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COMPLETE PERIODICITY OF PERPENDICULAR BILLIARDS IN RATIONAL TRIANGLES OF PRIME DENOMINATOR

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Abstract. Let Δ be a triangle with angles $\pi a_i/N$, $\gcd(a_0, a_1, a_2) = 1$, and N odd. The directions perpendicular to the sides of Δ form one orbit under the reflection group, and a companion note showed that this direction class is completely periodic whenever it has no *swapped pair*: two vertex-to-vertex billiard paths, each the reverse of the other, that meet no side at a right angle. Let p be the number of swapped pairs. We prove that **when N is prime there are no swapped pairs**, so the class is completely periodic and every regular orbit in it meets a side at a right angle. This answers the note's conjecture at prime N and proves Boshernitzan's Conjecture 2.2(b) for every rational triangle with prime denominator placed with a side horizontal. The proof has two halves: a residue modulo $1 - \zeta_N$ rules out paths between distinct vertices (at an odd prime power ℓ^k it forces ℓ to divide the third angle's numerator), and at prime N, where the first homology is one-dimensional over $\mathbb{Q}(\zeta_N)$, a closed path is homologous to its mirror image, which an orientation-reversing involution forbids unless they coincide.

At composite odd N swapped pairs exist, but if no signed sum of their lengths vanishes, every regular orbit of the class still meets a side at a right angle; so $p \leq 1$ never breaks complete periodicity at odd N, unlike at even N. This criterion applies with p unbounded: the triangles $(3, 6m + 2, 12m + 4)$ and $(3, 6m + 1, 12m + 5)$ have exactly $p = m$, with independent lengths for all practical m (computer-assisted). And it is sharp: on four explicit intervals of triangles a pair of mirror-image cylinders meets no side at a right angle (computer-assisted).

1. Introduction

A billiard path in a polygon that meets a side at a right angle retraces itself, and in a rational polygon every regular path leaving a side at a right angle is periodic [Bos92, Proposition 1.8], [VGS92, §5]. In a companion note [SPO] we studied, for a rational triangle Δ with angles $\pi a_0/N$, $\pi a_1/N$, $\pi a_2/N$ ($\gcd(a_0, a_1, a_2) = 1$), the *perpendicular class*: the directions perpendicular to the sides together with their images under the group generated by the reflections in the sides. When N is odd it is a single class C of $2N$ oriented directions, and every side is perpendicular to one of them.

A *slot* is a vertex together with a direction of C pointing into its angle, and its *prong* is the billiard path leaving the vertex in that direction, followed to the next vertex (if it reaches one). A *swapped pair* is a pair of distinct

prongs, each the reverse of the other, that meet no side at a right angle; p is their number. The note proved [SPO, Theorem 7.8] that $p = 0$ *implies that C is completely periodic*: every regular billiard path with direction in C is periodic. It also showed, by exact computation, that the converse fails at odd N, and conjectured [SPO, Conjecture 8.4]:

Conjecture 1.1 ([SPO, Conjecture 8.4])**.** *If N is odd and $p \geq 1$, then* $\gcd(a_i, N) \geq 3$ *for some i.*

The conjecture has content at composite N with every $\gcd(a_i, N) = 1$, and at prime N, where it predicts that swapped pairs never occur. We prove it at prime N.

Theorem A (Theorem 4.3)**.** *Let N be an odd prime. Then $p = 0$. Hence the perpendicular class is completely periodic, every regular billiard path with direction in it meets a side at a right angle, and Boshernitzan's Conjecture 2.2(b)* [Bos92] *holds for Δ placed with a side horizontal.*

Boshernitzan's Conjecture 2.2(b) asserts, for a rational triangle with odd least common denominator d (here $d = N$), that every regular orbit in a direction that is an odd multiple of $\pi/2d$ is periodic; with a side horizontal this set is the class C [SPO, Corollary 7.9, Remark 7.10]. It was verified by computer for $d \leq 16$ [Bos92] and, by an exhaustive search, up to $d = 189$ [DR22]. Delecroix [Del22] asked whether, when such a direction is completely periodic, every saddle connection crosses a side perpendicular to it; by the dictionary of [SPO, Lemma 7.5] that is $p = 0$, so Theorem A answers his question affirmatively at prime N. Figure 1 shows the dichotomy.

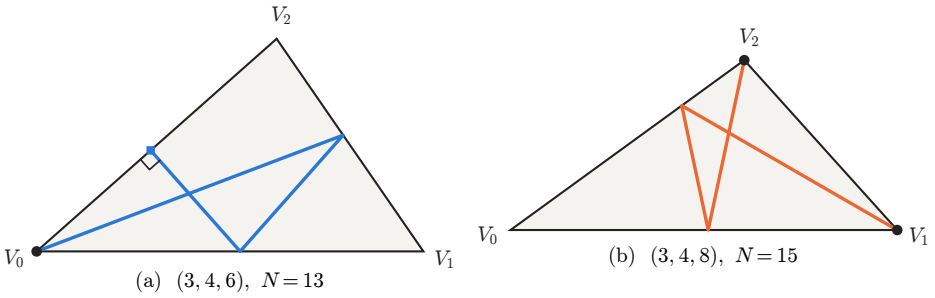


FIGURE 1. (a) In $(3, 4, 6)$, $N = 13$, the prong leaving V_0 reflects twice and then meets the side V_2V_0 at a right angle (marked), after which it retraces itself back to V_0; at prime N every prong does this (Theorem A). (b) In $(3, 4, 8)$, $N = 15$, the prong leaving V_2 reaches V_1 after two reflections, in the two sides at V_0, meeting no side at a right angle; with its reverse it is a swapped pair [SPO, Proposition 8.1].

The proof has two halves, and they use primality differently. A prong between two *distinct* vertices is ruled out by a residue: $2i$ times the holonomy

of a relative cycle, divided by $\zeta_N - 1$ and reduced modulo the prime above ℓ, remembers the numerator of the angle opposite its endpoints, up to sign, modulo ℓ, and a perpendicular reflection forces it to vanish. This works at every odd prime power $N = \ell^k$ and shows that a two-vertex prong forces $\ell \mid a_v$ at the third vertex (Proposition 3.2). A prong from a vertex to *itself* is a closed curve on the unfolded surface; at prime N the first homology is a one-dimensional $\mathbb{Q}(\zeta_N)$-vector space, so holonomy is injective on it and the prong is homologous to the reverse of its image under the involution ι that reverses the direction; and an orientation-reversing map fixing the base point cannot exchange two distinct homologous loops there (Proposition 4.2). Neither half needs a computer, and neither uses a lattice property: the infinitely many rational triangles that Mirzakhani and Wright show to have unfoldings with a dense $\mathrm{GL}(2,\mathbb{R})$-orbit are built with prime denominator [MW18, Theorem 1.4]. On the regular polyhedra, which are lattice surfaces, a congruence of the same kind and symmetry arguments in the spirit of the second half exclude geodesics from a vertex to itself [DDTY17], [Fuc16], [AAH22, Proposition 5.1], [Tro23]. (The clause of Theorem A on regular paths is Theorem B below with $p = 0$.)

At composite odd N swapped pairs do occur, and complete periodicity is then not automatic. But one pair cannot break it, and several can do so only through a coincidence among their lengths:

Theorem B (Theorem 5.3)**.** *Let N be odd and let $\lambda_1, \dots, \lambda_p$ be the lengths of the swapped pairs of the perpendicular class. If $\sum_i \varepsilon_i \lambda_i \neq 0$ for every $\varepsilon \in \{-1, 0, 1\}^p \setminus \{0\}$, then every regular billiard path with direction in the class meets a side at a right angle; the class is then completely periodic, and Conjecture 2.2(b) holds for Δ placed with a side horizontal. In particular this holds whenever $p \leq 1$.*

At even N the statement is false: the triangle $(7, 8, 15)$ has a perpendicular class with $p = 1$ and a minimal component ([DR22], [SPO, §9]). The proof (§5) is a balance of lengths around a region of the surface none of whose leaves meets a side at a right angle.

