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# A Polynomial Kontsevich–Zagier Theorem for Compact Exponential Periods on the Affine Line

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**Abstract**

Let $q, r \in \overline{\mathbb{Q}}[x]$, let $\Delta$ be a finite degree-zero zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$, and let $\Gamma$ be any finite piecewise $C^1$ one-chain in $\mathbb{C}$ with $\partial\Gamma = \Delta$. We consider compact exponential periods

$$\int_\Gamma r(x)e^{q(x)}\,dx$$

and the zero-dimensional exponential periods $e^\lambda$, $\lambda \in \overline{\mathbb{Q}}$. The companion linear theorem identifies every numerical linear relation among these periods with a formal consequence of bilinearity, functoriality, and twisted Stokes.

We prove the corresponding polynomial theorem for the commutative algebra generated by the same periods. Every polynomial numerical relation is generated by the companion linear relations, multiplication, and the identities

$$e^0 = 1, \qquad e^\lambda e^\mu = e^{\lambda+\mu}, \qquad e^\lambda \int_\Gamma re^q\,dx = \int_\Gamma re^{q+\lambda}\,dx.$$

Equivalently, the period map is injective on the tensor-generated formal algebra of compact exponential periods on the affine line.

The new input is a uniform description of the functional polynomial-relation ideal. After adjoining the finitely many exponential functions coming from endpoint values and constant shifts, the affine Picard–Vessiot group of the inhomogeneous moment system has a vector translation kernel and reductive homogeneous quotient. Horizontal projectors are Laurent polynomials in the adjoined exponentials. An explicit gauge built from the spectral projectors of the constant connection matrix separates the coordinates already in the base ring from a full translation block, and the latter coordinates are algebraically independent. Hence the complete functional relation ideal is generated by affine-linear relations. Beukers' lifting theorem transfers this statement to values, and the companion linear Kontsevich–Zagier theorem makes every specialized generator formal.

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# 1 Introduction and main theorem

Kontsevich and Zagier formulated their period conjecture as the assertion that all identities between integral presentations of ordinary periods follow from additivity, algebraic change of variables, and Stokes' formula [12]. In cohomological language this is the injectivity of a formal period map; see [9, Chapters 12–13] and [5]. Exponential periods replace an algebraic differential form $\omega$ by $e^f\omega$, where $f$ is a regular function, and replace the de Rham differential by

$$d_f = d + df \wedge .$$

General definitions and comparison results for exponential periods are developed in [4, 8]. We use *exponential period function* in the sense of Fresán and Jossen [7].

This paper is the nonlinear continuation of two companion papers. The first proves algebraic-value rigidity for

$$I_{q,\Delta}(z) = \int_\Gamma e^{zq(x)}\,dx$$

and supplies the homogeneous moment connection and its residue structure [13]. The second treats arbitrary polynomial differentials, different polynomials in the same relation, and zero-dimensional exponential periods, and proves the linear Kontsevich–Zagier theorem for compact exponential periods on the affine line [14]. The present paper adds multiplication and classifies all polynomial relations in the resulting tensor-generated algebra.

Throughout, put $k = \overline{\mathbb{Q}}$. For $q, r \in k[x]$ and a finite degree-zero zero-cycle

$$\Delta = \sum_\alpha d_\alpha[\alpha] \quad \text{on } \mathbb{A}^1(k),$$

choose a finite piecewise $C^1$ one-chain $\Gamma$ in $\mathbb{C}$ with $\partial\Gamma = \Delta$ and set

$$I_{q,r,\Delta}(z) = \int_\Gamma r(x)e^{zq(x)}\,dx. \tag{1.1}$$

The one-form in (1.1) has an entire primitive in $x$, so the function is independent of the chosen chain and is entire in $z$. The compact exponential period is $I_{q,r,\Delta}(1)$. A common nonzero algebraic evaluation parameter is already absorbed into the polynomial $q$.

Let $\mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$ be the formal linear period space from [14]. It is generated over $k$ by the symbols

$$\mathsf{P}(q, r, \Delta) \quad \text{and} \quad \mathsf{E}(\lambda), \qquad \lambda \in k,$$

modulo bilinearity, functoriality through smooth affine curves, and twisted Stokes. The companion theorem states that its period map

$$\mathsf{P}(q, r, \Delta) \longmapsto I_{q,r,\Delta}(1), \qquad \mathsf{E}(\lambda) \longmapsto e^\lambda$$

is injective.

