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COMMON PERIODIC POINTS OF COMMUTING INTERVAL MAPS

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Abstract. We construct pairs of commuting continuous self-maps of the compact interval with disjoint periodic-point sets. More precisely, every finite union of sets of positive multiples, including the empty union, is the set of integers n for which two commuting nth iterates have a common fixed point. In the other direction, we show that if either generator has zero topological entropy, the generators act as mutually inverse cyclic permutations on a common finite orbit. We also establish a common fixed-point theorem for two commuting maps that are each monotone on a finite partition, with constant pieces allowed.

1. Introduction

Throughout, $I = [0, 1]$ and composition is written from right to left. For a continuous map $f : I \to I$ put $\mathrm{Fix}(f) = \{x : f(x) = x\}$ and $\mathrm{Per}(f) = \bigcup_{n\geq 1} \mathrm{Fix}(f^n)$. For a commuting pair f, g define its *common return exponents* by

$$\mathcal{R}(f, g) = \{n \in \mathbb{N} : \mathrm{Fix}(f^n) \cap \mathrm{Fix}(g^n) \neq \varnothing\}.$$

A point is periodic for both generators if and only if it belongs to $\mathrm{Fix}(f^n) \cap \mathrm{Fix}(g^n)$ for some common n; hence $\mathrm{Per}(f) \cap \mathrm{Per}(g) = \varnothing$ if and only if $\mathcal{R}(f, g) = \varnothing$. Necessarily $n \in \mathcal{R}(f, g)$ implies $kn \in \mathcal{R}(f, g)$ for every $k \geq 1$.

Boyce [6] and Huneke [11] constructed commuting interval maps without a common fixed point. This left open the possibility of a common periodic point, formulated explicitly by Alikhani–Koopaei [1, Conjecture 1.1], revisited in [2], and included among the questions in [7, Problem 9.4]. Our first theorem answers that question.

Theorem 1.1 (Prescribed common return exponents)**.** *For every finite set $D \subset \mathbb{N}$, including $D = \varnothing$, there are commuting continuous maps $F, G : I \to I$ such that*

$$\mathcal{R}(F, G) = \bigcup_{d\in D} d\mathbb{N}.$$

In particular, there exist commuting continuous interval maps with $\mathrm{Per}(F) \cap \mathrm{Per}(G) = \varnothing$.

For a continuous map f, denote the ω-limit set of x by $\omega_f(x)$. The following is an independent positive result.

Theorem 1.2 (A common cycle in the fixed set of the composition)**.** *Let $f, g : I \to I$ be commuting continuous maps, and set $K = \mathrm{Fix}(fg)$. Suppose that for each $x \in K$ with infinite $\omega_f(x)$ there exist disjoint closed intervals $L_x, R_x \subset I$ with*

$$f(L_x) \subset R_x, \qquad f(R_x) \subset L_x, \qquad \omega_f(x) \subset L_x \cup R_x.$$

OpenAI's ChatGPT models discovered and proved all three main theorems and formalized them in Lean 4. With minor human direction, GPT 5.6 found the proof of Theorem 1.3 in about two weeks and formalized it in about two days. Given that proof, GPT Astra 6.2 Ultra was asked to determine whether two commuting continuous maps of the unit interval must share a periodic point. With an average of six agents working concurrently, it proved Theorems 1.1 and 1.2 in approximately 24 hours. Theorem 1.1 gives a negative answer. The same model formalized both theorems in Lean 4 in a total of 30 minutes.

Then K contains a finite orbit on which f and g act as mutually inverse cyclic permutations. In particular, this conclusion holds if $h_{\mathrm{top}}(f) = 0$ or $h_{\mathrm{top}}(g) = 0$.

A complementary regularity theorem shows that counterexamples to a common fixed point cannot be finitely piecewise monotone in both generators.

Theorem 1.3 (The finite-piece fixed-point theorem)**.** *Let $P, Q : I \to I$ be commuting continuous maps. If each is monotone (not necessarily strictly) on every member of a finite interval partition, then*

$$\mathrm{Fix}(P) \cap \mathrm{Fix}(Q) \neq \varnothing.$$

The strong-commutation version for strictly piecewise-monotone maps was established by Anušić and Mouron [4, Corollary 5.24]. Ordinary commutation is weaker. Theorem 1.3 covers ordinary commutation and permits constant branches. The contrast with Theorem 1.1 is sharp at the level of finite versus unrestricted complexity of the generators.

The proof of Theorem 1.1 occupies Sections 2 and 3: the first uses finite cell permutations, while the second constructs a compatible system of maps on increasingly fine partitions. The dynamical criterion in Theorem 1.2 is proved in Section 4, and Section 5 proves the piecewise-monotone fixed-point theorem using constant-slope factors and substitution dynamics.

The index of Lean proofs links Theorems 1.1–1.3 to their formal statements and proofs.

2. Prescribing nonempty common return exponents

Proof of Theorem 1.1 when $D \neq \varnothing$. Set $m = \sum_{d \in D} d$. Partition I into $2m$ equal closed cells, colored alternately $1, 2$ (with disjoint interiors). For each color, choose a permutation P of its m cells having exactly one cycle of length d for each $d \in D$. The two permutations together define a color-preserving permutation of all cells.

For each cell J, choose an affine embedding $s_J : I \to \mathrm{int}(J)$. For each cell J choose a continuous decoder $A_J : J \to I$ satisfying

$$A_J s_J = \mathrm{id}_I. \tag{1}$$

At an endpoint e of J shared with the neighboring, opposite-color cell K, prescribe

$$A_J(e) = s_{P(K)}(0). \tag{2}$$

The values at an exterior endpoint may be prescribed arbitrarily. Interpolation on the two components of $J \setminus s_J(I)$, joining the endpoint marks to 0 or 1 at the appropriate end of the core, produces a decoder with the stated properties.

For $i \in \{1, 2\}$ define $F_i : I \to I$ on its own color-i cells by $F_i|_J = A_J$. On a cell K of the other color require

$$F_i|_K = s_{P(K)}\, F_i\, A_K. \tag{3}$$

Here and below, restrictions and identities are equalities of maps with the indicated domains. To solve (3), consider the closed subset of $C(I, I)$ consisting of maps with the prescribed color-i branches. It is nonempty: at each mixed endpoint, the opposite-color branch is assigned the same required endpoint value, so the remaining branches may be interpolated continuously. The update in (3) preserves these endpoint values. Indeed, at a junction between J of color i and K of the other color, $A_K(e) = s_{P(J)}(0)$ and $F_i(s_{P(J)}(0)) = 0$ by (1). Thus the updated value on K is $s_{P(K)}(0) = A_J(e)$. The uniform distance between two updates is at most

$$q\|U - V\|_\infty, \qquad q := \max_J \mathrm{Lip}(s_J) < 1.$$

Banach's theorem supplies the unique continuous F_i.

If J has color i and $j \neq i$, then, on J,

$$F_j F_i = F_j A_J, \qquad F_i F_j = F_i s_{P(J)} F_j A_J = F_j A_J.$$

This proves $F_1 F_2 = F_2 F_1$ on every cell. Moreover, for a cell J of color i and $j \neq i$,

$$F_j(J) \subset s_{P(J)}(I) \subset \operatorname{int} P(J). \tag{4}$$

If x is fixed by both F_1^n and F_2^n, choose a cell J containing x and let j be its opposite color. Iterating (4) gives $x = F_j^n(x) \in \operatorname{int} P^n(J)$. The point x therefore lies in the interior of its original cell and $P^n(J) = J$. Thus the cycle length of J, which belongs to D, divides n.

Conversely, the affine contraction s_J has a unique fixed point $z_J \in \operatorname{int}(J)$. By (1) the map of the same color fixes z_J. The opposite-color relation is

$$F_j s_J = s_{P(J)} F_j,$$

so $F_j(z_J)$ is the unique fixed point of $s_{P(J)}$ and is therefore $z_{P(J)}$. Along a P-cycle of length d this gives a common point of return exponent d. All multiples of d are return exponents. The two inclusions prove the assertion. □

3. Commuting maps without common periodic points

The empty case cannot be obtained by omitting the cycles from the finite construction. Instead we arrange that a hypothetical common periodic point of exponent n lifts, with the *same* exponent, to grades where the required cycle lengths grow without bound.