Theorem B has content beyond $p \leq 1$ only if several swapped pairs with independent lengths occur, and the swapped prongs of the note's families [SPO, Propositions 8.1, 8.2] make p unbounded. We show the criterion applies to them:

Theorem C (Theorem 6.4)**.** *For every $m \geq 1$ the triangles $(3, 6m+2, 12m+4)$ and $(3, 6m+1, 12m+5)$, with $N = 18m + 9$, have exactly $p = m$, and the lengths of their swapped pairs are $R\cos(kA/2)$, $k = 1, 3, \dots, 2m-1$, $A = \pi/(6m+3)$, for a constant $R > 0$. If $m \leq \varphi(6m+3)/2$ — in particular for every $m < 18\,591\,072$, the first m where this fails — every regular billiard path with direction in the perpendicular class meets a side at a right angle.*

The lower bound $p \geq m$ is [SPO, Proposition 8.2]; the note left the upper bound open [SPO, Remark 8.3]. The upper bound is computer-assisted:

interval certificates, uniform in m for $m \geq 100$, and a rigorous finite check below (§6.3, Appendix B). The length law holds on every band of a general family of swapped prongs (Proposition 6.2), so Theorem B applies at any point of such a band whose swapped pairs are all band prongs, provided $m \leq \varphi(n)/2$ (Corollary 6.3); what Theorem C adds is that, on these two lines, they are.

Finally, the conclusion of Theorem B is genuinely stronger than complete periodicity, and its hypothesis cannot be dropped:

Theorem D (Theorem 6.5)**.** *There are four explicit pairs (n, J), $J \subset \mathbb{R}$ an interval and $n = 9, 15, 15, 27$, such that every rational triangle with angles $(\pi/n, B, \pi - \pi/n - B)$, N odd and $Bn/\pi \in J$ has, in its perpendicular class, a pair of cylinders, each the mirror image of the other, whose leaves meet no side at a right angle.*

At the thirteen such triangles where we computed the whole decomposition the class is completely periodic (§8), so these are completely periodic classes in which a regular orbit misses every perpendicular collision. Theorem D is computer-assisted (a certificate in ball arithmetic, Appendix C).

Scope. Prime N is a natural arithmetic class, as prime denominators are for right triangles, but it is not a large one: among primitive triangles with odd $N \leq X$ its share decays like $1/\log X$ (31% at $X = 1001$). The composite class where Conjecture 1.1 has content (every $\gcd(a_i, N) = 1$) is about as large (30% at $X = 1001$), and no swapped pair has been found there in 2 288 triangles (§8); §7 explains why the proof does not extend there (the two-vertex half does extend to prime powers, but the loop half does not, and at N with two prime factors neither does).

Organisation. §2 recalls the setting from [SPO]. §3 proves the two-vertex half, §4 the loop half and Theorem A, §5 Theorem B. §6 is about sharpness: the bands, the length law, Theorem C and Theorem D. §7 records where the arguments stop, §8 the computations, §9 open problems. The appendices hold the casework: the development of the bands (Appendix A) and the two certificates (Appendices B, C).

2. Setting

We use the setting of [SPO, §§2, 7], and recall what is needed. Table 1 collects the notation. Δ has vertices V_0, V_1, V_2, angle $\pi a_i/N$ at V_i, $\gcd(a_0, a_1, a_2) = 1$ and $N = a_0 + a_1 + a_2$; from §3 on, N is odd. D_N is the group generated by the linear parts of the reflections in the sides, the dihedral group of order $2N$.

The class. With a side horizontal the side directions are the multiples of π/N and their perpendiculars the odd multiples of $\pi/2N$. When N is odd these $2N$ oriented directions form one D_N-orbit C, on which D_N acts freely, and no side direction lies in C [SPO, §2]. A *slot* is a pair (V, u), V a vertex and $u \in C$ strictly inside the angle at V; there are N slots [SPO, Lemma

4.1]. The *prong* of a slot is the billiard path leaving V along u, followed until it first reaches a vertex. Write s' for the number of prongs that never reach a vertex and p for the number of swapped pairs (§1). A prong that meets a side at a right angle retraces itself and returns to its own vertex [SPO, proof of Theorem 4.2, step (iii)]. In particular a prong from V_u to V_w, $w \neq u$, never meets a side at a right angle, and is swapped with its reverse.

The surface. Let X be the unfolding of Δ: copies (Δ, g), $g \in D_N$, with (x, g) glued to (x, gr_j) for x on side j, r_j the reflection in the direction of side j. For N odd it is a translation surface [SPO, §7.1]. We mark every image of a vertex and write Σ for the set of marked points; a marked point *has type* i if it is an image of V_i. The points of type i are $\gcd(a_i, N)$ in number, with cone angle $2\pi a_i / \gcd(a_i, N)$. The side copies cut X into $2N$ triangles; they form a cell structure with $3N$ edges and vertex set Σ. Rotation ρ by $2\pi/N$ acts on X by $(x, g) \mapsto (x, \rho g)$, preserving types.

Fix $\omega \in C$ and let F_ω be the foliation of X in direction ω; the billiard paths with directions in C are the projections of its leaves. C is *completely periodic* if F_ω decomposes into cylinders. Let $\bar{r} \in D_N$ be the reflection with $\bar{r}\omega = -\omega$ and $\iota(x, g) \coloneqq (x, \bar{r}g)$. Then ι is an orientation-reversing isometric involution of X with linear part $\bar{r}$, preserving F_ω, types, and every marked point that is the only one of its type. At odd N its fixed set $\mathrm{Fix}(\iota)$ is a union of side copies perpendicular to ω, the *vertical arcs*, with their endpoints [SPO, Lemma 7.1]. (The names follow [SPO], which draws ω horizontally, so that side copies perpendicular to it are vertical.) Vertical arcs exist, for instance the copy (σ, g) of a side σ with g carrying the inward normal of σ to ω [SPO, §7.1].

What the note proves. We use four facts from [SPO, §7], stated for odd N.

(T) *Dichotomy* [SPO, Lemma 7.2]. A saddle connection e of F_ω either satisfies $\iota(e) = e$ and meets $\mathrm{Fix}(\iota)$ exactly at its midpoint, on a vertical arc (e is *blocked*), or $\iota(e) \neq e$ (e is *swapped*). (These are [SPO]'s types (2) and (3); its type (1), the *horizontal arcs* — side copies parallel to ω and fixed by ι — exists only at even N.)

(R) *Reversibility* [SPO, Lemma 7.3, Corollary 7.4]. Almost every point of a vertical arc lies on a closed leaf, and a saddle connection meeting the interior of a vertical arc has a cylinder on each side.

(D) *Dictionary* [SPO, Lemma 7.5]. The saddle connections of F_ω are the lifts of the vertex-to-vertex billiard paths with directions in C. The two lifts of one path are exchanged by ι, and coincide if and only if the path meets a side at a right angle. So the swapped saddle connections form exactly p pairs $\{\gamma_i, \iota\gamma_i\}$.

(P) [SPO, Proposition 7.6, Theorem 7.8]. C is completely periodic if and only if $s' = 0$, and every minimal component of F_ω has a swapped saddle connection on its boundary. In particular $p = 0$ implies complete periodicity.

Property ($\perp$)**.** A cylinder of F_ω is *invariant* if ι maps it to itself and *free* otherwise. Let $X^\circ \coloneqq X \setminus \Sigma$, $\mathcal{K}$ the closure in X° of the union of the invariant cylinders, and $\mathcal{F} \coloneqq X^\circ \setminus \mathcal{K}$ the *free region*. We say C has *property* ($\perp$) if every regular billiard path with direction in C meets a side at a right angle (it is then periodic [Bos92, Proposition 1.8], so ($\perp$) implies complete periodicity). By Lemma 5.2(e) below, ($\perp$) holds if and only if $\mathcal{F} = \emptyset$.