**Definition 1.1** (Tensor-generated formal algebra)**.** Let $\mathcal{A}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$ be the quotient of $\mathrm{Sym}_k(\mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1})$ by the ideal generated by

$$\mathsf{E}(0) = 1, \tag{1.2}$$

$$\mathsf{E}(\lambda)\mathsf{E}(\mu) = \mathsf{E}(\lambda + \mu), \tag{1.3}$$

$$\mathsf{E}(\lambda)\mathsf{P}(q, r, \Delta) = \mathsf{P}(q + \lambda, r, \Delta). \tag{1.4}$$

Multiplication is the formal tensor, or Fubini, product. The numerical assignments above define a homomorphism

$$\mathrm{per} : \mathcal{A}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1} \longrightarrow \mathbb{C}. \tag{1.5}$$

**Theorem 1.2** (Polynomial Kontsevich–Zagier theorem)**.** *The period map* (1.5) *is injective. Equivalently, every polynomial numerical relation among compact exponential periods on the affine line and zero-dimensional exponential periods is generated by*

(i) *the linear formal relations of [14];*

(ii) *multiplication;*

(iii) (1.2)–(1.4).

**Prior work and scope.** The exponential-motive framework, the deformation strategy, and the use of Siegel–Shidlovskii theory are established in [8, §§12.5–12.7] and [10, §§2–3]. In particular, Jossen's Theorem 3.2 already states the lifting of polynomial value relations. The homogeneous polynomial-phase Fourier–Laplace construction appears in [7, Proposition 5.1]. In a 2025 lecture reporting joint work in progress with Fresán, Jossen presented the quadratic marked-point model, its torus and vector translation group, and the arithmetic independence furnished by Siegel–Shidlovskii theory [11]. For the compact integral $F_a(z) = \int_0^a e^{-zx^2}\,dx$, the normalization $F_a(z)/a$ for $a \neq 0$ depends only on $a^2$; the raw integrals satisfy $F_{-a} = -F_a$. The quadratic model is therefore a predecessor of the translation-group description below. The present result gives the uniform affine-linear relation ideal for arbitrary polynomial phases, amplitudes, and boundary data, and identifies its specialized generators by formal period operations. It does not address the extra $\sqrt{\pi}$-independence question in the full quadratic rapid-decay period field.

The theorem is not an algorithmic relation theorem for one fixed tuple of $E$-functions. Fischler and Rivoal proved that, for any fixed finite tuple of $E$-functions, the ideals of functional and value relations can be computed algorithmically [6]; the exceptional-value and minimal-operator problems were developed by Adamczewski–Rivoal and Bostan–Rivoal–Salvy [1, 3]. The new point here is uniform and structural: for the entire geometric family (1.1), after adjoining the required exponential shifts, the functional ideal is generated by its linear part, and the companion paper identifies the specialized generators by formal period operations.

The proof has three steps. First, the moment connections from [13] remain semisimple after adjoining finitely many exponential functions, and every horizontal projector over that exponential field is a Laurent polynomial in those functions. Second, affine Picard–Vessiot theory splits the inhomogeneous moment coordinates into an affine-linear constrained part and a full translation part. The complete functional ideal is therefore linear. Third, Beukers' theorem lifts a numerical relation of arbitrary degree to the functional ideal; specialization of its linear generators is handled by [14].

## 2 Imported arithmetic and moment-theoretic input

### 2.1 $E$-functions and Beukers' lifting theorem

We use Beukers' normalization of Siegel's $E$-functions [2, Section 1].

**Definition 2.1** ($E$-function)**.** An entire function

$$F(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}$$

is an $E$-function if $a_n \in k$, the joint logarithmic height $h(a_0, \ldots, a_n)$ is $O(n)$, and $F$ satisfies a nonzero linear differential equation over $k(z)$.

**Theorem 2.2** (Beukers). *Let $F_1, \dots, F_N$ be $E$-functions forming a vector solution of*

$$\mathbf{F}' = A(z)\mathbf{F}, \qquad A(z) \in M_N(k(z)),$$

*and let $T \in k[z]$ clear the denominators of $A$. Let $\xi \in k$ satisfy $\xi T(\xi) \neq 0$. If a homogeneous polynomial $P \in k[X_1, \dots, X_N]$ satisfies*

$$P(F_1(\xi), \dots, F_N(\xi)) = 0,$$

*then there is a polynomial $Q \in k[z, X_1, \dots, X_N]$, homogeneous in the $X$-variables and of the same degree as $P$, such that*

$$Q(z, F_1(z), \dots, F_N(z)) \equiv 0, \qquad Q(\xi, X_1, \dots, X_N) = P(X_1, \dots, X_N).$$

This is the combination of [2, Theorems 1.3 and 3.2]. Adjoining the constant $E$-function 1 homogenizes an affine relation.

