A counterexample to the cocircuit diameter conjecture

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Abstract

A rationally realizable uniform rank-41 oriented matroid on 80 elements has cocircuit-graph diameter at least $42 > 41 = n - r + 2$, refuting the general cocircuit diameter conjecture. We build on the rank-21 example of Adler, De Loera, Klee and Zhang, the non-Hirsch polytope of Matschke, Santos and Weibel, Santos's product construction, and the source certification of Allamigeon, Canu and Strub. A count of negative coordinates confines every potentially shorter walk to two affine sections; projection to the product factors then gives the lower bound. Two rational perturbations make the resulting oriented matroid uniform. Lean 4 separately verifies the matrix existence and walk lower bound for a different realizing matrix; the printed perturbations are proved here, not checked in Lean.

This work has not yet been reviewed by experts. The proofs, the Lean formalization, the certificate checkers and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

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1 Introduction

Let $A \in \mathbb{R}^{n\times r}$ have rank r. Its sign covectors are $\operatorname{sgn}(Ax)$, $x \in \mathbb{R}^r$, with values in $\{-1,0,1\}^n$. Its *cocircuits* are the nonzero covectors of minimal support. Write $X^0 = \{i : X_i = 0\}$ and $S(X,Y) = \{i : X_i = -Y_i \neq 0\}$ for the separation set. When every r-row minor of A is nonzero, the associated oriented matroid is *uniform*; every cocircuit has $r-1$ zeros. Its *full cocircuit graph* $G^*(A)$ has all signed cocircuits as vertices, with distinct X, Y adjacent when $S(X,Y) = \varnothing$ and $|X^0 \cap Y^0| = r-2$. Rational realizability means A can be chosen over $\mathbb{Q}$. Cocircuits X, Y are *antipodal* if $Y = -X$, and nonantipodal otherwise. Write d_H for distance in H.

Adler, De Loera, Klee and Zhang's (ADKZ) Conjecture 1 states that every rank-r oriented matroid on n elements has cocircuit-graph diameter at most $n - r + 2$ [4, Conjecture 1]. Babson, Finschi and Fukuda's early farthest-pairs question [1, final remarks, pp. 13–14] and Felsner, Gómez, Knauer, Montellano-Ballesteros and Strausz's Question 4.2 [3] precede ADKZ's formulation. ADKZ report Fukuda's personal communication that the bound is folklore. Klee's report attributes its introduction to Fukuda and Terlaky [13, p. 2425].

Theorem 1. *There exists a rationally realizable uniform oriented matroid of rank 41 on 80 elements whose cocircuit graph has diameter at least* $42 > 41 = 80 - 41 + 2$.

ADKZ state a uniform rank-21 example on 40 elements with nonantipodal cocircuits at distance at least 21 [4, Proposition 8]. Their proof uses the 20-dimensional non-Hirsch polytope of Matschke, Santos and Weibel (MSW), with 40 facets [6, Theorem 1.3], obtained using Santos's construction [7, Theorem 2.6]. Allamigeon, Canu and Strub (ACS) proved its diameter 21 in Coq [8, §5, PDF p. 12]. We reuse their numerical inequality data and check a new certificate for the designated pair. Product amplification is standard [7, Lemma 6.1 and §6], where Santos cites Klee–Walkup, Proposition 1.3.

We must control walks that leave the product's nonnegative section. The lifted source matrix M_0 is not uniform: rows 5, 7–10 and 27–29 are supported on the first six columns (Appendix A). We first perturb it to a uniform rank-21 factor, preserving the pair at distance 21. Two factors give 80 inequalities in rank 41: the product pair is 42 edges apart; the conjectured bound is $80 - 41 + 2 = 41$. To exclude shorter cocircuit walks, Lemma 2 generalizes ADKZ's count of sign-coordinate changes [4, Lemma 5 and proof of Proposition 8]: a walk of at most 41 edges can visit at most one negative coordinate. Such a walk stays in the union of two sections. A second uniform perturbation preserves these sections; projection to the factors still requires $21 + 21$ edges.

2 A lower bound on walk length

Fix an inequality matrix $M \in \mathbb{R}^{n\times r}$ and put $C_0(M) = \{z : Mz \geq 0\}$ and $C_e(M) = \{z : (Mz)_e \leq 0,\ (Mz)_i \geq 0\ (i \neq e)\}$. Choose a rational linear functional c positive on every nonzero vector in these cones. The *affine section* of a cone C is $C \cap \{c(z) = 1\}$; positivity makes it bounded (Lemma 4(a)). The sections of C_0 and C_e are nonnegative and singly reversed, respectively. A row is *tight* when equality holds; the tight row labels form the *active set*. When M is uniform and the section is full-dimensional, we identify a section vertex z with the cocircuit $\operatorname{sgn}(Mz)$; its active set is its zero set (Lemma 3). A d-dimensional section is *simple* if each vertex lies on exactly d facets.