Remark 2.1. If N is odd, the perpendicular class contains no side direction [SPO, §2]. If N is even, each of the two perpendicular classes contains one (we use this only to say where oddness enters): for N even a class is the set of integer direction indices (in units of π/N) of one parity, and the side indices are 0, a_0, $a_0 + a_1$; if all three had the same parity then a_0, a_1 and $a_2 = N - a_0 - a_1$ would all be even, contradicting $\gcd(a_0, a_1, a_2) = 1$. So both parities occur among the side directions.

3. Two vertices: a residue

Throughout this section $N = \ell^k$ is a power of an odd prime ℓ. Put $\zeta \coloneqq e^{2\pi i/N}$, $z \coloneqq e^{i\pi/N}$, and let $\mathfrak{p} = (1-\zeta)$, the prime of $\mathbb{Z}[\zeta]$ above ℓ; it is totally ramified, so reduction is a ring map $\varepsilon\colon \mathbb{Z}[\zeta] \to \mathbb{F}_\ell$ with $\varepsilon(\zeta) = 1$. Since N is odd, $z = -\zeta^{(N+1)/2} \in \mathbb{Z}[\zeta]$ and $\varepsilon(z) = -1$.

Take Δ with circumdiameter 1, placed with side V_0V_1 horizontal. The side opposite V_v has length $\sin(\pi a_v/N)$, and every side copy in X has direction $\pi t/N$ for an integer t. So its holonomy h satisfies

$$2ih = z^t\big(z^a - z^{-a}\big) = z^{t-a}(\zeta^a - 1) \in \mathfrak{p}, \qquad a = a_v. \tag{3.1}$$

Let $\Lambda \subset \mathbb{C}$ be the group generated by the holonomies of the side copies; it contains the holonomy of every relative cycle in $H_1(X, \Sigma; \mathbb{Z})$, since the side copies are the edges of a cell structure with vertex set Σ. By (3.1), $2i\Lambda \subset \mathfrak{p}$, so

$$\varrho\colon \Lambda \to \mathbb{F}_\ell, \qquad \varrho(w) \coloneqq \varepsilon\Big(\frac{2iw}{\zeta - 1}\Big)$$

is a well-defined homomorphism, and on a side copy opposite V_v, by (3.1), $\varrho(h) = \varepsilon(z^{t-a})\,\varepsilon(1 + \zeta + \cdots + \zeta^{a-1}) = \pm a_v$.

Lemma 3.1. *$\varrho(gw) = \varrho(w)$ for every $g \in D_N$ and $w \in \Lambda$.*

Proof. A rotation multiplies w by a power of ζ, and $\varepsilon(\zeta) = 1$. A reflection is $w \mapsto \zeta^j \bar{w}$ for some j. Write $2iw = (\zeta - 1)q$ with $q \in \mathbb{Z}[\zeta]$; then $2i\zeta^j\bar{w} = -\zeta^j\,\overline{2iw} = -\zeta^j(\zeta^{-1} - 1)\bar{q} = \zeta^{j-1}(\zeta - 1)\bar{q}$, so $\varrho(\zeta^j\bar{w}) = \varepsilon(\zeta^{j-1}\bar{q}) = \varepsilon(\bar{q}) = \varepsilon(q)$: complex conjugation preserves $\mathfrak{p}$ and acts trivially on $\mathbb{Z}[\zeta]/\mathfrak{p} = \mathbb{F}_\ell$. □

The side copies of one side of Δ are its images under D_N, so by Lemma 3.1 ϱ takes the same value on all copies of the oriented side $V_x \to V_y$. Since the three oriented sides of Δ sum to zero there is a function ϕ on the vertex types with $\varrho(V_x \to V_y) = \phi(y) - \phi(x)$ ($\phi(0) = 0$), and each difference $\phi(w) - \phi(u)$, $u \neq w$, is ϱ of the side $V_u \to V_w$, which is $\pm a_v$, v the third type. Extending

symbol	meaning	where
Δ, V_i, a_i, N	the triangle, its vertices, angles $\pi a_i/N$, $N = a_0 + a_1 + a_2$	§2
D_N, ρ	the group of linear parts of the reflections in the sides; the rotation by $2\pi/N$	§2
C, ω	the perpendicular class; a fixed direction in it	§2
slot, prong, swapped pair	vertex with a direction of C into its angle; the path from it to the next vertex; two prongs, each the reverse of the other, meeting no side at a right angle	§1, §2
p, s'	number of swapped pairs; number of prongs that never reach a vertex	§2
X, Σ, type i	the unfolding, its marked points, the marked points over V_i	§2
F_ω, ι, $\bar{r}$, $\mathrm{Fix}(\iota)$	the foliation in direction ω; the flow-reversing involution, its linear part, its fixed set (the vertical arcs)	§2
blocked, swapped	a saddle connection with $\iota(e) = e$, or with $\iota(e) \neq e$	§2, (T)
X°, $\mathcal{K}$, $\mathcal{F}$	$X \setminus \Sigma$; closure of the invariant cylinders; the free region $X^\circ \setminus \mathcal{K}$	§2
$(\perp)$	every regular path in C meets a side at a right angle	§2
ζ, z, $\mathfrak{p}$, ε	$e^{2\pi i/N}$, $e^{i\pi/N}$, the prime $(1-\zeta)$, reduction modulo $\mathfrak{p}$	§3
Λ, ϱ	the group of holonomies of side copies; the residue $\varrho(w) = \varepsilon(2iw/(\zeta-1))$	§3
hol	the holonomy map on $H_1(X;\mathbb{Q})$	Lemma 4.1
$\{\gamma_i, \iota\gamma_i\}$, λ_i	the ith pair of swapped connections, its length	Thm 5.3
W, l_i, r_i	a component of $\mathcal{F}$; members of the ith pair on its frontier with W on their left, right	proof of Thm 5.3
q, m, n, A, β, d	band parameters: $n = q(2m+1)$, $A = \pi/n$, $\beta = \pi/q$, $d = 4m+2$	§6.1
t, Γ, $S_{q,m}$	angle $(K+t)A$ at V_1; the angle at V_2; the strip	§6.1
t_{up}, $\mathcal{B}_{q,m}$	the root of (6.1); the band $0 < t < t_{\mathrm{up}}$	Prop. 6.1
$W_{q,m,j}$, ψ_j, λ_j	the band words, their directions, their lengths	Props. 6.1, 6.2
star	a fan of copies of Δ about an image of V_0	§6.1

TABLE 1. Notation used throughout. A few letters are also reused locally, with the meaning fixed where they appear: h (a holonomy, §3; $(q-1)/2$, §6.1), t (a direction index, §3), K (the integer $(n-2m-3)/2$, §6.1), P (marked points P_u, §§3–4; with Q, margin factors, Appendix A.3), s (a run or star index, §6.1 and Appendix A) and ν (a slot index, §6.3).

ϕ to 0-chains on Σ by type, we get $\varrho \circ \mathrm{hol} = \phi \circ \partial$ on 1-chains of side copies, hence on $H_1(X, \Sigma; \mathbb{Z})$.

Proposition 3.2. *Let $N = \ell^k$ be an odd prime power and $c \in H_1(X, \Sigma; \mathbb{Z})$ with $\partial c = P_w - P_u$, where P_u, P_w are marked points of types $u \neq w$. If* hol c

is perpendicular to a side copy, then $\ell \mid a_v$, v the third type. In particular, a prong of the perpendicular class from V_u to V_w, $u \neq w$, exists only if $\ell \mid a_v$; at prime N there is none.

