The Heawood family is not the complete list of excluded minors for Colin de Verdière invariant at most 5

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Certificates and checkers: folders `data` and `checks` of
https://github.com/6/clank-lab/tree/heawood-family-mu5-v1/results/heawood-family-mu5

hexagon:2611.00727v1

Abstract

We show that two intrinsically knotted graphs G_1 and G_2 with 23 edges, on 11 and 12 vertices, have Colin de Verdière invariant $\mu = 6$ and no minor in the Heawood family, the 78 graphs obtained from K_7 and $K_{3,3,1,1}$ by ΔY- and $Y\Delta$-transformations. So the Heawood family is not the complete list of excluded minors for $\mu \leq 5$, as van der Holst proposed; Mattman and Pavelescu stated this proposal as a conjecture and derived from it that G_1 and G_2 have $\mu = 5$. More precisely, either the common minor H of G_1 and G_2, a 4-regular graph with 11 vertices and 22 edges, is an excluded minor for $\mu \leq 5$, or G_1 and G_2 both are. It follows that the following two conjectures of van der Holst cannot both hold: (a) a graph is 4-flat if and only if $\mu \leq 5$; (b) the Heawood family is the complete list of excluded minors for 4-flat graphs. The lower bound $\mu \geq 6$ is proved by explicit matrices certified in interval and exact rational arithmetic, and the minor exclusion by exhaustive enumeration.

This work has not yet been reviewed by experts. The proofs, the computer checks and the text were produced by AI models (from OpenAI and Anthropic), working under the author's direction.

Keywords: Colin de Verdière invariant, excluded minors, Heawood family, 4-flat graphs, intrinsically knotted graphs.
MSC 2020: 05C83, 05C50, 57M15, 05C10.

1 Introduction

1.1 Main result

All graphs are finite and simple, with vertex set $\{0, \ldots, n-1\}$. Let $G = (V, E)$ be a graph on n vertices. Following van der Holst, Lovász and Schrijver [3, 1.1], the *Colin de Verdière invariant* $\mu(G)$ is the largest corank of a real symmetric $n \times n$ matrix M such that

(i) for $i \neq j$, $M_{ij} < 0$ if $ij \in E$ and $M_{ij} = 0$ if $ij \notin E$, with no condition on the diagonal;

(ii) M has exactly one negative eigenvalue, of multiplicity one;

(iii) (the Strong Arnold Property) no nonzero real symmetric matrix X with $X_{ij} = 0$ whenever $i = j$ or $ij \in E$ satisfies $MX = 0$.

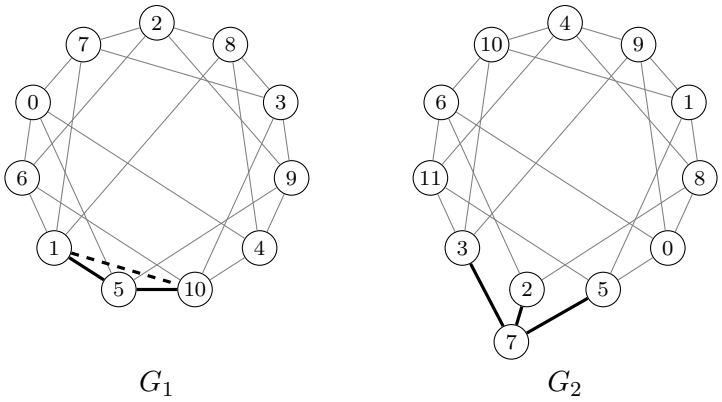
A *minor* of a graph is a graph obtained from it by deleting vertices and edges and contracting edges, merging parallel edges and removing loops. Since μ does not increase under taking minors [1], the graphs with $\mu \le k$ form a minor-closed class. An *excluded minor* for a minor-closed class is a graph outside the class all of whose proper minors lie in it; a graph is in the class if and only if it has no excluded minor as a minor. A graph is *t-apex* if deleting at most t vertices leaves a planar graph.

A ΔY-transformation replaces the three edges of a triangle by a new vertex adjacent to its three corners; a $Y\Delta$-transformation is the reverse, applied to a vertex of degree three. The *Heawood family* consists of the 78 graphs obtained from K_7 and $K_{3,3,1,1}$ by repeated ΔY- and $Y\Delta$-transformations [2, §1.1]: the 20 graphs of the K_7 family, with 21 edges, and the 58 of the $K_{3,3,1,1}$ family, with 22 edges.