**Lemma 2.3** (Distinct exponential functions). *If $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ are distinct, then $e^{\lambda_1 z}, \dots, e^{\lambda_s z}$ are linearly independent over $\mathbb{C}(z)$.*

*Proof.* Assume a nontrivial relation with the fewest nonzero terms. Differentiate and eliminate one term. Minimality forces the logarithmic derivative of a nonzero rational function to be a nonzero constant, which is impossible at infinity. □

## 2.2 The polynomial moment package

Let $q \in k[x]$ have degree $n \geq 2$, let $\Delta$ be a finite degree-zero zero-cycle, and put

$$U_j(z) = \int_\Gamma x^j e^{zq(x)}\, dx, \qquad 0 \leq j \leq n-2, \qquad \mathbf{U} = (U_0, \dots, U_{n-2})^{\mathsf{T}}.$$

The following is the exact form of the input proved in [13] and used in [14]. No affine normalization of $q$ is required.

**Theorem 2.4** (Polynomial moment package). *There are matrices $C_q, D_q \in M_{n-1}(k)$ and vectors $\mathbf{b}_\lambda \in k^{n-1}$, indexed by*

$$\Lambda = \{0\} \cup q(\operatorname{supp} \Delta),$$

*such that*

$$\mathbf{U}' = \left(C_q + \frac{D_q}{z}\right)\mathbf{U} + \frac{1}{z}\sum_{\lambda \in \Lambda} \mathbf{b}_\lambda e^{\lambda z}. \tag{2.1}$$

*Every coordinate of $\mathbf{U}$ is an $E$-function. The homogeneous module*

$$\mathbf{Y}' = \left(C_q + \frac{D_q}{z}\right)\mathbf{Y} \tag{2.2}$$

*is semisimple over $\mathbb{C}(z)$, and $C_q$ is semisimple. Moreover,*

$$\operatorname{Spec}(D_q) = \left\{-\frac{1}{n}, -\frac{2}{n}, \dots, -\frac{n-1}{n}\right\}. \tag{2.3}$$

*If $\lambda$ is an eigenvalue of $C_q$, if $\pi_\lambda$ is the corresponding spectral projector, and if*

$$\overline{D}_{q,\lambda} = \pi_\lambda D_q|_{\ker(C_q - \lambda I)},$$

*then*

$$\operatorname{Spec}(\overline{D}_{q,\lambda}) = \bigcup_{\substack{\tau : q'(\tau)=0 \\ q(\tau)=\lambda}} \left\{ -\frac{m_\tau}{m_\tau+1}, -\frac{m_\tau-1}{m_\tau+1}, \dots, -\frac{1}{m_\tau+1} \right\}, \tag{2.4}$$

*where* $m_\tau = \operatorname{ord}_\tau q'$.

For a finite direct sum of such systems, we write

$$\mathbf{U}' = A\mathbf{U} + \mathbf{f}, \qquad A = C + \frac{D}{z}, \qquad \mathbf{f} = \frac{1}{z}\sum_{\lambda\in\Lambda} \mathbf{b}_\lambda e^{\lambda z}. \tag{2.5}$$

The matrices $C, D$ are block diagonal. The spectra in (2.3) and (2.4) therefore remain contained in the open interval $(-1, 0)$.

# 3 Horizontal projectors over an exponential field

Let $\Gamma \subset k$ be a finitely generated additive subgroup containing $\Lambda$. Since $\Gamma$ is torsion-free, it is free abelian. Put

$$A_\Gamma = \mathbb{C}[\Gamma], \qquad R_\Gamma = \mathbb{C}[z] \otimes_{\mathbb{C}} A_\Gamma, \qquad K_\Gamma = \operatorname{Frac}(R_\Gamma). \tag{3.1}$$

The basis element $\varepsilon_\mu \in A_\Gamma$ is realized as $e^{\mu z}$. If $\omega_1, \dots, \omega_r$ is a $\mathbb{Z}$-basis of $\Gamma$, then

$$A_\Gamma = \mathbb{C}[e^{\pm\omega_1 z}, \dots, e^{\pm\omega_r z}]$$

is a Laurent polynomial ring. By Lemma 2.3, this realization is injective. The differential constants of $K_\Gamma$ are $\mathbb{C}$. The torus $\mathbf{T} = (\mathbb{C}^\times)^r$ acts differentially by rescaling the basis exponentials and fixes $\mathbb{C}(z)$.