3.1. **Graded cells, successors, and sections.** Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, let $v = (a, b) \in \mathbb{N}_0^2$, and set

$$I_v = I, \qquad M_v = 5^a 7^b, \qquad e_1 = (1, 0), \quad e_2 = (0, 1).$$

The index v is a grade: the spaces I_v are separate copies of I. Put $\sigma(t) = \operatorname{dist}(t, 2\mathbb{Z})$; this is the triangular folding map $\mathbb{R} \to I$, with period 2. At grade v define the closed cells

$$J_{v,\ell,k} = \chi_{v,\ell,k}(I), \qquad \chi_{v,\ell,k}(u) = \sigma\left(\frac{2k + (\ell + u)/4}{M_v}\right), \quad 0 \leq \ell \leq 3, \quad k \in \mathbb{Z}/M_v\mathbb{Z}. \tag{5}$$

Each $\chi_{v,\ell,k}$ is an affine homeomorphism onto its cell, possibly reversing orientation. These $4M_v$ cells tile I_v with disjoint interiors. For example, $\sigma(M_v x)$ maps every cell homeomorphically onto the ℓth quarter of I. Color $\ell = 0, 2$ by 1, and $\ell = 1, 3$ by 2. The distinguished quarter for color 1 is $\ell = 2$; for color 2 it is $\ell = 1$.

For a color-i cell J set

$$s_J : I_{v+e_i} \longrightarrow J, \qquad s_J(t) = \chi_J\left(\frac{1 + t}{3}\right). \tag{6}$$

Its image is the *core* $\operatorname{core} J$; the other two closed thirds are the *shoulders*. The core has length $|J|/3$, and in particular

$$\operatorname{Lip}(s_J) = \frac{1}{12 M_v} \leq \frac{1}{12}. \tag{7}$$

For $w \in \mathbb{N}_0^2$, the projection $\pi_{v,w}$ of a cell of grade $v + w$ to grade v keeps ℓ and reduces k modulo M_v. If $d = M_{v+w}/M_v$ (always odd), then the standard fold

$$\varphi_{v,w} : I_{v+w} \to I_v, \qquad \varphi_{v,w}(x) = \sigma(dx) \tag{8}$$

maps each fine cell, its core, and its shoulders onto the projected coarse cell and corresponding pieces, preserving the parameter u in (5). This follows from $\sigma(d\,\sigma(t)) = \sigma(dt)$ for odd positive d.

For a grade $v = (a, b)$ express k in Chinese-remainder coordinates

$$k_5 = k \pmod{5^a}, \qquad k_7 = k \pmod{7^b}.$$

Define a color-preserving *successor* P_v as follows:

$$(2, k_5, k_7) \longmapsto (2, k_5 + 1, 0) \qquad \text{(color 1)}, \tag{9}$$
$$(1, k_5, k_7) \longmapsto (1, 0, k_7 + 1) \qquad \text{(color 2)},$$

where active coordinates are reduced modulo 5^a or 7^b, respectively. Every other cell goes to the distinguished cell of its own color with residue zero. The only cycles of P_v have length 5^a in color 1 and 7^b in color 2; all other cells enter a cycle in one step. Directly from the coordinate rules,

$$\pi_{v,w} P_{v+w} = P_v \pi_{v,w}. \tag{10}$$

We also need a lift of a successor to a neighboring grade. If K is a cell of color $j \neq i$ at grade v, denote by $\widehat{P}_{v,i}(K)$ the distinguished color-j cell at grade $v + e_i$ whose active coordinate is the active coordinate of $P_v(K)$ and whose other coordinate is zero. Since increasing the ith coordinate does not change the modulus active for color j, this definition is unambiguous. It satisfies

$$\pi \widehat{P}(K) = \widehat{P}(\pi K) \tag{11}$$

for compatible reductions (the grades are understood). The three combinatorial identities (10), (11), and preservation of the color are the only properties of the successor labels used later.

3.2. **Decoders and explicit shoulder models.** A cell J of color i at grade v will have a continuous decoder

$$A_J : J \longrightarrow I_{v+e_i}, \qquad A_J s_J = \mathrm{id}_{I_{v+e_i}}. \tag{12}$$

We first prescribe the endpoint values. If an endpoint e of J is shared with an opposite-color cell K, set

$$A_J(e) = s_{\widehat{P}_{v,i}(K)}(0). \tag{13}$$

At a junction of two same-color cells, and at a global endpoint of I_v, set $A_J(e) = s_{K_*}(0)$, where K_* is the distinguished opposite-color cell of residue zero at grade $v + e_i$. The two cells meeting at a same-color junction have the same nondistinguished quarter and hence receive equal marks. Thus adjacent decoders have compatible endpoint marks whenever such compatibility will be needed.

The interpolation on each shoulder is chosen in a specific way. Suppose its outer endpoint is marked by $s_H(0)$ for a cell H in the decoder target. Write the mark in unfolded coordinates:

$$\beta = \frac{2k_H + (\ell_H + 1/3)/4}{M_{v+e_i}}, \qquad s_H(0) = \sigma(\beta). \tag{14}$$

Choose the displayed real representative of β. Let $E \in \{0, 1\}$ be the decoder value required at the adjacent end of the core (by (12), this is 0 or 1). Let B be the least integer $B \geq \beta$ satisfying $\sigma(B) = 1 - E$. Parametrize the physical shoulder affinely by $[\beta, B + 1]$, sending β to the outer endpoint and $B + 1$ to the core endpoint. In this parameter define

$$A_J(t) = \sigma(t). \tag{15}$$

The value at β is the prescribed mark, the value at $B + 1$ is E, and the final unit interval $[B, B + 1]$ maps once across I. Together with the inverse section on the core, this defines A_J continuously on the whole cell.

Here is a useful intrinsic form of the shoulder construction. Define

$$q_{\beta,B} : [\beta, \infty) \longrightarrow [\beta, B + 1] \tag{16}$$

by fixing $[\beta, B+1]$ and, for $t \geq B$, selecting the unique point of $[B, B+1]$ with the same σ-value as t. This is well-defined, is 1-Lipschitz, and satisfies

$$\sigma q_{\beta,B} = \sigma, \qquad q_{\beta,B}^{-1}(u) = \{u\} \quad (\beta \leq u < B). \tag{17}$$

The physical shoulder may be regarded as the quotient of the ray $[\beta, \infty)$ obtained by identifying points of the tail $[B, \infty)$ with equal σ-values. The quotient coordinate is $q_{\beta,B}$, followed by affine rescaling to the physical shoulder. The quotient description is what makes the inter-grade compatibility argument below work.

Lemma 3.1 (Compatibility of decoder marks)**.** *Let $w \in \{e_1, e_2, e_1+e_2\}$, let J' be a cell of grade $v+w$, and let $J = \pi_{v,w}J'$. On corresponding shoulders, write (β', B', E') and (β, B, E) for the parameters in (14)–(15). If $d = M_{v+w}/M_v$, then*

$$E' = E, \qquad d\beta' = \beta + 2m \quad \textit{for some } m \in \mathbb{Z}. \tag{18}$$

Proof. The core coordinate u is preserved under the standard fold, so the decoder's required core-end value is the same E. For a mixed junction, the incident opposite-color cell projects to the incident opposite-color cell at the coarse grade. The successor lift commutes with this projection by (11). At a same-color junction, its image is either a same-color junction or a global endpoint: in both cases the target mark is the distinguished opposite-color residue-zero section. Global endpoints behave the same way, since d is odd. Thus the cell H' defining the fine mark projects to the cell H defining the coarse mark, with the same quarter index. In the target grades the degree is again d, so the unfolded chart (5) gives

$$d\frac{2k_{H'} + (\ell_H + 1/3)/4}{M_{v+w+e_i}} = \frac{2k_H + (\ell_H + 1/3)/4}{M_{v+e_i}} + 2\frac{k_{H'} - k_H}{M_{v+e_i}}.$$

The last quotient is an integer because the projected residue is k_H. This is (18). □

3.3. **Admissible projection maps and their unfoldings.** The notation in the next lemma makes the grades explicit. If $w \in \mathbb{N}_0^2$, a map from I_{v+w} to I_v has *degree* $d = M_{v+w}/M_v$.

Definition 3.2. For $w \in \{e_1, e_2, e_1+e_2\}$, a continuous map $r : I_{v+w} \to I_v$ is *admissible* if it maps each fine cell, core, and shoulder into the respective projected piece and agrees with $\varphi_{v,w}(x) = \sigma(dx)$ at every endpoint of every fine cell, core, and shoulder.

The ordinary fold $\varphi_{v,w}$ is admissible. The following construction separates the two delicate assertions needed later: *control of the contraction* and *compatibility with composition*.

Lemma 3.3 (Canonical unfoldings)**.** *An admissible $r : I_{v+w} \to I_v$ of degree d has a unique continuous unfolded map $H_r : \mathbb{R} \to \mathbb{R}$ such that*

(i) $\sigma H_r(t) = r(\sigma t)$;
(ii) for every interval U bounded by consecutive lifts of fine cell or core endpoints, $H_r(U) \subset dU$;
(iii) $H_r(z) = dz$ at all such vertices.

