hexagon:2611.00733v2

TWO SHAPES OF MAXIMAL DEGREE 135 IN THE PLANE JACOBIAN PROBLEM: EXCLUSION OF THE DEGREE PAIRS $(135, 90)$ AND $(90, 135)$ WITH CORNERS $(9, 36)$ AND $(17/9, 4)$

DAVID NIEDBALA GIRAUDIN

Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, two shapes have starting corner $(9, 36)$ and final corner $(17/9, 4)$: the standard $(3, 2)$-pairs of degrees $(135, 90)$ and the standard $(2, 3)$-pairs of degrees $(90, 135)$. We show that no such pair exists, when it is normalised as in Corollary 5.21 of Guccione–Guccione–Valqui. The two sides of the Newton polygon are described by the results of Guccione–Guccione–Valqui. In suitable polynomial coordinates the pair is confined to an explicit finite system of polynomial equations. Its leading part is a pair of polynomials of degrees 12 and 8 in one variable with $3p_3q_2' - 2p_3'q_2$ constant, which is unique up to the symmetries of the system and has rational coefficients; the remaining equations are then solved by three steps of linear algebra over $\mathbb{Q}$, and the 29 compatibility conditions, polynomials in five unknowns, generate the unit ideal. The computation is exact and is reproduced by an ancillary script (SymPy, and Singular for one Gröbner basis). All statements are proved relative to the cited results. Nothing in this note has been checked by a third party.

1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L = K[x, y]$, $[P, Q] = P_xQ_y - P_yQ_x$ and (m, n) is $(3, 2)$ or $(2, 3)$. We use the notation of [1]: for a weight (ρ, σ), $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, $\mathrm{st}_{\rho,\sigma}$ and $\mathrm{en}_{\rho,\sigma}$ its end points; $L^{(l)} = K[x^{\pm 1/l}, y]$. *Standard (m, n)-pairs*, their *starting triple* and their *regular corners* are defined in [1, Definitions 4.3, 6.2, 5.5]. Put $t := x^2y^9$.

Standing data 1.1. (P, Q) is a standard (m, n)-pair in L with $[P, Q] = J \in K^\times$, $\deg P = 45m$, $\deg Q = 45n$, starting triple $\big((9, 36), (1, 0), (9, -2)\big)$, and $(-1, 1) < \mathrm{Succ}_P(1, 0) < (-1, 0)$ as in [1, Corollary 5.21(4)]. Thus:

(D1) the support of P lies in $\{(i, j) : 9i - 2j \le 9m\}$, that of Q in $\{9i - 2j \le 9n\}$, and

$$\ell_{9,-2}(P) = \lambda_P\big(x\,h(t)\big)^m, \qquad \ell_{9,-2}(Q) = \lambda_Q\big(x\,h(t)\big)^n, \qquad \lambda_P, \lambda_Q \in K^\times,$$

with $h \in K[t]$ monic of degree 4, $h(0) \neq 0$ [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1].

Moreover h has a single root: $h = (t - c)^4$ with $c \in K^\times$.

Theorem 1.2. *There is no pair (P, Q) as in Standing data 1.1, neither for $(m, n) = (3, 2)$ nor for $(m, n) = (2, 3)$.*

Remark 1.3. A root of multiplicity k of h which satisfies inequality (5.9) of [1, Proposition 5.18], that is $mk \ge \frac{9m}{7}$, generates the regular corner $(1, 0) + k(2/9, 1)$ of an (m, n)-pair in $L^{(9)}$, and such a root exists [1, Propositions 5.16, 5.18]; so $k \ge 2$. The last condition of the standing data is the case $k = 4$, with corner $(17/9, 4)$: these are the two shapes with corners $(9, 36)$, $(17/9, 4)$ of [2, Section 6] (the line through these two points has direction $(9, -2)$ and meets the horizontal axis at $(1, 0)$, and $(17/9, 4) = (1, 0) + 4(2/9, 1)$). If h has no root of multiplicity at least 3, then $k = 2$, which is impossible: the point $(13/9, 2)$ admits neither Case I (the

Date: 20 November 2026; revised 23 November 2026.
ORCID: 0009-0009-1526-1178.

