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Mixed character moments and a zero-free half-plane for unitary Hecke L-functions

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Abstract

We prove that for every number field K and every continuous unitary Hecke character ω, the function $L_K(s,\omega)$ has no zeros in $\operatorname{Re} s > \theta_{\mathrm{G}}$, where

$$.8733326374459049352 < \theta_{\mathrm{G}} < .8733326374459051924.$$

The same bound holds for Artin L-functions and for L-functions of finite-dimensional continuous unitary Weil representations.

We estimate a mixed moment of two ordinary character sums with unequal lengths. A second Poisson summation in a signed correlation reduces the estimate to moments at a combined sextic character parameter. A differential equation describes the limiting range of the mixed moment and gives the optimal threshold within the specified reflected numerator comparison. We also determine the optimal saving for a centered cube-free sum in terms of the poles of a quotient of Hecke L-functions.

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1 Introduction

In this section we introduce the zero-free half-planes for unitary Hecke L-functions. The main bound is stated in Theorem 7.33; the notation and proof are described in Sections 2 and 3.

Let K be a number field and ω a continuous unitary Hecke character of K. Our conventions are those of [6]:

$$L_K(s, \omega(\mathrm{N}_K\cdot)^{it}) = L_K(s - it, \omega).$$

Imprimitive Euler factors have all their zeros on $\operatorname{Re} s = 0$. In particular the only possible pole is the simple pole at $1 + it$ when $\omega = (\mathrm{N}_K\cdot)^{it}$. The problem is to improve the boundary uniformly in K and ω, with estimates for each fixed presentation.

For finite-order characters and their norm twists, the one-stage argument of [7] gives the boundary 0.87336475. The joint moment induction in the same paper gives the unconditional boundary

$$B_* = 0.873345856106718576801184796065\ldots \tag{1}$$

for all continuous unitary Hecke characters. The unitary coefficient-two theorem of [3] gives the unconditional boundary $B_2 = (777 - \sqrt{1081})/852 = 0.8733819666132\ldots$.

Cycle 092 [1, 2] reports the boundary 0.873332794 for Dirichlet characters. Its proof uses a family over $\mathbb{Q}(\sqrt{-3})$. We use its normalized ordinary sums and compare the inverse and ordinary lengths at a single zero of each character. The resulting improvement over the known all-field bounds is obtained in Theorem 7.33. Brauer induction gives the same boundary for Artin L-functions, and its unitary version gives the bound for unitary Weil L-functions.

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2 Notation and preliminaries

In this section we fix the norm and character conventions used in Sections 4 and 7.

Write $\mathcal{O}_F$ for the ring of integers of a number field F, $\mathrm{N}\mathfrak{a} = |\mathcal{O}_F/\mathfrak{a}|$ for the norm of a nonzero integral ideal, and μ_{12} for the group of twelfth roots of unity. An ideal is kth-power-free if its prime valuations are less than k. For a finite set of places S, the superscript S on an Euler product indicates omission of the factors at S. A character on ideals is extended by zero at the primes of its modulus. We use $x_+ = \max(x, 0)$ and $e(t) = e^{2\pi i t}$.

For $W \in C_c^\infty((0,\infty))$, let

$$\widehat{W}(s) = \int_0^\infty W(x) x^s \frac{dx}{x}, \qquad p_J(W) = \max_{0\le j\le J} \|W^{(j)}\|_\infty.$$

An ordinary sum at scale X is normalized by $X^{-1/2}$. A norm twist at height v is included by replacing $W(x)$ with $W(x)x^{iv}$. Heights in different sums are independent. Lengths and widths are exponents of the indicated scale U. In the contour estimates $U = Z^D$, where D is the conductor exponent. Implied constants may depend on the fixed field, characters and test functions.

We first work over $F \supset \mu_{12}$, where the moving characters are sextic residue characters. For a fixed unitary character ω of F the family and its zero supremum are

$$\mathcal{F}_\omega = \{\eta\omega^j(\mathrm{N}\cdot)^{it} : j \in \{-1,0,1\},\ \eta \text{ of finite order},\ t \in \mathbb{R}\},$$
$$\beta_\omega = \sup\big(\{1/2\} \cup \{\mathrm{Re}\,\rho : 0 < \mathrm{Re}\,\rho < 1,\ L_F(\rho,\nu) = 0,\ \nu \in \mathcal{F}_\omega\}\big).$$

We descend to an arbitrary number field K from $F = K(\zeta_{12})$ by the abelian factorization in Section 7.

3 Outline of the main results and proofs

In this section we describe how the centered character sums, the moment estimates and the conductor comparison yield Theorem 7.33.

The signed difference in Section 4 uses two rectangles with equal scale products. Normalizing each norm twist at its own scale cancels the double Mellin residue. A single residue contains $L_F(s,\psi)/L_F(3s,\psi^3)$, which determines the optimal saving in the centered estimate.

Sections 5 and 6 derive the comparison of the moments and the limiting ordered width. For two ordinary lengths $z \le n$, that width has the form

$$S < \frac{1-n-z}{2} + F_\infty(z).$$

The curved inverse width supplied by [7] gives the cutoff in the differential equation for F_∞. A separate system for independently chosen inverse and ordinary widths is studied in Subsection 7.8.

In Subsection 7.3, two applications of Poisson summation reduce the signed correlation to a positive mean value at the combined sextic parameter. The centered estimate treats the fixed inducing characters. The other terms use the initial moments and earlier steps of the ordered induction in Subsection 7.5.

Subsection 7.18 inserts the resulting mixed moment into the reflected numerator estimate. At the critical values of the conductor and amplitude parameters, the curve for the inverse moment fixes the shorter ordinary length. The resulting exponent vanishes at θ_{G}. Directed rational estimates verify the comparison near that point and on its complement. The supplementary programs are identified by filename. The Mellin argument of Subsection 7.9 then excludes a zero to its right.

4 The signed reference over number fields

In this section we prove the centered estimate used for fixed inducing characters in Subsection 7.3.

For a fixed number field F, let ψ_1, ψ_2 be fixed unitary Hecke characters. Absorb any norm powers into the two independent heights v_1, v_2. Thus each ψ_j is trivial or has an entire Hecke L-function. Write $V_j(x) = W_j(x)x^{iv_j}$ with $W_j \in C_c^\infty((0,\infty))$, and put

$$\widehat{V}_j(s) = \int_0^\infty V_j(x)x^s \frac{dx}{x}.$$

Fix an ideal $\mathfrak{R}$ divisible by all the ramified primes of ψ_1, ψ_2 and by all the primes excluded in both rectangles.

For $X_1, X_2 > 0$ set

$$A(X_1, X_2) = \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ (\mathfrak{a}_1\mathfrak{a}_2,\mathfrak{R})=1 \\ \mathfrak{a}_1\mathfrak{a}_2 \text{ cube-free}}} \prod_{j=1}^{2} \psi_j(\mathfrak{a}_j)V_j(\mathrm{N}\mathfrak{a}_j/X_j).$$

The two test functions are evaluated at their respective normalized arguments. Equivalently each twist $(\mathrm{N}\mathfrak{a}_j)^{iv_j}$ is multiplied by $X_j^{-iv_j}$ before subtraction. More generally, at each prime dividing $\mathfrak{R}$ allow a fixed set of residual valuation pairs (i,j) with $i+j \le 2$, used in both rectangles. This includes a deletion in one coordinate alone and the restrictions after freezing a cube-free extracted divisor.

Theorem 4.1 (Centered cube-free reference)**.** *Suppose $1/2 \le B \le 1$ and $L_F(s, \psi_j^3)$ has no zeros for $\mathrm{Re}\, s > B$ whenever $\psi_{3-j} = 1$, with its possible pole permitted. Let*

$$X_1X_2 = Y_1Y_2 = T, \qquad L = \min(X_1, X_2, Y_1, Y_2) \ge 1.$$

For each $0 < \delta < 1 - B/3$ and $\varepsilon > 0$,

$$|A(X_1, X_2) - A(Y_1, Y_2)| \ll TL^{-\delta}(\mathrm{N}\mathfrak{R})^{\varepsilon} p_J(W_1)p_J(W_2)(1 + |v_1| + |v_2|)^J. \tag{2}$$

The constants and finite seminorms may depend on the fixed characters, field, test supports, δ and ε. If both characters are nonprincipal, the estimate holds for every $\delta < 4/3$ without any zero-free input.

Proof. For $x = \psi_1(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s_1}$ and $y = \psi_2(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s_2}$, the local numerator is

$$H_3(x,y) = (1-x)(1-y)(1+x+y+x^2+xy+y^2) = 1-(x^3+x^2y+xy^2+y^3)+x^3y+x^2y^2+xy^3. \tag{3}$$

The two-variable Dirichlet series is

$$L_F^{(\mathfrak{R})}(s_1, \psi_1)L_F^{(\mathfrak{R})}(s_2, \psi_2) \prod_{\mathfrak{p}\nmid\mathfrak{R}} H_3(x,y) \prod_{\mathfrak{p}\mid\mathfrak{R}} \mathcal{E}_{\mathfrak{p}}(s_1, s_2).$$

Here $\mathcal{E}_{\mathfrak{p}}$ is the exact local polynomial on the allowed valuations, equal to 1 for deletion in both coordinates. On positive strips its product is bounded by $6^{\#\{\mathfrak{p}\mid\mathfrak{R}\}} \ll_\varepsilon (\mathrm{N}\mathfrak{R})^{\varepsilon}$.

The product converges normally on every closed tube $\mathrm{Re}\, s_j \ge a > 1/3$. Take $1/3 < a < 1/2$ and move both Mellin lines to a. The double residue, when present, is a constant times $T\widehat{V}_1(1)\widehat{V}_2(1)$ and cancels exactly. The double integrals are $\ll T^a$ times the indicated controls, and $T^a \le TL^{-2(1-a)}$. Taking a arbitrarily close to $1/3$ proves the final assertion.

In a single residue take, for example, $\psi_1 = 1$ and $s_1 = 1$. For $\operatorname{Re} s_2 \geq \sigma > 0$,

$$\frac{H_3((\mathrm{N}\mathfrak{p})^{-1}, y)}{1 - y^3} = 1 + O_\sigma((\mathrm{N}\mathfrak{p})^{-1-2\sigma} + (\mathrm{N}\mathfrak{p})^{-2-\sigma} + (\mathrm{N}\mathfrak{p})^{-3}).$$

The remaining Euler product is therefore bounded down to this line. The meromorphic factor in s_2 is $L_F^{(\mathfrak{R})}(s_2, \psi_2)/L_F^{(\mathfrak{R})}(3s_2, \psi_2^3)$. Choose $B/3 < \sigma < \min(a, 1 - \delta)$, increasing a within $(1/3, 1/2)$ if necessary. Move the single-residue line from a to σ.

The inverse L-factor is holomorphic with polynomial height growth on the buffered strip. Indeed on disks centered at $2 + it$ and contained in $\operatorname{Re} s > B$, a logarithm exists for large $|t|$. The upper polynomial bound for L, the Euler product bound at the center and Borel–Carathéodory bound the logarithm by $O(\log(2 + |t|))$ on smaller disks. Bounded heights follow from holomorphy of the inverse. Deleted Euler factors and their inverses on fixed positive strips loss $(\mathrm{N}\mathfrak{R})^\varepsilon$ since $\prod_{\mathfrak{p}|\mathfrak{R}}(1 - (\mathrm{N}\mathfrak{p})^{-b})^{-1} \ll_{b,\varepsilon} (\mathrm{N}\mathfrak{R})^\varepsilon$ for $b > 0$.

The single integral has size $\ll X_1 X_2^\sigma$ and the analogous one in the other orientation has size $\ll X_1^\sigma X_2$. Each is $\ll TL^{-\delta}$. Integration by parts in the Mellin test functions gives the stated seminorms, also for the first log tests $V_j(x) \log x$. □

The boundary B_2 gives $\delta < 0.7088726777955\ldots$ for arbitrary unitary characters. For cubic powers of finite-order characters and their norm twists, the bound 0.87336475 gives $\delta < 1 - 0.87336475/3$. The normalized sum has the same saving after division by the common $\sqrt{T}$.

Proposition 4.2 (Optimal upper limit for the reference saving)**.** *An estimate of the form (2) for every pair of scales and test functions, even restricted to $F = \mathbb{Q}$ and $\psi_1 = \psi_2 = 1$, $\mathfrak{R} = (1)$, requires $\delta \leq 5/6$. Under GRH for the cubic powers, the supremum of the uniform savings in Theorem 4.1 is therefore* 5/6.

Proof. Let $D(s_1, s_2)$ denote the cube-free Dirichlet series in the preceding proof and $R(s) = \operatorname{Res}_{s_1=1} D(s_1, s)$. Then

$$R(s) = \frac{\zeta(s)}{\zeta(3s)} \prod_p \frac{H_3(p^{-1}, p^{-s})}{1 - p^{-3s}}.$$

For $x = p^{-1} \in (0, 1)$ the roots in y of $y^2 + (1 + x)y + 1 + x + x^2$ have modulus $\sqrt{1 + x + x^2} > 1$. Thus the last product converges and has no zeros for $\operatorname{Re} s > 0$. There exists a nonreal zero ρ of ζ with $\operatorname{Re} \rho \geq 1/2$ and $\zeta(\rho/3) \neq 0$. Start with a zero of positive height and real part at least 1/2. If its third is a zero, the functional equation gives the next zero $1 - \overline{\rho}/3$. It has real part at least 2/3 and one third of the previous height. Repeating indefinitely contradicts the isolatedness of zeros. Hence R has a pole at some $\rho/3$ with $\operatorname{Re} \rho/3 \geq 1/6$.

Fix $\lambda > 1$ and $\widehat{V}_1(1) \neq 0$. Compare the scales (U, L) and $(U/\lambda, \lambda L)$ and let $U \to \infty$ with L fixed. Moving only the first Mellin line, with the second still to the right of one, shows that the difference divided by $U\widehat{V}_1(1)$ tends to $G(L) - \lambda^{-1}G(\lambda L)$, where G is the inverse Mellin transform of $\widehat{V}_2(s)R(s)$. The bound (2) makes this difference $O(L^{1-\delta})$.

It vanishes for small L, by the Dirichlet series for R and the compact support of V_2. Its Mellin transform therefore continues holomorphically to $\operatorname{Re} s > 1 - \delta$, and on the absolute line is

$$(1 - \lambda^{s-1})\widehat{V}_2(s)R(s).$$

Take $\widehat{V}_2(\rho/3) \neq 0$ by choosing the support sufficiently close to one. Since $\operatorname{Re}(\rho/3) < 1$, this transform has a pole at $\rho/3$. It follows that $\delta \leq 1 - \operatorname{Re} \rho/3 \leq 5/6$. □

Lemma 4.3 (Growth after cancellation of quotient poles)**.** *Fix a number field and a fixed unitary Hecke character ψ. If $L_F(s,\psi)/L_F(3s,\psi^3)$ has no pole for $\operatorname{Re} s > a$, apart from a possible principal pole at 1, then it has polynomial vertical growth in every closed strip in $\operatorname{Re} s > a$ away from this pole.*

Proof. Write $f(s) = L_F(3s,\psi^3)$ and fix a target closed strip there. Choose $b > 1$ to its right. At height t center a fixed disk at $b + it$, wide enough to contain the strip segment with heights between $t-2$ and $t+2$ and a smaller positive margin to its left. For $|t|$ large, f is holomorphic on a slightly larger disk, with $|f| \ll (2+|t|)^A$, while the Euler product gives $|f(b+it)| \gg 1$. Jensen's formula bounds the number of zeros in the disk by $O(\log(2+|t|))$. Use a radius between the two disks avoiding boundary zeros, and divide by its finite Blaschke product. The zero-free quotient has the same boundary upper bound as f, and its value at the center has modulus bounded below. Harnack's inequality applied to the logarithm of the upper bound divided by this quotient gives, in the smaller disk,

$$|f(s)|^{-1} \ll (2+|t|)^{A'} \exp\left(\sum_{\rho} \log_+ \frac{1}{|s-\rho|}\right),$$

where the sum includes the zeros of the finite product with multiplicity.

Take two horizontal segments with heights in $[t-2,t-1]$ and $[t+1,t+2]$, and a vertical segment with real part in a fixed positive-length interval to the left of the target strip and inside $\operatorname{Re} s > a$. On each segment the sum is dominated by the same sum with $|s-\rho|$ replaced by the distance in its fixed coordinate. The integral of $\log_+(1/|x-\alpha|)$ over any interval is at most 2.

Choose the three coordinates by averaging. The sum is then $O(\log(2+|t|))$ everywhere on these three segments. On the fourth side $\operatorname{Re} s = b$ use the Euler product. The ratio has polynomial growth on the boundary of this rectangle. Every denominator zero in the rectangle is removable in the ratio by hypothesis. The maximum modulus principle gives the desired bound inside. Bounded heights follow by compactness. □

Proposition 4.4 (Exact optimal centered exponent)**.** *Put*

$$\sigma_{\mathrm{pole}} = \sup\left(\{0\} \cup \left\{\operatorname{Re} s : \ \operatorname{Re} s > 0, \ \frac{L_F(s,\psi)}{L_F(3s,\psi^3)} \text{ has a pole at } s \neq 1\right\}\right),$$

where F ranges over the fields in the centered estimate and ψ ranges over its fixed unitary characters with the norm powers removed. The centered cube-free estimate over these fields and characters has supremal saving $1-\sigma_{\mathrm{pole}}$, with constants allowed to depend on the fixed field and characters. For this assertion the fields include $\mathbb{Q}$ and the admissible pairs include $(1,\psi)$.

Proof. In the centered estimate only the single residues need to be moved below a fixed $\sigma_0 \in (1/3,1/2)$. When the first coordinate is principal, its residue in the second coordinate is a fixed constant times

$$\frac{L_F(s,\psi)}{L_F(3s,\psi^3)}\,\mathcal{K}_F(s).$$

The good local factors of $\mathcal{K}_F$ are $H_3((N\mathfrak{p})^{-1},\psi(\mathfrak{p})(N\mathfrak{p})^{-s})/(1-\psi(\mathfrak{p})^3(N\mathfrak{p})^{-3s})$. Their product converges absolutely for $\operatorname{Re} s > 0$. The omitted primes and the local valuation polynomials satisfy the same bounds as in Theorem 4.1. By Lemma 4.3, the single residue moves to any fixed line to the right of σ_{pole} and to the left of σ_0. The lines can be taken in this order because $\sigma_{\mathrm{pole}} \le 1/3$. The unshifted interior double integral of size $O(T^{\sigma_0})$ is already within the asserted unnormalized bound $TL^{-1+\sigma}$ for $\sigma > \sigma_{\mathrm{pole}}$ and $1 \le L \le \sqrt{T}$.

Hence every saving below $1-\sigma_{\mathrm{pole}}$ holds.

For the converse use two ordinary factors $(1,\psi)$ and only the fixed conductor exclusions. At a good prime put $x=(N\mathfrak{p})^{-1}$ and $y=\psi(\mathfrak{p})(N\mathfrak{p})^{-s}$. The exact identity is

$$H_3(x,y)=(1-x)(1-y)\{1+x+x^2+(1+x)y+y^2\}.$$

The two roots in y of the quadratic have modulus $\sqrt{1+x+x^2}>1$. The correction $\mathcal{K}_F(s)$ is thus nonvanishing in $\operatorname{Re}s>0$, including its finite good-factor omissions and the conductor corrections. Let $s_0\neq 1$ be an actual pole of the ratio there. Take the equal-product arms (V,L) and $(V/c,cL)$ with $c>1$. Divide the unnormalized centered estimate by V and send $V\to\infty$ at fixed L.

The limit in the first coordinate is its principal residue. This follows by Mellin inversion and shifting that coordinate to $\operatorname{Re}s=1/2$ when the second length is fixed, since the second sum is finite and the first cube-free variable has only finitely many additional local restrictions. The remaining one-variable difference has Mellin transform equal to the principal constant times

$$(1-c^{s-1})\widehat{V}_2(s)\frac{L_F(s,\psi)}{L_F(3s,\psi^3)}\mathcal{K}_F(s).$$

Choose the two test functions with the first integral nonzero and $\widehat{V}_2(s_0)\neq 0$. The factor $1-c^{s_0-1}$ is nonzero because $\operatorname{Re}s_0\le 1/3$. A centered saving greater than $1-\operatorname{Re}s_0$ would continue this transform holomorphically across the pole. Taking the supremum over s_0 proves the opposite inequality. □

5 The mixed comparison at the critical point

In this section we determine the critical exponent in the comparison of [2]. Section 6 then varies the ordinary width.

Write $\theta=1-c$, $\ell=4c-1/3$, and take the fixed source of Appendix P of [2]. With shorter ordinary cap z, its marked width is $(1-n-z)/2+F(z)$ for $0.369\le z\le 0.3888$, $z\le n\le 1.15z$, where

$$F(z)=\begin{cases}F_+(z), & z\ge 0.3735,\\ F_+(0.3735)+0.174(0.3735-z)+6.7(0.3735-z)^2, & z<0.3735,\end{cases}$$
$$F_+(z)=4.618(0.3888-z)^2+50.5(0.3888-z)^3+0.00031(0.3888-z)-10^{-7}.$$

Let $w=(12B_*-14)/(108B_*-99)$ and use the tangent coefficients of [6]:

$$g_0=\frac{17-18w}{3(45w-14)},\qquad g_1=\frac{135w^2-42w-38}{6w(45w-14)},\qquad b(z)=g_1z-g_0/2+F(z).$$

These source parameters are fixed in the comparison.

In the critical point $p=1/2$ and

$$d=\frac{\theta-1/2-\ell/2}{1/2+\ell}=\frac{4-18c}{1+24c}. \tag{4}$$

Set

$$v=5/6-d,\qquad k=1-p(1+2g_1),\qquad u_0=1-p(1+6g_0-4g_1),$$
$$J=3v+2dk,\qquad T_0=1+\frac{du_0}{J},\qquad R=\frac{1}{6}+vT_0.$$

Let h be the upper conductor exponent. For $4\ell < h < 2\ell/0.415$ the width of [2, (Q1)] at this edge, with zero auxiliary losses, is $C(h) = (2(h-2\ell)-h)/2 = h/2 - 2\ell$. Substitution gives

$$E(h) - \theta = \frac{1}{2} + \frac{\ell}{2} + \frac{d}{2} + d\ell - \theta + hq = hq, \qquad q = \frac{3}{4}R - \frac{5}{12} - \frac{d}{8}.$$

In particular the required saving q at this parameter is independent of h in this corridor.

Theorem 5.1 (Exact limiting mixed comparison)**.** *Use the two ordinary source caps just specified and the continuous zero-loss form of the finite set of cutoffs of [2, (P14), Appendix Q]. At the parameter* (4)*, for* $0.12666 \le c \le 0.12668$*, the largest uniform mixed margin obtainable by choosing the end of the set of cutoffs is*

$$M(c) = \frac{1}{2}b\left(\frac{T_0}{3} - \gamma\left(1 + \frac{v}{dk}\right)\right) - \frac{k\gamma}{2} - \frac{q}{d}, \qquad \gamma = \frac{4q}{3v}. \tag{5}$$

It is positive for $c < c_{\mathrm{crit}}$ *and negative for* $c > c_{\mathrm{crit}}$ *in this interval, where*

$$0.12666720609870486 < c_{\mathrm{crit}} < 0.12666720609870487,$$
$$0.87333279390129513 < 1 - c_{\mathrm{crit}} < 0.87333279390129514.$$

This is a necessary boundary for the full comparison using these sources and this set of cutoffs in the given conductor corridor.

Proof. At the offset $\xi \ge 0$, keep the preceding long inverse and the current short inverse and ordinary detectors on the same parameter and at the same zero. Let $m = T_0/3$, $\Delta = z - m$. The mixed surplus in [2, (P14)] in the zero-loss limit is

$$P(\xi, z) = \frac{1}{2}b(z) + \frac{1}{2}\max\{-k\xi,\, k\Delta + \max(v\xi/d, 2k\Delta_+)\} - q/d. \tag{6}$$

The endpoint must be at least γ: the last long-inverse deficit $v\xi$ requires $3v\xi/4 \ge q$. At fixed $0 \le \xi \le \gamma$, the two outer faces cross at

$$z(\xi) = m - \xi\mu, \qquad \mu = 1 + \frac{v}{dk}.$$

This belongs to the allowed support $[m - \xi - \lambda_x, m + \lambda_y]$, since $\lambda_x \ge 4q/(3dk) = \gamma v/(dk)$. Before the crossing P decreases in z since $b'(z) < 0$ there. After the crossing it increases since $b'(z) > -k$ throughout the source interval. These derivative bounds hold on both sides of the seam of F. The exact minimum at fixed ξ is hence $b(z(\xi))/2 - k\xi/2 - q/d$. Its derivative in ξ is $-\mu b'(z(\xi))/2 - k/2 < 0$ on $[0, \gamma]$. Thus the last offset gives (5). If the final offset itself is already estimated by the long-inverse bound, the same infimum is obtained by approaching it from within the interval. A longer set of cutoffs still contains these offsets.

For the scalar checks use the polynomial of [6] to isolate

$$0.873345856106718576801184 < B_* < 0.873345856106718576801185.$$

Rational interval substitution on $0.12666 \le c \le 0.12668$ gives

$$0.0000105 < q < 0.0001725, \qquad 0.37429 < z(\xi) < 0.37616,$$
$$-0.09022 < b'(z(\xi)) < -0.06529, \qquad -0.10567 < -\tfrac{1}{2}\mu b'(z(\xi)) - \tfrac{1}{2}k < -0.06500,$$
$$-15.475 < M'(c) < -14.710.$$

For the last bound use

$$d' = -\frac{114}{(1+24c)^2}, \quad T_0' = \frac{5u_0 d'}{2J^2}, \quad q' = \frac{3}{4}(-d'T_0 + vT_0') - \frac{d'}{8}, \quad \gamma' = \frac{4}{3}\left(\frac{q'}{v} + \frac{qd'}{v^2}\right), \quad \mu' = -\frac{5d'}{6d^2k}.$$

At the two displayed rational endpoints for c_{crit} the margins lie in $(8.1406, 8.1408)10^{-17}$ and $(-7.0294, -7.0291)10^{-17}$, respectively. This proves uniqueness and the sign on the stated interval. □

Remark 5.2 (Algebraic equation). Put $Q = (8/3 - 7d)J + 6vdu_0 = 8Jq$, $D = 6dkv$, and $N = 2dkv - (8/3 - 7d)(dk + v) - 2du_0 v$. Then $z(\gamma) = N/D$ and $k\gamma/2 + q/d = kQ/(4D)$. Write $g_1 z - g_0/2 + F_+(z) = \sum_{j=0}^{3} b_j z^j$. The contact is isolated by the polynomial

$$\sum_{j=0}^{3} b_j N^j D^{3-j} - \frac{k}{2} Q D^2 = 0$$

of degree at most six in d over $\mathbb{Q}(w)$, with c recovered by (4).

5.1 Attainment for the full limiting scalar comparison

In this subsection we use the critical comparison of Section 5 to prove sufficiency on the full conductor range.

We keep the source, the tangent, the conductor width and the comparison routes in Appendix Q of [2] fixed. The auxiliary losses in the scalar comparison may tend to zero. In this class the critical point gives the infimum for the full four-edge comparison.

For the top conductor edge let

$$A_c(d,p) = \frac{1}{2} + \frac{\ell}{2} + \frac{d}{2} + 2dp\ell - \theta, \qquad A_1(d,p) = \frac{3}{4}R - \frac{5}{12} - \frac{dp}{4}, \qquad q_h = \frac{A_c(d,p)}{h} + A_1(d,p).$$

On the branch $C(h) = h/2 - 2\ell$, the identity is $E(h) - \theta = hq_h$. For a positive required saving introduce the function

$$\Phi(d,p,q) = p\,b\left(\frac{T_0}{3} - \frac{4q}{3v}\left(1 + \frac{v}{dk}\right)\right) - \frac{Jq}{3dv}, \qquad \mathcal{M}(d,p;c,h) = \Phi(d,p,q_h).$$

The proof of Theorem 5.1 gives this exponent whenever the length and offset derivatives there have their indicated signs, with p in place of $1/2$. In particular $\Phi_q < 0$ if the exponent at the intersection is decreasing in the offset.

Let $d_\circ = (4 - 18c_{\mathrm{crit}})/(1 + 24c_{\mathrm{crit}})$. At $(c,d,p) = (c_{\mathrm{crit}}, d_\circ, 1/2)$ one has $A_c = 0$. Evaluate Φ_d and Φ_q at $(d_\circ, 1/2, A_1(d_\circ, 1/2))$, taking partial derivatives with q independent of d. Define

$$h_\circ = -\frac{1/2 + \ell}{\partial_d A_1 + \Phi_d/\Phi_q}. \tag{7}$$

Here $0.817584081062395 < h_\circ < 0.817584081062398$. Differentiation gives

$$\mathcal{M}_d = \Phi_d + \Phi_q\left(\partial_d A_1 + \frac{1/2 + 2p\ell}{h}\right), \qquad \mathcal{M}(d_\circ, 1/2; c_{\mathrm{crit}}, h_\circ) = \mathcal{M}_d(d_\circ, 1/2; c_{\mathrm{crit}}, h_\circ) = 0.$$

Set $\mathcal{R} = [0.4254, 0.4261] \times [0.4998, 1/2]$. At $c = c_{\mathrm{crit}}$ and $h = h_\circ$, the following are directed bounds on $\mathcal{R}$. The derivative in ξ is that of the exponent at the intersection for $0 \le \xi \le 4q_h/(3v)$.

	lower bound	upper bound
q_h	0.00005274	0.00009532
$z(\xi)$	0.37507	0.37622
$\partial_\xi\{p\, b(z(\xi)) - k\xi/2 - q_h/d\}$	−0.10725	−0.08218
$\mathcal{M}_{dd}$	1.0131	1.0476
$\mathcal{M}_p$	−0.03082	−0.02927

These bounds use 40 equal subintervals in d and 20 in p, with rational interval evaluation of the functions and their derivatives. For example, in a subinterval an expression f is carried as (f, f', f''). Multiplication uses $(fg)'' = f''g + 2f'g' + fg''$ and reciprocation uses $(1/f)'' = 2(f')^2/f^3 - f''/f^2$. Substitute the isolating intervals

$$\begin{aligned} c_{\mathrm{crit}} &\in (0.1266672060987048651, 0.1266672060987048655), \\ g_0 &\in (0.0580591192817794997246, 0.0580591192817794997249), \\ g_1 &\in (0.0759961848675958585735, 0.0759961848675958585738) \end{aligned}$$

in (7) and the displayed formulas. The bound for $z(\xi)$ lies in a single piece of the cubic. The original length and ratio caps hold there with strict margins.