It satisfies

$$H_r(t + 2k) = H_r(t) + 2dk \qquad (k \in \mathbb{Z}). \tag{19}$$

If $r, \tilde{r}$ are admissible of the same grades, then

$$\sup_{t \in \mathbb{R}} |H_r(t) - H_{\tilde{r}}(t)| = \|r - \tilde{r}\|_\infty. \tag{20}$$

If r and s are composable admissible projections of odd degrees d_1, d_2, their unfoldings satisfy $H_{sr} = H_s H_r$.

Proof. The subdivision lifts of the cells and cores to $\mathbb{R}$ are aligned with the turning points of σ. For each subdivision interval U, the interval dU is contained in a single monotonicity interval of σ; on it σ is an isometry onto the corresponding coarse piece. Thus (i) and (ii) determine H_r uniquely on U. At a common endpoint z the admissible endpoint prescription gives $H_r(z) = dz$ from both sides, proving (iii) and continuity. Translation by 2 in the domain and by $2d$ in the range yields another lift with the same prescriptions, giving (19) by uniqueness. On each U, the two values $H_r(t)$ and $H_{\tilde{r}}(t)$ lie on the same monotonicity sheet of σ. Their distance consequently equals $|r(\sigma t) - \tilde{r}(\sigma t)|$. Taking suprema proves (20).

For composition, $H_s H_r$ has the required folded values and endpoint values. It also maps every fine subdivision interval into its appropriately dilated interval, because each of the two admissible maps preserves the corresponding cells, cores, and shoulders. Uniqueness of the chosen sheets then identifies it with H_{sr}. □

Lemma 3.4 (Shoulder descent and functoriality)**.** *Let J' be a color-j cell at grade $v + w$ and $J = \pi_{v,w} J'$. An admissible map $r : I_{v+w+e_j} \to I_{v+e_j}$ induces, for each shoulder, a continuous map from the shoulder of J' into the corresponding shoulder of J. The induced map agrees with the standard fold at the shoulder endpoints and obeys*

$$A_J \, \Theta(r) = r \, A_{J'} \tag{21}$$

on that shoulder. The construction respects composition: for composable admissible maps r, s,

$$\Theta(s)\Theta(r) = \Theta(sr). \tag{22}$$

With physical shoulder coordinates and the supremum norm, the dependence on r is Lipschitz with constant at most $1/12$.

Proof. Write the matching fine and coarse shoulder parameters as (β', B', E) and (β, B, E), and use Lemma 3.1 to choose m such that $d\beta' = \beta + 2m$. Let $H = H_r$, where here r is the map between the *decoder targets*, also of degree d. In the unfolded coordinates the induced map is

$$t \longmapsto q_{\beta,B}(H(t) - 2m), \qquad \beta' \le t \le B' + 1. \tag{23}$$

After this formula, apply the affine map from $[\beta, B + 1]$ to the physical coarse shoulder.

Indeed, β' is a subdivision vertex in the decoder target. Hence $H(\beta') = d\beta'$ and the expression inside q starts at β. Since H maps each subdivision interval into its d-dilate, it cannot pass backwards through a subdivision vertex: for every vertex z and every $t \ge z$, one has $H(t) \ge dz$. At the integer B' this gives

$$H(B') - 2m = dB' - 2m \ge \beta.$$

The integer on the left has the same parity as B' because d is odd. Since E agrees in the two shoulder models, its σ-value is $1 - E$; by the defining minimality of B, it is at least B. Therefore the whole fine tail $[B', \infty)$ maps into the coarse tail $[B, \infty)$. Formula (23) is continuous, and its initial value is β. At $B' + 1$ its σ-value is E, so the quotient coordinate is the terminal value $B + 1$. Thus the two physical endpoints are preserved. Moreover,

$$\sigma \, q_{\beta,B}(H(t) - 2m) = \sigma H(t) = r(\sigma t),$$

which is exactly (21).

We verify composition rather than appealing to an implicit choice of lifts. For two successive degrees d_1, d_2, the starting marks satisfy

$$d_1 \beta'' = \beta' + 2m_1, \qquad d_2 \beta' = \beta + 2m_2,$$

so $d_1 d_2 \beta'' = \beta + 2(m_2 + d_2 m_1)$. By (19), the unreduced ray map for the composite is

$$H_s(H_r(t)) - 2(m_2 + d_2 m_1) = H_s(H_r(t) - 2m_1) - 2m_2.$$

Taking the intermediate quotient before applying H_s changes this input only when it lies in the intermediate tail. On that tail, the two inputs have the same σ-value; their images under H_s also have the same σ-value, and both lie in the final tail. The final quotient therefore identifies them. Outside the intermediate tail, its quotient map is the identity. This proves (22) pointwise.

Finally, in (23) the translation $2m$ is independent of the admissible map; the unfolding is an isometry in the supremum norm by (20), and $q_{\beta,B}$ is 1-Lipschitz. The physical coarse shoulder has length $1/(12M_v) \le 1/12$, while the parameter interval $[\beta, B+1]$ has length at least 1. Hence its affine rescaling has Lipschitz constant at most $1/12$. This is the claimed bound. □

Proposition 3.5 (Compatible projections)**.** *For every $w \in \{e_1, e_2, e_1+e_2\}$ and every grade v there exists an admissible projection*

$$\mathsf{R}_v^{(w)} : I_{v+w} \longrightarrow I_v$$

such that, for every color-j cell J' at grade $v+w$, with $J = \pi_{v,w}J'$, one has

$$\mathsf{R}_v^{(w)} s_{J'} = s_J \mathsf{R}_{v+e_j}^{(w)}, \tag{24}$$

$$A_J \mathsf{R}_v^{(w)}|_{J'} = \mathsf{R}_{v+e_j}^{(w)} A_{J'}. \tag{25}$$

The degree-5 and degree-7 families form commuting squares:

$$\mathsf{R}_v^{(e_1)}\mathsf{R}_{v+e_1}^{(e_2)} = \mathsf{R}_v^{(e_2)}\mathsf{R}_{v+e_2}^{(e_1)} \quad \text{on } I_{v+e_1+e_2}. \tag{26}$$

Proof. Fix w. Consider the product, over all grades v, of the spaces of admissible maps $I_{v+w} \to I_v$, with metric

$$d(\mathbf{r}, \widetilde{\mathbf{r}}) = \sup_v \|r_v - \widetilde{r}_v\|_\infty.$$

The product is nonempty (it contains the standard folds), and is complete: each admissibility condition is closed under uniform convergence, and all maps take values in I.

Define an update Φ_w on this product cell by cell. For J' of color j at grade $v+w$, with coarse cell $J = \pi_{v,w}J'$, set on the core

$$(\Phi_w \mathbf{r})_v s_{J'} = s_J r_{v+e_j}. \tag{27}$$

On each shoulder use the map $\Theta(r_{v+e_j})$ of Lemma 3.4. On both ends of each piece these prescriptions agree with the standard fold; thus they join continuously across cores, shoulders, and neighboring cells. They still preserve the projected pieces, so Φ_w maps the admissible product into itself.

On a core, (27) and (7) yield a contraction factor at most $1/12$; on a shoulder the same factor follows from Lemma 3.4. Consequently,

$$d(\Phi_w \mathbf{r}, \Phi_w \widetilde{\mathbf{r}}) \le \frac{1}{12} d(\mathbf{r}, \widetilde{\mathbf{r}}). \tag{28}$$

There is a unique fixed family $\mathbf{r} = \{\mathsf{R}_v^{(w)}\}_v$. Its core definition gives (24), and the shoulder descent identity together with $A_J s_J = \mathrm{id}$ gives (25).