**Lemma 3.1** (Finite torus-orbit decomposition)**.** *Let $v \in K_\Gamma^m$. If the complex span of its $\mathbf{T}$-orbit is finite-dimensional, then*

$$v = \sum_{\mu\in S} e^{\mu z} r_\mu(z)$$

*for a finite set $S \subset \Gamma$ and vectors $r_\mu(z) \in \mathbb{C}(z)^m$.*

*Proof.* The finite-dimensional orbit span is a rational representation of the torus and hence is the direct sum of its character spaces. A vector of weight $\mu \in \Gamma$ becomes torus invariant after multiplication by $e^{-\mu z}$, and therefore belongs to $\mathbb{C}(z)^m$. □

**Theorem 3.2** (Horizontal endomorphisms after exponential base change)**.** *If $P \in M_N(K_\Gamma)$ satisfies*

$$P' = [A, P], \qquad A = C + \frac{D}{z}, \tag{3.2}$$

*then*

$$P(z) = \sum_{\mu\in S} e^{\mu z} P_\mu \tag{3.3}$$

*for a finite set $S \subset \Gamma$ and constant matrices $P_\mu \in M_N(\mathbb{C})$ satisfying*

$$[C, P_\mu] = \mu P_\mu, \qquad [D, P_\mu] = 0. \tag{3.4}$$

*In particular, every horizontal idempotent belongs to $M_N(A_\Gamma)$.*

*Proof.* The solution space in $M_N(K_\Gamma)$ of (3.2) has dimension at most $N^2$ over the constant field and is stable under $\mathbf{T}$. Thus the torus orbit of $P$ has finite-dimensional span. By Lemma 3.1,

$$P = \sum_{\mu \in S} e^{\mu z} R_\mu(z), \qquad R_\mu \in M_N(\mathbb{C}(z)).$$

For each weight,

$$R'_\mu + \mu R_\mu = [C, R_\mu] + \frac{1}{z}[D, R_\mu]. \tag{3.5}$$

A pole of $R_\mu$ at a nonzero finite point is impossible because differentiation raises its order by one. At zero, a leading term $z^{-m}R_{-m}$ with $m \geq 1$ would satisfy

$$-mR_{-m} = [D, R_{-m}].$$

Every eigenvalue of $\operatorname{ad} D$ is a difference of two numbers in $(-1, 0)$, hence lies in $(-1, 1)$; it cannot equal the negative integer $-m$. Thus $R_\mu$ is polynomial.

Suppose $R_\mu = R_d z^d + \cdots$ with $d \geq 1$. The leading coefficient gives

$$[C, R_d] = \mu R_d.$$

Because $C$ is semisimple, a nonzero block of $R_d$ maps a $C$-eigenspace $V_\alpha$ to $V_\beta$ with $\beta - \alpha = \mu$. Projecting the next coefficient equation to this block gives

$$dR_{d,\beta\alpha} = \overline{D}_\beta R_{d,\beta\alpha} - R_{d,\beta\alpha}\overline{D}_\alpha.$$

By (2.4), the spectrum of the operator on the right lies in $(-1, 1)$, so it cannot contain the positive integer $d$. Hence $d = 0$, and (3.5) yields (3.4). □

**Lemma 3.3** (Explicit simultaneous spectral gauge)**.** *Partition* $\operatorname{Spec}(C)$ *by*

$$\alpha \sim \beta \quad \Longleftrightarrow \quad \alpha - \beta \in \Gamma,$$

*and choose a representative* $\rho_B$ *in every equivalence class* $B$*. Let* $\pi_\alpha$ *be the spectral projector of* $C$ *at* $\alpha$*, and set*

$$G(z) = \sum_{\alpha \in \operatorname{Spec}(C)} e^{(\alpha - \rho_{[\alpha]})z} \pi_\alpha. \tag{3.6}$$

*Then* $G \in \mathrm{GL}_N(A_\Gamma)$*,* $G(0) = I$*, and every horizontal endomorphism* $P \in M_N(K_\Gamma)$ *satisfies*

$$G^{-1} P G = P(0). \tag{3.7}$$

*Proof.* All exponents in (3.6) belong to $\Gamma$. The orthogonal spectral projectors give

$$G^{-1}(z) = \sum_{\alpha \in \operatorname{Spec}(C)} e^{-(\alpha - \rho_{[\alpha]})z} \pi_\alpha, \qquad G(0) = G^{-1}(0) = I.$$

By Theorem 3.2, the block from $V_\alpha$ to $V_\beta$ is

$$\pi_\beta P(z) \pi_\alpha = \begin{cases} e^{(\beta - \alpha)z} \pi_\beta P(0) \pi_\alpha, & \beta - \alpha \in \Gamma, \\ 0, & \beta - \alpha \notin \Gamma. \end{cases}$$

In the first case $[\alpha] = [\beta]$, so conjugation by $G$ cancels the exponential factor; in the second case the block is zero. This proves (3.7) simultaneously for all $P$. The construction uses only differences within $\Gamma$, so no enlargement of the exponential coefficient ring is needed. □

The appearance of exponential coefficients is necessary. If two data differ by a constant, then

$$\mathbf{U}_{q+c}(z) = e^{cz}\mathbf{U}_q(z),$$

and the projector onto their graph has entries involving $e^{\pm cz}$. Thus the correct rigidity statement is Laurent-polynomial, not constant, after exponential base change.

