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A Linear Kontsevich–Zagier Theorem for Compact Exponential Periods on the Affine Line

Christopher D. Long Antoine-Auguste Le Blanc*

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galizur@gmail.com

Abstract

Let $q, r \in \overline{\mathbb{Q}}[x]$, let $\Delta = \sum_\alpha d_\alpha[\alpha]$ be a finite degree-zero zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$, and let Γ be any finite piecewise C^1 one-chain in $\mathbb{C}$ with $\partial\Gamma = \Delta$. We consider

$$I_{q,r,\Delta}(z) = \int_\Gamma r(x)e^{zq(x)}\,dx$$

and the compact exponential period $I_{q,r,\Delta}(1)$.

We define the formal exponential-period space on smooth affine curves using bilinearity, functoriality, and twisted Stokes, and consider the subspace generated by the affine-line periods above and by the zero-dimensional exponential periods e^β, $\beta \in \overline{\mathbb{Q}}$. We prove that the period map is injective on this subspace. Equivalently, every $\overline{\mathbb{Q}}$-linear numerical relation among these compact exponential periods is generated by the standard Kontsevich–Zagier operations.

The proof first reduces every polynomial differential uniquely modulo $R \mapsto R' + q'R$. Direct sums of the moment systems from the companion paper *Algebraic Exponential Integrals* are then used to lift a numerical relation by Beukers' theorem. A direct-sum residue argument makes the resulting relation submodule constant and forces an exponential-polynomial identity. A common finite tree and a Cauchy transform show that its combined inverse-branch densities vanish edge by edge. Finally, a simultaneous finite Galois cover realizes every such functional relation by functoriality in the formal period space.

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1 Introduction and main results

Kontsevich and Zagier formulated the period conjecture as the assertion that all identities between integral presentations of ordinary periods follow from additivity, algebraic change of variables, and Stokes' formula [18]. In cohomological language this is the injectivity of a formal period map; see [14, Chapters 12–13] and [7]. Exponential periods replace an algebraic differential form ω by $e^f\omega$, where f is a regular function. The de Rham differential becomes

$$d_f = d + df\wedge,$$

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and Stokes' formula takes the twisted form

$$d(Ge^f) = e^f(dG + G\,df).$$

General definitions and comparison results for exponential periods are developed in [6, 11]; the de Rham and rapid-decay realizations for irregular connections on curves go back to [4, 13]. We use the terminology *exponential period function* in the sense of Fresán and Jossen [10].

This paper proves the linear period principle for compact exponential periods on the affine line. Let $q, r \in \overline{\mathbb{Q}}[x]$, and let

$$\Delta = \sum_{\alpha \in S} d_\alpha[\alpha], \qquad S \subset \overline{\mathbb{Q}} \text{ finite}, \quad d_\alpha \in \overline{\mathbb{Q}}, \quad \sum_\alpha d_\alpha = 0.$$

Choose a finite piecewise C^1 one-chain Γ in $\mathbb{C}$ with $\partial\Gamma = \Delta$, and put

$$I_{q,r,\Delta}(z) = \int_\Gamma r(x)e^{zq(x)}\,dx. \tag{1.1}$$

The one-form in (1.1) has an entire primitive in x, so the function is independent of the chosen chain and is entire in z. The number of interest is $I_{q,r,\Delta}(1)$; a nonzero algebraic evaluation parameter is already absorbed into the polynomial q.

The formal period space used below is defined precisely in Section 2. It is generated by compact relative periods on smooth affine curves and zero-dimensional exponential periods, modulo bilinearity, functoriality, and twisted Stokes. Let $\mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$ denote its subspace generated by the symbols attached to (1.1) and by e^β, $\beta \in \overline{\mathbb{Q}}$.

Theorem 1.1 (Linear Kontsevich–Zagier theorem)**.** *The period map*

$$\mathrm{per} : \mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1} \longrightarrow \mathbb{C}$$

is injective. Thus every $\overline{\mathbb{Q}}$-linear relation among compact exponential periods on the affine line and zero-dimensional exponential periods is generated by bilinearity, functoriality through smooth affine curves, and twisted Stokes.

The arithmetic core is a transfer theorem for a finite family of different polynomials. The degree condition on s_j is the canonical twisted de Rham normal form established in Section 3.

Theorem 1.2 (Transfer and separation of linear relations)**.** *For $1 \le j \le N$, let $q_j, s_j \in \overline{\mathbb{Q}}[x]$ and let Δ_j be a finite degree-zero zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$. Assume*

$$n_j := \deg q_j \ge 2, \qquad \deg s_j \le n_j - 2,$$

and put

$$F_j(z) = I_{q_j,s_j,\Delta_j}(z).$$

Let $\beta_1, \dots, \beta_M \in \overline{\mathbb{Q}}$ be distinct. For $a_1, \dots, a_N, b_1, \dots, b_M \in \overline{\mathbb{Q}}$, one has

$$\sum_{j=1}^N a_j F_j(1) + \sum_{\nu=1}^M b_\nu e^{\beta_\nu} = 0 \tag{1.2}$$

if and only if

$$\sum_{j=1}^N a_j F_j(z) \equiv 0 \tag{1.3}$$

and

$$b_1 = \cdots = b_M = 0.$$

The proof also gives a finite geometric criterion. A common tree in the targets of the q_j carries inverse-branch densities $\psi_{j,e}$, defined in (7.1). For any fixed common tree representation, Theorem 1.2 strengthens to

$$(1.2) \quad \Longleftrightarrow \quad b_1 = \cdots = b_M = 0 \text{ and } \sum_j a_j \psi_{j,e} \equiv 0 \text{ for every open edge } e. \tag{1.4}$$

The vanishing density identities are then shown to be formal consequences of functoriality on one simultaneous algebraic cover. This is the step that turns the E-function relation theorem into a Kontsevich–Zagier statement.

Prior work and scope. The relative exponential-motive framework already accommodates algebraic boundary data [11, Chapter 1]. The strategy of deriving a parameter-dependent differential system and applying Siegel–Shidlovskii/Beukers appears in Jossen's report [16, §§2–3]; the compact-loop construction is developed in [11, §12.7]. The homogeneous polynomial-phase Fourier–Laplace construction is given in [10, Proposition 5.1]. Jossen's 2025 lecture, reporting joint work in progress with Fresán, treats the quadratic marked-point model and explains the independence of its compact integrals and endpoint exponentials, apart from an additional $\sqrt{\pi}$-direction in the full rapid-decay period problem [17]. The contribution here is the uniform linear transfer theorem for arbitrary polynomial phases and amplitudes, allowing different phases in one relation, and the realization of every resulting relation by the stated formal period operations. The simultaneous-cover argument supplies this last step; it is not a consequence of lifting alone.

The companion paper [19] treats $r = 1$ for one fixed polynomial and proves algebraic-value rigidity together with an exact description of constant-coefficient functional relations. The present paper adds arbitrary polynomial differentials, different polynomials in one relation, zero-dimensional exponential periods, and the formal-period interpretation. For any fixed finite tuple of E-functions, Fischler and Rivoal give an algorithm for the ideals of functional and value relations [9]; algorithms for exceptional algebraic values and the required minimal differential equations were developed by Adamczewski–Rivoal and Bostan–Rivoal–Salvy [1, 5]. The result here is instead uniform in the polynomial and boundary data and identifies the relations by a finite geometric criterion and by formal period operations. Delaygue's linear-independence theorem for E-values assumes a separation condition on associated finite singularity sets [8]; no such separation is imposed here.