Lemma 2. *Let G^* be the cocircuit graph of a uniform rank-$(d+1)$ oriented matroid on $2d$ elements. Suppose that X^0, Y^0 are complementary d-element sets and X, Y are positive elsewhere. For a walk of length L from X to Y, let k be the number of coordinates that are negative at any vertex of the walk. Then*

$$2d + 2k \leq 2L.$$

Proof. Adjacent cocircuits have d zeros each, sharing $d - 1$. Exactly one zero becomes nonzero and one nonzero becomes zero. All other nonzero entries agree because the separation set is empty. Thus exactly two coordinates change, each by absolute amount one. The complementary endpoint zero sets make every coordinate go from 0 to 1 or from 1 to 0. Its total absolute change is at least one, or at least three if it visits -1. Summing gives $2d + 2k \leq 2L$. □

For $d = 40$ and $L \leq 41$, the walk uses at most one fixed negative coordinate e, so every vertex lies in C_0 or C_e. If none is negative, any e works.

Lemma 3. *Let $M \in \mathbb{Q}^{n\times r}$ have rank $r \geq 2$ and be uniform. Let a rational linear functional c be positive on every nonzero vector of $C_0(M)$ and every $C_e(M)$. Suppose each section by $c(z) = 1$ is bounded and has dimension $r - 1$. Each section graph is the restriction of $G^*(M)$ to cocircuits satisfying its sign requirements $X_i \geq 0$ for C_0, or $X_e \leq 0$ and $X_i \geq 0$ for $i \neq e$ for C_e. For every row index e, every walk whose negative coordinates are contained in $\{e\}$ lies in the union of the graphs of the sections of $C_0(M)$ and $C_e(M)$.*

Proof. The $r - 1$ zero rows of a cocircuit are independent. Their kernel is a line; choosing the signs selects a ray. Positivity of c gives exactly one representative with $c(z) = 1$, and excludes the antipodal ray from that cone. This representative is a section vertex. Conversely, a section vertex needs $r -$

1 independent tight rows; uniformity forbids more, so its sign vector is a cocircuit. These $r-1$ independent active rows give local coordinates in the $(r-1)$-dimensional section, so it is simple. Two vertices sharing $r-2$ tight rows are the endpoints of their common one-dimensional face, and every edge has such endpoints. Conversely, full-graph adjacency between section vertices requires these $r-2$ shared zeros, so gives such a section edge. Separation is empty inside one of these cones; thus its graph is the induced cocircuit subgraph on those vertices. For an adjacent pair with no negative coordinates outside e, if either has a negative entry at e, the other has a nonpositive entry there: opposite nonzero signs are excluded by adjacency. Both then lie in C_e. Otherwise both lie in C_0. When both signs at e vanish, both vertices lie on the shared facet and the edge lies in both graphs. Every walk edge therefore belongs to one of the two section graphs. □

3 The construction

Put $P = C_0(M_0) \cap \{z : c(z) = 1\}$, with the rational matrix M_0 and functional c specified in Appendix A. The source certificate [12], detailed in Appendix B, gives a simple 20-dimensional P and $d_P(u, v) = 21$ for vertices u, v with active rows 1–20 and 21–40, respectively. The inequality data are `poly20dim21.ine` in Coq-Polyhedra (Allamigeon, Katz and Strub). ACS attribute the supplied inequalities to Weibel [8, §5, PDF p. 11, footnote 6]; Allamigeon, Id-Sahra and Strub likewise credit MSW for the 20- and 23-dimensional non-Hirsch polytopes they consider [11, §1]. We use the documented attribution for credit; the source file was not compared entrywise with MSW's table. The proof uses the exact matrix and certificate specified here; the supplement records the source version and comparisons. Clearing each row's reduced denominators by their least common multiple gives $M_0 \in \mathbb{Z}^{40\times 21}$, leaving every sign unchanged. Let $\sigma_e \in \{-1, 1\}^{40}$ have entries $\sigma_{e,i}$: σ_0 is all positive, and σ_e reverses row e for $e > 0$. Appendix A gives a rational row vector c and weights $\lambda_{e,i} \geq 1$ satisfying

$$c = \sum_{i=1}^{40} \lambda_{e,i}\sigma_{e,i}(M_0)_i, \qquad (0 \leq e \leq 40).$$

These exact identities, with $\operatorname{rank} M_0 = 21$, imply $c(z) > 0$ on every nonzero vector of each closed cone: its signed slacks are nonnegative and not all zero.