Proof. Let $\tau \in D_N$ be the reflection in the direction of the side copy. Then $\tau(\operatorname{hol} c) = -\operatorname{hol} c$, so by Lemma 3.1 $\varrho(\operatorname{hol} c) = \varrho(-\operatorname{hol} c) = -\varrho(\operatorname{hol} c)$, and $\varrho(\operatorname{hol} c) = 0$ since ℓ is odd. But $\varrho(\operatorname{hol} c) = \phi(w) - \phi(u) = \pm a_v$. So $\ell \mid a_v$. A prong from V_u to V_w lifts to a saddle connection, whose class has boundary $P_w - P_u$ and holonomy in a direction of C, which is perpendicular to a side copy. At prime N, $0 < a_v < N$. □

The proof uses neither straightness nor the number of points of each type, so it applies to any relative class, realised by a saddle connection or not. At N with two distinct prime factors it has no analogue (Remark 7.1). On the regular octahedron, Fuchs reads the vertex at which a geodesic ends off $p - q \bmod 3$, where $pu + qv$ is its holonomy in a basis of unit vectors at angle $\pi/3$ [Fuc16, §3]; this is the residue of $pu + qv$ modulo $(1 - \zeta_3)$. On the regular tetrahedron, Davis, Dods, Traub and Yang read it off $(a, b) \bmod 2$ for the holonomy $au + bv$, the residue modulo the inert prime 2, and exclude a geodesic from a vertex to itself because its holonomy would be divisible by 2, hence not primitive [DDTY17, Lemma 3.1, Corollary 3.8].

4. Loops, and prime denominators

In this section N is an odd prime. Then $\gcd(a_i, N) = 1$ for each i, so each type is a single marked point P_i, of cone angle $2\pi a_i$, fixed by ρ and by ι.

Lemma 4.1. *At odd prime N,* $\operatorname{hol}\colon H_1(X;\mathbb{Q}) \to \mathbb{C}$ *is injective.*

Proof. This is the eigenspace count for cyclic covers of the sphere branched over three points [AI88], [MW18, Corollary 6.2]; we include the short argument. $X/\langle\rho\rangle$ is the double of Δ, a sphere, and $X \to X/\langle\rho\rangle$ has degree N and is totally ramified over the three vertices. By Riemann–Hurwitz $\chi(X) = 2N - 3(N-1) = 3 - N$, so $\dim_{\mathbb{Q}} H_1(X;\mathbb{Q}) = N - 1$. The transfer shows that $1 + \rho_* + \cdots + \rho_*^{N-1}$ acts on $H_1(X;\mathbb{Q})$ through H_1 of the sphere, which is 0; so ρ_* satisfies Φ_N, and $H_1(X;\mathbb{Q})$ is a $\mathbb{Q}(\zeta)$-vector space, ζ acting as ρ_*, of dimension $(N-1)/(N-1) = 1$. Since ρ is a rotation by $2\pi/N$, $\operatorname{hol}(\rho_* c) = \zeta \operatorname{hol}(c)$, so hol is $\mathbb{Q}(\zeta)$-linear and its kernel is a $\mathbb{Q}(\zeta)$-subspace of a line. It is not everything (X has positive genus, and its absolute periods span $\mathbb{C}$ over $\mathbb{R}$), so it is 0. □

Proposition 4.2. *Let N be an odd prime and γ a saddle connection of F_ω from P_u to P_u. Then $\iota(\gamma) = \gamma^{-1}$; in particular γ is ι-invariant as a set, so blocked.*

Proof. Put $\gamma' \coloneqq \iota(\gamma)^{-1}$: since ι fixes P_u and has linear part $\bar{r}$, γ' is again a saddle connection from P_u to P_u in direction ω, with $\operatorname{hol}\gamma' = \operatorname{hol}\gamma$. Both are closed curves, and $\operatorname{hol}([\gamma] - [\gamma']) = 0$, so by Lemma 4.1 $[\gamma] = [\gamma']$ in $H_1(X;\mathbb{Q})$; this class is non-zero, as $\operatorname{hol}\gamma \neq 0$. Suppose $\gamma' \neq \gamma$.

(i) The configuration. Distinct leaves of F_ω have disjoint interiors, so $\gamma \cap \gamma' = \{P_u\}$. At P_u there are four germs: γ_{out}, γ_{in} (the start and the end of γ, as rays from P_u) and γ'_{out}, γ'_{in}. They are distinct: out-germs point along ω and in-germs along $-\omega$, and a saddle connection is determined by either of its end germs.

(ii) Two components. This step is the known fact that distinct homologous saddle connections separate the surface [EMZ03]; we need the shape of the splitting, so we prove it. If $X \setminus (\gamma \cup \gamma')$ were connected, a closed curve δ crossing γ once and missing γ' would have $\delta \cdot \gamma = \pm 1 \neq 0 = \delta \cdot \gamma'$, against $[\gamma] = [\gamma']$. Write $L\gamma$, $R\gamma$ for the two sides of γ (left and right of its direction) and $L\gamma'$, $R\gamma'$ likewise; each lies in one component of the complement. If Y is a component, the boundary of the 2-chain $\bar{Y}$ (oriented with Y on its left) is $(n_L - n_R)\gamma + (n'_L - n'_R)\gamma'$, where $n_L, n_R \in \{0, 1\}$ record which sides of γ lie in Y and n'_L, n'_R those of γ'. It is null-homologous, and $[\gamma] = [\gamma'] \neq 0$, so $(n_L - n_R) + (n'_L - n'_R) = 0$: every component contains as many right sides as left sides. Each component contains a side (X is connected), hence at least two of the four sides; there are at least two components; so there are exactly two, each containing one left and one right side: either $\{L\gamma, R\gamma\} \sqcup \{L\gamma', R\gamma'\}$ or $\{L\gamma, R\gamma'\} \sqcup \{R\gamma, L\gamma'\}$. The four germs cut a small disc about P_u into four sectors, each in the complement, and since two germs belong to each curve some sector lies between a germ of γ and a germ of γ'; it meets a side of each. That excludes the first splitting. So the components are $A \supset \{L\gamma, R\gamma'\}$ and $B \supset \{R\gamma, L\gamma'\}$.

(iii) The cyclic order. Orient the link of P_u counterclockwise. The sector just counterclockwise of an out-germ lies to the left of its curve, and the sector just counterclockwise of an in-germ to its right; so the sector counterclockwise of γ_{out}, γ_{in}, γ'_{out}, γ'_{in} lies in A, B, B, A respectively, and the sector clockwise of each in the other component. A sector between consecutive germs lies in one component, so the germ following a given germ, counterclockwise, must have on its clockwise side the component that the given germ has on its counterclockwise side. This leaves exactly two cyclic orders of the four germs:

$$\mathcal{O}_1 = (\gamma_{\text{out}}, \gamma_{\text{in}}, \gamma'_{\text{in}}, \gamma'_{\text{out}}), \qquad \mathcal{O}_2 = (\gamma_{\text{out}}, \gamma'_{\text{out}}, \gamma'_{\text{in}}, \gamma_{\text{in}})$$

(Figure 2); for instance, after γ_{out} must come γ_{in} or γ'_{out}. Now ι fixes P_u and maps the set of four germs to itself: $\gamma_{\text{out}} \mapsto \gamma'_{\text{in}}$ and $\gamma_{\text{in}} \mapsto \gamma'_{\text{out}}$, since $\iota(\gamma) = \gamma'^{-1}$, and vice versa. Being orientation-reversing, it carries the counterclockwise order of the germs to the clockwise order of their images. If the germs occur in the order $\mathcal{O}_1$, their images, read clockwise, are $(\gamma'_{\text{in}}, \gamma'_{\text{out}}, \gamma_{\text{out}}, \gamma_{\text{in}})$, so read counterclockwise they are $(\gamma_{\text{in}}, \gamma_{\text{out}}, \gamma'_{\text{out}}, \gamma'_{\text{in}})$, which is $\mathcal{O}_2$; likewise $\mathcal{O}_2$ goes to $\mathcal{O}_1$. But ι maps the configuration of the four germs at P_u onto itself, so the cyclic order actually realised must be carried to itself. It is neither $\mathcal{O}_1$ nor $\mathcal{O}_2$: a contradiction. Hence $\gamma' = \gamma$, that is $\iota(\gamma) = \gamma^{-1}$. □

On Platonic surfaces, Athreya, Aulicino and Hooper [AAH22, Proposition 5.1] and Troubetzkoy [Tro23] exclude closed saddle connections with a half-turn about the midpoint, which reverses the saddle connection and so would fix its endpoint. There the involution preserves orientation and maps the saddle connection to itself by the symmetry of the surface; here Lemma 4.1 supplies only a homologous partner, and the orientation-reversing ι is what forces it to coincide.