Let G_1 and G_2 be the graphs with these edge sets:

$$\begin{aligned} E(G_1) = \{&04, 05, 06, 07, 15, 16, 17, 18, 1\,10, 26, 27, 28, 29, 37, 38, 39, 3\,10,\\ &48, 49, 4\,10, 59, 5\,10, 6\,10\},\\ E(G_2) = \{&05, 06, 08, 09, 15, 18, 19, 1\,10, 26, 27, 28, 37, 39, 3\,10, 3\,11,\\ &48, 49, 4\,10, 4\,11, 57, 5\,11, 6\,10, 6\,11\}. \end{aligned}$$

G_1 has 11 vertices and 23 edges, nine of degree 4 and two of degree 5; G_2 has 12 vertices and 23 edges and arises from G_1 by a ΔY-transformation on the triangle $\{1, 5, 10\}$ (Figure 1). They are entries 90 and 332 (numbered from 0) of the list of 3028 obstructions to knotless embedding in Appendix A of Mattman and Pavelescu [8], and the graphs G_2 and G_3 of Kim and Mattman [5, §3]. Let H be the graph G_1 with the edge $\{1, 10\}$ deleted; it is 4-regular with 11 vertices and 22 edges. Mattman and Pierce found H by a computer search, as one of two 4-regular graphs with 11 vertices and 22 edges that are not 2-apex while all their proper minors are [9, §2] (their second edge list, with labels shifted by one). Kim and Mattman call it H_2 and show that, up to isomorphism, H is the only minor of G_1 or G_2 obtained by one edge deletion or contraction that is not 2-apex [5, §§3.1, 3.3].



G_1 G_2

Figure 1: Left, G_1; deleting the dashed edge $\{1, 10\}$ gives H. Right, G_2, obtained from G_1 by a ΔY-transformation on the thick triangle $\{1, 5, 10\}$; the new vertex is 7 and the thick edges are its three edges. Vertices of G_2 carry their labels in G_2. Drawn by `checks/figure.py` from `data/graphs.g6`, which also checks the drawing.

Theorem 1. *$\mu(G_1) = \mu(G_2) = 6$, and neither G_1 nor G_2 has a minor in the Heawood family.*

Corollary 2. *The Heawood family is not the complete list of excluded minors for the class of graphs with $\mu \le 5$. More precisely, if $\mu(H) = 6$ then H is an excluded minor for $\mu \le 5$, and otherwise G_1 and G_2 both are; in either case none of them is in the Heawood family.*

The lower bound $\mu \ge 6$ rests on two certificates, one explicit matrix for each graph, checked in exact rational and interval arithmetic, and the minor exclusion on an exhaustive enumeration (Section 6). The upper bound $\mu \le 6$ and the identification of the excluded minor are short arguments from known facts.

1.2 Background

Colin de Verdière proved that μ is minor-monotone and that $\mu(G) \le 3$ exactly when G is planar [1]; see [3, Theorem 2.4 and 1.4]. Lovász and Schrijver [7] and Robertson, Seymour and Thomas [11] showed that $\mu(G) \le 4$ exactly when G is linklessly embeddable, that is, embeddable in $\mathbb{R}^3$ with no two disjoint cycles linked, so the excluded minors for $\mu \le 4$ are those for linkless embedding, the Petersen family [3, 1.4]. As Georgakopoulos and Winter point out, the Petersen and Heawood families are both obtained from the Kuratowski graphs K_5 and $K_{3,3}$ "by forming suspensions, and then ΔY- and $Y\Delta$-transformations" [2, §1.1]; this analogy makes the Heawood family the natural candidate for $\mu \le 5$.