We justify the commuting-square identity carefully. Consider the two families of degree-35 maps

$$U_v = \mathsf{R}_v^{(e_1)}\mathsf{R}_{v+e_1}^{(e_2)}, \qquad V_v = \mathsf{R}_v^{(e_2)}\mathsf{R}_{v+e_2}^{(e_1)}.$$

Each family is admissible: the successive maps preserve the relevant projected pieces and the prescribed endpoint folds compose, since $\sigma(d_1\sigma(d_2 x)) = \sigma(d_1 d_2 x)$ for odd degrees. Substituting (24) twice shows that U and V satisfy the degree-35 update equation on every core. On shoulders, the update equation follows from (22); in particular, the even translations of the unfolded starting marks do not introduce an extra shift. Thus both U and V are fixed points of $\Phi_{e_1+e_2}$. By (28) its fixed family is unique, so $U_v = V_v$ for every v. □

3.4. **The decoder families and the commuting interval maps.**

Proposition 3.6. *For $i = 1, 2$ and all v there are continuous maps $D_{i,v} : I_v \to I_{v+e_i}$ such that*

$$D_{i,v}|_J = A_J \qquad (J \text{ has color } i), \tag{29}$$

$$D_{i,v}|_K = s_{\widehat{P}_{v,i}(K)} D_{i,v+e_j} A_K \qquad (K \text{ has color } j \neq i). \tag{30}$$

They satisfy the graded commutation and intertwining relations

$$D_{1,v+e_2} D_{2,v} = D_{2,v+e_1} D_{1,v}, \tag{31}$$

$$\mathsf{R}^{(w)}_{v+e_i} D_{i,v+w} = D_{i,v} \mathsf{R}^{(w)}_v, \qquad w \in \{e_1, e_2, e_1 + e_2\}. \tag{32}$$

Proof. For fixed i start with families of continuous maps $I_v \to I_{v+e_i}$ whose branches on color-i cells are the prescribed decoders and which match the values required at their endpoints. They exist by interpolation; the same-color junctions have equal decoder marks by construction. Update all the opposite-color branches by (30), keeping (29) fixed. At a mixed junction of a color-i cell J and an opposite-color cell K, the latter has $A_K(e) = s_{\widehat{P}_{v,j}(J)}(0)$. The inner $D_{i,v+e_j}$ is a decoder on the indicated color-i cell and sends this section value to 0. The updated branch on K therefore takes the value $s_{\widehat{P}_{v,i}(K)}(0) = A_J(e)$, as required. At a same-color junction, both cells have the same nondistinguished quarter and successor, hence the same outer section; their decoder endpoint marks also agree. The update is thus continuous. Its uniform contraction factor is at most $1/12$ by (7). Banach's theorem, applied simultaneously over all grades, gives the unique family $D_{i,v}$.

For (31), take a cell J of color i, with $j \neq i$. One composition restricts to $D_{j,v+e_i} A_J$. In the reverse composition, $D_j|_J$ begins with an outer section of a color-i cell, and the next D_i cancels that section by $A_H s_H = \mathrm{id}$. The resulting expression is again $D_{j,v+e_i} A_J$. This proves the identity on all cells.

For (32), the two sides agree on color-i cells by (25). On a color-j cell K with $j \neq i$, substitute (30) on both sides, and use (24) and (11) to pull their common outer section to grade v. After this outer section is removed, the difference is precisely the difference between the two sides of (32) one grade higher, at $v + e_j$. Taking the supremum of the norms of these differences over all grades gives $\Delta \leq (1/12)\Delta$, hence $\Delta = 0$. There is no circularity: this comparison is between already constructed continuous families. □

Define maps on each grade by

$$F_{i,v} := \mathsf{R}^{(e_i)}_v D_{i,v} : I_v \longrightarrow I_v. \tag{33}$$

Corollary 3.7. *The maps $F_{1,v}$ and $F_{2,v}$ commute at every grade, and for either $i, j \in \{1, 2\}$ one has*

$$D_{i,v} F_{j,v} = F_{j,v+e_i} D_{i,v}. \tag{34}$$

If J is a color-i cell and $j \neq i$, then

$$F_{j,v}(J) \subset s_{P_v(J)}(I_{v+e_i}) \subset \operatorname{int} P_v(J). \tag{35}$$

Proof. To compare $F_1 F_2$ and $F_2 F_1$, move the decoder D_i past the other projection using (32); then use the commuting square (26) for the projections and (31) for the decoders. Exactly the same rearrangement, without completing a square, gives (34). For (35), the opposite-color formula (30) places $D_{j,v}(J)$ inside the section of $\widehat{P}_{v,j}(J)$; applying $\mathsf{R}^{(e_j)}_v$ and (24) places the image inside the section of $P_v(J)$. Every section lies strictly inside its target cell. □

3.5. **No common periodic points.**

Proof of Theorem 1.1 for $D = \varnothing$. Set $F = F_{1,(0,0)}$ and $G = F_{2,(0,0)}$. They commute by Corollary 3.7. Suppose a point $x \in I_{(0,0)}$ is fixed by both nth powers, where $n \geq 1$. More generally, consider such a point at some grade v. Choose a cell J containing x and let i be its color, $j \neq i$. By (35),

$$x = F_{j,v}^n(x) \in \operatorname{int} P_v^n(J).$$

The cell containing x is therefore unique, x belongs to its interior, and $P_v^n(J) = J$. In particular, its successor cycle length divides n.

The union of color-j cells is forward invariant under $F_{i,v}$, again by (35), applied with the colors interchanged. The $F_{i,v}$-orbit of x returns to x and therefore cannot enter this union. Thus $F_{i,v}(x)$ has color i. Now put $x' = D_{i,v}(x) \in I_{v+e_i}$. By (34), x' is fixed by both $F_{1,v+e_i}^n$ and $F_{2,v+e_i}^n$. Moreover, $\mathsf{R}_v^{(e_i)}(x') = F_{i,v}(x)$, and $\mathsf{R}_v^{(e_i)}$ preserves colors. Since the latter point avoids the union of color-j cells, x' has color i. We may repeat this argument at grades $v + e_i, v + 2e_i, \ldots$, always preserving the exponent n and color i.

Starting at $v = (0, 0)$, the cell successor cycle at grade re_1 has length 5^r in color 1, while at grade re_2 it has length 7^r in color 2. We conclude that either $5^r \mid n$ for all $r \geq 0$ or $7^r \mid n$ for all $r \geq 0$, a contradiction. Therefore $\mathcal{R}(F, G) = \varnothing$. □

4. A common cycle under a limit-set hypothesis

Theorem 1.2 is complementary to the preceding construction. Alikhani–Koopaei's period theorem [1, Theorem 2.6] already implies the existence of a common periodic point when one commuting generator has zero entropy, because a zero-entropy interval map has only periods that are powers of two [12]. Our conclusion locates a common cycle in $\operatorname{Fix}(fg)$ and specifies the actions on it. For related entropy bounds and the consequences of shared odd-part periodic behavior, see Cánovas and Linero [8]. For comparison, Bogatyi and Frolkina [5, Theorem 1] show that $h_{\mathrm{top}}(fg) < \log 4$ forces a common *fixed* point.

Proof of Theorem 1.2. The interval fixed-point theorem gives $K = \operatorname{Fix}(fg) \neq \varnothing$. Commutativity implies that K is forward invariant under both f and g. On K,

$$f|_K\, g|_K = g|_K\, f|_K = \mathrm{id}_K;$$

thus $T = f|_K$ is a homeomorphism of K with inverse $g|_K$. It is enough to prove that T has a periodic point: on its finite orbit $g = T^{-1}$ automatically.

Suppose T has no periodic point. Every $\omega_f(x)$, $x \in K$, is then infinite: a finite nonempty omega-limit set is a periodic orbit, and it remains in K. For each $x \in K$ choose an exchanged pair of disjoint closed intervals L_x, R_x. Because the limit set is infinite, some iterate $f^{n_x}(x)$ lies in the interior of one of these intervals. By continuity this remains true for all points of a relative neighborhood of x in K. Compactness yields finitely many such neighborhoods and corresponding exchanged pairs (L_j, R_j), $j = 1, \ldots, N$; order each pair so that L_j lies to the left of R_j.

For each j choose a fixed point c_j of f strictly between L_j and R_j. Existence follows from the intermediate value theorem: f carries the left interval to the right one and conversely. No fixed point of f can lie in either member of any exchanged pair; hence every c_j is outside every $L_k \cup R_k$.

For $x \in K$, define the sign vector

$$s(x) = \big(s_1(x), \ldots, s_N(x)\big) \in \{-, +\}^N,$$

where $s_j(x)$ is the eventual sign of $f^{2n}(x) - c_j$. To see that the signs exist, choose one of the finite neighborhoods containing x. Once an even iterate of x enters one interval of its exchanged

pair, all later even iterates remain in that interval. This interval avoids every c_j, so each sign is eventually constant. The same argument at a fixed finite entrance time shows that the entire vector s is locally constant on K. Furthermore,

$$s(T^2x) = s(x), \qquad s(Tx) \neq s(x) :$$

the first equality removes two initial iterates; the second follows from the cut c_j associated with the pair eventually containing the orbit, since f exchanges that pair.

Choose a total order on the finite set $\{-,+\}^N$, and put

$$A = \{x \in K : s(x) < s(Tx)\}, \qquad B = K \setminus A.$$

Because s and $s \circ T$ are locally constant, A, B are disjoint clopen subsets of K. They cover K; moreover $T(A) = B$ and $T(B) = A$ by the displayed relations. Both are nonempty.

