixed (F.IV3), compute the quotient of the commuting matrices by the subgroup $\{AD + DA\}$. On the trivial and sign diagonal blocks the norm is respectively $2D$ and $-2D$. The commuting off-diagonal blocks between the trivial and sign summands are zero. Every commuting block involving a swap summand is a norm. For example a trivial-to-swap block is a row (a, a) and is obtained as $(d, e) + (e, d)$; a sign-to-swap block has the same calculation with a difference. For two swap summands, the commuting block is $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, and applying the norm to $\begin{pmatrix} d & e \\ f & g \end{pmatrix}$ gives $\begin{pmatrix} e+f & d+g \\ d+g & e+f \end{pmatrix}$. The reverse mixed blocks have the transposed calculations. Therefore the full integral quotient is exactly

$$\mathrm{End}_{\mathbb{Z}[A]}(L)/\{AD + DA\} \cong \mathrm{Mat}_p(\mathbb{F}_2) \oplus \mathrm{Mat}_q(\mathbb{F}_2). \tag{F.IV4}$$

No finite-index sublattice has been substituted for a norm image; the displayed block calculations are surjective integer formulas.

⟨1⟩5. The centralizer action on (F.IV4) is precisely independent similarity in the two matrix factors. Indeed, in a product of commuting block matrices, every contribution factoring through a swap block is a norm. The remaining diagonal blocks are just the reductions of the trivial and sign blocks of the centralizer matrix. If that matrix is an integral unit, its inverse shows that both reductions are invertible. Conversely block-diagonal unimodular changes on the trivial and sign summands induce every element of $\mathrm{GL}_p(\mathbb{F}_2) \times \mathrm{GL}_q(\mathbb{F}_2)$; elementary matrices give unimodular lifts. There is no additional identification between the two factors, since an intertwiner of a trivial and a sign lattice is zero over $\mathbb{Z}$.

⟨1⟩6. It remains a genuinely finite classification. Enumerate all 2^{n^2} matrices of size $n = 0, 1, 2, 3$ over $\mathbb{F}_2$ and conjugate by every invertible matrix of that size. Their respective numbers of similarity classes are $1, 2, 6, 14$. Taking the smallest bit-coded representative in each orbit gives a fixed canonical choice. The six integral involution types and their counts are

(p, q, r)	$(0,3,0)$	$(1,2,0)$	$(2,1,0)$	$(3,0,0)$	$(0,1,1)$	$(1,0,1)$
classes	14	12	12	14	2	2

and their sum is 56. A representative is (F.IV2) with B equal to the chosen 0–1 representatives in its trivial and sign blocks, and zero in every other block.

⟨1⟩7. This also gives an effective normalizer for an arbitrary input, not merely a list of representatives. Normalize its cusp and then its involution by the preceding Smith and glue constructions. Use the finite-field orbit lookup to normalize its two reductions. In dimensions at most three the binary lift of an invertible binary matrix is already unimodular: its determinant is odd and has absolute value at most two. The latter bound follows by

adjoining a row and column of ones and applying the 4×4 Hadamard bound. Finally solve the full integral linear equation $AD + DA = B - B_{\mathrm{can}}$ by Smith reduction. The preceding norm calculation proves solvability. Conjugating by $\begin{pmatrix} I & D \\ 0 & I \end{pmatrix}$ subtracts this norm from the upper-right block. The product of these explicitly recorded conjugators is integral unimodular and carries all three marked matrices to their canonical representative.

⟨1⟩8. Both matrices in (F.IV2) have a three-dimensional fixed space. Choose a primitive integral fixed vector v_i. Then $g_i(x) = M_i x + v_i/4$ has affine order four: g_i^2 is the nontrivial translation $v_i/2$. This order-two subgroup is free, and every nontrivial subgroup of a cyclic group of order four contains it, so the full action is free. For any other finite affine lift the affine order is still even, since its linear part has order two. The two-point orbifold quotient of 11.25 is therefore the nontrivial cyclic group of order $\gcd(m_1, m_2)$. Simple connectivity is impossible for every choice of twists and affine orders.

□

Theorem 11.33 (the full effective classification and all marked gluings)**.** [tag 02NU]

The admissible data and the topology of their gluings are classified as follows. For $k = 2$ and $k = 3$, every admissible ordered integral tuple has the canonical representative of 11.31, with an explicit simultaneous unimodular conjugator. Equality of these representatives is necessary and sufficient for simultaneous integral conjugacy. There is no fixed global height cutoff; the finite stopping bound for an input of height H is $(2H + 1)^{36(k+1)}$ candidates before the proved rejection filters. The separate involution normal form has exactly 56 classes and is computed completely in 11.32.

For every choice of free affine orders and integer numerators, integral clutch vector, and specified primitive Kuhn cusp marking, 11.25 and 11.29 compute the full fundamental group and every integral homology group of the gluing. The algorithms have no cutoff on affine orders, twist entries or lattice indices. The complete list of data with $\pi_1 = 1$ and $H_2 = H_3 = 0$ is the constructive parametrization of 11.24, with its exact lattice enumeration and necessary-and-sufficient finite tests. Each such filling has the homology and oriented smooth type stated there.

Proof. ⟨1⟩1. The input is finite, so the prime-subgroup test in 11.25 decides whether every claimed affine cyclic action is free and has the stated affine order. The integral cusp test and its adapted basis are 4.4. The complete linear admissibility filter and the canonical representative, including the proved finite stopping bound, are 4.5 and 11.30–38. The characteristic-polynomial filters used by that algorithm are conjugacy invariants, so none can remove a possible representative. The unbounded trace family (F.CN3) proves that no fixed finite box could replace the input-dependent bound.

⟨1⟩2. We spell out how the linear classification is used with a specified gluing. If R is its returned conjugator and P is the cusp marking, change the data by

$$M_i' = RM_iR^{-1}, \qquad v_i' = Rv_i, \qquad \kappa' = R\kappa, \qquad P' = RP. \tag{F.SY1}$$

The affine orders are unchanged. The coordinate map $x \mapsto Rx$ on the torus intertwines the finite quotient charts, since it sends $M_i x + v_i/m_i$ to $M_i' x' + v_i'/m_i$. At the cusp, its coordinates in the marked Kuhn model are unchanged: $(P')^{-1}x' = P^{-1}x$. Hence it intertwines every toric chart, translated face identification and collapse map of that fixed filling. It also carries the clutch path to the path with vector κ'. These maps agree on the collars and give a diffeomorphism of the two fully marked gluings. Thus homology can be computed in the original coordinates, while the canonical data and their group presentation are also returned. Independently replacing P' by another Smith marking is not this comparison.

⟨1⟩3. An affine order-one site is ordinary, even if its integer numerator has been recorded. Delete it by the explicit comparison of theorem 11.25: if it is the ith site, replace κ by $\kappa+(M_{i+1}\cdots M_k)^{-1}v_i$ before deletion. The cusp matrix and its given marking are unchanged. There cannot be an order-one site in an admissible two-point tuple: its other finite matrix would be the inverse of a nontrivial unipotent matrix. Nor can a three-point tuple have two such sites. Thus this reduction leaves precisely the genuine two- or three-point inputs of 11.29 and loses no case from the input class.

⟨1⟩4. For each remaining input, the exact unsaturated normal closure, cyclic or triangle-group extension and exceptional spherical Euler relation of 11.25 give the full fundamental group. The finite cap, complement and marked Kuhn chain maps of 11.27–35 give a finite integer Mayer–Vietoris cone through degree nine. Integer kernels and Smith reduction therefore give all $H_0,\dots,H_8$ with their complete torsion factors. Its independent abelianization check compares the resulting H_1 with the van Kampen answer.

⟨1⟩5. Finally 11.1–30 prove both directions of the classification. Genuine three-point orbifold quotients exclude simple connectivity. The two-point filters reduce every three-connected datum to the minimal order-$(5,4)$ spectrum; the rational normal form, finite stable-lattice and extension algorithms include every integral representation above it. The universal affine normalization includes every twist, and the six exact integral tests decide three-connectivity. Conversely every datum that passes them satisfies all three topological conditions by those tests. 11.24 computes its common integral homology and applies the almost-complex and smooth-classification arguments to identify its oriented type. There is no residual untested rational-height bound in this parametrization. The canonical-representative algorithm can remove duplicates from it if desired, but no finite global enumeration is asserted or needed for these complete per-input decisions.

□

12 Source obstructions and the punctured-base period decision

Theorem 12.1 (exclusion of the abelian-surface source)**.** [tag 0206]

The construction of Engel (2026), starting with a polarized algebraic family of abelian surfaces with a toric-rank-two cusp, taking a fibrewise degree-zero line bundle, and quotienting its punctured bundle by a lifted translation, cannot have $b_1 = b_2 = 0$ with the stipulated rank-three square-zero Kuhn cusp. The two sections need not be proportional.

Proof. ⟨1⟩1. Let W be the rank-four rational homology local system of the abelian surfaces. It is semisimple by (Deligne, 1971, Definition 4.2.4 and Proposition 4.2.5, p. 44; Theorem 4.2.6, p. 45). Its toric-rank-two cusp has $N_W^2 = 0$ and $\ker N_W = \operatorname{im} N_W$ of dimension two. Semisimplicity splits $W = W^\Gamma \oplus W'$. The image of N_W lies in W' whereas W^Γ lies in its kernel. The equality of image and kernel forces $W^\Gamma = 0$. The coinvariants vanish as well.

⟨1⟩2. The rank-six rational homology system T after the two Engel operations has a natural filtration

$$0 \subset L \subset E \subset T, \qquad L \simeq \mathbb{Q}, \quad E/L \simeq W, \quad T/E \simeq \mathbb{Q}.$$

The lower line is the circle in the punctured line bundle and the upper quotient is the lifted-translation generator. No polarization on T is assumed. The triviality of T/E gives $\operatorname{im} N_T \subset E$. Maximal square-zero cusp monodromy gives $\ker N_T = \operatorname{im} N_T$; thus every globally fixed vector lies in E. Since $W^\Gamma = 0$, the fixed vectors of E are exactly L. Hence $\dim T^\Gamma = 1$. The dual filtration, whose middle term is W^*, gives $\dim (T^*)^\Gamma = 1$ by the identical argument.

⟨1⟩3. Consequently the two ranks a, b in Theorem 4.6 are both one. Its Euler-parity calculation applies without requiring self-duality of T, and contradicts $b_1 = b_2 = 0$. This excludes the algebraic abelian-surface construction and, more generally, every such filtration with semisimple middle term. It does not exclude an arbitrary unpolarized torus period family. □

Theorem 12.2 (the unpolarized period domain has LSAP)**.** [tag 020A]
Let D_3 be the component of

$$\{F \in \mathrm{Gr}(3, \mathbb{C}^6) : F \cap \overline{F} = 0\}$$

containing the standard complex structure, so that $D_3 \cong GL^+(6,\mathbb{R})/GL(3,\mathbb{C})$. At every point, D_3 is dominated by a spanning tube of nine complex lines. In particular, it has the local spray approximation property (LSAP) and is Oka–1. These assertions concern the unpolarized domain.

Proof. ⟨1⟩1. Use the graph chart $F_Z = \mathrm{im}\binom{Z}{I_3}$. The open set

$$\Omega = \{Z \in M_3(\mathbb{C}) : \det(\mathrm{Im}\, Z) > 0\} \tag{F.O1}$$

lies in D_3 and contains iI_3. For a real column w, write $K_w x = w \times x$, put $q_w = w^t w$, and define

$$A_w = q_w I_3 - w w^t + K_w. \tag{F.O2}$$

Take the six vectors

$$e_1,\ e_2,\ e_3,\ e_1+e_2,\ e_1+e_3,\ e_2+e_3$$

and the corresponding six matrices A_w, followed by the three matrices $K_{e_1}, K_{e_2}, K_{e_3}$. Call these nine matrices $A_1, \dots, A_9$. Their symmetric parts are a basis of the six-dimensional space of real symmetric matrices: the first three give all diagonal matrices and the last three add the three distinct off-diagonal entries. The final three skew matrices form a basis of the skew matrices. Thus the nine directions are linearly independent over $\mathbb{R}$ and also over $\mathbb{C}$. For reference, the determinant of their nine row-major coordinate columns is -16.

⟨1⟩2. Each A_w is real normal, is zero on $\mathbb{R}w$, and has eigenvalues $q_w \pm i\sqrt{q_w}$ on $w^\perp \otimes \mathbb{C}$. For every real s its two transverse squared singular values are

$$(1+q_w s)^2 + q_w s^2 = q_w(q_w+1)\left(s + \frac{1}{q_w+1}\right)^2 + \frac{1}{q_w+1} \geq \frac{1}{3}, \tag{F.O3}$$

since $q_w \in \{1, 2\}$; the remaining squared singular value is one. For a pure skew direction K_{e_j} all singular values of $I_3 + sK_{e_j}$ are at least one. Their determinants are positive at $s = 0$ and never vanish, so they are positive for all real s.

⟨1⟩3. Let $\Lambda = \bigcup_{j=1}^{9}(iI_3 + \mathbb{C}A_j)$ and let $B_{1/4}$ be the open Frobenius ball of radius $1/4$ in $M_3(\mathbb{C})$. Then

$$T = \Lambda + B_{1/4} \subset \Omega. \tag{F.O4}$$

Indeed, the imaginary part of $iI_3 + tA_j + E$ is $I_3 + (\mathrm{Im}\, t)A_j + \mathrm{Im}\, E$. Its unperturbed least singular value is at least $1/\sqrt{3}$, whereas $\|\mathrm{Im}\, E\|_{\mathrm{op}} < 1/4$. It therefore stays invertible throughout the straight perturbation from $E = 0$, preserving the positive determinant. The set T is open and connected and is a union of affine translates of the spanning tree Λ. Thus it is a spanning tube in the precise sense of (Alarcón and Forstnerič, 2025, Definition 2.1, p. 4), not merely a neighborhood of one entire line. Its inclusion in D_3 is a local biholomorphism at iI_3.

⟨1⟩4. The real group $GL^+(6,\mathbb{R})$ acts holomorphically and transitively on D_3. Transporting this tube gives the same domination at every point. The observation preceding (Alarcón and Forstnerič, 2025, Proposition 7.13 and preceding paragraph, pp. 23–24) gives LSAP from this domination, and its part (b), or Theorem 2.2 of that paper, gives Oka–1. The exact program `fourfolds_oka_tubes.py` checks all nine integer matrices, their linear independence and the symbolic identity (F.O3).

□

Theorem 12.3 (holomorphic periods on every open base)**.** [tag 020B]

Let B be an open connected Riemann surface and $\rho : \pi_1(B) \to GL^+(6,\mathbb{R})$ any representation. Its associated flat D_3-bundle admits a holomorphic section. More generally, every continuous section which is holomorphic near a compact Runge set can be deformed through sections to a holomorphic section, with arbitrary uniform approximation there and any prescribed finite-order interpolation at finitely many points of that set. For an integral representation, this gives a holomorphic family of complex three-tori with the prescribed lattice local system and a zero section over B.

Proof. ⟨1⟩1. The flat bundle $\mathscr{D} = \widetilde{B} \times_\rho D_3 \to B$ is a holomorphic fibre bundle: its transition maps are constant real linear transformations, acting holomorphically on the Grassmannian. Its fibre has LSAP by Theorem 12.2, so the bundle map is an Oka–1 map by (Alarcón and Forstnerič, 2025, Proposition 7.13(d), p. 24). The lifting property in Definition 7.7 of that paper, applied to the identity map of B, gives precisely the stated section deformation and approximation, including the finite jets.

⟨1⟩2. There is a continuous section to start with. An open Riemann surface has the homotopy type of a one-dimensional CW complex; a bundle with connected fibre admits a section over its vertices and edges, and the homotopy lifting property transports this section to B. Connectedness holds for the chosen component D_3. Taking the Runge set empty therefore proves existence.

⟨1⟩3. For an integral representation let $\mathbb{V}_\mathbb{Z}$ be its lattice local system and let $\mathcal{F} \subset \mathbb{V}_\mathbb{Z} \otimes \mathcal{O}_B$ be the resulting holomorphic rank-three subbundle. Real transversality says that projection identifies the real vector bundle with the underlying real bundle of $\mathcal{Q} = (\mathbb{V}_\mathbb{Z} \otimes \mathcal{O}_B)/\mathcal{F}$. Hence the image of $\mathbb{V}_\mathbb{Z}$ is a full lattice in every fibre of $\mathcal{Q}$. Locally it is uniformly discrete and acts by holomorphic translations. The quotient $\mathcal{Q}/\mathbb{V}_\mathbb{Z} \to B$ is consequently a smooth proper holomorphic family of compact complex three-tori, with its evident zero section and the required marking.

□

Lemma 12.4 (Isolated extension points)**.** [tag 0202]

Let R be an open Riemann surface, $V \to R$ a holomorphic vector bundle, and $\mathscr{U} \subset \mathrm{Gr}_r(V)$ an open subset whose projection onto R is surjective. Let $S \subset R$ be finite. Suppose that over $R \setminus S$ this projection is a holomorphic fibre bundle with an LSAP fibre. Let $K \subset R$ be a compact Runge set containing S in its interior. Every continuous section of $\mathscr{U}$ which is holomorphic near K can be deformed through sections to a holomorphic section, with arbitrary uniform approximation on K and interpolation of any given finite jets at finitely many points of its interior. No fibre-bundle or Serre-fibration assertion is required at the points of S.

Proof. ⟨1⟩1. We give the relative version of the LSAP argument in (Alarcón and Forstnerič, 2025, Lemma 5.1, Theorem 6.1 and Proposition 7.13(d)). Enlarge K slightly inside the domain of holomorphicity, retaining the Runge property. Choose a strongly subharmonic Morse exhaustion of R starting with this compact set. All subsequent boundary arcs, attached

discs and critical one-handles can and will be chosen disjoint from S. On them $\mathscr{U}$ is an ordinary holomorphic bundle with LSAP fibre.

⟨1⟩2. The global spray on the old compact set can be taken to be vertical even though the fibres at S differ. Indeed, let f be the current holomorphic section on a Stein neighborhood W of that set. Put A equal to the finite interpolation set, including S, and choose an ideal sheaf $\mathcal{I}$ vanishing to one more than the prescribed jet orders at its points. Cartan's theorem A supplies finitely many holomorphic sections $T_1, \dots, T_N$ of $\operatorname{End}(V)|_W \otimes \mathcal{I}$ which generate $\operatorname{End}(V)$ on the compact attaching arcs, after shrinking W if needed. The formula

$$F(z,a) = \exp\left(\sum_{j=1}^{N} a_j T_j(z)\right) f(z) \tag{F.O5}$$

defines a holomorphic spray of sections of the Grassmannian bundle. Its differential in a surjects onto the vertical tangent space at every attaching point, since $\operatorname{End}(V_z) \to \operatorname{Hom}(f(z), V_z/f(z))$ is onto. For a sufficiently small it has range in $\mathscr{U}$ on a neighborhood of the old compact set, by openness and compactness. Every parameter value preserves all the chosen jets.

⟨1⟩3. For a noncritical extension, subdivide the new annular regions into small discs lying in bundle trivializations over $R \setminus S$. Restrict (F.O5) to a thin attaching disc. LSAP of the fibre extends this spray across the attached disc, approximating the old spray as closely as desired on the intersection. The spray gluing used in the proof of (Alarcón and Forstnerič, 2025, Lemma 5.1, equations (5.6)–(5.8)) now applies verbatim to vertical sprays: the inverse-function step is needed only on the attaching intersection, where (F.O5) is dominating. It produces reparametrizations of the two sprays and a glued section. The old part remains in $\mathscr{U}$ because it is evaluated in (F.O5); the new part remains in $\mathscr{U}$ because the LSAP spray takes values in the bundle fibre. The fixed jets survive the reparametrizations, being independent of the spray parameters on the old side.

⟨1⟩4. For a critical extension, first extend over the attached real arc using the given continuous section and the homotopy already constructed on the old compact set. Apply Mergelyan approximation with the same finite jets to a neighborhood of the union of the old set and this arc, and then apply the noncritical step. To see explicitly that this approximation is by sections, trivialize V holomorphically on the open Riemann surface: its underlying complex bundle is trivial over a graph spine, and the frame bundle therefore has a continuous section. Its fibre $GL(\operatorname{rank} V, \mathbb{C})$ has the dominating spray $(g, A) \mapsto g \exp A$, so (Forstnerič and Prezelj, 2002, Theorem 1.2, preprint p. 3) gives a holomorphic frame. The section is then a map to the fixed Grassmannian, to which (Alarcón and Forstnerič, 2025, Theorem 2.13, p. 9) applies. The base coordinate is kept unchanged. Close approximation on the compact arc and old set, followed by shrinking its neighborhood, preserves membership in the open subset $\mathscr{U}$. This is the critical case of Theorem 6.1 of that paper, with no approximation performed at a singular point of the projection: here the projection is a submersion everywhere, as $\mathscr{U}$ is open in a Grassmannian bundle.

⟨1⟩5. Choose successive errors summably small on the preceding compact sets and smaller than their distances to the complement of $\mathscr{U}$. The sections converge uniformly on every compact set to a holomorphic section with range in $\mathscr{U}$. The finite jets persist. The close homotopies on successive stages concatenate in the usual exhaustion construction. All new attachment regions lie outside S; the old section remains holomorphic on a full neighborhood of each point of S at every stage. This proves the lemma.

□

Theorem 12.5 (the bounded-cusp period decision)**.** [tag 0203]

Let $B^* = \mathbb{P}^1 \setminus \{p_0, p_1, \dots, p_k\}$, where $k \geq 1$, and let $\rho : \pi_1(B^*) \to GL^+(6, \mathbb{R})$ be a representation. Assume that the monodromy M_0 at p_0 satisfies $(M_0 - I)^2 = 0$ and $\operatorname{rank}(M_0 - I) = 3$. Then there is a holomorphic period family with this monodromy and nondegenerate real transversality everywhere on B^*, equivalently a nondegenerate imaginary part in each local normalized period chart. In a marking with $M_0 = \left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$ its cusp period is

$$Z(t) = \frac{\log t}{2\pi i} I_3 + C(t), \qquad C(t) \text{ holomorphic at } t = 0. \tag{F.O6}$$

Any prescribed finite jet of C at zero may be imposed. For an integral primitive cusp the marking can be integral. Consequently every admissible datum with three-connected gluing admits such a period family. No good-reduction condition at the other, still punctured, points is included in this assertion.

Proof. ⟨1⟩1. Write $N = M_0 - I$ and choose a real basis with $N = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$. Its integral version is Theorem 4.4. Choose the component of the period domain selected by the positive logarithmic model in (F.O6). This choice matters only when an orientation has been fixed independently of the marking; every element of $GL^+(6, \mathbb{R})$ preserves either component.

⟨1⟩2. Retain the cusp as a point of the *open* surface

$$B = \mathbb{P}^1 \setminus \{p_1, \dots, p_k\}.$$

The flat complex vector bundle on B^* has a holomorphic extension V_D across p_0: in a local coordinate t use the single-valued frame obtained by the change $\exp(-sN)$, where $s = (\log t)/(2\pi i)$. Changing the logarithm by $2\pi i$ is cancelled by M_0. This specifies the extension directly.

⟨1⟩3. Over B^* let $\mathscr{U}$ be the flat bundle of real-transverse planes in the chosen component. At p_0 adjoin only the planes transverse to $\ker N_{\mathbb{C}}$, an affine space of matrices $C \in M_3(\mathbb{C})$. This gives an *open* subset $\mathscr{U} \subset \mathrm{Gr}_3(V_D)$. Indeed, in the chart $F_C = \operatorname{im}\left(\begin{smallmatrix} C \\ I_3 \end{smallmatrix}\right)$ the flat period is $\exp(sN)F_C = F_{sI_3 + C}$. For $t \neq 0$ membership is exactly

$$\det\bigl(y(t) I_3 + \operatorname{Im} C\bigr) > 0, \qquad y(t) = -\frac{\log |t|}{2\pi}. \tag{F.O7}$$

For every fixed C_0, and uniformly for C in a sufficiently small bounded neighborhood of C_0, the matrix in (F.O7) is invertible with positive determinant when $|t|$ is sufficiently small. This follows, for example, from $y(t) > \|\operatorname{Im} C\|_{\mathrm{op}}$ and a straight-line homotopy to $y(t) I_3$. Thus a full neighborhood of $(0, C_0)$ lies in the adjoined set, proving openness. The projection is consequently a holomorphic submersion, although its fibre at p_0 is $\mathbb{C}^9$ rather than D_3.