Van der Holst introduced 4-flat graphs [4]: in the words of Georgakopoulos and Winter, "a graph G is 4-flat if every 2-dimensional regular CW complex having G as its 1-skeleton can be (piecewise linearly) embedded into $\mathbb{R}^4$" [2, §1]. Van der Holst proved that for every graph G of the Heawood family the complex obtained from G by attaching a disk to each cycle does not embed in $\mathbb{R}^4$ [4], so these graphs are not 4-flat. Two of van der Holst's conjectures concern us, as Georgakopoulos and Winter record them: (a) a graph is 4-flat if and only if $\mu \le 5$ [4, Conjecture 3], [2, Conjecture 1.4]; and (b) the Heawood family is the complete list of excluded minors for 4-flat graphs, the second statement of [2, Conjecture 1.1], which by their footnote 1 combines (a) with van der Holst's conjecture that the Heawood graphs are the excluded minors for $\mu \le 5$ [4, §4]. Georgakopoulos and Winter prove that each Heawood graph is an excluded minor for 4-flat graphs [2, Theorem 6.1] and note that all Heawood graphs have $\mu = 6$ [2, §1.1].

A graph is *intrinsically knotted* if every embedding of it in $\mathbb{R}^3$ contains a knotted cycle, and *minor-minimal intrinsically knotted* (MMIK) if moreover no proper minor is intrinsically knotted; the MMIK graphs are the excluded minors for knotless embedding. Mattman and Pavelescu state the classification for μ directly: "Conjecture 4 ([vdH]). The graphs in the K_7 and $K_{3,3,1,1}$ families are the obstructions to $\mu \le 5$" [8, p. 3]. Assuming it, they list the MMIK graphs with $\mu \ge 6$ through 24 edges, noting that "All the MMIK graphs G in our list are 3-apex" [8, §3]. Of the three MMIK graphs with 23 edges, found by Kim and Mattman [5], they write that "they do not have a minor in the K_7 family. By van der Holst's conjecture these three have $\mu = 5$" [8, §3]. Theorem 1 shows that two of them, G_1 and G_2, have $\mu = 6$.

1.3 What is new

The new step is the lower bound $\mu(G_i) \ge 6$. A corank-6 matrix with the sign pattern of G_i has to satisfy one polynomial equation for each non-adjacent pair of vertices, and it must also have the Strong Arnold Property; such matrices are found numerically and then certified rigorously. Minor exclusion and the upper bound are routine, and the non-2-apex minors of G_i follow from the work of Mattman and Pierce [9] and of Kim and Mattman [5] cited above; we recheck them (Lemma 8). Corollary 2 has the following consequence for 4-flat graphs.

Corollary 3. *Conjectures (a) and (b) of Section 1.2 cannot both hold: (a) a graph is 4-flat if and only if $\mu \le 5$; (b) the Heawood family is the complete list of excluded minors for 4-flat graphs.*

Proof. If (a) holds, the two classes coincide and so have the same excluded minors; by Corollary 2 these include a graph outside the Heawood family, so (b) fails. □

We do not decide which of (a) and (b) fails, whether G_1 is 4-flat, or the value of $\mu(H)$. The third 23-edge graph of [8, §3], entry 24 of its Appendix A, is not treated here.

2 Proof overview

For the lower bound we exhibit, for each $G \in \{G_1, G_2\}$, a matrix M satisfying (i)–(iii) with corank 6. We split the vertices into a set S of six vertices and the rest T, choose the $T \times T$ block C and most of the $S \times T$ block B as explicit rationals, and set the $S \times S$ block to $A = BC^{-1}B^{\mathsf{T}}$. Then the corank of M is at least 6 automatically (Lemma 4), and (ii) reduces to an inertia statement about C. The pattern condition (i) on A asks that $(BC^{-1}B^{\mathsf{T}})_{ij} = 0$ for each non-adjacent pair $i, j \in S$; we leave exactly as many entries of B free as there are such pairs and solve this square polynomial system numerically. Krawczyk's test proves that an exact solution lies in a box of radius 10^{-40} around the numerical one (Lemma 5); the remaining sign conditions hold on the whole box; and the Strong Arnold Property, an injectivity statement, is proved at the box center with a margin that survives the perturbation to the exact solution (Lemma 6).

For minor exclusion, every Heawood graph has at least 21 edges and G_i has 23, so a Heawood minor would arise from at most two deletions or contractions (Lemma 7), and these are enumerated.

