hexagon:2611.00736v1

Algebraic Exponential Integrals: Transcendence and Linear Relations

Christopher D. Long Antoine-Auguste Le Blanc*

August 2026

galizur@gmail.com

Abstract

Let $q \in \overline{\mathbb{Q}}[x]$ be nonconstant, and let $\Delta = \sum_\alpha d_\alpha[\alpha]$ be a finite degree-zero zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$. If Γ is any finite piecewise C^1 one-chain in $\mathbb{C}$ with $\partial\Gamma = \Delta$, put

$$I_{q,\Delta}(z) = \int_\Gamma e^{zq(x)}\,dx.$$

We prove that every nonzero function $I_{q,\Delta}$ is a purely transcendental E-function: for every $\xi \in \overline{\mathbb{Q}}^\times$, $I_{q,\Delta}(\xi)$ is transcendental. Equivalently,

$$I_{q,\Delta}(1) \in \overline{\mathbb{Q}} \quad\Longleftrightarrow\quad I_{q,\Delta} \equiv 0.$$

In particular, $\int_A^B e^{q(x)}dx$ is transcendental for every pair of distinct algebraic endpoints. For a fixed polynomial q, we also determine all $\overline{\mathbb{Q}}$-linear relations among finite families of these values: they are exactly the specializations of constant-coefficient functional relations, and the latter are characterized by vanishing of explicit inverse-branch densities on a finite tree in the target of q.

The proof combines Beukers' lifting theorem with a homogeneous moment connection. A Fourier–Laplace description proves semisimplicity, while an explicit residue calculation shows that every rational horizontal endomorphism is constant. An algebraic value therefore forces an exponential-polynomial identity. Constant-coefficient annihilation and a Cauchy-transform argument then force the original exponential period function to vanish identically.

2020 Mathematics Subject Classification. Primary 11J81; Secondary 14F10, 34M35.
Keywords. E-functions, exponential period functions, Siegel–Shidlovskii theorem, Fourier–Laplace transform, polynomial moments, Cauchy transforms, transcendence, linear relations.

1 Introduction and main results

Following Fresán and Jossen, we use *exponential period function* for a parameter-dependent integral of the form $\int_\gamma e^{-zf}\omega$ with algebraic data [9]. The present functions form a one-dimensional relative family: $X = \mathbb{A}^1$, the regular function is $-q$, the algebraic differential form is dx, and the integration chain has finite boundary. This is compatible with the general terminology of exponential periods [5, 10]. We refer simply to the polynomial q and to its values and critical values.

*Le Blanc cryptographic author identifier: b604812530b3535b23d6dbb0f5abe32d1ed1a9de35fb4f7d04d62cff08ea19c8

Definition 1.1 (Zero-cycles and exponential period functions)**.** A finite zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$ is a formal sum

$$\Delta = \sum_{\alpha \in S} d_\alpha [\alpha], \qquad S \subset \overline{\mathbb{Q}} \text{ finite}, \quad d_\alpha \in \overline{\mathbb{Q}}.$$

Its degree is $\deg \Delta = \sum_{\alpha \in S} d_\alpha$. Suppose $\deg \Delta = 0$. Choose a finite piecewise C^1 one-chain Γ in $\mathbb{C}$, with complex coefficients, such that $\partial \Gamma = \Delta$, and set

$$I_{q,\Delta}(z) = \int_\Gamma e^{zq(x)}\, dx. \tag{1.1}$$

For every z, the one-form $e^{zq(x)}dx$ has an entire primitive in x. Thus (1.1) is independent of the chosen Γ, and compactness of Γ shows that $I_{q,\Delta}$ is entire in z.

Theorem 1.2 (Algebraic-value rigidity)**.** *Let $q \in \overline{\mathbb{Q}}[x]$ be nonconstant, and let Δ be a finite degree-zero zero-cycle on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$. Then*

$$I_{q,\Delta}(1) \in \overline{\mathbb{Q}} \quad \Longleftrightarrow \quad I_{q,\Delta}(z) \equiv 0. \tag{1.2}$$

In the equivalent cases, $I_{q,\Delta}(1) = 0$.

The fixed evaluation point 1 entails no loss: a nonzero algebraic parameter is already part of the polynomial in the exponent.

Corollary 1.3 (Pure transcendence and algebraic zeros)**.** *If $I_{q,\Delta}$ is not identically zero, then*

$$I_{q,\Delta}(\xi) \notin \overline{\mathbb{Q}} \qquad (\xi \in \overline{\mathbb{Q}}^\times).$$

In particular, $I_{q,\Delta}$ has no nonzero algebraic zero. As an E-function, it is purely transcendental in the standard sense.

Proof. For $\xi \in \overline{\mathbb{Q}}^\times$,

$$I_{\xi q,\Delta}(z) = I_{q,\Delta}(\xi z).$$

The right side is nonzero as an entire function if and only if $I_{q,\Delta}$ is nonzero. Apply Theorem 1.2 to the polynomial ξq. □

Corollary 1.4 (A single algebraic exponential integral)**.** *Let $q \in \overline{\mathbb{Q}}[x]$ be nonconstant and let $A, B \in \overline{\mathbb{Q}}$ with $A \neq B$. Then, for every $\xi \in \overline{\mathbb{Q}}^\times$,*

$$\int_A^B e^{\xi q(x)}\, dx$$

is transcendental and nonzero.

Proof. For $\Delta = [B] - [A]$, one has $I_{q,\Delta}(0) = B - A \neq 0$, so $I_{q,\Delta}$ is not identically zero. Apply Corollary 1.3. □

For a fixed E-function, Adamczewski and Rivoal gave an algorithm that computes all nonzero algebraic arguments at which its value is algebraic [1]; Bostan, Rivoal, and Salvy made the required minimal-equation computations effective in practice [4]. The point of Theorem 1.2 is different: it gives a uniform answer for the entire family indexed by (q, Δ) and identifies exactly when the

function itself vanishes. Delaygue's Lindemann–Weierstrass theorem gives linear independence under a separation condition on the finite singularity sets of the associated G-series [7]. The functions attached to a fixed q arise from one common moment connection and can have overlapping singularity data; the results below require no such separation and describe the whole relation space.