The conjunction of compact chains with the affine-line polynomial setting is essential to the present argument. In this setting the Taylor coefficients of (1.1) are algebraic and therefore lie within the theory of E-functions. General curves introduce ordinary period coefficients, products lead to higher-dimensional periods, and noncompact rapid-decay chains include Gamma values. The precise boundary of the result is recorded in Section 10.

The paper is organized as follows. Section 2 defines the formal period space. Section 3 gives the canonical reduction of polynomial differentials. Sections 4 and 5 record the input from Paper I and prove direct-sum rigidity. Section 6 proves Theorem 1.2 by Beukers lifting and a common-tree Cauchy transform. Section 8 realizes the resulting functional relations formally on a simultaneous Galois cover. The main theorem and applications follow in Section 9.

2 The formal period space

For a smooth connected affine curve $Y/\overline{\mathbb{Q}}$, put

$$H_1^{\mathrm{rel}}(Y) = \varinjlim_{S \subset Y(\overline{\mathbb{Q}}) \text{ finite}} H_1(Y(\mathbb{C}), S; \overline{\mathbb{Q}}). \tag{2.1}$$

The transition maps are induced by inclusions of finite sets. Every class is represented by a finite piecewise C^1 one-chain with coefficients in $\overline{\mathbb{Q}}$ and boundary supported on algebraic points. Its boundary is a well-defined finite degree-zero zero-cycle on $Y(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$. A morphism $\rho : Y \to X$ induces a pushforward $\rho_* : H_1^{\mathrm{rel}}(Y) \to H_1^{\mathrm{rel}}(X)$.

Definition 2.1 (Compact curve-level formal exponential periods)**.** Let $\mathcal{P}_{\mathrm{curve}}^{\mathrm{KZ,cpt}}$ be the $\overline{\mathbb{Q}}$-vector space generated by symbols

$$\langle Y, f, \omega, \gamma \rangle,$$

where $Y/\overline{\mathbb{Q}}$ is a smooth connected affine curve, $f \in \mathcal{O}(Y)$, $\omega \in \Gamma(Y, \Omega^1_{Y/\overline{\mathbb{Q}}})$, and $\gamma \in H_1^{\mathrm{rel}}(Y)$, together with zero-dimensional symbols $\mathsf{E}(\beta)$ for $\beta \in \overline{\mathbb{Q}}$, modulo the following relations.

(K1) *Bilinearity.* The symbol is $\overline{\mathbb{Q}}$-linear in ω and in γ.

(K2) *Functoriality.* If $\rho : Y \to X$ is a morphism of smooth connected affine curves, $f \in \mathcal{O}(X)$, $\omega \in \Gamma(X, \Omega^1_{X/\overline{\mathbb{Q}}})$, and $\gamma \in H_1^{\mathrm{rel}}(Y)$, then

$$\langle Y, f \circ \rho, \rho^* \omega, \gamma \rangle = \langle X, f, \omega, \rho_* \gamma \rangle. \tag{2.2}$$

(K3) *Twisted Stokes.* If $G \in \mathcal{O}(Y)$ and $\partial\gamma = \sum_p n_p [p]$, then

$$\langle Y, f, dG + G\, df, \gamma \rangle = \sum_p n_p G(p) \mathsf{E}(f(p)). \tag{2.3}$$

The assignments

$$\mathrm{per}(\langle Y, f, \omega, \gamma \rangle) = \int_\gamma e^f \omega, \qquad \mathrm{per}(\mathsf{E}(\beta)) = e^\beta \tag{2.4}$$

define a well-defined $\overline{\mathbb{Q}}$-linear period map. Indeed, $e^f\omega$ is a closed holomorphic one-form, (2.2) is change of variables, and (2.3) is the identity $d(Ge^f) = e^f(dG + G\, df)$.

Let Δ be a finite degree-zero zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$. The long exact sequence of the pair $(\mathbb{A}^1(\mathbb{C}), \operatorname{supp}\Delta)$ shows that there is a unique class

$$\gamma_\Delta \in H_1^{\mathrm{rel}}(\mathbb{A}^1) \qquad \text{with} \qquad \partial\gamma_\Delta = \Delta,$$

because $H_1(\mathbb{A}^1(\mathbb{C}); \overline{\mathbb{Q}}) = 0$ and $\mathbb{A}^1(\mathbb{C})$ is connected.

Definition 2.2 (Affine-line subspace)**.** Let $\mathcal{P}_{\mathbb{A}^1}^{\mathrm{KZ,cpt}}$ be the subspace of $\mathcal{P}_{\mathrm{curve}}^{\mathrm{KZ,cpt}}$ generated by

$$\langle \mathbb{A}^1, q, r(x)\, dx, \gamma_\Delta \rangle, \qquad q, r \in \overline{\mathbb{Q}}[x],$$

for finite degree-zero zero-cycles Δ, and by the zero-dimensional symbols $\mathsf{E}(\beta)$. We abbreviate

$$\mathsf{P}(q, r, \Delta) = \langle \mathbb{A}^1, q, r(x)\, dx, \gamma_\Delta \rangle. \tag{2.5}$$

Then

$$\mathrm{per}(\mathsf{P}(q, r, \Delta)) = I_{q,r,\Delta}(1).$$

3 Twisted reduction of polynomial differentials

The first step is the one-variable twisted de Rham normal form. It is performed for the numerical period at $z = 1$.

Lemma 3.1 (Twisted polynomial reduction)**.** *Let $q \in \overline{\mathbb{Q}}[x]$ have degree $n \geq 2$. For every $r \in \overline{\mathbb{Q}}[x]$ there are unique polynomials $R, s \in \overline{\mathbb{Q}}[x]$ such that*

$$r = R' + q'R + s, \qquad \deg s \leq n - 2. \tag{3.1}$$

For every finite degree-zero zero-cycle $\Delta = \sum_\alpha d_\alpha[\alpha]$, one has in $\mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$

$$\mathsf{P}(q, r, \Delta) = \mathsf{P}(q, s, \Delta) + \sum_\alpha d_\alpha R(\alpha)\mathsf{E}(q(\alpha)). \tag{3.2}$$

Proof. Write $q'(x) = cx^{n-1} + O(x^{n-2})$ with $c \neq 0$. If r has degree $m \geq n - 1$ and leading term ax^m, then

$$R_0 = \frac{a}{c}x^{m-n+1}$$

has the property that $R_0' + q'R_0$ has leading term ax^m. Subtracting this polynomial lowers the degree, and iteration proves existence.

For uniqueness, suppose $R' + q'R + s = 0$ with $\deg s \leq n - 2$. If $R \neq 0$, then

$$\deg(R' + q'R) = \deg R + n - 1 \geq n - 1,$$

since the leading term of $q'R$ cannot cancel with R'. Hence $R = 0$ and then $s = 0$.