direction would be $(81,-56)$, with no admissible starting point in [1, Proposition 5.13]) nor Case II ($\gcd(13,2)=1$) of [1, Proposition 5.19], and Case III keeps the same point and leads to Case I or Case II. The remaining case $h=(t-c)^3(t-c')$, with corner $(15/9,3)$, is not excluded by this arithmetic; for it we rely on [2], where no shape with these two corners is listed. We found no exclusion of the two shapes of the standing data in the literature we searched; their novelty is not otherwise verified. The preprint [3] of the four authors discards all cases with $\max\{\deg P, \deg Q\} \leq 150$ except $(75,125)$, using a formula for the intersection number of a Jacobian pair; in the later version of that work [4] only an inequality is obtained, and the authors state that the infinite families cannot be discarded in this way. They treat in [5] the shapes of degree below 125 and obtain the bound 108.

2. The left side

Lemma 2.1. *There are $\gamma_2, \gamma_3 \in K$ such that*

$$\hat{P}(z,v) := P\big(zv^4+\gamma_2 v^2+\gamma_3 v^3,\, v^{-1}\big), \qquad \hat{Q}(z,v) := Q\big(zv^4+\gamma_2 v^2+\gamma_3 v^3,\, v^{-1}\big)$$

are polynomials in z, v. Give z the weight 1 and v the weight 2. Then $\hat{P}$ has weighted degree $9m$ with top form $\lambda_P\big(z(z^2-cv)^4\big)^m$, $\hat{Q}$ has weighted degree $9n$ with top form $\lambda_Q\big(z(z^2-cv)^4\big)^n$, and

$$\hat{P}_z\hat{Q}_v - \hat{P}_v\hat{Q}_z = -J\,v^2.$$

Proof. The argument is the one of [5, Section 4, Case $(7,21)$]; we give it in detail. Apply the flip ψ_1 of [1, Section 8], so that the weight $(9,-2)$ becomes $(\rho_0,\sigma_0)=(-2,9)$, with $\mathrm{st}_{\rho_0,\sigma_0} = m(36,9)$ and $\mathrm{en}_{\rho_0,\sigma_0} = m(0,1)$ for the flipped P, and $\frac{1}{m}\,\mathrm{st}(P) = \frac{1}{n}\,\mathrm{st}(Q) = (36,9)$. Let F be the element of [1, Theorem 2.6] for (ρ_0,σ_0); then $v_{-2,9}(F)=7$. If $\mathrm{st}(F)=(1,1)$ then $\mathrm{en}(F)\neq(1,1)$, since F is not a monomial [1, Remark 2.5], so $\mathrm{en}(F)\sim\mathrm{en}(P)\sim(0,1)$ and $\mathrm{en}(F)=(0,7/9)$, which is impossible; hence $\mathrm{st}(F)\sim(36,9)$, so $\mathrm{st}(F)=(28,7)=\frac{7}{9}(36,9)$ and $q=9$ in [1, Corollary 7.4]. That corollary applies to the flipped pair (its hypotheses hold with $(a/l,b)=(36,9)$, and $\rho_0+\sigma_0>0$), and to every pair obtained from it by a translation of the second variable by a Laurent monomial in the first which leaves ℓ_{ρ_0,σ_0} unchanged: such a pair lies in $L^{(1)}$, the degrees $v_{1,1}$, $v_{0,1}$ of the flipped pair are unchanged, and so is F (the element G_0 of [1, Theorem 2.6] can be kept, and $(0,7/9)$ is excluded in $L^{(1)}$ as well). It yields: for every weight (ρ,σ) between $(1,-4)$ and (ρ_0,σ_0) at which the pair has an edge, and such that the value of P is positive at all weights from (ρ,σ) to (ρ_0,σ_0), the edge form of P is a power R^{9m}.