There is classical overlap in special cases. For example,

$$\int_0^A e^{cx^n}\,dx = A\,{}_1F_1\!\left(\frac{1}{n};1+\frac{1}{n};cA^n\right),$$

so monomial exponents belong to the hypergeometric E-function setting; see, for example, Delaygue's discussion of hypergeometric values [7]. The no-algebraic-zero consequence also connects with recent work of Fischler and Rivoal on zeros of E-functions [8].

Prior work and the present contribution. The use of parameter-dependent exponential periods and Siegel–Shidlovskii theory is established in the work of Fresán and Jossen. Jossen's 2018 report develops polynomial-phase families with noncompact rapid-decay cycles and states the André–Beukers lifting principle for polynomial relations [13, §§2–3]. For compact loops on $\mathbb{G}_m$, Fresán and Jossen derive E-function systems from linearity, differentiation, and twisted Stokes [10, §12.7]. They also identify the absolute polynomial-phase connection with the Fourier–Laplace transform of the reduced polynomial direct image [9, Proposition 5.1].

In a 2025 lecture reporting joint work in progress with Fresán, Jossen presented the quadratic marked-point model, including $\int_0^a e^{-zx^2}\,dx$ at algebraic a and the endpoint exponentials [14]. He described its differential Galois group and explained the arithmetic independence obtained from Siegel–Shidlovskii theory, separating the additional $\sqrt{\pi}$-independence question for the full rapid-decay period field. Our contribution is the uniform rigidity theorem for compact algebraic boundary data and arbitrary polynomial phase, together with the complete fixed-q linear-relation criterion. The argument uses constant-coordinate endomorphism rigidity and inverse-branch densities, without assuming simple critical points or distinct critical values.

The proof has three parts. First, moments of (1.1) satisfy one explicit inhomogeneous E-function system. Second, the homogeneous connection is semisimple and every rational horizontal endomorphism is constant; this converts a polynomial-coefficient relation supplied by Beukers' theorem into a constant-coordinate relation. Third, an exponential-polynomial identity produces vanishing polynomial moments, and a Cauchy transform on a finite tree forces all inverse-branch densities to vanish.

Scope of the differential form. The fixed form dx is essential. For $q(x) = x(x-1)$ and $r(x) = 2x^2 - x + 1$, one has

$$r(x)e^{q(x)} = \frac{d}{dx}\big(xe^{q(x)}\big), \qquad \int_0^1 r(x)e^{q(x)}\,dx = 1.$$

Yet the function $\int_0^1 r(x)e^{zq(x)}\,dx$ is not identically zero: its value at $z = 0$ is $7/6$. Thus Theorem 1.2 does not extend unchanged to arbitrary polynomial amplitudes.

2 The moment system

We use the standard definition: an entire function $F(z) = \sum_{m\ge 0} a_m z^m/m!$ is an E-function if its coefficients are algebraic, $h(a_0,\dots,a_m) = O(m)$, and F satisfies a nonzero linear differential

equation over $\overline{\mathbb{Q}}[z]$.

Assume throughout this section that $n = \deg q \geq 2$, and put

$$V = \mathbb{C}[x]_{<n-1}.$$

For $0 \leq j \leq n-2$, divide

$$q(x)x^j = h_j(x)q'(x) + r_j(x), \qquad \deg r_j < n-1. \tag{2.1}$$

Define endomorphisms C, D of V by

$$\mathsf{C}(x^j) = r_j, \qquad \mathsf{D}(x^j) = -h_j'.$$

Let $C = \mathsf{C}^{\mathsf{T}}$ and $D = \mathsf{D}^{\mathsf{T}}$ in the monomial basis $1, x, \ldots, x^{n-2}$.

Lemma 2.1 (Residue spectrum)**.** *The matrix D is triangular, with*

$$\operatorname{Spec}(D) = \left\{-\frac{1}{n}, -\frac{2}{n}, \ldots, -\frac{n-1}{n}\right\}. \tag{2.2}$$

Proof. Comparison of the leading terms in (2.1) gives

$$h_j(x) = \frac{x^{j+1}}{n} + O(x^j).$$

Consequently, $-h_j'(x) = -(j+1)x^j/n + O(x^{j-1})$. □

Fix a chain Γ with boundary Δ, and put

$$U_j(z) = \int_\Gamma x^j e^{zq(x)}\,dx, \qquad \mathbf{U} = (U_0, \ldots, U_{n-2})^{\mathsf{T}}. \tag{2.3}$$

Let

$$\Lambda = \{0\} \cup q(\operatorname{supp}\Delta), \tag{2.4}$$

with repetitions removed, and define

$$\mathbf{h}(x) = (h_0(x), \ldots, h_{n-2}(x))^{\mathsf{T}}, \qquad \mathbf{b}_\lambda = \sum_{\substack{\alpha \in \operatorname{supp}\Delta \\ q(\alpha)=\lambda}} d_\alpha \mathbf{h}(\alpha) \quad (\lambda \in \Lambda).$$

Proposition 2.2 (Moment system)**.** *The moment vector satisfies*

$$\mathbf{U}' = \left(C + \frac{D}{z}\right)\mathbf{U} + \frac{1}{z}\sum_{\lambda\in\Lambda} \mathbf{b}_\lambda e^{\lambda z}. \tag{2.5}$$

Each U_j is an E-function.

Proof. Integration by parts in (2.1) gives, for $z \neq 0$,

$$U_j' = \int_\Gamma \left(r_j - \frac{h_j'}{z}\right) e^{zq}\,dx + \frac{1}{z}\sum_{\alpha\in\operatorname{supp}\Delta} d_\alpha h_j(\alpha) e^{zq(\alpha)},$$

which is (2.5). Both sides are meromorphic at 0, and the identity holds there by continuation.