Lemma 4 (Stability of bounded simple sections)**.** *Let $M \in \mathbb{R}^{n\times r}$ and let $\Sigma \subseteq \{-1, 1\}^n$ be finite. Suppose a linear functional c is positive on every nonzero vector of*

$$C_\sigma(M) = \{z : \sigma_i(Mz)_i \geq 0 \text{ for all } i\}, \qquad \sigma \in \Sigma.$$

(a) *For all M' in a sufficiently small neighborhood of M in $\mathbb{R}^{n\times r}$, the same positivity holds and the sections $C_\sigma(M') \cap \{c = 1\}$ are uniformly bounded.* (b) *For each σ whose original section has dimension $r-1$ and exactly $r-1$ independent active rows at every vertex, its perturbed section has the same dimension and face lattice, with the same row labels and nonzero signs. For two patterns σ, σ' satisfying these hypotheses, their perturbed sections intersect in the face where all rows on which σ and σ' differ are tight. This is the continuation of their original common face, and the two continuation maps agree on its vertices and edges.*

Proof. Positivity and boundedness. Failure of positivity along $M' \to M$ would give unit vectors in perturbed cones with $c \leq 0$; a convergent subsequence contradicts the original positivity. Unbounded normalized sections similarly give a unit limit in an original cone with $c = 0$. *Vertices.* For a full-dimensional original section, a point with all inequalities strict remains feasible, preserving dimension. Each original vertex continues uniquely through its nonsingular active-row system and $c = 1$. Every inactive signed slack was strictly positive; by continuity it remains so, giving exactly the original active set.

Suppose additional vertices appeared along $M' \to M$. Uniform boundedness gives a convergent subsequence with limit z. Finiteness of row subsets gives a further subsequence with a fixed set I of $r-1$ independent tight rows at the perturbed vertices. These rows need not a priori be independent at M. The original bounded face with I tight contains z by continuity of every signed inequality and tight equation. This nonempty bounded polyhedral face has a vertex w. Its active set contains I and has the same cardinality $r-1$, so equals I. Thus M_I has rank $r-1$, and $M_I z = 0$, $c(z) = 1$ uniquely determines w. For sufficiently small perturbations the system remains nonsingular; the alleged additional vertex is the continuation of w.

Faces. Identify each vertex with its unique continuation, retaining its active set and nonzero signs. For a row subset I_0, the face cut out by making I_0 tight has precisely the vertices whose active sets contain I_0. The continuation map preserves this vertex set and its emptiness, for each section satisfying the stated dimension and active-row hypotheses. Every face of an inequality polytope has this form and is the convex hull of its vertices. Thus equal faces, and inclusion between faces, are preserved in both directions. This gives a face-lattice bijection, preserving edges and their endpoint pairs.

Common faces. If two patterns differ on I_0, the intersection of their sections is the face obtained by making I_0 tight in either section. Every vertex of this face is a vertex of both sections: in a convex decomposition into section points, all signed slacks on I_0 are nonnegative and average to zero, so both points lie in the common face. A nontrivial decomposition would contradict its being a vertex there. A row is tight precisely when its linear form is zero, independently of the sign pattern. The active set I is thus the same in both sections, has exactly $r-1$ rows, and contains I_0. Its continuation is the unique solution of $M'_I z = 0$, $c(z) = 1$ in either section. The identifications, full sign vectors and common-face edges consequently agree. Intersecting the neighborhoods for the finitely many sections and vertex systems gives one neighborhood for every $\sigma \in \Sigma$. □

Set $F = M_0 + \varepsilon J$, where $J_{ij} = (i+1)^{j-1}$ has 21 columns. Each 21-row minor is a polynomial in ε with nonzero Vandermonde leading coefficient, so all are nonzero for sufficiently small rational $\varepsilon > 0$. Apply Lemma 4 with $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_{40}\}$: c stays positive on these 41 cones. Each vertex of P has exactly 20 independent tight rows, as the certificate checks. The section P of $C_0(M_0)$ is therefore simple, so Lemma 4 preserves its graph with the active row labels. Let $F \in \mathbb{Q}^{40\times 21}$ be the resulting uniform matrix, and U, V the unique continuations of u, v. Write H_0, H_e for the section graphs of $C_0(F), C_e(F)$, respectively.

Lemma 5. *Each reversed section $C_e(F) \cap \{c = 1\}$ has dimension 20, and every vertex has exactly 20 independent tight rows.*

Proof. All reversed sections are bounded by Lemma 4, applied above with all 41 patterns. (1) Every row is active at U or V. There the twenty independent active constraints are local coordinates on $c = 1$. Keeping one zero and making the other nineteen positive gives a 19-dimensional face. Thus every row defines a facet of the nonnegative section.