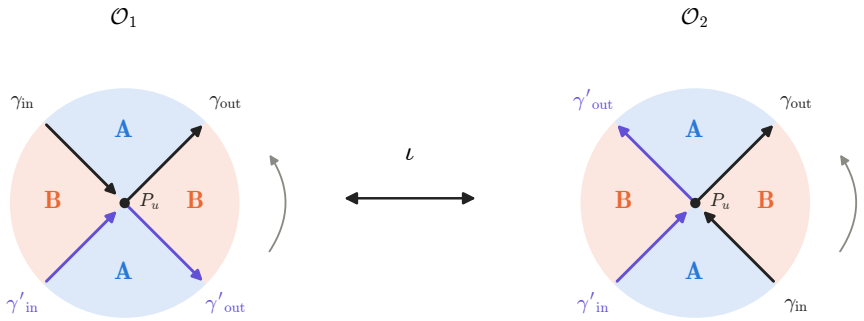


FIGURE 2. Step (iii) of the proof of Proposition 4.2, schematically. The link of P_u (of length $2\pi a_u$) is drawn as a circle, angles not to scale: in fact out-germs point along ω and in-germs along $-\omega$, on different sheets of the cone point. The sector just counterclockwise (grey arrow) of an out-germ lies to the left of its curve, that of an in-germ to its right; with step (ii) this puts each sector in A or B as shown. Only the cyclic orders $\mathcal{O}_1$ and $\mathcal{O}_2$ are consistent, and the orientation-reversing ι, which sends $\gamma_{\text{out}} \mapsto \gamma'_{\text{in}}$ and $\gamma_{\text{in}} \mapsto \gamma'_{\text{out}}$, exchanges them.

Theorem 4.3. *Let N be an odd prime. Then the perpendicular class has $p = 0$. It is completely periodic, has property ($\perp$) — every regular billiard path with direction in C meets a side at a right angle — and every saddle connection in a direction of C crosses a side perpendicular to it. Conjecture 1.1 holds at N, and so does Boshernitzan's Conjecture 2.2(b) for Δ placed with a side horizontal.*

Proof. Let γ be a prong that reaches a vertex. If it ends at a vertex of another type it does not exist, by Proposition 3.2. Otherwise its lift is a saddle connection from P_u to P_u (each type is one point), which is ι-invariant by Proposition 4.2; by (D) the prong meets a side at a right angle. So no prong is swapped: $p = 0$. By (P) the class is completely periodic, and by Theorem 5.3 below with $p = 0$ (whose proof does not use this theorem) it has property ($\perp$). By (T) and (D) every saddle connection is then blocked,

so it crosses a vertical arc. Conjecture 2.2(b) follows by [SPO, Corollary 7.9]. □

5. The length criterion

In this section N is odd. By (T), (D) and Remark 2.1, every saddle connection of F_ω is blocked or one of the $2p$ members of the swapped pairs.

Lemma 5.1. *Orient each saddle connection along ω. If e is a saddle connection, e^* the saddle connection $\iota(e)$ oriented along ω, then ι maps the left side of e to the left side of e^*, and the right to the right.*

Proof. In a flat chart with $\omega = \partial_x$ the map ι is $(x, y) \mapsto (-x + c, y + c')$, its linear part $\bar{r}$ fixing the direction perpendicular to ω. It sends $e = \{y = 0\}$ to $\{y = c'\}$ and the left side $\{y > 0\}$ to $\{y > c'\}$, the left side of $\{y = c'\}$ oriented along ω. □

Lemma 5.2. *(a) An open cylinder or minimal component of F_ω that meets* $\mathrm{Fix}(\iota)$ *is an invariant cylinder; every closed leaf of an invariant cylinder is mapped to itself and crosses* $\mathrm{Fix}(\iota)$ *twice. (b) Both regions beside a blocked saddle connection are invariant cylinders. (c) No swapped saddle connection meets* $\mathrm{Fix}(\iota)$ *in its interior. (d)* $\mathrm{Fix}(\iota) \cap X^\circ \subseteq \mathcal{K}$. *(e) Every regular billiard path with direction in C meets a side at a right angle if and only if $\mathcal{F} = \emptyset$.*

Proof. (a) By (R) almost every point of $\mathrm{Fix}(\iota)$ inside the region lies on a closed leaf, so the region is a cylinder; such a leaf passes through a fixed point of ι and ι maps it to a leaf through the same point, so to itself; so ι maps the cylinder to itself. On an invariant cylinder, with coordinates (θ, y), θ along the leaves and $y \in (0, w)$ across, ι is $(\theta, y) \mapsto (c - \theta, \phi(y))$ with ϕ an isometry of $(0, w)$, increasing by Lemma 5.1; so ϕ is the identity, every leaf is invariant, and ι reverses each leaf, fixing exactly two of its points. (b) The vertical arc through the midpoint crosses the connection and enters both regions; apply (a). (c) A point of $\mathrm{Fix}(\iota)$ in the interior of e also lies on $\iota(e)$, and distinct parallel saddle connections share no interior point. (d) A point of $\mathrm{Fix}(\iota) \cap X^\circ$ is an interior point of a vertical arc. It lies either in an open region, which meets $\mathrm{Fix}(\iota)$ and so is an invariant cylinder by (a), or on a saddle connection crossing the arc, which is blocked by (c) and so bounds invariant cylinders by (b). (e) A right-angle collision with a side is a crossing of a side copy perpendicular to ω, that is of a vertical arc. If $\mathcal{F} = \emptyset$ every regular leaf lies in an open invariant cylinder and crosses a vertical arc by (a). If $\mathcal{F} \neq \emptyset$, it contains a regular leaf (there are finitely many separatrices), which by (d) never meets $\mathrm{Fix}(\iota)$. □

Theorem 5.3. *Let N be odd and $\lambda_1, \dots, \lambda_p$ the lengths of the swapped pairs of the perpendicular class (λ_i the length of either prong of the ith pair). If $\sum_i \varepsilon_i \lambda_i \neq 0$ for every $\varepsilon \in \{-1, 0, 1\}^p \setminus \{0\}$, then the class has property ($\perp$): every regular billiard path with direction in the class meets a side at a right angle, the class is completely periodic, and every cylinder of F_ω is*

invariant. Conjecture 2.2(b) then holds for Δ placed with a side horizontal. In particular all of this holds whenever $p \leq 1$.

Proof. F_ω decomposes X into finitely many cylinders and minimal components, bounded by finitely many saddle connections. $\mathcal{K}$ is the union of the closed invariant cylinders (minus Σ), so $\mathcal{F}$ is a union of free cylinders, minimal components and whole saddle connections. Let W be a component of $\mathcal{F}$. $\mathcal{K}$ is non-empty (by (R) and Lemma 5.2(a)) and X° is connected, so W has non-empty frontier in X°. The frontier is a finite union of open saddle connections, each with W on one side and an invariant cylinder on the other; a connection with $\mathcal{F}$ on both sides lies in $\mathcal{F}$, hence in W. By Lemma 5.2(b) each frontier connection is swapped. For a frontier connection e write e_W for its side facing W.