⟨1⟩4. Start with a polynomial $C_0(t)$ realizing the desired finite jet. Formula (F.O7) makes it a holomorphic section on a small closed cusp disc K, which is Runge in B. This section extends continuously over B. To justify the relative extension, D_3 is connected and simply connected: it retracts onto $SO(6)/U(3)$, and $\pi_1(U(3)) \to \pi_1(SO(6))$ is onto, since a unitary rotation in one complex coordinate is a generator of the real two-plane rotation loop. A cellular decomposition of the surface outside the cusp disc now extends its boundary section over edges by connectedness and over two-cells by simple connectivity. The flat bundle is trivial over each cell. This argument uses no fibration property at p_0.

⟨1⟩5. Over $B \setminus \{p_0\}$ the fibre is the LSAP manifold D_3 from Theorem 12.2. Lemma 12.4, with $S = \{p_0\}$, produces a holomorphic section of $\mathscr{U}$ over all of B, with the chosen jet

at p_0. Since its value at p_0 lies in the affine chart just adjoined, it has a holomorphic matrix $C(t)$ there. Undoing the canonical frame gives exactly (F.O6), with a bounded holomorphic correction. Everywhere else membership in $\mathscr{U}$ is precisely nondegeneracy of the complex-torus periods. More explicitly, choose three members of a real lattice basis whose projected periods are complex-linearly independent, and normalize them to I_3. The remaining three periods form a matrix Z, and real transversality is exactly $\det \operatorname{Im} Z \neq 0$. This normalization persists locally.

⟨1⟩6. For admissible data with three-connected gluing, Theorem 11.1 excludes $k = 3$ and Proposition 11.9 proves orientation preservation of the finite matrices for $k = 2$. The cusp determinant is also one. Thus all three-connected data satisfy the hypotheses.

□

Part III
Shifted fibre products

13 Shifted fibre products of two six-spheres

In this part X and X' are two members of the family used in Part I, and $\varphi \in PGL_2(\mathbb{C})$ is chosen so that the three exceptional values of X' are disjoint from those of X. The fibre product $P = X \times_B \varphi^* X'$ is then a compact complex fivefold. We compute its fundamental group and integral homology, its stable tangent bundle and framed bordism class, its second homotopy module, a circle fibration with nine-dimensional fibre, and an explicit decomposition into ordinary framed handles. These results are not used in Part II.

Theorem 13.1 (a shifted second sphere)**.** [tag 02MN]

Let X' be a second canonical member and choose $\varphi \in \mathrm{PGL}_2(\mathbb{C})$ so that its three exceptional values are disjoint from those of X. The smooth fivefold $P = X \times_B \varphi^* X'$ has Euler characteristic zero and an infinite fundamental group with presentation

$$\begin{aligned} G = \langle a_1, a_2, b_1, b_2, h_1, h_2 \mid\ & h_1, h_2 \text{ central}, \quad a_1^3 = h_1,\ a_2^4 = h_1^{-1}, \\ & b_1^3 = h_2,\ b_2^4 = h_2^{-1}, \quad a_1 a_2 b_1 b_2 = 1 \rangle. \end{aligned} \tag{E.16}$$

Its abelianization is $\mathbb{Z}$, and its rational Betti numbers are

$$(1, 1, 1, 20, 63, 88, 63, 20, 1, 1, 1). \tag{E.17}$$

For fixed X, X', all admissible φ give the same oriented diffeomorphism type.

Proof. ⟨1⟩1. Theorem III.9.3 (p. 317) proves smoothness. At each special value the other factor is a smooth four-torus. The Euler characteristic of every fibre is therefore zero, and Euler additivity gives $\chi(P) = 0$. We apply van Kampen with the cusp relations. Over the six-punctured sphere the fibre group is $\Lambda_X \oplus \Lambda_{X'}$ and the base is free on five peripheral loops. Each monodromy acts on its own direct summand. Filling the first cusp kills its angular generators $\widehat{w}, \widehat{\delta}$ and its cusp meridian. Their normal closure under A_1 also kills $\widehat{u}$, since $A_1 \widehat{w} = \widehat{u} - \widehat{w}$. The same calculation in the other summand leaves only $h_1 = \widehat{\gamma}$ and $h_2 = \widehat{\gamma}'$. The finite monodromies now act trivially on them. The four finite filling relations are precisely those in (E.16), since $\gamma(v_1) = 1$ and $\gamma(v_2) = -1$. Both cusp collars are untwisted; their meridians are killed. The product of the remaining four meridians gives the last relation. These are all relations introduced by the six fillings, so van Kampen gives exactly (E.16).

⟨1⟩2. Quotienting by h_1, h_2 gives the orbifold group of the sphere with cone orders $3, 4, 3, 4$. Its orbifold Euler characteristic is $-5/6$, so its universal orbifold cover is the disc and the group is infinite. Equivalently one can build the Seifert T^2-bundle with these four primitive translation vectors; over the disc it is a trivial torus bundle. Its exact sequence shows that the central subgroup in (E.16) is $\mathbb{Z}^2$. The abelianized five-by-six relation matrix has Smith factors $(1, 1, 1, 1, 1)$, as printed in the code, proving $G_{\mathrm{ab}} = \mathbb{Z}$.

⟨1⟩3. For the rational homology there is no extra cusp kernel: each cusp is the original rank-two cusp times a smooth torus, and its specialization is an isomorphism onto invariants over $\mathbb{Q}$. The two monodromy groups act on independent summands. In exterior degrees $q = 0, \ldots, 8$, the H^0 and H^2 rows are both

$$(1, 2, 3, 4, 5, 4, 3, 2, 1),$$

and the H^1 row is

$$(0, 0, 18, 60, 86, 60, 18, 0, 0).$$

To verify this without a matrix-size guess, the local invariant polynomials of the first factor are $A(z) = 1 + 2z + 2z^2 + 2z^3 + z^4$ at a finite point and $C(z) = A(z) + 2z^2$ at its cusp. Hence the Euler polynomial of the direct-image rows is

$$-4(1+z)^8 + (4A(z) + 2C(z))(1+z)^4.$$

Formula (E.9), with six point strata, gives the displayed rows.

⟨1⟩4. Each invariant algebra of a factor has one generator in degrees zero through four. Its invariant-to-coinvariant map is nonzero in even degrees and zero in odd degrees. In the marking used above, this follows from the invariant form $6\gamma \wedge \delta + u \wedge w$ and its exterior products: the odd invariant classes pair to zero with every invariant complementary class, and the even ones have nonzero square or top pairing. Its four transgressions are all nonzero, as the factor is a sphere. The derivation rule therefore sends a tensor of bidegree (i, j) into bidegrees $(i-1, j)$ and $(i, j-1)$, with the first term nonzero exactly when $i > 0$ and j is even, and the second exactly when $j > 0$ and i is even. The nonzero column supports are disjoint except for pairs in a single row, so their scalar normalizations do not affect ranks. This gives

$$(r_1, \dots, r_8) = (1, 2, 2, 3, 3, 2, 2, 1).$$

The exact program forms these matrices and takes their ranks. There are no later differentials. Their diagonal sums give (E.17), which also satisfies Poincaré duality and Euler zero.

⟨1⟩5. Finally, the admissible set of φ is the complement of finitely many proper complex hypersurfaces in the connected complex manifold $\mathrm{PGL}_2(\mathbb{C})$, so it is path connected. The universal pullback over this set is a proper smooth submersion: transversality was checked pointwise above. Parallel transport along a path gives an oriented diffeomorphism of its fibres.

□

Proposition 13.2 (the shifted product's smooth type)**.** [tag 02M0]

The shifted product in (E.16) *is the closed oriented ten-manifold* $\mathcal{H}_{10}$ *specified by the* 2432 *ordinary framed handles of 13.19. Its regular-fibre complement is, relative to its entire marked boundary, the* 76*-handle six-manifold* $\mathcal{H}_6$ *of that theorem. The full attaching embeddings, their ordered normal coordinates, and every dual belt embedding are prescribed by finite coordinate formulas. The identification follows by an relative diffeomorphism of the cusp, core and finite caps, and then by the handle-by-handle comparison of 13.19 using Milnor's Theorems 1.4, 3.13 and 5.8.*

The smooth type is given by the explicit framed-handle presentation. 13.16's quadratic obstruction is consistent with this presentation.

Proof. ⟨1⟩1. The relative diffeomorphism of proposition 13.18 identifies the cusp, flat core and both finite caps while fixing the full outer torus marking. Apply the ordered block expansion of theorem 13.19 to this comparison: each full attaching embedding, including its normal coordinates, is carried to the displayed coordinate embedding. The compactly supported collar isotopies in that proof extend the comparison across each successive handle, with the protected boundary fixed. Taking its product with the two regular torus factors and gluing by the explicit normal-angle reversal gives the ten-dimensional comparison. The stated counts are the two expanded handle lists of theorem 13.19; every top handle uses its

prescribed attaching map. The remaining assertions are exactly the already established results theorem 2.5, III.9.3 (p. 317) , theorem 13.1, proposition 13.3, proposition 13.16.

□

Proposition 13.3 (the first integral torsion of the shifted product)**.** [tag 02MP]

Let $Z_\varphi = X \times_B \varphi^* X$ be the transverse fivefold in 13.1, with the two sets of three critical values disjoint. Its second integral homology is

$$H_2(Z_\varphi;\mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/12)^7.$$

Proof. ⟨1⟩1. Use subscripts one and two for the two rank-four cohomology local systems. At a critical value of one factor the other factor is a smooth torus bundle over a disc, hence a product there. Each local specialization is therefore the tensor product of the original specialization with the identity on the exterior algebra of $\mathbb{Z}^4$. The original finite reduced surfaces and the original cusp have free cohomology. Their specialization maps are injective; at the cusp they are integral isomorphisms onto invariants. For the latter, apply the support-function construction of 2.2 in two coordinates. It supplies the invariant deck classes b_1, b_2, the three classes $a_1b_1, a_2b_2, a_1b_2 + a_2b_1$, and their products. Together with b_1b_2 these give an integral kernel basis in degree two; their products give integral bases in degrees three and four. Both the central-fibre and invariant lattices have ranks $(1, 2, 4, 2, 1)$, as follows also from the normalization matrix in III.5.2 (p. 32) at $d = 1$. Thus the specialization is onto between free groups of equal ranks in every degree, proving the claimed isomorphism. Thus the only difference between $R^q g_*\mathbb{Z}$ and $\mathcal{J}_q = j_* \bigwedge^q(\mathcal{V}_1 \oplus \mathcal{V}_2)$ is their finite covering cokernel, with no point-supported kernel.

⟨1⟩2. The finite Wang calculation (E.I1), now with the original three-dimensional vertical lattice, gives

$$C_3^1 = \mathbb{Z}/3, \quad C_4^1 = \mathbb{Z}/4, \qquad C_3^2 = 0, \quad C_4^2 = \mathbb{Z}/2.$$

The invariant covector γ generates the first two cokernels, and the invariant two-form q_X generates the last, by the same even-intersection argument as in 2.4. Consequently the degree-two covering cokernel at all four finite values, combined, is

$$C^2 = \Big((\mathbb{Z}/3)^4 \oplus (\mathbb{Z}/4)^4 \oplus \mathbb{Z}/2\Big)^2.$$

The two monodromy groups act independently on the two factors. The three global invariant two-forms have integral basis $q_1, \gamma_1\gamma_2, q_2$. The first and third map onto the respective $\mathbb{Z}/2$ summands and vanish in the other summands. The middle form maps to the primitive γ coordinate of each of the four order-three or order-four blocks. After the two order-two summands have been removed, it therefore generates a direct cyclic summand of order twelve in $(\mathbb{Z}/12)^8$. Hence

$$\operatorname{coker}\{H^0(B, \mathcal{J}_2) \to C^2\} = (\mathbb{Z}/12)^7. \tag{E.20}$$

The program `fourfolds_shifted_integral.py` realizes this as an eighteen-row integer presentation; its Smith form has eleven unit factors and seven factors twelve.

⟨1⟩3. In degree two the global coboundary image is primitive. Thus the parabolic group, a subgroup of the free-group cohomology on the punctured base, is torsion free. Its rank is eighteen: the six local invariant ranks are $(16, 16, 18, 16, 16, 18)$, and its global invariant and coinvariant ranks are both three. The Euler formula on the six-punctured sphere gives $3 + 3 + 4 \cdot 28 - 100 = 18$. These ranks and the primitivity are computed with exact exterior matrices. The sheaf sequence $0 \to R^2 g_*\mathbb{Z} \to \mathcal{J}_2 \to C^2 \to 0$ and (E.20) now give

$$H^1(B, R^2 g_*\mathbb{Z}) = \mathbb{Z}^{18} \oplus (\mathbb{Z}/12)^7.$$

Here the quotient by its finite subgroup is free, so this abelian-group extension splits.

⟨1⟩4. The group $H^2(B, R^1g_*\mathbb{Z})$ is the free rank-two group of coinvariants, one primitive summand from each factor. Pulling back the integral degree-two transgression of the original sphere fibration through each projection kills these two summands separately and surjectively. Thus $E_\infty^{2,1} = 0$ in the Leray sequence of g. Its other middle group $E_\infty^{0,3}$ is free, since $R^3g_*\mathbb{Z}$ injects into $\mathcal{J}_3$. There are no higher differentials. Consequently $\operatorname{Tor} H^3(Z_\varphi;\mathbb{Z}) = (\mathbb{Z}/12)^7$. The universal coefficient theorem identifies this with $\operatorname{Tor} H_2(Z_\varphi;\mathbb{Z})$. Finally 13.1 gives $b_2 = 1$, proving the assertion. This argument does not determine the extensions in the higher integral Leray filtrations.

□

Theorem 13.4 (full integral homology of the shifted product)**.** [tag 02MQ]
For the transverse shifted product P of 13.1, write

$$A = (\mathbb{Z}/12)^7, \qquad B = (\mathbb{Z}/12)^{10} \oplus (\mathbb{Z}/2)^7, \qquad C = (\mathbb{Z}/12)^6 \oplus \mathbb{Z}/24 \oplus (\mathbb{Z}/2)^{17}.$$

Its complete integral homology is

i	$H_i(P;\mathbb{Z})$
$0, 1, 8, 9, 10$	$\mathbb{Z}$
2	$\mathbb{Z} \oplus A$
3	$\mathbb{Z}^{20} \oplus B$
4	$\mathbb{Z}^{63} \oplus C$
5	$\mathbb{Z}^{88} \oplus C$
6	$\mathbb{Z}^{63} \oplus B$
7	$\mathbb{Z}^{20} \oplus A$.

In particular the higher integral filtration extensions are nontrivial: the middle torsion includes a cyclic factor of order 24.

Proof. ⟨1⟩1. *A decomposition with free groups on every piece.* Choose a separating circle in $B = \mathbb{P}^1$ so that one closed disc contains the three critical values of X and the other contains the three critical values of X'. The smooth torus family of the other factor is trivial over each disc. Set

$$F = T^4, \qquad E = X \setminus \operatorname{int}(F \times D^2), \qquad E' = X' \setminus \operatorname{int}(F' \times D^2).$$

The product trivializations give the decomposition

$$P = (E \times F') \cup_{F \times F' \times S^1} (F \times E'). \tag{E.MV1}$$

The boundary identification preserves the two marked torus factors and reverses the normal circle, because the two base discs have opposite boundary orientations. The same regular-disc trivializations define both the boundary identifications of E, E' and (E.MV1).

⟨1⟩2. Since X, X' are six-spheres, the Mayer–Vietoris sequence of $S^6 = E \cup (F \times D^2)$ gives, integrally,

$$H_0(E) = \mathbb{Z}, \qquad H_k(E) = H_{k-1}(F) \quad (1 \le k \le 4), \qquad H_k(E) = 0 \quad (k \ge 5). \tag{E.MV2}$$

The isomorphism is the normal-circle sweep $\alpha \mapsto [S^1] \times \alpha$. Indeed, in degrees one through four the boundary map to $H_k(E) \oplus H_k(F)$ is an isomorphism, and its second component is the identity on the unswept fibre summand and zero on the swept summand. In degree five the boundary fundamental class is the image of the sphere fundamental class, so $H_5(E) = 0$. Thus all homology groups of the two pieces and their common boundary in (E.MV1) are free abelian. The same statements hold for E'.

⟨1⟩3. *Determining every boundary map integrally.* Let $V = H^1(F;\mathbb{Z})$, in the basis (γ, u, w, δ), and use the positively ordered volume $v = \gamma u w \delta$. The primitive global invariant classes of the original torus fibration are

$$I_0 = 1, \quad I_1 = \gamma, \quad I_2 = q = uw + 6\gamma\delta, \quad I_3 = \gamma uw, \quad I_4 = v.$$

The images of $H^k(E) \to H^k(F)$ are exactly

$$n_k\mathbb{Z}I_k, \qquad (n_1, n_2, n_3, n_4) = (12, 2, 2, 12). \tag{E.MV3}$$

We verify this integrally. On the disc the only local restrictions on a global invariant class are the finite-cover pullback lattices; the cusp specialization is an integral isomorphism onto its invariant lattice by 13.3. In degree one the finite images contain 3γ and 4γ, giving the intersection $12\mathbb{Z}\gamma$. In degree two the order-three image contains q, while the order-four image requires an even coefficient of q, by the finite integral lattices computed in the proof of theorem 2.4. In degree three use the perfect integral pairing downstairs: an invariant class is a pullback precisely when its pairings with every pulled-back degree-one class are divisible by the covering degree. The other degree-one generators are $2u + w + 3\delta$ and $u + w + 2\delta$. Their pairings with $t\gamma uw$ are, up to signs, $3t$ and $2t$. Thus the first cover imposes no further condition, and the second requires t even. In degree four the pullback degrees are three and four, giving $12\mathbb{Z}v$.

⟨1⟩4. These local restrictions are also sufficient for the image from E. For a local system on a disc with punctures, the restriction from its free-group H^1 to the direct sum of the local coinvariants is onto: each puncture has its own independent free generator. Consequently $H^2(D, j_*\mathcal{L}) = 0$. Replacing $j_*\mathcal{L}$ by the direct-image sheaf changes it only by point-supported cokernels, and so does not change this vanishing. The Leray sequence over the disc therefore makes $H^k(E) \to H^0(D, R^k f_*\mathbb{Z})$ onto. This proves (E.MV3)

⟨1⟩5. Write θ for the normal-circle covector. The dual of (E.MV2) identifies $H^k(E)$ with $\bigwedge^{k-1} V$, and its boundary restriction has the graph form

$$r_k(\alpha) = \phi_k(\alpha) + \theta\alpha, \qquad \phi_k : \bigwedge^{k-1} V \longrightarrow \bigwedge^{k} V. \tag{E.MV4}$$

The image of ϕ_k is exactly $n_k\mathbb{Z}I_k$ by (E.MV3). If $k + l = 5$, classes extending to E have zero product evaluated on ∂E. Therefore

$$(-1)^k\phi_k(\alpha)\beta + \alpha\phi_l(\beta) = 0 \quad \text{in } \bigwedge^{4} V. \tag{E.MV5}$$

Each I_k is primitive, and wedge pairing between the two complementary exterior powers is unimodular. Writing $\phi_k = f_k I_k$, equation (E.MV5), tested against a class pairing to one with I_k, proves that f_k is an integral multiple of $\alpha \mapsto \langle \alpha I_{5-k}, [F]\rangle$. The exact image (E.MV3) fixes the absolute multiple. Thus

$$\begin{aligned} \phi_k(\alpha) &= c_k\langle \alpha I_{5-k}, [F]\rangle I_k, \\ (c_1, c_2, c_3, c_4) &= (12\varepsilon, 2\eta, -2\eta, -12\varepsilon), \qquad \varepsilon, \eta \in \{1, -1\}. \end{aligned} \tag{E.MV6}$$

This proves every boundary map up to two signs. We allow those signs independently for E and E' below; the answer is the same for all sixteen possibilities.

⟨1⟩6. *The full integral Mayer–Vietoris matrices.* Let $i_p : H_p(F) \to H_p(E)$ be the transpose of ϕ_p in the sweep bases, with $i_0 = 1$; let j_{p+1} denote the sweep isomorphism for $0 \le p \le 3$, and zero for $p = 4$. The analogous maps of the second copy are i'_q, j'_{q+1}. For a fibre tensor $\alpha \otimes \beta$, with degrees p, q, the matrix D_k of (E.MV1) is

$$\alpha \otimes \beta \longmapsto (i_p\alpha \otimes \beta, -\alpha \otimes i'_q\beta), \qquad p + q = k. \tag{E.MV7}$$

For a swept tensor, place the normal circle first. Then

$$\theta \times \alpha \times \beta \longmapsto (j_{p+1}\alpha \otimes \beta, (-1)^p \alpha \otimes j'_{q+1}\beta), \qquad p+q = k-1. \tag{E.MV8}$$

The second sign contains the Koszul sign from moving the circle past α, the reversal of the two boundary-circle orientations, and the second minus in Mayer–Vietoris. The last two cancel. This specifies every integer entry of D_k.

⟨1⟩7. The executable `manifolds_shifted_mv.py` constructs the exterior bases, the four adjoint maps (E.MV6), and the full matrices (E.MV7)–(E.MV8). Exact Smith reduction gives the table below. The source ranks are $\binom{9}{k}$; a zero binomial is understood outside $0 \le k \le 9$.

k	target rank	source rank	rank D_k	unit factors	nonunit factors
0	2	1	1	1	−−
1	10	9	9	9	−−
2	28	36	27	20	$12^{[7]}$
3	64	84	53	36	$2^{[7]}, 12^{[10]}$
4	114	126	82	58	$2^{[17]}, 12^{[6]}, 24$
5	138	126	94	70	$2^{[17]}, 12^{[6]}, 24$
6	104	84	73	56	$2^{[7]}, 12^{[10]}$
7	44	36	35	28	$12^{[7]}$
8	8	9	8	8	−−
9	0	1	0	0	−−
10	0	0	0	0	−−

An exponent $[r]$ records repetition, not an integer power. All sixteen choices of the two signs in each copy give precisely this table.

⟨1⟩8. Finally the exact homology sequence gives

$$0 \longrightarrow \operatorname{coker} D_k \longrightarrow H_k(P) \longrightarrow \ker D_{k-1} \longrightarrow 0.$$

The last group is free, since the boundary homology is free. This sequence therefore splits as abelian groups, so the Smith factors in the table are the torsion of $H_k(P)$. Adding the free kernel ranks gives exactly the rational Betti vector in 13.1 and the integral groups asserted here. The resulting torsion also satisfies the required ten-dimensional duality $\operatorname{Tor} H_k(P) \cong \operatorname{Tor} H_{9-k}(P)$.

□

Proposition 13.5 (a unique transverse family type and its stable tangent bundle)**.** [tag `02MR`]

For fixed marked sphere fibrations, all Möbius transformations whose critical triples are disjoint give orientation-preservingly diffeomorphic shifted products. The same conclusion holds while either sphere varies in its proper smooth marked period family. For every such fivefold P there is an isomorphism of real vector bundles

$$TP \oplus \mathbb{R}^4 \simeq \mathbb{R}^{14}. \tag{E.MV9}$$

Thus all its positive-degree Stiefel–Whitney and Pontryagin classes vanish. For its given complex tangent bundle, $c_1(P) = c_2(P) = 0$ integrally. These assertions and 13.4 do not identify the complete smooth structure from its ordinary homology and fundamental group.

Proof. ⟨1⟩1. The allowed transformations form the complement in $\mathrm{PGL}_2(\mathbb{C})$ of the nine analytic hypersurfaces imposing an equality of one critical value from each triple. This complement is path connected: a piecewise smooth path between two points can be perturbed relative to its endpoints to avoid these sets, which have real codimension two. Equivalently, use local complex coordinate balls and perturb each segment there. Choose a smooth such path.

⟨1⟩2. The corresponding fibre products form a compact smooth family over its parameter interval. Indeed the vertical derivative of $f_1(x) - f_2(y)$ is surjective at every point, since its two factors cannot be critical at the same base value. The same surjectivity solves the equation for a lift of the interval coordinate vector field. A partition of unity supplies a global smooth lift; compactness makes its flow exist for the whole interval. This flow gives the claimed diffeomorphisms. The complex orientations vary continuously and hence are preserved. The identical argument applies to smooth proper period families of either factor, keeping the critical triples disjoint throughout.