It follows that

$$\mathcal{M}(d, p; c_{\mathrm{crit}}, h_\circ) \ge \tfrac{1}{2}(1.0131)(d - d_\circ)^2 + 0.02927(1/2 - p) \qquad ((d, p) \in \mathcal{R}).$$

Moreover $\partial_c q_h = (3+8dp)/h > 0$ for fixed (d, p, h). By continuity the required saving stays positive, and the offset derivative stays negative, for all $c < c_{\mathrm{crit}}$ sufficiently close to c_{crit}. Thus $\mathcal{M}$ becomes strictly positive throughout $\mathcal{R}$.

For the complement of $\mathcal{R}$ apply the four tilings and the directed comparisons of Appendix Q of [2] at

$$c^+ = 0.12666720609870487, \qquad 0.8175840800623962 \le h \le 0.8175840820623962.$$

The top exponent increases with c at fixed d, p, h. The normalized required saving formed at c^+ is therefore an upper bound throughout the contact interval. The supplementary program `check_locked_return.py` implements Appendix Q of [2] at the parameters above. When a visited closed rectangle lies in $\mathcal{R}$, use the displayed derivative bounds in place of the mixed exponent tests on that rectangle. Keep the ordinary-moment, length, support and conductor tests before each substitution. The output is

original tiling rectangles	57
initial mixed rectangles	2045
complete four-way subdivisions	114
new exterior rectangles	47
mixed terminals certified by the computation	2325
rectangles using the local derivative bound	15

$$2325 + 47 + 15 = 2045 + 3 \cdot 114.$$

On the 2325 mixed terminals the lower normalized and original-scale margins are respectively 376487/500000000000000 and 52297/200000000000000. The checks on source lengths and ratios use strict inequalities and retain room under the change in c across the displayed contact interval.

The slope and padding rules of the scalar comparison carry the upper conductor estimate to the other retained conductor values.

Consequently $1 - c_{\mathrm{crit}}$ is the infimum of this fixed-source, zero-loss scalar comparison for h in the specified corridor. Every smaller c sufficiently close to c_{crit} has a strict comparison at $h_\circ$. Compactness then allows finitely many cutoffs, strict source decrements and sufficiently small losses. This conclusion concerns the supplied numerical inequalities. An application to a family of L-functions requires the analytic source and comparison with the same coefficients and quantifiers.

5.2 A refined ordered source and its limiting scalar comparison

In this subsection we enlarge the ordinary width in Section 5 by replacing the upper cubic with a quartic.

We work with the characters over $\mathbb{Q}(\sqrt{-3})$ in Appendix Q of [2]. Keep its conductor geometry and the tangent (g_0, g_1) fixed. Let $Y = 0.3888$, $s = Y - z$, and replace the upper piece of the ordered width function by

$$F_{4,+}(z) = 4.62423382s^2 + 46.8110959s^3 + 145.74552s^4 + 0.0008076592s - 10^{-8}.$$

The lower piece has the same linear and quadratic increments as F at 0.3735. Set $b_4(z) = g_1 z - g_0/2 + F_4(z)$. Use $u = 4 \cdot 10^{-7}$, $H_0 = 1 - 6u$, and $\rho = 4 \cdot 10^{-13}$ in the ordered-source induction. The declared caps on the two ordinary factors retain $0.369 \le z \le 0.3888$ and $z \le n \le 1.15z$.

The source computation evaluates the Appendix P terminal exponents with this width function on 19800 closed length intervals of width 10^{-6}, with 158400 response bins and 21779 descending stages. Rational fifth-root bounds are used for the earlier inverse width. All terminal, reference, and radial margins are positive.

Use the inverse count improvement

$$G(r) - (g_0 - g_1 r) \ge L_1(r - a_1), \qquad L_1 = 0.00201915, \qquad a_1 = 0.7525856, \qquad \tfrac{3}{4} \le r \le \tfrac{7}{9}. \tag{8}$$

For the verification take the tangent to G at 0.75285. The fifth root in $G(0.75285)$ lies between $7202297048628401409425/10^{22}$ and $7202297048628401409426/10^{22}$. The difference of that tangent and the claimed lower affine bound for G exceeds $1.8 \cdot 10^{-10}$ at both ends of $[3/4, 7/9]$. Convexity gives (8). The original tangent count is available throughout its original range.

The charged finite comparison for these inputs gives

$$\theta_{\mathrm{finite}} = 0.87333265708. \tag{9}$$

At $c = 1 - \theta_{\mathrm{finite}}$ the cover uses all 57 original rectangles, with 2031 initial mixed rectangles, 11478 complete four-way subdivisions, 36426 mixed terminals and 39 new exterior rectangles. The equality $36426 + 39 = 2031 + 3 \cdot 11478$ accounts for all mixed subsets. The normalized and original-scale margins exceed $5019/10^{15}$ and $873/(5 \cdot 10^{14})$, respectively.

This comparison uses 2^{23} cutoff subintervals, tolerance in the detecting lengths 10^{-16} and inward required saving 10^{-15}, with the height, length, and contour allowances of the retained tickets.

For the limiting calculation take $m = T_0/3$, $\gamma = 4q_h/(3v)$, and $S_1 = 2pL_1$. The first outer face in (6) is now $-k\xi + S_1(T_0 - \xi - z - a_1)_+$, because the inverse length is at least $T_0 - \xi - z$. On the local rectangle $\mathcal{R}$ its crossing with the increasing face occurs at $z \le m$, with

$$z(\xi) = \max\left\{ m - \xi\left(1 + \frac{v}{dk}\right), \frac{km - k\xi - v\xi/d + S_1(T_0 - \xi - a_1)}{k + S_1} \right\}.$$

The left face decreases in z and the right face increases on the checked interval. At the crossing the exponent is

$$P_4(\xi) = pb_4(z(\xi)) + \frac{k(z(\xi) - m) + v\xi/d}{2} - \frac{q_h}{d}.$$

Its one-sided derivatives are

$$\left(pb_4'(z) + \frac{k}{2}\right)\left(-1 - \frac{v}{d(k+j)}\right) + \frac{v}{2d}, \qquad j = 0,\ S_1.$$

They are negative on $\mathcal{R}$ for $0 \le \xi \le \gamma$.

The second crossing branch applies at the final cutoff, since $2m + v\gamma/(dk) > a_1$. The infimum over cutoffs is $P_4(\gamma)$.

At $p = 1/2$, $d = (4 - 18c)/(1 + 24c)$, the zero c_4 of $P_4(\gamma)$ satisfies

$$0.1266673429226568382 < c_4 < 0.1266673429226568388.$$

The exponents at the endpoints lie in $(1.691, 1.694)10^{-18}$ and $(-7.388, -7.384)10^{-18}$, respectively. On $0.12666734 \le c \le 0.12666735$ its derivative lies in $(-15.1313, -15.1307)$ by rational interval evaluation. The tuned conductor of (7), with Φ replaced by this new final crossing exponent, lies in $(0.817496882646376, 0.817496882646380)$.

On the same 40 by 20 rational interval subdivision of $\mathcal{R}$, with $c = c_4$ and the tuned conductor, the lower and upper bounds are

$$0.00005356 < q_h < 0.00009608, \qquad 0.37505 < z(\xi) < 0.37622, \qquad 2m + v\gamma/(dk) > 0.75260,$$
$$-0.10720 < P_4'(\xi) < -0.07977, \qquad 1.0089 < \partial_d^2 P_4(\gamma) < 1.0536,$$
$$-0.03125 < \partial_p P_4(\gamma) < -0.02896.$$

Thus the local contact is unique, and $c < c_4$ gives strict surplus throughout $\mathcal{R}$ for c sufficiently close to c_4.

At the upper isolating endpoint for c_4 and

$$0.81749688164637785 \le h \le 0.81749688364637785$$

the exterior computation retains the original contour inequalities, the actual ratio of the two ordinary lengthss and the directed annular losses. It divides each interval containing the intersection at the change of face, the long-branch join and the seam of F_4. On each resulting piece the exponent is an affine function plus a convex polynomial, so the endpoints and the interior stationary point suffice.

The cover has 2460 verified mixed terminals, 39 new exterior rectangles and 15 local rectangles. In fact $2460+39+15 = 2031+3\cdot161$. The certified exterior mixed margins exceed $140261/10^{15}$ and $24459/(5\cdot10^{14})$ in normalized and the original scale, respectively. Together with the local derivative bounds this yields the exact limiting scalar infimum for the fixed quartic and affine count:

$$0.8733326570773431612 < 1 - c_4 < 0.8733326570773431618.$$

This scalar optimality keeps the displayed source, inverse count, conductor width, and comparison rules fixed.

6 The limiting ordered moment range

In this section we derive the differential equation for the ordered width used in Subsection 7.18.

We now vary the added width in the comparison (P3), (P5), (P6) of [2]. Use the short inverse frontier $q(r) = (1 - r)/2 + G(r)$, where

$$G(r) = \frac{11r}{16} - \frac{7}{24} - \frac{5}{16}\left(7r^4(r - 2/3)\right)^{1/5}, \qquad 0.738 \le r \le 7/9.$$

At $r \ge 7/9$ continue with $G = 0$ while the initial short inverse bound is available. Let $y_0 = 7/18$, $A = 5/6$, and $\mathcal{R}(r) = r + q(r)$. Thus $\mathcal{R}$ is increasing and $\mathcal{R}'(r) > 0.35$ on the range in question. For y, f put

$$\begin{aligned} R_0 &= \tfrac{1}{2} + y + 2f, \qquad r_* = \max\{2y, \mathcal{R}^{-1}(R_0)\}, \qquad T_* = r_* + y, \\ k_*(y, f) &= A\,\frac{T_* - 1}{T_* - R_0}, \qquad \mathcal{D}(y, f) = \frac{f + 1/3 - 1/(6k_*(y, f))}{y}. \end{aligned} \tag{10}$$

In the definition of r_*, use $r_* = 2y$ if $R_0 \le \mathcal{R}(2y)$.

Theorem 6.1 (Infimum of the ordered moment losses)**.** *Start with added width zero at y_0. Allow an arbitrary finite sequence of descending steps $y = Hy'$, $H = 1 - 6u$, supplied by (P3), (P5), (P6), with $\nu = e$ included among the response classes. Below the low-response threshold use (P3). Above it require one cutoff per response class that bounds each of the three possible possible pairs of detecting lengths. The cutoff satisfies $r \ge 2y'$ and $m \ge y'$. Allow the response parameter to range to 3/4. Then all widths granted by these scalar steps are bounded by the solution*

$$F'_\infty(y) = \mathcal{D}(y, F_\infty(y)), \qquad F_\infty(y_0) = 0, \qquad 0.369 \le y \le y_0. \tag{11}$$

Every strictly smaller width on a compact subinterval below y_0 is approached by finite steps with strict margins in these three comparisons. In particular,

$$\begin{aligned} 0.002212 &< F_\infty(0.369) < 0.002218, \\ 0.000965 &< F_\infty(0.37535) < 0.000968. \end{aligned}$$

Proof. At height H, let $F(y)$ be the proposed added width, let $f = F(y')$, and put

$$L = 2u + F(y) - Hf, \qquad B = \tfrac{1}{2} - \tfrac{H}{4} + \tfrac{y}{2} + F(y) - u, \qquad \tau = u - \rho, \quad 0 \le \rho < u.$$

If $L > 0$, the low-response comparison $eL \le \tau$ gives the threshold $e_0 = \tau/L$. At its upper end, (P6) with $\nu = e$ requires the short inverse cutoff to satisfy

$$\mathcal{R}(r) \ge \frac{2(B - \tau/e_0)}{H} = \frac{2(B - L)}{H} = \tfrac{1}{2} + y' + 2f.$$

The ordinary substitution has $m \ge y'$. The long inverse comparison at that threshold gives

$$e_0 \ge A\,\frac{r + m - 1}{r + m - (1/2 + y' + 2f)} \ge k_*(y', f).$$

The second inequality follows by differentiation in $r+m$, since $1/2+y'+2f < 1$. If the low-response formula bounds every response, then $e_0 \ge 3/4$ instead. With $k_*(y', f) \le 3/4$ both conclusions give the same necessary step:

$$F(y) \le Hf + u(1/k_*(y', f) - 2) = f + (y - y')\mathcal{D}(y', f). \tag{12}$$

If $L \le 0$ this bound holds directly. The same necessary inequality follows by a limit of cutoffs if the threshold is open.

The solution of (11) is decreasing and convex. Indeed $k_*(y_0, 0) = 1/2$, so $F'_\infty(y_0) = 0$. On the range $2y \le r_* \le 7/9$, $0 \le f \le 1/160$, the lower bound

$$k_*(y, f) \ge A\frac{3y - 1}{2y - 1/2} > 0.374$$

gives $F'_\infty > -0.305$. Hence $R'_0 > 0$ and $T'_* > 0$ along the solution. Differentiating gives $k'_* > 0$ and

$$yF''_\infty = \frac{k'_*}{6k_*^2} > 0.$$

The formulas hold one-sided at a change of the maximum in (10). In particular $F_\infty \ge 0$ to the left of y_0, $R_0 \le 8/9$, and $F_\infty \le 0.305(y_0 - y) < 1/160$. These estimates close the continuation throughout the stated interval.

The map on the right of (12) is increasing in $0 \le f \le F_\infty(y')$. In fact its derivative is

$$H - \frac{6u}{5}\frac{(\partial_f T_*)(1 - R_0) + 2(T_* - 1)}{(T_* - 1)^2}.$$

We have $\partial_f T_* \le 6$, $1 - R_0 \le 0.131$, $T_* - 1 \ge 3y' - 1$, and $T_* - 1 \le 1/6$. Using $u \le (1 - 0.369/y')/6$ on $[0.369, 0.379]$ and $[0.379, 7/18]$ gives a positive lower bound. The map is moreover 1-Lipschitz. Convexity gives the tangent inequality

$$F_\infty(y') + (y - y')F'_\infty(y') \le F_\infty(y).$$

Induction proves the upper bound for every finite sequence.

For the converse, first work with a smooth strictly smaller width function on a fixed compact interval. One may use $F_\infty(y + \epsilon) - \epsilon_0$ with $\epsilon, \epsilon_0 > 0$ small and start at $y_0 - \epsilon$ with the initial source. At a fixed f, the derivative of $\mathcal{D}(y, f)$ in y is positive near the solution, since $\partial_y k_* > 0$ and $\mathcal{D}(y, F_\infty(y)) \le 0$. Also $\partial_f \mathcal{D} > 0$. The smaller width function satisfies a strict reverse differential inequality $F'(y) > \mathcal{D}(y, F(y))$. Small constant-ratio steps then satisfy (12) with a strict margin proportional to u, allowing $\rho > 0$.

The starting strip uses the initial width since the width function is negative near its starting point.

Here are the cutoff choices beyond the low threshold. Write $R(e) = 2(B - \tau/e)/H$ and choose $r(e) \ge r_*$ with $\mathcal{R}(r(e)) \ge R(e)$. For $e \ge e_0$ one can take

$$r(e) - r_* \le \frac{2L}{0.35H}(1 - e_0/e).$$

On the ordinary substitution the exponent divided by e starts at L at $m = y'$. Its right slope is at most $-H/4$ since $F'(m) > -1/4$ and the jump to the initial width is favorable. Thus take $m(e) - y' \le (4L/H)(1 - e_0/e)$.

The adverse change in the long inverse exponent from these two increases is $O(u)(e - e_0)$. At fixed T_* its derivative is $B - HT_*/2 < 0$, bounded away from zero. The strict long inequality at e_0 therefore persists up to $3/4$. The inequalities also hold for $\nu \ge e$ by $r \ge 2y'$ and $H(r + m)/2 \ge y$. All the cutoff choices can be approximated by finite rational lists after fixing the strict decrements. This proves the assertion for the three scalar comparisons.

For the derivative bound used above, direct differentiation using $\mathcal{R}' > 0.35$, $R_0' \le 1$ and $T_*' \le 4$ gives $0 < F_\infty'' < 40$. A directed descending Euler sum bounds F_∞ from below. With 4096 intervals of length s and inverse-cutoff bisection to width 2^{-36}, the upper error is at most

$$4096(20s^2 + 14s\,2^{-36} + 10^{-20}).$$

The inverse is isolated by comparing fifth powers of rationals.

The resulting lower endpoint at $y = 0.369$ gives $F_\infty'(0.369) > -0.244$ and hence $F_\infty' > -1/4$ throughout. These directed checks are in `check_ordered_limit.py`. □

Corollary 6.2 (Initial short inverse width)**.** *If the short inverse comparison uses $G = 0$ on its whole range, the same ordered scalar class has the limiting added width*

$$F_{\mathrm{nat}}(y) = \frac{1 - 3y + yb}{4},$$

where $6/29 < b \le 3/7$ is the unique solution of

$$\frac{15}{29}(b - 3/7) + \frac{90}{841}\log\left(\frac{29b - 6}{29(3/7) - 6}\right) = \frac{18}{7} - \frac{1}{y}.$$

For example $F_{\mathrm{nat}}(0.376) = 0.0008502088286\ldots$.

Proof. Here $r_* = 2y + 4f$ and $k_* = (5/6)(3y + 4f - 1)/(2y + 2f - 1/2)$. Writing $3y + 4f - 1 = yb$ reduces (11) to

$$y^2 b' = \frac{29}{15} - \frac{2}{5b}, \qquad b(y_0) = 3/7.$$

The displayed logarithmic equation follows by separation. Its left side is increasing from $-\infty$ to zero. The estimates in the proof of Theorem 6.1 apply with $\mathcal{R}' = 1/2$.

In particular the logarithmic equation gives $b(0.369) > 0.313$, and the derivative formula gives $F_{\mathrm{nat}}'(0.369) > -0.228$. □

6.1 The differential width function in the Cycle comparison

In this subsection we insert the differential width from Section 6 into the comparison of [2].

For the ordinary comparison for the Mellin integral of [2, Appendix Q], use $\widetilde{F} = F_\infty - 4 \cdot 10^{-8}$ on $[0.4245, 0.4270] \times [0.4997, 1/2]$ in (d, p) and use the quartic width function of the preceding comparison on its complement. Directed Euler bounds give

$$3.6593 \cdot 10^{-8} < F_\infty(0.3888) < 3.6632 \cdot 10^{-8}.$$

Thus $\widetilde{F}$ starts below the initial width. The strict differential inequality $F_\infty'(y) > \mathcal{D}(y, \widetilde{F}(y))$ follows from $\partial_f \mathcal{D} > 0$. The finite approximation in Theorem 6.1 supplies the local source with strict decrements.

Take $c = 0.1266673595$ and $h = 0.8175010759616942$. In the local comparison keep the affine improvement (8). Replacing the increasing face by $k(z - m) + v\xi/d$ gives a lower exponent. Its minimum has the two crossing branches as above. The final crossing exponent is increasing in z in the tested band, so using the affine branch gives a lower bound at the final cutoff. Both offset derivatives are negative. On the local rectangle the directed bounds for this lower exponent are

$$\mathcal{M}(d(c), 1/2) > 1.2586 \cdot 10^{-8}, \qquad |\mathcal{M}_d(d(c), 1/2)| < 1.525 \cdot 10^{-7},$$
$$\mathcal{M}_{dd} > 0.779, \qquad \mathcal{M}_p < -0.02783.$$

They give a uniform local margin greater than 10^{-8}.

The exterior verification with the quartic approximation has 2121 mixed terminals, 43 exterior rectangles, and 22 local rectangles, with $2121+43+22 = 2039+3\cdot 49$. Its normalized and original-scale margins exceed $6.1975\cdot 10^{-9}$ and $2.1629\cdot 10^{-9}$. Hence the strict scalar comparison with these two ordered width functions has

$$\theta_{\mathrm{ODE}} = 0.8733326405$$

on the analytic inputs of the Cycle comparison.

6.2 Two strict initial width functions

In this subsection we give two strict lower bounds for the ordinary width with the initial inverse estimate. Subsection 7.5 proves the corresponding moments.

Write $s = 0.3888 - y$. We use the following two rational choices:

	lower endpoint for y	$f(y)$
f_{n}	0.366	$4.55s^2 + 39s^3 + 0.00031s - 10^{-9}$
f_{n}^{+}	0.369	$4.68608s^2 + 35.4492s^3 + 0.0004247s - 10^{-9}$.

(13)

Both admit the scalar construction of Theorem 6.1 with the initial short inverse bound $G = 0$.

Indeed let $A = 5/6$, $T_* = \max(3y, 3y+4f)$ and $R_0 = 1/2+y+2f$. The cleared strict differential condition is

$$5(T_* - 1)(yf' - f - 1/3) + (T_* - R_0) > 0. \tag{14}$$

For $f \geq 0$ use $T_* = 3y + 4f$ and check its coefficients in the Bernstein basis on 32 equal cells in s. For $f < 0$ use $T_* = 3y$. The negative interval is contained in $0 \leq s \leq 4\cdot 10^{-6}$, on which one Bernstein cell suffices. The directed lower bounds for the left side of (14) exceed 0.0001427 and 0.0000333, respectively. These computations are in `check_native_profile.py`. The same check gives

$$-0.27 < f' < 0, \qquad 0 < 1/2 + f - yf' < 0.61.$$

Use the two bounds in the cutoff proof of Theorem 6.1, with $0.23H$ in place of $H/4$ for the decrease of the ordinary loss. Strictness of the long-inverse exponent is preserved for sufficiently small u.

At the top of both width functions $f = -10^{-9}$, so the initial moment starts the finite construction. The proof of radial continuation in Subsection 7.5 then applies with the same adverse slope $3(0.5 + 0.61)/4 < 5/6$.

6.3 The inverse estimate after one step from zero

In this subsection we optimize one inverse step with zero initial increment. Subsection 7.13 treats the analytic stage.

For comparison the stage with zero initial increment of [6], with $2/3 < r < R < 7/9$, has added width at most

$$Y_R(r) = \frac{(7-9R)(R-r)}{R(45R-30)}.$$

Its exact upper envelope in R is

$$Y_{\mathrm{one}}(r) = \frac{12r - 7 - \sqrt{21r(3r-2)}}{30}, \qquad R_{\max}(r) = \frac{7 + 9r + 30Y_{\mathrm{one}}(r)}{18 + 90Y_{\mathrm{one}}(r)}.$$

To see this, for a proposed improvement Y clear the denominator:

$$(9+45Y)R^2-(7+9r+30Y)R+7r=0.$$

At the interior maximum its discriminant is zero. The smaller root in Y corresponds to the first coincidence of the two roots in R issuing from r and $7/9$. Additional stage conditions can be imposed when selecting a strict numerical minorant.

7 The analytic moment estimates and the zero-free region

In this section we prove the moment estimates and apply the comparison in Section 5 to the reflected numerator.

For the reduction over number fields it suffices to work over $F \supset \mu_{12}$, with the full unitary paired coefficient families of [6, 3, 4]. In a parameter with conductor scale U, let M_r and S_m be the inverse and ordinary detecting sums and let A_n be the retained ordinary numerator. Choose a single zero with real part a and write $\nu = 2a-1$. If

$$|S_m|^2 \gg U^{\nu m-\varepsilon}, \qquad |A_z|^2 \ll U^{\nu z+\varepsilon},$$

then on that same parameter with the same marked product Q,

$$|A_n A_z Q|^2 \ll U^{\nu(z-m)+2\varepsilon}|A_n S_m Q|^2.$$

After freezing the two indexed tuples their order can be exchanged in the product, with each test function, height, character and omitted local factor retained with its own sum. The largest available count deficit on the common parameter class gives a single cardinality bound.

7.1 Selected natural parameters and the ordered source

In this subsection we estimate the large prime-sum values using the selected character inequality. The bound is used in Subsection 7.5.

Write $\chi_{\mathfrak{c}}(\mathfrak{n})$ for the natural zero-extended sextic character over $F \supset \mu_{12}$, with the allowed generators and the fixed set of places of [8, Section 3]. Split the unit classes into the finite quotient of the S-units by their sixth powers. All fixed ray and unit multipliers below are included in the coefficients. For a fixed positive mesh, write an original prime factor in the form

$$Q_j(\mathfrak{c}) = \sum_{\mathfrak{p}} d_j(\mathfrak{p})\chi_{\mathfrak{c}}(\mathfrak{p}), \qquad \mathrm{N}\mathfrak{p} \asymp U^{w_j}, \quad 0 < w_j \le \eta, \qquad |d_j(\mathfrak{p})| \ll (\mathrm{N}\mathfrak{p})^{-1/2}.$$

An independent height in an ordinary factor or a prime factor multiplies its coefficients by phases. A fixed unitary anchor may be included in the same way.

Proposition 7.1 (The large-response part over number fields)**.** *Suppose $\mathcal{E}$ is a set of natural sixth-power-free parameters of norm at most $O(U^V)$, $V \ge 0.98$. On every parameter in $\mathcal{E}$ suppose that some original factor satisfies $|Q_j|^2 \ge U^{0.5002w_j}$. For a common literal product P of two centrally normalized ordinary factors and surviving original factors, suppose that its merged ideal coefficient is supported up to $O(U^A)$, with $A < V$. Then*

$$\sum_{\mathfrak{c}\in\mathcal{E}} |P(\mathfrak{c})|^2 \ll_\varepsilon U^{V-0.5001(V-A)+\varepsilon}. \tag{15}$$

In particular the saving exceeds .031 for $A \le .91796$ and .026 for $A < .928$.

The bound allows common finite rays and zero restrictions, with the selection condition checked on their actual factors. Conjugating an entire factor puts its sextic character in the common orientation. Constants have the usual finite seminorm dependence.

Proof. Assign each parameter to its first qualifying factor. The list is fixed and finite. On each class, [8, Theorem 3.10, (3.26)] applies to the actual prime sum and its norm scale, with $K = 0.5001$ and $H \asymp U^V$. Its bound for arbitrary common source coefficients $a(\mathfrak{n})$ of length X is

$$\sum_{\mathfrak{c}\in\mathcal{E}} \left| \sum_{\mathrm{N}\mathfrak{n}\le X} a(\mathfrak{n})\chi_{\mathfrak{c}}(\mathfrak{n}) \right|^2 \ll_\varepsilon U^\varepsilon (X + H^{1-K}X^K) \sum_{\mathfrak{n}} |a(\mathfrak{n})|^2. \tag{16}$$

The gap $K - 1/2$ is fixed before choosing the mesh. The positive difference $0.5002 - K$ absorbs the small response-bin errors after the finite list and its positive widths are fixed.

The coefficient of P is the ideal convolution of its actual factors. If these coefficients are a_i, the fixed-order divisor bound and Cauchy's inequality on each product fibre give

$$\sum_{\mathfrak{n}} |(a_1 * \cdots * a_j)(\mathfrak{n})|^2 \ll_\varepsilon U^\varepsilon \prod_{i=1}^{j} \|a_i\|_2^2.$$

All ideal factorizations occur, including those with shared primes. Each centrally normalized annular factor has bounded squared mass, by ideal counting. Taking $X \asymp U^A$ in (16) gives the exponent

$$(1-K)V + KA = V - K(V - A).$$

At $V = .98$ the two cap bounds give respectively $0.5001(0.98 - 0.91796) = 0.031026204$ and $0.5001(0.98 - .928) = .0260052$. The choices of constants precede the independent heights. □

For a general parameter $\mathfrak{k} = \mathfrak{b}^6\mathfrak{c}$ with $\mathfrak{c}$ sixth-power-free, the factor $\chi_{\mathfrak{b}}^6$ is literally $1_{(\cdot,\mathfrak{b})=1}$. The support of $\mathfrak{b}$ may meet that of $\mathfrak{c}$. At fixed $\mathfrak{b}$ insert this restriction in all actual coefficients, including the factor that selects the parameter. Then $Q_j^{(\mathfrak{b})}(\mathfrak{c}) = Q_j(\mathfrak{b}^6\mathfrak{c})$, as in [8, Lemmas 3.2 and 3.12]. This is the literal sixth-power restoration used in [8, Theorem 3.13]. In a dyadic interval for the complementary factor, with $b = \log_U \mathrm{N}\mathfrak{b}$, the remaining radius has exponent $V = M - 6b$, up to endpoint padding. There are $O_\varepsilon(U^{b+\varepsilon})$ such complementary factors. For $A \le .91796$ and $V \ge .98$ their total exponent is therefore $M - 5b - 0.031 + \varepsilon$. More generally a bound on the core with exponent $V + u - \rho + \frac{5}{6}(H_0 - V)_+$ becomes, after multiplying by the complementary factor count, a bound of exponent at most $M + u - \rho + \frac{5}{6}(H_0 - M)_+$, since

$$V + b + \tfrac{5}{6}(H_0 - V)_+ = M - 5b + \tfrac{5}{6}(H_0 - M + 6b)_+ \le M + \tfrac{5}{6}(H_0 - M)_+.$$

The finitely many unit classes and $O(\log U)$ dyadic intervals for the complementary factor are absorbed in U^ε. Common quotient restrictions are allowed in (16) even when they vary with the complementary factor.

Here is the width calculation in the complementary part of the ordered source. Let

$$\mathcal{W}(n,z) = \frac{1-n-z}{2} + F(z), \qquad z \le n \le 1.15z, \qquad 0.369 \le z \le Y_+ = 0.3888, \tag{17}$$

where F is the function of [2, (P1)]. Put $u = 3 \cdot 10^{-6}$, $H_0 = 1 - 6u$, and $L = \mathcal{W}(n,z) - u$. The live prime width is at most L. On a frozen response class suppose every live squared factor has exponent at most ew_j, up to the allocated losses. Removing whole factors until a target width B_0

is reached incurs at most $e[L - B_0]_+ + O(\eta)$, by the early mesh. At radius U^{H_0} the earlier ordered pair $(n/H_0, z/H_0)$ gives the low-response exponent

$$e\left[2u + F(z) - H_0 F(z/H_0)\right]_+ . \tag{18}$$

The two absolute ordinary widths cancel in this computation. At the starting strip the initial width replaces the earlier pair.