Finally, apply twisted Stokes on $\mathbb{A}^1$ to

$$d(Re^q) = e^q(R' + q'R)\,dx.$$

This gives (3.2). □

Lemma 3.2 (Polynomials of degree at most one)**.** *Let $q, r \in \overline{\mathbb{Q}}[x]$ and let $\Delta = \sum_\alpha d_\alpha[\alpha]$.*

(i) *If $q = \beta$ is constant, then*

$$\mathsf{P}(q, r, \Delta) = \left(\sum_\alpha d_\alpha R(\alpha)\right)\mathsf{E}(\beta),$$

where $R' = r$.

(ii) *If $q(x) = ax + b$ with $a \neq 0$, there is a unique $R \in \overline{\mathbb{Q}}[x]$ satisfying $r = R' + aR$, and*

$$\mathsf{P}(q, r, \Delta) = \sum_\alpha d_\alpha R(\alpha)\mathsf{E}(q(\alpha)). \tag{3.3}$$

Proof. The constant case follows from twisted Stokes with a polynomial primitive of r. In the linear case, the map

$$\overline{\mathbb{Q}}[x] \longrightarrow \overline{\mathbb{Q}}[x], \qquad R \longmapsto R' + aR,$$

preserves the degree filtration and induces multiplication by a on every associated graded piece. It is therefore an automorphism. Apply twisted Stokes once more. □

Corollary 3.3 (Formal normal form)**.** *Every element of* $\mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}$ *is represented by a finite* $\overline{\mathbb{Q}}$*-linear combination of symbols*

$$\mathsf{P}(q, s, \Delta), \qquad \deg q \geq 2, \qquad \deg s \leq \deg q - 2,$$

and zero-dimensional symbols $\mathsf{E}(\beta)$*.*

Remark 3.4. The reduction (3.1) is canonical for the period at $z = 1$. For the deformation (1.1), one has

$$d(Re^{zq}) = e^{zq}(R' + zq'R)\,dx,$$

so an unreduced differential does not retain the same decomposition as z varies. This is why Theorem 1.2 is stated for reduced polynomials s_j, while the general formal theorem first applies Lemma 3.1 at $z = 1$.

For example, take $q(x) = x(x-1)$, $r(x) = 2x^2 - x + 1$, and $\Delta = [1] - [0]$. Then $r = R' + q'R$ with $R = x$, so $s = 0$ and

$$\int_0^1 r(x)e^{q(x)}\,dx = \left[xe^{q(x)}\right]_0^1 = 1.$$

Nevertheless, $I_{q,r,\Delta}(0) = 7/6$, so the deformed function is not identically zero. The algebraic value at 1 is entirely an endpoint term; the reduced transfer theorem cannot be applied to the unreduced function.

4 The moment package from Paper I

We recall only the properties of the single-polynomial moment system needed below. An entire function $F(z) = \sum_{m\geq 0} a_m z^m/m!$ is an E-function if its coefficients are algebraic, their joint heights grow at most linearly, and F satisfies a nonzero linear differential equation over $\overline{\mathbb{Q}}[z]$.

Let $q \in \overline{\mathbb{Q}}[x]$ have degree $n \geq 2$, let Δ be a finite degree-zero zero-cycle, and choose a chain Γ with $\partial\Gamma = \Delta$. Put

$$U_j(z) = \int_\Gamma x^j e^{zq(x)}\,dx, \qquad 0 \leq j \leq n-2, \qquad \mathbf{U} = (U_0, \ldots, U_{n-2})^{\mathsf{T}}. \tag{4.1}$$

Theorem 4.1 (Moment package from Paper I)**.** *There are matrices* $C_q, D_q \in M_{n-1}(\overline{\mathbb{Q}})$*, a finite set*

$$\Lambda = \{0\} \cup q(\operatorname{supp}\Delta),$$

and vectors $\mathbf{b}_\lambda \in \overline{\mathbb{Q}}^{n-1}$ *such that*

$$\mathbf{U}' = \left(C_q + \frac{D_q}{z}\right)\mathbf{U} + \frac{1}{z}\sum_{\lambda\in\Lambda} \mathbf{b}_\lambda e^{\lambda z}. \tag{4.2}$$

Moreover:

(i) *every coordinate* U_j *is an* E*-function;*

(ii) *the homogeneous differential module*

$$\mathbf{Y}' = \left(C_q + \frac{D_q}{z}\right)\mathbf{Y}$$

is semisimple over $\mathbb{C}(z)$*;*

(iii) *every eigenvalue of D_q lies strictly between -1 and 0;*

(iv) *C_q is semisimple, and for every eigenvalue λ of C_q, every eigenvalue of the compression*

$$\pi_\lambda D_q|_{\ker(C_q-\lambda I)}$$

lies strictly between -1 and 0, where π_λ is the spectral projection of C_q.

Proof. This is the unnormalized moment system of [19, Proposition 2.2], together with semisimplicity from [19, Proposition 3.1]. The global residue spectrum is computed in [19, Lemma 2.1]; the semisimplicity of C_q and the compressed local spectra are established in the proof of [19, Proposition 3.2]. In particular, no affine normalization of q is required. □

We shall also use the following standard results.

Theorem 4.2 (Beukers, linear lifting)**.** *Let $F_1, \dots, F_s$ be E-functions forming a vector solution of*

$$\mathbf{F}' = A(z)\mathbf{F}, \qquad A(z) \in M_s(\overline{\mathbb{Q}}(z)),$$

and let $T(z) \in \overline{\mathbb{Q}}[z]$ be a common denominator of the entries of A. If $\xi \in \overline{\mathbb{Q}}$ satisfies $\xi T(\xi) \neq 0$ and

$$\alpha_1 F_1(\xi) + \cdots + \alpha_s F_s(\xi) = 0, \qquad \alpha_i \in \overline{\mathbb{Q}},$$

then there are polynomials $A_1(z), \dots, A_s(z) \in \overline{\mathbb{Q}}[z]$ such that

$$\sum_{i=1}^{s} A_i(z)F_i(z) \equiv 0, \qquad A_i(\xi) = \alpha_i.$$

Proof. This is the linear lifting theorem [3, Theorem 3.2 and Lemma 3.1]. No linear independence hypothesis on the coordinates is required. □

Lemma 4.3 (Distinct exponential functions)**.** *If $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ are distinct, then $e^{\lambda_1 z}, \dots, e^{\lambda_s z}$ are linearly independent over $\mathbb{C}(z)$.*

Proof. Choose a nontrivial relation with the fewest nonzero terms. Differentiate and eliminate one term. Minimality forces the logarithmic derivative of a nonzero rational function to be a nonzero constant, which is impossible at infinity. □

Theorem 4.4 (Lindemann–Weierstrass, linear form)**.** *If $\beta_1, \dots, \beta_s \in \overline{\mathbb{Q}}$ are distinct, then $e^{\beta_1}, \dots, e^{\beta_s}$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. See, for example, [2, Chapter 1]. □

5 Direct sums and constant relation submodules

Let $q_1, \dots, q_N \in \overline{\mathbb{Q}}[x]$ have degrees $n_j \geq 2$. For each j, let C_j, D_j be the matrices supplied by Theorem 4.1, and put

$$C = \bigoplus_{j=1}^{N} C_j, \qquad D = \bigoplus_{j=1}^{N} D_j, \qquad \mu = \sum_{j=1}^{N} (n_j - 1). \tag{5.1}$$

The direct sum of the homogeneous moment modules is semisimple. The next result shows that its rational horizontal endomorphisms are still constant, even when different polynomials have common critical values.