⟨1⟩3. View P as the transverse inverse image of the diagonal in $B \times B$ under $f_1 \times f_2$. Its real oriented normal two-plane bundle is the underlying bundle of g^*T_B, where $g : P \to B$ is the common base map. It is trivial: its first Chern class is

$$c_1(g^*T_B) = 2g^*[B]^\vee = 2\,\mathrm{pr}_1^*\, f_1^*[B]^\vee = 0,$$

because $H^2(X;\mathbb{Z}) = 0$, and complex line bundles are classified by their first Chern class. The tangent bundles of the two six-spheres become trivial after adding one real line each. Restriction of their product tangent bundle gives

$$(TP \oplus \mathbb{R}^2) \oplus \mathbb{R}^2 \simeq \mathbb{R}^{14},$$

proving (E.MV9) and the characteristic-class assertions for the real tangent bundle. Finally the holomorphic tangent sequence is

$$0 \longrightarrow TP \longrightarrow \mathrm{pr}_1^* TX \oplus \mathrm{pr}_2^* TX' \longrightarrow g^*T_B \longrightarrow 0.$$

Both sphere factors have integral $c_1 = c_2 = 0$, since their cohomology in degrees two and four vanishes. The last line has zero first Chern class, as just proved. Multiplying total Chern classes in this exact sequence gives $c_1(P) = c_2(P) = 0$ integrally.

□

Proposition 13.6 (the shifted product bounds with its natural stable framing)**.** [tag 02MS]

Equip the two oriented six-spheres with their standard bounding stable tangent framings. Trivialize the oriented two-plane bundle $f_1^*T\mathbb{P}^1$ on the first sphere, and equip the transverse shifted product P with the induced stable framing in (E.MV9)*. Then there is a compact smooth oriented eleven-manifold V with*

$$\partial V = P, \qquad TV \oplus \mathbb{R}^3 \simeq \mathbb{R}^{14}, \qquad [P] = 0 \quad \textit{in } \Omega_{10}^{\mathrm{fr}}. \tag{E.B1}$$

The boundary framing is the stated one. In particular its induced spin structure bounds.

Proof. ⟨1⟩1. Write $f_2 : S^6 \to \mathbb{P}^1$ for the shifted map of the second factor. By III.3.4 (p. 15) the first map $f_1 : S^6 \to \mathbb{P}^1$ is null-homotopic. It therefore extends to a smooth map $F : D^7 \to \mathbb{P}^1$, which can be chosen to equal f_1 on each sphere of a collar of the boundary. To see this with the boundary fixed, first use the null-homotopy to obtain a continuous extension constant in the collar direction. Smooth its interior in coordinate charts, with cutoffs supported away from the collar; on the overlap the extension is already smooth.

⟨1⟩2. Let C_2 be the three critical values of f_2. The restriction f_1 is transverse to each point of C_2, since the two critical triples are disjoint. Perturb F only in the interior to be transverse to all these points. The applicable finite simultaneous transversality statement is (Hirsch, 1976, Chapter 3, Section 2, Theorems 2.1 and 2.5). For the relative assertion, the proof of that theorem uses local perturbations multiplied by cutoff functions. Their supports may be restricted to the complement of the collar, where transversality has not already been secured; the open condition on the collar persists. Compactness permits finitely many such perturbations.

⟨1⟩3. Define the smooth fibre product

$$V = \{(x, y) \in D^7 \times S^6 : F(x) = f_2(y)\}. \tag{E.B2}$$

At a point whose common value is outside C_2, the differential df_2 is onto. At a point over C_2, the differential dF is onto. Consequently $dF - df_2$ is onto everywhere on (E.B2). The same argument holds on its boundary, using df_1 there. The implicit-function theorem, including its boundary version, makes V a smooth eleven-manifold with boundary exactly P. It is compact as a closed subset of $D^7 \times S^6$. The inverse-image orientation using the complex orientation of $T\mathbb{P}^1$ induces the complex orientation of P at the boundary, because the collar keeps the first map fixed.

⟨1⟩4. Let $g : V \to \mathbb{P}^1$ be the common map. The real normal two-plane bundle of (E.B2) is $g^*T\mathbb{P}^1$, identified by $dF - df_2$. This is the pullback from D^7 of $F^*T\mathbb{P}^1$, which is trivial because the disc is contractible. A choice of its oriented framing restricts on the boundary to the chosen framing of $f_1^*T\mathbb{P}^1$ up to homotopy: two oriented framings of this two-plane bundle differ by a map $S^6 \to SO(2)$, and $H^1(S^6; \mathbb{Z}) = 0$. The tangent exact sequence and the bounding stable framing of the second sphere now give

$$TV \oplus \mathbb{R}^2 \simeq T(D^7 \times S^6)|_V,$$
$$TV \oplus \mathbb{R}^3 \simeq \mathbb{R}^7 \oplus (TS^6 \oplus \mathbb{R})|_V \simeq \mathbb{R}^{14}.$$

At the boundary, the outward normal of D^7 supplies the additional real line in $TV|_P = TP \oplus \mathbb{R}$. The resulting framing is precisely the stable tangent framing obtained from the two sphere framings and the normal framing in 13.5. This proves (E.B1) by the definition of framed bordism. It also proves the assertion about the induced spin structure, by forgetting the framing.

□

Theorem 13.7 (framed null-bordism over the fundamental group)**.** [tag 02MT]
Let G be the group in (E.16). There is a closed smooth aspherical four-manifold N with

$$\pi_1(N) = G, \qquad \widetilde{N} \simeq \mathbb{H}^2 \times \mathbb{R}^2, \qquad H_*(N; \mathbb{Z}) \cong H_*(S^1 \times S^3; \mathbb{Z}), \tag{E.B3}$$

and a smooth classifying map $c : P \to N$ inducing the displayed identification of fundamental groups. With the natural stable framing of 13.6, one has

$$[P, c] = 0 \quad \text{in } \Omega_{10}^{\mathrm{fr}}(N). \tag{E.B4}$$

Thus P has a compact stably framed eleven-dimensional filling over BG. This is stronger than the filling of 13.6.

Proof. ⟨1⟩1. *The aspherical target.* The quotient maps of I.22.4 (p. 298) are smooth maps $e_i : S^6 \to S^3$ over the base, and III.3.4 (p. 15) gives the exact factorizations $f_i = h_i e_i$. Here $h_i : S^3 \to \mathbb{P}^1$ is the weighted Hopf map of orders $3, 4$, with the second map shifted by the chosen Möbius transformation. Define

$$N = \{(u, v) \in S^3 \times S^3 : h_1(u) = h_2(v)\}, \qquad c(x, y) = (e_1(x), e_2(y)). \tag{E.B5}$$

The critical values of each h_i are only its two exceptional circle values. They are disjoint, so the same transversality argument as in III.9.3 (p. 317) makes N a compact smooth four-manifold. The two circle actions on the weighted Hopf spheres give a principal T^2 bundle in the orbifold sense over

$$\mathcal{B} = S^2(3, 4, 3, 4).$$

At a cone of order m, its uniformizing chart is $(D^2 \times T^2)/C_m$, with the generator rotating the disc and translating one torus coordinate by a primitive fraction of denominator m. This local description is smooth and is the same finite-filling description as I.22.3 (p. 297) , with the unused circle kept.

⟨1⟩2. The orbifold has Euler characteristic $2 - 2(1 - 1/3) - 2(1 - 1/4) = -5/6$. It is good and has hyperbolic orbifold universal cover, by (Scott, 1983, Theorems 2.3–2.4 and the Euler-number discussion in Section 2). Pulling the principal torus bundle to $\mathbb{H}^2$ gives an ordinary principal T^2 bundle over a contractible base, hence the trivial bundle. Its universal cover is $\mathbb{H}^2 \times \mathbb{R}^2$. The local chart above shows that these orbifold pullbacks are ordinary covering maps of total spaces. This proves asphericity and injectivity of the fibre subgroup $\mathbb{Z}^2$.

⟨1⟩3. Using the marked fibre generators h_1, h_2 and the four positive peripheral generators gives exactly

$$\pi_1(N) = \langle a_1, a_2, b_1, b_2, h_1, h_2 \mid h_1, h_2 \text{ central}, \quad a_1^3 = h_1, \quad a_2^4 = h_1^{-1}, \\ b_1^3 = h_2, \quad b_2^4 = h_2^{-1}, \quad a_1 a_2 b_1 b_2 = 1 \rangle.$$

The map c sends the surviving regular-fibre generators $\widehat{\gamma}, \widehat{\gamma}'$ to h_1, h_2, respectively, and each peripheral generator to the one with the same name. This is because e_i is the literal circle quotient on these fibres and finite collars. Its cusp replacement extends over a solid torus and kills precisely the cusp meridian in this presentation. Comparing with the full van Kampen calculation in 13.1 therefore shows that c_* is the identity on the displayed presentation, and in particular is an isomorphism. As N is aspherical, c is a classifying map.

⟨1⟩4. For completeness the abelian relation matrix has Smith diagonal $(1, 1, 1, 1, 1)$; a primitive integer kernel vector in the order $(a_1, a_2, b_1, b_2, h_1, h_2)$ is $(4, -3, -4, 3, 12, -12)$. Thus $H_1(N) = \mathbb{Z}$. Every stratum of its base has a torus fibre, so Euler additivity gives $\chi(N) = 0$. Integral Poincaré duality and the universal coefficient theorem give $H_3(N) = \mathbb{Z}$ and no torsion in $H_2(N)$. Euler characteristic then forces $H_2(N) = 0$, proving the integral homology assertion in (E.B3).

⟨1⟩5. *The bordism computation before taking the inverse image.* For a based sphere, the stable splitting $(S^3)_+ \simeq S^0 \vee S^3$ identifies

$$\Omega_6^{\mathrm{fr}}(S^3) = \pi_6^{\mathrm{S}} \oplus \pi_3^{\mathrm{S}}. \tag{E.B6}$$

We use Pontryagin–Thom in its form for maps to a space, (Davis and Kirk, 2001, Definition 8.16 and Theorem 8.17). The first component of $[S^6, e_i]$ is obtained by forgetting the map, so it is zero for the bounding sphere framing. The second component is the stable class of e_i, equivalently the framed inverse image of a regular point. It is η^3 by III.3.3 (p. 14) . One can also check the splitting directly: the Pontryagin–Thom collapse of the framed sphere into its disjoint-basepoint suspension has degree one on the sphere summand; composing with e_i gives its stable homotopy class. Thus

$$[S^6, e_i] = (0, \eta^3).$$

Smashing the two sphere splittings gives

$$(S^3 \times S^3)_+ \simeq S^0 \vee S^3 \vee S^3 \vee S^6.$$

Under this splitting the external product $[S^6 \times S^6, e_1 \times e_2]$ has four components. The first three each contain a factor $[S^6] = 0$; the last is $\eta^3\eta^3 = \eta^6 = 0$. Indeed η^4 belongs to $\pi_4^{\mathrm{S}} = 0$, as recorded in (Isaksen et al., 2023, Corollary 1.3 and Table 1, pp. 112–113). Therefore

$$[S^6 \times S^6, e_1 \times e_2] = 0 \quad \text{in } \Omega_{12}^{\mathrm{fr}}(S^3 \times S^3). \tag{E.B7}$$

⟨1⟩6. *The comparison with the classifying map.* By (E.B7) there is a compact stably framed thirteen-manifold Z with boundary $S^6 \times S^6$ and a map $E : Z \to S^3 \times S^3$ extending $e_1 \times e_2$. Smooth it relative to the boundary and make it transverse to N there. Transversality on the boundary already holds: the derivative of its defining equation is $df_1 - df_2$, which is onto. Relative transversality therefore supplies the interior perturbation without changing that boundary.

⟨1⟩7. The normal bundle of $N \subset S^3 \times S^3$ is the restriction of $h_1^* T\mathbb{P}^1$ pulled back from the first sphere. This oriented two-plane bundle is trivial because $H^2(S^3; \mathbb{Z}) = 0$. Choose a framing on it before restriction to N. Its pullback under $e_1 \times e_2$ is a framing of $f_1^* T\mathbb{P}^1$, agreeing with that in 13.6 up to homotopy since $H^1(S^6; \mathbb{Z}) = 0$. It follows that

$$V = E^{-1}(N), \qquad E|_V : V \longrightarrow N$$

is a compact stably framed eleven-manifold with boundary (P, c) and the required boundary framing. This proves (E.B4).

⟨1⟩8. The program `shifted_group_bordism.py` does the group-matrix and Euler calculations. □

Proposition 13.8 (the fibre complement and a spinning obstruction)**.** [tag `02MA`]

Let $F = T^4$ be a regular fibre of the original sphere fibration and $E = S^6 \setminus \operatorname{int}(F \times D^2)$, with the marking of 13.4. Its group and peripheral homomorphism are

$$\begin{aligned} K = \pi_1(E) = \langle a, b, h \mid h \text{ central},\ a^3 = h,\ b^4 = h^{-1} \rangle, \\ g \longmapsto h, \qquad u, w, d \longmapsto 1, \qquad \mu \longmapsto ab. \end{aligned} \tag{E.S1}$$

Here μ is the normal meridian, with the peripheral orientation chosen so that $[a] = 4[\mu]$, $[b] = -3[\mu]$ and $[h] = 12[\mu]$. The image of the boundary group is the embedded copy $\langle h, \mu \rangle \simeq \mathbb{Z}^2$. In the unique homotopy class of normal framings for which every pushed-off loop of F has linking number zero, the image of g is

$$\lambda = h\mu^{-12}, \qquad \ker\{\pi_1(F) \longrightarrow K\} = \mathbb{Z}\langle u, w, d \rangle. \tag{E.S2}$$

In particular the peripheral structure is that of the $(3, 4)$ torus knot, with its regular Seifert fibre rather than its preferred longitude as the original g coordinate. Nevertheless the embedded four-torus $F \subset S^6$ is not the ordinary threefold closed spin of that classical knot, even allowing ambient orientation reversal and arbitrary changes of its normal framing. More strongly, it is not a single ordinary closed spin of any embedded $T^3 \subset S^5$.

Proof. ⟨1⟩1. Apply the van Kampen presentation in III.3.10 (p. 23) with the regular base disc removed. The cusp kills w, d and its peripheral loop. Conjugating w by the order-three generator also kills u. The two finite caps impose $a^3 = h$ and $b^4 = h^{-1}$; there is now no relation $ab = 1$, since the outer boundary has not been filled. The outer meridian is ab in the stated orientation. This proves (E.S1), including the peripheral map. Eliminating h gives $\langle a, b \mid a^3 b^4 = 1 \rangle$. Its quotient by h is $C_3 * C_4$. The word ab has infinite order in that free product, so a relation $h^r \mu^s = 1$ forces $s = 0$. Abelianization then forces $r = 0$. This proves peripheral injectivity. It also shows that λ has infinite order: its image in $C_3 * C_4$ is $(ab)^{-12}$.

⟨1⟩2. Write $\theta \in H^1(\partial E; \mathbb{Z})$ for the normal-circle covector and retain (γ, u, w, δ) for $H^1(F; \mathbb{Z})$. The generator $\xi \in H^1(E; \mathbb{Z})$ restricts to $\theta + 12\gamma$. Homotopy classes of normal framings differ by $H^1(F; \mathbb{Z})$. Consequently there is a unique framing for which the associated section $s : F \to \partial E$ satisfies $s^* \xi = 0$: it has $s^* \theta = -12\gamma$. Its induced peripheral map is exactly (E.S2). Simultaneously reversing the meridian conventions changes the displayed signs and preserves all the following assertions.

⟨1⟩3. For this section set

$$\mathcal{R} = \operatorname{im}\{H^2(E;\mathbb{Z}) \longrightarrow H^2(F;\mathbb{Z})\}, \qquad \operatorname{rad}(\mathcal{R}) = \{v \in H_1(F;\mathbb{Z}) : \iota_v\omega = 0 \text{ for all } \omega \in \mathcal{R}\}. \tag{E.S3}$$

This uses the canonical identification of the cohomology of a torus with the exterior algebra of its first cohomology. Equations (E.MV4)–(E.MV6) give classes whose boundary restrictions are θu and θw: their unswept terms are zero because $u\gamma uw = w\gamma uw = 0$. Restricting them along s proves

$$-12\gamma u,\ -12\gamma w\ \in \mathcal{R}. \tag{E.S4}$$

The simultaneous contraction kernel of these two forms is $\mathbb{Z}d$. Indeed their contractions force successively $\gamma(v) = u(v) = w(v) = 0$. Thus

$$\operatorname{rad}(\mathcal{R}) \subseteq \ker\gamma = \ker\{\pi_1(F) \longrightarrow K\}. \tag{E.S5}$$

Both the zero-linking framing and this inclusion are invariants of the embedded pair, independently of the torus coordinates.

⟨1⟩4. We define the comparison construction precisely. For a closed submanifold $L \subset \operatorname{int} D^n$, its ordinary closed spin is $L \times S^1$ in $D^n \times S^1 \subset \partial(D^n \times D^2) = S^{n+1}$. Start with the closed classical knot $K_0 \subset \operatorname{int} D^3$ and repeat this operation three times, each time placing the previous compact pair in the interior of a disc. This is the ordinary torus spinning construction reviewed in (Winter, 2008, Section 3, immediately before Theorem 3.4); no turning, twisting, or family of deformations is inserted. If Σ is a Seifert surface for K_0, the same three spins give the embedded Seifert hypersurface $\Sigma \times T^3$ with boundary $K_0 \times T^3$. Pushing the latter off using the normal of this hypersurface gives its zero-linking framing.

⟨1⟩5. Let v be the original knot-circle class and let v_1, v_2, v_3 be the spinning circles. Each pushed-off two-cycle $v \wedge v_i$ bounds the corresponding copy of $\Sigma \times S^1$ in a parallel copy of that hypersurface, disjoint from the knot. Hence every restriction of a complement two-class vanishes on $v \wedge H_1(F)$: v belongs to $\operatorname{rad}(\mathcal{R})$. On the other hand its pushed-off loop is the original preferred longitude and is nontrivial for $K_0 = T(3,4)$, whereas the spinning circles are trivial in the complement group. These peripheral claims also follow directly at each spin: its complement is

$$((D^n \setminus \operatorname{int}\nu L) \times S^1)\ \cup_{S^{n-1}\times S^1}\ (S^{n-1} \times D^2),$$

so van Kampen kills precisely the new spinning circle and preserves the old complement group. They agree with the primary peripheral calculation (Winter, 2008, Theorem 4.5) in the first spin. Thus the ordinary spun torus violates (E.S5). An identification of embedded pairs would preserve its two sides, giving a contradiction.

⟨1⟩6. For the stronger assertion, let $L = T^3 \subset S^5$ be arbitrary and suppose that its ordinary closed spin were our four-torus. Write E_L for its exterior. Deleting a ball disjoint from L changes cohomology only in degrees at least four. In the displayed spun-complement decomposition, with $n = 5$, the cap and intersection are $S^4 \times D^2$ and $S^4 \times S^1$. Mayer–Vietoris and the surjectivity onto the latter first cohomology therefore give, in degrees two and three, precisely the cohomology of $(E_L \setminus D^5) \times S^1$. Let s be the spinning-circle covector and v_s its primitive homology generator. Put $\mathcal{R}^j_L = \operatorname{im}\{H^j(E_L) \to H^j(L)\}$, using the zero-linking framing. One has $\mathcal{R}^1_L = 0$. Also $\mathcal{R}^3_L = 0$: the pushed-off L bounds a parallel Seifert hypersurface in its exterior, so every top-degree restriction evaluates to zero. The product restrictions consequently satisfy

$$\mathcal{R}^2 = \mathcal{R}^2_L, \qquad \mathcal{R}^3 = \mathcal{R}^2_L \wedge s. \tag{E.S12}$$

In particular $v_s \in \mathrm{rad}(\mathcal{R}^2)$, and contraction by v_s is injective on $\mathcal{R}^3$. For our four-torus, (E.S4) forces $v_s = \pm d$. But (E.MV4)–(E.MV6), now applied to $\alpha = \gamma\delta$ in degree three, show that $\mathcal{R}^3$ contains $\pm 2\gamma uw$. Its swept term vanishes in the zero-linking framing, and this is a nonzero class annihilated by d. This contradicts that injectivity. No ordinary closed spin is possible. Turned or deform-spun constructions insert additional data and are not excluded by this argument.

□

Theorem 13.9 (exact second-homotopy coinvariants)**.** [tag 02MB]
For the classifying map $c : P \to N = K(G,1)$ of 13.7, one has

$$c_* : H_3(P;\mathbb{Z}) \longrightarrow H_3(N;\mathbb{Z}) = \mathbb{Z} \quad \text{equal to zero,} \qquad \big(\pi_2(P)\big)_G \simeq \mathbb{Z}^2 \oplus (\mathbb{Z}/12)^6. \tag{E.S6}$$

The pullback of a generator of $H^3(N;\mathbb{Z})$ has exact order twelve in $H^3(P;\mathbb{Z})$. In particular $\pi_2(P)$ is not a projective $\mathbb{Z}G$-module. The fivefold cannot be diffeomorphic, or homotopy equivalent, to $N \times S^6$ connected-summed with any finite collection of $S^r \times S^{10-r}$, $2 \le r \le 8$, and any homotopy ten-sphere.

Proof. ⟨1⟩1. Use the separating base circle in 13.4 also in the target of 13.7. There M denotes the exterior of a regular orbit of the weighted Hopf map, namely the torus-knot exterior of I.22.3 (p. 297) . The two decompositions, and the restriction of c, are

$$\begin{array}{ccc} (E \times F') \cup (F \times E') & \longrightarrow & (M \times S^1) \cup (S^1 \times M') = N, \\ F \times F' \times S^1 & \longrightarrow & S^1 \times S^1 \times S^1. \end{array} \tag{E.S7}$$

On the common boundary the map retains exactly the covectors $\gamma_1, \gamma_2, \theta$. It is the literal quotient there. Each knot exterior has the integral homology of a circle, so the third homology and cohomology of each target piece in (E.S7) vanish.

⟨1⟩2. We first prove the homological statement. In the degree-two source of (E.MV7)–(E.MV8) write t_1, t_2, t_{12} for the coefficients of $\theta \times g_1$, $\theta \times g_2$, and $g_1 \times g_2$. The image of $i_2 : H_2(F) \to H_2(E)$ lies entirely in the d-sweep coordinate, by (E.MV6). Consequently the g_1-sweep coordinate in the first $H_2(E)$ summand of the target is just t_1, and the corresponding second coordinate is just t_2. Every vector in the kernel of the boundary matrix therefore has $t_1 = t_2 = 0$. Its first $H_1(E) \otimes \mathbb{Z}g_2$ coordinate then forces $12t_{12} = 0$, up to an immaterial orientation sign. These three coordinates are the only ones retained in $H_2(T^9) \to H_2(T^3)$. The target Mayer–Vietoris sequence injects $H_3(N)$ into $H_2(T^3)$; its source-piece third groups are zero. Naturality now proves $c_* H_3(P) = 0$.

⟨1⟩3. The cohomological calculation determines the extension. On the target boundary write $v = \gamma_1\gamma_2$. The generators of the two degree-two piece cohomology groups restrict, with consistent meridian orientations, to

$$\theta\gamma_2 + 12v, \qquad \theta\gamma_1 + 12v. \tag{E.S8}$$

Their image is a primitive rank-two subgroup of $H^2(T^3)$: the minor in the last two coordinates is a unit. Its quotient is freely generated by v. Since the degree-three piece groups are zero, the connecting class ∂v is a generator of $H^3(N;\mathbb{Z})$.

⟨1⟩4. In the source decomposition both $\theta\gamma_1$ and $\theta\gamma_2$ already belong to the image of the degree-two piece restrictions. This is (E.MV4) with $\alpha = \gamma$, since $\phi_2(\gamma) = 0$; the second normal circle is reversed, which only changes its sign. Subtracting one of these classes from the corresponding product of a degree-one class of E with γ_2 proves that $12v$ is a source boundary image. Conversely the coefficient of v in every such boundary image is divisible

by twelve. The pure degree-two terms from a single factor have no v coefficient, and the mixed terms use the restriction $\theta + 12\varepsilon\gamma$ of the degree-one generator, with $\varepsilon = \pm 1$. It follows that

$$c^*(\partial v) = \partial v \quad \text{has exact order 12.} \tag{E.S9}$$

This uses the Mayer–Vietoris image.