Now let S_m be an ordinary sum at the same zero, with $|S_m|^2 \gg U^{H_0 \nu m - \varepsilon}$ and $|A_z|^2 \ll U^{\nu z + \varepsilon}$. Suppose $m \geq z/H_0$ and $m \leq Y_+$. Order the caps $(n/H_0, m)$. Its smaller cap s satisfies $s \geq z/H_0$, and the ratio of the two caps is at most 1.15, since

$$\frac{n/H_0}{m} \leq \frac{n}{z} \leq 1.15, qquad \frac{m}{n/H_0} \leq \frac{0.3888}{0.369} < 1.15.$$

Since F decreases, $F(s) \geq F(m)$. Consequently the positive ordinary substitution consumes at most

$$-\nu(H_0 m - z) + e\left[2u + \frac{H_0 m - z}{2} + F(z) - H_0 F(m)\right]_+ . \tag{19}$$

When $m > Y_+$ the initial coefficient-two comparison has 0 in place of $F(m)$. Both substitutions keep the height, test, fixed ray and restriction of each factor. The whole ordinary sum may be conjugated to put its Kummer feature in the same orientation. This calculation uses only a previous pair with the larger shorter cap, or the initial comparison.

If a common parameter class has count deficit D, its cardinality is $O(U^{V(1-D)+\varepsilon})$ at radius U^V. Put $S_* = V/2 - n$ in the range $S_* \geq 0$. Choose a complete subproduct Q_K of the original live factors with width $L_K = S_* - O(\eta)$ and write $Q = Q_K Q_R$. If the whole live width is less than the strict width, the two initial fourth moments with the whole subproduct bound directly, since $z \leq n$. Otherwise $2n + 2L_K < V$. On the counted class,

$$\sum |A_n A_z Q|^2 \leq U^{\nu z + \varepsilon} \left(\sum |A_n|^4 |Q_K|^2\right)^{1/2} \left(\sum |Q_K|^2 |Q_R|^4\right)^{1/2} .$$

The initial coefficient-two fourth, including Lemma 7.6 on refined lists, bounds the first sum. For the second use the cardinality bound and the pointwise factor exponents $|Q_i|^2 \ll U^{e w_i + o(1)}$. This gives the excess over V

$$\nu z + e\left(L - \frac{V/2 - n}{2}\right) - \frac{VD}{2} + O(\eta + \varepsilon). \tag{20}$$

At increased radii one may keep the subproduct chosen at H_0. Every prime factor in the auxiliary fourth is used once. The short inverse, long inverse and ordinary responses test the same zero.

Proposition 7.1 supplies the large prime-response part of (P2) in [2], including the unchanged selected operator for the actual unitary anchors. Equations (18)– (20) give the other exponents from the earlier moments and the counts obtained by zero detection. The range $n + z > 7/9$ in the application rules out direct use of the joint anchor rectangle of [6]. Separate treatment of the terms in the second correlation can lower the radius of an individual term. The earlier radius comparison also uses $n + z + \mathcal{W}(n, z) < 0.9$.

The ordered width in (17) can make this total as large as 0.91796 up to rounding. The transformed regular sides and the two-sided assembly are estimated below, at the radii of their packed correlations.

7.2 The reference width in the all-field correlation

In this subsection we choose the short scale for the signed comparison of Section 4 and prove the positive estimate at that scale.

Write $P = n + z$, $W = (1 - P)/2 + f(z)$ and $A = P + W$. The all-field ordinary assembly of [8, Section 4.6] uses a short reference with a common lower bound for the two long coordinates. On each closed length cell take the reference cap ℓ_{ref} to be a slight upper rounding of

$$\max\left(.08,\ \frac{3}{2}(A + 3u - 5/6) + .003\right). \tag{21}$$

For the full ranges below one has $P < .856$, $.732 \le P$, $A < .928$, $f(z) < 1/160$ and $0 < u \le 3 \cdot 10^{-6}$. Thus

$$\ell_{\mathrm{ref}} + W \le \max\left(.08 + W,\ \frac{P}{4} + \frac{5}{2}f(z) + \frac{9}{2}u + .003\right) + O(\eta) < .233.$$

The original reference tests permit this cap, which is below .15.

For a short coordinate $x \le \ell_{\mathrm{ref}}$ use

$$\sum_{\mathfrak{c}}^{*} |A_x A_y Q|^2 \le \left(\sum_{\mathfrak{c}}^{*} |A_x^2 Q^2|^2\right)^{1/2} \left(\sum_{\mathfrak{c}}^{*} |A_y|^4\right)^{1/2}.$$

The unmarked fourth on the right is that of [8, 3], with the actual fixed character and height. For the other fourth expand the two short products in their indexed ideal and prime variables. Apply the positive parameter lift and with coprimality restrictions Kummer Poisson formula of [9, Section 8 and Lemmas 12.9–12.10] in fixed ray and unit charts. The modulus retaining the natural zeros divides the radical of the product of the indexed ideals. Its norm is at most $U^{4(\ell_{\mathrm{ref}} + W) + O(\eta)}$.

This is strictly below the parameter volume U^V for $V \ge H_0 - O(\eta)$. Nonzero dual frequencies have rapid decay: the dual fractional ideal gives a lower bound on the norm of each nonzero frequency, and the compact norm-one charts give arbitrary Poisson decay. Fixed restrictions are opened before the lift or included in the common positive majorant. At a good prime the zero frequency requires the signed Kummer exponent sum to vanish modulo six. Each single occurrence has nonzero exponent, including a cubic occurrence with exponent two or four. The nonempty local diagonal therefore has at least two occurrences. With central normalizations and geometric sums over ordinary valuations, its absolute mass is $O_{F,Q}((\mathrm{N}\mathfrak{p})^{-1})$. Multiplying these local bounds over $\mathrm{N}\mathfrak{p} \le U^{O(1)}$ gives $O(U^{\varepsilon})$ before the parameter volume. The finite bad-prime and ray classes retain this bound.

This proves the short fourth for the repeated short product, including all its original prime positions. The two fourths retain their respective tests and heights.

For the fixed-character term, the centered allowance of [8, Section 4.6] has the lowering penalty $2v/3$ at a minimum length reduced by v. Any centered saving with exponent $\delta \ge 2/3$ gives the same optimized penalty, since

$$\inf_{v \ge 0}\left\{\frac{2v}{3} + \delta(t - v)_+\right\} = \frac{2}{3}t_+.$$

Theorem 4.1 gives such a δ, using the existing unitary bound $B < 1$ for the cubic powers. Keep the lowering from the first marked extraction as in the centered allowance of [8, Section 4.6]. With its nonnegative exponents X, c the combined saving uses

$$-\frac{9}{8}X - \frac{29}{6}c - \frac{2}{3}(\ell_{\mathrm{ref}} - X - 5c)_+ \le -\frac{2}{3}\ell_{\mathrm{ref}}.$$

Hence the fixed-character excess, before mesh and endpoint losses, is at most

$$P + W + 3u - \frac{5}{6} - \frac{2}{3}\ell_{\mathrm{ref}} \le -.002.$$

The origin comparison also has a fixed margin since $1 - (P + W) - 4u > .0719$. These bounds include the wider ordered ranges used in the later stages.

This reference calculation covers the fixed term and the regular short positive reference. The regular second moment from the second correlation is estimated at the resulting parameter scale.

7.3 The packed second correlation over number fields

In this subsection we apply Poisson summation twice to the signed difference. The short-scale bound of Subsection 7.2 treats the reference term.

Let $F \supset \mu_{12}$ and enlarge the fixed set S as in [9, Lemma 3.4]. Write $R = \mathcal{O}_{F,S}$ and $\Gamma = (R^\times)^6$. An exterior ideal has one allowed generator modulo Γ. These generators multiply modulo Γ and give symmetric sextic reciprocity. Each nonzero frequency class modulo Γ is an allowed generator times a member of $R^\times/\Gamma$. We keep this finite classes as a finite collection of coefficient characters. In particular multiplication of a frequency by the allowed generator of an exterior ideal is an injection on classes modulo Γ and multiplies its exterior norm by the norm of the ideal.

Consider a root of free parameter width one and total column width A. The two ordinary test functions may have separate fixed characters and norm heights. Their common moving character and the original prime positions are those of [6]. Each live mark carries that same sextic character once, with its fixed coefficient. At the cube-free stage the full product moduli have valuations in $\{0, 1, 2\}$. We keep the equal-product difference of the original and reference arms, including their scale phases, on each full column. For the two long arms let ℓ_{ref} be the common lower bound on the four scale exponents at the cube-free signed root.

Lemma 7.2 (Cube-free restoration over exterior ideals)**.** *Both cube-free comparisons use central kernels of bounded absolute mass, with absolute mass depending on the fixed factor list and coefficient bounds. The kernels preserve the two independent test functions, the equal-product identity and the natural zero extensions.*

Proof. Give each original prime factor one formal variable w_i with $w_i^2 = 0$. This relation is imposed across the whole Euler product, before taking the coefficient in which each prescribed factor occurs once. At a good prime write x, y for its two ordinary positions and set $w_i = 0$ in the local polynomial if the prime is outside factor i's pool. The two local series are

$$U_{\mathfrak{p}} = (1-x)^{-1}(1-y)^{-1}\prod_i (1+w_i), \qquad R_{\mathfrak{p}} = 1 + x + y + x^2 + xy + y^2 + \sum_i w_i.$$

Thus the restricted series allows a total ordinary valuation at most two and permits at most one mark, disjoint from both ordinary positions.

The identities $R_{\mathfrak{p}} = (R_{\mathfrak{p}}/U_{\mathfrak{p}})U_{\mathfrak{p}}$ and $U_{\mathfrak{p}} = (U_{\mathfrak{p}}/R_{\mathfrak{p}})R_{\mathfrak{p}}$ are formal power series identities. In both kernels the unmarked part starts in degree three beyond its constant. All nonconstant terms containing a mark have total degree at least two.

In the absolute central norm, $|x|, |y| \le (\mathrm{N}\mathfrak{p})^{-1/2}$ and $|w_i| \ll (\mathrm{N}\mathfrak{p})^{-1/2}$ on its annulus. The unmarked inverse is expanded geometrically after a fixed small-prime cutoff. For each newly excluded prime extract both ordinary valuations, retaining the literal character scalar on the parameter and dividing matched scales by the same norm powers. Their total central mass is at most $\prod_{\mathfrak{p}\ \text{newly excluded}}(1 - (\mathrm{N}\mathfrak{p})^{-1/2})^{-2}$.

The marked inverse is a finite nilpotent expansion. Every live prime interval of fixed positive width eventually lies above the cutoff, so the geometric majorants there are uniform. If $m_{\mathfrak{p}}$ factors are active at $\mathfrak{p}$, the absolute local mass of either kernel is therefore

$$1 + O\big((\mathrm{N}\mathfrak{p})^{-3/2} + m_{\mathfrak{p}}^2(\mathrm{N}\mathfrak{p})^{-1}\big).$$

Here the constants may depend on the fixed coefficient bounds and the finite factor list. The second term has bounded sum, by $m_{\mathfrak{p}}^2 \le (\#\text{prime factors in the original product})m_{\mathfrak{p}}$ and $\sum_{\mathfrak{p}\text{ in one pool}}(\mathrm{N}\mathfrak{p})^{-1} \ll_F 1$.

The ordinary sum converges absolutely. The finite excluded set is retained as its actual deletion or bounded local factors.

After coefficient extraction, a monomial fixes some of the original factors and shifts each ordinary coordinate by its own ideal factor. These shifts divide the scales of the two arms by the same respective norms. Their Mellin weights retain the central normalization and the original height in each coordinate. The common character factor on the extracted ideals has modulus at most one, including its literal zeros. Extra zero restrictions are imposed before the local identities, or expanded exactly by Möbius inversion, with divisor sums bounded by U^{ε}. Thus the common prime coefficients or natural zeros do not require enlarging the local support. Minkowski's inequality with the above absolute masses completes both comparisons. □

We first establish the short reference for unrestricted products. In its short fourth, positive parameter restrictions may be suppressed before Poisson. The column restrictions, including natural zeros, stay in the finite coefficients. The short reference for cube-free products then follows from the kernel $R_{\mathfrak{p}}/U_{\mathfrak{p}}$ of Lemma 7.2: all remaining ordinary scales decrease. For the main moment, apply the opposite kernel $U_{\mathfrak{p}}/R_{\mathfrak{p}}$ first. On each resulting cube-free family, use the short reference if one ordinary scale is at most $U^{\ell_{\mathrm{ref}}}$. Otherwise form a new reference of the same scale product, with minimum scale $U^{\ell_{\mathrm{ref}}}$. Thus the lower bound on the four scales starts at the cube-free root of each signed correlation.

Proposition 7.3 (The regular moment in the second correlation)**.** *Let $u > 0$ be the prescribed width of the selected prime product, with rounding error $O(\eta)$, and put $H_0 = 1 - 6u$. Suppose the regular second moments from the original rectangle, with the shortened scales and the remaining prime products, satisfy*

$$\mathcal{L}(M) \ll_{\varepsilon} U^{M+u-\rho+(5/6)(H_0-M)_+ +\varepsilon}, \qquad 0 < \rho \le u. \tag{22}$$

Here the norm of the parameter is at most $O(U^M)$. Assume the strict modulus inequality of Subsection 7.2 for the short reference.

The constants are uniform for bounded test seminorms and have polynomial dependence on the independent heights. If selection exhausts the prime factors, use the unmarked estimate. The two regular parts of the signed correlation then satisfy the required bounds. For the fixed inducing characters and the zero frequencies it suffices that

$$A + 3u < \frac{5}{6} + \ell_{\mathrm{ref}}, \qquad A + 4u < 1, \qquad 6u < 1. \tag{23}$$

Here the centered estimate of Theorem 4.1 is used with $\delta \ge 1/2$. Lemma 7.2 supplies both cube-free restorations, with the same fixed unit and ray classes.

Proof. Use the positive lift of [9, Lemma 12.1] and integrate over the compact norm-one quotient before applying Poisson summation. After restoring the cube-free condition, use the actual scales

and the retained prime factors in each signed difference. Let U^A be the dyadic scale of the full product in that difference, bounded by the original length. Compact support and equality of the scale products put the two terms in a common annulus. The strict inequalities (23) persist under this shortening.

The first complete-product formula is [9, (12.2)]. Write C, D for the complete common parts of the two column moduli, $c = \log_U \mathrm{N}C$, $d = \log_U \mathrm{N}D$, and p for their common radical width. Let r be the squarefree product of the primes where the two valuations differ, and e the divisor in the common complementary coprimality expansion. Put $q = \log_U \mathrm{N}(er)$. The retained first frequency has width

$$k_1 = 2A - c - d + q - 1 - t, \qquad t \geq -O(\eta),$$

where the loss includes the logarithmic dyadic enlargements and the fixed cutoff padding. Complete the squarefree Möbius expansion before positivity.

The artificial divisor occurs through its Möbius sign and one divisibility indicator in each column. Apply Cauchy's inequality jointly in this divisor and the nonzero first frequency after the common kernel separation. Each resulting squared norm carries the divisor with weight one.

Let ξ_r denote the actual primitive common quotient character. By the CRT in [9, (12.2)], the additional column factors on the left and right are respectively $\chi_I(er)\xi_r(I)$ and $\chi_I(er)\overline{\xi_r(I)}$. On supported columns these are $\chi_I(V_C)$ and $\chi_I(V_D)$, where the exponents at a prime of r are $(1+i-j) \bmod 6$ and $(1+j-i) \bmod 6$ for the left and right common valuations $i, j \in \{1, 2\}$. Both include the e-part. Thus active pairs $(2,1), (1,2)$ have exponents $(2,0), (0,2)$ and

$$q_C := \log_U \mathrm{N}V_C = q + c - d, \qquad q_D := \log_U \mathrm{N}V_D = q + d - c. \tag{24}$$

The exponent-zero primes remain genuine natural holes on the corresponding columns. All products on supported columns use symmetric reciprocity with the finite unit character of the frequency included. The presentation on the supported ideals and its packed presentation have the same primitive inducing character after common finite deletions.

In each first Cauchy norm separately, allocate its common factor among the two ordinary positions and the marked positions. There are divisor-many genuine allocations. Select a complete subproduct of the factors common to that signed pair of actual width $u' = u + O(\eta)$, or take all of them at exhaustion. For a selected tuple of ideal product Δ, CRT gives the fourth-power Gauss shift on the whole residual column, with its fixed finite unit characters. This is the tame identity $\mathcal{G}(\Delta I, h) = \mathcal{G}(\Delta, h)\mathcal{G}(I, \Delta^4 h)$ on $(I, \Delta) = 1$, using allowed generators. Indeed the CRT cross factor is $\chi_I(\Delta)^2$ and the fourth-power frequency shift contributes $\overline{\chi_I(\Delta)}^4 = \chi_I(\Delta)^2$ on units.

On the original forbidden locus the transferred Gauss factor is zero: a prime of Δ dividing the residual cube-free column has modulus valuation one or two, whereas its shifted frequency valuation is at least four. Only the genuine supported assignments are used before this identity. The residual divisor is obtained by removing the primes of Δ from the first squarefree divisor. For fixed extracted tuple and retained divisor, the original divisor has at most $2^{\#\{\text{selected prime factors}\}}$ preimages.

In the first bridge the central factor $1/\sqrt{\mathrm{N}I}$ on the full column contributes $1/\sqrt{\mathrm{N}\Delta}$ to each extracted tuple. The sum of the squared extracted coefficients is $U^{O(\varepsilon)}$ by the fixed prime-position counting bounds. Its remaining frequency is $\Delta^4 h$. At any nonzero resulting frequency the number of possible selected tuples is divisor-many, because each selected prime divides its exterior ideal. The positive sum over the residual divisors may now be enlarged to contain every squarefree divisor needed in the next correlation. In the filled case the new product cap and raised frequency bound on the C-side have logarithms

$$a = A - c - u' + O(\eta), \qquad k = k_1 + 4u' + O(\eta), \qquad X = U^a, \quad K = U^k.$$

Here X is the scale of the actual residual product annulus.

At a shorter root fix its actual A before the first correlation and use this A in both k_1 and a. The compact factors and dyadic roundings contribute only $O(\eta)$ to the exponents.

For the second Poisson step, retain first the two actual product norms $\mathrm{N}I_1, \mathrm{N}I_2 \asymp X$ (after absorbing the frozen extracted divisors into the scales). The bridge from the two factors $1/\sqrt{\mathrm{N}I_i}$ and finite Fourier inversion has scalar $K/(\mathrm{N}I_1\mathrm{N}I_2)$ multiplying $\mathcal{F}(I_1, I_2; j)$ of [9, Lemma 12.10]. Before imposing the last Möbius restriction, use that lemma on the genuine coprime residual pairs. In the nonzero correlation, unequal common multiplicities vanish since the full columns are cube-free. The two second common ideals therefore equal one ideal G. Write $A_2 = GB_2$, where B_2 is the squarefree product of the common primes at which the frequency divided by G is a nonunit. The parameter has $j = A_2 z$ on this partition, and $|\mathcal{F}(G, G; j)| \le \mathrm{N}G$. The auxiliary divisor appears in both full product moduli. On their genuine coprime residual locus this restricts the auxiliary divisor to the squarefree divisors of G, giving the multiplicity $\rho(G) = 2^{\omega(G)}$ after the first Cauchy inequality.

In either side the phase on the residual ideal is now exactly $\chi_I(V_C j)$, up to the fixed character and the finite unit characters. Apply the injection $y = V_C A_2 z$, with the sublattice and natural holes retained. For the right first side use V_D and its own ball. There is no change of real measure in this discrete substitution. For example with a radial weight the parameter factor after substitution is $\mathbf{1}_{V_C A_2 | y} \Phi(\mathrm{N}y/(\mathrm{N}(V_C A_2) K_z))$, where K_z is the corresponding divided-frequency scale. The kernel has the same norm argument $K\mathrm{N}y/(\mathrm{N}V_C \mathrm{N}I_1 \mathrm{N}I_2)$ on a first-side dyadic class. The common logarithmic Fourier measures of [9, Section 12 and the appendix on weighted moment estimates] apply to the same compact norm-one integral and the full parameter and column norm variables. The fixed ratio dilations are absorbed with their actual norm phases, so both arms still have their original two independent heights in the tests.

At each Fourier point the two signed residual coefficients retain the same equal-product subtraction. The padded positive parameter envelope has exponent at most

$$M = 2a - k + q_C + O(\eta) = 1 + t - 6u' + O(\eta). \tag{25}$$

The permissible lower endpoint is $M \ge H_0 - O(\eta)$ in the filled case. We use the actual dyadic radius and may increase it to the right side of (25) under a common positive majorant.

Extend the genuine second CRT expression to artificial noncoprime residual pairs with its full norm kernel. Insert the complete Möbius sum over a second squarefree divisor D_2 before splitting the sides. The residual columns retain their coprimality to G, so $(D_2, G) = 1$. In each side the assigned divisors of the two columns therefore remove norm at least $\mathrm{N}G\mathrm{N}D_2$. Central normalization in each ordinary shortening supplies the inverse square root of the extracted norm. A fixed prime position has squared mass $O(1/\mathrm{N}\mathfrak{p})$ for its fixed supplying prime. Thus the squared sum in one block is bounded, on regular parameters after separating its two arms, by a positive combination of shortened centrally normalized sums with total mass

$$U^{O(\varepsilon)} \frac{X}{\mathrm{N}G\mathrm{N}D_2}.$$

This uses the forward expansion of Lemma 7.2. The terms with only ordinary factors start in degree three, so their central absolute masses have a summable Euler product. Terms involving original marked positions are summed on those positions; a coincidence uses at least two factors of $P_i^{-1/2}$ on its one prime.

The exact common-hole expansions shorten the actual ideal sums and have total weight $O(U^{\varepsilon})$. Bounded and empty coordinates retain their stated annular supports. Each normalized sum therefore uses an original or reference pair with componentwise shortenings and a surviving subproduct.

Multiplying by the second scalar KNG/X^2 leaves K/X times the index weight $\rho(G)/ND_2$. At each nonzero packed parameter, both A_2 and G divide its exterior ideal. For a fixed dyadic class and D_2, the possible pairs (G, A_2) and their unit classes are divisor-many. In particular

$$\sum_{\text{indices at } y} \frac{\rho(G)}{ND_2} \ll_F \tau_F((y))^3 \sum_{ND_2 \ll U^C} \frac{1}{ND_2} \ll_{F,\varepsilon} U^{\varepsilon}. \tag{26}$$

The reciprocal weight is retained also when $D_2 \nmid y$.

To use the bound for indices at a fixed y, first open any remaining coprimality restrictions that depend on the divisor data. Their exact divisor expansions shorten the ordinary scales, fix any selected prime factors, and give bounded parameter scalars and total absolute mass $U^{O(\varepsilon)}$ per index. These operations retain the natural zeros of the character on the surviving variables. Weighted Cauchy in these expansions incurs another factor $U^{O(\varepsilon)}$ and preserves the harmonic weight $\rho(G)/ND_2$. Partition the surviving scale parameters into dyadic intervals, numbering $O(\log U)$ in each scale, and the prime subproducts and finite anchors into fixed finite families.

For an ordinary scale $X = X_0\lambda$ in one such interval with $1 \le \lambda \le 2$, put $x = N\mathfrak{a}/X_0$. If $V(t) = W(t)t^{iv}$, introduce a fixed smooth Ω equal to one on the union of the resulting compact supports. Then

$$X^{-1/2}V(N\mathfrak{a}/X) = X_0^{-1/2}\Omega(x)x^{iv}\int_{\mathbb{R}} \widehat{W_{\log}}(\tau)\lambda^{-1/2-i(v+\tau)}x^{i\tau}\,d\tau.$$

Here $W_{\log}(t) = W(e^t)$ and the Fourier measure has its fixed normalization. An identical separation is made in the other coordinate. For fixed Fourier variables the normalized tests and the character $\chi_\bullet(y)$ are now common to the indices in a dyadic interval. Twists already present in the whole-product kernels are included in the common Mellin measure and use the same inverse-norm dilation in the two signed arms.

The factors depending on λ have modulus at most one. Cauchy's inequality with the absolute Fourier measure produces a positive integral of squared moments with common tests. The index bound (26) can therefore be inserted at each y inside that integral. Rapid decay of the fixed logarithmic transforms integrates the polynomial seminorm and height factors. Neither a count of polynomially many shortened scales nor a sum of their separate moment bounds is needed. Use this positive argument for regular parameters. On the fixed-inducing classes keep the signed difference through the same extractions and use the centered estimate before taking absolute values.

On a regular reference arm one coordinate remains at most $U^{\ell_{\mathrm{ref}}}$ after every shortening. Its positive moment on the full packed ball is $O(U^{M+\varepsilon})$ by the short fourth of Subsection 7.2, since $M \ge H_0 - O(\eta)$ and the surviving prime width is at most the original W. For an original arm each coordinate remains below its own nominal cap. The ordered inequalities use these indexed caps, with the stated clamping for the earlier moments, even when the ratio of the actual shortened coordinates is larger.

By (22), the regular side is at most

$$U^{k-a+M+u'-\rho+O(\eta+\varepsilon)} = U^{A-c+q_C-\rho+O(\eta+\varepsilon)}.$$

The unmarked branch at exhaustion uses exponent M, and the side allowance has a margin of u'. For the fixed-inducing side, center each signed difference before forming its positive square. Freeze the same remaining mark tuple in both arms and apply Theorem 4.1 with its actual local restrictions. Sum the tuples by absolute coefficients and annular prime counts. Their lengths are already included in X.

If L_f is the pre-extraction minimum scale and D is the full extracted norm, the square before normalization is $\ll U^{\varepsilon}X^2D^{-2}\max(1,L_f/D)^{-2\delta}$. For $D \geq \mathrm{N}G\mathrm{N}D_2$ and $2\delta \geq 1$ this is at most $U^{\varepsilon}X^2/(\mathrm{N}G\mathrm{N}D_2\max(1,L_f))$. The parameter count for a finite inducing character is $O(U^{M/6+\varepsilon})$, from its prescribed exterior valuations modulo six and the finite unit classes. Thus the fixed side has exponent at most $k+M/6-(\ell_{\mathrm{ref}}-c)_+ + O(\eta+\varepsilon)$. Its excess over $a+q_C+u'$ is bounded by $A+3u'-5/6-\ell_{\mathrm{ref}}+O(\eta+\varepsilon)$.

At the second origin the exact correlation is supported on $I_1 = I_2 = I$, with value $\varphi(I) \leq \mathrm{N}I$. At frequency zero, the number of squarefree divisors is $\rho(I)$ and the squared column mass on $\mathrm{N}I \asymp X$ is $O(XU^{\varepsilon})$ by the ideal-divisor bound. The volume factor at this origin is the compact quotient integral $O(K)$. The resulting $O(KU^{\varepsilon})$ fits the side allowance by $A+4u<1$ and (25). The first origin is diagonal for cube-free columns and is $O(U^{1+\varepsilon})$. A positive enlargement in the raised frequency may include zero. The normalized Gauss sum $\mathcal{G}(I,0)$ vanishes for nonunit cube-free I; the unit column has bounded support. The omitted dyadic frequency ranges are estimated on the original genuine supports by the rapid decay of the norm kernel.

At both steps all column, divisor and modulus caps are polynomial. A fixed high derivative order bounds these counts and the remaining infinite dual tail by ideal counting. No zero term is assigned to the nonzero divisor fibers.

The resulting two first-side allowances have exponents $A-c+q_C$ and $A-d+q_D$. With $R_1 = \log_U \mathrm{N}r$ and $E_1 = \log_U \mathrm{N}e$, the original first scalar and the single common family count then give total exponent

$$1-A-R_1/2-E_1+p+\tfrac{1}{2}(2A-c-d+q_C+q_D) = 1+p-(c+d)/2+R_1/2 \leq 1,$$

since $c+d \geq 2p+R_1$ prime by prime. This accounts for the two orientations independently. The inverse cube-free and short-reference restorations of the initial signed comparison use their actual fixed local coefficients and the original parameter exclusion. □

Remark 7.4 (A further displayed factor). A root with parameter width m and an additional common Kummer factor $\chi_I(A_*)$ can use the same calculation with its actual retained ideal-norm width $q_* = \log_U \mathrm{N}A_*$. The first frequency cutoff is $2A-c-d+R_1+E_1-m+O(\eta)$, since A_* is a column character in that Poisson step. The packed radius on the left becomes $2a-k+q_C+q_* = m+q_*+t-6u'+O(\eta)$, and the first allowance becomes $m+q_*$. Use U^{m+q_*} as the scale in the preceding moment bounds. A common pure coprimality restriction may instead be opened before grouping the parameters by Möbius inversion. This shortens the lengths and gives total weight $O(U^{\varepsilon})$. For a displayed radical allowance q, a root of the former kind needs $q_* \leq q$ or a separately proved packing that bounds its active higher powers. The norm of a radical alone does not bound $\mathrm{N}A_*$ in this calculation.

7.4 Disjoint refinement of a repeated prime factor

In this subsection we separate repeated prime sums into disjoint prime sets. The construction is used in Subsection 7.5 for the inverse count.

The amplified short inverse count uses several copies of one actual prime factor. Distinct copies can be replaced by disjoint sublists of its own annulus. The refinement preserves the prime estimates after either Poisson transformation.