3 Certificates for $\mu \ge 6$

Order the vertices so that S comes first and write

$$M = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}, \qquad |S| = 6,\ |T| = n - 6.$$

Lemma 4. *If C is invertible and $A = BC^{-1}B^{\mathsf{T}}$, then M has the eigenvalue 0 with multiplicity 6 plus that of C, and M has as many negative and as many positive eigenvalues as C.*

Proof. With $P = \left(\begin{smallmatrix} I & 0 \\ C^{-1}B^{\mathsf{T}} & I \end{smallmatrix}\right)$ one has $M = P^{\mathsf{T}} \left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix}\right) P$, and P is invertible. Sylvester's law of inertia gives the claim. □

The entries of B on non-edges are 0. On the edges between S and T, all but q entries are fixed negative rationals and the remaining q are unknowns $u = (u_1, \dots, u_q)$, where q is the number of non-adjacent pairs in S. For each such pair $i < j$ let $f_{ij}(u) = b_i C^{-1} b_j^{\mathsf{T}}$, where b_i is the row of B indexed by i. Write $f = (f_{ij}) \colon \mathbb{R}^q \to \mathbb{R}^q$ and J for its Jacobian matrix.

The following is Krawczyk's test [6] in the form of Moore [10], with inclusion in the interior, which makes the choice of Y free.

Lemma 5. *Let $X \subset \mathbb{R}^q$ be a box with sides of positive lengths $w_1, \dots, w_q$, let $y \in X$, let Y be a real $q \times q$ matrix, and let $\mathcal{J}$ be an interval matrix containing $J(x)$ for every $x \in X$. If the interval vector $K = y - Yf(y) + (I - Y\mathcal{J})(X - y)$ lies in the interior of X, then f has a zero in X.*

Proof. Let $J \in \mathcal{J}$ and $D = I - YJ$. The interval vector K contains $y - Yf(y) + Dd$ for every $d \in X - y$, and as d ranges over this box the k-th coordinate of Dd ranges over an interval of length $\sum_j |D_{kj}| w_j$. Since K lies in the interior of X, this is less than w_k for every k. So D has norm less than 1 for the norm $\max_k |v_k|/w_k$ on $\mathbb{R}^q$, and $YJ = I - D$ and hence Y are invertible. For $x \in X$, $f(x) = f(y) + \bar{J}(x - y)$, where $\bar{J}$ is the average of J along the segment from y to x; it lies in $\mathcal{J}$ because interval matrices are convex. So $g(x) = x - Yf(x) = y - Yf(y) + (I - Y\bar{J})(x - y)$ lies in $K \subset X$. By Brouwer's fixed-point theorem g has a fixed point x^*, and $Yf(x^*) = 0$ gives $f(x^*) = 0$. □

Here y is a point near the center of X, Y a floating-point approximation of $J(y)^{-1}$, and $f(y)$ and $\mathcal{J}$ are enclosed in interval arithmetic.

Lemma 6. *Let W be the space of real symmetric matrices X with $X_{ij} = 0$ whenever $i = j$ or $ij \in E$, with coordinates $x_{ij} = X_{ij}$ for the non-edges $i < j$, so that $\|X\|_F = \sqrt{2}\,|x|$. Suppose that $\|M_0 X\|_F \geq \sqrt{\varepsilon}\,|x|$ for all $X \in W$, for some $\varepsilon > 0$. If $\|M - M_0\|_F < \sqrt{\varepsilon}/2$, then M has the Strong Arnold Property.*

Proof. For nonzero $X \in W$, $\|MX\|_F \geq \|M_0 X\|_F - \|M - M_0\|_F \|X\|_F > \sqrt{\varepsilon}\,|x| - \frac{\sqrt{2}}{2}\sqrt{\varepsilon}\,|x| > 0$. □

The hypothesis on M_0 says that the Gram matrix of the linear map $X \mapsto M_0 X$ on W, in the coordinates x, minus εI is positive definite; for a rational M_0 this is an exact computation.

Proof of $\mu(G_i) \geq 6$. For G_1 take $S = \{1, 2, 3, 4, 5, 10\}$, which spans five edges, so $q = 10$; for G_2 take $S = \{3, 5, 6, 8, 9, 10\}$, which spans three edges, so $q = 12$. The certificate files give C, the fixed entries of B, and a rational point c; let X be the box of radius 10^{-40} about c in every coordinate. The following are verified.