**Lemma 3.4** (Semisimplicity after exponential base change)**.** *The homogeneous module in* (2.5) *remains semisimple after extension of scalars from* $\mathbb{C}(z)$ *to* $K_\Gamma$*.*

*Proof.* Put $F = \mathbb{C}(z)$, let $L_h/F$ be a Picard–Vessiot field for the homogeneous system, and let $H_h$ be its Galois group. Semisimplicity of the faithful solution representation implies that $H_h$ is reductive. The extension $K_\Gamma/F$ is Picard–Vessiot with torus group $\mathbf{T}$. Choose a joint Picard–Vessiot realization, with no new constants, and write $L = L_h K_\Gamma$ and $\mathcal{G} = \mathrm{Gal}(L/F)$. Restriction gives surjective morphisms

$$\mathcal{G} \longrightarrow H_h, \qquad \mathcal{G} \longrightarrow \mathbf{T}.$$

Let $\mathcal{N}$ be the kernel of the second morphism. Restriction to $L_h$ identifies $\mathcal{N}$ with a closed normal subgroup of $H_h$: normality follows from the surjectivity of the first morphism, and injectivity follows because an automorphism fixing both $L_h$ and $K_\Gamma$ fixes their compositum. A closed normal subgroup of a reductive algebraic group in characteristic zero is reductive. Now $\mathcal{N} = \mathrm{Gal}(L/K_\Gamma)$ is the Picard–Vessiot group of the base-changed homogeneous system, so its solution representation is semisimple. The restriction and fixed-field identifications used here are the Picard–Vessiot correspondence of [15, Chapter 1]. □

## 4 Affine Picard–Vessiot linearity

**Lemma 4.1** (Meromorphic homogeneous-solution exclusion)**.** *Let*

$$y' = \left(B(z) + \frac{D}{z}\right) y,$$

*where* $B$ *is holomorphic at zero and* $\mathrm{Spec}(D) \cap \mathbb{Z} = \varnothing$*. A single-valued meromorphic solution at zero is identically zero. In particular, the operator*

$$\mathscr{L}(v) = v' - Av$$

*is injective on* $K_\Gamma^N$ *for the direct-sum moment connection.*

*Proof.* If a nonzero meromorphic solution has leading term $z^m(v + O(z))$, where $m \in \mathbb{Z}$ and $v \neq 0$, comparison of the coefficient of $z^{m-1}$ gives $mv = Dv$. This contradicts the hypothesis. Every element of $K_\Gamma$ is single-valued and meromorphic at zero, and (2.3) excludes integral residue eigenvalues. □

**Lemma 4.2** (Affine Picard–Vessiot exact sequence)**.** *Let* $K$ *be a differential field with algebraically closed constant field* $\mathbb{C}$*. Let* $Y' = AY$ *have reductive Picard–Vessiot group* $H$ *and constant solution space* $V$*, and let* $u' = Au + f$*. The Picard–Vessiot group* $\widetilde{H}$ *of the augmented system fits into an exact sequence*

$$1 \longrightarrow W \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1, \tag{4.1}$$

*where* $W$ *is an* $H$*-stable vector subspace of* $V$*. If* $V = W \oplus V_0$ *is* $H$*-stable, the translation term projected to* $V_0$ *is a regular algebraic* 1*-cocycle* $H \to V_0$*, and this cocycle is a coboundary.*

*Proof.* Choose a fundamental matrix

$$\widetilde{Y} = \begin{pmatrix} Y & u \\ 0 & 1 \end{pmatrix}.$$

Every $\sigma \in \widetilde{H}$ has a unique affine form

$$\sigma(Y) = Y h_\sigma, \qquad \sigma(u) = u + Y c_\sigma.$$

The restriction map onto $H$ is surjective: an element of $K(Y)$ fixed by its image is fixed by all of $\widetilde{H}$ and hence lies in $K$. The kernel is a closed algebraic subgroup of a vector group, therefore a vector subspace $W$; conjugation makes it an $H$-submodule.