Theorem 5.1 (Direct-sum endomorphism rigidity)**.** *If $P(z) \in M_\mu(\mathbb{C}(z))$ satisfies*

$$P' = \left[C + \frac{D}{z}, P\right], \tag{5.2}$$

then P is constant and commutes with both C and D.

Proof. A pole of order $m \geq 1$ at a point $z_0 \neq 0$ is impossible: differentiation raises its order to $m+1$, whereas the right side of (5.2) has order at most m.

Suppose that P has a pole of order $m \geq 1$ at 0 and write

$$P = z^{-m} P_{-m} + O(z^{-m+1}), \qquad P_{-m} \neq 0.$$

The coefficient of z^{-m-1} gives

$$-mP_{-m} = [D, P_{-m}]. \tag{5.3}$$

Every eigenvalue of D lies in $(-1, 0)$ by Theorem 4.1(iii). Hence every eigenvalue of $X \mapsto [D, X]$ is a difference of two numbers in $(-1, 0)$ and lies in $(-1, 1)$. It cannot equal the negative integer $-m$. Thus P has no finite pole and is a polynomial matrix.

Assume that

$$P(z) = P_d z^d + P_{d-1} z^{d-1} + \cdots, \qquad d \geq 1, \quad P_d \neq 0.$$

Comparing the coefficients of z^d and z^{d-1} in (5.2) gives

$$[C, P_d] = 0, \tag{5.4}$$

$$dP_d = [C, P_{d-1}] + [D, P_d]. \tag{5.5}$$

The matrix C is semisimple. For an eigenvalue λ of C, let π_λ be its spectral projection and put

$$P_{d,\lambda} = \pi_\lambda P_d \pi_\lambda, \qquad \overline{D}_\lambda = \pi_\lambda D|_{\ker(C - \lambda I)}.$$

Equation (5.4) makes P_d block diagonal for the C-eigenspace decomposition. Compressing (5.5) to the λ-eigenspace eliminates the C-commutator and yields

$$dP_{d,\lambda} = [\overline{D}_\lambda, P_{d,\lambda}]. \tag{5.6}$$

For at least one λ, the block $P_{d,\lambda}$ is nonzero.

The matrix $\overline{D}_\lambda$ is the direct sum of the compressed blocks belonging to all critical points of all q_j with critical value λ. By Theorem 4.1(iv), all of its eigenvalues lie in $(-1, 0)$. Therefore every eigenvalue of $X \mapsto [\overline{D}_\lambda, X]$ lies in $(-1, 1)$, contradicting (5.6) because d is a positive integer. Hence $d = 0$.

Thus P is constant. Substitution in (5.2) gives $[C, P] = [D, P] = 0$. □

Corollary 5.2 (Constant relation submodules)**.** *Let $\mathcal{M}$ be the row differential module with connection*

$$\boldsymbol{\phi} \longmapsto \boldsymbol{\phi}' + \boldsymbol{\phi}\left(C + \frac{D}{z}\right).$$

Every differential submodule $R \subseteq \mathcal{M}$ has the form

$$R = W \otimes_{\mathbb{C}} \mathbb{C}(z)$$

for a constant subspace $W \subseteq \mathbb{C}^{1\times\mu}$ invariant under right multiplication by C and D.

Proof. The direct-sum module and its dual are semisimple. Hence R has a horizontal complement and a rational horizontal idempotent projector. In row coordinates the projector acts by right multiplication by a matrix $P(z)$ satisfying (5.2). By Theorem 5.1, P is constant and commutes with C, D. Take W to be its constant row image. □

6 Transfer of linear relations

Fix data as in Theorem 1.2. For each j, choose a chain with boundary Δ_j and let $\mathbf{U}_j$ be its moment vector (4.1). Put

$$\mathbf{U} = \bigoplus_{j=1}^{N} \mathbf{U}_j.$$

Since $\deg s_j \le n_j - 2$, there is a constant row $v \in \overline{\mathbb{Q}}^{1\times\mu}$ such that

$$F(z) := \sum_{j=1}^{N} a_j F_j(z) = v\mathbf{U}(z). \tag{6.1}$$

Let $\Lambda \subset \overline{\mathbb{Q}}$ be the finite set consisting of 0, all values $q_j(\alpha)$ with $\alpha \in \operatorname{supp}\Delta_j$, and all β_ν. Put

$$\mathcal{E}_\Lambda = \operatorname{span}_{\mathbb{C}(z)}\{e^{\lambda z} : \lambda \in \Lambda\}.$$

The direct-sum inhomogeneous system has the form

$$\mathbf{U}' = \left(C + \frac{D}{z}\right)\mathbf{U} + \frac{1}{z}\sum_{\lambda\in\Lambda} \boldsymbol{\eta}_\lambda e^{\lambda z}, \tag{6.2}$$

Proposition 6.1 (From a numerical relation to an exponential polynomial)**.** *If* (1.2) *holds, then*

$$F(z) = \sum_{\lambda\in\Lambda} p_\lambda(z) e^{\lambda z} \tag{6.3}$$

for polynomials $p_\lambda \in \overline{\mathbb{Q}}[z]$.

Proof. Adjoin the functions $e^{\lambda z}$, $\lambda \in \Lambda$, to (6.2). The resulting vector consists of E-functions and satisfies a homogeneous first-order system over $\overline{\mathbb{Q}}(z)$ with common denominator z. By Theorem 4.2, the relation at $z = 1$ lifts to

$$\boldsymbol{\phi}(z)\mathbf{U}(z) + \sum_{\lambda\in\Lambda} c_\lambda(z) e^{\lambda z} = 0, \qquad \boldsymbol{\phi}(1) = v, \tag{6.4}$$

where $\boldsymbol{\phi}$ and the c_λ have coefficients in $\overline{\mathbb{Q}}[z]$.

Define

$$\mathcal{R} = \left\{\boldsymbol{\psi} \in \mathbb{C}(z)^{1\times\mu} : \boldsymbol{\psi}\mathbf{U} \in \mathcal{E}_\Lambda\right\}. \tag{6.5}$$

This is a differential submodule of the dual homogeneous moment module. Indeed, if $A = C + D/z$ and $\mathbf{f} = z^{-1}\sum_\lambda \boldsymbol{\eta}_\lambda e^{\lambda z}$, then

$$(\boldsymbol{\psi}\mathbf{U})' = (\boldsymbol{\psi}' + \boldsymbol{\psi}A)\mathbf{U} + \boldsymbol{\psi}\mathbf{f}.$$

The left side and the last term belong to $\mathcal{E}_\Lambda$, so $\boldsymbol{\psi}' + \boldsymbol{\psi}A \in \mathcal{R}$.