Write

$$U_j(z) = \sum_{m\ge 0} u_{j,m}\frac{z^m}{m!}, \qquad u_{j,m} = \int_\Gamma x^j q(x)^m\,dx,$$

and

$$q(x)^m = \sum_{k=0}^{nm} \gamma_{m,k} x^k.$$

A polynomial primitive gives the exact formula

$$u_{j,m} = \sum_{k=0}^{nm} \gamma_{m,k} \frac{\displaystyle\sum_{\alpha\in\operatorname{supp}\Delta} d_\alpha \alpha^{j+k+1}}{j+k+1}. \tag{2.6}$$

Thus every $u_{j,m}$ is algebraic.

We record the arithmetic growth explicitly. Let K be a number field containing the coefficients of q and all α, d_α. Uniformly over embeddings $\sigma : K \hookrightarrow \mathbb{C}$, integration of a primitive along the segment from 0 to $\sigma(\alpha)$ gives

$$|\sigma(u_{j,m})| \le A^{m+j+2} \tag{2.7}$$

for a constant $A \ge 1$. Choose an integer $d \ge 1$ such that d times every coefficient of q, every α, and every d_α is an algebraic integer. For $m \ge 0$, set

$$\delta_m = d^{(n+1)m+j+2}\operatorname{lcm}(1,2,\dots,nm+j+1). \tag{2.8}$$

Formula (2.6) shows that $\delta_m u_{j,\ell}$ is an algebraic integer for every $0 \le \ell \le m$: indeed, $d^\ell\gamma_{\ell,k}$, dd_α, and $d^{j+k+1}\alpha^{j+k+1}$ are integral, while $k \le n\ell \le nm$, and the total exponent of d needed for each term satisfies $\ell + j + k + 2 \le (n+1)m + j + 2$. Since $\log\operatorname{lcm}(1,\dots,N) = O(N)$, (2.7) and (2.8) imply

$$h(u_{j,0},\dots,u_{j,m}) = O(m).$$

Finally, (2.5) shows that the $\overline{\mathbb{Q}}(z)$-span of the U_j and the finitely many functions $e^{\lambda z}$ is stable under differentiation. Hence each U_j satisfies a nonzero scalar differential equation over $\overline{\mathbb{Q}}[z]$. This proves all defining properties of an E-function. □

When $q(x) = ax + b$ is linear, the explicit identity

$$I_{q,\Delta}(z) = \frac{1}{az}\sum_{\alpha\in\operatorname{supp}\Delta} d_\alpha e^{zq(\alpha)} \qquad (z \ne 0) \tag{2.9}$$

shows directly that $I_{q,\Delta}$ is also an E-function.

We use the following linear form of Beukers' refinement of the Siegel–Shidlovskii theorem [3, Theorem 3.2 and Lemma 3.1]. No linear independence hypothesis on the coordinates is required.

Theorem 2.3 (Beukers, linear lifting)**.** *Let $\mathbf{F} = (F_1,\dots,F_N)^{\mathsf{T}}$ be a vector of E-functions satisfying $\mathbf{F}' = A(z)\mathbf{F}$ with $A(z) \in M_N(\overline{\mathbb{Q}}(z))$, and suppose that A has no pole at $z = 1$. For every relation*

$$a_1F_1(1) + \cdots + a_NF_N(1) = 0, \qquad a_i \in \overline{\mathbb{Q}},$$

there are polynomials $p_i \in \overline{\mathbb{Q}}[z]$ such that

$$\sum_{i=1}^{N} p_i(z)F_i(z) = 0, \qquad p_i(1) = a_i \quad (1 \le i \le N).$$

3 The homogeneous moment connection

The homogeneous part of (2.5) is

$$\mathbf{Y}' = \left(C + \frac{D}{z}\right)\mathbf{Y}. \tag{3.1}$$

The Fourier–Laplace description used below is the polynomial direct-image construction of [9, Proposition 5.1], with the sign and dual conventions made explicit. We recall it in the moment coordinates needed for the subsequent residue argument; the finite-monodromy proof of semisimplicity does not require genericity of q.

Proposition 3.1 (Semisimplicity)**.** *The differential module over $\mathbb{C}(z)$ defined by (3.1) is semisimple.*

Proof. Let Σ be the finite set of critical values of q, and put $V_0 = \mathbb{A}^1_t \setminus \Sigma$. Over V_0, the finite map $q : q^{-1}(V_0) \to V_0$ defines a rank-n permutation local system $\mathcal{L}$. Its monodromy group is finite, so $\mathcal{L}$ is semisimple by Maschke's theorem. The trace map splits it as

$$\mathcal{L} \simeq \mathbb{C}_{V_0} \oplus \mathcal{L}^0, \qquad \mathcal{L}^0 = \ker(\mathrm{Tr} : \mathcal{L} \to \mathbb{C}_{V_0}).$$

The map q is finite, hence proper and small. The small-map intermediate-extension identity [6, Remark 4.2.4] gives

$$Rq_*\mathbb{C}_{\mathbb{A}^1_x}[1] \simeq \mathrm{IC}(\mathcal{L}). \tag{3.2}$$

Intermediate extension preserves direct sums and takes irreducible local systems to simple perverse sheaves. Thus the semisimplicity of $\mathcal{L}$ makes this perverse sheaf semisimple. Under the Riemann–Hilbert correspondence and its compatibility with proper direct image, the regular holonomic $\mathcal{D}$-module $q_+\mathcal{O}_{\mathbb{A}^1_x}$ is therefore semisimple; the constant summand of $\mathcal{L}$ corresponds to $\mathcal{O}_{\mathbb{A}^1_t}$ [12].