⟨1⟩5. Put $A = (\mathbb{Z}/12)^7$ and $L = (\pi_2(P))_G$. The low-degree homology sequence of the universal-cover fibration, together with 13.4 and 13.7, gives

$$0 \longrightarrow \mathbb{Z}z \longrightarrow L \longrightarrow \mathbb{Z} \oplus A \longrightarrow 0. \tag{E.S10}$$

The zero on the left uses $c_* H_3(P) = 0$, just proved. To determine this extension, replace c by its mapping cylinder and use relative groups for (N, P). The induced map on fundamental groups is an isomorphism and the target has zero second homotopy group. On universal covers the relative pair is two-connected; relative Hurewicz gives $H_3(\widetilde{N}, \widetilde{P}) = \pi_2(P)$ and all lower relative groups vanish. The relative covering homology sequence therefore identifies $H_3(N, P) = L$ and $H_2(N, P) = 0$. The universal coefficient theorem and the relative cohomology sequence identify the map

$$H^3(N, P) = \mathrm{Hom}(L, \mathbb{Z}) \longrightarrow H^3(N) = \mathbb{Z}$$

with evaluation on z. Its image is $12\mathbb{Z}$ by (E.S9).

⟨1⟩6. The free summand of the quotient in (E.S10) splits. Lift its seven order-twelve generators to $x_1, \dots, x_7$ in the remaining extension. There are integers e_i, defined modulo twelve, with the complete abelian presentation

$$\langle z, x_1, \dots, x_7 \mid 12x_i = e_i z \ (1 \le i \le 7) \rangle. \tag{E.S11}$$

Evaluation of homomorphisms to $\mathbb{Z}$ on z has image $(12/\gcd(12, e_1, \dots, e_7))\mathbb{Z}$. Thus that gcd is one. The relation matrix in (E.S11) has Smith factors $1, 12, 12, 12, 12, 12, 12$, giving $\mathbb{Z} \oplus (\mathbb{Z}/12)^6$. Adding the split free summand proves (E.S6).

⟨1⟩7. If $\pi_2(P)$ were projective over $\mathbb{Z}G$, it would be a direct summand of a free module. Taking coinvariants preserves that splitting, so L would be a projective, hence free, abelian group. The torsion in (E.S6) contradicts this. Finally $N \times S^6$ has zero second homotopy group. Connected sum in dimension ten with a simply connected sphere product adds its second homotopy as a free induced $\mathbb{Z}G$-module, as seen by lifting the connected-sum necks and using their simply connected boundary spheres. Only $r = 2, 8$ contribute, with one free generator each; a homotopy sphere contributes none. Their second homotopy modules are therefore free, so none is homotopy equivalent to P.

⟨1⟩8. The program `maps_shifted_pi2.py` checks the simultaneous contraction kernel, the three homological boundary coordinates, and the exact-order-twelve image certificate for all sixteen independent boundary-adjoint sign choices in 13.4.

□

Theorem 13.10 (central localizations of the homotopy module)**.** [tag `02MC`]

Take the central generators h_1, h_2 of G from 13.7 and put

$$R = \mathbb{Z}G, \qquad R_i = R[(h_i - 1)^{-1}], \qquad R_{12} = R[(h_1 - 1)^{-1}, (h_2 - 1)^{-1}]. \tag{E.S13}$$

These are flat central localizations. As modules over the indicated localized group rings, the second homotopy group satisfies

$$\begin{aligned} R_1 \otimes_R \pi_2(P) &\simeq R_1/(h_2 - 1), \\ R_2 \otimes_R \pi_2(P) &\simeq R_2/(h_1 - 1), \\ R_{12} \otimes_R \pi_2(P) &= 0. \end{aligned} \tag{E.S14}$$

More generally, $H_j(P;R_1)$ is zero outside $2 \le j \le 7$; in those six degrees it is respectively a direct sum of

$$1,\ 3,\ 4,\ 4,\ 3,\ 1 \tag{E.S15}$$

copies of $R_1/(h_2-1)$. The analogous statement holds with the indices interchanged. All homology with coefficients in R_{12} vanishes. For the fibre complement of 13.8, with $S = \mathbb{Z}K[(h-1)^{-1}]$, one likewise has

$$S \otimes_{\mathbb{Z}K} \pi_2(E) \simeq S, \qquad H_4(E;S) \simeq S, \tag{E.S16}$$

and its other localized homology groups vanish. Moreover, if $q = (h_1-1)(h_2-1)$, then

$$q^2\pi_2(P) = 0, \qquad \dim_{\mathbb{Q}}(\pi_2(P) \otimes \mathbb{Q}) = \infty. \tag{E.S20}$$

Consequently the universal cover has no finite homotopy type. There is no fibration of P over N with compact manifold fibre and inducing an isomorphism on fundamental groups.

Proof. ⟨1⟩1. *The cusp module with its marked deck transformation.* In the original cusp, the angular generators w, d vanish and $\pi_1(N_0) = \mathbb{Z}\langle g, u\rangle$. In E their images are respectively $h, 1$; in the first factor of P they are $h_1, 1$. Thus the connected cover of this cusp occurring inside the universal cover is the cover $\widehat{W}$ of its central fibre in which the g deck transformation is z, whereas u remains a loop. Write $A = \mathbb{Z}[z, z^{-1}]$. Its ordinary homology modules are

$$H_0(\widehat{W}) = H_1(\widehat{W}) = A/(z-1), \quad H_2(\widehat{W}) = A \oplus (A/(z-1))^2, \quad H_3(\widehat{W}) = 0, \quad H_4(\widehat{W}) = A. \tag{E.S17}$$

Here is an integral normalization calculation of this input. The dual complex of the cover is an infinite cylinder, whose ordinary homology is $A/(z-1)$ in degrees zero and one and zero in degree two. The normalization degree-four row is a free copy of A. In degree two, the homology map from the three double curves to the four component classes is the transpose, up to the unimodular intersection marking and possible inversion of z, of

$$B(z) = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & z \\ -z & -z & -1 & -z \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z-1 & 0 & 0 \\ 0 & 0 & z-1 & 0 \end{pmatrix}. \tag{E.S18}$$

For example multiply on the left by $\left(\begin{smallmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -z & 0 & 1 \end{smallmatrix}\right)$, then use as new columns $e_1, e_4 - e_1, e_3 - e_1, e_2 - e_1$. Both operations are unimodular over A. The transpose is injective and its cokernel is the displayed degree-two module. All strata have only even homology, and the normalization direction has length two, so no higher differential is possible. There is no degree-two extension from the dual complex, whose second homology is zero. This proves (E.S17). Ordinary chains in this locally finite normalization have compact support and use direct sums. The matrix computation is done by the program `manifolds_cusp_alexander.py`.

⟨1⟩2. *Localization and the vanishing of every other piece.* The central subgroup $\langle h_1, h_2\rangle$ is an injected $\mathbb{Z}^2$ by 13.7. The group ring is free over its central Laurent subring, by choosing coset representatives. In particular $h_i - 1$ and their products are non-zero-divisors. Inverting them is central localization and is flat; it commutes with homology of the universal-cover chain complex. The analogous assertions hold for h in K.

⟨1⟩3. Use the regular torus bundle over a graph and the six closed cap neighborhoods in 13.1. For R_1 coefficients, the first regular torus has its g_1 loop acting by h_1. The torus chain complex is the Koszul complex of its commuting loop actions. Its $h_1 - 1$ term is invertible, providing a chain contraction. Thus every regular fibre has zero homology with these coefficients. The homology sequence of a bundle over a graph, and the same calculation over each boundary circle, show that the whole regular piece and every cap boundary are acyclic. Each cap of the second factor is a product with a first-factor regular torus over a disc, so it too is acyclic.

⟨1⟩4. At a finite cap of the first factor, the affine generator a_j has $a_j^{m_j} = h_1^{\pm 1}$ in its image in G. The connected cover of the reduced flat surface corresponding to its map onto $\langle a_j \rangle$ is $T^3 \times \mathbb{R}$. The m_j-th power acts by an integral translation with linear part $A_j^{m_j} = I$, and therefore acts identically on all of its homology. Thus $h_1 - 1$ annihilates every such homology module before localization. Keeping the other regular torus does not change that assertion, and induction to G commutes with the central action. Both first-factor finite caps are therefore acyclic after localization as well.

⟨1⟩5. *The one surviving cap and induction.* The first cusp is $N_{0,1} \times F_2$, since the two critical triples are disjoint. Its image in G is precisely $H = \langle h_1, h_2 \rangle$. A connected component of its preimage in the universal cover is

$$\widehat{W} \times \mathbb{R} \times T^3$$

up to deformation retraction: g_1 acts on $\widehat{W}$ by $z = h_1$, and $g_2 = h_2$ translates the real factor. It acts trivially on the homology of the retained three-torus. All components together give induction from $\mathbb{Z}H$ to R. By (E.S17), inverting $h_1 - 1$ leaves one free $\mathbb{Z}[h_1^{\pm 1}, (h_1 - 1)^{-1}]$ summand in cusp degrees two and four and nothing else. The integral product homology with T^3 has no torsion correction. Inducing the result gives

$$H_j(N_{0,1} \times F_2; R_1) \simeq (R_1/(h_2 - 1))^{\binom{3}{j-2} + \binom{3}{j-4}}, \tag{E.S19}$$

where a binomial coefficient is zero outside its usual range. All other pieces and their intersections are acyclic, as proved above, so Mayer–Vietoris identifies this with $H_j(P; R_1)$. This proves (E.S15) and (E.S19), and interchanging the factors proves the second version. Inverting $h_2 - 1$ also kills (E.S19), proving the double-localization assertion. Since $\widetilde{P}$ is simply connected, Hurewicz identifies its second homology with $\pi_2(P)$; flat localization now gives (E.S14).

⟨1⟩6. For E the same decomposition has only one cusp and two finite caps. Every other piece is acyclic after $h - 1$ is inverted by the identical argument. Its cusp image is $\langle h \rangle$, so inducing (E.S17) after localization gives one free S summand in degrees two and four. This proves (E.S16).

⟨1⟩7. *A uniform annihilator without localization.* Put $I = (h_1 - 1, h_2 - 1) \subset R$. Before localization, the homology of a regular eight-torus with R coefficients is

$$(R/I) \otimes_{\mathbb{Z}} \bigwedge^{j} \mathbb{Z}^6 \quad (0 \le j \le 6). \tag{E.S21}$$

Indeed the two nontrivial loop images are the independent central generators h_1, h_2. Their differences from one form a regular sequence in the central Laurent ring, and R is free over that ring. The two-variable Koszul complex therefore has only its degree-zero quotient as homology; the other six loop differentials are zero. This proves (E.S21). The homology sequence of the regular bundle over a graph has only base degrees zero and one. Both filtration quotients are annihilated by I, by (E.S21), so I^2 annihilates the homology of the regular piece. The same two-stage Wang argument applies to every boundary mapping torus.

⟨1⟩8. At a finite cap its connected lift is a product of six torus directions and two real directions up to deformation retraction. Each central deck generator acts trivially on homology: the one belonging to that cap is the finite affine power already checked above, and the other translates the regular real direction. Hence I annihilates its induced homology. At the first cusp, h_2 translates the real factor of the other regular torus, so $h_2 - 1$ annihilates all of its homology; the second cusp is annihilated by $h_1 - 1$. Thus q annihilates the homology of every piece and every intersection, since $q \in I^2$. The decomposition has a regular piece

and disjoint caps with disjoint boundary collars, so its Mayer–Vietoris sequence gives in every degree

$$0 \longrightarrow \operatorname{coker} D_j \longrightarrow H_j(P;R) \longrightarrow \ker D_{j-1} \longrightarrow 0.$$

Both end groups are killed by q, and therefore the middle group is killed by q^2. Since $H_2(P;R) = \pi_2(P)$, this proves the first assertion of (E.S20) integrally.

⟨1⟩9. *Infinite rational rank.* If $\pi_2(P) \otimes \mathbb{Q}$ were finite dimensional, localizing its linear automorphism h_1 at $h_1 - 1$ would remain finite dimensional. Explicitly, this removes the generalized eigenspace with eigenvalue one and inverts an automorphism on the remaining finite-dimensional quotient. But (E.S14), tensored with $\mathbb{Q}$, contains a copy of $\mathbb{Q}[h_1^{\pm 1}, (h_1 - 1)^{-1}]$, as follows by taking the identity coset in the free module over the central Laurent subring and setting $h_2 = 1$. That vector space is infinite dimensional, a contradiction. This proves the second assertion and rules out a finite homotopy type of the universal cover. A fibration over the aspherical N inducing a fundamental-group isomorphism would have simply connected fibre, and pulling it to $\widetilde{N}$ would identify that fibre up to homotopy with $\widetilde{P}$. A compact manifold fibre has finite homotopy type, so such a fibration is impossible.

⟨1⟩10. The quotient $R_1/(h_2 - 1)$ is nonzero, since the two central generators are independent, and it is annihilated by the non-zero-divisor $h_2 - 1$. A projective R_1-module is a direct summand of a free module and has no such nonzero annihilated element. Thus (E.S14) also gives an independent proof of the nonprojectivity in 13.9.

□

Theorem 13.11 (a circle fibration and infinite-order return)**.** [tag 02MD]
The shifted fivefold admits a smooth fibre bundle

$$Y^9 \longrightarrow P \xrightarrow{\rho} S^1 \tag{E.S22}$$

with connected closed fibre, inducing the primitive character

$$\chi(a_1, a_2, b_1, b_2, h_1, h_2) = (4, -3, -4, 3, 12, -12) \tag{E.S23}$$

on the presentation of 13.7. Its fibre group is the fundamental group of the oriented principal circle bundle of Euler number -1 over a surface of genus six. Here the base has its complex orientation and the positive circle is $h_1 h_2$; reversing that circle changes the Euler number to $+1$. In particular

$$\pi_1(Y) = \ker \chi, \qquad H_1(Y;\mathbb{Z}) = \mathbb{Z}^{12}. \tag{E.S24}$$

The return diffeomorphism of (E.S22) has infinite order even among unbased homotopy classes of self-equivalences of Y, although its induced outer automorphism of $\pi_1(Y)$ has exact order twelve. No connected finite cover of P is homotopy equivalent to a product $S^1 \times Z$.

Proof. ⟨1⟩1. *A closed one-form with a nonzero derivative at every point.* Use the smooth map $c = (e_1, e_2) : P \to N$ in 13.7, not an arbitrary representative of its classifying homotopy class. The manifold N has the locally free T^2 action induced by the two weighted Hopf actions. Take a closed one-form representing the primitive integral class χ on N and average it over this compact torus. Its average α is closed, represents the same integral class, and is invariant. If V_i is the fundamental vector field with time-one orbit h_i, Cartan's formula gives $d(\alpha(V_i)) = 0$. Integrating along an ordinary orbit therefore gives

$$\alpha(V_1) = 12, \qquad \alpha(V_2) = -12. \tag{E.S25}$$

The integrality of the periods integrates α to a smooth map $\rho_N : N \to \mathbb{R}/\mathbb{Z}$.

⟨1⟩2. Choose the two cusp replacements defining e_i in disjoint base discs, themselves disjoint from the other factor's critical values. At every point of P, at least one factor is a regular torus at which its e_i is the literal character quotient of I.22.1 (p. 295) . In a local flat marking of that factor, the constant vertical vector g_i maps to V_i. Holding the other factor fixed gives a tangent vector to P, and

$$(c^*\alpha)(g_i) = 12 \quad \text{or} \quad -12.$$

Thus $c^*\alpha$ is nowhere zero. The map $\rho = \rho_N c$ is a proper submersion. Equivalently, choose a vector field W with $(c^*\alpha)(W) = 1$; its complete flow supplies local trivializations and a return diffeomorphism. The fibre is connected because χ is primitive: the image in $\pi_1(S^1)$ of a connected component of a bundle's total space has index equal to the number of fibre components. The homotopy sequence now proves $\pi_1(Y) = \ker \chi$.

⟨1⟩3. *The exact surface cover and its circle Euler class.* Write $\Gamma = G/\langle h_1, h_2\rangle$ for the oriented orbifold group of $S^2(3,4,3,4)$. The character modulo twelve descends to the surjection

$$\overline{\chi} : \Gamma \longrightarrow \mathbb{Z}/12, \qquad (a_1, a_2, b_1, b_2) \longmapsto (4, -3, -4, 3). \tag{E.S26}$$

Its restriction to every cone group is injective. Hence its kernel Γ_0 is torsion free and defines a degree-twelve ordinary surface cover of the orbifold. Its Euler characteristic is $12(-5/6) = -10$, so that surface is Σ_6. The image of $\ker \chi$ in Γ is exactly Γ_0: a lift of an element of Γ_0 has character a multiple of twelve, which can be cancelled by a power of h_1. Its central kernel is the primitive subgroup

$$\ker \chi \cap \langle h_1, h_2\rangle = \mathbb{Z}\langle h_1 h_2\rangle. \tag{E.S27}$$

⟨1⟩4. The circle Euler number can be fixed without an ambiguous Seifert sign convention. On either weighted Hopf sphere put $z_1 = \cos r\, e^{i\varphi_1}$ and $z_2 = \sin r\, e^{i\varphi_2}$. For the positive action with weights $(4,3)$ the smooth invariant connection is

$$\vartheta = \tfrac{1}{4}\cos^2 r\, d\varphi_1 + \tfrac{1}{3}\sin^2 r\, d\varphi_2.$$

It evaluates to one on the generator with period 2π. With the boundary orientation of the unit ball in $\mathbb{C}^2$,

$$\frac{1}{(2\pi)^2}\int_{S^3} \vartheta \wedge d\vartheta = \frac{1}{12}. \tag{E.S28}$$

For the associated complex line whose unit circle is this positive circle bundle, the convention agreeing with the tautological Hopf line is $c_1 = -[d\vartheta/(2\pi)]$. Thus each orbifold circle Euler number is $-1/12$. Both shifted base identifications preserve the complex orientation. Pullback to Σ_6 multiplies the two Euler numbers by twelve, giving $(-1,-1)$; this is also the covering formula of (Scott, 1983, Theorem 3.6). In the integral fibre basis

$$v = (1,1), \qquad w = (0,1)$$

the determinant is one, the Euler vector is $-v$, and $\chi(v) = 0$, $\chi(w) = -12$. Consequently the circle kernel has Euler number -1 with generator $h_1 h_2$.

⟨1⟩5. One can see this bundle directly in N. Invariance of α implies $\rho_N((s_1,s_2)x) = \rho_N(x) + 12(s_1 - s_2)$, where $s_i \in \mathbb{R}/\mathbb{Z}$. A level contains twelve diagonal circles over an ordinary base point. The peripheral permutations (E.S26) give exactly the connected cover Σ_6; they have no stabilizers at the cone points. The diagonal circle acts freely, since at every cone only one of the two Hopf actions has a nontrivial stabilizer. The level is therefore the circle bundle just computed. This identifies its fundamental group with $\ker \chi$ and gives a three-dimensional aspherical model. The homology sequence of a circle bundle of Euler number ± 1 gives its first homology $\mathbb{Z}^{12}$, proving the second assertion in (E.S24).

⟨1⟩6. *Faithful central action and the return obstruction.* The centre of G is exactly $\langle h_1, h_2\rangle$: the quotient is a nonelementary oriented hyperbolic orbifold group with trivial centre. This centre acts faithfully on $\pi_2(P)$. Indeed if $h_1^a h_2^b$ acts as the identity, localize first at $h_1 - 1$ and apply 13.10. On $R_1/(h_2 - 1)$ its action is multiplication by h_1^a. This module is free over the central ring $\mathbb{Z}[h_1^{\pm 1}, (h_1 - 1)^{-1}]$ after choosing coset representatives, so this is the identity only if $a = 0$. The symmetric localization then gives $b = 0$.

⟨1⟩7. If $H < G$ has finite index, its image in Γ is again nonelementary. Its centralizer in Γ is trivial: commuting with two hyperbolic elements with distinct axes fixes at least three boundary points and hence is the identity. It follows that $Z(H) = H \cap Z(G)$. A homotopy product $S^1 \times Z$ for the corresponding finite cover would supply an infinite central element acting trivially on π_2, namely the circle factor; this contradicts the faithful action just proved. If a positive power of the return of (E.S22) were homotopic to the identity, its mapping torus would be homotopy equivalent to $S^1 \times Y$. That mapping torus is the corresponding finite cyclic cover of P, so this too is impossible.

⟨1⟩8. Finally take $t = a_1 a_2$, for which $\chi(t) = 1$, as a stable letter. The image of $\ker \chi$ in Γ is Γ_0, so the same centralizer argument gives $C_G(\ker \chi) = Z(G)$. The automorphism $\mathrm{Ad}(t^n)|_{\ker \chi}$ is inner precisely when t^n differs from an element of $\ker \chi$ by a central element. Since $\chi(Z(G)) = 12\mathbb{Z}$, this holds precisely for $12 \mid n$. Thus the outer action has order twelve, while the return has infinite homotopy order. This return is the circle fibration of the shifted ten-manifold; it is distinct from the six-sphere parameter returns computed in IV.2.54 (p. 102) .

⟨1⟩9. The integral character, cone orders, primitive kernel basis and Euler calculation are computed by the program `maps_shifted_circle.py`.

□

Theorem 13.12 (the cyclic-cover splice and its separating torus twist)**.** [tag 02ME]

Let $E_{12} \to E$ be the connected cyclic cover defined by reduction modulo twelve of the positive meridian character. The circle fibration of 13.11 may be chosen with fibre

$$Y = (E_{12} \times T^3) \cup_{T^8} (T^3 \times E_{12}), \tag{E.S29}$$

with the following exact boundary marking. Write θ_i' for the lifted normal circle and retain the two regular-torus markings. Then

$$\theta_1' = \gamma_2 - \gamma_1, \qquad \theta_2' = \gamma_1 - \gamma_2. \tag{E.S30}$$

There is a return representative ψ whose twelfth power is the separating torus twist

$$\begin{gathered} T(x, s) = (x + \eta(s)(g_1 + g_2), s) \quad \text{on } T^8 \times [-1, 1], \\ T = \mathrm{id} \ \text{ elsewhere.} \end{gathered} \tag{E.S31}$$

Here η is zero near -1 and one near 1, and circle coordinates have period one. In particular (E.S31) is the identity near both ends of its collar. Its rational cohomological variation satisfies

$$(T^* - I)^2 = 0, \qquad \operatorname{rank}(T^* - I)|_{H^j(Y;\mathbb{Q})} = \begin{cases} 1 & j = 4, 5, \\ 0 & j \neq 4, 5. \end{cases} \tag{E.S32}$$

The rational Betti numbers of the fibre are

$$(b_0, \dots, b_9)(Y) = (1, 12, 121, 407, 693, 693, 407, 121, 12, 1). \tag{E.S33}$$

Every eigenvalue of ψ^* is a twelfth root of unity. At eigenvalue one there is exactly one Jordan block of size two in each of degrees four and five; all its other blocks, and all blocks at other eigenvalues, have size one. Thus the infinite order is already visible on ordinary rational cohomology.

Proof. ⟨1⟩1. *Adapting the fibration to the splice.* Cut the base along the circle used in 13.4, with the two critical triples on opposite sides. Over the first disc the second factor is a regular marked T^4, and over the second disc the first factor is regular. Let $\xi_E : E \to \mathbb{R}/\mathbb{Z}$ represent the positive meridian class. Its restriction to the boundary can be chosen exactly

$$\xi_E = \theta + 12\gamma. \tag{E.S34}$$

Indeed this is its integral restriction by 13.8; homotopy extension makes the representative equal to that linear map on a collar. The two maps

$$\rho_A = \xi_{E_1} - 12\gamma_2, \qquad \rho_B = 12\gamma_1 - \xi_{E_2} \tag{E.S35}$$

agree exactly on the common boundary, where $\theta_2 = -\theta_1$. Smooth them with this common product collar. The regular-factor circle derivative is -12 on the first piece and 12 on the second, so they give a proper submersion throughout, including both cusp pieces. These are also permissible choices in theorem 13.11: on the quotient N, an invariant closed one-form differs from these representatives only by the differential of a single-valued function on each of the two pieces. Absorb that function by translation in the unused regular circle. Thus this construction uses the same quotient and primitive character.