(2) A facet's affine span in $c = 1$ spans a linear hyperplane in $\mathbb{R}^{21}$. Two rows defining the same facet therefore vanish on the same hyperplane and are proportional, contradicting a nonzero 21-row minor of the uniform matrix F. Distinct rows define distinct facets.

(3) At a relative-interior point of facet e, any other tight row would vanish on the facet's affine span: a nonnegative affine form vanishing at an interior point vanishes identically there. This would make its facet equal to facet e, since both are facets by (1), contradicting (2). Thus only row e is tight. Crossing its hyperplane within $c = 1$ makes row e negative and all others positive, giving a 20-dimensional reversed section.

(4) At every reversed-section vertex at least twenty independent constraints are tight. Uniformity forbids 21 tight rows, whose common kernel would be zero. Thus exactly twenty rows are tight and independent. □

By Lemma 3, H_0, H_e are subgraphs of $G^*(F)$, identified by sign vectors. Their union is the induced subgraph on the cocircuits satisfying either set of sign inequalities. H_0 inherits P's graph and row labels, giving $d_{H_0}(U, V) = 21$. Their active labels still partition the 40 facets, so the cocircuits corresponding to U, V have complementary zero sets and are positive elsewhere.

Lemma 6. *Every U–V walk in $G^*(F)$ has length at least 21.*

Proof. This is ADKZ's Proposition 8 count applied to F. For a walk of length at most 20, Lemma 2 with $d = 20$ gives $k = 0$. Its vertices are nonnegative, so Lemma 3 puts it in H_0, contradicting $d_{H_0}(U, V) = 21$.

□

Choose rational coordinates $(s, x) \in \mathbb{R} \times \mathbb{R}^{20}$ with $s = c(z)$, so the affine sections lie in $s = 1$. After this rational invertible change of coordinates we may assume that $F = (\beta\ R)$ with $\beta \in \mathbb{Q}^{40}$ and $R \in \mathbb{Q}^{40\times 20}$, so $F(s, x) = \beta s + Rx$. The change preserves all sign vectors and nonvanishing maximal minors. In product coordinates (s, x, y), the inequality matrix is

$$B = \begin{pmatrix} \beta & R & 0 \\ \beta & 0 & R \end{pmatrix} \in \mathbb{Q}^{80\times 41}.$$

The section of $C_0(B)$ is a product of two 20-dimensional polytopes. Its vertices are pairs of factor sign vectors; each edge moves one factor. For a flip in the first factor, the reversed section graph is $H_e \square H_0$. We use only the inclusion

$$(H_0 \square H_0) \cup (H_e \square H_0) \ \subseteq\ (H_0 \cup H_e) \square H_0.$$

Here $\square$ is the Cartesian product of graphs. The second-factor case is symmetric.

For the second perturbation, set $W \in \mathbb{Q}^{80\times 41}$ with $W_{ij} = (i + 1)^{j-1}$ and seek $A = B + \delta W$ with rational $\delta > 0$ sufficiently small.

Lemma 7. *For some rational $\delta > 0$, the matrix $A = B + \delta W$ is uniform. The functional s is positive on every nonzero vector of $C_0(A)$ and all $C_e(A)$. Their sections by $s = 1$ are bounded, have dimension 40, and have the same graphs and common faces as the corresponding sections for B, identifying vertices by active rows together with their sign vectors.*

Proof. Fix F and s before choosing δ. Every factor section has dimension 20, by the construction of P for H_0 and Lemma 5 for H_e. Uniformity of F gives exactly 20 independent active rows at each vertex, with all other signed inequalities strict. All 81 sections (the nonnegative section and one reversal for each of the 80 rows) thus have dimension 40. Fixing $s = 1$ makes the two factor blocks independent. Each product-section vertex is a pair of factor-section vertices. The active equations in the x and y variables form two nonsingular blocks of size 20, giving exactly 40 independent active equations. The remaining signed inequalities are strict. On each product cone the two factor vectors belong to $C_0(F)$ or one $C_e(F)$ and share the coordinate s. Positivity of the factor functional forces $s \geq 0$; if $s = 0$, both factor vectors vanish. Thus $s > 0$ on every nonzero vector of all 81 product cones. Their sections in the common hyperplane $s = 1$ are bounded. The product may be nonuniform. For any 41-row set I, $\det(B_I + \delta W_I)$ has degree 41: its top coefficient $\det W_I$ is the Vandermonde determinant of 41 distinct integers and is nonzero. All finitely many maximal-minor polynomials are therefore nonzero for sufficiently small $\delta > 0$. Use Σ consisting of the all-positive pattern and each of its 80 single reversals. Lemma 4 applies: the functional s remains positive, the sections remain uniformly bounded, and their labeled graphs and shared faces are preserved. □