In the original variables, consider the weights $(-s,1)$ with $s<4$. The point $(9m,36m)=\mathrm{en}_{9,-2}(P)$ is the starting point of the edge of P at $\mathrm{Succ}_P(1,0)$: an edge with a direction strictly between $(9,-2)$ and $(1,1)$ starting at $(9m,36m)$ would end at a point with $v_{1,1}>45m=\deg P$ [1, Remark 1.8], and by the condition on $\mathrm{Succ}_P(1,0)$ there is no direction of P in $](1,0),(-1,1)]$; then [1, Proposition 3.7] applies. As long as $(9m,36m)$ is the starting point of an edge of weight $(-s,1)$ with $s<4$, the value of P there is $9m(4-s)>0$, the point $(9m,36m)$ is leading at all weights from $(9,-2)$ to $(-s,1)$, where the value of P is therefore positive as well, and the edge form is a power R^{9m} with R a $(-s,1)$-homogeneous element of $K[x,y^{\pm1}]$ whose only possible monomials are xy^4 and y^{4-s}; so the form is $(xy^4-\gamma y^{4-s})^{9m}$ up to a constant, $s\in\mathbb{Z}$, and $s\geq 2$ by the condition on $\mathrm{Succ}_P(1,0)$, that is $s\in\{2,3\}$. The translation $x\mapsto x+\gamma y^{-s}$ removes this edge and leaves unchanged the leading forms at the weights between $(9,-2)$ and $(-s,1)$, while the $(-s,1)$-form becomes the monomial $x^{9m}y^{36m}$ ([1, Proposition 3.9], applied after the flip); so the next edge starting at $(9m,36m)$, if any, has a larger s. After at most two such translations ($s=2$, then $s=3$; we write γ_2,γ_3 for the constants, which may be 0) no edge of weight $(-s,1)$, $s<4$, starts at $(9m,36m)$. Since the weight $(-4,1)$ lies within a half-turn of $(9,-2)$, the point $(9m,36m)$ is then still leading for $(-4,1)$ [1, Remark 2.10, Proposition 3.7]; hence every monomial x^iy^j of the translated P satisfies $j\leq 4i$, that is, $x^iy^j=z^iv^{4i-j}$ is a monomial in $z=xy^4$, $v=y^{-1}$.

For Q: at a weight $(-s,1)$, $s\in\{2,3\}$, where P has an edge as above, [1, Proposition 7.3] with $T_0=Q$ gives $[\ell(Q),\ell(P)]=0$ and proportional values, so $\ell(Q)\sim R^{9n}$ (the element $R=xy^4-\gamma y^{4-s}$, $\gamma\neq 0$, is not a proper power) and the same translation applies. For the pair

obtained after the translations and $3 \le s < 4$, the values of P and Q are positive, because $(9m, 36m)$ and $(9n, 36n)$ are in the supports; the leading forms commute by [1, Proposition 1.13] (the sum of the values exceeds $1 - s$), so $\ell_{-s,1}(Q)$ is proportional to a power of the monomial $\ell_{-s,1}(P)$ [1, Proposition 2.1(2)], hence a monomial at a point $(i, 4i)$; by (D1) such a point of the support has $i \le 9n$, and $(9n, 36n)$ is in the support, so $j - si \le 9n(4 - s)$ on the support of the translated Q. Letting s tend to 4 gives $j \le 4i$ there.

A translation $x \mapsto x + \gamma y^{-s}$, $s \in \{2, 3\}$, replaces $x^i y^j$ by monomials $x^{i-r} y^{j-sr}$ of $(9, -2)$-weight $9i - 2j - (9 - 2s)r$, so (D1) holds for the translated pair with the same leading forms. Since $x^i y^j = z^i v^{4i-j}$ has $9i - 2j = i + 2(4i - j)$, the bound $9i - 2j \le 9m$ is the weighted degree bound, and for $x = zv^4$, $y = v^{-1}$ one has $t = z^2 v^{-1}$ and $x(t - c)^4 = z(z^2 - cv)^4$, which gives the top forms. The Jacobian follows from $dx \wedge dy = -v^2\, dz \wedge dv$. □

3. The right side

Write $X := x^{1/9}$, so that $L^{(9)} = K[X^{\pm 1}, y]$.

Lemma 3.1. *There are $\lambda \in K^\times$ with $\lambda^9 = c$ and $\lambda_3 \in K$ such that, with*

$$Y := \eta + \lambda X^{-2} + \lambda_3 X^{-3}, \qquad P_1(X, \eta) := P\big(X^9, Y\big), \qquad Q_1(X, \eta) := Q\big(X^9, Y\big),$$

and with the weight $W(X^i \eta^j) := i - 4j$, every monomial of P_1 has $W \le m$ and every monomial of Q_1 has $W \le n$.