We now use the commutativity and additivity properties of the fixed-point index for maps of Euclidean neighborhood retracts [9]. Isolate A and B by disjoint relatively open neighborhoods in I, containing no other fixed points of fg. Since $f(A) = B$ and $g(B) = A$, the local commutativity formula for $g \circ f$ near A and $f \circ g$ near B gives

$$\operatorname{ind}(fg, A) = \operatorname{ind}(fg, B). \tag{36}$$

(The notation means the index in any sufficiently small isolating neighborhood; the local commutativity formula applies to f and g on neighborhoods of these compact fixed sets.) Additivity and normalization for an interval self-map give

$$1 = \operatorname{ind}(fg, I) = \operatorname{ind}(fg, A) + \operatorname{ind}(fg, B),$$

contrary to (36), since indices are integers. Therefore T has a periodic point.

For the last assertion, the zero-entropy structure theorem for interval omega-limit sets, in the form recorded in [12, Proposition 5.24], supplies exchanged intervals for every infinite omega-limit set when $h_{\mathrm{top}}(f) = 0$. If $h_{\mathrm{top}}(g) = 0$, interchange the roles of f and g; the finite orbit obtained is the same sort of mutually inverse orbit in $\operatorname{Fix}(fg)$. □

5. Common fixed points for piecewise-monotone maps

We turn to Theorem 1.3. Here a *piecewise-monotone* map is continuous and monotone on each member of a finite interval partition; constant pieces are allowed. A piecewise-linear map has *constant slope magnitude* C if every nondegenerate linear piece has slope $+C$ or $-C$, where $C > 0$.

The fixed set of a piecewise-monotone map need not have finitely many components. For example, if C_0 is the middle-thirds Cantor set, then the increasing interval map $x \mapsto x + \frac{1}{2}\operatorname{dist}(x, C_0)$ fixes exactly C_0. Consequently, the proof cannot rely on the number of components of individual fixed-point sets (compare [7]). Instead, we first obtain a simultaneous constant-slope factor, then construct an expanding path from a minimal counterexample. Finiteness of its critical orbits leads to two substitutions, whose rigidity yields a common fixed point. The argument draws on the constant-slope semiconjugacy theorem [3], fixed-point and entropy results in [5], and Durand's theorem on substitutions [10].

5.1. **Minimal invariant intervals and a simultaneous constant-slope factor.** Assume, toward a contradiction, that P, Q satisfy the hypotheses of Theorem 1.3 but have no common fixed point. A common forward-invariant closed interval means an interval J with $P(J) \subset J$ and $Q(J) \subset J$. The set of its ordered endpoint pairs is closed in the compact square I^2. Minimize its length. The minimizer is nondegenerate, since a degenerate one would be a common fixed point. Its images under P and Q are again common forward-invariant intervals by commutativity. Minimality of the length implies that both maps carry it *onto* itself. After an affine change

of variable, we may therefore assume that $P, Q : I \to I$ are surjective and there is no proper nonempty common forward-invariant interval. We call this last property *minimality of the pair*; it is not minimality in the usual pointwise dynamical sense.

Let $H = PQ$. By [5, Theorem 1], a pair without a common fixed point satisfies $h_{\mathrm{top}}(H) \geq \log 4 > 0$. Put

$$\lambda = e^{h_{\mathrm{top}}(H)} > 1.$$

We recall how one obtains a semiconjugacy that works simultaneously for P and Q, rather than merely for H.

Let $\mathscr{M}$ denote the real vector space of finite signed nonatomic Borel measures on I. For a continuous piecewise-monotone map T define its branchwise pullback $L_T\nu$ by prescribing, on an interval $[x, y]$ lying in one monotonicity branch,

$$(L_T\nu)([x, y]) = \nu(T([x, y])). \tag{37}$$

This defines a measure first on each branch via its continuous distribution function; adjoining branches causes no ambiguity because the measures have no atoms. Refining a branch partition does not alter the result. Constant branches contribute zero. The chain rule on common refinements gives

$$L_{ST} = L_T L_S. \tag{38}$$

In particular, L_P, L_Q, L_H commute, and $L_P L_Q = L_Q L_P = L_H$.

The constant-slope semiconjugacy theorem [3] supplies a nondecreasing surjection $\theta_0 : I \to I$ such that on every monotonicity branch of H,

$$|\theta_0(Hy) - \theta_0(Hx)| = \lambda|\theta_0(y) - \theta_0(x)|.$$

The nonatomic probability measure determined by its continuous distribution function is an eigenmeasure of L_H with eigenvalue λ. Thus the space

$$E = \ker(L_H - \lambda \mathrm{id}) \subset \mathscr{M}$$

contains a probability measure.

Lemma 5.1 (Finite-dimensional eigenmeasure space)**.** *The space E is finite-dimensional. Its set of positive probability measures is a nonempty compact convex set in the norm $\|F_\nu\|_\infty$, where $F_\nu(x) = \nu([0, x])$.*

Proof. Choose a monotonicity partition $0 = t_0 < \cdots < t_s = 1$ of H. The linear evaluation map

$$E \longrightarrow \mathbb{R}^{s+1}, \qquad \nu \longmapsto (F_\nu(Ht_0), \ldots, F_\nu(Ht_s))$$

is injective. Indeed, suppose every listed value is zero. The identity $L_H\nu = \lambda\nu$ expresses $\lambda F_\nu(x)$ as the signed sum of the masses of images of the completed monotonicity branches preceding x, together with the mass of the final partial image. All completed terms vanish, because both endpoints of each image occur among the displayed values. The last term is $\pm F_\nu(Hx)$, because its other endpoint also occurs there. Hence

$$\lambda\|F_\nu\|_\infty \leq \|F_\nu\|_\infty,$$

which forces $\nu = 0$. Thus $\dim E \leq s + 1$. The conditions $\nu \geq 0$ and $\nu(I) = 1$ are closed in this finite-dimensional space. On this set $0 \leq F_\nu \leq 1$, so it is bounded and therefore compact. Convexity is immediate. □

Proposition 5.2 (Simultaneous constant-slope reduction)**.** *There is a nondecreasing surjective semiconjugacy $\theta : I \to I$ of the pair (P, Q) onto commuting, surjective piecewise-linear maps $(\overline{P}, \overline{Q})$ with constant slope magnitudes $C, D > 1$, respectively. The factor pair still has no proper nonempty common forward-invariant interval and no common fixed point.*

Proof. By (38), L_P and L_Q preserve E, and their product on E is $\lambda\mathrm{id}$. On the compact convex set of probability measures in E, consider

$$\mathcal{T}(\nu) = \frac{L_P\nu}{(L_P\nu)(I)}.$$

The denominator is at least 1: the images of the monotonicity branches cover I by surjectivity, and $L_P\nu(I)$ is the sum of their nonnegative masses. Thus $\mathcal{T}$ is continuous and maps the compact convex set into itself. Brouwer's fixed-point theorem (on its finite-dimensional affine hull) yields a probability measure μ and $C > 0$ such that

$$L_P\mu = C\mu, \qquad L_Q\mu = D\mu, \qquad CD = \lambda.$$

Let $\theta(x) = \mu([0, x])$. It is a continuous nondecreasing surjection. A fiber of θ is a closed interval of zero μ-measure. Split such a fiber at the finitely many turns of P. By the eigenmeasure identity, the image under P of each resulting subinterval has zero μ-measure. Their union is an interval, so P sends the entire fiber into a fiber of θ. The same argument applies to Q. Both maps consequently descend to continuous, commuting maps $\overline{P}, \overline{Q}$ under θ. The branchwise increment identity (37) gives

$$|\theta(Py) - \theta(Px)| = C|\theta(y) - \theta(x)|$$

on each monotonicity branch of P, with the analogous formula for Q and D. Thus the descended maps are piecewise-linear of constant slope magnitudes C, D. They are surjective, since P, Q and θ are.

If a closed interval is forward invariant for both factor maps, its full preimage under θ is a common forward-invariant closed interval upstairs. Minimality therefore passes to the factor. A common fixed point of the factor pair would be a proper common invariant singleton, which is impossible. Finally, a surjective interval map of constant slope magnitude c has total variation at least 1, hence $c \geq 1$. If $c = 1$, it is either the identity or the reflection $x \mapsto 1 - x$. Each commutes with an interval map only when there is a common fixed point: for the identity take any fixed point of the other generator, while for the reflection the point $1/2$ is fixed by every commuting map. Hence $C, D > 1$. □

5.2. **Minimal factors and invariant return intervals.** We shall repeatedly use the following elementary property of monotone semiconjugacies.