Proof of Theorem 1. Lemma 7 gives a uniform A, positivity of s, and bounded sections of dimension 40. Thus Lemma 3 applies. In the section of $C_0(A)$ choose the vertices labeled (U, U) and (V, V). Their zero sets are complementary 40-element sets and both are positive elsewhere. Suppose there were a full-graph walk of length $L \leq 41$ between them. Lemmas 2 and 3 confine its vertices and edges to the sections of $C_0(A)$ and one fixed $C_e(A)$. Lemma 7 transports the walk to B, preserving active rows and sign vectors. The common-face clause of Lemma 4(b) makes the two section maps agree on every shared vertex and edge. If the fixed coordinate e belongs to the first factor, the resulting walk is in $(H_0 \cup H_e)\square H_0$. The first-factor projection lies in $H_0 \cup H_e$ and the second in H_0. By Lemma 3 applied to F, both H_0 and H_e are subgraphs of $G^*(F)$. Project the whole walk to each factor and delete consecutive repetitions. Each edge moves exactly one factor, so the lengths of the projected walks sum to L. Lemma 6 requires at least 21 first-factor edges, and $d_{H_0}(U, V) = 21$ requires at least 21 second-factor edges. Thus $L \geq 42$, a contradiction. The second-factor case is identical. Every 41-row minor of A is nonzero, so its rank is 41. Its real sign covectors give the standard realizable oriented matroid, uniform by these minors and rationally realizable since A is rational. □

Remark 8. The distance between these two vertices is exactly 42. The two 21-edge paths in P, continued to H_0, give the path

$$(U, U), \ldots, (V, U), \ldots, (V, V)$$

in the product section. Lemma 7 preserves its edges after the second perturbation. This gives the upper bound 42, matching the lower bound in the proof of Theorem 1.

4 Formal verification

The Lean theorem `ADKZPricing.Generated.source_frozen_endpoint` asserts that there is $A' \in \mathbb{Q}^{80\times 41}$ with every 41-row minor nonzero and real signed cocircuits X, Y having complementary 40-element zero sets and positive entries elsewhere, such that no walk of at most 41 edges joins them. This is Theorem 1 for a different matrix from Section 3. The walks allow arbitrary signed intermediate cocircuits and repetitions; antipodal sign vectors are distinct vertices.

In the formal definitions, cocircuits are nonzero covectors minimal under conformality: a conformal face Z of X has $Z_i \in \{0, X_i\}$ for every i. Adjacency is defined by composition, $(X \circ Y)_i = X_i$ if $X_i \neq 0$ and Y_i otherwise: for distinct X, Y, the compositions $X \circ Y$ and $Y \circ X$ must agree and be a covector having exactly X, Y as cocircuit conformal faces. In the uniform case, Lean proves that conformality-minimality agrees with support-minimality and that this adjacency agrees with Section 1: empty separation set and 39 common zeros. Thus the formal walk bound has the stated full-graph meaning; the passage from real matrix sign covectors to the standard realizable oriented matroid is the usual mathematical interpretation.

Lean uses a different positive functional for the same M_0 and the same product-and-perturb strategy. Its rational perturbations are chosen nonconstructively; the endpoint theorem specifies A' existentially, without printing its entries. The printed Vandermonde perturbations are not checked in Lean; their proofs and Remark 8's upper bound are written. The formal statement, axioms and reproduction instructions are given in [12].

5 Further questions

The diameter of the full cocircuit graph of this example is unknown. What is the least counterexample rank? What sharp bound replaces $n - r + 2$?

Fukuda's linear-bound question remains [5, §3.7.2, PDF p. 16]; see also [2, Open Problem 5].

Appendix A An exact positive-functional certificate

This appendix specifies the source matrix M_0 and a positive functional for the cones obtained by reversing at most one row. Row and column labels are one-based: $1 \le i \le 40$, $1 \le j \le 21$; $e = 0$ denotes the nonnegative cone and $e = 1, \dots, 40$ reverses row e.