⟨1⟩2. A level of $\xi_E - 12\gamma_2$ is the pullback of the degree-twelve circle cover along ξ_E, times the three remaining regular circles. The pullback is E_{12}. Its boundary is connected because the normal meridian has character one. The covering of that boundary has $\theta = 12\theta'$ and unchanged γ, u, w, δ. Equation (E.S34) in the level condition gives precisely (E.S30). This proves (E.S29) with its smooth boundary maps.

⟨1⟩3. Choose the vector field integrating (E.S35) to be $-g_2/12$ on the first piece off its collar and $g_1/12$ on the second. On the common regular collar interpolate by

$$W_s = \frac{\eta(s)g_1 - (1-\eta(s))g_2}{12}. \tag{E.S36}$$

It has $d\rho(W_s) = 1$, and it preserves the collar coordinate s and the original base point. Its flow therefore defines a return ψ. At time twelve it is the identity off the collar, since the two indicated regular circles have period one. On the collar it translates by $\eta(s)g_1 - (1-\eta(s))g_2$; removing the integral translation $-g_2$ gives (E.S31). This comparison uses a globally regular factor on each piece.

⟨1⟩4. *The variation map and the invariant boundary calculation.* Put $D = T^8$ and let $r : H^j(Y;\mathbb{Q}) \to H^j(D;\mathbb{Q})$ be restriction. Denote the Mayer–Vietoris connecting map by $\partial : H^{j-1}(D;\mathbb{Q}) \to H^j(Y;\mathbb{Q})$. Writing $v = g_1 + g_2$, the collar pullback of a translation invariant form is

$$T^*\omega = \omega + d\eta \wedge \iota_v\omega.$$

Average its restriction to the torus and change the global representative by an exact form to make this equality applicable on the collar. The difference represents the connecting class, and hence, with the connecting sign chosen by the increasing collar coordinate,

$$T^* - I = \partial\, \iota_v\, r. \tag{E.S37}$$

As $r\partial = 0$, its square is zero.

⟨1⟩5. The computation of (E.S37) needs no further information on $H^*(E_{12})$. Its cyclic deck action is a translation on the boundary torus, hence acts trivially on boundary cohomology. Every noninvariant deck eigenspace therefore restricts to zero. Rational transfer identifies the invariant part with $H^*(E)$, whose marked restriction is 13.4. Its pullback to (E.S30) is

$$R_{k,1}(a) = \phi_k(a) + 12(\gamma_2 - \gamma_1)a, \qquad R_{k,2}(b) = \phi_k(b) + 12(\gamma_1 - \gamma_2)b. \tag{E.S38}$$

Retain the other factor's three covectors u, w, δ, and take exterior products to obtain the two subspaces $U_j^k \subset \bigwedge^k \mathbb{Q}^8$. The image of r is exactly $U_1^k \cap U_2^k$, and the kernel of ∂ is $U_1^{k-1} + U_2^{k-1}$. Consequently the rank in (E.S32) is exactly the rank of

$$U_1^k \cap U_2^k \xrightarrow{\iota_v} \bigwedge^{k-1} \mathbb{Q}^8/(U_1^{k-1} + U_2^{k-1}). \tag{E.S39}$$

⟨1⟩6. Here are the full dimensions for this small rational calculation. Let d_k be twice the degree-k dimension of $H^*(E) \otimes H^*(T^3)$ and let $r_k = \dim(U_1^k + U_2^k)$:

k	0	1	2	3	4	5	6	7	8
d_k	2	8	20	44	70	68	36	8	0
r_k	1	8	19	34	47	46	27	8	0

(E.S40)

Contraction in (E.S39) has rank one only for $k = 4, 5$ and is zero in every other degree. All entries are computed by exterior products of (E.S38) in `maps_shifted_return.py`. One can normalize the remaining sign in ϕ_2 by the ring structure: with $r_1(1) = \theta + 12\gamma$, the equality $r_1(1)r_2(\delta) = r_3(a)$ forces the coefficient $c_2 = -2$, and then $c_3 = 2$, in the notation of 13.4.

⟨1⟩7. *The finite-cover cohomology needed for the full Betti numbers.* The full rational cohomology of E_{12} is computed as follows. Let $\mathcal{L}_\lambda$ be the rank-one system on E whose meridian acts by λ, with $\lambda^{12} = 1$. On a regular torus it is trivial. Its three base transports in vertical degree q are

$$\lambda^4 \bigwedge^q A_1^{-t}, \qquad \lambda^{-3} \bigwedge^q A_2^{-t}, \qquad \bigwedge^q M_0^{-t}. \tag{E.S41}$$

The exponents carry the affine meridian character.

⟨1⟩8. At a finite cap, averaging on its finite root cover identifies the cap cohomology with the invariant part of (E.S41). At the unchanged cusp the system is trivial, because g has character twelve and u has character zero. Its rational specialization is an isomorphism onto the monodromy invariants in every degree. The cusp Betti numbers are $(1, 2, 4, 2, 1)$ by its normalization calculation. In degree two, the integral surjectivity of the specialization on homology (proof of proposition 13.3) gives injectivity on cohomology; the monodromy-invariant subspace has the same dimension four. Its image therefore includes uw and $\gamma\delta$. In degree one the image is $\langle \gamma, u \rangle$; their products with those two degree-two classes give the two independent invariant classes in degree three and the top class in degree four. Thus all the specialization maps have the asserted ranks, with no additional skyscraper summand. The proof of proposition 13.3 gives these specialization assertions integrally in all degrees.

⟨1⟩9. It follows that $R^q f_* \mathcal{L}_\lambda = j_* L_q$ over the disc containing the three critical values. On this disc $H^p(j_* L_q) = 0$ for $p \geq 2$. To check this directly, retract the punctured disc to a free graph on three loops. Restriction of a graph cocycle to the three local coinvariant spaces is onto: its values on the three free generators can be chosen independently. The exact restriction sequence therefore kills $H^2(j_* L_q)$. If $d_q = \binom{4}{q}$, g_q is the common fixed dimension of (E.S41), and $k_{i,q}$ are their three individual fixed dimensions, this same graph calculation gives

$$\dim H^0(j_* L_q) = g_q, \qquad \dim H^1(j_* L_q) = 2d_q + g_q - \sum_i k_{i,q}. \tag{E.S42}$$

The Leray sequence has only these two base degrees and degenerates. Exact exterior-matrix

ranks give the following table; a row applies to each character of the indicated order:

$\mathrm{ord}(\lambda)$	b_0	b_1	b_2	b_3	b_4	b_5
1	1	1	4	6	4	0
2	0	0	4	6	4	0
3	0	0	3	4	3	0
4	0	0	3	4	3	0
6	0	1	5	6	5	1
12	0	1	4	4	4	1

(E.S43)

Here $b_j = \dim H^j(E;\mathcal{L}_\lambda)$ and $b_6 = 0$. For the nontrivial characters all g_q are zero; for the trivial one they are all one. The program `maps_shifted_cover.py` computes (E.S41)–(E.S43) over the exact cyclotomic fields, represented by rational companion matrices. It prints every local fixed dimension. Summing the rows with multiplicities $1, 1, 2, 2, 2, 4$ gives

$$(b_0, \dots, b_6)(E_{12}) = (1, 7, 46, 56, 46, 6, 0). \tag{E.S44}$$

Both sides have Euler characteristic 24, as required by the degree-twelve covering of E.

⟨1⟩10. *The Betti and Jordan calculations on the fibre.* The return on each of the two pieces of (E.S29), up to isotopy allowing its boundary to move, is its cyclic deck transformation times the identity of T^3. This follows also by moving the interpolation in (E.S36) entirely into the other piece's collar. It acts trivially on the cohomology of D. The generalized eigenvalue-one part of the Mayer–Vietoris sequence therefore uses just the invariant part $H^*(E)$ in (E.S40). Its dimensions are

$$(1, 0, 1, 19, 45, 45, 19, 1, 0, 1). \tag{E.S45}$$

Every other eigenspace is the direct sum of the two piece eigenspaces, because the boundary has none. In particular those eigenspaces are semisimple. Their combined Poincaré polynomial is

$$2(1+t)^3(6t + 42t^2 + 50t^3 + 42t^4 + 6t^5), \tag{E.S46}$$

by (E.S43)–(E.S44). Adding (E.S45) proves (E.S33).

⟨1⟩11. Finally the Wang sequence for $P = M_\psi$ and the rational Betti numbers in 13.5 give the numbers of Jordan blocks at eigenvalue one as

$$(1, 0, 1, 19, 44, 44, 19, 1, 0, 1). \tag{E.S47}$$

The only possible block sizes are one and two: the twelfth power is (E.S31), and (E.S37) squares to zero. Subtracting (E.S47) from (E.S45) gives exactly one block of size two in each of degrees four and five. This also independently checks the direct variation ranks (E.S39). All other eigenvalues and blocks are already described by (E.S43) and (E.S46).

□

Theorem 13.13 (integral fibre groups and the coefficient of the torus twist)**.** [tag `02MF`]

The groups $H^*(E_{12};\mathbb{Z})$ are torsion free, with the ranks in (E.S44). For the fibre Y, the only torsion in cohomology is

$$H^3(Y;\mathbb{Z})_{\mathrm{tors}} = H^7(Y;\mathbb{Z})_{\mathrm{tors}} = (\mathbb{Z}/6)^2. \tag{E.S48}$$

Equivalently, its integral homology has the ranks (E.S33) and torsion $(\mathbb{Z}/6)^2$ exactly in degrees two and six. The twelfth return $T = \psi^{12}$ acts as the identity on all integral cohomology except in degrees four and five. In each of those two free groups it has, in an integral basis, the form

$$\begin{pmatrix} 1 & 72 \\ 0 & 1 \end{pmatrix} \oplus I_{691}. \tag{E.S49}$$

For $n \geq 1$, let P_{12n} be the connected cyclic cover of P of degree $12n$ defined by χ. Its Betti numbers, independent of n, are

$$(1, 13, 133, 528, 1099, 1384, 1099, 528, 133, 13, 1). \tag{E.S50}$$

Its integral homology torsion is exactly

$$H_j(P_{12n}; \mathbb{Z})_{\mathrm{tors}} = \begin{cases} (\mathbb{Z}/6)^2, & j = 2, 3, 6, 7, \\ \mathbb{Z}/(72n), & j = 4, 5, \\ 0, & \textit{otherwise.} \end{cases} \tag{E.S51}$$

In particular these covers are pairwise nonhomotopy-equivalent.

Proof. ⟨1⟩1. *The integral cohomology of the cover.* Apply the proof of (E.S42) with the coefficient lattice $\mathbb{Z}[C_{12}]$, using its regular permutation matrix P_{12}. In vertical degree q set

$$\begin{gathered} T_{1,q} = P_{12}^{4} \otimes \bigwedge^{q} A_1^{-t}, \quad T_{2,q} = P_{12}^{-3} \otimes \bigwedge^{q} A_2^{-t}, \\ T_{0,q} = I_{12} \otimes \bigwedge^{q} M_0^{-t}, V_q = \mathbb{Z}^{12\binom{4}{q}}. \end{gathered} \tag{E.S52}$$

This coefficient system computes $H^*(E_{12}; \mathbb{Z})$. At the first finite cap the cover has four components, each the ordinary degree-three torus root cover; at the second there are three ordinary degree-four torus root covers. An invariant vector is determined by its value on one sheet of such an orbit, with the other values its integral monodromy translates. This identifies cap cohomology with the full primitive invariant lattice. The cusp has twelve unchanged components and the integral specialization of the proof of proposition 13.3. Consequently its direct-image sheaf is exactly the integral j_*L_q, including at every critical point.

⟨1⟩2. The three graph-cocycle values again surject onto the three local coinvariant groups, integrally. Thus $H^2(j_*L_q) = 0$, and its remaining cohomology is the kernel and cokernel of the exact lattice map

$$\begin{gathered} V_q \longrightarrow \bigoplus_{i=0}^{2} \operatorname{im}(T_{i,q} - I), \\ v \longmapsto ((T_{i,q} - I)v)_i. \end{gathered} \tag{E.S53}$$

Use Hermite column bases for the three image lattices. The source and target ranks, and the rank of (E.S53), are respectively

q	0	1	2	3	4
$\operatorname{rank} V_q$	12	48	72	48	12
$\operatorname{rank} \bigoplus \operatorname{im}(T_{i,q} - I)$	17	92	126	92	17
rank(E.S53)	11	47	71	47	11

(E.S54)

Every nonzero Smith entry is one. Hence $H^0(j_*L_q)$ has rank one and $H^1(j_*L_q)$ is free with ranks $(6, 45, 55, 45, 6)$. The two-column Leray filtration has free end groups and therefore splits as abelian groups. This proves the torsion-free assertion and (E.S44) integrally. The full integer matrices and the exact Hermite/Smith check are in `maps_shifted_cover_integral.py`.

⟨1⟩3. *The integral images.* Each piece $A = E_{12} \times T^3$ in (E.S29) has free integral homology. By Poincaré–Lefschetz duality, $H^{k+1}(A, \partial A; \mathbb{Z})$ is free. The cokernel of $H^k(A; \mathbb{Z}) \to H^k(\partial A; \mathbb{Z})$ embeds in that free group, so the boundary image is a primitive lattice. Its rational span has already been computed in (E.S38). Therefore the integral image is precisely the saturation of that displayed image in $\bigwedge^k \mathbb{Z}^8$.

⟨1⟩4. Choose saturated column bases $\overline{A}_k, \overline{B}_k$ for the two images. The combined matrix $[\overline{A}_k\ \overline{B}_k]$ has the ranks in the second row of (E.S40). All its nonzero Smith entries are one except that, in each of degrees two and six, exactly two entries are six. The Mayer–Vietoris exact sequence has a free kernel term on its right, since all piece cohomology is free. Thus its abelian extension splits, and these Smith entries give exactly (E.S48). The free ranks remain (E.S33).

⟨1⟩5. *The integral variation, including possible torsion components.* Formula (E.S37) holds integrally. One can use the singular prism of the loop of translations instead of differential forms: the action map $S^1 \times T^8 \to T^8$ has cohomology formula

$$a \longmapsto 1 \otimes a + [S^1]^* \otimes \iota_v a.$$

The difference between the two collar homotopies is exactly this loop, so its connecting class is $\partial \iota_v r(a)$. This proves the integral formula also on torsion and still gives $(T^* - I)^2 = 0$.

⟨1⟩6. For complete lattice control, take an integral kernel basis of $[\overline{A}_k\ -\overline{B}_k]$. Its first block, multiplied by $\overline{A}_k$, is a basis of the common boundary lattice. Contract it by v and reduce modulo $[\overline{A}_{k-1}\ \overline{B}_{k-1}]$ using the integral Smith transforms. In degrees four and five the result has rank one and unique nonzero Smith entry 72. In degree three its two possible components modulo six are both zero; in degree seven the common boundary image is zero. All the remaining reductions are zero. These computations are performed, with the unimodular transforms verified, in `maps_shifted_fibre_integral.py`. In degrees four and five the connecting cokernel is free and embeds as a primitive summand of $H^k(Y)$, because the following Mayer–Vietoris kernel is free. Thus 72 is the divisibility in the whole cohomology group, not just in a chosen sublattice.

⟨1⟩7. A rank-one integral map of Smith entry 72 has the form $72u\ell$, where u and ℓ are primitive. Its square is zero precisely when $\ell(u) = 0$. Extend u to a basis of $\ker \ell$ and choose a vector on which ℓ is one. This gives (E.S49). The same calculation proves identity on every other integral cohomology group, including its torsion.

⟨1⟩8. *The finite-cover groups and the two potentially nontrivial extensions.* The cover P_{12n} is the mapping torus of T^n. Since $(T^* - I)^2 = 0$, its variation is $n(T^* - I)$, with Smith entry $72n$ in degrees four and five. For every k the integral Wang sequence gives

$$0 \longrightarrow \operatorname{coker}(T^{n*} - I)_{k-1} \longrightarrow H^k(P_{12n}; \mathbb{Z}) \longrightarrow \ker(T^{n*} - I)_k \longrightarrow 0. \tag{E.S55}$$

The ranks immediately give (E.S50). Whenever the right group is free, (E.S55) splits as an abelian sequence. In particular degrees four and eight have torsion $(\mathbb{Z}/6)^2$, contributed by $H^3(Y)$ and $H^7(Y)$ respectively, while degrees five and six have torsion $\mathbb{Z}/(72n)$. Every other such degree has no torsion.

⟨1⟩9. The only right groups in (E.S55) that are not free occur in degrees three and seven. Instead use the nondegenerate integral torsion linking pairing of the closed oriented ten-manifold P_{12n}: cohomological torsion in degree k is Pontryagin dual to that in degree $11 - k$. The already fixed torsion in degree eight therefore forces the degree-three torsion to be $(\mathbb{Z}/6)^2$, and the fixed degree-four torsion forces the same answer in degree seven. The free ranks are already known. This resolves precisely the two possibly absorbing extensions. The universal coefficient theorem now translates these cohomology groups to (E.S51). The order $72n$ in the middle groups distinguishes the covers for distinct n.

□

Theorem 13.14 (the full complement module and its peripheral map)**.** [tag `02MG`]

Let $K = \pi_1(E)$ have the marking (E.S1), and put

$$R = \mathbb{Z}K, \qquad Q = K/\langle h\rangle = C_3 * C_4, \qquad \overline{R} = \mathbb{Z}Q.$$

Use right deck modules; conversion to left modules applies the involution $g \mapsto g^{-1}$ to every coefficient. Then

$$\pi_2(E) \simeq R \oplus \overline{R}^{\,2} \oplus L, \qquad L = \langle x, t \mid t(a-1) = t(b-1) = 0,\ x(1+b) + 3t = 0\rangle_{\overline{R}}. \tag{E.C1}$$

Equivalently, replacing x by $x + t$ and eliminating t gives

$$L \simeq \overline{R}/((b^2-1)\overline{R} + (1+b)(a-1)\overline{R}). \tag{E.C2}$$

There is a nonsplit exact sequence

$$0 \longrightarrow \mathbb{Z}\langle t\rangle \longrightarrow L \longrightarrow \mathbb{Z}_- \otimes_{\mathbb{Z}C_4} \overline{R} \longrightarrow 0, \tag{E.C3}$$

where b acts as -1 on $\mathbb{Z}_-$ and the first term has trivial Q action. Its extension class is the nonzero element of an order-two extension group. In the connected lift of the peripheral T^5 to the universal cover of E, the degree-two classes uw, ud, wd map respectively to

$$t, \quad 0, \quad 0. \tag{E.C4}$$

Thus the central torsion summand of the module is annihilated by $h-1$ itself. Formula (E.C1) is a full integral module calculation, including the central value $h = 1$. There is also a chain-level peripheral comparison. For the splitting constructed below, let e generate the free R summand and let y, z generate the two copies of $\overline{R}$. The projection of the peripheral degree-two chain map to (E.C1) is

cell	gu	gw	gd	uw	ud	wd
value	0	0	$6t$	t	0	0

(E.C19a)

and, writing Θ for the outer base circle with group image $\mu = ab$, use the cone convention: the notation $v\Theta$ denotes the suspended cell $\Theta \times v$, with the interval factor first. In this convention,

$g\Theta$	$eh - 2x - 6yb + z(5-2b)$
$u\Theta$	$x - yb + zb$
$w\Theta$	$-y(1+b) + z$
$d\Theta$	$x + 2t + z(b-1)$.

(E.C19b)

These are values on chains, not assertions that all the displayed cells are cycles in the cover. In particular the free coefficient h on $g\Theta$ is nonzero even though the free summand receives zero from the lifted peripheral homology in (E.C4).

Proof. ⟨1⟩1. We first give the local chain comparison; the homology maps alone do not determine (E.C1).

⟨1⟩2. *The cusp boundary and its derived attachment.* Write $A = \mathbb{Z}[h, h^{-1}]$ and $q = h - 1$. Let $\widehat{W}$ be the gamma cover used in 13.10, and let $\widehat{M}$ be the corresponding boundary cover. Ordinary chains with the deck action give finite free A chain models. The homology of $\widehat{W}$ is

$$H_0 = H_1 = A/(q), \quad H_2 = A \oplus (A/(q))^2, \quad H_3 = 0, \quad H_4 = A. \tag{E.C5}$$

For the boundary, start with the torus chain complex $C = A \otimes \bigwedge\langle g, u, w, d\rangle$, whose differential contracts g and multiplies by q. The cellular monodromy has degree-one matrix

$$F(M_0)g = g - hd, \quad F(M_0)u = u + w, \quad F(M_0)w = w, \quad F(M_0)d = d. \tag{E.C6}$$

Its higher matrices are exterior powers. The mapping-torus complex is the cone of $I-F(M_0)$, with differential

$$d(c,e) = \big(dc + (I - F(M_0))e, -de\big). \tag{E.C7}$$

For example $d(g\theta) = hd - q\theta$ and $d(u\theta) = -w$. Cancelling these two unit pairs leaves the degree-one classes u and θ, with resolutions q and a unit times q^2. In degree two the four surviving classes are $uw, ud, w\theta, d\theta$, each with resolution q. In particular

$$H_0(\widehat{M}) = A/(q), \quad H_1(\widehat{M}) = A/(q^2) \oplus A/(q), \quad H_2(\widehat{M}) = (A/(q))^4. \tag{E.C8}$$

Geometrically this is the Heisenberg three-manifold times a circle: the deck transformation shears the meridian by the d circle. The homology map to (E.C5) is the identity in degree zero, projection to u in degree one, and the primitive projection $(uw, ud, w\theta, d\theta) \mapsto (U, D, 0, 0)$ in degree two. The last assertion uses the integral specialization of the proof of proposition 13.3 and the collapsed meridian tori.

⟨1⟩3. The extension is determined by a relative intersection calculation. Let S_i denote the normalized hexagon component with gamma index i. In its standard Picard basis H, E_1, E_2, E_3, the six boundary curves are

$$E_1,\ H - E_1 - E_2,\ E_2,\ H - E_2 - E_3,\ E_3,\ H - E_1 - E_3. \tag{E.C9}$$

The first opposite pair is self-glued in the surviving u direction; the other two pairs join S_i to its two gamma neighbours. Put $C_0 = -H + E_1$, the primitive generator of the free kernel of the normalization matrix $B(h)$ in 13.10. The divisor S_i restricts on its own normalization to $2C_0$: the total central divisor is principal, and the two self-gluing preimages cancel their terms in the normal restriction. It restricts to $-C_0$ on each neighbouring component and to zero on every other component. Thus the free part of the equivariant intersection map is

$$H_4(\widehat{W}) = A \longrightarrow H_4(\widehat{W}, \widehat{M}) \longrightarrow A, \qquad 2 - h - h^{-1} = -h^{-1}q^2. \tag{E.C10}$$

Poincaré–Lefschetz duality over A, with its usual involution, gives the same coefficient, up to a unit, for the free part of $H_2(\widehat{W}) \to H_2(\widehat{W}, \widehat{M})$. The latter relative group is A: the universal-coefficient sequence in degree four has $\mathrm{Hom}_A(H_4, A) = A$, zero H_3, and no higher extension terms. Indeed every module in (E.C5) has a projective resolution of length at most one. The relative sequence therefore includes

$$0 \longrightarrow A \xrightarrow{\text{unit } q^2} A \longrightarrow A/(q^2) \longrightarrow 0. \tag{E.C11}$$

Consequently the boundary map has a unit component in $\mathrm{Ext}^1_A(A/(q^2), A) = A/(q^2)$. Every unit of $A/(q^2) = \mathbb{Z}[\varepsilon]/(\varepsilon^2)$ lifts to a unit $\pm h^n$ of A.

⟨1⟩4. This determines the low-degree chain map, as follows. All the modules in (E.C5) and (E.C8) have projective dimension at most one. Their complexes split in the derived category into their two-term projective resolutions: the obstructions to splitting consecutive truncations are extension groups in degree at least two and vanish. In degrees relevant to H_2 of a mapping cone, a map between these split complexes consists of the displayed homology maps and extension components from H_i to H_{i+1}. Normalize the component in (E.C11) by a unit on the free H_2 summand of the target. Its components to the two torsion H_2 summands can then be removed by target maps $A \to (A/(q))^2$: reduction of the unit modulo q is ± 1. The remaining components from the u summand are removed by target degree-one extension maps from $H_1(\widehat{W})$ to $H_2(\widehat{W})$. Finally the degree-zero component is removed using the isomorphism on H_0 and a target extension map from H_0 to H_1.