Use a Fourier variable τ and the Weyl-algebra convention

$$t \longmapsto -\partial_\tau, \qquad \partial_t \longmapsto \tau. \tag{3.3}$$

This is the inverse convention to the transform with kernel $e^{t\tau}$ in [17, §1.3.b]; our convention corresponds to the kernel $e^{-t\tau}$. Fourier transformation is an exact equivalence on holonomic Weyl modules and preserves simple objects and direct sums. The transform of $\mathcal{O}_{\mathbb{A}^1_t}$ is supported at $\tau = 0$, hence disappears after localization to $\mathbb{C}(\tau)$. A nonzero generic localization of a simple Weyl module remains irreducible. Indeed, its $\mathbb{C}[\tau] \setminus \{0\}$-torsion is a Weyl submodule: if $p(\tau)m = 0$, then $p(\tau)^2\partial_\tau m = 0$. Thus a simple module whose localization is nonzero embeds in that localization. Any nonzero $\mathbb{C}(\tau)$-differential submodule meets the original module after clearing a denominator, and simplicity then forces equality. It follows that the generic Fourier transform of $q_+\mathcal{O}_{\mathbb{A}^1_x}$ is semisimple.

For completeness, we identify this generic transform explicitly. Factor q as its graph embedding followed by projection to the t-line. A graph generator δ_q satisfies

$$(t - q(x))\delta_q = 0, \qquad (\partial_x + q'(x)\partial_t)\delta_q = 0.$$

After (3.3), the transformed generator satisfies

$$(\partial_\tau + q(x))\widehat{\delta}_q = 0, \qquad (\partial_x + \tau q'(x))\widehat{\delta}_q = 0.$$

In particular, $\partial_x(p\widehat{\delta}_q) = (p' - \tau q' p)\widehat{\delta}_q$: the relation on the generator induces a minus sign in the differential on coefficient polynomials. The generic direct image along the x-line is therefore the degree-zero cohomology of

$$\mathbb{C}(\tau)[x] \xrightarrow{\partial_x - \tau q'(x)} \mathbb{C}(\tau)[x],$$

with τ-connection induced by $\partial_\tau - q(x)$. The displayed map is injective: for nonzero p, the term $-\tau q' p$ has larger x-degree than p'. Hence the generic transform is

$$\mathbb{C}(\tau)[x]/(\partial_x - \tau q'(x))\mathbb{C}(\tau)[x], \qquad \nabla_{\partial_\tau}[p] = [\partial_\tau p - qp]. \tag{3.4}$$

This is the algebraic Fourier–Laplace direct-image construction of [17, §1.3.b], with the inverse convention fixed in (3.3).

Pulling (3.4) back by $\tau = -z$ gives

$$\mathcal{M}_q = \mathbb{C}(z)[x]/(\partial_x + zq'(x))\mathbb{C}(z)[x], \qquad \nabla_{\partial_z}[p] = [\partial_z p + qp]. \tag{3.5}$$

The connection descends to this quotient because

$$[\partial_z + q, \partial_x + zq'] = q' - q' = 0. \tag{3.6}$$

The classes of $1, x, \dots, x^{n-2}$ form a basis: the relation $[zq'p] = -[p']$ reduces every higher monomial, while degree comparison proves independence. In this basis, (2.1) gives the connection matrix $\mathsf{C} + \mathsf{D}/z$. The period system (3.1) is the corresponding dual system, and duality preserves semisimplicity. □

Proposition 3.2 (Rational endomorphism rigidity)**.** *If $P(z) \in M_{n-1}(\mathbb{C}(z))$ satisfies*

$$P' = \left[C + \frac{D}{z}, P\right], \tag{3.7}$$

then P is constant. The horizontal endomorphism algebra of (3.1) is therefore

$$Z(C) \cap Z(D).$$

The same constancy conclusion holds for the dual module.

Proof. A pole at $z_0 \neq 0$ is impossible: differentiation raises its order by one, whereas the right side of (3.7) does not. At 0, a leading term $z^{-m}P_{-m}$ with $m \geq 1$ would give

$$-mP_{-m} = [D, P_{-m}].$$

By (2.2), every eigenvalue of $\operatorname{ad} D$ lies strictly between -1 and 1, so the negative integer $-m$ cannot occur. Thus P has no finite pole and is polynomial.

It remains to exclude positive degree. Transpose (3.7) and write $Q = P^{\mathsf{T}}$. The operator C is multiplication by q on $\mathbb{C}[x]/(q')$. For a critical point τ, put

$$m_\tau = \operatorname{ord}_\tau q', \qquad \lambda_\tau = q(\tau),$$

and define

$$g_{\tau,k}(x) = \frac{q'(x)}{(x-\tau)^k}, \qquad 1 \leq k \leq m_\tau.$$

The Chinese remainder theorem shows that all the $g_{\tau,k}$ form a basis of V. Moreover,

$$\mathsf{C} g_{\tau,k} = \lambda_\tau g_{\tau,k},$$

because

$$q g_{\tau,k} = \frac{q-\lambda_\tau}{(x-\tau)^k} q' + \lambda_\tau g_{\tau,k}.$$

Thus C is semisimple, with eigenspaces indexed by critical values.

Let E_λ be the λ-eigenspace of C, let π_λ be its spectral projection, and put

$$\overline{D}_\lambda = \pi_\lambda \mathsf{D}|_{E_\lambda}.$$

Fix τ with $\lambda_\tau = \lambda$, and write $m = m_\tau$. Define

$$F_{\tau,0} = 0, \qquad F_{\tau,k} = \operatorname{span}_{\mathbb{C}}\{g_{\tau,1}, \dots, g_{\tau,k}\} \quad (1 \le k \le m).$$

We claim the explicit flag relation

$$\overline{D}_\lambda g_{\tau,k} + \frac{m+1-k}{m+1} g_{\tau,k} \in F_{\tau,k-1} \qquad (1 \le k \le m). \tag{3.8}$$

To prove it, write $y = x - \tau$ and $q - \lambda = a y^{m+1} + O(y^{m+2})$, with $a \neq 0$. Euclidean division gives

$$\mathsf{D} g_{\tau,k} = -\left(\frac{q-\lambda}{(x-\tau)^k}\right)',$$

so

$$\begin{aligned} g_{\tau,k} &= (m+1) a y^{m-k} + O(y^{m-k+1}), \\ \mathsf{D} g_{\tau,k} &= -(m+1-k) a y^{m-k} + O(y^{m-k+1}). \end{aligned}$$

Their combination in (3.8) therefore vanishes at τ to order at least $m-k+1$. If $\sigma \neq \tau$ has the same critical value, then $(x-\tau)^{-k}$ is regular at σ, while $q - \lambda$ vanishes there to order $m_\sigma + 1$. Hence $\mathsf{D} g_{\tau,k}$ vanishes at σ to order at least m_σ, and its component in that factor of $\mathbb{C}[x]/(q')$ is zero.