After choosing an $H$-stable complement, the projection of $c_\sigma$ to $V_0$ is independent of the lift of $h_\sigma$ and satisfies

$$c_0(hk) = c_0(h) + h c_0(k).$$

Because a reductive algebraic group over $\mathbb{C}$ is linearly reductive, the associated extension of the trivial representation by $V_0$ splits. Hence there is $v_0 \in V_0$ with

$$c_0(h) = v_0 - h v_0.$$

□

**Theorem 4.3** (Linearity of the functional relation ideal)**.** *For the finite direct-sum moment vector in (2.5), the kernel of*

$$\Theta_{\Gamma,\mathbb{C}} : R_\Gamma[X_1, \dots, X_N] \longrightarrow \mathscr{O}(\mathbb{C}), \qquad X_i \longmapsto U_i(z), \tag{4.2}$$

*is generated by polynomials affine-linear in the variables $X_i$. More precisely, there are $S \in \mathrm{GL}_N(A_\Gamma)$, an integer $0 \le t \le N$, and $g_1, \dots, g_t \in R_\Gamma$ such that, with $\mathbf{V} = S\mathbf{U}$,*

$$V_i = g_i \quad (1 \le i \le t), \qquad V_{t+1}, \dots, V_N \text{ are algebraically independent over } K_\Gamma. \tag{4.3}$$

*Proof.* Put $K = K_\Gamma$. Let $L_h = K(Y)$ be a Picard–Vessiot field for the homogeneous system and let $L = K(Y, \mathbf{U})$ be the Picard–Vessiot field of the augmented system. By Lemma 3.4, $H = \mathrm{Gal}(L_h/K)$ is reductive. Apply Lemma 4.2 and let $W$ be the pure-translation subspace of the constant solution space.

Choose an $H$-equivariant projector $E$ with image $W$. Then

$$P = Y E Y^{-1}$$

is fixed by $H$, belongs to $M_N(K)$, and satisfies $P' = [A, P]$. By Theorem 3.2, $P \in M_N(A_\Gamma)$. By Lemma 3.3, $G^{-1} P G = P(0)$. Since $P(0)$ is a constant idempotent, choose $S_0 \in \mathrm{GL}_N(\mathbb{C})$ with

$$S_0 P(0) S_0^{-1} = \mathrm{diag}(0_t, I_{N-t}).$$

Set $S = S_0 G^{-1} \in \mathrm{GL}_N(A_\Gamma)$. This explicit choice gives

$$S P S^{-1} = P_0 = \mathrm{diag}(0_t, I_{N-t}). \tag{4.4}$$

After a constant change of columns in $Y$, we may also assume that $E = P_0$. If $B = SY$, then $B E B^{-1} = P_0 = E$, so $B$ commutes with $E$ and is block diagonal:

$$B = \mathrm{diag}(B_0, B_W).$$

Write $\mathbf{V} = S\mathbf{U} = (\mathbf{V}_0, \mathbf{V}_W)^{\mathsf{T}}$. The translation term on the first block is the cocycle from Lemma 4.2. Thus, for some constant vector $v_0$,

$$\mathbf{a} = \mathbf{V}_0 + B_0 v_0$$

is fixed by the augmented Galois group and belongs to $K^t$. The matrices $S$ and $S^{-1}$ are entire and invertible at zero. For the transformed homogeneous connection,

$$A_S = (S' + SA)S^{-1} = B'B^{-1}$$

is block diagonal. Thus

$$h = \begin{pmatrix} B_0 v_0 \\ 0 \end{pmatrix}, \qquad y = S^{-1}h$$

satisfy $h' = A_S h$ and $y' = Ay$. Moreover, $B_0 v_0 = \mathbf{a} - \mathbf{V}_0$ is single-valued and meromorphic at zero: $\mathbf{a} \in K^t$ is meromorphic there, and $\mathbf{V}_0$ is entire. The same is true of $y$, so Lemma 4.1 gives $y = 0$, hence $B_0 v_0 = 0$. It follows that

$$\mathbf{V}_0 \in K^t. \tag{4.5}$$

Let $Q = I - P$. Equation (4.4) and (4.5) imply $Q\mathbf{U} \in K^N$. Horizontality of $Q$ gives

$$(Q\mathbf{U})' = A(Q\mathbf{U}) + Q\mathbf{f}. \tag{4.6}$$

The forcing term $Q\mathbf{f}$ has only finitely many torus weights. The operator $\mathscr{L}$ in Lemma 4.1 commutes with the torus and is injective on $K^N$. Consequently the torus orbit of $Q\mathbf{U}$ has finite-dimensional span. By Lemma 3.1,

$$Q\mathbf{U} = \sum_{\mu \in T} e^{\mu z} \mathbf{r}_\mu(z), \qquad \mathbf{r}_\mu \in \mathbb{C}(z)^N.$$

Comparing weights in (4.6) gives

$$\mathbf{r}'_\mu = (C - \mu I)\mathbf{r}_\mu + \frac{D}{z}\mathbf{r}_\mu + \frac{\mathbf{d}_\mu}{z}. \tag{4.7}$$

Pole-order comparison excludes poles away from zero. A leading term $z^{-m}v$, $m \geq 1$, at zero would give $-mv = Dv$, impossible by (2.3). Hence every $\mathbf{r}_\mu$ is polynomial, so

$$Q\mathbf{U} \in R_\Gamma^N. \tag{4.8}$$

It follows that the first $t$ coordinates of $\mathbf{V} = S\mathbf{U}$ are the functions $g_i \in R_\Gamma$ in (4.3).