⟨1⟩5. Each change just described is a unit on the diagonal of the split complex and is triangular with extension entries off the diagonal. It is therefore invertible. Replacing the split resolutions by the original finite free complexes represents it by a chain homotopy equivalence. The change is applied to the cusp target and its entire incoming boundary map simultaneously. The other vertices and all maps into the regular core are unchanged. It thus gives a chain homotopy equivalence of the global mapping-cone complex that is the identity on the external peripheral complex. Tensoring with R preserves these comparisons, since R is free over A.

⟨1⟩6. One resulting cusp model in degrees zero through three has ranks $(1, 2, 4, 2)$, generators

$$(1); \quad (g, u); \quad (gu, E, U, D); \quad (R_U, R_D),$$

and differentials $dg = q$, $d(gu) = qu$, $dR_U = qU$, $dR_D = qD$. The map (E.C7) to this model is, through degree two,

	g	u	w	d	θ
i_1	g	u	0	0	0

,

	gu	gw	gd	uw	ud	wd
i_2	gu	$-hD$	E	U	D	0

, $\qquad i_2(v\theta) = 0.$ (E.C12)

The signs and the factor h satisfy the chain equation directly. At $h = 1$, choose $E = -\overline{C}_1 - \overline{C}_2$, $U = \overline{C}_2 + \overline{C}_3$, and $D = \overline{C}_2$.

⟨1⟩7. *Finite caps and the global complex.* For any marked monodromy $A = \left(\begin{smallmatrix} 1 & 0 \\ t & D \end{smallmatrix}\right)$ use the chain lift $F(A) = \left(\begin{smallmatrix} 1 & 0 \\ ht & D \end{smallmatrix}\right)$ and its exterior powers. These lifts multiply exactly and commute with contraction by qg^*.

⟨1⟩8. This exterior lift represents the equivariant torus map. The gamma cover is $\mathbb{R}_g \times T^3$, and the marked linear map is $(g, v) \mapsto (g, Dv + gt)$. At every integer value of g the translation is integral, so the vertex torus map is exactly D. In a single unit strip, concentrate the loop of translations $v \mapsto v + st$ at its terminal endpoint by an equivariant cellular approximation relative to the two vertex tori. The translation prism on the minimal cellular complex of T^3 is exterior multiplication by t: its value on each product torus is the homology class of the literal translation sweep, and all differentials in that minimal torus complex are zero. The terminal vertex has deck coefficient h. Consequently

$$v \longmapsto \bigwedge D(v), \qquad g \wedge v \longmapsto g \wedge \bigwedge D(v) + ht \wedge \bigwedge D(v), \tag{E.C20}$$

which is exactly the exterior lift above. There is only this one strip parameter. Composing two such maps adds the translation loops by $t_A + D_A t_B$, the same strict multiplication law as $F(A)F(B) = F(AB)$. A regular boundary with deck image s is the cone of $sI - F(A)$. This convention gives the group relation $s\lambda s^{-1} = A\lambda$.

⟨1⟩9. At finite cap j, take its adapted first fibre vector $n_j v_j$, where

$$(m_1, n_1, v_1) = (3, 1, (1, 2, -4, 0)), \qquad (m_2, n_2, v_2) = (4, -1, (-1, -3, 3, 0)). \tag{E.C13}$$

It has gamma coordinate one and is fixed by A_j. The other three coordinates form $L_0 = \langle u, w, d\rangle$, with monodromy $D_j = A_j|_{L_0}$. The cap is the mapping torus of this three-torus, and its complex has differential $sI - \bigwedge^r D_j$ on the swept cells. The fibre-to-cap map is the identity on $\bigwedge L_0$; on the adapted long-circle factor it is the integral norm matrix $N_{m_j n_j}(s, D_j)$ determined by

$$(sI - D_j)N_{m_j n_j}(s, D_j) = (h - 1)I, \qquad s = a \text{ or } b. \tag{E.C14}$$

For positive n_j the norm is $\sum_{r=0}^{m_j-1} s^r D_j^{m_j-1-r}$, and for $n_j = -1$ it is minus s^{-m_j} times that sum. In phase degree r, use $N_{m_j n_j}(s, \bigwedge^r D_j)$, rather than the exterior power of the

norm matrix. The boundary base-circle generator maps to the cap circle, while the cell containing both that meridian and the adapted long circle maps to zero. These are winding maps in the mapping-torus coordinates. The linear three-torus return fixes its origin; hence there is no additional translation prism in this comparison.

⟨1⟩10. Form the core over a bouquet of three circles with deck images $a, b, 1$ and monodromies A_1, A_2, M_0. Attach the two finite complexes by (E.C14) and the cusp by (E.C12), using their double mapping cylinders. Through degree three the resulting free R complex has ranks $(4, 20, 49, 62)$. Its entries and unit cancellations are computed exactly by the program `maps_complement_module.py`, in the integral normal form h^r times an alternating word in a, a^2 and b, b^2, b^3.

⟨1⟩11. The final reduction is as follows. Cancelling degree-zero and degree-one unit pairs leaves ranks $(1, 2, 34, 62)$, with $d_1 = (a - 1, b - 1)$. The only nonzero columns of d_2 are F and $F(b - 1)$, where

$$F = \begin{pmatrix} 1 + a + a^2 \\ h(1 + b + b^2 + b^3) \end{pmatrix}. \tag{E.C15}$$

The first component of F is injective on R: multiplication by $a - 1$ converts it to $h - 1$, and R is free over the Laurent ring A. Thus removing the F column after subtracting its multiple gives a free basis of $\ker d_2$ of rank33. The corresponding row of d_3 is zero by the chain equation. Unit reductions of the remaining presentation leave five generators (x, t, y, z, e). One generator e occurs in no relation. Three pure t relations are

$$tq = 0, \qquad t(1 - a^{-1}) = 0, \qquad t(b - a^{-1}) = 0. \tag{E.C16}$$

They generate exactly the augmentation relations on t: multiply the second by a to get $t(a - 1) = 0$, and subtract it from the third to get $t(b - 1) = 0$. Every other pure t coefficient has augmentation zero and is consequently redundant. Reducing the remaining t coefficients by augmentation, three relations give successively $yq = 0, zq = 0, xq = 0$. The final three relations are

$$x(1 + b) + 3t = 0, \quad x(b + b^2) + 3t = 0, \quad x(1 + b + b^2 + b^3) + 6t = 0. \tag{E.C17}$$

The last two are respectively the first multiplied by b, and the first multiplied by $1 + b^2$. These identities prove (E.C1).

⟨1⟩12. The invariant subgroup generated by t injects in L. One way to see this is to start with the exact sequence of right $\mathbb{Z}C_4$ modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[C_4/C_2] \longrightarrow \mathbb{Z}_- \longrightarrow 0,$$

where the first map is the sum of the two cosets. Induce to Q and push out its first term by the augmentation to the trivial module $\mathbb{Z}$. The resulting presentation is (E.C1), after the change $x \mapsto x + t$ which replaces the coefficient 3 by 1. Pushout preserves the displayed injection. Restriction to C_4 shows nonsplitting: an invariant correction of the lift x changes this odd coefficient by an even number. Equivalently the standard cyclic resolution computes $\mathrm{Ext}^1_{\mathbb{Z}C_4}(\mathbb{Z}_-, \mathbb{Z}) = \mathbb{Z}/2$. Induction–restriction gives the same extension group for (E.C3).

⟨1⟩13. Finally insert the fibre cycles uw, ud, wd in the core before any cancellation and carry their coordinate columns through the same basis changes. The five-generator presentation gives respectively

$$t, \qquad t(a^{-1} - a), \qquad t(a^{-1} - 1). \tag{E.C18}$$

Relations (E.C16) reduce these to (E.C4). The external boundary cover has homotopy type T^3 and these are its three degree-two generators. This also checks the ordinary coefficient independently: after taking K coinvariants, $2x + 3t = 0$, so t is twice a primitive class, exactly the coefficient2 in 13.4.

⟨1⟩14. For the remaining peripheral data, use the boundary word abc, where the cusp generator c has deck image1. On swept fibre chains its map to the three core edges has blocks

$$bI, \qquad F(A_1), \qquad F(A_1)F(A_2). \tag{E.C21}$$

The identity

$$(aI - F(A_1))b + F(A_1)(bI - F(A_2)) + F(A_1)F(A_2)(I - F(M_0)) = (ab - 1)I$$

checks its chain equation. This is the usual prism subdivision of the peripheral word, with its given ribbon marking. Tracking all ten degree-two columns through the same reductions, and using only $t(a - 1) = t(b - 1) = 0$ and $xq = yq = zq = 0$, gives (E.C19a)–(E.C19b).

□

Theorem 13.15 (the full shifted homotopy module)**.** [tag 02MH]

Use the right-module convention of 13.14. Write

$$\begin{gathered} R = \mathbb{Z}G, \qquad q_i = h_i - 1, \qquad S = \mathbb{Z}\Gamma, \\ \Gamma = G/\langle h_1, h_2\rangle = \langle a, b, c, d \mid a^3 = b^4 = c^3 = d^4 = abcd = 1\rangle. \end{gathered} \tag{E.P1}$$

where a, b belong to the first complement, c, d to the second, and $\mu = ab = (cd)^{-1}$. The full integral second-homotopy module is

$$\pi_2(P) \simeq R/q_2R \ \oplus \ R/q_1R \ \oplus \ T_L \oplus T_R. \tag{E.P2}$$

The last two modules are inflated from the following explicit six-generator presentations over S. Put $N_a = 1 + a + a^2$, $N_b = 1 + b + b^2 + b^3$, and define N_c, N_d similarly.

For T_L use generators p_i, r_i for $0 \le i \le 2$, put $v_i = p_i - r_i$, and define

$$\begin{aligned} \Omega_L &= v_0(1 + d - d^2 - d^3) + v_1(d^3 - d) + v_2(d^2 + d^3), \\ p_iN_a = 0, &\qquad r_iN_b = 0 \quad (0 \le i \le 2), \\ \Omega_L(a - 1) = \Omega_L(b - 1) &= \Omega_L(c - 1) = \Omega_L(d - 1) = 0. \end{aligned} \tag{E.P3}$$

For T_R use generators B_i, C_i, put $w_i = B_id + C_i$, and define

$$\begin{aligned} \Omega_R &= w_0a(1 - b - b^2 + b^3) + w_1a(b^2 - 1) - w_2a(1 + b^3), \\ B_iN_c = 0, &\qquad C_iN_d = 0 \quad (0 \le i \le 2), \\ \Omega_R(a - 1) = \Omega_R(b - 1) &= \Omega_R(c - 1) = \Omega_R(d - 1) = 0. \end{aligned} \tag{E.P4}$$

These are complete presentations: six generators and ten relations in each block. In particular

$$q_1q_2\pi_2(P) = 0, \qquad \ker(q_1 : \pi_2(P) \to \pi_2(P)) \cap \ker(q_2 : \pi_2(P) \to \pi_2(P)) = T_L \oplus T_R. \tag{E.P5}$$

This sharpens the annihilator of theorem 13.10 and determines the central-intersection summand. Its coinvariants are $(\mathbb{Z}/12)^6$, so (E.P2) recovers $\pi_2(P)_G = \mathbb{Z}^2 \oplus (\mathbb{Z}/12)^6$.

Let $\Pi = \pi_1(Y)$ and let $\Sigma \simeq \pi_1(\Sigma_6)$ be its quotient by $z = h_1h_2$, as in 13.11. Then

$$\pi_2(Y) \simeq (\mathbb{Z}\Pi)^{24} \oplus \mathrm{Res}^{\Gamma}_{\Sigma}(T_L \oplus T_R), \qquad \pi_2(Y)_\Pi \simeq \mathbb{Z}^{110} \oplus (\mathbb{Z}/6)^2. \tag{E.P6}$$

The surface-group summand is inflated to Π, so z acts trivially on it.

Proof. ⟨1⟩1. *A small chain presentation with the peripheral map.* The calculation in 13.14 permits the following model through degree three for the first complement. Its bases have ranks $(1, 2, 6, 6)$, with degree-one basis (a, b) and degree-two basis (r, x, t, y, z, e). The first two differentials are

$$d_1 = (a-1, b-1), \qquad d_2(r) = \begin{pmatrix} N_a \\ h_1 N_b \end{pmatrix}, \quad d_2(x) = d_2(t) = d_2(y) = d_2(z) = d_2(e) = 0. \tag{E.P7}$$

The six degree-three columns are the six relations

$$t(a-1), \quad t(b-1), \quad xq_1, \quad yq_1, \quad zq_1, \quad x(1+b) + 3t. \tag{E.P8}$$

The degree-two norm N_a in (E.P7) is written with the same expression as in (E.P3), but here $a^3 = h_1$, not 1. The second complement is the identical model with (a, b, h_1) replaced by (c, d, h_2).

⟨1⟩2. The ordinary one-relator complex C_K is the subcomplex on degrees zero and one and the cell r. It is an aspherical model: it is the double mapping cylinder of the degree-three and degree-four maps of circles, and its universal cover is a tree of contractible lines and strips. Equivalently it is the standard Seifert spine of the torus-knot exterior. Projection of (E.P7)–(E.P8) onto C_K retains the peripheral map. In degree one this map is

$$g \longmapsto (N_a, 0), \qquad u, w, d \longmapsto 0, \qquad \Theta \longmapsto (b, 1). \tag{E.P9}$$

Its degree-two r coefficient is $b-1$ on the swept cell $g\Theta$ and zero on the other nine cells. Its remaining components are exactly (E.C19).

⟨1⟩3. Here and below the small complex is only a presentation through the degree needed for H_2. Replacing redundant degree-three columns by the six generators of their image in (E.P8) changes no degree-two cycle or boundary. Reducing a peripheral degree-two column modulo that image changes it by a chain homotopy with values in degree three. Products with a torus and the mapping cones below therefore retain the same degree-two homology. For the comparison with an aspherical target one may first keep the original, full complex: its projection to C_K extends by zero in degrees at least three, since the Fox column in (E.P7) is injective and hence the r row of d_3 is zero. Only after taking the kernel is its image in degree two shortened as in (E.P8).

⟨1⟩4. *The exact kernel complex for the splice.* Let $T(s)$ be the four-torus chain complex whose first circle has deck image s and whose other three circles have trivial image. Its differential contracts that first circle and multiplies by $s-1$. The two vertex complexes for the splice are

$$V_1 = C_{E_1} \otimes T(h_2), \qquad V_2 = C_{E_2} \otimes T(h_1). \tag{E.P10}$$

We have reversed the two product factors in the second vertex only for notation; the Koszul signs below account for that reversal. The boundary complex B is the nine-torus complex with ordered coordinates

$$(g_1, u_1, w_1, d_1, g_2, u_2, w_2, d_2, \Theta), \quad \text{deck images } (h_1, 1, 1, 1, h_2, 1, 1, 1, \mu). \tag{E.P11}$$

Its differential is the usual exterior contraction by those deck images minus one.

⟨1⟩5. The source signs are as follows. Write a boundary product cell as $\alpha \wedge \beta$ with an optional final Θ, where α uses the first four coordinates and β the second four. Put $\epsilon = 1$ when Θ occurs and 0 otherwise. Its first vertex map applies the 13.14 peripheral map to (α, Θ), keeps β, and has sign $(-1)^{\epsilon(|\alpha|+|\beta|)}$. The second map applies that peripheral map to (β, Θ_2),

keeps α, and has sign $(-1)^{|\alpha|(|\beta|+\epsilon)+\epsilon|\beta|}$. If $\epsilon = 1$, multiply its coefficient on the right by $-\mu$. This is the exact cellular map of $\Theta_1 \mapsto \Theta_2^{-1}$, since $\mu_2 = cd = \mu^{-1}$. The suspended cells in (E.C19) have the interval first; these signs convert from the ordered exterior cells in (E.P11). Denote the two maps by j_1, j_2.

⟨1⟩6. The chain complex of the splice is the cone of $(j_1, -j_2) : B \to V_1 \oplus V_2$. Project each C_E to C_K and each $T(s)$ to its first circle, and project B to its three coordinates g_1, g_2, Θ. The result is the complex of the aspherical graph of spaces

$$(K(K_1, 1) \times S^1) \ \cup_{T^3} \ (K(K_2, 1) \times S^1). \tag{E.P12}$$

It is the model N of 13.7. Both peripheral maps on groups are injective by 13.8; their common group is $\langle h_1, h_2, \mu \rangle = \mathbb{Z}^3$. Consequently its universal cover is a Bass–Serre tree of contractible vertex and edge spaces and is contractible. Its chain complex has ranks $(2, 7, 9, 5, 1)$, including its degree-four cell.

⟨1⟩7. The indicated projection is a degreewise split surjection of the full chain complexes. The peripheral Fox coefficient in (E.P9) makes the boundary square commute; higher degrees commute as well by the product construction and the zero higher groups of C_K. Let D be its kernel. The target has zero positive-degree homology, so the long exact sequence gives $H_2(D) \simeq H_2(\widetilde{P}) = \pi_2(P)$. Using the small models for the relevant images, the ranks of D through degree three are

$$(0, 6, 40, 123). \tag{E.P13}$$

Its six degree-one generators are the unused regular-factor circles at the two vertices. The six shifted boundary circles u_i, w_i, d_i map to them by unit entries, because each circle maps to zero in its own complement and identically to the opposite regular torus. Cancel these six unit pairs. This leaves a presentation of $H_2(D)$ with34 generators and123 relations.

⟨1⟩8. Here is also an explicit block definition of that matrix. In each V_i keep the cells killed by projection to $C_{K_i} \otimes C(S^1)$; their ranks in degrees two and three are17 and45. In B_2 keep the33 cells containing at least one of $u_1, w_1, d_1, u_2, w_2, d_2$. In those ordered bases the matrix is

$$\mathcal{M} = \begin{pmatrix} (d_{V_1})_3 & 0 & (j_1)_2 \\ 0 & (d_{V_2})_3 & -(j_2)_2 \end{pmatrix} : R^{45} \oplus R^{45} \oplus R^{33} \longrightarrow R^{17} \oplus R^{17}. \tag{E.P14}$$

Every entry is specified by (E.P7)–(E.P11) and (E.C19); (E.P14) itself is a complete finite presentation of the desired module. The formula includes the derived peripheral comparison.

⟨1⟩9. *Reduction to the compact presentation.* The program `maps_splice_module.py` checks both peripheral chain equations and $d_2 d_3 = 0$, cancels the indicated unit pairs, and reduces (E.P14) to 16 generators and52 nonzero columns. It computes in the amalgam

$$G = (K_1 \times \mathbb{Z}) \underset{\mathbb{Z}^3}{*} (\mathbb{Z} \times K_2), \tag{E.P15}$$

not in a finite quotient. To specify its normal form, the torus-knot factor uses h^r times an alternating word in a, a^2 and b, b^2, b^3. Modulo the peripheral subgroup, strip initial ab and $b^3a^2 = (ab)^{-1}$, allowing their last letter to be a partial power; choose b^3 in the sole length-one tie with a. Inverting these left coset representatives gives right coset representatives. The usual alternating amalgam normal form, followed by a final element of $\mathbb{Z}^3$, is unique. These elementary word operations are implemented exactly, including both central exponents.

⟨1⟩10. In the resulting16-generator presentation two generators occur only in their respective single relations $q_2e_1 = 0$ and $q_1e_2 = 0$. They give the first two direct summands of (E.P2). For the other 14 generators, two generators t_0, t_1 have trivial action of a, b, c, d. There are three generators initially killed by q_2 whose pure relations include h_1N_b; the identity $h_1N_b(b-1) = 1 - h_1$ kills them by q_1 as well. There are three initially killed by q_1 with pure N_c relations, and three more with pure h_2N_d relations. They are killed by q_2. The remaining generators already have both central relations. Thus the whole14-generator summand is a module over S, without imposing an additional quotient. The pure norm columns are exact differences of the recorded columns and are checked before setting either central element equal to1.

⟨1⟩11. For a readable final reduction, the14 generators separate into two seven-generator blocks. The first has generators $U_0, U_1, U_2, X, Y, Z, t_1$, with t_1 invariant, $U_iN_b = 0$, and

$$\begin{aligned}(U_0 + X - Yd + Zd)N_a &= 0,\\ (U_1 - Y(1+d) + Z)N_a &= 0,\\ (U_2 + X + 2t_1 + Z(d-1))N_a &= 0,\\ X(1+d) + 3t_1 &= 0.\end{aligned} \tag{E.P16}$$

The second has generators B_i, C_i, t_0, with $B_iN_c = C_iN_d = 0$, t_0 invariant, and the single additional relation $t_0 = \Omega_R$. These formulas follow by collecting the displayed16-generator columns.

⟨1⟩12. In the first block put

$$\begin{aligned} v_0 = X - Yd + Zd, \quad v_1 = -Y(1+d) + Z,\\ v_2 = X + 2t_1 + Z(d-1), \qquad r_i = U_i, \quad p_i = U_i + v_i.\end{aligned} \tag{E.P17}$$

The linear part of this change on (X, Y, Z) is

$$\begin{pmatrix} 1 & 0 & 1 \\ -d & -1-d & 0 \\ d & 1 & d-1 \end{pmatrix}, \quad \det = 1,$$

over $\mathbb{Z}[d, d^{-1}]$. It is therefore an integral group-ring basis change. Its inverse gives

$$X = v_0(1-d^2) + v_1(d^2-d) + (v_2 - 2t_1)d^2.$$

Substitute in the last relation of (E.P16), using the invariance of t_1. The result is $t_1 = \Omega_L$. Eliminate t_1 with this unit coefficient. Its four invariance relations become the four relations in (E.P3). Eliminating t_0 in the second block gives (E.P4). This proves the compact presentation in (E.P2)–(E.P4), including the integral central-intersection information.

⟨1⟩13. For the two cyclic axis summands, the other central element is a non-zero-divisor: their group rings are free over its Laurent polynomial ring. Thus neither contributes to the common kernel of q_1, q_2. This proves (E.P5). On coinvariants, the four invariance relations in each block vanish, while the norm relations become three copies of multiplication by3 and three copies of multiplication by4. Each block therefore has coinvariants $(\mathbb{Z}/12)^3$. This is an independent recovery of 13.9.

⟨1⟩14. *Restriction to the circle fibre.* The universal covers of P and $Y \times \mathbb{R}$ are Π-equivariantly homotopy equivalent by 13.11. The map $\Pi \to G/\langle h_i\rangle$ is injective, since $\chi(h_i) = \pm 12$, and has index12. Hence each of the first two group-ring summands in (E.P2) is free of rank12 over $\mathbb{Z}\Pi$. The last two factor through the index12 surface subgroup $\Sigma < \Gamma$ and have trivial z action. This proves the module formula in (E.P6).

⟨1⟩15. For its exact coinvariants use the regular 12-sheet permutation representation with exponents $(4,-3,-4,3)$ on (a,b,c,d). Evaluating the compact12-by20 presentation gives an integral144-by240 matrix. Its Smith form has 56 unit entries and two entries6; its free cokernel rank is86. Adding the24 free coinvariants proves the last formula in (E.P6).

□

Proposition 13.16 (the cusp quadratic attachment)**.** [tag 02MI]

Keep the right-module presentation of 13.14 and write

$$R=\mathbb{Z}K,\qquad M=\pi_2(E)=eR\oplus T,\qquad T=\overline{R}^{\,2}\oplus L.$$

For an abelian group A, let $\Gamma(A)$ be Whitehead's universal quadratic group, with quadratic generator $q(a)$ and symmetric bilinear cross effect $[a,b]=q(a+b)-q(a)-q(b)$. In particular $[a,a]=2q(a)$. There is a lift of the normalized cusp component $S=\mathbb{CP}^2\#3\overline{\mathbb{CP}}^{\,2}$ to the universal cover $\widetilde{E}$ whose fundamental class has image

$$\pm([e(h-1),t]-12q(t))\quad \textit{under}\quad H_4(\widetilde{E};\mathbb{Z})\longrightarrow\Gamma(M). \tag{E.Q1}$$

The sign is solely the orientation chosen on the normalized component. Its mixed component in

$$\Gamma(M)=\Gamma(eR)\oplus(eR\otimes_{\mathbb{Z}}T)\oplus\Gamma(T) \tag{E.Q2}$$

is nonzero. Consequently E is not homotopy equivalent to $E_0\#(S^2\times S^4)$ for any connected compact six-manifold E_0, even without requiring the equivalence to preserve the boundary. Thus the free R summand of its second homotopy group does not come from a sphere-product connected summand. Likewise the nine-dimensional fibre Y is not homotopy equivalent to $Y_0\#(S^2\times S^7)$ for any connected closed nine-manifold Y_0, despite the twenty-four free $\mathbb{Z}\Pi$ summands in (E.P6).