By Corollary 5.2,

$$\mathcal{R} = W \otimes_{\mathbb{C}} \mathbb{C}(z)$$

for a constant C, D-invariant row subspace W. The row $\boldsymbol{\phi}$ in (6.4) belongs to $\mathcal{R}$ and is regular at 1. Choose a constant matrix T_W whose columns span the annihilator of W. Then $\boldsymbol{\phi}(z)T_W = 0$, and evaluation at 1 gives $vT_W = 0$. Thus $v \in W$. Hence the constant row v belongs to $\mathcal{R}$, and

$$F = v\mathbf{U} \in \mathcal{E}_\Lambda. \tag{6.6}$$

Choose a constant full-row-rank matrix S whose rows form a basis of W, and put $\mathbf{V} = S\mathbf{U}$. The invariance of W gives constant matrices C_W, D_W satisfying

$$SC = C_W S, \qquad SD = D_W S.$$

Thus

$$\mathbf{V}' = \left(C_W + \frac{D_W}{z}\right)\mathbf{V} + \frac{1}{z}\sum_{\lambda\in\Lambda} \mathbf{b}_{W,\lambda} e^{\lambda z}. \tag{6.7}$$

Every component of $\mathbf{V}$ lies in $\mathcal{E}_\Lambda$. By Lemma 4.3, there are unique rational vectors $\mathbf{r}_\lambda(z)$ such that

$$\mathbf{V}(z) = \sum_{\lambda\in\Lambda} \mathbf{r}_\lambda(z) e^{\lambda z}. \tag{6.8}$$

Comparison of the independent exponential coefficients in (6.7) gives

$$\mathbf{r}_\lambda' = (C_W - \lambda I)\mathbf{r}_\lambda + \frac{D_W}{z}\mathbf{r}_\lambda + \frac{\mathbf{b}_{W,\lambda}}{z}. \tag{6.9}$$

A pole of $\mathbf{r}_\lambda$ at $z_0 \neq 0$ is impossible because its derivative has one order more than every term on the right. If

$$\mathbf{r}_\lambda = z^{-m}u + O(z^{-m+1}), \qquad m \geq 1, \quad u \neq 0,$$

then the coefficient of z^{-m-1} in (6.9) gives $-mu = D_W u$. The eigenvalues of D_W are among those of D and lie in $(-1, 0)$, so this is impossible. Thus every $\mathbf{r}_\lambda$ is a polynomial vector, and (6.6) gives (6.3) with $p_\lambda \in \mathbb{C}[z]$.

Finally, the coefficients of the p_λ are algebraic by a finite Hermite-interpolation argument. If every p_λ is zero, there is nothing to prove. Otherwise choose an integer $\ell \geq 1$ such that $\deg p_\lambda < \ell$ for every λ, write

$$p_\lambda(z) = \sum_{k=0}^{\ell-1} c_{\lambda,k} z^k, \qquad h = \ell|\Lambda|,$$

and differentiate (6.3) at zero. For $0 \leq m < h$,

$$F^{(m)}(0) = \sum_{\lambda\in\Lambda}\sum_{k=0}^{\ell-1} c_{\lambda,k} \left.\frac{d^k}{dt^k} t^m\right|_{t=\lambda}.$$

The coefficient matrix is the transpose of the matrix of the Hermite jet-evaluation map

$$\overline{\mathbb{Q}}[t]_{<h} \longrightarrow \bigoplus_{\lambda\in\Lambda} \overline{\mathbb{Q}}^{\ell}, \qquad H \longmapsto \left(H^{(k)}(\lambda)\right)_{\lambda,\, 0\leq k<\ell}.$$

This map is injective: an element of its kernel is divisible by $\prod_{\lambda\in\Lambda}(t-\lambda)^\ell$, whose degree is h. Since the source and target both have dimension h, the matrix is invertible over $\overline{\mathbb{Q}}$. Every $F^{(m)}(0)$ is algebraic, so the unique solution has $c_{\lambda,k} \in \overline{\mathbb{Q}}$. Hence $p_\lambda \in \overline{\mathbb{Q}}[z]$. □

7 A common tree and the Cauchy-transform criterion

We next show that a reduced linear combination of the functions under consideration cannot be a nonzero exponential polynomial.

Lemma 7.1 (Connectivity of a polynomial branch tree)**.** *Let $q \in \mathbb{C}[x]$ be nonconstant, let Σ be its set of finite critical values, and choose $c \in \mathbb{C} \setminus \Sigma$. For each $\sigma \in \Sigma$, choose a simple arc A_σ from c to σ whose interior avoids Σ, with pairwise disjoint interiors. Put*

$$T_\Sigma = \bigcup_{\sigma \in \Sigma} A_\sigma,$$

with $T_\Sigma = \{c\}$ when Σ is empty. Then $q^{-1}(T_\Sigma)$ is connected.

Proof. If Σ is empty, then q is linear. Otherwise let $E = q^{-1}(c)$. For each σ, a standard loop based at c that runs along A_σ, circles σ, and returns along the other side has a monodromy permutation g_σ of E. The local model $u \mapsto u^r$ shows that the lifts of A_σ whose initial sheets belong to one cycle of g_σ meet at the corresponding point above σ. It follows that the connected components of $q^{-1}(T_\Sigma)$ meet E in the orbits of the group generated by the g_σ.

The loops just described generate $\pi_1(\mathbb{C} \setminus \Sigma, c)$, and $q^{-1}(\mathbb{C} \setminus \Sigma) = \mathbb{C} \setminus q^{-1}(\Sigma)$ is connected. Thus their monodromy action on E is transitive. Hence all points of E lie in one component of $q^{-1}(T_\Sigma)$, and every other point is joined to E by a lifted subarc. □

Lemma 7.2 (Common tree representation)**.** *Let $q_1, \dots, q_N \in \overline{\mathbb{Q}}[x]$ be nonconstant, and let Δ_j be finite degree-zero zero-cycles. There is a finite embedded polygonal tree $T \subset \mathbb{C}$, with algebraic vertices, such that:*

(i) *T contains every finite critical value of every q_j and every value $q_j(\alpha)$ with $\alpha \in \operatorname{supp} \Delta_j$;*

(ii) *for each j there is a finite one-chain Γ_j in $q_j^{-1}(T)$, with coefficients in $\overline{\mathbb{Q}}$, such that $\partial \Gamma_j = \Delta_j$;*

(iii) *over every open edge of T, the chain Γ_j is a finite $\overline{\mathbb{Q}}$-linear combination of analytic inverse branches of q_j.*

Proof. Choose an algebraic point c that is neither a critical value nor an endpoint value for any q_j. Join c to all distinct critical and endpoint values by simple polygonal arcs with pairwise disjoint interiors, using algebraic auxiliary vertices. Their union is a finite tree T.

For fixed j, the union of the arms ending at the critical values of q_j has connected inverse image by Lemma 7.1. Every other arm has regular interior, and all its lifts attach above c; adjoining them preserves connectivity. Thus $q_j^{-1}(T)$ is connected.

Choose $x_{j,*} \in q_j^{-1}(c)$ and write $\Delta_j = \sum_\alpha d_{j,\alpha}[\alpha]$. For each endpoint α, choose an edge path $P_{j,\alpha}$ in $q_j^{-1}(T)$ from $x_{j,*}$ to α, and put

$$\Gamma_j = \sum_\alpha d_{j,\alpha} P_{j,\alpha}.$$

The coefficients at $x_{j,*}$ cancel because $\deg \Delta_j = 0$, so $\partial \Gamma_j = \Delta_j$. Over the interior of any edge, all q_j are unramified, giving the asserted inverse-branch decomposition. □

Orient every open edge e of T. Write the inverse branches of q_j over e as

$$x_{j,e,1}, \dots, x_{j,e,n_j}, \qquad q_j(x_{j,e,i}(t)) = t,$$

and let $f_{j,e,i} \in \overline{\mathbb{Q}}$ be the coefficient with which Γ_j traverses the corresponding lift. Define

$$\psi_{j,e}(t) = \sum_i f_{j,e,i} \frac{s_j(x_{j,e,i}(t))}{q_j'(x_{j,e,i}(t))} \tag{7.1}$$

and

$$\Psi_e(t) = \sum_{j=1}^{N} a_j \psi_{j,e}(t). \tag{7.2}$$

These functions are analytic on the open edge and have integrable algebraic singularities at its endpoints. At a point of ramification index r, an inverse-branch derivative is $O(|t-t_0|^{1/r-1})$.