The integer source matrix

Let u_j denote the jth coordinate row of $\mathbb{Z}^{21}$ and let m_i be row i of M_0. For the rows in the following table set

$$m_i = s_i(u_1 + 100\eta_i u_2) + a_i u_3 + b_i u_4 + p_i u_5 + q_i u_6 + T_i, \qquad \eta_i = \begin{cases} 1 & i \le 20, \\ -1 & i \ge 21. \end{cases}$$

i	s_i	a_i	b_i	p_i	q_i	T_i
1	1	0	0	21	-7	$u_{17} + u_{18}$
2	1	0	0	16	-15	$u_{18} + u_{19}$
3	1	0	0	0	-32	$u_{19} + u_{20}$
4	1	0	0	-16	-15	$u_{20} + u_{21}$
5	1	0	0	-21	-7	0
6	1	0	0	-20	-4	u_{21}
7	1	0	0	0	32	0
8	1	0	0	20	-4	0
9	100	3	-2	0	-3000	0
10	100	-3	-2	0	3000	0
11	2000	-3	7	0	63600	$2000u_{16}$
21	1	30	0	0	0	$u_9 + u_{10}$
22	1	4	-15	0	0	$u_{10} + u_{11}$
23	2	0	-33	0	0	$2u_{11} + 2u_{12}$
24	1	-1	-16	0	0	$u_{12} + u_{13}$
25	2	-55	0	0	0	$2u_{13} + 2u_{14}$
26	1	-17	18	0	0	u_{14}
27	1	0	38	0	0	0
28	1	22	17	0	0	0
29	5	-50	0	1	-1	0
30	100	2999	0	-12	-20	$100u_9$
31	10000	299999	0	0	100	$10000u_8$
32	20000	-549000	0	4	25	$20000u_7$

The remaining seventeen rows are specified by two triangular recipes. Define

$$\alpha = 2000u_1 + 200000u_2 + 3u_3 + 7u_4 - 63600u_6 + 2000u_{15},$$
$$(t_7, \dots, t_{14}) = 2(10^{10}, 10^{10}, 10^{10}, 10^{13}, 10^{14}, 10^{14}, 10^{14}, 10^{14}).$$

Then

$$m_{12} = \alpha + \sum_{k=7}^{14} t_k u_k, \qquad m_{j+6} = \alpha + \sum_{k=7}^{j-1} t_k u_k - t_j u_j \quad (7 \le j \le 14).$$

Likewise define

$$\gamma = 2000u_1 - 200000u_2 - 54000u_3 + 4u_5 - 25u_6,$$
$$(h_{15}, \dots, h_{21}) = 2(10^8, 10^{10}, 10^{10}, 10^{10}, 10^{11}, 10^{11}, 10^{12}).$$

Then

$$m_{33} = \gamma + \sum_{k=15}^{21} h_k u_k, \qquad m_{j+19} = \gamma + \sum_{k=15}^{j-1} h_k u_k - h_j u_j \quad (15 \le j \le 21).$$

Empty sums are zero, and all entries not specified by these recipes are zero. These recipes give exactly the 40×21 integer matrix obtained by clearing each source row's reduced denominators by their least common multiple.

The functional and weights

Set $c(z) = 40000 \sum_{j=1}^{21} v_j z_j$, where the integers v_j are

j	v_j	j	v_j	j	v_j
1	7 889 496 505 278 816	8	16 770 653 889 890	15	11 008 963 417
2	-788 643 250 921 389 998	9	16 773 144 135 505	16	193 944 140 058
3	-322873814	10	2 918 818 204 673 914	17	570 600 902 642
4	73 889 176 985 687 161	11	2 039 103 229 854 373	18	574 335 730 279
5	0	12	1 964 406 684 567 592	19	186 741 363 964
6	-8 399 586 269 177	13	2 477 284 371 671 404	20	112 044 818 677
7	16 778 123 544 418	14	2 402 587 826 384 623	21	373 482 727 181

Let $\sigma_{0,i} = 1$ for every i; for $1 \le e \le 40$ let $\sigma_{e,e} = -1$ and $\sigma_{e,i} = 1$ for $i \neq e$. For each e, specify an increasing list $B_e = (\ell_1, \dots, \ell_{21})$ as follows:

$$K = \{2, 5, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 25, 27, 33, 35, 36, 37, 38, 39, 40\},$$
$$B_e = \operatorname{sort}\big((K \setminus R_e) \cup S_e\big).$$

In the table, $\mathcal{D} = \{2, 4, 6, 8, 10, 11, 22, 23, 24, 26, 28, 29, 30, 31, 32\}$. Each displayed comma-separated entry denotes a set, and a multi-label row assigns the same sets to every listed e.