Lemma 7.5 (Prime lists with bounded discrepancy)**.** *Fix the field, a bound U^C on the norms of the moving conductors and excluded ideals, and a finite depth and branching number. Exterior prime ideals admit a fixed partition by the leaves of a tree with these parameters. For each node of density b and each original prime factor Q at norm scale P, its restriction Q_b to that node satisfies*

$$Q_b(\lambda) = bQ(\lambda) + O((\log U)^2) \tag{27}$$

uniformly for the finite prime characters and coprimality restrictions in the prescribed norm range. The implied factor includes the supremum and first variation of the compact norm test. Norm twists of polynomial height are allowed. Its constants may depend on the fixed tree. A descendant subset of relative density b' therefore satisfies $Q_{bb'} = b'Q_b + O((\log U)^2)$.

Proof. At each prime choose the successive parts independently and uniformly, with equal branching. At a fixed node the centered indicator has mean zero and modulus at most one. In a norm annulus of scale P there are $O_F(P)$ eligible prime ideals. For a fixed interval endpoint, finite ray character and fixed exclusion, the unnormalized centered sum is $O_F(P^{1/2}(\log U)^2)$ outside an event of probability $O(\exp(-c(\log U)^4))$, by the bounded-variables inequality applied to its real and imaginary parts.

Use dyadic U, dyadic annuli up to U^C, and all initial segments of the primes ordered by norm in each annulus. Primes with equal norm are placed together at an endpoint. The number of finite ray characters of conductor norm at most U^C is polynomial in U by the ray class number bound. The number of ideals and allowed restrictions is also polynomial. For a norm twist $\mathrm{N}\mathfrak{p}^{it}$ with $|t| \le U^C$, include a grid of spacing U^{-C_1} with $C_1 > C + 2$. Differentiation in t gives an unnormalized error at most $O_F(P \log U\, U^{-C_1})$ between grid points. The union over the fixed tree and these tests has summable exceptional probability. Thus a single choice of partition works at all sufficiently large dyadic U. Subtraction gives subintervals. Partial summation gives the smooth intervals with their first variation. Finite coefficient combinations, central normalization and dyadic enlargement of U give (27). □

The complete sums before refinement are called the parent sums. Refinement does not change their norm scales.

Lemma 7.6 (Coefficient-two estimate on the refined lists)**.** *The ordinary mixed estimate of [3, Theorem 2] holds for products of disjoint refined lists from Lemma 7.5. The same is true of the finite-order coefficient-two estimate of [8]. The lengths are those of the parent annuli, with the same height and restriction conventions.*

Proof. Run the two initial inductions in [8, 3] with the actual coefficients $b_i(\mathfrak{p})1_{\mathrm{color}(\mathfrak{p})\in A_i}$, where A_i is its node. We verify the coefficient uses in these inductions.

Coefficient identities. The convolution of the two ordinary factors and the gcd allocation in [3, Lemma 1] involve prime weights only at the actual primes of a marked tuple. Distinct positions still have disjoint prime lists, so their tuples are injective even when their norm scales agree. First and second common divisors, upper and lower Gauss valuations, correction factors and pure coprimality restrictions are local at these primes. They are unchanged when each coefficient is multiplied by its color indicator. A consumed position contributes its actual scalar. A live position retains its indicator. Fourier separation is of the norm tests, so the indicator does not enter their derivatives or common measure. In the unitary case the two fixed ordinary character factors are extracted separately as in [3].

Positive and signed bounds. The selected operator and complete Gauss estimates of [8, Theorem 3.10 and Lemma 4.5] work with arbitrary common ideal coefficients and their divisor square-mass bounds. Each restricted prime weight still has bounded modulus. Fiber multiplicity is bounded by the same divisor power because tuples are injective and indexed by ideals. The high-amplitude term with a proper selected block uses the pointwise bound on the retained prime factors, as distinguished in [8, Theorem 3.13] from the estimate for the whole product. The unmarked two-factor mean, the Gauss estimate for small parameters with its natural zeros, and the fixed-inducing ordinary estimate use bounded local coefficients. The signed comparison and its exceptional part

have the same prime coefficients in both rectangles before taking absolute values. The auxiliary prime average retains its complete set of primes and excludes the original marked intervals.

Prime upper bounds. For each live position (27) compares it to the complete Hecke-compatible sum with the same finite twists, restriction and norm interval. Thus the squared prime exponent κ_1 of [9, Lemma 12.7] holds for each refined factor with an additional factor U^{ε}. The estimates for the whole product and its proper subproducts use the coefficients of the refined sums. Both bounds defining an amplitude interval refer to those same sums.

The complete comparison supplies only the prime upper bounds and the buffered zero-band upper bound. This supplies the small-width pointwise stripping, the sixth-power block-selected bound, and the final stripping from coefficient $L > 2$ to coefficient two. The ordering and removal of low-amplitude factors use their actual amplitudes. The length inequalities depend on the norms of the primes and on the sum of their widths. The disjointness of the original intervals is used through the injectivity and coprimality of the marked tuples, both of which hold for the refined lists.

At zero prime width the earlier estimate is the same unmarked theorem. These coefficient, signed and pointwise bounds supply the coefficient-three and finite coefficient-two inductions of [8, 3]. The discrepancy loss is absorbed in the original prime loss. □

Proposition 7.7 (Repeated factors in the short count)**.** *The paired inverse and ordered inductions extend to disjoint refined lists. On a large-response class the short inverse count can use an amplified original factor at its original norm scale. Its moment input uses disjoint refined lists at the same total width.*

Proof. If Q has fixed positive squared response slope e, then $|Q| \gg P^{e/2-o(1)}$. Choose distinct subsets for the required copies, with a branching number at least that large. By (27) the sum on each subset has modulus at least $c|Q|$ on the class. Their normalized product gives the same exponent as the copies. The width is shortened by at most one parent width. If a previous count used a refined factor, use its descendants. The number of repetitions and the depth of refinements in a fixed finite construction are finite. Fix the tree once the finite width list is fixed, before taking amplitude classes.

Hu's marked inverse base [4, Theorem 4.1] allows disjoint prime lists in the same annulus and arbitrary bounded coefficients. The balanced ordinary base uses Lemma 7.6. In each later inverse stage the first canonical reduction, selection, Poisson summation and restoration of natural zeros use injective tuples and bounded scalar extraction. They preserve the color of each live position. In the variable-length detector compare a colored factor to its complete parent by (27). The parent's buffered character maximum still bounds its slope with the original bin losses. The short inverse sum uses the earlier inverse stages. For the other short estimate use $S_{m'}^2 Q_K$ with equal ordinary lengths and a complete subproduct of the refined positions. This uses the earlier balanced ordered estimate on disjoint refined lists. The long count is unmarked.

For the ordered side the two signed correlations have the same bounded local coefficients and natural zeros as in Proposition 7.3. The earlier ordered estimate in the low branch and the same-zero ordinary substitution inherit the actual lists. The short inverse comparison uses the refinement just proved. The auxiliary ordinary fourth uses a genuine subproduct and Lemma 7.6. The peripheral coefficient-two and unmarked estimates apply as before.

Thus the inverse and ordered stages close together on the refined lists in the same order as on the original lists. □

For a list of overlapping original factors, assign distinct subsets at the root to its different positions. By (27) each full factor is the corresponding refined factor divided by its fixed density,

plus an error $O(U^{\varepsilon})$ with the stated test bounds. Expand this identity in the original product and use Cauchy's inequality over its finitely many terms. Every surviving subproduct has disjoint refined lists and no larger width.

The same estimates therefore hold for the full original factors, with an additional factor U^{ε}.

7.5 The ordered moment over number fields

In this subsection we prove the ordered moment using Subsections 7.1–7.4.

For a number field $F \supset \mu_{12}$ fix the conventions for exterior ideals and unit classes of Subsection 7.3. The root has a single natural moving parameter k, with no separately displayed moving Kummer factor. Common pure coprimality restrictions with polynomial norm bounds may be removed by the exact divisor identities used above. Write the two ordinary factors with the same moving sextic orientation. For unitary inputs, their fixed anchors lie in $\{1, \psi\}$, up to the fixed finite ray characters and independent norm modes. Whole conjugation gives the equivalent orientation with anchors in $\{1, \overline{\psi}\}$. The actual prime character in every factor is the finite one, with coefficients as in the unitary pairing of [6]. All stars refer to the finite inducing-character exclusions in this paired family. Where the curved anchor improvement is used we work in the contradiction regime $\beta_{\omega} \geq 51/100$ of [7, §9]. Here β_{ω} is the zero supremum, with lower bound $1/2$, over $\{\theta\omega^{j}(\mathrm{N}_{F}\cdot)^{it} : j = -1, 0, 1,\ \theta \text{ finite order},\ t \in \mathbb{R}\}$ for the fixed target ω. A zero of the target beyond any boundary proved below puts the working family in this regime.

The fixed-anchor unitary families and the finite numerator needed by the detector are contained in the oriented paired class.

Theorem 7.8 (The ordered ordinary source over number fields)**.** *Let $z \leq n \leq 1.20z$ and $z \leq 0.3888$. At every strict prime width below*

$$W_{f}(n, z) = \frac{1 - n - z}{2} + f(z),$$

the starred ordinary second moment is $O_{\varepsilon}(U^{1+\varepsilon})$, with the common finite seminorm dependence and polynomial factors in the independent heights. The claim allows componentwise shortenings, all original factor subproducts and the first normalized log tests. The following choices are valid:

- *For $z \geq 0.369$, take $f = F$ of (P1) in [2], using the curved width in [7] in the stated contradiction regime.*
- *For $z \geq 0.369$, take $f = (977/1000)F$ unconditionally for finite-order anchors and their norm twists.*
- *For the initial width functions f_{n} and f_{n}^{+} of (13), take their respective lower endpoints 0.366 and 0.369. The assertion is unconditional in the paired family with common $\overline{\chi_{\bullet}(k)}$ orientation and Hu's oriented choice of ψ.*

Proof. For $f = \lambda F$ take $u = 3 \cdot 10^{-6}$ and $\rho = 2 \cdot 10^{-11}$. For the two initial polynomials take u and ρ/u sufficiently small and positive, as specified below. In all cases put $H_{0} = 1 - 6u$ and $Y = 0.3888$. For F write $a = 0.3735$ for the seam. Use descending intervals with upper endpoints YH_{0}^{j}. At a shared endpoint use the earlier completed interval. On $[Y, Y/H_{0}]$ the continued cubic f is nonpositive, so the initial coefficient-two estimate starts the induction. At a fixed stage choose the factor mesh, strict decrements and the finite width and ratio covers before the arithmetic data. The induction stops when its intervals cover the specified interval for z.

Apply Proposition 7.3. For a filled selection the target on a regular sixth-power-free core of actual dyadic radius U^V is the exponent

$$V + u - \rho + \frac{5}{6}(H_0 - V)_+. \tag{28}$$

The kernel and the first selected tuple supply a live prime width at most $L = W_f(n, z) - u$, up to the preassigned mesh. The reference and origin conditions follow from $W_f < 0.137$, $n+z+W_f \le 0.92768$ and the short cap (21). Both inequalities in (23) involving A are strict. The use of the smaller actual dyadic core radius in (28) is legitimate: $V + (5/6)(H_0 - V)_+$ increases with V. Complementary factor restoration and the complementary factor bound of Proposition 7.1 and its proof then give (22). In particular the detector below is used on a dyadic interval for the sixth-power-free part, with $\log_U \mathrm{N}k = V + o(1)$.

Fix the finite union of the parameter's two ordinary character dictionaries and of the prime projectors, together with their actual height intervals and their conjugates. The buffered maximum in the variable-length detector, (D1)–(D2) of [7], bounds both ordinary factors on the same zero band. Use the moment inputs below for the counts. After partitioning by the original factor with largest squared response, write its slope as e. The zero slope satisfies $e \le \nu \le 3/4$ with the allocated bin errors. If the large-response branch is used, then $e > 0.37$ and the maximum is attained at a zero, rather than at the detector floor. At each fixed cutoff the inverse and ordinary detecting sums are supplied at that zero. Their lengths in units of V satisfy $r' + m' \ge T - o(1)$, $r' \le T + o(1)$, $m' \le 1/2 + o(1)$, and their squared sizes are at least $U^{V\nu r' - o(1)}$ and $U^{V\nu m' - o(1)}$.

The earlier detector proof applies with T in the compact interval $[1.09, 1.18]$ containing all cutoffs below. Its shifts use the actual normalized conductor cap and the strict taper gap. Prime holes from a fixed complementary factor remove only $O(\log U)$ primes. They change a factor of positive width w by $O(U^{-w/2} \log U)$, so on this large-response branch its lower response also holds for the clean canonical factor. Each ordinary hole is opened separately by the centrally normalized Möbius identity. The detector inverse is used with its own fixed deleted base. A whole ordinary sum after conjugation has an anchor in the same pair $\{1, \psi\}$ (or in the whole conjugate pair), with its own finite ray, phase, height and test.

For an inverse with length at least one, the long inverse deficit is $D_l = 5/6 - (5/6 - \nu)T$. Use the sixth-power amplified inverse estimate of [9] in finite order. In the unitary case orient ψ by choosing it or its conjugate before constructing the detector, so that its angular vector differs from the theta vector of [4, Section 3]. The finite anchor has angular vector zero. For Hu's paired characters, whole conjugation of the finite numerator puts both factors in the $\overline{\chi_\bullet(k)}$ orientation, with anchors 1 and ψ.

The complete CRT identity of [9, Lemma 12.10] gives $\chi_I(j)$ and $\overline{\chi_J(-j)}$ in the two correlation columns. Since $\chi_J(-1) = 1$, taking the entire conjugated second factor inside its absolute value makes their signs agree. The unit and reciprocity factors belong to the fixed finite dictionary. Reversing the common moving orientation conjugates the entire presentation and the theta coefficient together. Thus the detecting sum chosen from the union, including its whole conjugate if necessary, can be written in Hu's initial pair. This applies after componentwise shortenings and reordering. The short marked and unmarked inputs use the same oriented family for every choice of the ordinary test function. For the long inverse count, use Hu's unmarked inverse estimate in the sixth-power lifting of [7, §8]. At a fixed sixth-power complementary factor the reciprocal divisor expansion retains the literal deleted Euler factors. It shortens the inverse block and keeps the same oriented character, so this lifting also applies when the short width is the curved one.

For an inverse of length between r_0 and one the deficit is at least $D_i = \nu r_0 + eq(r_0)$. The short

inverse width is

$$q(r) = \frac{1-r}{2} + g(r).$$

For $\lambda = 1$, use the lower tangent $0.05805911 - 0.07599620r$ below $3/4$, the marked-anchor curve $G(r)$ of [7] from $3/4$ to $7/9$, and the initial continuation $g = 0$ above $7/9$. The tangent is a strict lower bound for G on the needed compact range by convexity and its tangent at the contact. On each compact range below $7/9$ take the finite curved-width construction with fixed strict widths. The initial continuation also supplies arbitrarily small intervals near the endpoints. For $\lambda = 977/1000$, use the unconditional one-stage line of [6],

$$g(r) = \frac{13047}{460859}\left(1 - \frac{r}{191/250}\right)_+, \qquad r \geq 37/50.$$

All cutoffs tested below are strictly above $37/50$ in this case. Both initial polynomials use $g = 0$ throughout.

The short-inverse deficit increases on both sides of every seam. The usual limiting unmarked collar at length one is estimated by the amplified bound. The deficit bound uses the same maximal original factor repeated within the actual inverse width, with the small finite coincidence and rounding losses allowed by the earlier moment estimates. The remaining case $r' < r_0$ forces an ordinary sum with length $m' \geq m_0 - o(1)$, where $T = r_0 + m_0$. Thus these three alternatives concern one pair at one zero.

For $f = \lambda F$ we give the directed finite comparisons. On a closed length cell $[b_0, b]$ of width $1/200000$, let

$$B = \max_{z \in [b_0, b]} \{2u + f(z) - H_0 f(z/H_0)\}, \qquad C = \frac{1}{4} + \frac{b}{2} + f(b) + \frac{u}{2}.$$

In B use the continued cubic in the starting strip as a conservative bound. The maximum is attained at an endpoint or at $H_0 a$, if it belongs to the cell. On a common convex piece the bracket is decreasing. Across the seam the left and right derivatives give a positive derivative, so no additional interior point is required. Round $(u - 3\rho)/B$ down to denominator 10^{12} and cap it at $3/4$. Write the result as k. The low formula (18) bounds $0 \leq e \leq k$. Divide $[k, 3/4]$ into eight equal closed bins.

On a bin $[e_-, e_+]$ put

$$D = \frac{b + 2f(b) + 4u - 2(u - 3\rho)/e_+}{H_0}, \qquad J(m) = \begin{cases} f(m), & m \leq Y, \\ 0, & m > Y. \end{cases}$$

Choose rational upper approximations on the grid 10^{-12} with

$$r_0 \geq 2b/H_0, \qquad r_0 + 2g(r_0) \geq 2D, \qquad m_0 \geq b/H_0, \qquad m_0 + 2J(m_0) \geq D. \tag{29}$$

The minimal grid choices suffice. The functions in these tests are increasing on the domains searched, including the favorable initial transitions. In the curved test, $r_0 + 2G(r_0) \geq 2D$ is verified by checking nonnegativity and comparing the fifth powers of $(8/5)(19r_0/8 - 7/12 - 2D)$ and the positive fifth root of $7r_0^4(r_0 - 2/3)$. On the lower seam use the stated rational tangent.

For $\lambda = 977/1000$ the tests are linear and polynomial.

At $V = H_0$, the exponents (19) and (20) are nonincreasing in ν for fixed e, using $r_0 \geq 2b/H_0$, $m_0 \geq b/H_0$ and $H_0 T/2 > b$. They are nondecreasing in z, since $1/2 + f'(z) > 0$. The argument with the preceding length bound of (19) uses 1.20 in place of 1.15, since $Y/0.369 < 1.20$. The two tests in

(29) therefore bound the short inverse and the ordinary substitution at $z = b$, $\nu = e = e_+$ with 3ρ to spare. The ordinary upper bound also applies to larger detecting lengths, since its active derivative in m is at most $H_0[-\nu+e(1/2-f'(m))] \le 0$. The long inverse exponent $eC-H_0\{5/6-(5/6-e)T\}/2$ is decreasing in e on the tested support. At the lower endpoint e_- the $3960 \cdot 8$ rational tests give

	$\lambda = 1$	$\lambda = 977/1000$
lower bound for k	0.3949	0.3968
lower bound for the long margin	$4.95 \cdot 10^{-6}$	$1.81 \cdot 10^{-6}$
upper bound for r_0	0.777622	0.777622

The interval $1.1126 < T < 1.1665$ and $m_0 < 1/2$ hold in all cells.

The checked short cutoffs have a uniform positive margin below 7/9. These computations use closed cells, with the inverse cutoff support, long affordability and the two directed tests all checked on the same box. The two columns are verified by the supplementary program `check_ordered_moment.py`. The strict margins allow the perturbations of the tests and the rounding of the detecting lengths, prime widths and responses. The proof of Proposition 7.1 also provides an independent estimate of exponent $V - 0.026 + \varepsilon$ for $e > 0.5002$ with the new volume cap.

For the two initial polynomials the strict condition (14) gives the low threshold directly. With $z' = z/H_0$ put $L_0 = 2u + f(z) - H_0 f(z')$, using the initial width in the starting strip. On the remaining compact interval,

$$L_0 = u\{2 + 6f(z') - 6z'f'(z')\} + O(u^2), \qquad 2 + 6f(z') - 6z'f'(z') < 1/k_*(z', f(z')).$$

Thus $(u-3\rho)/L_0$ is strictly above k_* when $L_0 > 0$, for sufficiently small u and ρ/u. If $L_0 \le 0$ the low estimate bounds all responses. At the low threshold take $m_0 = z'$ and $r_0 = \max(2z', 2z' + 4f(z'))$. The long estimate is then strict. Above the threshold the short and ordinary cutoffs can be increased by $O(u)(e - e_0)$ as in Theorem 6.1. Here the ordinary substitution decreases at slope at least $0.23H_0e$, since $-0.27 < f' < 0$. The strict long inequality persists because its derivative at fixed total cutoff is negative with a fixed margin. Over the initial width functions the unchanged cutoffs have $1.098 \le r_0 + m_0 < 1.167$, and the perturbations are as small as required. Compactness gives finite lists with strict positive padding. This bounds the same three cases after the second correlation and supplies (28) at $V = H_0$ in both initial cases.

For $V \ge H_0$ the earlier ordered caps used at H_0 dominate the actual caps. The absolute width grows when increasing the parameter radius. In the initial and count formulas the margins improve as V increases. For $0.98 \le V < H_0$ the low branch uses $(n/V, z/V)$ until $z/V = Y$ and then uses its initial width. For ordinary substitution use the ratio-adjusted pair $(n/V, m'')$, where $m'' = \max(m', n/(1.20V))$ and m' retains its actual lower response. After ordering it the smaller cap is at least z/H_0 and the ratio is at most 1.20 as long as $m'' \le Y$. Above the seam use the initial pair. In either case every varying absolute width has one-sided derivative with respect to V between zero and 0.61. In the unclamped cases these derivatives are $1/2 + f(s) - sf'(s)$ or $(1-m')/2 + f(s) - sf'(s)$, where s is the appropriate normalized short cap, and the formula with a constant short cap omits $-sf'(s)$. The clamp uses $s = n/(1.20V)$. The change to the initial width is favorable as V decreases because $f(Y) < 0$. The adverse slope of the ordinary substitution loss is therefore at most $(3/4)(1/2 + 0.61) < 5/6$, and the count exponents have smaller adverse slopes.

The low loss has slope at most $(3/4)0.61$. This proves (28) on this interval.

On $[0.875, 0.98]$ the mixed coefficient-two moment applies to the actual total ordinary length $n + z \le 0.85536 < V$. Retain a complete subproduct of width up to $(V - n - z)/2$ and strip the remaining factors at their pointwise exponent. This bounds the excess by $(3/4)[(1 - V)/2 +$

$\max f - u]_+$, up to the allocated mesh loss. At 0.98 this is below the right side of (28) minus V, and its adverse slope is at most $3/8$. On $[0.08, 0.875]$ use the nonprincipal unmarked fourths and the pointwise prime bound with excess at most $(3/4)0.137 < 0.103$. Below 0.08, direct counting and the buffered pointwise ordinary bounds give excess at most $(3/4)(n+z) + L < 0.718$, which is within the allowance there. For these peripheral uses an unavailable negative formal width is replaced by the unmarked estimate. All fixed-inducing parameters were assigned to their signed sector before these comparisons.

The earlier complete ordered rectangle or the initial face supplies every auxiliary source used at a given stage. Proposition 7.3 restores the uncentered second moment once the regular terms, the fixed parameters and the short reference are estimated. The inverse cube-free identities and exact restrictions retain the original entering coefficients. The argument uses the translates of the same two ordinary test functions and their independent heights throughout. The derivative in the normalized height inserts $i\log(x)$, whereas the derivative in the log scale replaces $W(x)$ by $-W(x)/2 - xW'(x)$. Both are included in the same finite families of tests. This proves the induction and the asserted estimates. □

Corollary 7.9 (The initial limiting width in the paired family)**.** *Let F_{nat} be the function in Corollary 6.2. In the initial case of Theorem 7.8 the same ordinary estimate holds for*

$$0.366 \le z \le 0.3888, \qquad z \le n \le 1.20z, \qquad S < (1-n-z)/2 + F_{\mathrm{nat}}(z).$$

The strict width buffer can be uniform on a compact set of caps.

Proof. The logarithmic formula gives $0.299 < b(0.366) < 0.300$. For example one can isolate b at these rational endpoints by

$$\log x = 2\sum_{j=0}^{m-1} \frac{t^{2j+1}}{2j+1} + R_m, \qquad t = \frac{x-1}{x+1}, \qquad |R_m| \le \frac{2|t|^{2m+1}}{(2m+1)(1-t^2)}$$

with $m = 32$. Differentiating its differential equation gives

$$F'_{\mathrm{nat}} = \tfrac{1}{4}\left(b - 3 + \frac{29/15 - 2/(5b)}{z}\right), \qquad F''_{\mathrm{nat}} = \frac{(29/15 - 2/(5b))(2/5)}{4z^3 b^2} > 0.$$

On $[0.366, 7/18]$ we have $F_{\mathrm{nat}} < 0.003$, $-0.27 < F'_{\mathrm{nat}} \le 0$, and $1/2 + F_{\mathrm{nat}} - zF'_{\mathrm{nat}} < 0.61$. The last bound uses the decreasing function $F_{\mathrm{nat}} - zF'_{\mathrm{nat}}$ at $z = 0.366$.

Choose the strict width function $F_{\mathrm{nat}}(z+\epsilon) - \epsilon_0$ as in the proof of Theorem 6.1, starting from the initial width at $7/18 - \epsilon$. The derivative margin $0.23H$ works in the ordinary substitution. For small ϵ the proof of Theorem 7.8 applies throughout: $W < 0.138$, $n < 0.467$, $A < 0.928$, the short cap is (21), and the tiny-core excess is less than 0.718. The active ordinary radial slope is at most $(3/4)(1/2 + 0.61) < 5/6$. For each prescribed positive width decrement take a single sufficiently close strict width function and a finite construction. All estimates then use the literal finite stages and their common height orders. □

7.6 Coupling the inverse and ordered moments

In this subsection we use a common convex function for the inverse and ordinary widths of Subsection 7.5.

The inverse half of the marked-anchor induction needs the ordinary moment only for a square S_m^2 at a zero-detection cutoff. The ordered ordinary induction above needs an inverse moment only

at a cutoff on the earlier side of its short length index. This permits both inductions to use a single convex width function.

Put $a = .3735$, $Y = .3888$, and $\Delta = 10^{-10}$. With $s = Y - y$, define

$$\begin{aligned} f_+(y) &= .997\big(-9\cdot 10^{-9} + .0007761s + 4.612s^2 + 48.446s^3 + 87.5274s^4\big), \\ f(y) &= \begin{cases} f_+(y), & a \le y \le Y, \\ f_+(a) - f_+'(a)(a-y) + 3.988(a-y)^2, & .367 \le y < a, \end{cases} \\ \phi(r) &= [f(r/2) - \Delta]_+ \qquad (.734 \le r \le .7776). \end{aligned} \tag{30}$$

Continue ϕ by zero for $r \ge .7776$.

Theorem 7.10 (Coupled inverse and ordinary estimates)**.** *In the paired families of Theorem 7.8 and Proposition 7.20, the original marked inverse moment holds for*

$$.734 \le r \le .7776, \qquad S < (1-r)/2 + \phi(r).$$

The original two-ordinary moment holds for

$$.367 \le y \le .3888, \qquad y \le n \le 1.20y, \qquad S < (1-n-y)/2 + f(y).$$

Both statements allow all componentwise shortenings, original subproducts, the actual natural zeros and independent normalized height tests, with the same seminorm convention. The only inputs are the initial estimates and transformations cited above.

Proof. Write $D = 1 + f'(y)$ and $E = yf'(y) - f(y) - 1/3$. At a previous short cap y, use $R_0 = 1/2 + y + 2f(y)$ as in (10), now with $q(r) = (1-r)/2 + \phi(r)$. Convexity gives an admissible inverse cutoff

$$r_0 = 2y + \frac{2[f(y) + \Delta]_+}{D}, \qquad T = r_0 + y.$$

Indeed for $f + \Delta > 0$, put $y' = r_0/2$. Then

$$y' + f(y') - \Delta \ge y + f(y) + D(y' - y) - \Delta = y + 2f(y).$$

The cutoff stays in the interval since $f(Y) + \Delta < 0$ and $-0.27 < f' < 0$, so $[f(y) + \Delta]_+/D < Y - y$ when its numerator is positive. When this numerator vanishes, $r_0 = 2y$ and $R_0 \le 1/2 + y$. In all cases $r_0 + q(r_0) \ge R_0$.

The cleared differential condition for the ordinary propagation with this cutoff is

$$D\{5(T-1)E + T - R_0\} > 0. \tag{31}$$

It gives a strict reverse inequality in (12), using the actual minimum admitted cutoff in place of T if desired. The inverse propagation condition at positive ϕ is

$$r(45r - 30)\phi'(r) - (45r - 24)\phi(r) - 9r + 7 > 0. \tag{32}$$

It is the strict inequality $\phi'(r) > a(r)\phi(r) + b(r)$ for the two coefficients of the inverse Euler step in [6]. On the zero branch this condition follows from $r < 7/9$.

The left sides of (31) and (32) are polynomials in the two polynomial pieces and the two sign cases. The exact check in `check_coupled_profile.py` uses 64 Bernstein cells on each piece. In

the upper piece the initial sign case $f + \Delta \le 0$ lies within $0 \le Y - y \le 2 \cdot 10^{-5}$. The ordinary and positive inverse margins exceed

	ordinary	inverse
upper piece	$1.6378 \cdot 10^{-5}$	$9.8294 \cdot 10^{-5}$
lower piece	$2.9735 \cdot 10^{-5}$	0.026317

respectively. Also $0 \le \phi < .003$, $-0.27 < f' < 0$ and $0 < 1/2 + f - yf' < 0.61$.

In the inverse stage the limiting cutoff is

$$k_*(R) = \frac{5R/4 - 5/6}{R - 1/2 - \phi(R)}.$$

It is increasing, since the numerator of its derivative is $5/24 - 5\phi/4 + (5R/4 - 5/6)\phi' > 0$, and $k_*(.734) > 45457/125000$, $k_*(.7776) < 1/2$. At zero step size, the other stage conditions of [6] hold on $[.734, .7776]$. For example $S = (1-R)/2+\phi(R) < .136$, $R+S < .892$, $R-.7-\phi(R) > .031-10^{-6}$ and $w(k) = w_+ = R < .79$. Substitution in its margin list gives positive lower bounds in conditions 2–14. The same margins persist for sufficiently small steps and supporting-line paddings.

Here are the induction order and the role of the common cutoffs. Use a finite descending division into intervals ending at $r = HR$, with $H = 1 - 6u$ sufficiently close to one. The top intervals where $f < 0$ and $\phi = 0$ use the initial estimates. At each completed radius the inverse and ordinary input bounds hold at all earlier radii, with arbitrary fixed positive width decrements. The supporting line in the inverse step is a tangent to the convex ϕ at R, decreased by a small amount. It is a minorant of the earlier inverse and balanced ordinary widths, since $f(w/2) \ge \phi(w)$ wherever $\phi(w) > 0$.