Theorem 7.3 (Combined moment rigidity)**.** *Let $0 \neq H \in \mathbb{C}[t]$. If*

$$\sum_{j=1}^{N} a_j \int_{\Gamma_j} s_j(x) q_j(x)^m H(q_j(x))\, dx = 0 \qquad (m \geq 0), \tag{7.3}$$

then

$$\Psi_e \equiv 0 \qquad \text{for every open edge } e \subset T. \tag{7.4}$$

Consequently,

$$\sum_{j=1}^{N} a_j I_{q_j,s_j,\Delta_j}(z) \equiv 0. \tag{7.5}$$

Conversely, (7.5) *implies* (7.4)*.*

Proof. For $w \in \mathbb{C} \setminus T$, define

$$\mathcal{C}_H(w) = \sum_{j=1}^{N} a_j \int_{\Gamma_j} \frac{s_j(x) H(q_j(x))}{w - q_j(x)}\, dx. \tag{7.6}$$

For sufficiently large $|w|$, the geometric expansion of the denominator converges uniformly on all chains. Equation (7.3) therefore gives $\mathcal{C}_H(w) = 0$. The complement of a finite plane tree is connected, so the identity theorem yields

$$\mathcal{C}_H \equiv 0 \qquad \text{on } \mathbb{C} \setminus T. \tag{7.7}$$

Changing variables $t = q_j(x)$ on each lifted edge gives the absolutely convergent representation

$$\mathcal{C}_H(w) = \sum_{e} \int_e \frac{H(t)\Psi_e(t)}{w - t}\, dt. \tag{7.8}$$

Fix an interior point t_0 of an edge e with $H(t_0) \neq 0$. Choose a small disk B such that $B \cap T$ is a compact subarc $\eta \Subset e$ containing t_0. All relevant inverse branches extend holomorphically across B, so $\rho(t) = H(t)\Psi_e(t)$ is holomorphic there. The contribution of $T \setminus \eta$ extends holomorphically across B, while

$$\int_\eta \frac{\rho(t)}{w - t}\, dt = \rho(w) \int_\eta \frac{dt}{w - t} - \int_\eta \frac{\rho(t) - \rho(w)}{t - w}\, dt. \tag{7.9}$$

The second term extends holomorphically across η. The two lateral values of the first differ by $2\pi i \varepsilon_e \rho(t_0)$, where $\varepsilon_e \in \{\pm 1\}$ depends on the orientation. Both lateral values of $\mathcal{C}_H$ vanish by (7.7); hence $H(t_0)\Psi_e(t_0) = 0$. Analyticity and the finiteness of the zero set of H give (7.4).

Finally,

$$\sum_j a_j I_{q_j,s_j,\Delta_j}(z) = \sum_e \int_e e^{zt} \Psi_e(t)\, dt = 0.$$

Conversely, functional vanishing makes every Taylor coefficient at $z = 0$ zero. This is (7.3) with $H = 1$, and the preceding argument again gives (7.4). □

The proof is a common-tree version of the one-polynomial Cauchy-transform criterion in [19]; it is closely related to the polynomial moment and algebraic Cauchy-transform methods of [20, 21, 12].

Corollary 7.4 (Exponential-polynomial exclusion)**.** *If*

$$F(z) = \sum_{j=1}^{N} a_j I_{q_j,s_j,\Delta_j}(z)$$

is an exponential polynomial, then $F \equiv 0$.

Proof. If $F \equiv 0$, there is nothing to prove. Otherwise write

$$F(z) = \sum_{\lambda \in \Lambda} p_\lambda(z) e^{\lambda z}$$

with Λ finite and at least one p_λ nonzero. Put

$$L = 1 + \max_{p_\lambda \neq 0} \deg p_\lambda, \qquad \mathscr{D} = \prod_{\lambda \in \Lambda} (\partial_z - \lambda)^L, \qquad H(t) = \prod_{\lambda \in \Lambda} (t - \lambda)^L.$$

Then $\mathscr{D}F = 0$, and differentiation under the integral sign gives

$$0 = \sum_j a_j \int_{\Gamma_j} s_j(x) H(q_j(x)) e^{z q_j(x)}\, dx.$$

Its Taylor coefficients at $z = 0$ are precisely (7.3). Apply Theorem 7.3. □

Proof of Theorem 1.2. Assume (1.2). By Proposition 6.1, the function $F = \sum_j a_j F_j$ is an exponential polynomial. By Corollary 7.4, $F \equiv 0$. The numerical relation now reduces to

$$\sum_{\nu=1}^{M} b_\nu e^{\beta_\nu} = 0,$$

and the distinctness of the β_ν together with Theorem 4.4 gives $b_1 = \cdots = b_M = 0$. The converse follows by evaluating the functional relation at $z = 1$. □

Corollary 7.5 (Geometric criterion for numerical relations)**.** *For any common tree representation from Lemma 7.2, the numerical relation* (1.2) *holds if and only if*

$$b_1 = \cdots = b_M = 0 \qquad \text{and} \qquad \sum_j a_j \psi_{j,e} \equiv 0 \quad \text{for every open edge } e.$$

Proof. Combine Theorem 1.2 with both directions of Theorem 7.3. □

Corollary 7.6 (Linear independence after adjoining zero-dimensional exponential periods)**.** *Let* $F_j = I_{q_j,s_j,\Delta_j}$ *satisfy the reduced hypotheses of Theorem 1.2, and let* $\beta_1, \dots, \beta_M \in \overline{\mathbb{Q}}$ *be distinct. If* $F_1, \dots, F_N$ *are linearly independent over* $\overline{\mathbb{Q}}$ *as entire functions, then*

$$F_1(1), \dots, F_N(1), e^{\beta_1}, \dots, e^{\beta_M}$$

are linearly independent over $\overline{\mathbb{Q}}$.

Proof. Apply Theorem 1.2 to an arbitrary numerical relation. □

Remark 7.7 (Algebraic evaluation points). No extra evaluation parameters are needed in the statements. For $\xi \in \overline{\mathbb{Q}}^{\times}$,

$$I_{q,s,\Delta}(\xi z) = I_{\xi q,s,\Delta}(z).$$

Thus separate nonzero algebraic arguments for a finite family are absorbed individually into the corresponding polynomials.

8 Functional relations in the formal period space

The density criterion is analytic. We now show that every vanishing density identity is already a consequence of functoriality and bilinearity in the formal period space.