1. C has the pattern (i) of the subgraph induced by T; the fixed entries of B are negative; every edge between S and T carries a fixed entry or an unknown, and every non-edge the entry 0.
2. The test of Lemma 5 succeeds on X, so there is $u^* \in X$ with $f(u^*) = 0$.
3. On all of X, each unknown is negative and $(BC^{-1}B^{\mathsf{T}})_{ij} < 0$ for each edge ij inside S.
4. C is invertible with exactly one negative eigenvalue: its characteristic polynomial has rational coefficients and only real roots, so Descartes' rule of signs counts its positive and negative roots exactly. The inertia is $(1, 0, 4)$ for G_1 and $(1, 0, 5)$ for G_2.
5. For M_0 the matrix built from c and $\varepsilon = 10^{-6}$, the hypothesis of Lemma 6 holds, and $\|M - M_0\|_F$ is at most 1.6×10^{-38} for every M built from a point of X.

Let M^* be the matrix built from u^*. By (1)–(3) it satisfies (i), since $A = BC^{-1}B^{\mathsf{T}}$ vanishes on the non-adjacent pairs in S by $f(u^*) = 0$. By (4) and Lemma 4 it has corank exactly 6 and one negative eigenvalue, which is (ii). By (5) and Lemma 6 it satisfies (iii). Hence $\mu(G_i) \geq 6$. □

4 Excluding Heawood minors

Lemma 7. *Let G have m edges and let F be a minor of G with at least $m - k$ edges. Then F is obtained from G by at most k edge deletions and edge contractions followed by deleting isolated vertices.*

Proof. A minor is obtained by deleting and contracting edges and deleting vertices. Deleting a vertex of positive degree can be replaced by deleting its edges and then the isolated vertex. Every deletion or contraction of an edge lowers the number of edges by at least one, and deleting an isolated vertex does not change it. So at most k edge operations occur, and since isolated vertices take part in no edge operation, their deletions can be postponed to the end. □

Proof that G_i has no Heawood minor. Every Heawood graph has 21 or 22 edges and no isolated vertex. By Lemma 7 with $k = 2$, a Heawood minor of G_i would be isomorphic to a graph obtained from G_i by at most two edge deletions or contractions, with isolated vertices removed and parallel edges merged, that has at least 21 edges. The bound of 21 edges loses no Heawood graph; it excludes some graphs, since contracting an edge of a triangle removes two edges. Up to isomorphism there are 354 graphs with at least 21 edges obtained in this way from G_1 and 455 from G_2, and none is isomorphic to a graph of the Heawood family. □

The family itself is regenerated from K_7 and $K_{3,3,1,1}$ by closing under ΔY and under those $Y\Delta$-transformations whose result is simple; this gives 20 graphs with 21 edges and 58 with 22 edges, the counts of [2, §1.1] and [8, §3].

5 Proofs of Theorem 1 and Corollary 2

Proof of Theorem 1. Sections 3 and 4 give $\mu(G_i) \geq 6$ and the absence of Heawood minors. For the upper bound, deleting a vertex lowers μ by at most one [3, Theorem 2.7] and planar graphs have $\mu \leq 3$ [3, 1.4], so a graph that becomes planar after deleting t vertices has $\mu \leq t + 3$. Deleting $\{0, 1, 2\}$ leaves a planar graph in both G_1 and G_2, so $\mu(G_i) \leq 6$. □

By the same bound a 2-apex graph has $\mu \leq 5$, and every minor of a 2-apex graph is 2-apex: if $G - X$ is planar with $|X| \leq 2$, deleting an edge or vertex keeps this, contracting an edge avoiding X contracts it in $G - X$, and contracting an edge uv with $u \in X$ into a vertex w gives a graph in which deleting w and $X \setminus \{u, v\}$ leaves $G - (X \cup \{v\})$. A graph is 2-apex if and only if the graph obtained by removing its isolated vertices is, since isolated vertices do not affect planarity.

Lemma 8. *Up to isomorphism, the minors of G_1 that are not 2-apex are G_1 and H, and the minors of G_2 that are not 2-apex are G_2 and H.*

Proof. For a graph F write F° for F with its isolated vertices removed. Starting from G_i, apply every single edge deletion and contraction, remove isolated vertices, keep the results that are not 2-apex, and repeat. The search finds, up to isomorphism, exactly G_i and H; for G_2 the second graph has 11 vertices and 22 edges and is isomorphic to H.