Proof. ⟨1⟩1. We use the natural, deck-equivariant part of Whitehead's exact sequence

$$H_4(\widetilde{E})\xrightarrow{\omega_E}\Gamma(\pi_2(E))\longrightarrow\pi_3(E)\longrightarrow H_3(\widetilde{E})\longrightarrow 0. \tag{E.Q3}$$

The exactness and identification of its quadratic term are (Whitehead, 1950, Theorems 1 and 5); the direct-sum rule (E.Q2) is proved there in Section 2. These statements apply to the simply connected universal cover, with no finiteness requirement. Naturality specifies the element in (E.Q1).

⟨1⟩2. For the normalization calculation take the negatives of the complex-oriented Picard homology generators and denote them by (H,E_1,E_2,E_3). Their intersection matrix is still $\operatorname{diag}(1,-1,-1,-1)$. This common reversal matches the phase-cell orientations of the cusp basis: the toric line with class $-(uw+6\gamma\delta)$ has complex curve degrees $(-6,0,1)$, whereas its values on the corresponding phase classes are $(6,0,-1)$; the support function rs similarly fixes the middle sign. The exact wall-degree calculation in III.5.9 (p. 39) gives this comparison. With these orientations the six conductor classes are

$$E_1,\ H-E_1-E_2,\ E_2,\ H-E_2-E_3,E_3,\ H-E_1-E_3.$$

The first opposite pair is self-glued in the surviving u direction. In the gamma cover its relation is therefore $E_1=H-E_2-E_3$. The other two pairs connect adjacent gamma translates. Their relations, after eliminating H, are $(h-1)E_2=(h-1)E_3=0$. Thus the image of normalization homology is, with these curve generators,

$$AE_1\oplus(A/(h-1))E_2\oplus(A/(h-1))E_3,\qquad A=\mathbb{Z}[h,h^{-1}],\qquad H=E_1+E_2+E_3. \tag{E.Q4}$$

This also follows directly from the transpose of the matrix $B(h)$ of 13.10, with the intersection signs of the Picard basis. It is the whole H_2 of the gamma cover: the degree-zero normalization row is the cellular complex of an infinite cylinder, whose second homology is zero.

⟨1⟩3. We fix the comparison with the cusp basis of (E.C12). In the ordinary central fibre the first conductor class $\overline{C}_1$ is the self-u pair; the other two are the images of E_2, E_3, in either order. The chosen cusp basis is

$$E = -\overline{C}_1 - \overline{C}_2, \qquad U = \overline{C}_2 + \overline{C}_3, \qquad D = \overline{C}_2.$$

The two torsion coordinates in (E.Q4) are determined by this ordinary comparison, because each is annihilated by $h-1$. The normalization map is onto the whole group in (E.Q4), and E_2, E_3 generate its entire A-torsion submodule. Consequently the coefficient of E_1 in the free quotient is a generator of A, not merely a nonzero element: it is a unit $\pm h^n$. The ordinary comparison fixes its sign. Translating the entire chosen normalization component by h^{-n} makes this coefficient -1 and leaves both torsion coordinates fixed. The common Picard sign convention above changes every generator simultaneously. The resulting normalization map in the basis used for the full peripheral comparison is

$$E_1 \longmapsto -E - D, \qquad E_2 \longmapsto D, \qquad E_3 \longmapsto U - D, \qquad H \longmapsto -E + U - D. \quad \text{(E.Q5)}$$

Exchanging E_2, E_3 has no effect below.

⟨1⟩4. The intersection form of S is $\mathrm{diag}(1,-1,-1,-1)$. Its usual four-cell description, or the Hopf invariant of its two-dimensional classes, therefore gives

$$\omega_S[S] = q(H) - q(E_1) - q(E_2) - q(E_3).$$

Substitute $H = E_1 + E_2 + E_3$ and then (E.Q5). Bilinearity of the cross effect gives

$$\omega_S[S] \longmapsto [E_1, E_2] + [E_1, E_3] + [E_2, E_3] = -[E, U] - 2q(D). \quad \text{(E.Q6)}$$

The normalization map $S \to E$ lifts to $\widetilde{E}$ because S is simply connected. Naturality of (E.Q3) shows that the last expression is the image of its four-dimensional fundamental class.

⟨1⟩5. The global cancellation also tracks the three cusp two-cycles in addition to the peripheral chains. It gives, in the generators (x, t, y, z, e) of 13.14,

$$E \longmapsto (1-h)e + 6t, \qquad U \longmapsto t, \qquad D \longmapsto 0. \quad \text{(E.Q7)}$$

For clarity, before the last relation reductions its t coefficient for E is $-3h + 4ha + 4ha^2 + h^2$, whose augmentation is six; its free coefficient is $1-h$. Its coefficient for D is $2h^{-1}a^2 - 1 - a$, which has augmentation zero and hence annihilates the invariant generator t. The other coordinates vanish. The program `maps_cusp_quadratic.py` checks these identities using the identical integral group-ring cancellations as 13.14. Applying Γ to (E.Q7) in (E.Q6) gives

$$-[(1-h)e + 6t, t] = [e(h-1), t] - 12q(t),$$

as asserted. In ordinary coinvariants $t = 2v$ and $E = 12v$ for a primitive class v. The quadratic image is then $-48q(v)$, consistent with the degree-two cup square evaluated on the normalization; this is a check, not the proof of (E.Q1).

⟨1⟩6. We finish with the connected-sum obstruction, allowing every possible free summand. Multiplication by $h-1$ is injective on R: a finitely supported group-ring element constant along an infinite h orbit is zero. It annihilates T, so

$$\mathrm{Ann}_M(h-1) = T, \qquad \mathrm{Hom}_R(T, R) = 0. \quad \text{(E.Q8)}$$

Suppose that a copy fR is a direct summand of M. Write $f = eu + t_0$ and let a retraction send e to v. It vanishes on T, and hence $vu = 1$.

⟨1⟩7. The ring $\mathbb{Z}K$ is directly finite. One elementary verification uses the index-twelve subgroup of K: its quotient by the central $\mathbb{Z}$ is the free group of the punctured genus-three surface, so it is $F_6 \times \mathbb{Z}$. This subgroup, and hence K, is residually finite. In every finite-group quotient the regular representation is finite dimensional over $\mathbb{Q}$, so $vu = 1$ implies $uv = 1$. Residual finiteness separates the finite support of $uv - 1$ and proves the same equality in R. Thus u is a unit, the kernel of the retraction is exactly T, and every free splitting is obtained by replacing e by $eu + t_0$.

⟨1⟩8. Under such a change the mixed component of (E.Q1) is $fu^{-1}(h-1) \otimes t$. It is nonzero: t has infinite order by (E.C3), and $u^{-1}(h-1)$ is nonzero. The central generator may also be reversed by a group automorphism, which multiplies $h - 1$ by a unit and does not affect the argument.

⟨1⟩9. If $E \simeq E_0 \# (S^2 \times S^4)$, its universal cover has, through dimension four, the homotopy type of $\widetilde{E}_0$ wedged with one copy of $S^2 \vee S^4$ for each deck element. This follows by lifting the connected-sum balls; deleting or regluing their boundary five-spheres does not alter the four-skeleton. The new two-spheres generate a free R summand. The four-spheres have trivial attaching maps, and every remaining four-dimensional Whitehead image lies in $\Gamma(\pi_2(E_0))$. There can therefore be no mixed component between this free summand and its complement. This contradicts (E.Q1)–(E.Q8) for every choice of the summand, and proves the stated obstruction.

⟨1⟩10. For the assertion about Y, first take the maps from the two complements to the splice. The program `maps_splice_quadratic.py` tracks the four cycles (t_0, e_0, t_1, e_1) through every unit cancellation used in 13.15. In its sixteen-generator presentation they become four unit vectors. The subsequent seven-generator block changes therefore give

$$t_0 = \Omega_R, \qquad t_1 = \Omega_L, \qquad e_0 \in R/q_2R, \quad e_1 \in R/q_1R \tag{E.Q9}$$

with exactly the axes in (E.P2). Both t_i are invariant under G: the four displayed relations for each Ω in (E.P3)–(E.P4) annihilate it by $a-1, b-1, c-1, d-1$. On Γ-coinvariants their respective images are

$$-2(B_2 + C_2), \qquad 2(p_2 - r_2), \tag{E.Q10}$$

and each has order exactly six: the relevant two coordinates have independent orders three and four in (E.P3)–(E.P4). In particular an integer annihilating either t_i in the full module must be divisible by six.

⟨1⟩11. Put $A_\Pi = \mathbb{Z}\Pi$ and $z = h_1h_2$. Choose the twelve coset representatives giving the free restriction of each axis in (E.P6), and denote the resulting twenty-four generators by f_j. The normalized cusp surfaces and all their chosen deck translates are maps to $\widetilde{P}$. Transfer them to $\widetilde{Y}$ using the Π-equivariant homotopy equivalence $\widetilde{P} \simeq \widetilde{Y} \times \mathbb{R}$. On the first axis h_2 is the identity, so multiplication by $h_1 - 1$ equals multiplication by $z - 1$; the analogous statement holds on the second axis. Naturality of (E.Q3) therefore supplies, for every j, an element in the image of the Whitehead map of $\widetilde{Y}$ of the form

$$\pm([f_j(z-1), t_{i(j)}] - 12q(t_{i(j)})), \tag{E.Q11}$$

where $i(j) = 0$ on the first twelve axes and 1 on the remaining twelve. Invariance of t_i under G removes any extra coset action in this formula.

⟨1⟩12. Write $\pi_2(Y) = F \oplus T_Y$, with $F = A_\Pi^{24}$ and $(z-1)T_Y = 0$ as in (E.P6). Suppose a free copy fA_Π is a direct summand, with a retraction $r : \pi_2(Y) \to A_\Pi$ satisfying $r(f) = 1$.

Multiplication by $z-1$ is injective on A_Π, so $r(T_Y)=0$. Write $r(f_j)=v_j$ and $f=\sum_j f_j u_j+t$; then

$$\sum_j v_j u_j = 1. \tag{E.Q12}$$

In the splitting $fA_\Pi \oplus \ker r$, the mixed component of (E.Q11) is $fv_j(z-1)\otimes t_{i(j)}$. If all these mixed components were zero, every integer coefficient of $v_j(z-1)$ would be divisible by six, by (E.Q10). The group-ring basis makes this implication immediate even if t_i has larger or infinite order. Multiplication by $z-1$ is also injective on $(\mathbb{Z}/6)[\Pi]$: along an infinite z orbit a finitely supported coefficient function with zero difference is zero. Hence every v_j belongs to $6A_\Pi$, contradicting (E.Q12).

⟨1⟩13. Finally, a connected sum with $S^2\times S^7$ has, through dimension four in its universal cover, the original space wedged with a free deck orbit of two-spheres. All images of its four-dimensional Whitehead map consequently have zero mixed component with that free summand. The preceding argument rules this out for every free summand of the module. This proves the assertion for Y.

□

Proposition 13.17 (the relative smooth cusp model)**.** [tag 02MJ]

For the original marked cusp, let

$$Z(q)=\frac{\log q}{2\pi i}B_0+C(q),\qquad B_0=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

where $C(q)$ is holomorphic and bounded on a disc about zero. For a sufficiently small closed disc the six-dimensional cusp tube is orientation-preservingly diffeomorphic, relative to its full marked torus-bundle boundary, to the unitless toric quotient defined by $C=0$. The latter has the following finite normal-crossings plumbing description.

Start with the canonical complex line bundle of $S=\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ and its six toric curves in cyclic order

$$\begin{aligned}&C_0=E_1, && C_1=H-E_1-E_2, && C_2=E_2,\\ &C_3=H-E_2-E_3, && C_4=E_3, && C_5=H-E_1-E_3.\end{aligned} \tag{E.T1}$$

Pair C_i with C_{i+3} for $i=0,1,2$. Both the surface-normal line and the canonical line restricted to every C_i have degree -1. In the resulting $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ coordinates (z,a,b), with $z=0$ at the endpoint meeting C_{i+1} and b the surface-normal coordinate, the pairing is

$$(z,a,b)\longmapsto(z^{-1},zb,za). \tag{E.T2}$$

The function giving the cusp parameter is $q=zab$ in this coordinate chart. Formula (E.T2), not merely the induced map of zero sections, specifies the normal framing of each plumbing. The two triple points are the endpoint orbits $\{0,2,4\}$ and $\{1,3,5\}$, where endpoint i means $C_i\cap C_{i+1}$. Ordered local complex triples undergo a cyclic permutation around each orbit, with threefold composition the identity. The plumbing is formed with collars and rounded corners; its distinguished outer boundary is $|q|=\eta$.

Proof. ⟨1⟩1. We first construct the comparison by a smooth family. Set $C_s(q)=(1-s)C(q)$ for $0\le s\le 1$. The affine toric cover is fixed. On its dense torus the deck translation for $\lambda\in\mathbb{Z}^2$ becomes

$$(x,q,s)\longmapsto(q^{B_0\lambda}\exp(2\pi i C_s(q)\lambda)x,\ q,\ s). \tag{E.T3}$$

The exponential is componentwise. The powers of q permute the toric charts by translation of the A_2 triangulation, while the exponential factors are nowhere-zero analytic units on every chart. Thus (E.T3) defines a smooth family of holomorphic deck actions on a fixed

smooth toric covering space. On a central stratum the lattice translation permutes its bounded fan cells without fixed points. Off the central fibre, freeness and properness are exactly the nondegeneracy of the real period lattice.

⟨1⟩2. These assertions hold uniformly after shrinking η. Indeed put $T = -\log|q|/(2\pi)$, and let K bound $\|\operatorname{Im} C_s(q)\|$ on the compact parameter interval and closed disc. For $T > K + 1$ the matrix $TB_0 + \operatorname{Im} C_s(q)$ is invertible, since B_0 is orthogonal. A fundamental parallelogram has logarithmic position $B_0a + T^{-1}\operatorname{Im} C_s(q)a$ with $a \in [0,1]^2$; it meets a uniformly bounded number of toric charts. More explicitly choose a triangle containing the unperturbed point B_0a and let P be its unimodular vertex matrix. Its barycentric coordinates t_j are nonnegative, and the corresponding chart coordinates satisfy $\log|z_j| = t_j \log|q| + O(2\pi\|P^{-1}\|K\sqrt{2})$. There are only finitely many such triangles, so with $c = 2\pi\sqrt{2}\max\|P^{-1}\|$ every chosen chart coordinate has modulus at most $\exp(cK)$, uniformly in q, s and all compact phases. Their closed polydiscs have compact closure in the affine charts. These finitely many compact chart images prove compactness of the quotient over $[0,1]$ and $|q| \le \eta$. The central charts and their incidence relations are fixed. Projection of this quotient to s is consequently a proper smooth submersion, including on every normal-crossings stratum. Its only boundary is $|q| = \eta$ together with the parameter end faces, and its boundary restriction is also a submersion.

⟨1⟩3. On $|q| = \eta$ prescribe the marked real coordinates $x = \exp(2\pi i(Z_s(q)a + b))$, $(a,b) \in T^4$. They give a smooth product trivialization in s and precisely the prescribed mapping-torus marking. Extend its horizontal vector field over a collar, and then over the compact family. Local toric product charts provide extensions tangent to every central stratum; a partition of unity preserves that property and the lift of ∂_s. Integrating this field proves the asserted relative diffeomorphism. Its orientation is the complex orientation on each fibre. This is relative Ehresmann transport in a smooth family.

⟨1⟩4. It remains to identify the endpoint $C = 0$ with the framed plumbing data. In the lattice $\mathbb{Z}^2$ use cyclic rays

$$(1,0),\ (0,1),\ (-1,1),\ (-1,0),\ (0,-1),\ (1,-1). \tag{E.T4}$$

The star of the vertex zero in the A_2 triangulation consists of the six triangles spanned by zero and consecutive rays. The corresponding three-dimensional cones have primitive rays $(v,1)$ and determinant one. Their common divisor at the central ray is the toric surface S. A principal central fibre has normal line

$$\mathcal{O}_S(-C_0 - \cdots - C_5) = K_S. \tag{E.T5}$$

The cyclic surface fan in (E.T4) also proves this identification directly: it is the fan of the three torus-fixed blow-ups of $\mathbb{CP}^2$, and each neighboring-ray relation gives $C_i^2 = -1$. In the Picard basis (E.T1), $K_S = -3H + E_1 + E_2 + E_3$, so $K_S \cdot C_i = -1$.

⟨1⟩5. Near a double curve the two adjacent unimodular charts give the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Its two base charts have coordinates

$$(z,a,b), \qquad (w,A,B) = (z^{-1}, za, zb), \qquad q = zab = wAB. \tag{E.T6}$$

The period translation taking a neighboring component back to the chosen component exchanges the two normal directions. It reverses the chosen endpoint coordinate on the opposite toric curve. In the first chart this is exactly (E.T2); in the target infinity chart it is simply (z,b,a), so it extends smoothly over the endpoint. The map is involutive, preserves q, and its complex Jacobian determinant is 1. These equalities specify the normal-disk attachment, including its sign and angular twisting.

⟨1⟩6. The endpoint $C_i \cap C_{i+1}$ is sent to $C_{i+2} \cap C_{i+3}$. At the first endpoint write the ordered ambient coordinates as the surface-normal coordinate, the C_i-normal coordinate and the C_{i+1}-normal coordinate. The next endpoint has their order cyclically permuted. After three crossings the same coordinates return, not just the same unframed planes. Thus the two endpoint orbits give ordinary triple charts $q = z_0 z_1 z_2$, with no additional clutching loop at either triple point.

⟨1⟩7. Equivalently, for a triangle with vertex matrix P whose columns are $(v_i, 1)$, translation by $n \in \mathbb{Z}^2$ replaces it by $L_n P$, where

$$L_n = \begin{pmatrix} 1 & 0 & n_1 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{E.T7}$$

It acts as the identity in the corresponding translated unimodular coordinates, since $(L_n P)^{-1} L_n P = I$. This verifies the preceding plumbing and triple identifications in the toric charts. More explicitly, order the star matrix as $P_i = ((0,1),(r_i,1),(r_{i+1},1))$ and reverse its columns when using the coordinate order (z,a,b). For every i the matrix of translation by $-r_i$ from this chart to the opposite following-endpoint chart is

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{E.T9}$$

Its monomials are exactly (E.T2). In normal-first order, $P_{i+2}^{-1} L_{-r_i} P_i$ is the cyclic permutation matrix. Thus the framing and triple-cycle assertions hold for all six fan identifications. Taking collars of the component core, three double-curve annuli and two triple charts, and rounding their outer corners, gives the asserted finite plumbing. These are subcharts of the smooth toric quotient, so the construction is Hausdorff and the collar identifications are the indicated smooth ones.

⟨1⟩8. Finally the period translations for g, u are $-e_2, e_1$. On $q = \eta e^{-2\pi i\theta}$ the marked expression is $x = \exp(2\pi i((-\theta + iT)B_0 a + b))$. At $\theta = 1$ it agrees with $\theta = 0$ and b replaced by $b - B_0 a$. This is exactly the inverse endpoint convention for

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{E.T8}$$

The comparison therefore preserves the full outer marking. The six unimodular charts, all normal degrees, the plumbing Jacobian and involution, both triple cycles and (E.T8) are checked exactly in `maps_cusp_smooth_plumbing.py`.

□

Proposition 13.18 (the explicit smooth plumbing presentation)**.** [tag `02ML`]

The following finite smooth plumbing presentation uses only standard pieces and specified attaching maps. Its conversion into ordinary framed handles and the relative smooth identification are given in 13.19.

Let $F = \mathbb{R}^4/\mathbb{Z}^4$, with basis (g,u,w,d), and use the two matrices A_j and fixed vectors v_j of I.22.1 (p. 295) . Set

$$(m_1, m_2) = (3,4), \quad (\kappa_1, \kappa_2) = (1,-1), \quad w_j = \kappa_j v_j, \qquad L = \ker \gamma,$$

so that $\Lambda = \mathbb{Z}w_j \oplus L$ *and* $A_j w_j = w_j$*. On* L *the two matrices are*

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{E.U1}$$

Define the standard smooth four-manifolds

$$\mathcal{S}_j = (T^3 \times \mathbb{R})/((l, \sigma + 1) \sim (B_j^{-1} l, \sigma)), \qquad \mathcal{N}_j = \mathcal{S}_j \times D_r^2. \tag{E.U2}$$

The cap boundary is identified with M_{A_j} *by the following explicit map. Write* $x = \phi w_j + l$ *and use the interval marking* $(x, 1) \sim (A_j^{-1} x, 0)$ *on* M_{A_j}*. Then*

$$[x, \theta] \longmapsto \left([l, \sigma = m_j \kappa_j \phi + \theta],\ r = e^{2\pi i \kappa_j \phi} \right). \tag{E.U3}$$

All angles on the right are taken in their stated mapping-torus quotients. In the ordered angular variables (ϕ, θ) *and* (σ, α)*, where* $r = e^{2\pi i \alpha}$*, the map is*

$$\begin{pmatrix} \sigma \\ \alpha \end{pmatrix} = \begin{pmatrix} m_j \kappa_j & 1 \\ \kappa_j & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix}, \qquad \begin{pmatrix} \phi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & \kappa_j \\ 1 & -m_j \end{pmatrix} \begin{pmatrix} \sigma \\ \alpha \end{pmatrix}. \tag{E.U4}$$

These are integral unimodular matrices.

To define a compact six-manifold $\mathcal{E}_{\mathrm{pl}}$*, take the flat* F*-bundle over an oriented disc with three open interior discs removed. Its three marked inner loop monodromies are* A_1, A_2, M_0*, with* $A_1 A_2 M_0 = I$*. Equivalently cut the planar surface along three disjoint paths to its outer boundary, take the product with* F*, and reglue the two banks by these three integral linear diffeomorphisms. This specifies smooth band attachments, including their product normal coordinates. Fill the first two boundary components by* (E.U2)–(E.U4)*, and the third by the finite canonical-bundle plumbing of 13.17, using its complete marked* M_0 *boundary map. The remaining boundary is marked as* $F \times S^1_\mu$*, with the outer meridian orientation of* (E.S1)*. The core uses its zero section.*

For the regular-fibre complement there is an orientation-preserving diffeomorphism

$$E \ \cong\ \mathcal{E}_{\mathrm{pl}} \quad \text{relative to } F \times S^1_\mu. \tag{E.U5}$$

Consequently the shifted fivefold has the explicit smooth presentation

$$P \ \cong\ (\mathcal{E}_{\mathrm{pl}} \times T^4) \ \cup_{(x_1,x_2,\theta)\mapsto(x_1,x_2,-\theta)} \ (T^4 \times \mathcal{E}_{\mathrm{pl}}). \tag{E.U6}$$

Every piece in this presentation is a standard torus product, a linear three-torus mapping torus times an oriented disk, or the canonical disk bundle of the explicitly specified toric blow-up with the normal-disk plumbings (E.T2)*. All attachment maps have been specified on their normal coordinates.*

Proof. ⟨1⟩1. We verify first that (E.U2)–(E.U4) are the finite caps, including the normal framing. The original cap is

$$(F \times D_s^2)/((x, s) \sim (A_j x + v_j/m_j, e^{-2\pi i/m_j} s)). \tag{E.U7}$$

The good-reduction real flat marking is smooth on the entire root disc, so these are smooth coordinates even at $s = 0$. Because $\gamma A_j = \gamma$ and $\gamma(v_j) = \kappa_j = \pm 1$,

$$r = s\, e^{2\pi i \kappa_j \gamma(x)} \tag{E.U8}$$

is invariant under the displayed action. It is single-valued on F, has the same absolute value as s, and is complex linear on every normal disk. The induced map $[x, s] \mapsto ([x], r)$ identifies (E.U7) smoothly with $S_j \times D^2$. An inverse is obtained by choosing any representative x and setting $s = re^{-2\pi i \kappa_j \gamma(x)}$; invariance makes this independent of the representative in the quotient.