Under the Chinese remainder decomposition, $F_{\tau,k-1}$ consists precisely of the classes supported in the τ-factor and divisible there by y^{m-k+1}. Projection by π_λ removes all factors over other critical values. The preceding vanishing statements thus prove (3.8). In particular, every $F_{\tau,k}$ is $\overline{D}_\lambda$-invariant. On each $F_{\tau,m}$ the matrix in the ordered basis $g_{\tau,1}, \dots, g_{\tau,m}$ is triangular, with diagonal

$$-\frac{m}{m+1}, -\frac{m-1}{m+1}, \dots, -\frac{1}{m+1}. \tag{3.9}$$

As τ ranges over the critical points with value λ, these lists give the full spectrum of $\overline{D}_\lambda$, with multiplicities. In particular, all its eigenvalues lie in $(-1, 0)$, including when distinct critical points have the same value and different ramification indices.

Suppose

$$Q(z) = Q_d z^d + Q_{d-1} z^{d-1} + \cdots, \qquad d \ge 1, \quad Q_d \neq 0.$$

The top two coefficients of the transposed equation give

$$[Q_d, \mathsf{C}] = 0, \qquad d Q_d = [Q_{d-1}, \mathsf{C}] + [Q_d, \mathsf{D}]. \tag{3.10}$$

The first identity makes Q_d block diagonal for the eigenspace decomposition of C. Choose a critical value λ for which $Q_{d,\lambda} \neq 0$. Projecting the second identity to this block yields

$$dQ_{d,\lambda} = [Q_{d,\lambda}, \overline{D}_\lambda].$$

After triangularizing $\overline{D}_\lambda$, the eigenvalues of the commutator operator on the right are pairwise differences of the numbers in (3.9). They all lie strictly between -1 and 1, so none equals the positive integer d. This contradiction proves that $d = 0$.

A constant solution of (3.7) commutes with both C and D, and the converse is immediate. The calculation for a right-acting endomorphism of the dual gives the same equation. □

Corollary 3.3 (Constant splitting)**.** *Every differential submodule of the dual of* (3.1) *has the form*

$$W \otimes_{\mathbb{C}} \mathbb{C}(z)$$

for a constant subspace $W \subseteq \mathbb{C}^{1\times(n-1)}$ *invariant under right multiplication by* C *and* D*.*

Proof. By Proposition 3.1, a differential submodule has a differential complement and hence a horizontal idempotent projector. By Proposition 3.2, this projector is constant. Its image is the required subspace W; substituting the constant projector into (3.7) shows that W is C- and D-invariant. □

4 From an algebraic value to an exponential polynomial

Put

$$\mathcal{E} = \operatorname{span}_{\mathbb{C}(z)}\{e^{\lambda z} : \lambda \in \Lambda\}.$$

Proposition 4.1. *If* $U_0(1) \in \overline{\mathbb{Q}}$*, then*

$$U_0(z) = \sum_{\lambda\in\Lambda} p_\lambda(z)e^{\lambda z} \tag{4.1}$$

with $p_\lambda \in \overline{\mathbb{Q}}[z]$*.*

Proof. Adjoin the exponential coordinates $e^{\lambda z}$ to the moment vector. By (2.5), the resulting vector of E-functions satisfies a homogeneous block system whose only finite pole is at $z = 0$. Let $a = U_0(1) \in \overline{\mathbb{Q}}$. Applying Theorem 2.3 to $U_0(1) - a \cdot 1 = 0$ gives a polynomial row $\boldsymbol{\phi}(z)$ and polynomials $c_\lambda(z)$ such that

$$\boldsymbol{\phi}(z)\mathbf{U}(z) + \sum_{\lambda\in\Lambda} c_\lambda(z)e^{\lambda z} = 0, \qquad \boldsymbol{\phi}(1) = (1, 0, \dots, 0). \tag{4.2}$$

Define

$$\mathcal{R} = \{\boldsymbol{\psi} \in \mathbb{C}(z)^{1\times(n-1)} : \boldsymbol{\psi}\mathbf{U} \in \mathcal{E}\}.$$

If $A(z) = C + D/z$ and $\mathbf{f} = z^{-1}\sum_\lambda \mathbf{b}_\lambda e^{\lambda z}$, then

$$(\boldsymbol{\psi}\mathbf{U})' = (\boldsymbol{\psi}' + \boldsymbol{\psi}A)\mathbf{U} + \boldsymbol{\psi}\mathbf{f}.$$

Since $\mathcal{E}$ is stable under differentiation, this proves that $\mathcal{R}$ is a differential submodule of the dual homogeneous connection. By Corollary 3.3, $\mathcal{R} = W \otimes_{\mathbb{C}} \mathbb{C}(z)$ for a constant C- and D-invariant subspace W. Equation (4.2) puts $\boldsymbol{\phi}$ in $\mathcal{R}$. Every constant linear form annihilating W therefore

annihilates $\boldsymbol{\phi}(z)$ identically, and hence also annihilates $\boldsymbol{\phi}(1)$. Thus $\boldsymbol{\phi}(1) \in W$, without specializing any rational basis. Hence the constant row $\mathbf{e}_0 = (1, 0, \ldots, 0)$ belongs to W, and $U_0 = \mathbf{e}_0\mathbf{U}$ belongs to $\mathcal{E}$.