Proof. In $\tilde z := x^{2/9} y$ one has $\ell_{9,-2}(P) = \lambda_P x^m (\tilde z^9 - c)^{4m}$, so every root λ of $\tilde z^9 = c$ has multiplicity $4m \ge \frac{m}{l} \frac{a\rho + b l \sigma}{\rho + \sigma} = \frac{9m}{7}$ (with $l = 1$, $(a, b) = (9, 36)$, $(\rho, \sigma) = (9, -2)$). By [1, Proposition 5.18] the automorphism $\varphi(y) = y + \lambda x^{-2/9}$ gives an (m, n)-pair $(\varphi P, \varphi Q)$ in $L^{(9)}$ with a regular corner at $A_1 := (1, 0) + 4(2/9, 1) = (17/9, 4)$, of direction $(\rho', \sigma') < (9, -2)$. We follow this corner as in the proof of [1, Proposition 5.19]. Case II is impossible, because [1, Proposition 5.14(4)] requires $\gcd(17, 4) > 1$; this holds at every step below, since it depends only on the point A_1. In Case III, [1, Proposition 5.17] gives $\rho' \mid 9$, and the translation $y \mapsto y + \lambda' x^{\sigma'/\rho'}$ produces an (m, n)-pair in $L^{(9)}$ with a regular corner at the same point A_1 and a smaller direction. After finitely many steps Case I occurs, for a pair (P_1, Q_1) and a direction (ρ, σ). There $v_{\rho,\sigma}(P_1) + v_{\rho,\sigma}(Q_1) = \rho + \sigma$ [1, Proposition 1.13], while $\operatorname{en}(P_1) = m A_1 = \frac{m}{n} \operatorname{en}(Q_1)$ [1, Corollary 5.7(2)]. Hence $m(\frac{17}{9}\rho + 4\sigma) = \frac{m}{5}(\rho + \sigma)$, that is $(\rho, \sigma) = (9, -4)$, $v(P_1) = m$, $v(Q_1) = n$. The directions of the Case III steps decrease strictly from $(9, -2)$ and stay larger than the direction $(9, -4)$ of the final Case I corner, which is determined by the point A_1; they have $\rho' \in \{1, 3, 9\}$. The only such direction is $(3, -1)$, so there is at most one such step, a translation $y \mapsto y + \lambda_3 x^{-1/3}$. Finally $v_{9,-4}(x^{i/9} y^j) = i - 4j$. □

4. Common coordinates

Let $\mathcal{A}$ be the set of formal sums $\sum c_{ij} X^i \eta^j$ ($i \in \mathbb{Z}$, $j \ge 0$) such that, for every r, only finitely many terms have $W = i - 4j \ge r$. It is a ring containing $K[X^{\pm 1}, \eta]$; for $f \ne 0$ let $W(f)$ be the largest weight of its terms and $\ell(f)$ the sum of the terms of that weight. Then $\ell(fg) = \ell(f)\ell(g)$ and $W(fg) = W(f) + W(g)$, and $1 + \varepsilon$ is invertible in $\mathcal{A}$ when $W(\varepsilon) < 0$.

Lemma 4.1. *Let γ_2, γ_3, λ, λ_3 be as in Lemmas 2.1 and 3.1, and put*

$$\zeta := z^2 - c\, v - \kappa_1 z - \kappa_0, \qquad \kappa_1 := 9\lambda^3 \lambda_3, \quad \kappa_0 := -9\lambda^6 \lambda_3^2.$$

Then $K[z, v] = K[z, \zeta]$. Write $\hat P = \sum p_{ab} z^a \zeta^b$ and $\hat Q = \sum q_{ab} z^a \zeta^b$. Then $p_{ab} \ne 0$ implies $a \le m$ and $a + 2b \le 9m$; $q_{ab} \ne 0$ implies $a \le n$ and $a + 2b \le 9n$. Moreover the part of $\hat P$ with $a + 2b = 9m$ is $\lambda_P z^m \zeta^{4m}$, the part of $\hat Q$ with $a + 2b = 9n$ is $\lambda_Q z^n \zeta^{4n}$, and

$$\hat P_z \hat Q_\zeta - \hat P_\zeta \hat Q_z = \frac{J}{c^3} \big(z^2 - \kappa_1 z - \kappa_0 - \zeta\big)^2.$$