The balance. Cut X along the union Γ of the saddle connections of F_ω; the closure of $W \setminus \Gamma$ in the cut surface is a compact surface with piecewise geodesic boundary, and its fundamental chain, pushed to X, is a 2-chain whose boundary is $\sum_{e_W=Le} e - \sum_{e_W=Re} e$ (each e oriented along ω; a connection with W on both sides cancels, and a marked point where W touches itself contributes nothing to a 1-chain). With $\lambda(e) \coloneqq |\operatorname{hol} e|$, so that $\operatorname{hol} e = \lambda(e)\,\omega$, integrating the closed form dz over this boundary gives

$$\sum_{e_W=Le} \lambda(e) = \sum_{e_W=Re} \lambda(e). \tag{5.1}$$

For the ith pair $\{\gamma_i, \iota\gamma_i\}$ let l_i (resp. r_i) be the number of its frontier connections whose left (resp. right) side faces W; the two members have the same length λ_i, so (5.1) reads $\sum_i (l_i - r_i)\lambda_i = 0$.

Case $\iota W \neq W$. Each invariant cylinder is mapped to itself, so $\iota\mathcal{K} = \mathcal{K}$, $\iota\mathcal{F} = \mathcal{F}$, ι permutes the components of $\mathcal{F}$, and $\iota W \cap W = \emptyset$. By Lemma 5.1, if the left side of γ_i faces W then the left side of γ_i^* faces ιW, so not W: hence $l_i, r_i \leq 1$, and the hypothesis forces $l_i = r_i$ for all i. The frontier is non-empty and consists of swapped connections, so some i has $l_i + r_i > 0$, hence $l_i = r_i = 1$: W faces the left side of e and the right side of e', with $e, e' \in \{\gamma_i, \gamma_i^*\}$. If $e' = e$ then W lies on both sides of e, which is then not a frontier connection. If $e' = e^*$ then ιW faces the right side of e (Lemma 5.1), so e has W on its left and $\iota W \subset \mathcal{F}$ on its right, and again is not a frontier connection.

Case $\iota W = W$. Then ι permutes the frontier sides of W, and by Lemma 5.1 $l_i, r_i \in \{0, 2\}$. The hypothesis, applied to $\varepsilon_i = (l_i - r_i)/2$, forces $l_i = r_i$, and as before some i has $l_i + r_i > 0$, so $l_i = r_i = 2$: W faces both sides of γ_i, which is then not a frontier connection.

Both cases are contradictory, so $\mathcal{F} = \emptyset$. Then there is no free cylinder and no minimal component (each would be a non-empty open subset of $\mathcal{F}$), and every regular leaf lies in an invariant cylinder; apply Lemma 5.2(a). With a side horizontal the class C is the direction set of Conjecture 2.2(b) [SPO, Corollary 7.9], so complete periodicity gives it. □

Remark 5.4 (where oddness enters). Twice. First, X is a translation surface on which ω orients every leaf, so that Lemma 5.1 holds and dz is global; and each billiard path has exactly two lifts, so swapped pairs are not forced to come in pairs of equal length. At even N the argument runs on the translation surface S_Δ (the unfolding, of which the half-translation surface X is the quotient by the half-turn at even N [SPO, §7.1]), but there every swapped pair lifts to two pairs of the same length, and the hypothesis fails as soon as $p \geq 1$; on the half-translation surface itself Lemma 5.1 fails at the horizontal arcs, where ι exchanges the two sides. Second, by Remark 2.1 no side direction lies in C, so every saddle connection that is not blocked is a lift of a swapped prong, and the hypothesis concerns prong lengths only. At even N the statement is false (Remark 7.4).

Remark 5.5. Nothing in the proof uses that Δ is a triangle: given the polygon analogues of (T), (R), (D), the description of $\mathrm{Fix}(\iota)$ and the odd half of Remark 2.1, it holds verbatim for rational polygons with odd N. Those analogues, and polygons with odd N where the conclusion fails (so the hypothesis cannot be dropped), are the subject of a forthcoming paper. (Since (5.1) holds in $H_1(X, \Sigma; \mathbb{Z})$, the same proof works under the alternative hypothesis of no signed relation among the classes of the $2p$ swapped connections; neither hypothesis is known to imply the other.)

6. Sharpness

6.1. **Bands of swapped prongs.** At composite odd N most swapped prongs we know belong to the following families, and every triangle with $p \geq 2$ in our censuses lies on one of their bands (§8); other swapped prongs exist (isolated ones, and families of other shapes). The prongs of a family are *loops* at V_2: from V_2 the path winds around q successive images of V_0 — fans of copies of Δ, which we call *stars* — and returns to an image of V_2 (Figure 3). They are thus of the kind that Proposition 4.2 excludes at prime N; at composite N the homology argument fails (Remark 7.2).

Let $q \geq 3$ be odd and $m \geq 1$, and put

$$n = q(2m+1), \quad A = \pi/n, \quad \beta = \pi/q, \quad h = \tfrac{q-1}{2}, \quad K = \tfrac{n-2m-3}{2},$$

and $d = 4m+2$, so that $dA = 2\beta$. The *strip* $S_{q,m}$ is the set of triangles with angle A at V_0 and $B = (K+t)A$ at V_1, $0 < t < 1$. Since $(K+1)A = \pi/2 - \beta/2$, the angle at V_2 is $\Gamma = \pi/2 + \beta/2 - tA$. We measure directions with V_0V_1 along the positive real axis and V_2 above it. Label the sides $0 = V_0V_1$, $1 = V_1V_2$, $2 = V_2V_0$ and write a bounce word as runs of the letters $0, 2$ (reflections in the two sides at V_0, so a run turns the path around an image of V_0) separated by single letters 1 (which move it to the next star). For $j = 0, \dots, m-1$ let $W_{q,m,j}$ be the word with q runs, of lengths

$$2j+1, \quad r_1 + 2j, \dots, r_h + 2j, \quad f_{h+1} - 2j, \dots, f_{q-2} - 2j, \quad 4m - 1 - 2j,$$

where $r_s = sd + 2$ and $f_s = (q-s)d - 2$: the first run is $(02)^j 0$, the *rising* runs are $(20)^{(r_s+2j)/2}$ $(1 \leq s \leq h)$, the *falling* runs are $(02)^{(f_s-2j)/2}$

$(h < s \le q-2)$, and the last run is $(02)^{2m-1-j}0$. Each inner run turns the path through an angle $\equiv 2s\beta + 2A \pmod{2\pi}$, so the condition that the word closes becomes a sum over the q-th roots of unity, which vanishes (Appendix A.2).

Proposition 6.1. *Let $t_{\mathrm{up}} \in (0,1)$ be the unique root of*

(6.1)
$$\sin(\tfrac{3\beta}{2} - A)\sin((1-t)A)\cos(\tfrac{\beta}{2} - tA) = \sin(tA)\cos(\tfrac{\beta}{2} + A - tA)\sin(\tfrac{\beta}{2} - A).$$

Then $t_{\mathrm{up}} > 2/3$, and for every $t \in (0, t_{\mathrm{up}})$ and every $j = 0, \dots, m-1$ the billiard path leaving V_2 in direction $\psi_j = -\pi/2 + (t-j)A$ has bounce word $W_{q,m,j}$ and ends at V_2, with no vertex between and no side met at a right angle. If N is odd, ψ_j lies in the perpendicular class, and these are m distinct swapped pairs: $p \ge m$.

We call the part $0 < t < t_{\mathrm{up}}$ of the strip the *band* $\mathcal{B}_{q,m}$. The proof of Proposition 6.1 develops the word and writes every *crossing condition* — that the developed line crosses each side copy on its way in the interior — in closed form; it is in Appendix A.