e	R_e (delete from K)	S_e (add to K)
$0, 1$	$\varnothing$	$\varnothing$
$\mathcal{D}$	2	1
3	2, 19, 25	8, 12, 34
5	2, 5, 7, 38	1, 6, 12, 34
7	2, 7	1, 11
9	5, 7, 33, 38, 39	1, 4, 6, 11, 12
12	2, 5, 27	3, 4, 26
13	2, 5, 13, 16, 21, 25	4, 6, 12, 28, 30, 34
14	2, 14, 25, 27	3, 12, 28, 32
15	2, 5, 15, 21, 25, 27	4, 12, 26, 28, 31, 34
16	2, 5, 16, 21, 25	3, 12, 30, 31, 34
17	2, 5, 17, 18	3, 6, 22, 26
18	2, 5, 13, 18, 19	3, 4, 12, 22, 26
19	2, 5, 19, 25, 27	3, 6, 12, 26, 34
20	2, 5, 16, 18, 20	3, 4, 12, 22, 24
21	2, 5, 13, 15, 21, 25	1, 6, 12, 22, 32, 34
25	5, 7, 25, 33, 38, 39	1, 4, 6, 12, 32, 34
27	2, 5, 19, 27	1, 6, 26, 34
33	2, 5, 7, 25, 33, 37	1, 4, 8, 11, 12, 24
34, 38, 39	5, 33, 38, 39	1, 4, 6, 12
35	2, 5, 7, 33, 35, 38	1, 3, 6, 8, 11, 12
36	5, 36, 39	1, 6, 12
37	5, 36, 37	3, 4, 12
40	2, 35, 38, 40	1, 3, 6, 12

Write $c^{\mathsf{T}} = 40000(v_1, \dots, v_{21})^{\mathsf{T}}$ and define

$$N_e = [\, \sigma_{e,\ell_1} m_{\ell_1}^{\mathsf{T}} \ \cdots \ \sigma_{e,\ell_{21}} m_{\ell_{21}}^{\mathsf{T}} \,], \qquad \Delta_e = \det N_e,$$
$$d_e = c^{\mathsf{T}} - \sum_{i=1}^{40} \sigma_{e,i} m_i^{\mathsf{T}}.$$

Let $N_e[r \leftarrow d_e]$ mean replace column r of N_e by d_e, and put

$$\lambda_{e,i} = \begin{cases} 1 + \det(N_e[r \leftarrow d_e])/\Delta_e, & i = \ell_r \ (1 \le r \le 21), \\ 1, & i \notin B_e. \end{cases}$$

Evaluating these integer determinants exactly gives

$$\Delta_e \neq 0 \quad (0 \le e \le 40), \qquad \frac{\det(N_e[r \leftarrow d_e])}{\Delta_e} > 8 \quad (0 \le e \le 40,\ 1 \le r \le 21).$$

Thus all weights are at least 1 (basis weights exceed 9), and Cramer's rule gives $c = \sum_{i=1}^{40} \lambda_{e,i}\sigma_{e,i}m_i$ for every e. Also $\Delta_0 \neq 0$ gives $\operatorname{rank} M_0 = 21$. If $z \neq 0$ has all signed slacks $\sigma_{e,i}m_i z \ge 0$, full column rank makes at least one slack positive, so $c(z) > 0$. Every datum needed for these finite checks is printed in this appendix. An exact checker of these formulas is included in [12].

For example, the first row is $m_1 = u_1 + 100u_2 + 21u_5 - 7u_6 + u_{17} + u_{18}$. The supplement's exact checker reconstructs every entry from these recipes, compares them with the rational inequality data, and verifies the displayed weight identities. The positive functional used in Lean differs from the one printed here.

Appendix B Completeness of the vertex certificate

The following local certificate criterion explains the completeness fact used in Section 3. The large finite list is in [12]; completeness is a consequence, not a premise supplied by the list. The labeled pair, potential, path and local source data are supplied only in the supplement and checked there; their large tables are not printed here. The pair has the complementary-zero-set form used in [4, proof of Proposition 8].

Lemma 9. *Let $\mathcal{Q} = \{x \in \mathbb{R}^d : Lx \ge \zeta\}$ be full-dimensional. Suppose a nonempty finite set $\mathcal{V} \subset \mathcal{Q}$ is given, each $v \in \mathcal{V}$ having exactly d tight rows, whose matrix L_v is invertible. For each of these rows j, suppose a point $v_j \in \mathcal{V}$ satisfies*

$$L_v(v_j - v) = g_j \mathbf{e}_j, \qquad g_j > 0,$$

where $\mathbf{e}_j$ is the jth coordinate vector. Then $\mathcal{V}$ is the complete vertex set, $\mathcal{Q}$ is simple, and its neighbors at v are precisely the points v_j.

Proof. For a linear objective f, choose $v \in \mathcal{V}$ maximizing f on this finite set. For every $x \in \mathcal{Q}$, the vector $\kappa = L_v(x - v)$ has nonnegative coordinates and

$$x - v = \sum_{j=1}^{d} \frac{\kappa_j}{g_j}(v_j - v).$$

Since $f(v_j - v) \le 0$, this proves $f(x) \le f(v)$: the list contains a global maximizer for every linear objective.