Proof. Since $c \ne 0$, $v = (z^2 - \kappa_1 z - \kappa_0 - \zeta)/c$, so $K[z, v] = K[z, \zeta]$; giving ζ the weight 2, this substitution does not increase the weighted degree of Lemma 2.1, which gives the bounds on $a + 2b$, and since $z^2 - cv = \zeta + \kappa_1 z + \kappa_0$ the top forms become $\lambda_P z^m \zeta^{4m}$ and $\lambda_Q z^n \zeta^{4n}$. In $\mathcal{A}$

the element Y of Lemma 3.1 is $\lambda X^{-2}(1+\varepsilon)$ with $W(\varepsilon) < 0$; it is invertible. The assignment $v \mapsto Y^{-1}$, $z \mapsto X^9Y^4 - \gamma_2 Y^2 - \gamma_3 Y$ defines a ring morphism $K[z,v] \to \mathcal{A}$ which sends $\hat{P}$ to P_1 and $\hat{Q}$ to Q_1, because $P(x,y) = \hat{P}\big((x - \gamma_2 y^{-2} - \gamma_3 y^{-3})y^4, y^{-1}\big)$ in $K[x, y^{\pm 1}]$. Expanding (ancillary script), the image of z has $W = 1$ and leading term $\lambda^4 X$, and the image of ζ has $W = 0$ and leading term $9\lambda^7 \eta X^4$: the terms of weights 2 and 1 and the constant term cancel by the choice of $c = \lambda^9$, κ_1, κ_0. Hence the image of $z^a\zeta^b$ has leading term a non-zero multiple of $\eta^b X^{a+4b}$, of weight a, and these monomials are distinct for distinct (a,b). Therefore, for a non-zero polynomial, the value of W on the image of $\sum c_{ab} z^a \zeta^b$ is $\max\{a : c_{ab} \neq 0\}$. Lemma 3.1 gives the bounds m and n. Finally $dz \wedge d\zeta = -c\, dz \wedge dv$, so the bracket in (z,ζ) is $-\frac{1}{c}$ times the bracket in (z,v), which is $-Jv^2$ by Lemma 2.1. □

5. The finite system

Let $(U,V) := (\hat{P},\hat{Q})$ if $(m,n) = (3,2)$ and $(U,V) := (\hat{Q},\hat{P})$ if $(m,n) = (2,3)$. Writing again z for $z - \kappa_1/2$ (this replaces κ_1 by 0, changes κ_0, and preserves all the statements of Lemma 4.1, the leading terms of the images and the weighted degrees being unchanged), and x for ζ, a pair as in the standing data yields in both cases a solution of the following system.

(S1) $U = z^3u_3 + z^2u_2 + zu_1 + u_0$ and $V = z^2v_2 + zv_1 + v_0$ with $u_i, v_i \in K[x]$, $\deg u_3 = 12$, $\deg u_2 \leq 12$, $\deg u_1 \leq 12$, $\deg u_0 \leq 13$, $\deg v_2 = 8$, $\deg v_1 \leq 8$, $\deg v_0 \leq 8$.

(S2) $U_zV_x - U_xV_z = J'\,(z^2 - \kappa - x)^2$ with $J' \in K^\times$, $\kappa \in K$.

Indeed the bounds $a \leq 3$, $a + 2b \leq 27$ for U and $a \leq 2$, $a + 2b \leq 18$ for V give the shape of U and V with $\deg u_1 \leq 13$ and $\deg v_0 \leq 9$, and the top forms $\lambda z^3 x^{12}$, $\lambda' z^2 x^8$ of Lemma 4.1 show that $\deg u_3 = 12$, $\deg v_2 = 8$ and that u_1 has no term of degree 13 and v_0 no term of degree 9. For $(m,n) = (2,3)$ the bracket changes sign, which is absorbed in J'. Since $\{z^a f, z^{a'} g\} = (a\, fg' - a' f' g)\, z^{a+a'-1}$ for $f, g \in K[x]$, where $\{F,G\} := F_zG_x - F_xG_z$, the coefficients of $z^4, z^3, z^2, z, 1$ in (S2) are

$$3u_3v_2' - 2u_3'v_2 = J', \tag{1}$$

$$3u_3v_1' - u_3'v_1 + 2u_2v_2' - 2u_2'v_2 = 0, \tag{2}$$

$$3u_3v_0' + 2u_2v_1' - u_2'v_1 + u_1v_2' - 2u_1'v_2 = -2J'(\kappa + x), \tag{3}$$