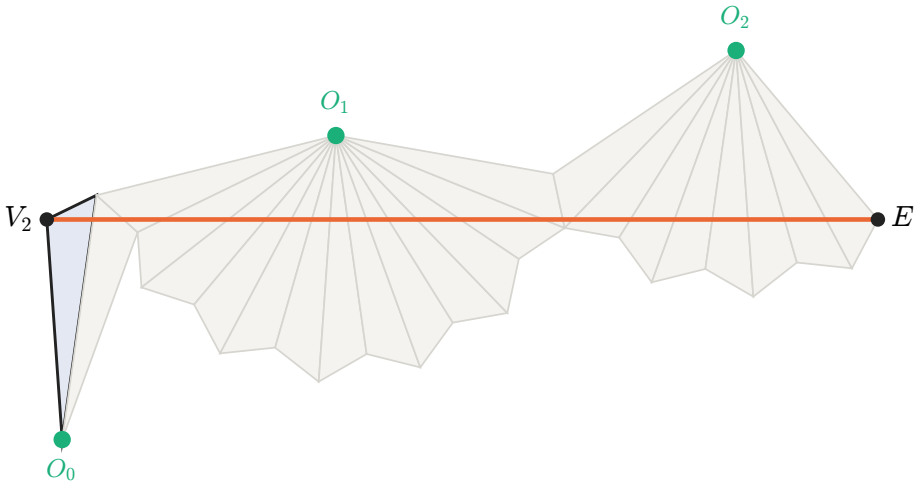


FIGURE 3. The band prong $W_{3,2,0}$ of $(3, 14, 28)$, $N = 45$ ($q = 3$, $m = 2$, $t = 2/3$), developed along its word $0 \mid (20)^6 \mid (02)^3 0$ from V_2 to the image E of V_2, and drawn with the prong horizontal; the base copy is shaded. The copies form three stars, fans of 2, 13 and 8 copies about the images O_0, O_1, O_2 of V_0. The prong is realised because every side copy it crosses has its two ends on opposite sides of the prong.

At $q = 3$, $t = 2/3$ and $t = 1/3$ the triangles are $(3, 6m+2, 12m+4)$ and $(3, 6m+1, 12m+5)$, and the words $W_{3,m,j}$ reversed are those of [SPO, Proposition 8.2]; the proposition extends that construction to every odd q

and to the whole band. Every odd composite n and every factorisation $n = q(2m+1)$, $q, 2m+1 \geq 3$, gives a band at $A = \pi/n$, and distinct factorisations give disjoint strips (distinct K). On a band, $\gcd(a_0, N) = N/n \geq 3$, consistent with Conjecture 1.1.

6.2. **The length law.**

Proposition 6.2. *In the setting of Proposition 6.1, with circumdiameter* 1*, the prong $W_{q,m,j}$ has length*

$$\lambda_j = R\cos\big((2m-1-2j)A/2\big), \qquad R = \frac{2\sin B\,\cos(tA)}{\sin(\beta/2)}.$$

Hence the m band prongs have distinct lengths, and these are linearly independent over $\mathbb{Q}$ if and only if $m \leq \varphi(n)/2$.

Proof. The endpoint E_j of the developed prong is computed in Appendix A (§A.3), where it is shown that $(E_j - V_2)e^{-i\psi_j} = \lambda_j$, real and positive. The values $\cos(kA/2)$, $k = 1, 3, \ldots, 2m-1$, are distinct and positive. For odd k, $\cos(k\theta)/\cos\theta$ is a polynomial of degree $(k-1)/2$ in $\cos 2\theta$, so $\lambda_j/(R\cos(A/2))$ is a rational polynomial of degree $m-1-j$ in $c = \cos(\pi/n)$. These m polynomials have degrees $0, \ldots, m-1$, so the lengths are $\mathbb{Q}$-independent if and only if $1, c, \ldots, c^{m-1}$ are, that is, if and only if $m \leq [\mathbb{Q}(c) : \mathbb{Q}] = \varphi(n)/2$ (n odd). □

Corollary 6.3. *Let Δ lie on the band $\mathcal{B}_{q,m}$, with N odd and $m \leq \varphi(n)/2$. If every swapped pair of Δ is one of the m band prongs, then $p = m$ and the perpendicular class has property* ($\perp$).

Proof. Proposition 6.2 and Theorem 5.3. □

The condition $m \leq \varphi(n)/2$ is mild. It reads $\varphi(n) \geq (n-q)/q$, and $\varphi(n)/n = \prod_{\ell \mid n}(1 - 1/\ell) > 1/3$ unless n has at least eight distinct (odd) prime factors; so it holds for every $n < 3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23 = 111\,546\,435$. For $q = 3$ it first fails at that n, that is at $m = 18\,591\,072$.

The hypothesis of Corollary 6.3 is not automatic. A second band can cross the realised part of a first, and then the triangle has swapped pairs outside the run; at some such points two swapped pairs have *equal* length, and they bound free cylinders (§6.4). What we can prove is that on the two lines of the note's families the band prongs are all.

6.3. **The note's families.**

Theorem 6.4. *For every $m \geq 1$, in the perpendicular class of $\Delta_1 = (3, 6m+2, 12m+4)$ and of $\Delta_2 = (3, 6m+1, 12m+5)$ ($N = 18m+9$) every prong meets a side at a right angle except the m band prongs $W_{3,m,j}$ ($t = 2/3$, resp. $1/3$) and their reverses. So $p = m$, the swapped-pair lengths are those of Proposition 6.2, and if $m \leq \varphi(6m+3)/2$ — in particular for $m < 18\,591\,072$ — the class has property* ($\perp$).

The proof is computer-assisted; Appendix B gives it in full. In outline: the $N - 2m = 16m + 9$ slots other than those of the band prongs and their reverses fall, for $m \geq 7$, into finitely many *word families* (28, resp. 24), within each of which the bounce word is affine in m and in the position ν of the slot along its vertex. Each family's crossing conditions (§6.1) are signs of finite sums of cosines in $x = \pi/2N$ and $y = \nu x$, with coefficients and frequencies independent of m: m enters only through x. So all $m \geq 100$ at once are the region $0 < x \leq \pi/3618$ of the (x, y)-plane, and finitely many interval computations (Taylor models of the conditions, which degenerate at $x = 0$, divided by the right power of x) certify them there; for $m < 100$ they are checked slot by slot, with rigorous error bounds. Each family ends with a right-angle collision, an exact identity. Four boundaries between consecutive families per triangle are transcendental, and there one needs that no slot lies *on* the boundary: a Mann-type argument on vanishing sums of roots of unity shows the boundary function never vanishes at an integer point (Proposition B.1). Once the prongs of these $16m + 9$ slots are known to meet a side at a right angle, $p = m$ follows by counting.

Theorem 6.4 closes [SPO, Remark 8.3]. It gives infinite families with p unbounded on which complete periodicity, and property ($\perp$), are proved although $p \neq 0$.

6.4. **Free cylinders.**

Theorem 6.5. *For each row $(n, |c|, J)$ of the table below, every primitive triangle with angles $(\pi/n, B, \pi - \pi/n - B)$, N odd and $Bn/\pi \in J$, has two free cylinders in its perpendicular class, exchanged by ι, whose cores have $|c|$ bounces and direction ψ given in the table. So property ($\perp$) fails on infinitely many triangles.*

n	$\lvert c\rvert$	*direction* $\psi - \pi/2$	J
9	576	$11B - 16A$	[2.02159990, 2.02191910]
15	712	$-5B - 14A$	[4.01917575, 4.01943080]
15	756	$-11B + 8A$	[4.02988894, 4.03003593]
27	1408	$-3B + 18A$	[8.02604251, 8.02619503]

Proof (outline; Appendix C). For each row an explicit cyclic word c closes, identically in B on the line $A = \pi/n$, by a translation in the stated direction, which lies in the perpendicular class at every rational point; a ball-arithmetic certificate shows that the strip of parallel paths with itinerary c has positive width for $Bn/\pi \in J$. So c is the core of a cylinder. The involution ι reverses the leaves and keeps the side labels, so ι of this cylinder has itinerary c reversed, and c is not a cyclic rotation of its reverse; a maximal cylinder contains no marked point, so its leaves share one cyclic itinerary, and the two cylinders are distinct. □

By Theorem 5.3 these triangles have swapped pairs with a signed relation among their lengths. At the two points where we examined the boundary,

$(137, 277, 819)$ ($n = 9$) and $(103, 414, 1028)$ ($n = 15$), the relation is the simplest possible: each free cylinder is bounded by one connection from each of two swapped pairs of exactly equal holonomy. We have not found a minimal component at odd N (§8).