Lemma 8.1 (Simultaneous finite Galois cover)**.** *Let $q_1, \dots, q_N \in \overline{\mathbb{Q}}[x]$ be nonconstant, let Σ be the union of their finite critical-value sets, and put $U = \mathbb{A}^1_t \setminus \Sigma$. There exist a finite Galois extension $L/\overline{\mathbb{Q}}(t)$, a smooth connected affine curve $Y/\overline{\mathbb{Q}}$, and a finite morphism*

$$\pi : Y \longrightarrow \mathbb{A}^1_t$$

with the following property. For every j and every $\overline{\mathbb{Q}}(t)$-embedding

$$\sigma : \overline{\mathbb{Q}}(x_j) \hookrightarrow L, \qquad t = q_j(x_j),$$

there is a morphism $\rho_{j,\sigma} : Y \to \mathbb{A}^1_x$ satisfying

$$q_j \circ \rho_{j,\sigma} = \pi. \tag{8.1}$$

The morphism π is finite étale over U. If $B \subset U(\mathbb{C})$ is a nonempty, connected, simply connected open set, and $\widetilde{B}$ is a connected component of $\pi^{-1}(B)$, then the functions

$$\rho_{j,\sigma} \circ (\pi|_{\widetilde{B}})^{-1}$$

are precisely the analytic inverse branches of q_j on B.

Proof. For each j, put $K_j = \overline{\mathbb{Q}}(x_j)$ with $t = q_j(x_j)$. The extension $K_j/\overline{\mathbb{Q}}(t)$ is finite separable of degree $\deg q_j$. Inside a fixed algebraic closure of $\overline{\mathbb{Q}}(t)$, let K_j^{gal} be the Galois closure of $K_j/\overline{\mathbb{Q}}(t)$, and take

$$L = K_1^{\mathrm{gal}} \cdots K_N^{\mathrm{gal}}$$

to be their compositum itself. This is a finite Galois extension of $\overline{\mathbb{Q}}(t)$. Let Y be the normalization of $\mathbb{A}^1_t$ in L. Since L is a field, Y is connected; normalization is finite over the affine line, so Y is affine; and a normal curve in characteristic zero is smooth. Let π be the normalization morphism.

For every embedding $\sigma : K_j \hookrightarrow L$, the element $\sigma(x_j)$ is integral over $\overline{\mathbb{Q}}[t]$, because it satisfies the monic equation obtained from $q_j(X) - t = 0$ after division by the leading coefficient. Hence $\sigma(x_j)$ is regular on Y and defines $\rho_{j,\sigma}$. Applying σ to $q_j(x_j) = t$ proves (8.1).

Each polynomial cover is finite étale over U. Its Galois closure, and then their finite compositum, remains unramified there. Hence π is finite étale over U. Over a simply connected open set B, every connected component of $\pi^{-1}(B)$ maps biholomorphically to B. The $\deg q_j$ embeddings give distinct holomorphic roots of $q_j(X) - t$ on that component and therefore exhaust the inverse branches. □

Theorem 8.2 (Functional relations are formal)**.** *Under the hypotheses of Theorem 1.2, if*

$$\sum_{j=1}^{N} a_j I_{q_j,s_j,\Delta_j}(z) \equiv 0,$$

then

$$\sum_{j=1}^{N} a_j \mathsf{P}(q_j, s_j, \Delta_j) = 0 \qquad \text{in } \mathcal{P}^{\mathrm{KZ,cpt}}_{\mathbb{A}^1}. \tag{8.2}$$

Proof. Choose a common tree and graph chains Γ_j as in Lemma 7.2. By the converse part of Theorem 7.3, every combined density Ψ_e vanishes.

For an oriented edge e, let $\gamma_{j,e,i}$ be the lifted branch chain corresponding to $x_{j,e,i}$. Then

$$\Gamma_j = \sum_{e,i} f_{j,e,i}\gamma_{j,e,i}.$$

The relative class on $\mathbb{A}^1$ is determined by its boundary, so bilinearity gives

$$\mathsf{P}(q_j, s_j, \Delta_j) = \sum_{e,i} f_{j,e,i}\mathsf{P}(q_j, s_j, \partial\gamma_{j,e,i}). \tag{8.3}$$

Take the simultaneous cover $\pi : Y \to \mathbb{A}^1_t$ from Lemma 8.1. Fix an edge e. Choose a thin simply connected neighborhood B_e of its interior containing no critical value, and a component $\widetilde{B}_e$ of $\pi^{-1}(B_e)$. We justify that the lifted edge defines an admissible relative chain. Since π is finite, it is proper, so the lift of the edge interior has compact closure in $\pi^{-1}(\overline{e})$. At a point over an endpoint t_0, a local holomorphic coordinate u gives

$$t - t_0 = u^r$$

for the ramification index $r \geq 1$. This model shows that the lift has a unique limit at each endpoint. Reparametrizing a terminal straight segment by an r-th power makes its lift C^1 up to that endpoint. Doing this at both ends gives a finite piecewise C^1 chain η_e. Its endpoints lie in finite fibers over the algebraic vertices of T; since π is defined over $\overline{\mathbb{Q}}$, these endpoints belong to $Y(\overline{\mathbb{Q}})$. Thus $[\eta_e] \in H_1^{\mathrm{rel}}(Y)$. Orient it so that $\pi_*\eta_e = \overline{e}$ with the chosen orientation.

The morphisms from Lemma 8.1, restricted to $\widetilde{B}_e$, give all inverse branches. Relabeling them, we obtain morphisms $\rho_{j,e,i} : Y \to \mathbb{A}^1_x$ such that

$$\rho_{j,e,i*}\eta_e = \gamma_{j,e,i}. \tag{8.4}$$

Define the global regular differential

$$\Omega_e = \sum_{j,i} a_j f_{j,e,i}\rho^*_{j,e,i}(s_j(x)\,dx) \qquad \text{on } Y. \tag{8.5}$$

On the interior of η_e, with $t = \pi(y)$, one has

$$\Omega_e = \left(\sum_{j,i} a_j f_{j,e,i}\frac{s_j(x_{j,e,i}(t))}{q_j'(x_{j,e,i}(t))}\right) dt = \Psi_e(t)\,dt = 0.$$

A regular differential on a connected smooth curve that vanishes on a nonempty analytic arc vanishes identically. Therefore $\Omega_e = 0$ on Y.

Functoriality, (8.1), (8.4), and bilinearity now give

$$\sum_{j,i} a_j f_{j,e,i} \mathsf{P}(q_j, s_j, \partial\gamma_{j,e,i}) = \langle Y, \pi, \Omega_e, [\eta_e] \rangle$$
$$= 0.$$

Each displayed identity is a separate relative-chain relation, so the lifts η_e may be chosen independently for different edges. Summing these identities and using (8.3) proves (8.2). □

9 Proof of the formal-period theorem and consequences

Proof of Theorem 1.1. Let an element in the kernel of the period map be represented by

$$\sum_{j=1}^{N} a_j \mathsf{P}(q_j, r_j, \Delta_j) + \sum_{\nu=1}^{M} b_\nu \mathsf{E}(\beta_\nu). \tag{9.1}$$

Use Lemma 3.2 to replace every term with $\deg q_j \le 1$ by zero-dimensional symbols. For every remaining term, apply Lemma 3.1 and collect equal exponents. The kernel element is formally equal to

$$\sum_{j=1}^{N'} a_j \mathsf{P}(q_j, s_j, \Delta_j) + \sum_{\nu=1}^{M'} c_\nu \mathsf{E}(\gamma_\nu), \tag{9.2}$$

where $\deg q_j \ge 2$, $\deg s_j \le \deg q_j - 2$, and the γ_ν are distinct.