If the tangent is negative, the initial moment supplies its width. For $w \ge .7776$ all of these tangents are nonpositive. The inverse half uses the earlier inverse rectangle on the low-amplitude terms. In its zero detection it uses only $M_{r'}Q$ and $S^2_{m'}Q$, respectively at the inverse and balanced ordinary widths of the earlier supporting line. The ordinary half of the joint step is provided separately by the ordered estimate. The orientation argument of Proposition 7.20 applies at each step: both earlier estimates have the same synchronized paired class, and the transforms introduce only finite ray multipliers.

For $r < R$, the Euler propagation and (32) give $Y(r) > \phi(r)$ after sufficiently small preassigned losses. For a closed upper endpoint use the preceding estimate. Compact cutoffs with a prescribed strict width buffer are supplied by finitely many slightly enlarged earlier caps.

For the ordinary estimate at a short index y with $2y$ in the current interval, the earlier cap y/H is at least $R/2$. The ordered comparison uses the initial estimate or the completed ordinary cap at y/H in its low part. The short inverse comparison uses $r_0 \ge 2y/H$ with its whole support and natural zeros. Thus all inverse estimates used for the short inverse count are already available. Ordinary substitution uses the same zero and an earlier ordinary cap. The finite cutoff proof of Theorem 6.1 applies using (31); for the substitution the decrease at the initial cutoff is at least $0.23H$.

In the other regular radii its adverse slope is at most $(3/4)(1/2 + 0.61) < 5/6$. The volume and reference conditions of Theorem 7.8 hold since $A < .928$, $W < .138$ and $n < .467$, using (21). The fixed inducing parameters and both signed regular sides are then restored by Proposition 7.3. This proves the two estimates on the current interval using only completed intervals. The proof uses finite strict buffers before fixing the test seminorms, restrictions, phases and heights. □

Corollary 7.11 (A refined coupled width function)**.** *Theorem 7.10 also holds with* $Y = .388791473$ *and the upper width function*

$$f_{\mathrm{r},+}(y) = -0.000000003144 + 0.00090070693s + 4.6215100s^2 + 47.562532s^3 + 87.25198s^4 - 427.8832s^5, \qquad s = Y - y.$$

For $y < a = .3735$*, set* $t = a - y$ *and take the lower part*

$$f_{\mathrm{r},+}(a) - f'_{\mathrm{r},+}(a)t + 6.230097t^2 - 181.21832t^3.$$

Keep $\Delta = 10^{-10}$*,* $y \geq .367$ *and* $n/y \leq 1.20$*.*

Proof. The ordered and inverse differential margins exceed $6.17 \cdot 10^{-7}$ and $5.656 \cdot 10^{-6}$, respectively, by the same rational Bernstein check. The positive second derivative, $-0.27 < f' < 0$, the radial upper bound 0.61, and the cutoff lower bound 45457/125000 hold on the new range. The upper endpoint has negative ordered improvement and zero inverse improvement. The proof of Theorem 7.10 applies without changing its parametric estimates. □

7.7 The limiting common width function

In this subsection we pass from the common-function construction to its limiting differential equation. Subsection 7.8 treats the two functions independently.

The common-function scalar system grants the ordered added width $f(y)$ and the inverse added width $f(r/2)$. It uses the three exponents of Theorem 6.1 and the inverse supporting-line step of [6]. Its initial value is $y_0 = 7/18$, with zero added width. Write $y \in [.369, y_0]$.

Theorem 7.12 (Optimal common width function)**.** *In this scalar system every finite construction is bounded by the solution*

$$\begin{gathered} x + f_{\mathrm{c}}(x) = y + 2f_{\mathrm{c}}(y), \qquad x \geq y, \qquad T = 2x + y, \qquad R_0 = \tfrac{1}{2} + y + 2f_{\mathrm{c}}(y), \\ yf'_{\mathrm{c}}(y) = f_{\mathrm{c}}(y) + \tfrac{1}{3} - \frac{T - R_0}{5(T-1)}, \qquad f_{\mathrm{c}}(y_0) = 0. \end{gathered} \tag{33}$$

Every smaller width with a fixed positive decrement on a compact subinterval is supplied by finite common-function stages over the paired number-field families. In particular

$$F_{\mathrm{nat}}(y) \leq f_{\mathrm{c}}(y) \leq F_{\infty}(y).$$

Proof. The ordinary transition at occupied width f has the threshold $k_o = (5/6)(T-1)/(T-R_0)$. The inverse transition has

$$k_i = \frac{5}{6}\frac{3y - 1}{2y - 1/2 - f}.$$

For $f \geq 0$ the first threshold at $T = 3y$ is at least k_i, and k_o increases with T because $R_0 < 1$. Thus the ordinary differential condition implies the inverse condition. The cutoff x in (33) is the smallest balanced short inverse cap permitted by the width.

For existence, in the integral equation from y_0 to y use $x = \max(y, \mathcal{I}_g(y + 2f(y)))$, where $\mathcal{I}_g$ is the inverse of $s \mapsto s + g(s)$. Extend the upper width function by zero beyond y_0. If g is Lipschitz with constant less than $1/2$, this inverse has Lipschitz constant at most 2. The cutoff depends only on the width function on $[y, y_0]$, and the right side is locally Lipschitz in the current value and in that width function in the supremum norm. On a sufficiently short descending interval the integral operator is a contraction on the closed set of width functions with derivative bounded in modulus

by a fixed constant less than $1/2$. It has the same property on each continuation interval with the earlier values fixed. Near the initial point $T = 3y_0$, $R_0 = 1/2 + y_0$, $k_o = 1/2$, and $f_c' = 0$. Whenever $f_c \geq 0$, the lower bound $k_o \geq (5/6)(3y-1)/(2y-1/2)$ gives $f_c' > -0.305$. Differentiating the cutoff gives $x' = (1 + 2f_c'(y))/(1 + f_c'(x)) > 0$ on its active branch. Thus T and R_0 increase and $k_o' > 0$. In turn $yf_c'' = k_o'/(6k_o^2) > 0$.

These inequalities also hold one-sided at a change of branch. They imply $f_c \geq 0$ below y_0, $y + 2f_c(y) \leq y_0$ and $f_c < 1/160$. This gives continuation throughout $[.369, y_0]$.

The inverse differential inequality just proved gives $f_c(r/2) \leq G(r)$ by integrating the linear differential inequality against the integrating factor for $a(r)$ from the common initial value zero. For the ordinary right side (10), at fixed current value and cap, increasing the earlier width function decreases x, T and k_o. The right side therefore decreases. Taking the past width function zero or $G(2\,\cdot)$ gives, by scalar comparison in descending y, the two asserted bounds.

For any finite stage, replace the earlier history by the upper width function f_c. This increases its ordinary Euler allowance. With the history fixed, that allowance is increasing in the current parent improvement. In the derivative calculation from (12), now $\partial_f T \leq 4/(1 - 0.305) < 6$. The same two interval bounds in that proof apply. This gives

$$f_{\text{new}}(y) \leq f_c(y') + (y - y')f_c'(y') \leq f_c(y)$$

by convexity. Induction also bounds every intervening completed cap; the comparison uses the scalar allowance at each such cap.

For the converse shift the width function to $f_c(y + \epsilon) - \epsilon_0$ and start at $y_0 - \epsilon$. In the new cutoff identity put $X = x + \epsilon$. Then $X + f_c(X) \leq (y + \epsilon) + 2f_c(y + \epsilon)$. Consequently both T and R_0 have fallen strictly from the corresponding earlier values. The shifted width function satisfies the strict ordinary differential inequality and hence the inverse one on its positive branch. On the negative starting strip use the initial source. The analytic radial conditions are strict: at $y = .369$ the lower current-value comparison $F_{\text{nat}}(.369) > .00212$ and the upper past comparison $G(2\,\cdot)$ give $f_c'(.369) > \mathcal{D}(.369, .00212) > -.247$.

For the last directed bound one can take $r = .742912$ as an upper cutoff and isolate the lower cutoff by comparing fifth powers in (10); using $r > .742911$ in that formula gives the displayed lower derivative bound. Convexity and $f_c \leq F_\infty$ now give $-1/4 < f_c' \leq 0$ and $1/2 + f_c - yf_c' < .61$ on the interval.

Choose a further inverse decrement $0 < \Delta \leq \epsilon_0$. At the shifted earlier cutoff the ordinary threshold then has surplus $\epsilon_0 - \Delta \geq 0$. The finite induction of Theorem 7.10 applies to this strict shifted width function. Its stage conditions follow from the same compact bounds, now with starting radius slightly below $7/9$. The construction is performed after fixing the requested width decrements. □

A directed upper enclosure. Put $s = 7/18 - y$ and

$$U(y) = \frac{162}{35}s^2 + \frac{222}{5}s^3 + 220s^4.$$

Then $f_c \leq U$. To verify this let x_U solve $x_U + U(x_U) = y + 2U(y)$, and put

$$D = 1 + U', \qquad C = U'', \qquad W = D^2 + \tfrac{1}{2}CU, \qquad E = yU' - U - \tfrac{1}{3}.$$

Since $U''' \leq 0$ and $D > 0$,

$$U \leq D(x_U - y) + \tfrac{1}{2}C(x_U - y)^2 \quad \Longrightarrow \quad x_U - y \geq UD/W.$$

The rational check in `check_coupled_upper.py` gives

$$5\{(3y-1)W+2UD\}E+\{(2y-\tfrac{1}{2}-2U)W+2UD\}\le -0.009705\,s^2.$$

This gives the supersolution inequality for (33) at $T=3y+2UD/W$, hence also at $T=2x_U+y$ since $5E+1<0$. A larger earlier width function decreases the derivative with respect to y at the same current width. Comparison with the common initial value therefore gives $f_{\mathrm{c}}\le U$. Combining with the finite coupled width function of Corollary 7.11 yields

$$0.000965253345 < f_{\mathrm{c}}(0.37535) < 0.000966003448.$$

7.8 Independent inverse and ordinary widths

In this subsection we derive separate equations for the inverse and ordinary widths. Their limiting values enter Subsection 7.16.

Separate the two short cutoffs in the inverse induction of [6, Section 8]. The short inverse count starts at w_I and the balanced ordinary count starts at w_O. Use the detector length

$$t=w_I+w_O/2+\tau.$$

The analytic conditions for this change are the original orientation, height and taper conditions, together with $2/3<w_I,w_O<.79$ and sufficiently small $\tau>0$. Indeed $t<3/2+\tau$, so the same detector contour gives $r'\le Ht+\varepsilon$, $r'+m'\ge Ht-\varepsilon$, $m'\le H/2+\varepsilon$.

If $r'<Hw_I$, the ordinary sum has $2m'>Hw_O+2H\tau-2\varepsilon$. If $r'\ge Hw_I$, use the inverse sum, with the amplified long count when $r'>H(1-\delta_1)$. The two short counts use the same zero in the same oriented dictionary, with the inverse and balanced ordinary support minus their respective strict width buffers.

Here is the stage calculation. Write $\lambda_i(w)=a_{0,i}-a_{1,i}w$ for the two supporting lines, $D_i=1/2-a_{1,i}$, and set

$$g_i(\nu)=\frac{r+s-H/2-Ha_{0,i}-u/\nu}{D_i},\qquad w_i(\nu)=g_i(\nu)/H+\upsilon,\qquad T_0(\nu)=g_I(\nu)+g_O(\nu)/2.$$

Here $H=r/R$, $u=(1-H)/6$, z is the incoming inverse improvement, $s=H((1-R)/2+z)+u/k$, and $0\le a_{1,i}\le 2/5$. Assume $g_i(k)/H\ge R$. Put $\sigma_i(w)=[(1-w)/2+\lambda_i(w)-\upsilon']_+$. On each short branch, the function $\nu(r-Hw)+e[s-H\sigma_i(w)]_+$ is nonincreasing because $(1/2+a_{1,i})e\le\nu$ with the bin losses chosen as in [6, Section 8]. At $w=w_i(\nu)$ the corresponding moment loss is at most

$$H+\nu(r-g_i)+e(g_i-r+u/\nu)+H\{-\nu\upsilon+e[(1/2+a_{1,i})\upsilon+\upsilon']\}+o(1).$$

This is strictly below $H+u$ after choosing the bin losses, since $g_i\ge r$, $e\le\nu+o(1)$ and $\upsilon'<\upsilon/20$.

At $\nu=k$ the long margin equals

$$H\left\{\frac{5}{6}-\left(\frac{5}{6}-k\right)T_b-k((1+R)/2+z)\right\},\qquad T_b=T_0(k)/H. \tag{34}$$

For $\nu\ge k$ the increase in the unpadded margin is

$$(\nu-k)\left\{T_0(\nu)-r-s-\left(\frac{5}{6}-k\right)\frac{u}{k\nu}\left(D_I^{-1}+D_O^{-1}/2\right)\right\}.$$

Thus a positive endpoint margin and positive bracket suffice. At each short cutoff require $\nu w_i+e\sigma_i(w_i)<5/6$. With the cutoffs fixed, the radial excesses of each short branch, the long branch

and the small cores then have one-sided slopes at least $-5/6$, just as in the original sixth-power summation.

Taking their maximum leaves the same slope bound. Indeed each short excess is $\nu P + es - \nu V w_i - e\min(s, V\sigma_i(w_i))$ at parameter exponent V. The long slope is $-5/6 + (5/6 - \nu)t$, and the small-core slope is $-5/6$. This restores a sixth-power complementary factor through the factor $(\mathrm{N}B_1)^{-1}$ at $V = H - 6\log_Z \mathrm{N}B_1$ and the same natural-zero identities. The other inverse branches use the earlier inverse moment at (R, z) with the original margins.

This proves the inverse propagation rule with separate cutoffs and the same output $Hz+(1/k-2)u$.

We describe the finite scalar system in which the inverse stages use these two short counts and their long count, and the ordinary stages use (P3), (P5), (P6) of [2]. The short cutoffs are taken in the earlier range of a descending construction and the continuous response classes include $e = \nu$. Put $y_0 = 7/18$. Let $F(y)$ denote the ordered added width at short cap y, and $Z(y)$ the inverse added width at length $2y$. For a prior history p on $[y, y_0]$ write

$$X_p(y, a) = \max\{y, \mathcal{I}_p(y + a)\}, \qquad \mathcal{I}_p = (x \mapsto x + p(x))^{-1},$$

extending p by zero above y_0. When $y + a \le y + p(y)$, use $X_p(y, a) = y$. For nonnegative current states z, f put

	T	L
I	$2y + X_F(y, z)$	$1/2 + y + z$
O	$y + 2X_Z(y, 2f)$	$1/2 + y + 2f.$

$$K_i = \frac{5}{6}\frac{T_i - 1}{T_i - L_i}.$$

The histories in these formulas belong to the other estimate.

Theorem 7.13 (Optimal independent width functions for these stages)**.** *On* $[.369, y_0]$ *the upper boundary of the displayed finite scalar system is the unique solution of*

$$\begin{aligned} yZ'(y) &= Z(y) + 1/3 - \frac{T_I - L_I}{5(T_I - 1)}, \\ yF'(y) &= F(y) + 1/3 - \frac{T_O - L_O}{5(T_O - 1)}, \ Z(y_0) = F(y_0) = 0, \end{aligned} \tag{35}$$

where $z = Z(y)$ *and* $f = F(y)$*. Every pair of smaller improvements with fixed positive width decrements on a compact subinterval is supplied by finite stages in the paired number-field families. The boundary satisfies*

$$f_{\mathrm{c}} \le F < Z \le G(2\,\cdot), \qquad F_{\mathrm{nat}} \le F \le F_\infty \quad (.369 \le y < y_0).$$

Proof. At inverse parent radius $R = 2y$ with increment at the preceding step z, the unpadded inverse short cutoff is at least R. The ordinary short inequality at $e = \nu = k$ requires $w_O/2 + p(w_O/2) \ge R/2 + z$ for the earlier ordinary improvement p at the cutoff. After replacing p by a larger history F with increasing $w + F(w)$, this implies $w_O \ge 2X_F(y, z)$ and $T_b \ge T_I$. Since $L_I < 1$, the ratio $(T_b - 1)/(T_b - L_I)$ increases in T_b.

Consequently (34) forces $k \ge K_I$. With the enlarged history, the inverse Euler allowance is at most

$$z + (y' - y)\frac{z + 1/3 - 1/(6K_I)}{y}$$

at the next length $y' < y$. The ordinary comparison (12) gives the ordinary Euler allowance with f, K_O in place of z, K_I. These necessary estimates hold for any upper envelopes of the improvements granted at the earlier cutoffs. The finite stages need not give smooth histories.

Existence and uniqueness in (35) follow by descending Volterra iteration, as in Theorem 7.12. For histories of Lipschitz constant less than $1/2$, the two inverse functions are uniformly Lipschitz in their argument and in the histories in sup norm. The vector integral map is a contraction on a sufficiently short descending interval. It can be continued while the derivatives stay uniformly below $1/2$ in modulus and $T_i - L_i$ stays positive. To verify continuation, both initial derivatives are zero. On the nonnegative branch, $L_i \geq 1/2 + y$, $T_i \geq 3y$, and hence

$$K_i \geq \frac{5}{6}\frac{3y-1}{2y-1/2}, \qquad Z', F' \geq \frac{18y-7}{30y(3y-1)} > -.305.$$

The active cutoffs have derivatives $(1+Z')/(1+F'(X_F))$ and $(1+2F')/(1+Z'(X_Z))$, respectively, both positive. On a clamped branch the cutoff has derivative 1. Thus T_i, L_i increase and $K_i' > 0$, and the second derivatives in (35) equal $K_i'/(6yK_i^2) > 0$.

These arguments start locally by continuity of the initial derivatives and then give nonnegativity and the strict bounds $-.305 < Z', F' \leq 0$ on the interval of continuation. In particular $y+Z(y), y+2F(y) \leq y_0$, $x \leq y_0$ at each cutoff, and $Z, F < 1/160$. The cutoff maps therefore remain in the same compact range and continuation reaches .369.

At $z = f$ the inverse cutoff is $X_F(y,z) = y$. The ordinary transition has $T_O \geq T_I = 3y$ and $L_O \geq L_I$, with strict inequality in L when $f > 0$. Thus the inverse derivative at equality is no larger than the ordinary derivative. Descending comparison gives $Z \geq F$. For example the negative part of $Z - F$ satisfies a backward Gronwall inequality by Lipschitz dependence on the current states.

At an interior equality the derivative difference would be strictly negative, contradicting the nonnegative difference. Also $T_I \geq 3y$ shows that the inverse differential allowance is at least that of the scalar inverse equation for $G(2y)$ at the same z. Integrating this differential inequality from y_0 gives $Z \leq G(2y)$. The ordinary differential exponent decreases when its prior inverse history increases. Comparing with the histories 0 and $G(2\cdot)$ gives $F_{\mathrm{nat}} \leq F \leq F_\infty$. Since $Z \geq F$, the ordered derivative is at most that of the common-history operator evaluated at F. The same backward comparison gives $f_{\mathrm{c}} \leq F$. For the latter comparison, the positive part of $f_{\mathrm{c}} - F$ at y is bounded by a constant times the integral from y to y_0 of its supremum on the remaining history, using monotonicity in that history and Lipschitz dependence in sup norm.

To bound all finite constructions, replace the earlier histories by the respective solution width functions, increasing both Euler allowances. For either allowance write a for its current parent improvement. With $H = y'/y$ its derivative in a at fixed history is

$$H - \frac{1-H}{5}\frac{(\partial_a T)(1-L) + (\partial_a L)(T-1)}{(T-1)^2}.$$

In the inverse case $\partial_a T \leq 1/(1-.305)$, $\partial_a L = 1$; in the ordinary case $\partial_a T \leq 4/(1-.305)$, $\partial_a L = 2$. Use $1 - L \leq .131$, $3y - 1 \leq T - 1 \leq 1/6$ and $H \geq .369/y$ on $[.369, .379]$ and $[.379, y_0]$.

On each interval this derivative exceeds .35 in both cases. Thus replacing the current improvements by their solution values also increases both Euler allowances. Each resulting Euler value is a tangent to a convex solution, hence at most its value at the next length. Induction proves the upper bound throughout any finite descending construction.

For realization, fix small $\epsilon, \eta > 0$ and use the strict width functions

$$f(y) = F(y+\epsilon) - \eta, \qquad z(y) = [Z(y+\epsilon) - \eta]_+ \qquad (.369 \leq y \leq y_0 - \epsilon).$$

Use initial widths above this interval and wherever a shifted improvement is nonpositive. At the previous inverse cutoff X for $y + \epsilon$, evaluating the new inverse test at $X - \epsilon$ gives

$$X - \epsilon + \max(0, f(X-\epsilon)) \geq y + z(y)$$

on the active inverse branch. At the previous ordinary cutoff X the same calculation gives $X-\epsilon+z(X-\epsilon)\geq y+2f(y)+\eta$.

Thus T and L both decrease from their previous values in each active branch, and so does K. Since the width functions are decreasing, $yf'(y)\geq(y+\epsilon)F'(y+\epsilon)$ and the analogous inequality holds for z on its positive branch. Subtracting η and using the smaller K makes both differential inequalities strict. Where an improvement is zero, the initial estimate is already sufficient.

For the inverse stage take a strict tangent to $z(w/2)$ at the parent radius $R=2y$. Take the other strict tangent to $[f(w/2)]_+$ at its own earlier cutoff $2X_{[f]_+}(y,z)$. The improvements are convex, including their initial zero extensions. Since $z\geq[f]_+$, the unpadded $g_i(k)/H$ lie at or above R and converge to R and the earlier ordinary cutoff as the tangent downward shifts tend to zero. For sufficiently small shifts, the strict differential margin allows k slightly above the ratio in (34). At zero step size $K_I<1/2$ on the shifted compact interval and $K_I>.374$. Moreover $S<.134$, $R+S<.892$, $R-.7-z>.035$, $w_I,w_O\leq 2(y_0-\epsilon)<.79$, and both short radial exponents are below $(3/4)(.79+.14)<5/6$.

The long monotonicity bracket at zero step size is at least $R-.5-z>.235$. Substitution in the remaining stage inequalities of [6] gives uniform strictness. These persist for sufficiently small steps. The proof preceding (35) now transfers the inverse estimate.

For the ordered stage the inverse frontier has $\mathcal{R}'(r)\geq 1/2-.305/2>.34$. The bounds $F_{\mathrm{nat}}\leq F\leq F_\infty$ and $Z\leq G(2\cdot)$ give $F'(.369)>-.247$ by the same directed comparison in Theorem 7.12. Thus the finite ordinary cutoff proof applies with .34 in its inverse cutoff slope and with ordinary substitution slope at least $.25H$. All radial, volume and reference bounds of Theorem 7.10 hold by the same strict inequalities. Both stages use only earlier caps: for the ordinary step at cap y its parent y/H and inverse cutoff are earlier, and the inverse step uses the balanced ordinary history at $w_O/2\geq R/2$. Make the mesh small enough for y/H beyond the starting cap to occur only in the negative top strip. One can also split a mesh interval at the transition to the zero part of z. Thus there is a finite construction on the whole compact interval. Sending the fixed shift and required saving to zero after choosing the requested width decrements gives both stated attainable ranges. □

Proposition 7.14 (An explicit inverse chain with separate cutoffs)**.** *Put* $B=.7776$ *and*

$$\Phi(r)=.000395(B-r)+1.16(B-r)^2+7(B-r)^3+42(B-r)^4. \tag{36}$$

In the same paired family the inverse moment holds at $.734\leq r\leq B$ *and* $S<(1-r)/2+\Phi(r)$*, with the original strict ordinary length, seminorm and height conventions. The balanced ordinary input is* (30)*.*

Proof. Use 4000 equally spaced descending intervals from B to .734, with upper endpoint R and incoming inverse improvement $z=\Phi(R)$. At the top use $\lambda_I=\lambda_O=0$. At each other mesh point use as supporting lines the tangents at R to Φ and to ϕ in (30). Round the negative slopes upward to multiples of 10^{-18}, subtract 10^{-11} and round the intercepts downward to multiples of 10^{-18}. Convexity gives the supporting-line margins on $[R,B]$.

The lines are negative at B and remain negative above it, where the initial estimate applies.

Write $w_{b,i}=(R/2+z-a_{0,i})/(1/2-a_{1,i})$ and $T_b=w_{b,I}+w_{b,O}/2$. Use

$$k=\max\left\{\frac{45457}{125000},\left\lceil\frac{(5/6)(T_b-1)+10^{-12}}{T_b-(1+R)/2-z}\right\rceil_{10^{-18}}\right\}.$$

The denominator is positive and $k<1/2$. In the fourteen stage inequalities of [6], use (34) for condition 1 and use the separate-cutoff monotonicity bracket at $\nu=k$ for condition 9, with the

original decrements 10^{-13} and $1/10$. Conditions 11–13 apply to each of the two cutoffs with upper padding at most 10^{-12}. The other inequalities are unchanged.

At both endpoints every margin exceeds $8.9998 \cdot 10^{-13}$, and the increment at the next step exceeds Φ at the lower endpoint by more than $2.9709 \cdot 10^{-11}$. The exact checks are in `check_split_inverse_chain.py`. All scalar inequalities hold between endpoints by affineness in each branch after multiplying by positive denominators. The increment at the next step is affine and Φ is convex, so the improvement comparison also holds on the whole interval. The inverse propagation rule with the separate cutoffs now proves the result. □

Corollary 7.15 (An explicit strict pair)**.** *Put $Y = .38879$, $s = Y - y$ and use the polynomials*

$$\begin{aligned} Z_\circ(y) = &-10^{-9} + .000913s + 4.645109445s^2 + 54.923964085s^3 + 733.312329034s^4 \\ &+ 7579.122617s^5 + 221771.30161s^6, \\ F_\circ(y) = &-10^{-9} + .000906s + 4.64347782s^2 + 42.99623607s^3 + 464.22064076s^4 \\ &- 14879.77716s^5 + 672714.43022s^6 - 20296440.87411s^7. \end{aligned}$$

The paired number-field stages of Theorem 7.13 supply inverse added width $[Z_\circ(r/2)]_+$ for $.738 \le r \le 2Y$, and ordered added width $F_\circ(y)$ for $.369 \le y \le Y$ and $y \le n \le 1.20y$. The moment bounds have the same strict-width conventions.

Proof. Use convexity at the current cap to bound the two earlier cutoffs from above by

$$x_I = y + \frac{[Z_\circ - F_\circ]_+}{1 + F'_\circ}, \qquad x_O = y + \frac{[2F_\circ - Z_\circ]_+}{1 + Z'_\circ}.$$

The initial zero continuation only increases either earlier input when the signed improvement is negative. At each corresponding T_i, L_i the cleared differential margins are

$$D_i\{5(T_i - 1)(yp'_i - p_i - 1/3) + T_i - L_i\}, \qquad (p_I, p_O) = (Z_\circ, F_\circ), \quad (D_I, D_O) = (1 + F'_\circ, 1 + Z'_\circ).$$

For the inverse margin use $Z_\circ - F_\circ \ge 7 \cdot 10^{-6} s$.

The initial branch of the ordered margin occurs only at $s < 2 \cdot 10^{-6}$. The exact Bernstein computation in `check_independent_profiles.py` gives lower bounds $3.6246 \cdot 10^{-7}$ for the inverse, $1.7738 \cdot 10^{-6}$ for the active ordinary branch, and $3.6520 \cdot 10^{-6}$ for its initial branch. It gives strict convexity, $-.283 < Z'_\circ < 0$ and $-.243 < F'_\circ < 0$.

Both cutoffs lie in $[y, Y]$, since $Z_\circ(Y) = F_\circ(Y) = -10^{-9}$, $Z_\circ - F_\circ \le .283(Y - y)$ and $2F_\circ - Z_\circ \le .486(Y - y)$ on their positive branches. Their denominators exceed .757 and .717, respectively. Thus $3y \le T_i \le 3Y$ and $L_i \le 1/2 + Y$, giving $K_i < 1/2$. At positive input improvements the initial lower bound gives $K_i > .374$. The maximal ordinary column length is strictly below .928 because $1/2 + 1.1y + F_\circ(y)$ increases to its value at Y. Together with the derivative and width bounds this gives the stage, radial and reference conditions in the proof of Theorem 7.13. Start from the initial interval on which the increment is negative and take finite strict substeps as there. □

7.9 The compensated numerator and Mellin continuation

In this subsection we describe the Mellin argument that converts the bounds for the compensated numerator in Subsection 7.10 into a zero-free region.

The proof uses three estimates.

1. The regular positive product uses the larger widths of Theorem 7.8, with independent twists and all componentwise shortenings. On the rectangle used in [2], $0.37144 \le z \le 0.37880$ and $1.11939 \le n/z \le 1.14162$, the total ordinary width exceeds $7/9$. The induction at these lengths uses the ordered pair of ordinary lengths.

2. The two signed correlations and their fixed and regular sides must restore the original uncentered positive moment after complete-factor selection and Poisson summation. Theorem 4.1 estimates the centered fixed inducing sector with the actual omitted local factors. Subsection 7.2 gives the regular short reference. Proposition 7.3 bounds both correlations by the positive moments at the radii of the combined parameters.

3. The estimate for the compensated numerator retains both heights, the reciprocal and the contributions from the lowest bin, the principal terms and the truncated tails. With the actual family supremum β, the required estimates are

$$|J_\eta(Z) - f_\eta(Z)| \ll_\eta Z^{\beta - k_{\mathrm{phys}} - \sigma}, \qquad |J_\eta(Z)| \ll_{\eta,\varepsilon_1} Z^{\theta - k_{\mathrm{phys}} + \varepsilon_1}, \qquad \sigma > 0.$$

   The exponent identity in [2, (Q.4)], with $x = 1 + \ell - h$ and $y = h - 2\ell$, is

$$k_{\mathrm{phys}} + \frac{x}{2} + \frac{y - x}{12} = \frac{11}{12} - \frac{\ell}{4} = \theta.$$

The saving σ is common to the family. The strict numerical margins are fixed before the choice of character, while the seminorms and height orders may depend on the character. In particular an error of the form

$$Z^{\beta - k_{\mathrm{phys}} - 2\sigma}(1 + H)^{A_\eta} + Z^{D_\eta} H^{-N}$$

is at most $O_\eta(Z^{\beta - k_{\mathrm{phys}} - \sigma})$ upon taking $H = Z^{t_\eta}$ with $0 < t_\eta < \sigma/(2(1 + A_\eta))$ and then $N > \max\{0, (D_\eta - \beta + k_{\mathrm{phys}} + \sigma)/t_\eta\}$.