Lemma 5.3 (Fixed points pass through monotone factors)**.** *Suppose $\pi : I \to I$ is continuous, nondecreasing and surjective, $\pi T = \overline{T}\pi$, and $T, \overline{T}$ are continuous interval maps. Then*

$$\mathrm{Fix}(\overline{T}) = \pi(\mathrm{Fix}(T)). \tag{39}$$

Proof. The inclusion from right to left is immediate. If $y \in \mathrm{Fix}(\overline{T})$, the fiber $\pi^{-1}(y)$ is a nonempty closed interval, and T carries it into itself. The intermediate value theorem gives a fixed point of T in that fiber. □

Take a minimal constant-slope counterexample (P_0, Q_0) from Proposition 5.2. Let

$$\nu_0 = \#\,\mathrm{Fix}(P_0Q_0), \qquad L = (2\nu_0)!.$$

The number ν_0 is finite and positive: P_0Q_0 is piecewise linear of constant slope magnitude greater than 1 and, as an interval self-map, has a fixed point. Consider all minimal constant-slope counterexample pairs obtained as nondecreasing factors of (P_0, Q_0). Their numbers $\#\,\mathrm{Fix}(P^{2L})$ are finite positive integers, so one can choose a pair (P, Q) minimizing this number. Write

$$A = P^L, \qquad B = Q^L.$$

Any further nondecreasing factor, followed if necessary by Proposition 5.2, is among the pairs compared in this minimization.

Lemma 5.4 (No common return interval)**.** *There is no proper nondegenerate closed interval $J \subset I$ with $A(J) = J = B(J)$.*

Proof. Suppose such a J exists, and consider the finite family of intervals

$$\mathscr{J} = \{P^i Q^j(J) : 0 \le i, j < L\},$$

with duplicate members identified. The relations $P^L(J) = Q^L(J) = J$ and commutativity show that P, Q permute this family. Moreover, their permutations preserve and reflect intersections. For example, $J_1 \cap J_2 \neq \varnothing$ implies $P(J_1) \cap P(J_2) \neq \varnothing$. Conversely, if $P(J_1) \cap P(J_2) \neq \varnothing$, applying P^{L-1} and using $P^L(J_i) = J_i$ gives $J_1 \cap J_2 \neq \varnothing$. The argument for Q is identical.

Consequently the finite intersection graph of $\mathscr{J}$ is vertex transitive: its vertices are the intervals, two vertices are adjacent when they intersect, and the commuting translations P, Q act transitively. Every connected vertex-transitive interval graph is complete. To see this, choose a vertex whose interval has the smallest right endpoint. All its neighbors contain that endpoint and therefore meet each other; by vertex transitivity the neighbors of every vertex form a clique. A connected graph in which every vertex has a clique as its neighborhood is complete, since a shortest path between nonadjacent vertices would contain two nonadjacent neighbors of an intermediate vertex. If our intersection graph were connected, its intervals would therefore meet pairwise and have a common intersection by the one-dimensional Helly property. That intersection would be a proper nonempty common forward-invariant interval for P, Q, contrary to minimality.

The graph thus has at least two components. By the same argument each component is a clique. The union of its member intervals is consequently a compact interval; different component unions are disjoint. These component intervals are permuted by both P and Q. Collapse every component interval to a point, using a continuous nondecreasing piecewise-linear surjection $\pi : I \to I$ whose slope is the same positive constant on all complementary gaps. Since each nontrivial fiber is mapped into another fiber by P and by Q, both maps descend through π to commuting continuous piecewise-monotone maps. Minimality passes to this quotient, so the quotient pair has no common fixed point. Apply Proposition 5.2 to obtain a further minimal constant-slope factor.

We show that the resulting factor has strictly fewer fixed points of the $2L$th power of its first generator. Since $A : J \to J$ is onto, A^2 has at least two fixed points in J. For completeness, if an onto continuous map $T : [u, v] \to [u, v]$ has more than one fixed point this is obvious. If it has only one fixed point z, then $u < z < v$, $T(x) > x$ to the left of z, and $T(x) < x$ to its right; otherwise surjectivity fails at one endpoint. Choose $a < z < b$ with $T(a) = v$ and $T(b) = u$, and then $c \in [a, z)$ with $T(c) = b$. Since $T^2(c) = u \le c$ and $T^2(u) \ge u$, the intermediate value theorem yields a fixed point of T^2 in $[u, c]$, distinct from z. Apply this to $T = A|_J$.

The first collapse identifies these two distinct fixed points of $A^2 = P^{2L}$. Under any subsequent nondecreasing semiconjugacy, the fixed-point image identity (39) shows that no new fixed points of the descended $2L$th iterate can appear. Thus the final factor has strictly fewer than $\#\operatorname{Fix}(P^{2L})$ fixed points, contradicting the choice of (P, Q). □

5.3. **An expanding path and multiplicative independence.** Let the constant slope magnitudes of P, Q be $C, D > 1$, so A, B have slope magnitudes

$$a = C^L, \qquad b = D^L,$$

respectively. The finite set $K = \operatorname{Fix}(PQ)$ has at least two points: if K consisted of one point, commutativity would force both P and Q to fix it. By Lemma 5.3 and the choice of the original factor, $\#K \le \nu_0$.

Choose consecutive points of K. On the open gap between them, $PQ(x) - x$ has constant sign. If this sign is positive, choose the left endpoint p and set $\delta = +1$; otherwise choose the right endpoint and set $\delta = -1$. Because PQ has constant slope magnitude $CD > 1$, locally on the indicated side,

$$PQ(p + \delta t) = p + \delta CD\, t \qquad (0 \le t \ll 1). \tag{40}$$

We briefly explain why large equal powers of the two maps fix this *one-sided direction*, and not merely the base point. Each generator permutes the points of K, with the other as inverse, and is locally strictly monotone on either side of every point of K. It therefore acts on the finite set of one-sided germs based at K. On the subset of germs fixed by PQ, the induced actions of P, Q are inverse permutations. The selected germ in (40) belongs to this subset. There are at most $2\nu_0$ such germs, so $L = (2\nu_0)!$ fixes every germ in the subset under both generators. In particular,

$$A(p + \delta t) = p + \delta at, \qquad B(p + \delta t) = p + \delta bt \qquad (0 \le t \ll 1). \tag{41}$$

Set $R = AB$ and $s = ab$. For $t \ge 0$ define

$$\Psi(t) = R^n\left(p + \frac{\delta t}{s^n}\right) \tag{42}$$

with any sufficiently large integer n. This is independent of n: increasing n by one first applies R within the neighborhood where $R(p + \delta u) = p + \delta su$, and then applies the previous iterate. The map $\Psi : [0, \infty) \to I$ is continuous, locally piecewise linear and locally of unit speed. Commutativity and (41) give the exact scaling identities

$$\Psi(at) = A\Psi(t), \qquad \Psi(bt) = B\Psi(t) \qquad (t \ge 0). \tag{43}$$

The image closure $J = \overline{\Psi([0, \infty))}$ is a nondegenerate compact interval, since Ψ is locally of unit speed. Both A and B map J onto itself by (43), because multiplication of the ray parameter by a or b is onto $[0, \infty)$. Lemma 5.4 therefore gives

$$\overline{\Psi([0, \infty))} = I. \tag{44}$$

Lemma 5.5 (Independence of the two slopes)**.** *The numbers a, b are multiplicatively independent: there are no positive integers m, n with $a^m = b^n$.*

Proof. If $a^m = b^n$, then (43) implies $A^m\Psi(t) = B^n\Psi(t)$ for every t. By (44) and continuity, $A^m = B^n$ on I. Consequently $P^{Lm} = Q^{Ln}$. We recall the standard inverse-limit obstruction to an equality of positive powers in a commuting pair without a common fixed point.

Suppose $P^i = Q^j$ with $i, j > 0$, and form

$$X = \varprojlim(I, PQ).$$

Because PQ is onto, this is an arc-like continuum: finite coordinate projections give interval approximations with fibers of arbitrarily small diameter in the product metric. The coordinatewise lifts S, T of P, Q commute on X and satisfy $ST = TS =$ the invertible inverse-limit shift; thus each of S, T is a homeomorphism. We also have $S^i = T^j$.

Write $(i, j) = d(u, v)$ with u, v coprime and choose $r, s \in \mathbb{Z}$ satisfying $rv + su = 1$. Define

$$V = S^u T^{-v}, \qquad U = S^r T^s.$$

Then $V^d = \mathrm{id}$, U commutes with the finite cyclic group $\langle V \rangle$, and

$$S = U^v V^s, \qquad T = U^u V^{-r}.$$

The finite-group fixed-point theorem for commuting maps on arc-like continua [5, Theorem 3] supplies a point of X fixed by U and V. It is fixed by both S and T. Its zeroth coordinate is a

common fixed point of P, Q, contrary to assumption. Therefore no positive powers of P and Q can agree, and a, b are independent. □

5.4. **Finite critical orbits and two substitutions.** For a surjective piecewise-linear map $T : I \to I$ with nonzero slope magnitude, let $\mathrm{Crit}(T)$ consist of its turning points and the two endpoints of I, and put $\mathrm{CV}(T) = T(\mathrm{Crit}(T))$. These sets are finite.