Its numerical value is zero. If $N' = 0$, the Lindemann–Weierstrass theorem immediately gives $c_1 = \cdots = c_{M'} = 0$. If $N' > 0$, Theorem 1.2 gives

$$\sum_{j=1}^{N'} a_j I_{q_j, s_j, \Delta_j}(z) \equiv 0 \qquad \text{and} \qquad c_1 = \cdots = c_{M'} = 0.$$

In the latter case the functional relation is zero in the formal period space by Theorem 8.2; in either case the exponential sum is zero by ordinary linearity. Hence (9.2), and therefore (9.1), is zero. □

9.1 Algebraic values with polynomial differentials

Theorem 9.1 (Algebraic-value normal form)**.** *Let $q \in \overline{\mathbb{Q}}[x]$ have degree $n \ge 2$, let $\Delta = \sum_\alpha d_\alpha[\alpha]$ be a finite degree-zero zero-cycle, and let $r \in \overline{\mathbb{Q}}[x]$. Write uniquely*

$$r = R' + q'R + s, \qquad \deg s \le n - 2,$$

and put

$$c_\lambda = \sum_{\substack{\alpha \\ q(\alpha) = \lambda}} d_\alpha R(\alpha) \qquad (\lambda \in \overline{\mathbb{Q}}).$$

Then

$$I_{q,r,\Delta}(1) \in \overline{\mathbb{Q}}$$

if and only if

$$I_{q,s,\Delta}(z) \equiv 0 \qquad \text{and} \qquad c_\lambda = 0 \quad (\lambda \ne 0). \tag{9.3}$$

In that case

$$I_{q,r,\Delta}(1) = c_0. \tag{9.4}$$

Proof. Twisted reduction gives

$$I_{q,r,\Delta}(1) = I_{q,s,\Delta}(1) + \sum_{\lambda} c_\lambda e^\lambda.$$

If the left side is an algebraic number a, regard $-a$ as the coefficient of e^0 and apply Theorem 1.2 after collecting equal exponents. The reduced function must vanish identically, and Lindemann–Weierstrass gives $c_\lambda = 0$ for $\lambda \neq 0$ and $c_0 = a$. The converse follows immediately from the displayed reduction. □

For $r = 1$ and $\deg q \geq 2$, one has $R = 0$ and $s = 1$, so Theorem 9.1 recovers the algebraic-value rigidity of Paper I. The linear case is already contained in Lemma 3.2. Thus polynomial differentials add precisely the boundary terms forced by twisted Stokes.

9.2 A terminal-value independence criterion

Proposition 9.2 (Terminal-value criterion)**.** *For $1 \leq j \leq N$, let $F_j(z) = I_{q_j,s_j,\Delta_j}(z)$ satisfy the reduced hypotheses of Theorem 1.2. Suppose that for every j there is an endpoint $\alpha_j \in \operatorname{supp} \Delta_j$ such that:*

(i) *the coefficient d_{j,α_j} of $[\alpha_j]$ in Δ_j is nonzero;*

(ii) *$q_j'(\alpha_j)s_j(\alpha_j) \neq 0$;*

(iii) *$\lambda_j = q_j(\alpha_j)$ is not a critical value of any q_k, and is not equal to $q_k(\alpha)$ for any $\alpha \in \operatorname{supp} \Delta_k$ except the designated endpoint $(k, \alpha) = (j, \alpha_j)$.*

Then $F_1, \dots, F_N$ are linearly independent over $\overline{\mathbb{Q}}$ as entire functions. Consequently their values at 1 remain linearly independent after adjoining any finite family of zero-dimensional exponential periods with distinct algebraic exponents.

Proof. Choose the common tree so that each λ_j is the terminal vertex of its own arm, and construct the graph chains using simple paths. Let e_j be the terminal open edge of this arm. Since λ_j is a regular value of every q_k, each lift of the closed arm is a leaf edge of $q_k^{-1}(T)$.

For $k = j$, the leaf ending at α_j occurs in the simple path from the base vertex to α_j with coefficient d_{j,α_j}. It cannot occur in a simple path to any other endpoint, because entering and leaving a leaf would repeat its attachment vertex. By (iii), no graph chain from any other datum uses a leaf above the same terminal arm. Hence the combined density on e_j for a relation $\sum_k a_k F_k$ is

$$a_j d_{j,\alpha_j} \frac{s_j(x_j(t))}{q_j'(x_j(t))},$$

where $x_j(t)$ is the branch ending at α_j.

If $\sum_k a_k F_k \equiv 0$, then Theorem 7.3 makes this density identically zero. Conditions (i) and (ii) imply $a_j = 0$. This holds for every j, proving functional independence. The numerical assertion follows from Corollary 7.6. □

Corollary 9.3 (An explicit family)**.** *For every $m \geq 1$, the numbers*

$$1, \quad \int_0^1 e^{x^2}\,dx, \quad \int_0^1 e^{2x^2}\,dx, \quad \dots, \quad \int_0^1 e^{mx^2}\,dx$$

are linearly independent over $\overline{\mathbb{Q}}$. They remain linearly independent after adjoining finitely many $e^{\beta_1}, \dots, e^{\beta_s}$ with distinct nonzero algebraic exponents.

Proof. Apply Proposition 9.2 with $q_j(x) = jx^2$, $s_j = 1$, and $\Delta_j = [1] - [0]$. The designated terminal values are the distinct regular values j. The zero-dimensional exponential periods are handled by Corollary 7.6. □

10 Scope of the theorem

The word *linear* refers to relations among period values, not to the degree of the polynomial. The theorem permits arbitrary polynomial regular functions and polynomial differentials on the affine line, but its scope has three essential boundaries.

First, the chains are compact and the source is $\mathbb{A}^1$. For a finite map from a general algebraic curve, the Taylor coefficients

$$\int_\Gamma f^m \omega$$

are generally ordinary one-periods rather than algebraic numbers. Their linear arithmetic is governed by the theory of one-periods [15], but Beukers' E-function specialization theorem no longer applies directly.

Second, products of the periods are higher-dimensional exponential periods:

$$\left(\int_{\Gamma_1} e^{q_1}\omega_1\right)\left(\int_{\Gamma_2} e^{q_2}\omega_2\right) = \int_{\Gamma_1\times\Gamma_2} e^{q_1 \boxplus q_2}\, \omega_1 \boxtimes \omega_2.$$

Thus algebraic relations of degree greater than one require a higher-dimensional linear theorem. On the differential-equation side, tensor constructions widen the residue spectrum and may introduce integral differences, so Theorem 5.1 does not extend formally.

Third, noncompact rapid-decay chains require the homology of irregular connections. The perfect de Rham–rapid-decay pairing is available in broad generality [4, 13], but even the elementary integrals

$$\int_0^\infty x^m e^{-x^n}\, dx = \frac{1}{n}\Gamma\left(\frac{m+1}{n}\right)$$

lead to unresolved algebraic-relation questions for Gamma values. No arithmetic specialization theorem for this noncompact setting is used here.

Finally, allowing arbitrary dimensions and setting the regular function in the exponential equal to zero recovers ordinary periods. The unrestricted exponential-period conjecture therefore contains the ordinary Kontsevich–Zagier period conjecture. The present theorem concerns exactly the compact affine-line subspace of Definition 2.2.

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