$$2u_2v_0' + u_1v_1' - u_1'v_1 - 2u_0'v_2 = 0, \tag{4}$$

$$u_1v_0' - u_0'v_1 = J'(\kappa + x)^2. \tag{5}$$

Lemma 5.1. *If* (S1)–(S2) *has a solution, it has one with* $u_0(0) = v_0(0) = 0$, *with no term of degree* 8 *in* u_2, *and with* $u_3 = p_3$, $v_2 = q_2$, $J' = 2^{19} \cdot 27 \cdot 19$, *where*

$$q_2 = x^8 + 21x^6 + 22x^5 + \tfrac{1183}{8}x^4 + 423x^3 + \tfrac{6721}{16}x^2 + \tfrac{13679}{8}x + \tfrac{268777}{256},$$

$$p_3 = x^{12} + \tfrac{63}{2}x^{10} + 33x^9 + \tfrac{6195}{16}x^8 + 981x^7 + \tfrac{40989}{16}x^6 + \tfrac{78783}{8}x^5 + \tfrac{3531639}{256}x^4 + \tfrac{617825}{16}x^3 + \tfrac{31456311}{512}x^2 + \tfrac{11753097}{256}x + \tfrac{291742453}{4096}.$$

One has $p_3^2 - q_2^3 = 884736\,(16x^5 + 248x^3 + 296x^2 + 1001x + 4426)$.

Proof. The system is preserved by: multiplying U and V by non-zero constants; adding constants to U and V; replacing U by $U - \alpha V$; the translation $x \mapsto x + \tau$ (with κ replaced by $\kappa + \tau$); and the substitution $(z,x) \mapsto (\mu z, \mu^2 x)$ (with κ replaced by κ/μ^2). So we may assume that u_3 and v_2 are monic, that v_2 has no term of degree 7, that $u_0(0) = v_0(0) = 0$ and that u_2 has no term of degree 8. Consider equation (1) for monic u_3, v_2 of degrees 12, 8, with no term of degree 7 in v_2. The substitution $x \mapsto \nu x$, followed by division by ν^{12} and ν^8, preserves these conditions and multiplies J' by ν^{-19} and the coefficient b_6 of x^6 in v_2 by ν^{-2}. For $J' = 1$ the ideal of the equation has dimension 0 and degree 19, so that there are at most 19 solutions, and $b_6^{19} = 3^{13}7^{19}/(2^{38}19^2)$ on each of them (Gröbner basis, Singular [6], ancillary script); since $b_6 \neq 0$, the 19 substitutions with $\nu^{19} = 1$ give 19 distinct solutions from one, so the solutions with $J' = 1$ form a single orbit, and therefore all solutions with $J' \neq 0$ form a single orbit under $x \mapsto \nu x$, $\nu \in K^\times$. The pair (p_3, q_2) is a solution, with $J' = 2^{19} \cdot 27 \cdot 19$ (ancillary script). Hence

$b_6 \neq 0$ for every solution, and using $(z, x) \mapsto (\mu z, \mu^2 x)$ we may assume $b_6 = 21$; the solutions with $b_6 = 21$ are $(p_3(x), q_2(x))$ and $(p_3(-x), q_2(-x))$, the second with $J' = -2^{19} \cdot 27 \cdot 19$. The substitution $(z, x) \mapsto (iz, -x)$, $i^2 = -1$, followed by the multiplication of U by i and of V by -1, preserves the system and the normalisations and exchanges the two. □

Proposition 5.2. *The system* (S1)–(S2)*, normalised as in Lemma 5.1, has no solution.*