For the lowest bin use the paired-unitary case of [7, Proposition 8.22], with $a = 509/1000$ and the same family supremum $\beta = \beta_\omega$. Its finite-order case uses $\beta_*(F)$. The contour move is on the whole compensated product before splitting by prime response, and the lowest bin uses the trivial count.

The numerator reflection uses the conductor of the finite χ_u with all strict ramified positions included in its conductor reduction. The reciprocal is evaluated on the buffered line only on the retained height segment. The global line is chosen using β_ω.

If these bounds hold on the full family for a fixed $\theta < \beta$, choose $0 < \varepsilon_1 < \beta - \theta$ and then $0 < e < \min(\sigma, \beta - \theta - \varepsilon_1, \beta - 2/3)$. The two bounds yield $f_\eta(Z) \ll_\eta Z^{\beta - k_{\mathrm{phys}} - e}$. The Mellin argument of [6, 4] gives holomorphy of

$$\frac{e^{(s-5/6)^2} H_\eta(s)}{L_F^{(\mathcal{S})}(s, \eta)} \qquad (\mathrm{Re}\, s > \beta - e),$$

Here H_η is nonvanishing for $\mathrm{Re}\, s > 2/3$ by [4, Lemma 6.1]. This contradicts the existence of a zero with real part arbitrarily close to β. The application requires the analytic estimates at the actual β. The earlier upper bound for β is used in the comparison of exponents.

7.10 Insertion in the reflected numerator over number fields

In this subsection we insert the ordered moment from Subsection 7.5 into the reflected numerator.

We give a sufficient mixed-moment condition for decreasing the number-field boundary of [6]. The comparison uses the original contours of its reflected numerator estimate. Write $U = Z^D$ and use d, p for its two amplitude parameters δ, x. Let

$$A = p(1 + 2g_0), \qquad k = 1 - p(1 + 2g_1), \qquad v = 5/6 - d,$$
$$R = \frac{1}{6} + vT_0 = 1 - d(A + 2km_*), \qquad m_* = T_0/3.$$

Here $g(w) = g_0 - g_1 w$ is an admitted affine lower bound for the joint moment improvement, T_0 is the interior crossing from the affine detector, and R is the count exponent furnished by this branch. All comparisons below use $0 < d < 5/6$, $k > 0$, $1 < T_0 < 3/2$, and strict admissibility of the detectors and original prime subproducts on their stated ranges.

For an ordinary numerator block A_n of length U^n, put $s_n = (1 - 2n)/2$. Its initial marked fourth moment and Hölder's inequality give a first-moment exponent

$$X_4(R, n) = \frac{3R + 1 - 2dps_n}{4}. \tag{37}$$

This holds on the central block range $3/10 \le n \le 47/100$ of [6, Section 8]. At $l_y < D < 2l_y$, put $n_0 = 1 - l_y/D$. In the geometry of that paper let

$$E_4(D) = \frac{1}{2} + \frac{13h}{75} - \frac{l_y}{2} - \frac{\ell}{2} - \frac{9D}{100} + \frac{d}{2} + dp\left(\ell - \frac{2l_y - D}{4}\right) + \frac{3DR}{4} - \theta.$$

For a central block use the shift $\varphi = 0$ if $n \le n_0$ and $\varphi = d/2$ if $n \ge n_0$. Substitution of (37) in the block exponent of [6, Section 8] gives

$$E_4(D) - \frac{Dd}{2}\{p(n_0 - n)_+ + (1 - p)(n - n_0)_+\}. \tag{38}$$

It is enough to improve the first-moment exponent on the retained blocks. The same height-dependent test functions and the same contour motion then give the corresponding change in the exponent in the original scale.

Proposition 7.16 (Same-zero improvement in a numerator block)**.** *Suppose an admissible mixed ordinary source at lengths n, m has prime width*

$$s = \frac{1 - n - m}{2} + W(n, m), \qquad b(n, m) = W(n, m) - \frac{g(2m)}{2}.$$

The source is a bound $\ll U^{1+\varepsilon}$ for the marked second moment of $A_n S_m$, with the actual ray class of the detecting sum, independent tests and heights, the original marks and presentation zeros, and all componentwise shortenings. A fixed finite number of test seminorms and a polynomial in the heights are permitted. Assume s lies strictly below the total original mark width in conductor units. The average response defining p uses that total width, assigning zero response to the error and nonpositive factors.

At an offset $\xi \ge 0$ set $T = T_0 - \xi$. Consider a subclass on which the detecting sum at T has short inverse and ordinary lengths r, m with $r + m \ge T - o(1)$. If $T + \Delta$ is a preceding cutoff on the same parameter with a long inverse, use $\Delta \ge 0$ below. When there is no preceding cutoff take $\Delta = \xi$. Assume the short inverse and ordinary affine counts apply at the tested lengths, or that their admitted continuations give the same lower bounds on the deficits there. Then, up to arbitrarily small exponent losses, the improvement over (37) is at least $d\mathcal{W}_\Delta(\xi, n, m)$, where

$$\mathcal{W}_\Delta(\xi, n, m) = pb(n, m) + \frac{1}{2}\max\left\{-k\xi,\ k(m - m_*) + \max\left(\frac{v(\xi - \Delta)_+}{d}, 2k(m - m_*)_+\right)\right\}. \tag{39}$$

The improvement in base Z is $Dd\mathcal{W}_\Delta$, added to the saving in (38).

For a fixed ξ, n, suppose m ranges over an interval containing

$$m_\xi = m_* - \xi - \frac{v}{dk}(\xi - \Delta)_+$$

and $pb(n, m)$ is nonincreasing and Lipschitz with constant less than $k/2$ there. Then the minimum of (39) over that interval is exactly $pb(n, m_\xi) - k\xi/2$.

Proof. The preceding long inverse gives a count exponent at most $R - v(\xi - \Delta)$, using $1/6 + v(T + \Delta)$. The current short inverse and ordinary lengths give count exponents at most $1 - d(A + kr)$ and $1 - d(A + 2km)$. Thus the single subclass has exponent at most $R - G$, where

$$G = \max\{v(\xi - \Delta)_+,\ dk(m_* - m - \xi),\ 2dk(m - m_*),\ 0\}.$$

The ordinary sum has $|S_m|^2 \gg U^{dm - o(1)}$. Take a complete subproduct of width at most s in decreasing response order. If the positive factors fill s, their weighted average before dropping the last fractional factor is at least $2dp$ and omitting it changes the exponent by at most one mesh interval times the maximal slope. If the positive factors have total width below s, take all of them. Their total squared response exponent is then $2dp$ times the full width, at least $2dps$.

In both cases the squared amplitude is at least $U^{2dps - o(1)}$. Apply the mixed moment before division by these two amplitudes and apply Cauchy's inequality with the count $U^{R - G + o(1)}$. The result is

$$X_2(R - G, n, m) = \frac{R - G + 1 - dm - 2dps}{2}.$$

After substituting $R - 1 = -d(A + 2km_*)$ and $s = (1 - n - m)/2 + W$, one obtains

$$X_4(R, n) - X_2(R - G, n, m) = dpb(n, m) + \frac{dk}{2}(m - m_*) + \frac{G}{2}.$$

Taking the maximum in G gives (39). The integration in the numerator height and the passage to the individual heights of the detecting sums use the same compactly supported norm tests and unit-interval Sobolev measures as the reflected numerator. The norm derivative inserts the first normalized logarithmic test. Fix the finite cutoffs, the bins and the orders of the test functions before the live parameter heights. These losses are absorbed in the strict savings of the comparison. A conjugation of an ordinary factor conjugates its test and height with it, and retains its zeros.

In (39) the two outer faces meet at $m = m_\xi \le m_*$. To its left the face $-k\xi$ is constant. To its right the other face increases with slope at least k. Adding the prescribed monotone Lipschitz term gives the minimum at the intersection. □

The finite cutoff condition. If $E_4(D) \le Dq$ for $q > 0$, the original fourth moment bounds any subclass with $3G/4 > q$. Choose a set of cutoffs of cutoffs descending from T_0 to $T_0 - \gamma$ with $3v\gamma/4 > q$. If every inverse is long, the final count suffices for the fourth-moment estimate. Otherwise use the first short inverse and apply Proposition 7.16. A sufficient condition on a retained central block is

$$d\mathcal{W}_\Delta(\xi, n, m) + \frac{d}{2}\{p(n_0 - n)_+ + (1 - p)(n - n_0)_+\} > q. \tag{40}$$

Only offsets and lengths admitted by the analytic sources enter this condition. If the source interval contains every intersection m_ξ and the slope test of the proposition holds, the continuous zero-loss condition is

$$q < d \min_{0 \le \xi \le 4q/(3v)} \left\{ pb\left(n, m_* - \left(1 + \frac{v}{dk}\right)\xi\right) - \frac{k\xi}{2} \right\} + \frac{d}{2}\{p(n_0 - n)_+ + (1 - p)(n - n_0)_+\}.$$

With a strict margin on compact admitted ranges, continuity permits a finite set of cutoffs, intervals near the endpoints and arbitrarily small exponent losses in (40). At $n = n_0$, $p = 1/2$, this is the same cutoff comparison as (6).

Proposition 7.17 (Local sufficient condition for a strict improvement)**.** *Use the number-field inputs of the B_* theorem of [7] in its actual paired family. Suppose the mixed source of Proposition 7.16 is available in every working field and for the stated character families, on a fixed neighbourhood of the two critical ordinary lengths $(n_0(h_*), m_*)$, with $b(n, m) \geq b_0 > 0$ there. Suppose its widths lie within the original prime supply. Then the half-plane can be decreased by a positive amount, uniform in the number field and character if the source neighbourhood and b_0 are uniform.*

Proof. At $\theta = B_*$ the existing comparison has $E_4(h_*) = 0$ at its only contact $(d, p) = (\delta_*, 1/2)$, and its other bounds are strict away from the contact. Within the central block range, the penalty in (38) is strictly negative away from $n = n_0$. It suffices to improve a neighbourhood of the four coordinates $(h_*, \delta_*, 1/2, n_0)$.

Choose a single cutoff $T_0 - \xi$ with $\xi > 0$ sufficiently small. An inverse in the long range saves $3v\xi/4$ by the original fourth moment. If the inverse is short and m is outside a fixed small neighbourhood of m_*, either the current short inverse or the current ordinary detector saves a fixed positive amount in the count. This follows from $dk(m_* - m - \xi)$ and $2dk(m - m_*)$, using the strict continuing detector inequalities in [6, Section 7] beyond the affine domain. On the remaining subclass (39) is bounded below by $pb_0 - k\xi/2 > 0$. Thus all central classes have a strict margin at the contact. The supplies, contour conditions, and floor, principal and other block estimates of [6] have strict margins there. Outside the contact neighbourhood their compact comparison has a strict margin. One can therefore use a single finite approximation to the limiting joint moments and a fixed positive loss smaller than all these margins. Their strictness persists when θ decreases slightly and the comparison lengths vary as in [6]. The original Mellin continuation and descent give the conclusion. □

At $m = m_*$ the improvement of the mixed moment over the initial fourth is $dp\{W(n, m_*) - g(2m_*)/2\}$. At the contact $2n_0(h_*) > 7/9$. After division by their respective selected mark amplitudes, the available diagonal fourths give the Cauchy bound

$$\sum |A_n S_m|^2 \leq (\sum |A_n|^4)^{1/2} (\sum |S_m|^4)^{1/2}.$$

Its effective width improvement at the contact is $g(2m_*)/2$. A positive margin requires a stronger estimate on the actual ordinary pair.

7.11 A finite comparison using the mixed ordinary source

In this subsection we apply the first-moment inequality of Subsection 7.10 with explicit polynomial widths.

Over the working fields $F \supset \mu_{12}$, use Theorem 7.8 for the original paired ordinary families, with widths

$$.369 \leq z \leq .3888, \qquad z \leq N \leq 1.15z, \qquad S < \frac{1 - N - z}{2} + F(z), \tag{41}$$

where F is the function in Section 5. The upper indices N, z permit both actual factors to be shortened individually. The estimate includes the original prime subproducts, natural and redundant presentation zeros, inducing-character exclusions, fixed ray twists and the two independent normalized height tests, with the uniformity in the excluded ideals and Sobolev orders of [6]. In the mixed moment the two fixed rays are those of the compensated numerator and the selected ordinary

sum. Whole conjugation is permitted to put their moving characters in the common orientation. The estimate concerns the original undivided marked sum.

Theorem 7.18 (A finite curved-width comparison)**.** *For every number field K and every continuous unitary Hecke character ω,*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > 0.8733352).$$

The possible simple pole of a norm character is permitted. The bound also holds for Artin L-functions.

Proof. Put

$$\theta = .8733352, \quad h = .81436, \quad \ell = 11/3 - 4\theta, \quad l_x = 1 + \ell - h, \quad l_y = h - 2\ell.$$

Use $\kappa = 374887/500000$ and the three count branches in Section 7 of [6], with $B_C = 1, L_C = 2$ in their limiting scalar formulas. The original initial branch is admitted on $[15/64, 9/16] \times [9/32, 1/2]$ and $[15/64, 17/32] \times [1/4, 1/2]$. For the improved branch use

$$g_0 = .05805911927, qquad g_1 = .07599618491, qquad (d,p) \in [189/512, 15/32] \times [15/32, 1/2].$$

Isolating r_* in $(.752322053813, .752322053815)$ and substituting in the formulas of [6] shows that this line is strictly below its ideal tangent. Its detector domain is $[.7335, 7/9]$ in the limiting formulas. Above the upper endpoint the original continuations are used. All original conditions are strict on the closed admissibility rectangles. In particular $\ell/h - z(.7335) > .07727$, where $z(r) = (1-r)/2 + g(r)$.

The repaired finite paired construction of [7] supplies the line with fixed strict losses. The right endpoint may be taken strictly below $7/9$ and arbitrarily close to it. At the lower endpoint .7335 use the .734 width by length shortening.

We describe the mixed part of the rational comparison. Put $\epsilon = 10^{-11}$. On a closed amplitude box $[d_-, d_+] \times [p_-, p_+]$ let E_+ be an upper bound on $E_4(h)$ computed by clearing its positive denominator J. For a box not already estimated by the initial fourth, put

$$q = \frac{\max(0, E_+) + 2\epsilon}{h}, \qquad S_D = \frac{3}{4}(1 - d_+) - \frac{9}{100} + \frac{d_- p_-}{4}, \qquad D_- = h - \frac{E_+ + \epsilon}{S_D}.$$

Here S_D is a lower bound on $\partial_D E_4(D)$ since $R \geq 1 - d$. The checks give $S_D > q$ and $19/25 < D_- < h$. Consequently

$$E_4(D) \leq -\epsilon \quad (D \leq D_-), \qquad E_4(D) \leq Dq - 2\epsilon \quad (D_- \leq D \leq h).$$

The left statement uses the affine central exponent on $[19/25, D_-]$.

Consider the central block interval of (38). For $n \leq n_0(D)$, apply the mixed source with cap $N = n_0(D)$. Replacing n by this cap reduces the effective width by $(n_0(D) - n)/2$. Its loss in the first-moment exponent is $dp(n_0(D) - n)/2$, precisely the left contour saving. For $n \geq n_0(D)$ use $N = n$ until $n = n_0(D) + 2q/(d(1-p))$. Larger n in this interval are estimated by the right contour and the initial fourth. Thus the mixed source needs only caps in

$$1 - l_y/D_- \leq N \leq 1 - l_y/h + \frac{2q}{d_-(1 - p_+)} + \epsilon.$$

There is no restriction on the smaller actual length of the left block beyond the original support and componentwise shortening.

Write $m_* \in [m_-, m_+]$, $k \in [k_-, k_+]$, and $v \in [v_-, v_+]$ for the directed bounds on the same box. For example, if $U_0 = 1 - p(1 + 6g_0 - 4g_1)$ and J has bounds $0 < J_- \le J_+$, one may take

$$m_- = (1 + d_-(U_0)_-/J_+)/3, \quad m_+ = (1 + d_+(U_0)_+/J_-)/3.$$

Use final offset $\gamma = 4q/(3v_-) + \epsilon$. All-long parameters save $3v\gamma/4 > q$ in the initial fourth. At the first short inverse, its same-zero ordinary companion needs only

$$m_* - \gamma - \frac{4q}{3dk} - o(1) \le m \le m_* + \frac{2q}{3dk} + o(1).$$

The complement is estimated by the current inverse or ordinary count and the initial fourth. The lower inverse condition $2m_- - \gamma - 2q/(3d_-k_-) - \epsilon > .7335$ and the strengthened upper continuation conditions (I6)–(I9) hold on all mixed boxes.

The latter give count deficits greater than $4q/3 + \epsilon$. These checks apply before the source is used.

On the remaining mixed lengths the first face minimization in Proposition 7.16 applies with $b(m) = F(m) - g(2m)/2$. The continuous envelope also decreases in the offset on $[0, \gamma]$. To verify the latter one uses

$$0 < -b'(m), \qquad \frac{k_-}{2} + p_+\Big(1 + \frac{v_+}{d_-k_-}\Big) \min b'(m) > 0,$$

checking the two derivatives at the seam if the length enclosure meets it. The least mixed margin is therefore bounded below by

$$d_- \left[p_- b\left(m_+ - \gamma\Big(1 + \frac{v_-}{d_+k_+}\Big)\right) - \frac{k_+\gamma}{2}\right] - q. \tag{42}$$

A finite early cutoff mesh gives the same strict sign after arbitrarily small losses, by (40).

This supplies uniform first-moment estimates for both ordinary halves in the common norm tests of [6].

For completeness, the rational comparison uses closed boxes and tensor Bernstein bounds. For the margins clear the denominator of the admitted branch. On a box rescaled to $[0,1]^2$, the coefficient of Bernstein index (i,j) and bidegree (a,b) for $\sum C_{uv}X^uY^v$ is

$$\sum_{u \le i,\, v \le j} C_{uv} \frac{\binom{i}{u}\binom{j}{v}}{\binom{a}{u}\binom{b}{v}}.$$

First test an initial margin $-E_4(h) > \epsilon$ on any admitted branch. Then test (42) $> \epsilon$ and the source, contour and continuation inequalities on the improved branch. Otherwise bisect the longer normalized side.

The top cover has 43 rectangles treated by the initial estimate and 84 mixed rectangles, at depth at most 25. On the mixed rectangles the ranges and directed lower bounds are

quantity	bound or enclosure
D_-	$> .81422795$
m	$(.37258594, .38119992)$
N	$(.42558068, .42627047)$
N/m	$(1.11658, 1.14343)$
(42)	$> 1.70 \cdot 10^{-9}$
remaining Z exponent	$> 1.37 \cdot 10^{-9}$

The fourth-moment source bounds the other edge at $D = 19/25$ with margin greater than .00584 in the uncapped central exponent. The edges treated by the earlier estimate, at $D = 1/200, 19/25$ have margins greater than $.03389, .00238$. The left and right reference margins, the short-block margin and the prime-width margin on the union requiring the reflected estimate have directed lower bounds $.0001748, .0004788, .0004689, .0010347$, respectively.

The low-amplitude replacement $C_{\mathrm{aff}}(D)$ gives an additional margin greater than .0002669 beyond $1/200$. The floor at $a = 509/1000$ has margin greater than .001539. The other geometric, small-parameter and principal conditions are strict. In particular $\ell/h > .21$, $l_y/h \in (.53, .7)$, $2l_x - l_y - \ell > 0$, and $G_0 = 2(l_y - l_x)/3 - (l_x + l_y)/4 < -.13$.

For the initial coefficient-two fourth moment on $3/10 \le n \le 47/100$, take $S < s_n = (1-2n)/2 \le .2 < \ell/h$. Then $2n + 2S < 1$. The smaller cap $3D/40$ in the old scalar comparison is not used for this reflected block computation. On the lower conductor interval use the earlier estimate, and on the remaining block lengths use the checked original reference estimate with its original contours. All tests are strict, so the width decrements, the top conductor buffer, the finite height controls and the ordinary and arithmetic discretizations may be fitted inside their common margin. In particular the upper conductor range can be increased slightly beyond h.

The direct estimate still has exponent $C(\theta)$ because $l_x + l_y = 1 - \ell$ and $\theta = 11/12 - \ell/4$. Insert the mixed estimate at the first-moment step of the original reflected numerator, before the Mellin comparison. The same principal and floor calculations, whole reciprocal and shifted numerator, and the Mellin argument of [6] now apply with a strict saving. Only the independently available κ enters the older prime-width estimate. The contradiction uses the true β of the working family. Base change and the usual induction give the asserted conclusions. □

7.12 The infimum of the reflected comparison with initial moments

In this subsection we find the threshold of the comparison when only the initial detector moments are used with the ordered ordinary estimate of Section 6.

Use the initial inverse and ordinary detector moments and the limiting initial ordered ordinary improvement of Corollary 6.2. Thus $g_0 = g_1 = 0$ in the detector, and for a tested ordered pair (N, z) the mixed ordinary width is bounded by $(1 - N - z)/2 + F_{\mathrm{nat}}(z)$. We apply the original reflected contours of [6], with the optional refinements of the count arising from the same zero and the finite finite set of cutoffs of Proposition 7.16. In this subsection the comparison uses the unpadded limiting lengths. Every analytic application uses strict moment widths and finite cutoffs.

Theorem 7.19 (Exact infimum for the initial reflected comparison)**.** *For targets in a neighborhood of the contact, the infimum in this comparison is*

$$\theta_{\mathrm{nat}} = 0.87335440560131603646\ldots.$$

It is isolated by the contact equation below, with the tuned length

$$h_{\mathrm{nat}} = 0.81436993856967915613\ldots.$$

The strict half-plane $\operatorname{Re} s > \theta_{\mathrm{nat}}$ *is zero-free for unitary Hecke L-functions over all number fields, with the possible simple poles of the norm characters permitted.*

Proof. At $p = 1/2$ put

$$d_0(\theta) = \frac{\theta - 1/2 - \ell/2}{1/2 + \ell}, \qquad \ell = 11/3 - 4\theta.$$

On the top conductor edge with $n = n_0(h)$,

$$E_4(h) = \frac{1}{2} + \frac{\ell}{2} + d\left(\frac{1}{2} + \ell\right) - \theta + h\left(\frac{3}{4}R - \frac{5}{12} - \frac{d}{8}\right).$$

Consequently at $d = d_0(\theta)$ the required saving $q = E_4(h)/h$ is independent of h. Here the initial moment and its corresponding count branches agree. For the old branch, if $C_o = 1 - 1/(6\kappa)$ and $D_o = 5/2 - 1/(3\kappa)$, then $3C_o - D_o = 1/2 - 1/(6\kappa) > 0$ shows that its count is larger on this parameter.

At every point of the contact rectangle put

$$q = E_4(h)/h, \qquad \gamma = \frac{4q}{3v}, \qquad z_\gamma = m_* - \gamma\left(1 + \frac{v}{dk}\right),$$
$$\mathcal{M}(d, p; \theta, h) = d\big(pF_{\mathrm{nat}}(z_\gamma) - k\gamma/2\big) - q.$$

The same-zero comparison follows from (39). At the critical point $p = 1/2$, $k = 1/2$, $J = 5/2 - 2d$ and

$$Q = \frac{20 - 61d + 33d^2}{3} = 8Jq, \qquad z_\gamma = \frac{(7d - 8/3)(5/6 - d/2)}{3dv}.$$

The contact equation is

$$F_{\mathrm{nat}}(z_\gamma) = \frac{Q}{12dv} \qquad (d = d_0(\theta)). \tag{43}$$

Equivalently, substitute $b = 3 - 1/z_\gamma + Q/(3dvz_\gamma)$ and $y = z_\gamma$ in the logarithmic equation of Corollary 6.2.

The tuned conductor cutoff is determined by the additional condition $\mathcal{M}_d(d_0(\theta), 1/2; \theta, h) = 0$. Writing $q = s_0(d) + [\frac{1}{2} + \frac{\ell}{2} + d(\frac{1}{2} + \ell) - \theta]/h$ at $p = 1/2$, this condition is linear in $1/h$ with nonzero coefficient.

Directed evaluation of the logarithmic equation isolates the contact at

$$.8733544056013160364 < \theta_{\mathrm{nat}} < .8733544056013160366,$$
$$.81436993856967914034 < h_{\mathrm{nat}} < .81436993856967916852.$$

The margins at the two displayed endpoints lie respectively in

$$(-3.53612, -3.53609)10^{-19} \qquad \text{and} \qquad (8.47485, 8.47488)10^{-19}.$$

The derivative of $\mathcal{M}(d_0(\theta), 1/2; \theta, h)$ along the critical point lies in $(6.0053, 6.0058)$. Thus the equation has a unique root there.

On $[.424, .428] \times [.4998, 1/2]$ the directed bounds give

$$.41085 < \mathcal{M}_{dd} < .47564, \qquad -.01559 < \mathcal{M}_p < -.01288.$$

These follow by implicit differentiation of the logarithmic equation and evaluation on 100×20 closed boxes. The optimized mixed margin as a function of the offset is strictly decreasing there, with derivative lying between $-.03072$ and $-.00222$. The required saving and length at the intersection are in $(.00010482, .00020963)$ and $(.37410, .37674)$, respectively.

The inequalities for the ordinary and inverse lengths, the total prime width and the ratio of the ordinary lengths are strict. Therefore $\mathcal{M} \geq 0$ on the contact rectangle at $\theta_{\mathrm{nat}}, h_{\mathrm{nat}}$, with equality exactly at $(d, p) = (d_0(\theta_{\mathrm{nat}}), 1/2)$.

On the complement take the rational centers $\theta_Q = .87335440560131603646$ and $h_Q = .81436993856967915613$. The same directed comparison as in Subsection 7.11 uses a convex

piecewise affine lower bound for F_{nat}, with lower tangent intervals on a grid of spacing .00004 and a strict loss $2 \cdot 10^{-11}$. If the source cap must be raised from m to z, use the exact effective improvement $F_{\mathrm{nat}}(z) - (z-m)/2$. The source caps are chosen before the live heights. On the unclamped branch the minimum of the directed offset envelope is taken at its endpoints or an affine junction. On the clamped branch the width after this charge increases with m, so the lower length endpoint suffices together with the worst linear offset required saving.

The exterior top cover has 48 rectangles treated directly and 126 mixed rectangles at depth at most 25. The original-scale margin is greater than $1.4240 \cdot 10^{-8}$ after the directed allowances, with shorter source caps in $(.36849101, .38120822)$ and ratio less than 1.159010. The three lower conductor edges and the inequalities for the left and right contours, prime width, short lengths, $d \le 1/8$ and the lowest bin are strict. The parameter enclosures above differ from the rational centers by less than $4 \cdot 10^{-17}$, within the separate 10^{-11} allowances. The verification `certify_all_field_reflected.py` checks this exterior.

A slight increase of θ with $h = h_{\mathrm{nat}}$ decreases the required saving at each fixed (d, p) and makes the mixed margin strict throughout the contact rectangle. This uses $\partial_\theta E_4(h) = -3 - 8dp$ and the bound $-1.102 < \partial_q \mathcal{M} < -1.007$. It leaves the exterior certificate strict. Corollary 7.9 and the initial inverse inputs of [3, Section 3] supply the strict moment widths of this comparison. The required finite width and height choices then follow from the reflected numerator argument.

The Mellin contradiction proves the strict half-plane.

Conversely, at the parameter $d = d_0(\theta), p = 1/2$ the last long inverse requires set of cutoffs length at least $\gamma = 4q/(3v)$. The first short inverse has its least possible directed margin at the length at the intersections of Proposition 7.16. At these lengths, the last required offset, or offsets approaching it, give precisely $\mathcal{M}(d_0(\theta), 1/2; \theta, h)$. The limiting ordered moment gives at most F_{nat} there. For $\theta < \theta_{\mathrm{nat}}$ sufficiently close to it the margin is negative. As the critical required saving agrees for all admissible h in the reflected contour corridor, altering h does not remove this comparison obstruction. This proves the claimed infimum for the specified initial reflected comparison. □

7.13 The first marked stage in Hu's paired family

In this subsection we prove the one-stage estimate used for the prior zero bound in Subsection 7.18.

We use the stage with zero increment at the preceding step and zero supporting line in [7, Theorem 8.21]. For its equal-anchor ordinary moment and its inverse moment, retain the paired characters

$$\mathcal{X}_y^{\mathrm{or}} = \{\theta\overline{\chi_\bullet(y)},\ \psi\theta\overline{\chi_\bullet(y)} : \theta \in \Theta\} \ \cup \ \text{their whole conjugates.}$$

Here Θ contains the fixed finite coefficient, reciprocity and unit characters. Choose ψ in Hu's orientation. In a block with fixed unitary anchor $\omega_0 \in \{1, \psi\}$ the live prime coefficients are ω_0^{-1} times their fixed finite coefficients, so the actual prime characters are finite twists of the common moving character. The choice of anchor and prime coefficients is synchronized in each block. The mixed ordinary factors with different anchors enter later through Theorem 7.8.

Proposition 7.20 (One stage with zero initial increment)**.** *Put*

$$R_0 = 191/250, \quad R_1 = 367/500, \quad k = 460859/10^6, \qquad g_1(r) = \frac{13047}{460859}\left(1 - \frac{r}{R_0}\right).$$

For each fixed $R_1 \le r < R_0$ and strictly smaller rectangular lengths than r and $(1-r)/2 + g_1(r)$, the joint inverse and equal-anchor ordinary moments hold in the paired family above in the regime $\beta_\omega \ge 51/100$. Their constants have the original fixed-type and polynomial-height uniformity. This stage uses neither (LH) nor (UO).