Lemma 5.6 (Critical values under composition)**.** *For surjective piecewise-linear maps S, T with nonzero slope magnitudes,*

$$\mathrm{CV}(ST) = \mathrm{CV}(S) \cup S(\mathrm{CV}(T)). \tag{45}$$

Proof. Every turn of ST is either a turn of T, whose image under the composition lies in $S(\mathrm{CV}(T))$, or a preimage under T of a turn of S, whose image lies in $\mathrm{CV}(S)$. Endpoint values also lie in the displayed union. Conversely, a turn of T survives the composition: near such a point the image of T approaches its turning value from one side, on which S is strictly monotone, so ST still turns there. Every interior turning point of S has a crossing preimage under T, because T is surjective and strictly piecewise monotone; such a crossing produces a turn of ST. Finally, every preimage of an endpoint of I under T is a turn or a domain endpoint, which accounts for the endpoint critical values of S. This proves both inclusions. □

Let

$$X_m = \bigcup_{0 \le i < m} A^i \,\mathrm{CV}(A), \qquad Y_n = \bigcup_{0 \le j < n} B^j \,\mathrm{CV}(B).$$

Commutativity and Lemma 5.6 applied to $A^m B^n = B^n A^m$ give

$$X_m \cup A^m Y_n = Y_n \cup B^n X_m, \qquad B(X_m) \setminus X_m \subset A^m \,\mathrm{CV}(B). \tag{46}$$

The second inclusion is the first equality with $n = 1$.

Lemma 5.7 (Finite joint critical orbit)**.** *The set*

$$\bigcup_{i,j \ge 0} A^i B^j \big(\mathrm{Crit}(A) \cup \mathrm{Crit}(B)\big)$$

is finite.

Proof. We first show that $X_\infty = \bigcup_{m \ge 1} X_m$ is finite. Otherwise, since it is the union of the forward A-orbits of finitely many critical values, it contains infinite A-orbits belonging to finitely many *tail classes*. Two infinite forward orbits are in the same tail class if some forward iterates meet. Take $w \in X_\infty$ with infinite A-orbit. The point Bw also has infinite A-orbit: otherwise commutativity would map infinitely many distinct terms of the orbit of w into a finite set under B, contrary to the fact that a nonzero-slope piecewise-linear map is finite-to-one.

If $Bw \notin X_\infty$, choose m_0 with $w \in X_m$ for every $m \ge m_0$. The inclusion in (46) then gives $Bw \in A^m \,\mathrm{CV}(B)$ for every $m \ge m_0$. Since $\mathrm{CV}(B)$ is finite, some critical value must produce Bw at two different positive iterates; thus Bw is periodic under A. This contradicts the infinitude of its A-orbit. Therefore $Bw \in X_\infty$. Commutativity also shows that B induces a map on the finite collection of infinite A-tail classes. Some class returns to itself under a positive power B^r. Choose w in that class. There exist $u, v \ge 0$ such that

$$A^u B^r(w) = A^v(w). \tag{47}$$

Applying A^t for every $t \ge 0$ yields equality of the two piecewise-linear maps $A^u B^r$ and A^v on the infinite set $\{A^t(w) : t \ge 0\}$. Two distinct points of this set lie in one common linearity interval of both maps. Comparing the absolute slopes on that interval gives $a^u b^r = a^v$, contradicting Lemma 5.5. Thus X_∞ is finite. By symmetry, $Y_\infty = \bigcup_{n \ge 1} Y_n$ is finite.

We record the finite-orbit consequences carefully. If $z \in B(X_\infty) \setminus X_\infty$, the second inclusion of (46), after stabilization of X_m, puts z in $A^m \operatorname{CV}(B)$ for every sufficiently large m. Repeating the same starting critical value shows that z is A-periodic. Thus, for some $k \geq 1$,

$$B(X_\infty) \subset X_\infty \cup \operatorname{Fix}(A^k). \tag{48}$$

The symmetric argument, increasing k if necessary, gives

$$A(Y_\infty) \subset Y_\infty \cup \operatorname{Fix}(B^k). \tag{49}$$

The fixed-point sets in these formulas are finite, because A^k, B^k are piecewise linear with slope magnitudes strictly greater than 1. They are invariant under both A and B by commutativity.

Finally,

$$B(\operatorname{Crit}(A)) \subset \operatorname{Crit}(A) \cup \operatorname{CV}(B), \qquad A(\operatorname{Crit}(B)) \subset \operatorname{Crit}(B) \cup \operatorname{CV}(A). \tag{50}$$

For the first inclusion, if a point of $\operatorname{Crit}(A)$ is also a turn or endpoint for B, its image lies in $\operatorname{CV}(B)$. Otherwise B is locally one-to-one there; the turn of $BA = AB$ forces a turn of A at its B-image. This proves the inclusion, and the second is symmetric.

The finite set

$$W = \operatorname{Crit}(A) \cup \operatorname{Crit}(B) \cup X_\infty \cup Y_\infty \cup \operatorname{Fix}(A^k) \cup \operatorname{Fix}(B^k)$$

is forward invariant under both maps, by (48)–(50) and the definitions of X_∞, Y_∞. It contains all the critical points, so their joint forward orbit is finite. □

Enlarge the finite invariant set W from the preceding proof by the point p fixed by A, B in (41). The resulting finite set of *vertices* is still forward invariant under A, B. Its complementary intervals, with their two possible orientations, are called *directed edges*. Because the turns of A, B are vertices and vertices map to vertices, each map replaces every directed edge by a finite word of directed edges. Denote these edge substitutions by τ_A, τ_B.

Lemma 5.8 (A common substitutive itinerary)**.** *The expanding path Ψ of (42) traces an infinite word $\mathbf{w}$ of complete directed edges, starting at p, with*

$$\tau_A(\mathbf{w}) = \mathbf{w} = \tau_B(\mathbf{w}). \tag{51}$$

On the subalphabet of edges occurring in $\mathbf{w}$, the incidence matrices of τ_A, τ_B have spectral radii a, b. A positive integer power of each of these radii is a Perron number.

Proof. Every turn of Ψ has a vertex value. Indeed, Ψ is built by applying iterates of $R = AB$ to a short linear path from the vertex p. By (45), the critical values of such iterates belong to the joint forward orbit of the critical values of A, B, which is contained in W. Thus, starting from p, Ψ crosses complete directed edges. Their minimum length is positive, so there cannot be infinitely many crossings in finite parameter time.

Write $\mathbf{w}$ for their sequence. The scaling identities (43) say that the path traced by $A\Psi(t)$ as t increases is the same as the path traced by $\Psi(at)$. The first path has edge word $\tau_A(\mathbf{w})$ and the second has edge word $\mathbf{w}$. This proves the first equality in (51); the second is identical. Every edge appearing in the word has all of its substitution edges appearing there as well, so the subalphabet is invariant under both substitutions.

Let $\ell_e > 0$ be the geometric length of the underlying interval of a directed edge e. Each substitution changes geometric length by its map's constant slope magnitude. Consequently the strictly positive vector (ℓ_e) is a right eigenvector of each row-source incidence matrix, with eigenvalue a, respectively b. A nonnegative matrix with a strictly positive eigenvector has that eigenvalue as its spectral radius. Furthermore the geometric length of $\tau_A^n(e)$ is $a^n \ell_e$, and similarly for B; thus both substitutions grow on every occurring letter. Each nonnegative integral incidence matrix has an irreducible diagonal block attaining the spectral radius. Taking a positive power

that removes the period of such a block makes the corresponding diagonal block primitive. Hence a positive power of a, respectively b, is the Perron root of a primitive integral matrix, and is a Perron number. □

Choose powers $r, s \geq 1$ for which a^r, b^s are Perron numbers. They are still multiplicatively independent by Lemma 5.5. The word $\mathbf{w}$ is fixed by τ_A^r and τ_B^s, whose incidence spectral radii are a^r and b^s. Durand's theorem [10] for sequences substitutive with respect to multiplicatively independent Perron numbers implies that $\mathbf{w}$ is ultimately periodic. Because its letters record *directed* edges, whose geometric lengths are fixed, the path Ψ is eventually periodic in its real arclength parameter.