The pure translations act on the remaining block by

$$\mathbf{V}_W \longmapsto \mathbf{V}_W + B_W c, \qquad c \in \mathbb{C}^{N-t}.$$

If a polynomial over $K$ vanished at $\mathbf{V}_W$, all its translates would vanish. Since $B_W$ is invertible, the affine substitution $X \mapsto \mathbf{V}_W + B_W c$ is injective over $L$, and a polynomial vanishing for every $c \in \mathbb{C}^{N-t}$ is zero. Thus the coordinates of $\mathbf{V}_W$ are algebraically independent over $K$.

Finally, $\mathbf{X} \mapsto S\mathbf{X}$ is an automorphism of $R_\Gamma[\mathbf{X}]$. In the transformed coordinates the kernel of (4.2) is exactly

$$(V_1 - g_1, \ldots, V_t - g_t),$$

because $R_\Gamma$ embeds in $\mathscr{O}(\mathbb{C})$ and the remaining coordinates are algebraically independent over its fraction field. Pulling this ideal back by $S$ proves the theorem. □

**Corollary 4.4** (Descent to algebraic coefficients)**.** *Suppose the data are defined over $k$. Let*

$$R_{\Gamma,k} = k[z] \otimes_k k[\Gamma].$$

*The kernel of*

$$R_{\Gamma,k}[X_1, \dots, X_N] \longrightarrow k[[z]], \qquad X_i \longmapsto U_i(z),$$

*is generated by finitely many polynomials affine-linear in the $X_i$.*

*Proof.* The Taylor coefficients of all moment and exponential functions lie in $k$. The natural map

$$\mathbb{C} \otimes_k k[[z]] \longrightarrow \mathbb{C}[[z]]$$

is injective: in a finite tensor sum, choose the complex scalar coefficients linearly independent over $k$ and compare Taylor coefficients. Flatness therefore identifies the complex kernel with the scalar extension of the kernel over $k$, and the same holds for their affine-linear parts. Apply Theorem 4.3 and descend the ideal equality by faithful flatness. The ring $R_{\Gamma,k}[\mathbf{X}]$ is Noetherian, so finitely many affine-linear elements suffice. □

## 5 Specialization and the formal polynomial theorem

Let

$$\mathscr{E}_k = k[(k, +)]$$

be the group algebra of the additive group of $k$, with basis $\varepsilon_\lambda$. Every element has finite support. At the numerical level we evaluate $\varepsilon_\lambda$ as $e^\lambda$.

**Theorem 5.1** (Polynomial value transfer)**.** *Let $\mathbf{U} = (U_1, \dots, U_N)^{\mathsf{T}}$ be a finite direct sum of moment vectors with algebraic data. Let*

$$P \in \mathscr{E}_k[X_1, \dots, X_N]$$

*and suppose*

$$P\big|_{\varepsilon_\lambda = e^\lambda,\, X_i = U_i(1)} = 0. \tag{5.1}$$

*Then there are finitely many polynomials*

$$L_\nu(z, \mathbf{X}) \in \big(k[z] \otimes_k k[\Gamma]\big)[\mathbf{X}], \tag{5.2}$$

*for a finitely generated additive subgroup $\Gamma \subset k$, such that*

(i) *each $L_\nu$ is affine-linear in the moment variables;*

(ii) *its functional image under $\varepsilon_\lambda \mapsto e^{\lambda z}$ and $X_i \mapsto U_i(z)$ is identically zero;*

(iii) *$P$ belongs to the ideal generated by the specializations $L_\nu(1, \mathbf{X})$ in $k[\Gamma][\mathbf{X}]$.*

*Proof.* Only finitely many exponential indices occur in $P$. Let $\Gamma$ be the additive subgroup generated by them and by all endpoint values in the moment systems. Adjoin to $\mathbf{U}$ the constant function 1 and the finitely many exponential coordinates needed for $P$ and for the forcing terms. Homogenize $P$ with the constant coordinate. By Theorem 2.2 at $z = 1$, the relation lifts to a homogeneous polynomial functional relation with coefficients in $k[z]$ and the same total degree.