Choose a full-row-rank constant matrix S whose rows form a basis of W, and put $\mathbf{V} = S\mathbf{U}$. There are constant matrices C_W, D_W such that

$$SC = C_W S, \qquad SD = D_W S.$$

Every component of $\mathbf{V}$ belongs to $\mathcal{E}$. The functions $e^{\lambda z}$ are linearly independent over $\mathbb{C}(z)$, so there is a unique expansion

$$\mathbf{V} = \sum_{\lambda \in \Lambda} \mathbf{r}_\lambda(z) e^{\lambda z}, \qquad \mathbf{r}_\lambda \in \mathbb{C}(z)^{\dim W}.$$

Equating exponential coefficients in the restricted moment system gives

$$\mathbf{r}'_\lambda = \left(C_W - \lambda I + \frac{D_W}{z}\right)\mathbf{r}_\lambda + \frac{S\mathbf{b}_\lambda}{z}. \tag{4.3}$$

A pole away from 0 is impossible by order comparison. If $\mathbf{r}_\lambda = z^{-m}v + O(z^{-m+1})$ at 0, with $m \geq 1$, then (4.3) gives

$$-mv = D_W v.$$

The eigenvalues of the restriction D_W are among those of D, hence lie strictly between -1 and 0 by (2.2); none is a negative integer. Thus every $\mathbf{r}_\lambda$ is polynomial, and so are the coefficients p_λ in (4.1).

It remains to show that their coefficients are algebraic. Choose an integer $d \geq 0$ bounding the degrees of all the p_λ, write $p_\lambda(z) = \sum_{k=0}^{d} a_{\lambda,k} z^k$, and put $M = (d+1)|\Lambda|$. Differentiating (4.1) at 0 gives

$$U_0^{(m)}(0) = \sum_{\lambda \in \Lambda} \sum_{k=0}^{d} a_{\lambda,k} \left.\frac{d^k}{dt^k} t^m\right|_{t=\lambda} \qquad (0 \leq m < M). \tag{4.4}$$

The square coefficient matrix has algebraic entries and is invertible. Indeed, its transpose is the Hermite evaluation map

$$\overline{\mathbb{Q}}[t]_{<M} \longrightarrow \overline{\mathbb{Q}}^M, \qquad f \longmapsto \left(f^{(k)}(\lambda)\right)_{\lambda \in \Lambda,\, 0 \leq k \leq d}.$$

A polynomial in its kernel is divisible by $\prod_{\lambda \in \Lambda}(t - \lambda)^{d+1}$, of degree M, so must vanish. Thus the map is injective and hence invertible. All derivatives $U_0^{(m)}(0)$ are algebraic by (2.6); solving (4.4) yields $a_{\lambda,k} \in \overline{\mathbb{Q}}$. Therefore $p_\lambda \in \overline{\mathbb{Q}}[z]$. □

5 Moments, Cauchy transforms, and functional relations

We first isolate the geometric representation used by the analytic argument.

Lemma 5.1 (Tree representation)**.** *Let $q \in \mathbb{C}[x]$ be nonconstant and let Δ be a finite degree-zero zero-cycle on $\mathbb{C}$ with complex coefficients. There exist a finite embedded tree T in the t-plane, containing all finite critical values of q and all values $q(\alpha)$ for $\alpha \in \operatorname{supp}\Delta$, and a one-chain $\widetilde{\Gamma}$ in the graph $q^{-1}(T)$ such that $\partial\widetilde{\Gamma} = \Delta$.*

Orient every open edge e of T. Over e, write $\widetilde{\Gamma}$ as a linear combination of analytic inverse branches $x_{e,i}$ of q, with coefficients $f_{e,i}$, and set

$$\psi_e(t) = \sum_i f_{e,i} x'_{e,i}(t) = \sum_i \frac{f_{e,i}}{q'(x_{e,i}(t))}. \tag{5.1}$$

Then ψ_e is analytic on the open edge, has at worst integrable singularities at its endpoints, and for every entire function F,

$$\int_\Gamma F(q(x))\,dx = \sum_e \int_e F(t)\psi_e(t)\,dt, \tag{5.2}$$

where Γ is any one-chain with boundary Δ.

Proof. Choose a regular value c of q, distinct from all endpoint values, and join c to the distinct finite critical values and endpoint values by simple analytic arcs with pairwise disjoint interiors. Their union is a finite tree T. The inverse image of the critical-value arms is the standard polynomial constellation. It is connected because the monodromy of the polynomial covering on a regular fiber is transitive [15, §2.1]. The remaining arms attach to this connected subgraph, so $q^{-1}(T)$ is connected.

Subdivide the graph so that every point of $\operatorname{supp}\Delta$ is a vertex. Fix a vertex x_* and choose an edge path P_α from x_* to each $\alpha \in \operatorname{supp}\Delta$. Then

$$\widetilde{\Gamma} = \sum_{\alpha \in \operatorname{supp}\Delta} d_\alpha P_\alpha$$

has boundary Δ, because $\deg\Delta = 0$. The inverse-branch description on open edges gives (5.1). Near a ramified endpoint of index r, one has $x'_{e,i}(t) = O(|t-t_0|^{1/r-1})$, which is integrable.

The one-form $F(q(x))dx$ is entire in x, so its integral depends only on the boundary. Replacing Γ by $\widetilde{\Gamma}$ and changing variables $t = q(x)$ edge by edge gives (5.2). □

Theorem 5.2 (Moment rigidity and the edge-density criterion)**.** *In the setting of Lemma 5.1, let $0 \neq H \in \mathbb{C}[t]$. If*

$$\int_\Gamma q(x)^m H(q(x))\,dx = 0 \qquad (m \geq 0), \tag{5.3}$$

then every density ψ_e in (5.1) vanishes identically. Consequently,

$$I_{q,\Delta}(z) \equiv 0.$$

For any fixed tree representation,

$$I_{q,\Delta} \equiv 0 \quad \Longleftrightarrow \quad \psi_e \equiv 0 \text{ for every open edge } e \subset T. \tag{5.4}$$