Proof (exact computation). With $u_3 = p_3$, $v_2 = q_2$ and $J' = 2^{19} \cdot 27 \cdot 19$, each of (2), (3), (4) is, coefficient by coefficient, a system of linear equations with rational matrix: (2) in the coefficients of u_2, v_1; then (3) in those of v_0, u_1 and in κ; then (4) in those of u_0; the right-hand sides depend on the solutions of the previous steps. Reducing each matrix to echelon form expresses the pivot unknowns through the others, and the rows without pivot give *conditions*, polynomials with rational coefficients in the unknowns which remain free, which vanish on every solution. Carrying this out (ancillary script): (2) leaves the coefficients $v_{1,7}$, $v_{1,8}$ of x^7, x^8 in v_1 free and gives no condition; (3) leaves the coefficients $u_{1,11}$, $u_{1,12}$ of x^{11}, x^{12} in u_1 and κ free and gives no condition; (4) determines u_0 and gives 8 conditions; finally the coefficients of (5) give 21 conditions. These 29 polynomials, of degrees 3 and 4 (they are not claimed to be linearly independent), generate the unit ideal of $\mathbb{Q}[v_{1,7}, v_{1,8}, u_{1,11}, u_{1,12}, \kappa]$ (Gröbner basis, SymPy). □

As an independent confirmation, the 82 equations of the normalised system in its 56 unknowns, without any elimination, also generate the unit ideal (Gröbner basis, Singular; optional step of the script). As a check of the procedure, the script applies the same elimination to a pair (U, V) with $u_3 = p_3$, $v_2 = q_2$ and randomly chosen u_2, u_1, u_0, v_1, v_0, with the right-hand side of (S2) replaced by its bracket: the conditions then vanish at the chosen values, and no longer do so after the right-hand side is modified.

Proof of Theorem 1.2. A pair as in the standing data gives a solution of (S1)–(S2) by Lemmas 2.1–4.1, hence a normalised one by Lemma 5.1, contradicting Proposition 5.2. □

6. Scope

- **Proved here, relative to the cited results:** the reduction of Theorem 1.2 to Proposition 5.2 (Lemmas 2.1–5.1). The inputs are (D1), which rests on [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1], and [1, Theorem 2.6, Remarks 1.8, 2.5, 2.10, Corollaries 5.7, 7.4, Propositions 1.13, 2.1, 3.7, 3.9, 5.13, 5.14, 5.16, 5.17, 5.18, 5.19, 7.3], as indicated. The condition on $\mathrm{Succ}_P(1, 0)$ in the standing data is the one of [1, Corollary 5.21(4)], which is established there for the standard pairs attached to a minimal counterexample; the theorem is stated under this condition.
- **Verified by exact computation:** Proposition 5.2; the statements on equation (1) in Lemma 5.1; the arithmetic of Remark 1.3 and Lemma 3.1; the leading terms in Lemma 4.1 (ancillary file `verify_135.py` in the folder `anc`; SymPy, and Singular for the number of solutions of (1), for which the script writes the input file). The result depends on the correctness of these computations.
- **Not claimed:** a statement on the plane Jacobian conjecture itself; on the pairs with the same starting triple whose polynomial h has more than one root (see Remark 1.3); on the third shape of maximal degree 135 with starting corner $(9, 36)$ in [2], which has the corners $(9, 24)$ and $(11/3, 8)$; a new bound on the degree of a counterexample. The completeness of the list of [2] is not re-examined.
- [1, Proposition 7.3 and Corollary 7.4], on which Lemma 2.1 rests, are stated in [1] with the indication that their proofs mimic those of Proposition 7.1 and Corollary 7.2; they are applied here to Laurent translates, as in [1, Section 8] and [5]. The references [2] and [5] are arXiv preprints.
- The argument has been prepared by the author with AI assistance (Claude, Anthropic). Nothing in this note has been checked by a third party; it has not been validated by professionals.

References

[1] C. Valqui, J. A. Guccione, J. J. Guccione, *On the shape of possible counterexamples to the Jacobian Conjecture*, J. Algebra **471** (2017), 13–74, doi:10.1016/j.jalgebra.2016.08.039; arXiv:1401.1784v3. Author order as in the journal; statement numbers refer to the arXiv version v3.

[2] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Some algorithms related to the Jacobian Conjecture*, arXiv:1708.07936v1.

[3] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *The Jacobian Conjecture: discarding infinite families using intersection numbers*, arXiv:1708.09367v1 (2017).

[4] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *The Jacobian conjecture: approximate roots and intersection numbers*, Pro Mathematica **30** (2019), no. 60, 51–89.

[5] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Increasing the degree of a possible counterexample to the Jacobian Conjecture from 100 to 108*, arXiv:2204.14178.

[6] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *Singular 4-4-1 — A computer algebra system for polynomial computations*, `https://www.singular.uni-kl.de` (2025).

Independent researcher, France