Let x be any vertex and choose d independent tight rows I there. The objective $f(y) = -\sum_{i \in I} L_i y$ is maximized precisely where all these rows are tight, because their slacks are nonnegative. Its unique feasible maximizer is x. A listed maximizer must therefore equal x. Conversely each listed point is a vertex by its nonsingular tight system, so the list is complete and every vertex has exactly d tight rows. Each such row defines a distinct facet locally. Releasing one row while retaining the other $d-1$ defines an edge. The distinct vertices v, v_j lie on that one-dimensional face and hence are its two endpoints. Every edge at a simple vertex arises this way. □

Here are the exact formal interfaces for these inputs, in the Lean development linked below the title; all names have prefix `ADKZPricing`. For the literal matrix M_0, `Generated.source_valid` proves `IntegerSourceValid` for the supplied 36,425 rays, their 20-element bases and their neighbors. Its field hypotheses are checked finite arithmetic: positive normalizing coordinates, feasible rays, injective and exact active labels, and a neighbor retaining every basis row except the released one. The resulting `Generated.sourceCertificate` is a `SourceVertexCertificate`. For this certificate, the theorems `SourceVertexCertificate.covers_active_labels`, `SourceVertexCertificate.active_count` and `SourceVertexCertificate.basis_nonsingular` give

$$\mathrm{Vert}(P) = \{w_v : v \in \mathcal{V}\}, \qquad |I(w_v)| = 20, \qquad \mathrm{rank}(M_0)_{I(w_v)} = 20,$$

where $\mathcal{V}$ is the supplied list, w_v its ray normalized by $c(w_v) = 1$, and $I(w_v)$ its tight-row set. In particular, P is simple. The normalization changes only the scale of each ray, preserving vertices, facets and their incidence; Lemma 9 is the corresponding written local-to-global argument.

The theorem `msw_integral_real_vertices` verifies the endpoints with tight rows 1–20 and 21–40. The theorems `Generated.source_local_potential` and `Generated.source_specified_gap` verify the local potential inequalities and endpoint gap at least 21. Applying `SourceVertexCertificate.potential_edge` and `SourceVertexCertificate.specified_distance_lower_bound` therefore gives a 1-Lipschitz potential on every geometric edge and $d_P(u, v) \geq 21$, with no assumed completeness or distance premise. The separate exact checkers `check_fraction.py` and `check_rational.py` (folder `checks` of [12]) verify the supplied 21-edge path, giving $d_P(u, v) = 21$; this upper bound is not a Lean distance-equality theorem. These source facts and Appendix A's identities are the complete computational input to the written construction. No enumeration of the final 80-element cocircuit graph is used.

Appendix C Earlier farthest-pair questions

The diameter conjecture is related to earlier questions about which pairs of cocircuits are farthest apart; we distinguish those questions here. For uniform oriented matroids, Babson, Finschi and Fukuda prove antipodal distance $n - r + 2$ [1, Lemma 2] and equality exactly for antipodal pairs in uniform ranks two and three [1, Lemma 6]. ADKZ answer Finschi's Open Problem 2 and refute the equality half of [3, Question 4.2] with their nonantipodal example [4, Proposition 8 and preceding paragraph]. Their proof already uses a pair at distance 21 in the source polytope; here an explicit path gives equality for the source matrix specified in Appendix A. Finschi's Open Problem 4 asks whether a uniform cocircuit graph can have an antipodal pair whose distance differs from the diameter, or a nonantipodal pair attaining the diameter [2, p. 90]; this is the farthest-pairs question in [1, final remarks, pp. 13–14]. ADKZ's lower-bound argument answers it affirmatively once the source example is made uniform, as in Lemmas 5–6: if the diameter is 21, the nonantipodal pair attains it; if larger, antipodal pairs are not farthest. Here every antipodal pair is shorter than some nonantipodal pair. This refutes the diameter half of Question 4.2 and ADKZ's Conjecture 1.

A *crabbed path* from X to Y has $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$ at every vertex Z, where $Z^+ = \{i : Z_i = 1\}$ and $Z^- = \{i : Z_i = -1\}$. ADKZ exhibit a realizable rank-four example on nine elements in which a shortest path is shorter than every crabbed path [4, Theorem 9].

Adjacent work concerns e-positive circuit graphs of oriented matroid polyhedra and embracing exchange distance (Bérczi, Knauer, Montejano and Nádor [9, Corollary 1.4 and Definition 2.1]), smaller non-Hirsch polytopes (Świrszcz and collaborators [10, §4.1]), and polyhedron certification (Allamigeon, Id-Sahra and Strub [11, abstract and §1]).

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