Replace every monomial in exponential coordinates by the corresponding single group-algebra basis element, using

$$\prod_j e^{\lambda_j z} = e^{(\sum_j \lambda_j) z},$$

and dehomogenize at the constant coordinate. We obtain

$$\widetilde{P} \in \big(k[z] \otimes_k k[\Gamma]\big)[\mathbf{X}]$$

with zero functional image and specialization $P$ at $z = 1$. By Corollary 4.4, the functional kernel is generated inside this polynomial ring by finitely many affine-linear elements $L_\nu$. Expressing $\widetilde{P}$ in their ideal and evaluating at $z = 1$ proves the result. □

*Remark* 5.2 (Algebraic evaluation parameters). A common parameter $\xi \in k^\times$ is redundant. Replacing every polynomial $q$ by $\xi q$ and every exponential index $\lambda$ by $\xi\lambda$ converts a relation at $z = \xi$ into the statement of Theorem 5.1 at $z = 1$.

*Proof of Theorem 1.2.* Let $x \in \mathcal{A}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$ have zero numerical period. It is represented by a polynomial in finitely many affine-line symbols and zero-dimensional exponential symbols. Apply the canonical twisted polynomial reduction and the treatment of constant and linear polynomials from [14]. Thus $x$ is represented by a polynomial in finitely many reduced symbols

$$\mathsf{P}(q, s, \Delta), \qquad \deg q \geq 2, \quad \deg s \leq \deg q - 2,$$

and the symbols $\mathsf{E}(\lambda)$. Each reduced period is a constant $k$-linear combination of coordinates in a moment vector.

Apply Theorem 5.1. The numerical polynomial relation lies in the ideal generated by specializations at $z = 1$ of affine-linear functional relations. A term $\varepsilon_\mu X_i$ specializes to $e^\mu U_i(1)$, which is the affine-line period obtained by replacing the corresponding polynomial $q$ by $q + \mu$. Hence every specialized generator is a numerical linear relation among compact affine-line periods and zero-dimensional exponential periods. By the linear Kontsevich–Zagier theorem of [14], its class in $\mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$ is zero. Its arbitrary polynomial multiples are therefore zero in the symmetric-algebra quotient $\mathcal{A}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$. Thus $x = 0$, proving injectivity.

□

**Corollary 5.3** (Complete polynomial relation criterion)**.** *For finitely many compact affine-line exponential periods $P_1, \dots, P_m$ and every polynomial $Q \in k[X_1, \dots, X_m]$, one has*

$$Q(P_1, \dots, P_m) = 0$$

*if and only if the relation is generated by the linear formal relations of [14], multiplication, and* (1.2)–(1.4)*.*

**Corollary 5.4** (Tensor-separable higher-dimensional periods)**.** *Consider finitely many product integrals*

$$J_a = \int_{\Gamma_{a,1} \times \cdots \times \Gamma_{a,d_a}} \exp\left(\sum_{j=1}^{d_a} q_{a,j}(x_j)\right) \prod_{j=1}^{d_a} r_{a,j}(x_j)\, dx_j,$$

*where the $q_{a,j}, r_{a,j}$ are algebraic polynomials and every factor chain is compact with algebraic boundary. Every polynomial numerical relation among the $J_a$ is generated, after Fubini, by the one-dimensional formal relations in Theorem 1.2.*

*Proof.* Fubini writes each $J_a$ as a product of compact affine-line exponential periods. Substitute those products and apply Theorem 1.2. □

## 6 Scope and relation to existing work

The theorem concerns the tensor-generated algebra of compact exponential periods on the affine line. It allows arbitrary polynomial $q$, arbitrary polynomial differential $r(x)\,dx$, arbitrary degree-zero algebraic zero-cycles, and polynomial relations of arbitrary degree. Its content is not that Beukers' theorem lifts polynomial relations: that lifting is classical. Nor is it the effective computation of the relation ideal of a fixed tuple, which is available in [6]. The new structural assertion is that, uniformly for this family and after adjoining the required exponential shifts, the entire functional ideal is generated by affine-linear relations; [14] then identifies those relations formally.

The restriction to the affine line and compact chains is essential to the present arithmetic argument. Taylor coefficients of (1.1) are algebraic because polynomial differentials on $\mathbb{A}^1$ have polynomial primitives. On a general algebraic curve they are ordinarily periods, so Beukers' theorem no longer applies directly. Genuinely coupled higher-dimensional exponential periods are also outside the result. The corollary above treats only those obtained from products of the one-dimensional objects. Setting the exponential function equal to one in arbitrary dimension already recovers the ordinary Kontsevich–Zagier conjecture. Noncompact rapid-decay chains introduce the arithmetic of Gamma values and require irregular specialization beyond the present framework.

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