5.5. **Rigidity of the periodic tail.** Let $\psi : \mathbb{R} \to I$ be the periodic extension of the tail of Ψ, and let $\tau > 0$ be its least period. It is nonconstant and locally piecewise linear of unit speed. Its image J is a nondegenerate interval; by (43), $A(J) = J = B(J)$, since the image of every sufficiently late tail is exactly J. Lemma 5.4 gives

$$\psi(\mathbb{R}) = I. \tag{52}$$

For all sufficiently large t, the scaling identities read $\psi(at) = A\psi(t)$ and $\psi(bt) = B\psi(t)$. Since $\psi(t + \tau) = \psi(t)$, the numbers $a\tau$ and $b\tau$ are eventual periods of ψ, and hence global periods. They are integer multiples of its least period, so

$$a, b \in \mathbb{N}, \qquad a, b > 1. \tag{53}$$

Now translate a given parameter by a sufficiently large integer multiple of τ. Because a, b are integers, this translation leaves both sides of each scaling identity unchanged. We obtain

$$\psi(at) = A\psi(t), \qquad \psi(bt) = B\psi(t) \qquad (t \in \mathbb{R}). \tag{54}$$

We use a rigidity principle for periodic unit-speed paths. A proof is included because it is the step that turns the eventual periodicity into global circle symmetries.

Lemma 5.9 (Rigidity of expanding periodic paths)**.** *Let $u, v : \mathbb{R} \to I$ be nonconstant periodic, locally piecewise-linear unit-speed paths. Suppose that, for a continuous map $T : I \to I$ and $r > 1$,*

$$u(rt) = T(u(t)), \qquad v(rt) = T(v(t)) \quad \text{for every } t \in \mathbb{R}.$$

If $v(t) = u(\epsilon t + c)$ on some nonempty open interval, with $\epsilon \in \{\pm 1\}$, then the same identity holds globally with possibly a different translation constant. In particular, if u, v are surjective, they agree globally up to a translation and possibly reversal of the parameter.

Proof. The scaling identities propagate the local agreement: for each $n \geq 0$,

$$v(t) = u(\epsilon t + r^n c) \quad \text{on } r^n J, \tag{55}$$

where J is the original open interval. Choose the midpoint t_n of $r^n J$. The intervals $r^n J$ have length tending to infinity. Pass to a subsequence for which t_n modulo a period of v, and $\epsilon t_n + r^n c$ modulo a period of u, converge to phases x and y. For any fixed $h \in \mathbb{R}$, $t_n + h \in r^n J$ for all sufficiently large n. Letting $n \to \infty$ in (55) therefore gives

$$v(x + h) = u(y + \epsilon h) \qquad (h \in \mathbb{R}).$$

This is a global parameter isometry. If both paths are surjective, choose a common value outside the finite sets of turning values of the paths over one period. Each path crosses this value with slope $+1$ or -1 at some regular parameter, so local agreement under a parameter isometry exists. Apply the first assertion. □

Lemma 5.10 (Integer slopes and affine circle lifts)**.** *The original slope magnitudes C, D are positive integers. With the parameter circle normalized to $\mathbb{R}/\mathbb{Z}$, there are integers k, ℓ with $|k| = C$, $|\ell| = D$ and constants $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ such that*

$$P\psi(t) = \psi(kt + \alpha), \qquad Q\psi(t) = \psi(\ell t + \beta). \tag{56}$$

The lifts commute modulo 1*.*

Proof. Consider $u(t) = \psi(t)$ and $v(t) = P\psi(t/C)$. Both are surjective periodic locally unit-speed paths: away from finitely many turning parameters, the slope magnitude of v is $C \cdot (1/C) = 1$. By commutativity of P and A, and by (54),

$$v(at) = P\psi(at/C) = PA\psi(t/C) = Av(t).$$

Lemma 5.9 gives $v(t) = \psi(\epsilon t + c)$ on all of $\mathbb{R}$. Because $C\tau$ is a period of v, and v has the same period group $\tau\mathbb{Z}$ as ψ, it follows that $C\tau \in \tau\mathbb{Z}$; hence C is an integer. Rescaling the parameter yields $P\psi(t) = \psi(\epsilon C t + c)$. The same argument with Q and B shows that D is an integer and gives the second identity in (56).

For the normalized circle, commutativity downstairs implies that

$$\psi(k\ell t + k\beta + \alpha) = \psi(k\ell t + \ell\alpha + \beta)$$

for every t. Thus the two constant offsets differ by a period of ψ, proving that the affine lifts commute modulo 1. □

We finish the proof of the finite-piece theorem. Because ψ is surjective and locally unit-speed, it has a strict local maximum, at which its two one-sided slopes are $+1$ and -1. Thus ψ has local reflection symmetry about that parameter. Applying Lemma 5.9 to the two paths $u = v = \psi$, with the common dilation a and map $T = A$, promotes this local reflection to a global reflection ρ of the parameter circle satisfying

$$\psi\rho = \psi. \tag{57}$$

The reflection is unique modulo periods: two distinct reflection symmetries would compose to a nontrivial translation symmetry, contradicting minimality of the period.

For either affine lift $L(t) = kt + \alpha$ of (56), there is a reflection ρ' with $L\rho = \rho' L$. Since

$$\psi\rho' L = \psi L\rho = P\psi\rho = P\psi = \psi L$$

and L is onto the circle, $\psi\rho' = \psi$. Uniqueness forces $\rho' = \rho$. Thus both affine lifts commute with the reflection. Choose its fixed point as the origin of the unit circle, so $\rho(t) = -t$. A lift $L(t) = kt + \alpha$ commutes with ρ if and only if $2\alpha = 0 \pmod 1$, and therefore

$$\alpha, \beta \in \{0, \tfrac{1}{2}\}, \qquad (k-1)\beta = (\ell - 1)\alpha \pmod 1. \tag{58}$$

Consider the three reflection orbits on the circle

$$\{0\}, \qquad \{\tfrac{1}{4}, \tfrac{3}{4}\}, \qquad \{\tfrac{1}{2}\}.$$

At least one is invariant under both lifts. Here is the complete parity check. If $\alpha = \beta = 0$, both preserve $\{0\}$. If $\alpha = 0, \beta = 1/2$, (58) requires k odd. The first lift then preserves all three orbits; the second preserves the middle orbit when ℓ is odd and preserves $\{1/2\}$ when ℓ is even. The case $\alpha = 1/2, \beta = 0$ is symmetric. If both offsets are $1/2$, the congruence requires k, ℓ to have the same parity: both preserve the middle orbit if they are odd, and both preserve $\{1/2\}$ if they are even.

The ψ-image of each such reflection orbit is a single point, by (57). If that orbit is preserved by both affine lifts, its image is fixed by P and Q by (56). This contradicts the assumption that the factor pair has no common fixed point, completing the proof of Theorem 1.3. □

References

[1] A. Alikhani–Koopaei, *On common fixed and periodic points of commuting functions*, Internat. J. Math. Math. Sci. **21** (1998), 269–276. doi:10.1155/S0161171298000386.

[2] A. Alikhani–Koopaei, *On common fixed points, periodic points, and recurrent points of continuous functions*, Internat. J. Math. Math. Sci. **2003** (2003), no. 39, 2465–2473. doi:10.1155/S0161171203205366.

[3] Ll. Alsedà and M. Misiurewicz, *Semiconjugacy to a map of a constant slope*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), 3403–3413. doi:10.3934/dcdsb.2015.20.3403.

[4] A. Anušić and C. Mouron, *Strongly commuting interval maps*, Fund. Math. **257** (2022), 39–68. doi:10.4064/fm993-11-2021.

[5] S. A. Bogatyi and O. D. Frolkina, *A common fixed point of commuting mappings of a tree*, Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. (2002), no. 6, 3–10; English translation in Moscow Univ. Math. Bull. **57** (2002), no. 6, 1–8. https://www.mathnet.ru/eng/vmumm1423.

[6] W. M. Boyce, *Commuting functions with no common fixed point*, Trans. Amer. Math. Soc. **137** (1969), 77–92. doi:10.1090/S0002-9947-1969-0236331-5.

[7] J. S. Cánovas, *Combining the dynamics of discrete dynamical systems*, in *Frontiers in the Study of Chaotic Dynamical Systems with Open Problems*, E. Zeraoulia and J. C. Sprott, eds., World Scientific, 2011, Chapter 9.

[8] J. S. Cánovas and A. Linero, *On topological entropy of commuting interval maps*, Nonlinear Anal. **51** (2002), no. 7, 1159–1165. doi:10.1016/S0362-546X(01)00885-9.

[9] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology **4** (1965), 1–8.

[10] F. Durand, *Cobham's theorem for substitutions*, J. Eur. Math. Soc. **13** (2011), 1799–1814. doi:10.4171/JEMS/294.

[11] J. P. Huneke, *On common fixed points of commuting continuous functions on an interval*, Trans. Amer. Math. Soc. **139** (1969), 371–381. doi:10.1090/S0002-9947-1969-0237724-2.

[12] S. Ruette, *Chaos on the Interval*, University Lecture Series, vol. 67, American Mathematical Society, Providence, 2017. doi:10.1090/ulect/067.