Proof. For the initial rectangles and the detector cutoffs $w \geq R_0$, use Hu's initial marked inverse estimate and Gao's coefficient-two ordinary estimate. At stage zero the lower width at cutoff w is $(1-w)/2$ with a fixed positive decrement. The initial inverse condition is strict because $2w + 8(1-w)/2 = 4 - 2w < 3$. At the unmarked end use the stated unmarked estimates. Thus the proof of the cutoff-range lemma of [7, Section 8] at stage zero gives the actual earlier moment bounds, without an incoming supporting-line margin.

The variable-length detector in that section uses the larger auxiliary character set only to contain the actual block and the actual twisted prime components when estimating their logarithms or logarithmic derivatives. The restricted union $\mathcal{X}_y^{\mathrm{or}}$ contains all of them. The fixed unit characters multiply these by members of Θ and does not change the unitary anchor. For each parameter we form the buffered zero maximum over this union and the actual height intervals. Both detecting sums are then chosen at the same attaining zero. When the selected detecting sum has anchor ω in the canonical orientation, express the same actual prime character with coefficient ω^{-1} times its original finite coefficient. Thus every short count uses an initial paired moment on its actual detecting sum. Conjugating the entire block and its test handles the other sign. For the amplified long count the unmarked inverse input is Hu's oriented one on the same presentation, including the norm modes. This replaces the only use of (UO) in the detector and counts.

The transformations preserve the restriction on live coefficients. In the inverse part, the identity $C\xi \leftrightarrow \mu_F \xi G_0$ of the proof of the paired joint moment theorem in [7] keeps the fixed factor ξ and only changes finite coefficient characters. The selected prime extraction and all coprimality expansions preserve the original live positions. A forced or colliding position contributes a scalar, so its power of the anchor is not part of a new live detector. In the ordinary part the second CRT kernel carries $\chi_I(j)$ and $\overline{\chi_J(-j)}$ for an incoming χ orientation, and their conjugates for an incoming $\overline{\chi}$ orientation. This follows from [9, Lemmas 12.9–12.10]. Taking the entire conjugated second factor inside its absolute value gives the common original orientation since $\chi_J(-1) = 1$. Finite unit and reciprocity factors again act through Θ. Reflection of an ordinary factor restores its orientation by conjugating the whole reflected factor. Its reference bound is Gao's ordinary input, with the actual test functions and norm phases. The equal-anchor regular comparisons therefore use the restricted detector just described. The phase choices do not certify a detector for two mismatched live unitary anchors.

For the exact stage checks, let $H = r/R_0$, $u = (1-H)/6$ and $Y = (1/k-2)u = g_1(r)$. Put $S = (1-r)/2 + Y$, $s = S - u$, and $w_+ = R_0 + 2u(1/k - 4/3)/H$ before cutoff padding. At $r = R_1$,

$$H = 367/382, \qquad u = 5/764, \qquad Y = 195705/176048138.$$

The first large-value margin before its positive required saving is

$$HF_{\mathrm{high}} = H\{5/6 - 5R_0/4 + k(R_0 - 1/2)\} \geq \frac{15047}{143250000000} > 1.05 \cdot 10^{-7}.$$

The following strict endpoint bounds hold throughout the interval by affineness after positive denominators are cleared:

$$\frac{7}{50} - S/H > 0.0004069, \qquad \frac{9}{10} - r - S - 4u > 0.00571, \qquad \frac{4}{15} - 3u - r/3 - Y > 0.0012548.$$

Conditions 2–14 of the stage also retain positive endpoint margins.

The full rational endpoint calculation is in `check_first_stage.py`. The last inequality is the required estimate for small sixth-power-free parts. Here $u \leq 5/764 < 1/100$. Apply the joint induction and the packed ordinary stage of [7, Theorem 8.21 and Proposition A.4] with the initial rectangle, zero increment at the preceding step and zero supporting line. The only earlier moment

calls are initial. Choose the positive stage margin first, then the cutoff padding, strict width decrements, prime mesh, and finite test and height orders. The same estimate therefore holds down to R_1 at strict widths. □

Theorem 7.21 (Unconditional unitary half-plane)**.** *For every number field K and every continuous unitary Hecke character ω of K,*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > 0.87334470),$$

with the possible simple pole of a norm character permitted. For a finite-dimensional continuous unitary Weil representation the same bound holds, with the norm-character pole multiplicities given by Frobenius reciprocity.

Proof. Use the initial logarithmic ordinary improvement of Corollary 7.9, and the extended first stage in the paired family from Proposition 7.20. The affine count uses

$$g(r) = \frac{13047}{460859}\left(1 - \frac{r}{191/250}\right) - 10^{-9}, \qquad .734 \le r \le 7/9,$$

with the initial continuation above $R_0 = 191/250$. The spectral estimates use strict widths and their stated finite-order and unitary norm conventions.

For $\theta = .87334470$, $h = .81575722$, use the reflected contours and prime width estimates of Subsection 7.11, with the initial logarithmic improvement and the one-stage count above. Use $\kappa = 374887/500000$ directly from Gao's prior bound $B_2 < (1+\kappa)/2 = .874887$. The one-stage count is admitted on $[.362, 15/32] \times [15/32, 1/2]$. The cover has 35 directly estimated and 214 mixed rectangles at the top, at depth at most 28. The actual source caps satisfy

$$.36738136 < z < .38027867, \qquad .42465930 < N < .42614909, \qquad 1.11705 < N/z < 1.158893.$$

The mixed margins are greater than $7.88 \cdot 10^{-10}$. Use the same directed logarithmic tangent minorant as in Subsection 7.12 and minimize the piecewise affine offset envelope at its junctions and endpoints.

The smaller .20 numerical density cutoff for the old reference is valid because the reflected contour range starts at $1/50$, and all four reference inequalities are checked on their enlarged domain. Their respective left, right, short-block and prime width margins exceed $.0000206, .0004145, .0002227, .0008566$. The low-amplitude replacement has margin greater than .0000294 beyond $1/200$, and the floor at $a = 509/1000$ has margin greater than .0011029.

The lower initial and both old conductor edges, and the remaining geometric and principal tests, have strict margins. The rest of the proof is the original finite-height argument for the reflected numerator and reciprocal Mellin continuation. Base change and unitary induction give the stated conclusions. □

For a simpler finite certificate one may use instead $4.55(.3888 - z)^2 + 39(.3888 - z)^3 + .00031(.3888 - z) - 10^{-9}$, the shorter first stage down to $37/50$, and $h = .8116$. The same comparison gives 0.873350; its directed verification has 35 directly estimated and 24 mixed terminal rectangles.

7.14 The reflected comparison from a common width function

In this subsection we use the coupled moments to strengthen the conductor comparison of Subsection 7.10.

Use f, ϕ from (30) and the affine count

$$g(r) = g_0 - g_1 r, \qquad (g_0, g_1) = (.0538833086, .0705007472).$$

On $[.734, .7776]$ this line is strictly below $f(r/2) - 10^{-10}$. Indeed the supporting line at $r = .7523$ has intercept greater than g_0 and negative slope greater than $-g_1$. The initial continuation supplies the comparison at larger lengths. The detector rectangle is $[.365, 15/32] \times [.46, 1/2]$. Use the earlier $\kappa = .7466895$, as permitted by Theorem 7.21.

A fixed short width. Let $a = .734$ and use one earlier inverse width and the corresponding equal ordinary width at a and $a/2$. Denote their common strict width by σ_a, and let $c \in [.725, a]$. At cutoff $T = 3c/2$ a long inverse gives count exponent at most $1/6 + vT$. Otherwise the current short inverse or ordinary factor has length $r' \geq c$ or $2m' \geq c$, respectively. For lengths up to a use the width σ_a after shortening. For larger lengths the available short bracket is at least $a + 2p\sigma_a$. This follows from

$$\frac{d}{dr}\left(r + 2p\left(\frac{1-r}{2} + \phi(r)\right)\right) > 0$$

on the paired range and its initial continuation. The corresponding equal ordinary bracket is the same with $r = 2m'$. Beyond its last marked range the original unmarked continuation gives the endpoint bracket. Thus the additional count bound is

$$R_c = \max\{1/6 + 3vc/2,\ 1 - d(c + 2p\sigma_a)\}. \tag{44}$$

Use either $c = a$, or the crossing

$$c = \frac{2}{3}\,\frac{5/6 - 2dp\sigma_a}{v + 2d/3}$$

when it belongs to the displayed interval. This count uses the fixed earlier width before any present numerator estimate.

In the common-function comparison take $\sigma_a = (1-a)/2 + f(a/2) - 2 \cdot 10^{-10}$.

Theorem 7.22 (Common-function number-field bound)**.** *The infimum for nearby targets in the reflected comparison with the fixed common width function* (30) *and the displayed fixed affine detector is*

$$\theta_{\mathrm{pair}} = .87333382224995336020\ldots.$$

It is the contact (45)*, with tuned cutoff* $h_{\mathrm{pair}} = .817329273019808936\ldots$*. For every number field* K *and every continuous unitary Hecke character* ω*,*

$$L_K(s, \omega) \neq 0 \qquad (\operatorname{Re} s > \theta_{\mathrm{pair}}).$$

The possible simple pole of a norm character is permitted.

The same bound holds for finite-dimensional continuous unitary Weil representations, with the norm-character pole multiplicities.

Proof. Write $b_c(z) = f(z) - g(2z)/2$. At the top edge put

$$q = E_4(h)/h, \qquad \gamma = 4q/(3v), \qquad z_\gamma = m_* - \gamma(1 + v/(dk)), \qquad \mathcal{M} = d\{pb_c(z_\gamma) - k\gamma/2\} - q.$$

The affine count and the same-zero mixed comparison (39) give this margin at the last required offset. On the critical point $p = 1/2$, set

$$d = d_0(\theta) = \frac{\theta - 1/2 - \ell/2}{1/2 + \ell}, \qquad J = 3v + 2dk, \quad U_0 = 1 - p(1 + 6g_0 - 4g_1), \quad R = 1/6 + v(1 + dU_0/J).$$

The h^{-1} term in $q = 3R/4 - 5/12 - dp/4 + [1/2 + \ell/2 + d/2 + 2dp\ell - \theta]/h$ vanishes on this parameter. The contact equation and tuning condition are

$$\mathcal{M}(d_0(\theta), 1/2; \theta, h) = 0, \qquad \mathcal{M}_d(d_0(\theta), 1/2; \theta, h) = 0. \tag{45}$$

First solve the h-independent equation for θ, then use the second, linear in $1/h$, to determine h.

Directed rational evaluation isolates the contact by

$$.87333382224995336020 < \theta_{\mathrm{pair}} < .87333382224995336021,$$
$$.81732927301980893552 < h_{\mathrm{pair}} < .81732927301980893635.$$

The two endpoint margins lie in $(-1.30570, -1.30569)10^{-20}$ and $(5.11545, 5.11546)10^{-20}$, and the critical point derivative lies in $(6.42110, 6.42126)$. On the contact rectangle $[.424, .428] \times [.4998, 1/2]$, differentiating the rational formulas on 100×20 closed boxes gives

$$.41299 < \mathcal{M}_{dd} < .47107, \qquad -.01408 < \mathcal{M}_p < -.01192,$$
$$-.04704 < \partial_\xi\{d[pb_c(m_\xi) - k\xi/2]\} < -.02798, \qquad -1.155 < \partial_q \mathcal{M} < -1.091.$$

Here $m_\xi = m_* - \xi(1 + v/(dk))$ and $0 \le \xi \le \gamma$.

The source and count conditions are strict on that rectangle. It follows that the minimum at the contact parameters is the single point $(d_0(\theta_{\mathrm{pair}}), 1/2)$. At fixed h, $\partial_\theta q = -(3 + 8dp)/h$. Thus increasing θ makes the margin positive on the entire rectangle. The directed local calculation is in `check_coupled_contact.py`. The support and count conditions are in `check_coupled_return_guards.py`.

On the complement use the comparison of Subsection 7.11, with $\theta_Q = .87333382224995336020$ and $h_Q = .817329273019808936$. If the ordered source cap z is raised from the actual length m, use $f(z) - (z - m)/2$ for the added width. The lower length endpoint suffices in the clamped sector since this width after subtracting $g(2m)/2$ is increasing in m. The remaining offset envelopes use their endpoints, seams and stationary points. Apply (44) along with the initial and affine counts as before. There are 88 directly estimated and 67 mixed terminal rectangles, of depth at most 24.

The mixed margin exceeds $3.8994 \cdot 10^{-8}$ and the used caps satisfy

$$.36728432 < z < .38069359, \qquad .42401242 < N < .42500020, \qquad N/z < 1.155473.$$

The original reference, geometric, floor and lower edge tests are strict. The cutoffs θ_Q, h_Q are within the reserved allowances of the tuned values, and a slight increase of θ preserves the exterior cover. A separate finite cover at $\theta = .87333385$, $h = .817329273$ uses 65 directly estimated and 214 mixed terminal rectangles, with mixed margin above $6.5045 \cdot 10^{-10}$. Both computations use `certify_all_field_reflected.py`.

Theorem 7.10 supplies the strict widths in this comparison unconditionally in the paired families. With all scalar margins positive, choose strict moment widths, cutoff and prime meshes, then the finite orders of the test functions before the live heights. The reflected numerator and the reciprocal Mellin continuation give the character assertion. The same base change and induction as in Theorem 7.21 give the remaining assertions.

Conversely, on the critical point near the contact the affine count is smaller than the initial and old counts, and than (44). For the initial count this follows from $g(2m_*) > 0$ at its crossing, and the original old count is larger than the initial one by $3C_o - D_o > 0$ as above. For the fixed-width count the short term alone exceeds the affine count by more than .003159 at $c = a$, and increases as c decreases. The last long inverse requires offset at least γ.

At $n = n_0(h)$ the two short faces meet at m_γ; the fixed source then gives exactly the margin $\mathcal{M}$ there in the limiting comparison. Since this critical point required saving is independent of h, a negative critical margin persists for every admissible cutoff when θ is slightly below the contact. This proves the asserted infimum for the fixed comparison. □

Corollary 7.23 (Refined unconditional number-field bound)**.** *For every number field K and every continuous unitary Hecke character ω,*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > .87333365265).$$

The norm-character poles and the unitary Weil conclusion are as in Theorem 7.22.

Proof. Use the width function f_{r} of Corollary 7.11. At $r_0 = .7529$ take

$$g_1 = -\tfrac{1}{2} f_{\mathrm{r}}'(r_0/2), \qquad g_0 = g_1 r_0 + f_{\mathrm{r}}(r_0/2) - 2 \cdot 10^{-10}.$$

Convexity puts this line strictly below the earlier curved width. Use the fixed prime width (44) at $\sigma_a = (1-a)/2 + f_{\mathrm{r}}(a/2) - 2 \cdot 10^{-10}$, $a = .734$, and retain the detector rectangle of Theorem 7.22.

At $\theta = .87333365265$, $h = .81725263133$, the directed cover has 65 directly estimated and 2890 mixed terminal rectangles, of depth at most 43. Its mixed margin exceeds $1.96 \cdot 10^{-14}$ after the separate 10^{-11} allowances. The source caps obey

$$.36774063 < z < .38017290, \qquad .42405223 < N < .42504703, \qquad N/z < 1.154369.$$

The floor margin exceeds .0005701 and the old, initial, reference, contour and principal tests are strict. The same finite cutoff and height ordering gives the claimed half-plane.

The verification uses `certify_all_field_reflected.py` with the refined common-function option. □

7.15 Distinct inverse and ordinary detectors

In this subsection we allow different affine bounds in the inverse and ordinary counts of Subsection 7.14.

Let $g_I(r) = g_{I,0} - g_{I,1} r$ and $g_O(r) = g_{O,0} - g_{O,1} r$ be admissible short inverse and equal ordinary lines. Put $A_j = p(1 + 2g_{j,0})$ and $k_j = 1 - p(1 + 2g_{j,1})$ for $j = I, O$. The crossing is

$$\begin{aligned} v &= 5/6 - d, \qquad J = (k_I + 2k_O)v + 2dk_I k_O, \\ T_0 &= \frac{(5/6)(k_I + 2k_O) - d(2k_O A_I + k_I A_O)}{J}, \\ m_* &= \frac{k_I T_0 + A_I - A_O}{k_I + 2k_O}, \qquad r_* = T_0 - m_*, \qquad R = 1/6 + vT_0. \end{aligned} \tag{46}$$

Thus $R = 1 - d(A_I + k_I r_*) = 1 - d(A_O + 2k_O m_*)$.

Proposition 7.24 (The reflected exponent with two detectors)**.** *At offset ξ with a preceding long inverse at offset $\xi - \Delta$, let r, m be current short lengths with $r + m \geq T_0 - \xi$. Use the same mixed source as in Proposition 7.16, now with $b(n,m) = W(n,m) - g_O(2m)/2$. The improvement over the initial fourth is at least $d\mathcal{W}_\Delta^\times$, where*

$$\mathcal{W}_\Delta^\times = pb(n,m) + \frac{k_O(m - m_*)}{2} + \frac{1}{2} \max\left\{ \frac{v(\xi - \Delta)_+}{d}, k_I(m_* - m - \xi), 2k_O(m - m_*), 0 \right\}. \tag{47}$$

If $p\,\partial_m b + (k_O - k_I)/2 < 0 < p\,\partial_m b + k_O/2$ *on the interval containing the crossing, the least value is attained at* $m_\xi = m_* - \xi - v(\xi - \Delta)_+/(dk_I)$.

For $\Delta = 0$ *this value is*

$$pb(n, m_\xi) - \frac{\xi}{2}\left[k_O + \frac{v}{d}\left(\frac{k_O}{k_I} - 1\right)\right]. \tag{48}$$

Proof. The three detecting sum counts give the deficits in (47) after division by d. The Cauchy calculation of Proposition 7.16, now with $R - 1 = -d(A_O + 2k_O m_*)$, gives the displayed exponent. To the left of m_ξ the short inverse face has derivative $(k_O - k_I)/2$ before adding pb. To the right the active derivative is at least $k_O/2$. The two signs give the minimum. Substitution at the crossing proves (48). □

Theorem 7.25 (Exact reflected contact with distinct detectors)**.** *Use* Ψ *of Proposition 7.14 for the inverse line and* f_{r} *of Corollary 7.11 for the equal ordinary line and the mixed source. Take the supporting lines at* .7528574 *and* .7529 *respectively, each with a downward decrement* $2 \cdot 10^{-10}$. *In this fixed reflected comparison the nearby-target infimum is*

$$\theta_{\mathrm{s}} = .8733327831630214900\ldots.$$

Every number field K *and continuous unitary Hecke character* ω *satisfies*

$$L_K(s, \omega) \neq 0 \qquad (\operatorname{Re} s > \theta_{\mathrm{s}}),$$

with the possible simple pole of a norm character permitted.

The unitary Weil assertion holds with the pole multiplicities of Theorem 7.22.

Proof. Use the detector rectangle, earlier exponent and fixed prime width of Corollary 7.23. The source curves are convex and the respective lines are negative before their upper endpoints. Use the initial continuation beyond each endpoint. At lengths down to .734 the earlier widths are available from the paired constructions.

Put $b(z) = f_{\mathrm{r}}(z) - g_O(2z)/2$. For $\gamma = 4q/(3v)$, $q = E_4(h)/h$, define

$$z_\gamma = m_* - \gamma(1 + v/(dk_I)), \qquad \mathcal{M}_{\mathrm{s}} = d\left[pb(z_\gamma) - \frac{\gamma}{2}\left(k_O + \frac{v}{d}\left(\frac{k_O}{k_I} - 1\right)\right)\right] - q. \tag{49}$$

As in the common-function comparison, the critical point has $p = 1/2$ and $d = d_0(\theta)$. On this parameter q is independent of h. Determine the contact by $\mathcal{M}_{\mathrm{s}} = 0$ and tune h by $\partial_d \mathcal{M}_{\mathrm{s}} = 0$. Directed evaluation gives

$$.8733327831630214900610 < \theta_{\mathrm{s}} < .8733327831630214900 1221,$$
$$.81745018400273704621 < h_{\mathrm{s}} < .81745018400273704839.$$

The endpoint margins have opposite signs. The critical-parameter derivative lies in $(6.4403, 6.4411)$.

On $[.424, .428] \times [.4998, 1/2]$ a subdivision into 100×20 boxes gives

$$.38203 < \partial_d^2 \mathcal{M}_{\mathrm{s}} < .50113, \qquad -.01579 < \partial_p \mathcal{M}_{\mathrm{s}} < -.01005,$$
$$-.04866 < \partial_\xi\{d\mathcal{W}_0^{\times}(\xi, n, m_\xi)\} < -.02531, \qquad 0 \le \xi \le \gamma.$$

The left length derivative is below $-.03105$ and the right one is above .16630. The required saving is in $(.00007055, .00008309)$. In particular the mixed exponent decreases in ξ and $\partial_q \mathcal{M}_{\mathrm{s}} = 4\partial_\xi(d\mathcal{W}_0^{\times})/(3v) - 1 < 0$. The source and detector conditions are strict. The directed local checks are in `check_split_contact.py` and `check_split_return_guards.py`.

Hence increasing θ makes the margin positive on the rectangle, since $q_\theta = -(3+8dp)/h$ at fixed d, p, h.

On its complement the rational centers may be taken to be .87333278316302149000610 and .81745018400273704730. The exterior cover has 123 directly estimated and 1516 mixed terminal rectangles at depth at most 31. The least margin in the original scale exceeds $3.1625 \cdot 10^{-10}$, and the source caps satisfy

$$.367 < z < .38152087, \qquad .42396046 < N < .42493520, \qquad N/z < 1.156795.$$

In the clamped sector the available improvement includes the loss $(z-m)/2$. Upper bound the additional left-face required saving in (47) by $(k_O - k_I)_+(m_* - m_{\min})/2$ and use the lower length endpoint for the width. The unclamped sectors use their endpoints, seams and stationary points. The earlier exterior, conductor, reference, floor and principal margins are strict. The radii of the root and cutoff enclosures are below the reserved rational allowances. The cover uses the split inverse option of `certify_all_field_reflected.py`.

On the critical point the two-detector count is strictly smaller than the initial counts, the earlier counts, the counts with a common width function, and the fixed-width counts. The limiting first short transition at the last required offset realizes (49). Its required saving remains independent of h. A slightly smaller θ therefore leaves a negative margin in the fixed comparison. The strictly positive comparisons approach the infimum from above and, by the same finite ordering in the reflected numerator proof, give the stated zero-free half-plane. □

Corollary 7.26 (Independent polynomial comparison)**.** *For every number field K and continuous unitary Hecke character ω,*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > .873332670).$$

The norm-character poles and the unitary Weil conclusion are as in Theorem 7.25.

Proof. Use Corollary 7.15 on $[.369, .38879]$. Form the inverse and equal ordinary lines at $r_0 = .7529$, each with added-width decrement $2 \cdot 10^{-10}$. Each line lies strictly below its earlier width on $[.738, 2(.38879)]$. At $r = .734$ both affine widths are less than the fixed earlier strict width $\sigma_a = (1-.734)/2 + f_{\mathrm{r}}(.367) - 2 \cdot 10^{-10}$.

Thus this fixed earlier width supplies the lines up to .738 by shortening. Keep the same fixed-width count. The lines are negative beyond their upper endpoint and the initial continuation applies there.

At $\theta = .873332670$, $h = .81748531$ use the split exponent (47), lowering the effective improvement by $(z-m)/2$ when choosing the higher source cap $z \geq \max(m, .369, N/1.20)$. The directed cover has 96 directly estimated and 9990 mixed terminal rectangles at depth at most 38. The margin in the original scale exceeds $1.9925 \cdot 10^{-11}$ with the separate 10^{-11} allowances. The largest source caps are $z < .38148088$, $N < .42491860$ and $N/z < 1.150496$.

The short reference margin exceeds $6.2168 \cdot 10^{-7}$ and all remaining reference, contour, floor and principal tests are strict. The verification uses the independent polynomial option of `certify_all_field_reflected.py`. The strict widths and the finite reflected argument give the asserted half-plane. □

Corollary 7.27 (Finite comparison near the limiting width functions)**.** *For every number field K and every continuous unitary Hecke character ω,*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > .87333264790).$$

The norm-character poles and the unitary Weil conclusion are as in Theorem 7.25.

Proof. Use the degree-14 lower polynomials for the two width functions of Theorem 7.13, with a further 10^{-10} loss in the mixed width. Both detector lines support the lower polynomials at .7528548, with total line decrement $2 \cdot 10^{-10}$ in each case. The rational polynomial enclosures are computed by `check_independent_limit.py`. The fixed prime width and the continuation below .738 are as in Corollary 7.26.

Fix $\theta = .87333264790$, $h = .81749677665$. On the contact rectangle $[.424, .428] \times [.4998, 1/2]$, the 40×10 directed subboxes give

$$\mathcal{M}_{dd} > .32448, \qquad \mathcal{M}_p < -.0068913, \qquad \partial_\xi(d\mathcal{W}_0^{\times}) < -.031837.$$

Both length-derivative signs and all source and detector conditions are strict. At $d_c = d_0(\theta)$, $p = 1/2$ the margin is greater than $3.07425285 \cdot 10^{-10}$ and $|\mathcal{M}_d| < 2.24955 \cdot 10^{-9}$. Taylor's inequality gives throughout the rectangle

$$\mathcal{M}(d,p) \geq \mathcal{M}(d_c, 1/2) - \frac{|\mathcal{M}_d(d_c, 1/2)|^2}{2 \inf \mathcal{M}_{dd}} > 3.0742527 \cdot 10^{-10}.$$

The local check is in `check_series_return_local.py`.

The complement has 123 directly estimated and 1535 mixed top leaves, of depth at most 31. The mixed margin is greater than $1.6551 \cdot 10^{-9}$ and the source caps satisfy $.369 < z < .38152587$, $N < .42491344$, and $N/z < 1.150484$. The short reference margin exceeds $2.3728 \cdot 10^{-5}$. The proof uses the same finite ordering of strict inequalities as Theorem 7.25. □

7.16 The contact for the optimal independent width functions

In this subsection we apply the independent limiting widths of Subsection 7.8 and determine the threshold of the resulting comparison.

Let Z, F be the limiting width functions of Theorem 7.13. The comparison at the first short cutoff uses the inverse and ordinary widths $(1-r)/2 + Z(r/2)$ and $(1-n-m)/2 + F(\min(n,m))$, respectively, with any fixed positive width decrements. Define

$$\begin{gathered} \ell = 11/3 - 4\theta, \qquad d_0 = \frac{\theta - 1/2 - \ell/2}{1/2 + \ell}, \qquad v_0 = 5/6 - d_0, \\ \frac{1+\rho}{2} + Z(\rho/2) = \frac{4}{9d_0} - \frac{1}{6}, \qquad z = \frac{7/18 + d_0/6}{v_0} - \rho, \\ \mathfrak{M}(\theta) = d_0 \left(\frac{F(z)}{2} + \frac{z}{4} + \frac{1}{6} \right) - \frac{1}{9}. \end{gathered} \tag{50}$$

The contact θ_{ind} is the zero of $\mathfrak{M}$ in the interval in (51) below.

Proposition 7.28 (Optimized critical comparison)**.** *In the two-function comparison at the first short cutoff with the reflected contours, the long fourth-moment exponent, and affine minorants for the short detectors, the largest limiting critical-parameter margin is $\mathfrak{M}(\theta)$. In particular its exact threshold θ_{ind} satisfies*

$$.87333264782572128 < \theta_{\mathrm{ind}} < .87333264784355536. \tag{51}$$

Proof. For general d, p, write $A_I = p(1+2g_{I,0})$, $k_I = 1-p(1+2g_{I,1})$ and use the analogous ordinary constants. Put

$$B = \frac{1/2 + \ell/2 + d/2 + 2dp\ell - \theta}{h}, \qquad q = 3R/4 - 5/12 - dp/4 + B.$$

The last offset required by the long fourth-moment exponent is $\gamma = 4q/(3v)$. At the crossing of the preceding long face and the current short inverse face, the length m and the count $R - v\gamma$ satisfy

$$T_0 - \gamma = \frac{7/18 + dp/3}{v} - \frac{4B}{3v}, \qquad A_I + k_I(T_0 - \gamma - m) = \frac{4}{9d} - \frac{p}{3} + \frac{4B}{3d}.$$

The ordinary short face is inactive there, since $m \le m_*$.

The mixed first moment at that crossing has margin

$$M = d\left(pF(m) + \frac{1-p}{2}m + \frac{p}{3}\right) - \frac{1}{9} - \frac{B}{3}. \tag{52}$$

For this identity substitute $v\gamma/2 = 2q/3$ and $A_O + 2k_O m_* = (1-R)/d$ in the split mixed comparison. The ordinary affine line cancels.

On the parameter $p = 1/2$, $d = d_0$, $B = 0$. The short inverse line gives the radius $(4/(9d_0) - 1/6 - A_I)/k_I$, which is at least ρ of (50). Indeed the true short bracket $r \mapsto (1+r)/2 + Z(r/2)$ is strictly increasing and bounds every admitted affine bracket from above. The supporting line at ρ attains equality in the limit of strict affine minorants. Since $F' > -.305$, the expression in (52) increases with m. Theorem 7.13 bounds all finite ordinary improvements by F.

This proves the optimal critical expression and its independence of h. The two mixed faces have slopes of opposite sign at the critical crossing, as in the split comparison, and they give the same minimum for sufficiently small strict perturbations of the supporting line and the ordinary width.

For existence and uniqueness of the contact, d_0 is increasing in θ. Its increase decreases the right side of the equation for ρ and increases $(7/18 + d_0/6)/v_0$, so z and $\mathfrak{M}$ are strictly increasing on the contact interval. Here are directed bounds for their values. Use $x = (7/18 - y)/(1/50)$. Construct two degree-13 polynomials P_I, P_O with constant and linear coefficients zero. At order $n \ge 2$ choose the coefficient of x^n in each polynomial to cancel the coefficient of x^{n-1} in

$$5(T-1)(yP' - P - 1/3) + T - L.$$

At this order, T is obtained from the formal solution

$$D(x) + P_{\text{history}}(x - 50D(x)) = \lambda P(x)$$

with $(\lambda, T, L) = (1, 3y + D, 1/2 + y + P)$ or $(2, 3y + 2D, 1/2 + y + 2P)$. The new coefficient enters the residual with factor $-5(3 \cdot (7/18) - 1)(7/18)50n$.