⟨1⟩2. In the integral basis (w_j, u, w, d) the affine generator is

$$(\phi, l) \longmapsto (\phi + \kappa_j/m_j, B_j l). \tag{E.U9}$$

Thus $\sigma = m_j\kappa_j\phi$ gives exactly the mapping torus in (E.U2). This also supplies a direct inverse to the product description: choose a representative (l, σ), put $x = (\kappa_j\sigma/m_j)w_j + l$ and $s = re^{-2\pi i\sigma/m_j}$. Replacing that representative by the defining mapping-torus relation changes (x, s) by the finite affine generator and hence not its class.

⟨1⟩3. The boundary parametrization of the finite cap is

$$[x, \theta] \longmapsto [x + \theta v_j/m_j, e^{-2\pi i\theta/m_j}]. \tag{E.U10}$$

Substitute (E.U10) into (E.U8)–(E.U9). The coefficient of θ in the normal angle is $-1/m_j + \kappa_j^2/m_j = 0$, while the base angle is $m_j\kappa_j\phi + \theta$. This proves (E.U3) and (E.U4). At $\theta = 1$ that base angle advances by one, precisely replacing l by $B_j^{-1}l$; this checks the mapping-torus seam. A lattice change of ϕ changes the base angle by $m_j\kappa_j$, and $B_j^{m_j} = I$, so it introduces no further ambiguity.

⟨1⟩4. The function of the cap that gives its original base parameter is, in these smooth coordinates,

$$q = r^{m_j}e^{-2\pi i\sigma}. \tag{E.U11}$$

On the marked boundary it has angle $m_j\alpha - \sigma = -\theta$, the original clockwise base convention. A positive normal meridian at fixed (l, σ) has $(\Delta\phi, \Delta\theta) = (\kappa_j, -m_j)$. Since $\kappa_j w_j = v_j$, it is exactly $t_{v_j}\rho_j^{-m_j}$, recovering the filling relation $\rho_j^{m_j} = t_{v_j}$. The seam, framing and meridian identities are checked exactly by the program `maps_finite_cap_framings.py`.

⟨1⟩5. Over the punctured planar core, the family has the smooth real flat marking with linear monodromies A_j and its zero section. The logarithmic finite gluing has exactly the drift in (E.U10); after subtracting it, the remaining translation has a single-valued real lift and is removed by the collar isotopy already proved in III.3.9 (p. 20) . At the cusp use the relative smooth diffeomorphism of 13.17. Each comparison has precisely the prescribed boundary value and can be made product on a smaller collar. They therefore glue to the relative diffeomorphism (E.U5). In particular the normal framing of the unfilled ordinary fibre is unchanged.

⟨1⟩6. Finally use the two hemispheres in the shifted fibre-product decomposition of 13.4. On each hemisphere the other factor is a regular marked torus bundle over a disc, hence a product preserving its zero section and flat marking. Their common boundary has unchanged torus coordinates and opposite normal angles. Taking the product of (E.U5) with those regular factors and gluing gives (E.U6), with its induced complex orientation.

□

Theorem 13.19 (ordinary framed handles and the smooth type)**.** [tag `02MM`]

The following finite ordinary framed-handle prescription defines a compact oriented six-manifold $\mathcal{H}_6$ with marked boundary $T^4 \times S^1$, and a closed oriented ten-manifold $\mathcal{H}_{10}$. There are orientation-preserving diffeomorphisms

$$E \cong \mathcal{H}_6 \quad \textit{relative to the full marked boundary,} \qquad P \cong \mathcal{H}_{10}. \tag{E.H0}$$

The respective numbers of ordinary handles, in increasing index, are

$$(2, 5, 14, 22, 21, 10, 2), \qquad (4, 31, 119, 290, 485, 574, 485, 290, 119, 31, 4).$$

The attaching embeddings and their normal coordinates are given below. The balls and complex normal charts carry their complex orientations; orient the product blocks by their displayed coordinate embeddings. For example the dense block below has orientation $-d\arg X \wedge d\arg Y \wedge d\mu_1 \wedge d\mu_2 \wedge d\operatorname{Re} q \wedge d\operatorname{Im} q$. *This fixes the orientation convention even when the chosen order of angle and moment coordinates differs from the complex coordinate order.*

1. The two triple-point balls. *Use two complex coordinate balls, denoted* V_+ *and* V_-, *with ordered coordinates* $Z = (Z_0, Z_1, Z_2)$ *and* $W = (W_0, W_1, W_2)$. *The boundary faces are rounded coordinate faces of*

$$|Z_i| \le \epsilon_v, \quad |Z_0Z_1Z_2| \le \eta, \qquad |W_i| \le \epsilon_v, \quad |W_0W_1W_2| \le \eta. \tag{E.H1}$$

They are smooth six-balls. Throughout the prescription, rounding is in the nonnegative radii, preserves the angular coordinates, and is fixed on the attaching collars. Positive changes of the radii only give the corresponding collar isotopy. The complex products in (E.H1) *are the same function* q. *The coordinate formulas may first be read on manifolds with corners; apply the same radial collar reparametrization on the two sides of each attaching face before smoothing their union. This convention also specifies the maps at a corner meeting* $|q| = \eta$.

2. The three annular blocks, with their endpoint framings. *For* $i = 0, 1, 2$ *use a block* $S^1 \times D^1 \times D^4$ *with coordinates*

$$z = \exp(s + 2\pi i\theta), \quad -R \le s \le R, \qquad (a, b) \in D^4, \qquad q = zab. \tag{E.H2}$$

The normal four-ball here is a rounded radial domain $|a|, |b| \le \epsilon_e$, $|ab| \le \eta/|z|$; *its positive radial trivialization identifies it with* D^4. *The ends attach to framed tubes about the following coordinate circles. A table entry* $(Z_j; Z_k, Z_l)$ *means that the curve coordinate is* Z_j *and the two ordered complex normal coordinates are* Z_k, Z_l; *the same convention applies to* W.

i	$z = 0$ *end:* $(z; a, b)$	$z = \infty$ *end:* $(w; A, B)$
0	$(Z_2; Z_1, Z_0)$	$(W_0; W_1, W_2)$
1	$(W_2; W_1, W_0)$	$(Z_1; Z_2, Z_0)$
2	$(Z_0; Z_2, Z_1)$	$(W_1; W_2, W_0)$

(E.H3)

The full-dimensional endpoint map between the two end charts is

$$(w, A, B) = (z^{-1}, za, zb). \tag{E.H4}$$

In particular the second attaching circle has angle $-\theta$, *and its two complex normal vectors each rotate by* $+2\pi\theta$. *The factors* $|z|$ *are the positive radial rescalings. Every coordinate circle of the two balls occurs exactly once in* (E.H3). *The maps are embeddings of* $S^1 \times \partial D^1 \times D^4$.

3. The dense torus block and its full sector maps. *Attach* $T^2 \times D^2 \times D^2_q$ *along* $T^2 \times \partial D^2 \times D^2_q$. *Here is a completely explicit choice of its base disk and the attaching map. Use the rays* r_i *of* (E.T4) *and the hexagon*

$$\mathcal{P} = \{v \in \mathbb{R}^2 : \langle v, r_i\rangle \ge -1\}, \quad \operatorname{Vert}(\mathcal{P}) = \{(-1,-1), (0,-1), (1,0), (1,1), (0,1), (-1,0)\}. \tag{E.H5}$$

On the positive real part of $(\mathbb{C}^*)^2$, *use the coordinates given by the toric moment map*

$$\mu(X, Y) = \frac{\displaystyle\sum_{v \in \operatorname{Vert}(\mathcal{P}) \cup \{0\}} |X^{v_1}Y^{v_2}|^2 v}{\displaystyle\sum_{v \in \operatorname{Vert}(\mathcal{P}) \cup \{0\}} |X^{v_1}Y^{v_2}|^2}. \tag{E.H6}$$

Choose a small inward parallel hexagon, with its corners rounded symmetrically, $\mathcal{P}_\epsilon \subset \operatorname{int}\mathcal{P}$. *The inverse image is* $T^2 \times \mathcal{P}_\epsilon$; *its two circle coordinates are* $\arg X, \arg Y$. *On the boundary disk circle*

put (X,Y) equal to the positive inverse of μ, multiplied by these two phases. On the six sectors of that boundary, the attaching map into the preceding blocks is the following monomial map, with the whole complex normal coordinate q included:

i	n_i	a_i	a_{i+1}
0	$q/(XY)$	X	Y
1	q/Y	XY	X^{-1}
2	qX	Y	$(XY)^{-1}$
3	qXY	X^{-1}	Y^{-1}
4	qY	$(XY)^{-1}$	X
5	q/X	Y^{-1}	XY

$$q = n_i a_i a_{i+1}. \tag{E.H7}$$

Use the cyclic identifications $(n_2,a_2,a_3)=(Z_1,Z_2,Z_0)$, $(n_4,a_4,a_5)=(Z_2,Z_0,Z_1)$ and the analogous two cyclic permutations of W for vertices $3,5$. Vertices $0,1$ are Z,W. The chart at an edge is $(z,a,b)=(a_{i+1},a_i,n_i)$, with (E.H4) at its other end. These rules specify (E.H7) also where a sector meets an annular block. The transition on a common sector is the monomial map $P_{i+1}^{-1}P_i$, where $P_i=((0,1),(r_i,1),(r_{i+1},1))$; the sum of its exponent rows is $(1,1,1)$, so it preserves $n_i a_i a_{i+1}$. Thus the sector maps specify the normal framing as well as the core map.

4. Expansion into ordinary handles. *We use the following fixed elementary rule on every parameterized block in this prescription:*

$$T^r \times D^k \times D^{n-r-k} \quad \text{attached on} \quad T^r \times \partial D^k \times D^{n-r-k}. \tag{E.H8}$$

Cut each ordered parameter circle into its two closed semicircles, with small endpoint collars. The first semicircle is its zero handle and the second is its one handle. For a bit vector $\varepsilon\in\{0,1\}^r$, the resulting ordinary handle is of index $k+|\varepsilon|$. Attach by increasing $|\varepsilon|$, then lexicographically. On the original attaching face use the displayed full-dimensional attaching embedding of the block; on a newly exposed parameter face use the identity endpoint map in that circle. The normal coordinates are the remaining ordered semicircle coordinates followed by the original normal coordinates. Corner rounding and the permutation putting negative coordinates first complete an ordinary framed attaching embedding $S^{k+|\varepsilon|-1}\times D^{n-k-|\varepsilon|}$. For one parameter it is the literal two-piece decomposition in (Vogel, 2009, Section 4.1, equation (16), p. 103).

Apply (E.H8) first to (E.H2), with $(r,k,n)=(1,1,6)$, and then to the core, with $(r,k,n)=(2,2,6)$. Together with the two balls this gives cusp handle counts $(2,3,4,2,1)$.

5. The torus bands and the finite caps. *Take the marked flat torus bundle of 13.18 over the disc with three holes, regarding the cusp boundary as the incoming boundary. Cut along the two marked disjoint arcs from the two finite boundary circles to the outer circle. The cut surface is an annulus; its torus bundle is the collar of the already constructed cusp boundary. Restoring the two strips attaches two blocks*

$$T^4 \times D^1 \times D^1. \tag{E.H9}$$

To fix their maps without a planar drawing, use the oriented ribbon polygon whose successive based inner boundary words are A_1,A_2,M_0 and whose outer word is their product. The negative bank of the jth cut has coordinate (x,t), the positive bank has (A_jx,t), and the strip uses the product normal coordinate. Reversing a bank orientation replaces that matrix by its inverse. This is the cut-polygon definition of the flat bundle in 13.18; the identity $A_1A_2M_0=I$ specifies the unfilled outer boundary. There is no affine term. Thus (E.H9) has its full attaching embedding, including the transverse interval, fixed by these linear maps. Expand each by (E.H8), with $(r,k,n)=(4,1,6)$. The unused boundary circles are precisely the two finite boundaries and the marked ordinary boundary. For clarity, the initial bank coordinates on the already attached cusp boundary are the explicit coordinates of proposition 13.17:

$$\begin{aligned} q=\eta e^{-2\pi i\theta}, \qquad T=-\log\eta/(2\pi),\\ (X,Y)=\exp\big(2\pi i((-\theta+iT)B_0a+b)\big), \qquad a=(g,u),\ b=(w,d). \end{aligned} \tag{E.H9a}$$

The exponential is componentwise; at an interval seam use $b \mapsto b - B_0 a$. Equations (E.H7) and (E.H9a) therefore give the bank embeddings into the existing handle boundary, including their derivatives.

For the jth finite cap use its already explicit $\mathcal{S}_j \times D^2_r$ and the entire boundary embedding (E.U3). The mapping torus $\mathcal{S}_j$ has the following product block construction: a block $T^3 \times D^1$ over the first semicircle of σ, followed by a block $T^3 \times D^1$ over the second semicircle, with endpoint maps I and B_j^{-1}. The first is a three-parameter zero handle, the second a three-parameter one handle. As a filling relative to $\mathcal{S}_j \times S^1_r$, the normal disk supplies two negative directions. Consequently the relative cap consists of

base-circle block	(r,k,n)	*full attaching data*
0	$(3,2,6)$	(E.U3), *the first semicircle, and the r-disk*
1	$(3,3,6)$	(E.U3), *endpoint maps I, B_j^{-1}, and the r-disk.*

(E.H10)

More explicitly an attaching face in the normal direction is sent to (E.U3) at $r = e^{2\pi i\alpha}$, and an attaching face in a base direction is the identical face of the preceding base block, crossed with D^2_r. At the base seam use B_j^{-1} on l and the identity on r. The inverse angular formula (E.U4) supplies the full map back to the boundary being filled. In particular the normal meridian is $v_j - m_j\rho_j$, not the base meridian. Apply (E.H8) to the two blocks in (E.H10). Each cap adds the ordinary relative counts $(1,4,6,4,1)$ in indices $2,\dots,6$. This completes the definition of $\mathcal{H}_6$.

6. The closed ten-dimensional handlebody. *For each handle of $\mathcal{H}_6$, take its product with the four ordered torus circles and apply (E.H8). This gives $\mathcal{H}_6 \times T^4$ with its full boundary framing. On the other side take the dual relative handle decomposition of $T^4 \times \mathcal{H}_6$: reverse the handle order and exchange the negative and positive disks of every handle. A handle of index j becomes one of index $10 - j$, and its attaching map is the belt embedding of the original handle. Glue its initial boundary by the explicit map*

$$(x_1, x_2, \theta) \longmapsto (x_1, x_2, -\theta). \tag{E.H11}$$

All subsequent dual attaching maps are obtained by this fixed exchange of the two coordinate disks, in reverse order, on the previously specified embeddings. Use the factor order E, T^4 on the first half and T^4, E on the second; their interchange has sign $(-1)^{24} = 1$. The boundary-angle reversal in (E.H11) is the orientation-reversing boundary gluing. This defines $\mathcal{H}_{10}$ entirely by ordinary handles. If

$$c(t) = 2 + 5t + 14t^2 + 22t^3 + 21t^4 + 10t^5 + 2t^6, \qquad c_{10}(t) = (c(t) + t^6 c(t^{-1}))(1+t)^4, \tag{E.H12}$$

the coefficients are the counts in the statement. The expanded lists of all 76 and 2432 handles are produced by the program `maps_ordinary_handles.py`.

Proof. ⟨1⟩1. The main geometric point is that the first three steps give *smooth blocks of the cusp tube*. We give the collar argument explicitly. Work in the smooth toric quotient of 13.17. Choose disjoint small coordinate neighborhoods of the two triple points. The three incident coordinate axes in each neighborhood are exactly the six ends in (E.H3). Remove smaller endpoint disks from each normalized double curve. Its remaining annulus has the two complex normal lines $\mathcal{O}(-1)|_{\text{annulus}}$, which are trivialized by (a, b) in (E.T6). The resulting normal domain in the tube is exactly the domain in (E.H2). Every ray in $\mathbb{C}^2$ meets its rounded boundary once and transversely. Indeed the three defining radius functions have positive radial derivative wherever the corresponding boundary face is active. Positive radial rescaling therefore gives a smooth family of four-balls as z ranges over this compact annulus. Choose the radial change to be constant linear scaling on a common sufficiently small ball about the origin and interpolate monotonically only away from that ball; it is then smooth at the origin as well as at the boundary and preserves the stated normal

frames. The same argument for the coordinate and product radius functions in (E.H1) gives six-balls at the triple points. Rounding can be chosen constant on the end attaching collars.

⟨1⟩2. At vertex i the normal-first coordinates are (n_i, a_i, a_{i+1}). At vertex $i+2$ the deck identification is their cyclic permutation, and the threefold composition is the identity by 13.17. At the two ends of C_i the coordinates are $(z, a, b) = (a_{i+1}, a_i, n_i)$ and $(w, A, B) = (a_{i-1}, a_i, n_{i-1})$. The ordinary $\mathcal{O}(-1)^2$ transition is (E.H4). Substituting the two cyclic permutations gives exactly (E.H3). This checks all six ends.

⟨1⟩3. We next choose the stratum collars so that their remaining interface is the entire attaching face of the dense block. On the normalized toric surface take a small collar of its hexagonal toric boundary, choosing the corner neighborhoods first and the edge strips second. Choose the strips narrower than the corner collars and invariant under the opposite-edge inversion and the two cyclic triple identifications. The uncovered moment region is the rounded interior disk $\mathcal{P}_\epsilon$. This can be done directly with six inward parallel affine inequalities in (E.H5), smoothing each corner inside its already chosen corner neighborhood. Thus the common frontier in the smooth part of the normalization is exactly $T^2 \times \partial\mathcal{P}_\epsilon$.

⟨1⟩4. The map (E.H6) gives the claimed coordinates: in logarithmic positive coordinates it is the gradient of a log-sum-exponential function. Its Hessian is twice the covariance matrix of the seven affinely spanning exponent vectors, hence positive definite. Strict convexity and the growth in each supporting direction give a diffeomorphism onto $\operatorname{int}\mathcal{P}$. The inverse on the compact disk and its frontier is smooth. The canonical line on this free torus locus is trivial; more specifically the toric function q is a nonzero toric factor times its normal coordinate. Hence q *itself*, including its phase, trivializes that normal line. The portion of the truncated tube over this disk is therefore exactly

$$(T^2 \times \mathcal{P}_\epsilon) \times D_q^2, \quad \text{with interface} \quad (T^2 \times \partial\mathcal{P}_\epsilon) \times D_q^2. \tag{E.H13}$$

In extending the preceding moment cuts off the zero fibre, hold their equations constant in this q coordinate on an open neighborhood of the compact frontier. On their ends use the product cuts of the annular and triple charts. A collar interpolation between these choices is an interpolation of positive radial cut functions, fixed on each already chosen end collar. Their radial derivatives remain positive, so its flow is a collar isotopy. The cuts can consequently be chosen to agree with (E.H13) exactly. In particular no twisting of the q disk is introduced on the six sector seams.

⟨1⟩5. These are blocks in a sufficiently small truncated neighborhood of the compact central fibre. Their central parts cover that fibre: two triple points, the three open double annuli and the remaining normalized surface core. Compactness from 13.17 permits one common sufficiently small η: the union of their open chart neighborhoods contains the compact zero fibre, so properness implies that it contains the entire inverse image of a sufficiently small closed q-disc. The nested cuts partition this entire inverse image. If a larger originally chosen tube is used, the difference is an exterior collar: on the compact annular region without critical points choose a vector field lifting the radial direction of q, and integrate it. On the prescribed outer collar choose the lift in the marked real torus coordinates of 13.17. It is then the identity on all those marked coordinates during radial collar transport.

⟨1⟩6. The embedding in (E.H13) is (E.H7). To check this, the exponent matrix for a unimodular chart is P_i^{-1}, applied to (X, Y, q). This gives each of the six displayed rows. Consecutive changes are $P_{i+1}^{-1}P_i$ and telescope to the identity. The translations identifying the two triple orbits are exactly those already used in (E.H3). All six normal coordinate multipliers in (E.H7) are thereby fixed, and the sector seams are equal maps on overlaps. Positive

rescaling of the chosen radial domains leaves these monomials unchanged before the collar reparametrization. This completes the smooth, fully framed identification of the cusp with steps 1–3.

⟨1⟩7. For completeness the handle expansion rule is itself a geometric equality. For one circle, write $S^1 = I_0 \cup I_1$. First attach $I_0 \times D^k \times D^{n-k-1}$; after rounding it is a k-handle. The remaining block is $(I_1 \times D^k) \times D^{n-k-1}$, a $(k+1)$-handle. Its attaching face consists of the original negative face and its two endpoint faces in the first handle. The stated maps on these three faces agree on their corners because they are restrictions of the same product coordinates. Iterate this argument on the ordered circle factors. It proves (E.H8), including all newly created attaching faces and their normal framings.

⟨1⟩8. The planar core relative to its incoming circle has two ordinary surface one-handles. Its flat bundle is a product over each strip, so restoring the two cut strips is exactly (E.H9). The zero section and the full lattice marking fix the normal coordinates on these attachments. Each cap in (E.H10) is the smooth product of 13.18. Its relative handle claim can also be checked by the function $-|r|^2 + \varepsilon f(l, \sigma)$, with a cutoff near the incoming boundary: choose f by a two-critical-point function of σ, perturbed on its two critical tori by the product Morse function on T^3. On the remaining arcs its base derivative dominates the perturbation, so there are no further critical points. The indices are two plus the base indices, as in (E.H10). Summing the blocks gives

$$(2,3,4,2,1,0,0) + 2(0,1,4,6,4,1,0) + 2(0,0,1,4,6,4,1) = (2,5,14,22,21,10,2).$$

Every comparison has its prescribed boundary value, so the relative diffeomorphism of 13.18 gives the first assertion of (E.H0). Taking the products, the dual relative handles and the boundary flip of 13.18 gives the second. This is an identification by explicit full-dimensional embeddings of ordinary handles.

⟨1⟩9. The exponent matrices, endpoint placements, angular rotations, sector seams, total linear monodromy and handle lists are checked by the program `maps_ordinary_handles.py`; the handle counts give Euler numbers 2 and 0.

⟨1⟩10. We finish by stating precisely why these data determine the smooth type, including its top handles. The relevant relative uniqueness results are Milnor's Theorem 3.13, which identifies an elementary cobordism from its full characteristic embedding, Theorem 1.4, which makes smooth composition unique with the marked boundary fixed, and the isotopy-extension Theorem 5.8 Milnor (1965). In the present notation the characteristic embedding is the whole map $S^{j-1} \times D^{n-j} \longrightarrow \partial W$, not just the attaching sphere or its homology class. Its normal coordinates are part of (E.H3)–(E.H11). The elementary consequence needed here can also be seen directly. If ϕ_0, ϕ_1 are joined by the isotopy of full attaching embeddings, extend the velocity field on their images to a slightly larger boundary collar and multiply it by a cutoff which is zero on the protected marked boundary. Its flow extends the attaching isotopy to an ambient boundary isotopy. Extending this flow across a product collar of W, and using the identity on the new coordinate handle, identifies $W \cup_{\phi_0} (D^j \times D^{n-j})$ with $W \cup_{\phi_1} (D^j \times D^{n-j})$. The same cutoff argument applies to the small radial corner changes above. This derives the relative, framed version from the stated isotopy-extension theorem.

⟨1⟩11. Apply this comparison successively in the displayed block and sector order. The initial ball comparisons are their coordinate identifications. At each next stage the attaching map and the prescribed attaching map agree in the computed normal coordinates, or differ by the explicitly described positive collar isotopy. The comparison of that stage is consequently the identity on the new coordinate handle and the product-collar extension on the previous stages. Theorems 3.13 and 1.4 give an inductive relative diffeomorphism of

the whole six-manifold. The final ordinary boundary coordinates are unchanged. The same argument, with the belt embeddings and the boundary flip, applies to every handle on the dual ten-dimensional side. In particular a top handle keeps its entire attaching map, so no homotopy-sphere ambiguity arises. The computed ordered full framed attaching data, under this relative uniqueness theorem, are the classifying data used for (E.H0).

□

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