Proof. Every differential $q(x)^m H(q(x))dx$ is polynomial and has a polynomial primitive, so (5.3) remains true after replacing Γ by $\widetilde{\Gamma}$. For $s \in \mathbb{C} \setminus T$, define

$$\mathcal{C}_H(s) = \int_{\widetilde{\Gamma}} \frac{H(q(x))}{s - q(x)}\,dx.$$

For sufficiently large $|s|$, the geometric series for $(s-q(x))^{-1}$ converges uniformly on the compact chain, and (5.3) gives $\mathcal{C}_H(s) = 0$. The complement of a finite embedded tree is connected, so the identity theorem yields

$$\mathcal{C}_H \equiv 0 \qquad \text{on } \mathbb{C} \setminus T. \tag{5.5}$$

By (5.2),

$$\mathcal{C}_H(s) = \sum_e \int_e \frac{H(t)\psi_e(t)}{s-t}\,dt. \tag{5.6}$$

Fix an interior point t_0 of an edge e with $H(t_0) \neq 0$, and choose a compact analytic subarc $\eta \Subset e$ containing t_0. Contributions to (5.6) from outside η extend holomorphically across a small disk about t_0. Put $\rho(t) = H(t)\psi_e(t)$ and

$$J(s) = \int_\eta \frac{\rho(t)}{s-t}\,dt.$$

If a, b are the oriented endpoints of η, then locally

$$J(s) = \rho(s)\big(\log(s-a) - \log(s-b)\big) - \int_\eta \frac{\rho(t)-\rho(s)}{t-s}\,dt.$$

The second term is holomorphic across η, while the two lateral branches of the logarithmic term differ by $2\pi i\varepsilon_e\rho(t_0)$, where $\varepsilon_e \in \{\pm 1\}$. By (5.5), both lateral values of the full Cauchy transform are zero. Hence $H(t_0)\psi_e(t_0) = 0$, and therefore $\psi_e(t_0) = 0$. The point t_0 was arbitrary outside the finite zero set of H; analyticity gives $\psi_e \equiv 0$ on every open edge.

Taking $F(t) = e^{zt}$ in (5.2) now gives $I_{q,\Delta} \equiv 0$. Conversely, if $I_{q,\Delta} \equiv 0$, its Taylor coefficients give (5.3) with $H = 1$, so the preceding argument forces all densities to vanish. The reverse implication in (5.4) follows directly from (5.2). □

This argument is a local Cauchy-transform form of techniques used in the polynomial moment problem and for Cauchy-type integrals of algebraic functions [15, 16, 11].

6 Proofs and linear relations

Proof of Theorem 1.2. The reverse implication is immediate. Assume that $I_{q,\Delta}(1) = \gamma \in \overline{\mathbb{Q}}$.

Suppose first that $q(x) = ax + b$ is linear. Formula (2.9) at $z = 1$ gives

$$\sum_{\alpha \in \operatorname{supp}\Delta} d_\alpha e^{q(\alpha)} - a\gamma e^0 = 0. \tag{6.1}$$

The values $q(\alpha)$ are pairwise distinct. Combine the two coefficients of e^0 if one endpoint satisfies $q(\alpha) = 0$. The linear form of the Lindemann–Weierstrass theorem [2, Chapter 1] then makes every coefficient in (6.1) zero. Thus $d_\alpha = 0$ for all endpoints with $q(\alpha) \neq 0$. If $q(\alpha_0) = 0$, then $d_{\alpha_0} = a\gamma$, while $\deg\Delta = 0$ gives $d_{\alpha_0} = 0$ and hence $\gamma = 0$. If no such endpoint exists, the coefficient of e^0 gives $\gamma = 0$ directly. Therefore $\Delta = 0$ and $I_{q,\Delta} \equiv 0$.

Now suppose $\deg q \geq 2$. By Proposition 4.1,

$$U_0(z) = I_{q,\Delta}(z) = \sum_{\lambda \in \Lambda} p_\lambda(z) e^{\lambda z}.$$

If $U_0 \equiv 0$, there is nothing to prove. Otherwise set

$$N = 1 + \max_{p_\lambda \neq 0} \deg p_\lambda, \qquad L = \prod_{\lambda \in \Lambda} (\partial_z - \lambda)^N,$$

and

$$H(t) = \prod_{\lambda \in \Lambda} (t - \lambda)^N.$$

Then $LU_0 = 0$, and differentiation under the integral sign gives

$$0 = LU_0(z) = \int_\Gamma H(q(x))e^{zq(x)}\,dx.$$

Comparing Taylor coefficients at $z = 0$ gives (5.3). Apply Theorem 5.2. □

Corollary 6.1 (Transfer of linear relations)**.** *Let $q \in \overline{\mathbb{Q}}[x]$ be nonconstant, let $\Delta_1, \dots, \Delta_r$ be finite degree-zero zero-cycles on $\mathbb{A}^1(\overline{\mathbb{Q}})$ with coefficients in $\overline{\mathbb{Q}}$, and put $I_j(z) = I_{q,\Delta_j}(z)$. For $a_0, \dots, a_r \in \overline{\mathbb{Q}}$,*

$$a_0 + \sum_{j=1}^{r} a_j I_j(1) = 0 \tag{6.2}$$

if and only if

$$a_0 = 0, \qquad \sum_{j=1}^{r} a_j I_j(z) \equiv 0.$$

Consequently,

$$\dim_{\overline{\mathbb{Q}}} \operatorname{span}_{\overline{\mathbb{Q}}}\{1, I_1(1), \dots, I_r(1)\} = 1 + \dim_{\overline{\mathbb{Q}}} \operatorname{span}_{\overline{\mathbb{Q}}}\{I_1, \dots, I_r\}.$$

Proof. Put $\Delta = \sum_j a_j\Delta_j$. Relation (6.2) says $I_{q,\Delta}(1) = -a_0 \in \overline{\mathbb{Q}}$. By Theorem 1.2, $I_{q,\Delta} \equiv 0$, and then its value at 1 gives $a_0 = 0$. The converse and the dimension identity are immediate. □

The same equivalence holds with the common evaluation point 1 replaced by any $\xi \in \overline{\mathbb{Q}}^\times$: replace q by ξq and then rescale the function variable.