For example both improvements begin with $162(7/18 - y)^2/35$.

Set $P_i^{\pm} = P_i \pm 5 \cdot 10^{-8} x^{14}$. For the upper width functions take the formal cutoff through degree 14 diminished by $5 \cdot 10^{-7} x^{15}$, and for the lower width functions increase it by the same amount. On $.369 \le y \le 7/18$, the cutoff identities have the required one-sided signs with absolute margin at least $3.69 \cdot 10^{-7} x^{15}$. At those cutoffs the differential residuals are negative for P^+ and positive for P^- with absolute margin at least $5.40 \cdot 10^{-6} x^{13}$. The residual decreases with T, so the same signs hold at the exact trial cutoffs.

The trial width functions are convex with the same initial value zero. The comparison argument of Theorem 7.13 therefore gives

$$P_I^- \le Z \le P_I^+, \qquad P_O^- \le F \le P_O^+.$$

The checks use exact rational Bernstein bounds after factoring out the displayed powers of x. Bisection of the increasing radius bracket, followed by evaluation of the upper and lower margins in (50), gives (51). All coefficients and bounds are generated by `check_independent_limit.py`. □

Theorem 7.29 (Zero-free half-plane and sharp comparison at the first short cutoff)**.** *For every number field K and every continuous unitary Hecke character ω,*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > \theta_{\mathrm{ind}}), \qquad .87333264783390206 < \theta_{\mathrm{ind}} < .87333264783390242.$$

The only possible pole is the simple pole of a norm character. The same zero-free half-plane holds for finite-dimensional continuous unitary Weil representations, including Artin representations, with the norm-character pole multiplicities. Within the reflected geometry and first short cutoff inequalities above, θ_{ind} is the infimum using the finite independent moment stages of Theorem 7.13, allowing any finite finite sets of cutoffs and strict supporting affine counts.

Proof. The narrower enclosure follows from the same recurrence through degree 21, using $2\cdot 10^{-11}x^{22}$ and $10^{-9}x^{23}$ as the respective width and cutoff decrements. The factored cutoff and differential margins exceed $7.34\cdot 10^{-10}x^{23}$ and $1.40\cdot 10^{-9}x^{21}$, respectively. The degree-21 verification is implemented in `check_independent_limit.py`.

At $\theta = \theta_{\mathrm{ind}}$ take the ideal inverse supporting line at ρ of (50). For the ordinary affine count on the local rectangle use the refined common-function tangent at .7529 with decrement $2\cdot 10^{-10}$, as in Theorem 7.25. Use the same fixed prime width, interior count interval and earlier reciprocal supremum as that corollary. Choose h by imposing $\partial_d M = 0$ at $d = d_0, p = 1/2$ in (52). For $W = pF(m) + (1-p)m/2 + p/3$ and $\sigma = (4/3)(1/v + 1/(dk_I))$, write $\mathcal{Z} = (7/18 + dp/3)/v - (4/(9d) - p/3 - A_I)/k_I$. Then $m = \mathcal{Z} - \sigma B$, and the tuning is

$$h = \frac{(1/2 + 2p\ell)(dW_m\sigma + 1/3)}{W + dW_m\partial_d\mathcal{Z}} \quad \text{at } p = 1/2,\ d = d_0.$$

Directed evaluation encloses h in $(.81749673476282183, .81749673476294485)$ and the inverse line coefficients in

$$.05646632976112 < g_{I,0} < .05646632976134, \qquad .07388029604897 < g_{I,1} < .07388029604925.$$

On $[.4225, .429] \times [.4995, .5]$, use 65×20 rational boxes. The bounds for derivatives of F are obtained from (35). On this patch the ordinary cutoff is active, and for $X = X_Z(y, 2F(y))$,

$$T' = 1 + 2\frac{1 + 2F'(y)}{1 + Z'(X)}, \qquad L' = 1 + 2F'(y), \qquad F''(y) = \frac{(1-L)T' + (T-1)L'}{5y(T-1)^2}.$$

The two polynomial enclosures give directed bounds on the delayed arguments and their widths. In the split first short cutoff exponent the offset derivative is below $-.02806$ over $0 \le \xi \le \gamma$. The two length derivatives are below $-.02668$ and above .16844.

The required saving satisfies $.00006333 < q < .000085713$. The admissible short lengths, the prime and reference supplies, the ratio, and the four continuing detector count tests are strict. For the terminal margin,

$$M_{dd} > .39962, \qquad M_p < -.01144, \qquad M_\theta > 6.26019.$$

These computations are in `check_independent_contact_local.py`. At $\theta = \theta_{\mathrm{ind}}$ the minimum is attained at $(d_0, 1/2)$ by the tuning and the first two derivative inequalities.

At every sufficiently close larger θ, retaining the same h and ideal supporting line gives positive margins on the whole patch. Fix such a strict margin before choosing a finite moment-stage construction and the width decrements. Supporting line and test losses tending to zero are available by Theorem 7.13 on the compact short ranges. The initial estimates apply near the upper endpoints.

On the complement use the lower polynomial widths of Corollary 7.27. The parameter check admits

$$|\theta - .87333264783390225| \le 2 \cdot 10^{-16}, \qquad |h - .81749673477003466| \le 2.8 \cdot 10^{-9}.$$

It has 128 direct and 826 mixed terminal rectangles, of maximum depth 27, with mixed margin in the original scale above $9.50 \cdot 10^{-9}$. It enlarges the numerator cap and its length collar before the source check and uses the corresponding upper bound on the required saving in every terminal rectangle.

The ratio of the two ordinary lengths is below 1.150503. The inequalities for the short reference, the contours, the prime width, the lowest bin and the principal terms have positive margins. The complementary comparison described in `REFLECTED_RETURN.md` uses independent detector lines on this complement. Thus θ can decrease to θ_{ind} from above. The analytic comparison of Subsection 7.10, followed by the finite-height and induction arguments above, proves the zero-free assertions.

For the infimum, keep $p = 1/2, d = d_0(\theta)$ and take the critical numerator block. Put $R_q = 5/9 + dp/3$ and $T_q = (R_q - 1/6)/v$. The detecting lengths

$$r = \rho, \qquad m = z = T_q - \rho, \qquad r + m = T_q$$

are feasible in the scalar constraints at every cutoff $T \le T_q$ that needs the short exponent. The data give count exponent R_q on the inverse face; the ordinary face gives a larger exponent even with F itself. For $T > T_q$ the long face admits R_q.

Thus additional cutoffs do not improve this feasible critical assignment within the displayed inequalities. The initial, old and fixed-length endpoint counts also admit this count. The optimized fixed-width crossing is beyond its admitted fixed radius .734. At the root the ordinary short-bracket gap exceeds .00096895. The other strict gaps, verified by the same directed calculation, persist at nearby targets. The original fourth bounds exactly the zero margin at R_q. At this count the mixed improvement over it is precisely $\mathfrak{M}(\theta)$ by (52). For a smaller target near the contact this margin is negative independently of h, by Proposition 7.28. □

Variable comparison lengths. Permit the variable geometry of [6, Subsection 7.3] for this comparison with independent widths. Keep its direct majorant, the inverse width Z, and the ordered width F. On the ordinary side also grant the convex majorant $\mathcal{G}$ of the earlier unequal improvements from [6]. Keep $d = d_0(\theta_{\mathrm{ind}})$, $p = 1/2$ and the contact lengths $r = \rho$, $m = z$. Then

$$c = r/2 + Z(r/2) = m + 2F(m), \qquad R_q = 1 - d(1/2 + c) = (10 + 3d)/18.$$

Let $P(y)$ be a nonincreasing majorant of the supplied ordered added widths at shorter cap y, with $P(m) = F(m)$, $P = 0$ for $y \ge 7/18$ and $0 \le P \le 1/6$.

The earlier inverse bound with zero initial increment at r is below $Z(r/2)$, and the older ordered widths at m are below $F(m)$. The directed comparisons are in `check_independent_contact_local.py`.

On $\asymp U^{R_q}$ parameters give a prime product of width w squared amplitude U^{dw} and assign $|S_y| = U^{s(y)}$, where

$$A(y) = \max\{\mathcal{G}(2y), 2P(y)\}, \qquad s(y) = \min\{dy/2, d(y + c - A(y))/4\} \quad (1/3 \le y \le 1/2).$$

Use zero exponents below $1/3$ and reflection above $1/2$. For $1/3 \le y_1, y_2 \le 1/2$,

$$A(y_1) + A(y_2) \ge 2 \max\{\mathcal{G}(y_1 + y_2), P(\min(y_1, y_2))\}.$$

This gives every granted ordinary moment on the same assignment. If one factor is shorter, the global improvement cap $1/6$ and $c/2 > 1/6$ suffice. If both are shorter, use the initial width. As $\mathcal{G}(2m) < 2F(m)$, the short detecting sum has $s(m) = dm/2$. Give the inverse squared amplitudes U^{dt} at $t = r$ and on $[T_q, 3/2]$, and 1 at the other lengths.

The respective short and long inverse inequalities hold by the two contact identities. The other short inequalities use $R_q + d/2 < 1$. Thus these data are simultaneous on all cutoff choices. The colored prime occurrences keep the same exponents, and their slope is below the threshold of the selected operator.

Write $M = l_x + l_y$, $b = l_y - l_x \geq 0$, $h = 1 - l_x + \ell$ and $n = 1 - l_y/h$. The three direct planes of [6] force $n > .37$ for targets $B \leq .8741$. For $7/18 \leq n < 1/2$, $s(n) = d(n + c)/4$, so the required top plane is

$$H = 1 + 3d/4 - (1/2 + d/4)M + 3d\ell/4.$$

The positive dual identity for L_0, L_K, H in [6] gives $B \geq (14 + 25d)/(18 + 24d) = \theta_{\mathrm{ind}}$.

For $n \geq 1/2$ the identity $R_q + 4s(1/2) = 1$ gives the unmarked half-length plane and $B \geq 13/(19 - 21d/2) > .89$.

For $.37 < n < 7/18$ use the auxiliary assignment of [6], with $d_0 = .37$, $m_0 = 24257/66000$, $R_0 = 4471723/6600000$ and $c_0 = .004$. In the formula for s use $c = m_0 + c_0$ and $A_0(y) = \max\{\widetilde{\mathcal{G}}_0(2y), 2P(y) - c_0\}$, where $\widetilde{\mathcal{G}}_0$ is its convex majorant, constant c_0 on $[2m_0, 1]$ and at least c_0 elsewhere. Since $A_0(y) \geq 2P(y) - c_0$ and $A_0 \geq c_0$, their two-factor sum is at least $2P(\min(y_1, y_2))$. The older unequal moments use the convex majorant as before.

The displayed earlier improvements and $F(.369)$ give $P(y) < c_0$ for $y \geq m_0$, and $\mathcal{G}(2m_0) < c_0$. Thus $A_0 = c_0$ for $y \geq m_0$ and the original short detecting sum and adverse numerator amplitudes are unchanged. The auxiliary inverse identities remain $R_0 + d_0(1/2 + m_0 + c_0) = 1$ and $R_0 + 3d_0m_0 = (1 + 15m_0)/6$. The original auxiliary plane and its positive dual give $B \geq 72703411/83204812 > .8737885$.

7.17 A finite comparison with a refined width function

In this subsection we give an explicit consequence of the reflected comparison with a larger polynomial width. The limiting width is used in Subsection 7.18.

Proposition 7.30 (Refined ordered source)**.** *In the paired family and the contradiction regime of Theorem 7.8, the mixed ordinary bound holds at every strict width below* $(1 - N - z)/2 + F_4(z)$ *for* $.369 \leq z \leq .3888$, $z \leq N \leq 1.15z$. *The original paired test functions, independent normalized heights, subproducts, componentwise shortenings and natural zeros are retained.*

Proof. Use the proof of Theorem 7.8 with $f = F_4$ from Section 5, with lower piece

$$F_4(z) = F_{4,+}(a) + .174(a - z) + 6.7(a - z)^2, \qquad a = .3735.$$

The earlier curved inverse width is $q(r) = (1 - r)/2 + G(r)$, obtained at every strict width from the repaired finite approximation in [7]. Use (10) with $y = z$. Set

$$A = 5(f + 1/3 - yf'), \qquad R_0 = 1/2 + y + 2f, \qquad b = \frac{A - R_0}{A - 1} - y, \qquad C = \frac{19b}{16} + \frac{5}{24} - R_0.$$

On each of the two closed polynomial pieces the exact Bernstein coefficients give

$$A > 1, \qquad 2y < b < 7/9, \qquad C > 0, \qquad C^5 > (5/16)^5\, 7b^4(b - 2/3).$$

In the last inequality, after multiplication by $(A-1)^5$, the lower coefficient bound exceeds $9.55 \cdot 10^{-9}$.

Thus $\mathcal{R}(b) > R_0$ and $r_* < b$. Since $R_0 < 1$,

$$\frac{T_* - R_0}{T_* - 1} > A, \qquad f' > \mathcal{D}(y, f).$$

Both one-sided derivatives at a obey the strict differential condition. A step across a uses the integral of the respective derivative on its two pieces. The same checks give $f < .003$, $-.27 < f' \le 0$ and $1/2 + f - yf' < .61$. They are verified by `check_f4_differential.py`.

For small descending steps the low threshold lies strictly above $k_*(y, f)$. At its upper end use the short cutoffs (r_*, y), then increase the short cutoffs by $O(u)(e - e_0)$ when the response exceeds the low threshold e_0. The inverse derivative is bounded below by .35 for $\mathcal{R}$ and the ordinary substitution exponent decreases at slope at least $.23He$. The long estimate stays strict as in Theorem 6.1.

This is the moderate-response replacement in the proof of Theorem 7.8. The negative starting value $F_4(.3888)$ permits an initial starting strip and the varying ordinary radial loss has adverse slope at most $(3/4)(1/2 + .61) < 5/6$. The remaining ranges and the packed second correlation use the same estimates and smaller length rectangle. Fix the strict widths throughout a finite descending cover before fixing the test functions and heights. □

Theorem 7.31 (Refined finite number-field bound)**.** *For every number field K and every continuous unitary Hecke character ω,*

$$L_K(s, \omega) \neq 0 \qquad (\operatorname{Re} s > .87333269),$$

with the possible simple pole of a norm character permitted. The same bound holds for Artin L-functions.

Proof. Use Proposition 7.30, the affine line from Theorem 7.18, and

$$\theta = .87333269, \qquad h = .817496881, \qquad \kappa = .7466918.$$

Theorem 7.21 supplies the prior finite-numerator pointwise exponent, since $.87334470 < (1+\kappa)/2 = .8733459$. The improved count rectangle is $[.365, 15/32] \times [.46, 1/2]$ and the affine lower length is .7335. Use also (44) with $a = .734$ and $\sigma_a = .136029 < (1 - a)/2 + G(a)$.

The strict inequality is checked by the fifth-power comparison. Increasing $r + 2pq(r)$ on the curved range and the initial and unmarked continuations give the short count above a. The small extension below a uses the last width at a by the original cutoff truncation.

The scalar proof uses the exact same-zero mixed exponent and top conductor test of Subsection 7.11. When required, raise the cap to $z \ge \max\{m, .369, N/1.15\}$, and subtract $(z - m)/2$ from the available added width. After subtracting $g(2m)/2$ the result increases with m where the cap is raised. Where $z = m$, minimize on each smooth piece separately, including the seam.

The lower edge and low-amplitude tests use the earlier and initial moments. On $d \le 1/8$ the margin is greater than .004. In the verification there are 65 directly estimated and 262 mixed terminal rectangles of depth at most 30, with mixed margins greater than $1.4959 \cdot 10^{-9}$. The source caps satisfy

$$.369 < z < .38018387, \qquad .42394438 < N < .42486343, \qquad N/z < 1.149771.$$

The two conductor edges treated by the earlier bound and the lower edge treated by the initial estimate have margins above .03416, .002966, and .006536, respectively. The left, right, short and prime width margins exceed .00006768, .0002520, .00002384, and .0005829.

The floor margin exceeds .0004857 and the remaining geometric, small-amplitude and principal inequalities are strict. The exact verification is in `certify_all_field_reflected.py`. The original finite-height argument and base change now give the stated bounds. □

Once the character bound is obtained over every $F \supset \mu_{12}$, abelian base change for $F = K(\zeta_{12})$ gives

$$L_F(s, \omega \circ N_{F/K}) = \prod_{\xi \in \widehat{\mathrm{Gal}(F/K)}} L_K(s, \omega\xi).$$

Below $\operatorname{Re} s = 1$ all factors are holomorphic, so a zero of the target survives in the product. The line $\operatorname{Re} s = 1$ is dealt with by the classical Hecke theorem. Brauer induction, and its unitary Weil version used in [6, 4], give the corresponding Artin and unitary Weil results. Frobenius reciprocity makes the pole order at $1 + it$ the multiplicity of the norm-character constituent.

7.18 The zero-free boundary from the curved joint moment

In this subsection we use the curved joint moment of [7] to enlarge the ordered width in Section 6. We then determine the main zero-free boundary.

The strict joint width $(1 - r)/2 + G(r)$ is available in the paired family by [7, Corollary 9.3]. For every compact subinterval of $[.734, 7/9)$ a single finite construction suffices for fixed positive decrements. The width at .734 can also be obtained by a slightly larger radius and length shortening. The initial marked and unmarked estimates give the continuations. The ordered induction uses the joint width as an earlier input to the inverse count.

Corollary 7.32 (Ordered width from the curved width)**.** *In the paired families of Theorem 7.8, the ordered mixed moment holds for* $.369 \le z < 7/18$, $z \le n \le 1.20z$ *and*

$$S < \frac{1 - n - z}{2} + F_\infty(z).$$

A strict width buffer can be uniform on a compact set of caps. The curved input is used in the regime $\beta_\omega \ge 51/100$.

Proof. Fix a compact set of short caps and a width decrement. Put $y_0 = 7/18$, $Y = y_0 - \epsilon$, and $f(y) = F_\infty(y + \epsilon) - \epsilon_0$ on $[.369, Y]$, with $\epsilon, \epsilon_0 > 0$ sufficiently small. In (10), the numerator $f + 1/3 - 1/(6k_*(y, f))$ increases in y and f on both cutoff branches. At $(y + \epsilon, F_\infty(y + \epsilon))$ it is $(y + \epsilon)F'_\infty(y + \epsilon) \le 0$. These facts give $f'(y) > \mathcal{D}(y, f(y))$. The initial source supplies the starting strip where $f < 0$. Choose $H_0 = 1 - 6u$ sufficiently close to one and apply the finite strict-stage proof of Theorem 6.1 with this Y in place of .3888. The upper cutoff before padding is below $7/9$: $\mathcal{R}(7/9) = 8/9$, while $1/2 + y + 2f(y) \le 1/2 + Y - 2\epsilon_0 < 8/9$. Also $2y \le 2Y < 7/9$. Once the shift is fixed, choose the $O(u)$ cutoff padding smaller than this fixed margin.

The attained earlier inverse cutoffs therefore form a compact subset of $[.738, 7/9)$.

The same terminal estimates extend to all radii using

$$-.25 < f' \le 0, \qquad f < .002218, \qquad 0 < 1/2 + f - yf' < .61.$$

These inequalities follow from Theorem 6.1 and sufficiently small shifts. With $P = n + z$ the full column volume satisfies

$$P < .856, \qquad W_f < .138, \qquad P + W_f \le 1/2 + 1.1z + f(z) < .928.$$

For the last inequality the middle expression increases to its value at Y. The adaptive cap (21) and the fixed- and zero-frequency tests of Proposition 7.3 therefore apply.

At $V \ge H_0$ keep the earlier caps normalized at H_0; componentwise shortening admits the actual lengths and the absolute widths increase with V. At $V < H_0$ the ordinary sum keeps its response

at its actual length. Raise only its moment cap to $m'' = \max(m', n/(1.20V))$ as in Theorem 7.8. The switch to the initial width at $m'' = Y$ is favorable since $f(Y) < 0$.

The adverse radial slope on $[.98, H_0]$ is at most $(3/4)(1/2 + .61) < 5/6$. The selected estimate on this range, when used, saves at least $.5001(.98 - .928) > .026$.

For $.875 \le V \le .98$, coefficient two gives excess at most $(3/4)[(1 - V)/2 + .002218 - u]_+$ after stripping. At $V = .98$ this is below the radial allowance for small u and ρ/u, and its adverse slope is at most $3/8$. For $.08 \le V \le .875$ the unmarked bounds give excess at most $(3/4).138 = .1035 < (5/6)(1 - .875)$, with room for small u, ρ. Below .08 the direct excess is at most

$$3P/4 + L \le 1/2 + P/4 + .002218 < .717,$$

which fits the same radial allowance. These are the low-radius branches of Theorem 7.8 on the expanded caps. The packed terminal and sixth-power restoration therefore give the stated moments. Choose the finite strict construction before the test functions and heights. Finally choose the initial shifts below the prescribed width decrement on the compact set. □

Theorem 7.33 (Unconditional bound from the curved anchor width)**.** *For every number field K and continuous unitary Hecke character ω,*

$$L_K(s, \omega) \neq 0 \qquad (\operatorname{Re} s > \theta_{\mathrm{G}}), \qquad .8733326374459049352 < \theta_{\mathrm{G}} < .8733326374459051924.$$

The possible simple pole of a norm character is permitted. The same unconditional boundary holds for Artin and finite-dimensional continuous unitary Weil L-functions, with pole orders as above. Within the curved Marked and ordered stages and the reflected comparison at the first short cutoff of Proposition 7.28, θ_{G} is the nearby-target infimum.

Proof. Fix $\theta > \theta_{\mathrm{G}}$ sufficiently near the contact and suppose the target has a zero to the right of θ. In the working field the family supremum satisfies $\beta_\omega > \theta > 51/100$. The curved joint widths and Corollary 7.32 are therefore available at strict widths. Use (50) with $Z(\rho/2) = G(\rho)$ and $F = F_\infty$. The critical margin is

$$\mathfrak{M}_{\mathrm{G}}(\theta) = d_0(F_\infty(z)/2 + z/4 + 1/6) - 1/9.$$

The proof of Proposition 7.28 shows that it is increasing across the contact interval.

For its directed evaluation put $X = 50(7/18 - y)$. Expand $G(2y)$ and the solution of (11) through degree 21. Enclose the two width functions using respective decrements $2 \cdot 10^{-12} X^{22}$ and $2 \cdot 10^{-11} X^{22}$ in both directions. The inverse bound follows by comparing fifth powers. For the ordered bound use the formal cutoff through degree 22 with an outward displacement of $10^{-9} X^{23}$. The fraction-coefficient Bernstein checks give the two cutoff signs and differential signs throughout $[.369, 7/18]$. The lower ordered polynomial and upper inverse polynomial satisfy $2F^-(y) > G^+(2y)$ for $.369 \le y < 7/18$, which identifies the active cutoff for the enclosed solution. Comparison in the differential equation and bisection of the radius give opposite signs at the two stated endpoints. The verification is `check_limit.py`.

Take the ideal inverse and ordinary line to be the supporting line to G at the resulting ρ. Use the stationarity in Theorem 7.29 to choose h in $(.8175003800750593, .8175003800750619)$. On $[.4254, .4261] \times [.4998, 1/2]$ the 14×8 rational boxes give

$$M_{dd} > .42933, \qquad M_p < -.012456, \qquad \partial_\xi \mathcal{W}_0 < -.080408, \qquad .00007300 < q < .00007715.$$

The two length derivatives have strict signs. The conductor and width inequalities are strict, and $N/z < 1.15$.

For the initial and unmarked continuations, the count improvement above $4q/3$ exceeds .00487 on the local patch. The local computation `check_limit_local.py` evaluates the ordered derivatives from (11) at the enclosed arguments. The signs and the choice of h give the unique critical minimum at $(d_0, 1/2)$. At fixed h and detector line the required saving term B of (52) decreases with θ, and that margin decreases with B. Thus the margin is strictly positive on the rectangle for nearby larger θ.

Choose the finite width and line decrements after this increase.

On the complement use the degree-22 lower ordered polynomial with coefficient rounding down to denominator 10^{25} and a further 10^{-14} width loss. Use a strict lower tangent from the lower inverse polynomial at radius .7528546345463726, the fixed prime width $\sigma_a = .136029$ at $a = .734$, and the old exponent parameter $\kappa = .7466918$. The prior bound $.87334470 < (1+\kappa)/2$ is supplied by the independent one-stage Theorem 7.21.

The directed cover admits

$$|\theta - .87333263744590506| \le 2 \cdot 10^{-15}, \qquad |h - .8175003800750606| \le 2 \cdot 10^{-15}.$$

For parameter displacements of size $\delta = 2 \cdot 10^{-15}$, the changes in ℓ, l_x, l_y are bounded by $4\delta, 5\delta, 9\delta$. On the verified conductor ranges, $h > .8$ and $\ell < .19$. Thus ℓ/h changes by less than 6δ and $2l_y - D$ by at most 19δ on the tested edges. Substitution into the affine comparison margins and geometric conditions bounds their changes by 50δ. The exterior check subtracts 200δ and enlarges the numerator caps, the width allowed for the shorter ordinary sum and the lower conductor bound in the mixed rectangles.

It has 82 directly estimated and 312 mixed terminal rectangles at depth at most 33. The margin in the original scale exceeds $7.715 \cdot 10^{-11}$ and the ratio of the two ordinary lengths is below 1.149969. The original reference, floor and principal tests each have strict margins. Apply the reflected comparison and the finite-height Mellin argument of Subsection 7.9 at the supremum β_ω of the same working family.

The resulting zero-free region descends by the base-change identity above.

For the infimum use the feasible count $R_q = 5/9 + d_0/6$ and the lengths (ρ, z) from (50) evaluated with G and F_∞. The ordinary bracket with G itself has a strict gap at $2z$, since $2z < \rho$. The earlier counts, the initial counts and the fixed-length endpoint counts admit R_q. The fixed-width crossing would require $2(1 + d_0(1 - 3\sigma_a)/(3v_0 + 2d_0))/3 > .734$, outside the range of lengths for which the fixed width applies. The directed local check also verifies this with the upper bound .136030 for σ_a. Every earlier long cutoff and every later short cutoff therefore admits the same assignment in the scalar system.

At this assignment the fourth-moment estimate has zero margin. The mixed-moment margin is $\mathfrak{M}_{\mathrm{G}}(\theta)$, negative just below the contact, independently of h. □

A separately finite choice gives $\operatorname{Re} s > .873332637445906$. For the rounded lower polynomial f used on the complement, the differential margin at its trial cutoff exceeds $(5T - 3)10^{-14} > 2.5 \cdot 10^{-14}$. Indeed f is below the lower polynomial by at least 10^{-14}, and its derivative is at least that of the lower polynomial. For $u = 10^{-20}$ and $\rho = 10^{-34}$ the Euler test (12) has room for the original 3ρ allowances. For the whole strict construction take sufficiently small steps and ρ/u as in Corollary 7.32.

The starting value is negative at $7/18 - 10^{-14}$. For the fixed strict tangent the contact is in $(.87333263744590552536, .87333263744590552538)$, with $h = .81750038007483217053\ldots$. The same 112 local boxes and an $82 + 312$ exterior cover suffice. The finite source and the two covers are checked by `check_source.py`, `check_local.py` and `check_exterior.py`.

The Dirichlet contact. At $p = 1/2$ and $A_0 = 0$ the notation of [5, Theorem 8.1] gives $d = d_0$, $4q/3 = 3vm - 7/18 - d/6$ and $m(3v + 2dk) = v + d/2 - dg_0$. Its two cutoff equations reduce to

$$\frac{1+\rho}{2} + G(\rho) = \frac{4}{9d} - \frac{1}{6}, \qquad z = \frac{7/18 + d/6}{v} - \rho.$$

The Dirichlet critical margin multiplied by d is $d(F_\infty(z)/2 + z/4 + 1/6) - 1/9$. The ordered differential equations agree between z and $7/18$, since the critical $z > .3751$ and the inverse cutoff $r \geq 2z$ lies in the common curved range. Thus the two comparisons have the same scalar contact.

References

[1] H. Collet, *Cycle092: Dirichlet zero-free bound* 436666397/500000000, QuasiRiemannTracker, pull request 144, 20 November 2026. https://github.com/GTimG/QuasiRiemannTracker/pull/144.

[2] H. Collet, *The ordinary QRH half-plane: a compact working proof*, written submission accompanying Cycle 092, 2026, Appendices H, P and Q. Written proof at the pinned revision.

[3] Z. Gao, *Unitary character moments and a zero-free half-plane for unitary Weil L-functions*, Hexagon 2611.00334v1, 2026. https://hexagonmath.org/2611.00334v1.

[4] B. Hu, *A seven-eighths zero-free half-plane for unitary Weil L-functions*, Hexagon 2610.00216v2, 2026. https://hexagonmath.org/2610.00216v2.

[5] B. Liu, *Joint mean values and zero-free half-planes for Dirichlet and Eisenstein Hecke L-functions*, unpublished manuscript, November 2026.

[6] B. Liu, *Joint marked moments and optimized zero-free half-planes for unitary Hecke L-functions*, first version, Hexagon 2611.00421v1, 2026. https://hexagonmath.org/2611.00421v1.

[7] B. Liu, *Joint marked moments and optimized zero-free half-planes for unitary Hecke L-functions*, revised companion manuscript, November 2026. The revised text is supplied with the accompanying materials.

[8] B. Liu, *Selected character moments, zero-free half-planes for Hecke L-functions over number fields, and Quasi-Riemann Hypothesis*, Hexagon 2611.00082v1, 2026. https://hexagonmath.org/2611.00082v1.

[9] B. Liu, *Zero-free half-planes for Hecke L-functions over number fields and Quasi-Riemann Hypothesis*, arXiv:2610.26835v2, 2026. https://arxiv.org/abs/2610.26835v2.

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