Let F be a minor of G_i that is not 2-apex. First, F° is isomorphic to a graph found by the search. Indeed, by Lemma 7 there are graphs $G_i = F_0, F_1, \ldots, F_r$, each obtained from the previous one by an edge deletion or contraction, with $F_r^\circ = F^\circ$. Each F_j has F as a minor, so it is not 2-apex, and neither is F_j°; and F_{j+1}° is obtained from F_j° by one edge deletion or contraction followed by removing isolated vertices. By induction on j the search reaches every F_j°. So F° is isomorphic to G_i or H, and F has at least 22 edges. Second, F has no isolated vertex. By Lemma 7 with $k = 1$, F is obtained from G_i by at most one edge deletion or contraction followed by deleting isolated vertices. The minimum degree of G_1 is 4 and that of G_2 is 3, and deleting or contracting one edge lowers no degree by more than one, except that the vertex formed by contracting uv has degree at least $\max(\deg u, \deg v) - 1$. So the graph before the last step has minimum degree at least 2, there is nothing to delete, and $F = F^\circ$. □

Proof of Corollary 2. Let $i \in \{1, 2\}$. Since $\mu(G_i) = 6$ and the class $\mu \leq 5$ is closed under minors, G_i has a minor O that is an excluded minor for $\mu \leq 5$. Then O is not 2-apex, so by Lemma 8 it is isomorphic to G_i or H. If $\mu(H) = 6$, the proper minors of H are 2-apex by Lemma 8, so H is an excluded minor. If $\mu(H) \leq 5$, every proper minor of G_i is H, a minor of H or 2-apex, so G_i is an excluded minor, for $i = 1$ and $i = 2$. These graphs are minors of G_1 or G_2, so by Theorem 1 none of them lies in the Heawood family. □

6 Computational verification

The certificates and programs are in the folders `data` and `checks` of the published files (linked below the title). The certificate for G_i lists its graph, the split (S, T), the rational block C, the fixed entries of B, the positions of the unknowns and the rational center c. The checker reads only the certificate. It recomputes the non-adjacent pairs in S from the graph; runs the test of Lemma 5 at 300 bits in interval arithmetic, with C^{-1} enclosed from exact rationals; checks the signs of items (1) and (3) over the box; computes the characteristic polynomial of C exactly; and verifies the Gram condition of Lemma 6 exactly, bounding $\|M - M_0\|_F$ over the box. A second checker, written separately from a description of the format, repeats these steps with different methods: Krawczyk's test in exact rational arithmetic using the constant Hessian of the quadratic system f, inertia by an LDL^{T} factorization and by Descartes' rule, and the Strong Arnold Property in two formulations, each with a Weyl perturbation bound. Both checkers accept both certificates, and each rejects its own suite of deliberately corrupted certificates, listed in the checker documentation.

The Heawood family is generated by two separate programs, which agree on the 78 graphs and the split $20 + 58$. Minor exclusion is checked twice: by the enumeration of Lemma 7 with isomorphism tests, and by a search for branch sets of each Heawood graph in G_i, cross-checked by deletion and contraction; both find the Heawood minors of three planted control graphs. The 3-apex and 2-apex statements, the minimum degrees used in Lemma 8 and Figure 1 are checked by the same programs. The whole check takes about half a minute on one core. The numerical search that produced the matrices is not part of the proof.

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Appendix A The graphs and certificates in graph6

In the graph6 format, G_1 is `J?bFF‘wNA{?`, G_2 is `K?BD@jguAsBo` and H is `J?bFF‘wN?{?`. In G_1 the split is $S = \{1, 2, 3, 4, 5, 10\}$, $T = \{0, 6, 7, 8, 9\}$, and the ten unknowns are entries of B on edges between S and T; in G_2 it is $S = \{3, 5, 6, 8, 9, 10\}$, $T = \{0, 1, 2, 4, 7, 11\}$, with twelve unknowns. The graph G_2 also arises from G_1 by a ΔY-transformation on the triangle $\{1, 6, 10\}$. Deleting any of 105 vertex triples of G_1, and any of 130 of G_2, leaves a planar graph.