Corollary 6.2 (Geometric description of the relation space)**.** *In the setting of Corollary 6.1, choose one tree T containing all critical values of q and all values $q(\operatorname{supp}\Delta_j)$. For each j, choose a graph chain representing Δ_j as in Lemma 5.1, and denote its density on an open edge e by $\psi_{j,e}$. Then*

$$a_0 + \sum_{j=1}^{r} a_j I_j(1) = 0$$

if and only if

$$a_0 = 0, \qquad \sum_{j=1}^{r} a_j \psi_{j,e} \equiv 0 \quad \text{for every open edge } e \subset T.$$

Thus all $\overline{\mathbb{Q}}$-linear relations among the values are determined by a finite family of inverse-branch identities on T.

Proof. The graph chain for $\Delta = \sum_j a_j\Delta_j$ is the same linear combination of the chosen graph chains, and its density on e is $\sum_j a_j\psi_{j,e}$. Combine Corollary 6.1 with the exact criterion (5.4). □

Corollary 6.3 (Linear independence over distinct regular values)**.** *Let $q \in \overline{\mathbb{Q}}[x]$ be nonconstant, and let $a_0, \dots, a_r \in \overline{\mathbb{Q}}$ be such that $q(a_0), \dots, q(a_r)$ are pairwise distinct regular values of q. Then, for every $\xi \in \overline{\mathbb{Q}}^\times$,*

$$1, \quad \int_{a_0}^{a_1} e^{\xi q(x)}\,dx, \quad \dots, \quad \int_{a_0}^{a_r} e^{\xi q(x)}\,dx$$

are linearly independent over $\overline{\mathbb{Q}}$.

Proof. Absorb ξ into q. It suffices to prove that the functions

$$I_j(z) = \int_{a_0}^{a_j} e^{zq(x)}\,dx, \qquad 1 \le j \le r,$$

are linearly independent. A functional relation produces a degree-zero zero-cycle

$$\Delta = \sum_{j=1}^{r} c_j[a_j] - \left(\sum_{j=1}^{r} c_j\right)[a_0]$$

whose exponential period function vanishes identically. Choose the tree so that each $q(a_i)$ is the endpoint of a separate terminal arm. Over the arm ending at $q(a_i)$, regularity and distinctness imply that the only nonzero terminal boundary coefficient can occur on the lift ending at a_i. The density there is therefore

$$\frac{d_i}{q'(x_i(t))},$$

where d_i is the coefficient of $[a_i]$ in Δ. By (5.4), this density vanishes, so $d_i = 0$ for every i. Hence all $c_j = 0$. Apply Corollary 6.1. □

Acknowledgments and AI-assisted research disclosure

This manuscript was developed through interactive work between Christopher D. Long and ChatGPT 5.6 Pro. ChatGPT 5.6 Pro assisted in proof discovery, organization, symbolic checking, reference verification, adversarial auditing, and drafting. Claude Opus 5 contributed to the semisimple-projector strategy and independent audits. The AI systems are not authors. The authors bear full responsibility for all statements, proofs, citations, and any remaining errors. This draft has not yet undergone independent peer review or formal verification.

References

[1] B. Adamczewski and T. Rivoal, *Exceptional values of E-functions at algebraic points*, Bull. Lond. Math. Soc. **50** (2018), no. 4, 697–708. doi:10.1112/blms.12168.

[2] A. Baker, *Transcendental Number Theory*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990.

[3] F. Beukers, *A refined version of the Siegel–Shidlovskii theorem*, Ann. of Math. (2) **163** (2006), no. 1, 369–379. doi:10.4007/annals.2006.163.369.

[4] A. Bostan, T. Rivoal, and B. Salvy, *Minimization of differential equations and algebraic values of E-functions*, Math. Comp. **93** (2024), no. 347, 1427–1472. doi:10.1090/mcom/3912.

[5] J. Commelin, P. Habegger, and A. Huber, *Exponential periods and o-minimality*, arXiv:2007.08280v4 (2025), to appear in Ann. Inst. Fourier (Grenoble).

[6] M. A. A. de Cataldo and L. Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633. doi:10.1090/S0273-0979-09-01260-9.

[7] É. Delaygue, *A Lindemann–Weierstrass theorem for E-functions*, J. Reine Angew. Math. **820** (2025), 75–85. doi:10.1515/crelle-2024-0090.

[8] S. Fischler and T. Rivoal, *Zeros of E-functions and of exponential polynomials defined over* $\overline{\mathbb{Q}}$, arXiv:2503.20345 (2025).

[9] J. Fresán and P. Jossen, *A non-hypergeometric E-function*, Ann. of Math. (2) **194** (2021), no. 3, 903–942. doi:10.4007/annals.2021.194.3.7.

[10] J. Fresán and P. Jossen, *Exponential motives*, preliminary book manuscript, 301-page version consulted October 5, 2026. http://javier.fresan.perso.math.cnrs.fr/expmot.pdf.

[11] L. Gavrilov and F. Pakovich, *Moments on Riemann surfaces and hyperelliptic Abelian integrals*, Comment. Math. Helv. **89** (2014), no. 1, 125–155. doi:10.4171/CMH/314.

[12] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-Modules, Perverse Sheaves, and Representation Theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Boston, MA, 2008. doi:10.1007/978-0-8176-4523-6.

[13] P. Jossen, *Special values of E-functions as exponential periods*, lecture report for *Periods and L-values of Motives*, Simons Foundation, May 2018, 11 pp. Lecture report.

[14] P. Jossen, *Examples around the exponential period conjecture*, CIRM, recorded lecture, February 20, 2025. doi:10.24350/CIRM.V.20308403.

[15] F. Pakovich and M. Muzychuk, *Solution of the polynomial moment problem*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 3, 633–657. doi:10.1112/plms/pdp010.

[16] F. Pakovich, N. Roytvarf, and Y. Yomdin, *Cauchy-type integrals of algebraic functions*, Israel J. Math. **144** (2004), 221–291. doi:10.1007/BF02916714.

[17] C. Sabbah, *Fourier transformation and Stokes structures*, lecture notes, version of February 13, 2020, https://perso.pages.math.cnrs.fr/users/claude.sabbah/livres/sabbah_chicago1205.pdf.