7. Where the arguments stop

Remark 7.1 (two primes). Let N be odd with at least two prime factors, $u \neq w$ types with $\gcd(a_u, N) = \gcd(a_w, N) = 1$ (so P_u, P_w are single points, fixed by ρ), and e a side copy from P_u to P_w. For $x \coloneqq \Phi_N(\rho)\, e = \sum_j c_j \rho^j e$, c_j the coefficients of the cyclotomic polynomial, $\operatorname{hol} x = \Phi_N(\zeta) \operatorname{hol} e = 0$ and $\partial x = \Phi_N(1)(P_w - P_u) = P_w - P_u$, because $\Phi_N(1) = 1$. So every vector of the absolute period lattice — perpendicular-class vectors included — is the holonomy of a relative class with boundary $P_w - P_u$, whatever a_v: no function of holonomy and boundary, equivariant or not, obstructs a two-vertex prong there. (At $N = \ell^k$ the same chain has $\partial x = \ell(P_w - P_u)$, and $\Phi_N(1) = \ell$ is exactly the index Proposition 3.2 detects.) Any proof of the two-vertex half of Conjecture 1.1 at such N must use that the prong is straight.

Remark 7.2 (composite N, loops). At composite N with every $\gcd(a_i, N) = 1$ the proof of Proposition 4.2 goes through except at one step: each type is still one point fixed by ι, and steps (i)–(iii) are topological once $[\gamma] = [\gamma']$ is known, but $H_1(X; \mathbb{Q})$ has an isotypic part for each $d \mid N$, $d > 1$, holonomy sees only $d = N$, and γ need not be homologous to γ'. Nor is there an algebraic substitute: on the line $A = \pi/n$ of a band, the word $W_{q,m,0}$ developed formally at the coprime points $(1, k, n-1-k)$ closes in a perpendicular-class direction, and its class has a non-zero component in a lower isotypic part at each of the 25 scalene such points we checked (and at 22 isosceles ones); only the failure of the word to be realised there prevents a swapped loop. So here too the missing input is realisation.

Remark 7.3 (prime powers). At $N = \ell^k$, Proposition 3.2 gives the two-vertex half of Conjecture 1.1. The loop half survives in part; we sketch the argument. For $d \mid N$, $1 < d < N$, the ζ_d-part of $H_1(X; \mathbb{Q})$ is the new part of $H_1(X/\langle \rho^d \rangle; \mathbb{Q})$, and $X/\langle \rho^d \rangle$ is a cyclic cover of degree d of the sphere branched over the vertices V_i with $d \nmid a_i$. If $\ell^{k-1} \mid a_v$ for some v, each such cover is branched over at most two points, hence a sphere, so $H_1(X; \mathbb{Q})$ is its ζ_N-part. By Riemann–Hurwitz $\chi(X) = 2N - 2(N-1) - (N - \ell^{k-1})$, so $\dim_{\mathbb{Q}} H_1(X; \mathbb{Q}) = N - \ell^{k-1} = \varphi(N)$: the ζ_N-part has dimension 1 over $\mathbb{Q}(\zeta_N)$, and holonomy is injective as in Lemma 4.1. Then the proof of Proposition 4.2 applies to loops based at the two types $u \neq v$, which are single points. It does not give $p = 0$: since $\ell \mid a_v$, Proposition 3.2 does not exclude prongs between the two single-point types, and prongs from a point of type v (several points, not all fixed by ι) are not covered. When no a_v is divisible by ℓ^{k-1}, swapped loops can lie in the lower parts (we found them at $N = 81$).

Remark 7.4 (even N). At even N the perpendicular classes contain side directions (Remark 2.1), X is a half-translation surface, and the conclusions of Theorems 4.3 and 5.3 both fail: $(7, 8, 15)$, $N = 30$, has a non-completely-periodic class with $p = 1$ [DR22], [SPO, §9]. On the translation surface S_Δ the involution ι is still orientation-reversing, and the local step (iii) of Proposition 4.2 applies verbatim.

Remark 7.5 (the side directions). Theorem 4.3 concerns the perpendicular class only. The directions parallel to the sides (the multiples of π/N) need not be completely periodic even at prime N: Boshernitzan's non-closing example, a direction in $(2, 3, 6)$, $N = 11$ [Bos92, §2], is a side direction [SPO, Remark 7.10], and Delecroix and Rüth found minimal components in side directions [DR22]. Exactly, as in §8: for $(2, 3, 6)$ the side directions give three cylinders and two components carrying no periodic orbit (about 30% of the transversal), the perpendicular class seven cylinders; $(1, 3, 9)$, $N = 13$, behaves in the same way (four cylinders and about 54% without periodic orbits, against eight cylinders). The method has no counterpart there. The reflection fixing a side direction ω' also induces an orientation-reversing isometric involution of X preserving $F_{\omega'}$, but its fixed set consists of side copies parallel to ω' — leaves, not transversals — so neither (R) nor the dictionary (D) has an analogue.

8. Computations

Everything in §§3–5 is proved by hand; Proposition 6.1 uses exact symbolic identities checked by computer algebra, and Theorems 6.4, 6.5 are computer-assisted. The computations below are checks, not inputs. Prongs were traced exactly, in the cyclotomic field of the triangle, and cylinder decompositions computed exactly from the interval exchange of the unfolded surface, as in [SPO, §9].

- *Theorem 4.3.* On every scalene primitive triangle with $N = 19, 23, 29, 31$ (4 797 slots, none capped) every prong returns to its own vertex after one right-angle collision; on 5 119 stored triangles with prime $N \leq 99$, $p = 0$. No two-vertex prong occurs at the primes $17 \leq N \leq 47$ or at $N = 9, 25, 27, 49, 81, 121, 125$ (about 386 000 slots, traced to 3 000 bounces; about 4% reach that cap), while the nine two-vertex prongs known at the composite control values $N \leq 75$ are found again (twelve are known in all). Every perpendicular-class loop traced at prime N (119, in nine triangles with $7 \leq N \leq 17$) is homologous to its ι-partner, as Lemma 4.1 predicts; none of the 80 closed swapped loops known at composite $N \leq 99$ is.
- *Conjecture 1.1.* Among the 10 588 scalene triangles with $53 \leq N \leq 99$ odd, 50 have $p \geq 1$, every one with some $\gcd(a_i, N) \geq 3$ (the range $N \leq 51$ is [SPO, §9]). Over the stored censuses ($N \leq 99$ and $N = 105$, 13 180 triangles), $p \geq 1$ occurs on 0 of the 2 288

triangles with composite N and every $\gcd(a_i, N) = 1$, on 52 of 867 with some $a_i \mid N$ (the band lines of §6.1) and on 42 of 4 906 with some $\gcd(a_i, N) \geq 3$, $a_i \nmid N$.

- *Theorem 5.3.* Every primitive triangle with odd $N \leq 61$ (3 162) has property ($\perp$), as do the 50 triangles with $p \geq 1$ above. The hypothesis of Theorem 5.3 holds, by an exact test in $\mathbb{Z}[\zeta_{4N}]$, on all 68 triangles with $p \geq 1$ in the censuses $N \leq 61$ and $53 \leq N \leq 99$ ($p \leq 5$) and on 74 points of the families of §6.1 with $N \leq 297$ ($p \leq 16$). As a check that the test can fail: at even $N \leq 24$, 55 of 87 completely periodic perpendicular directions have free cylinders.
- *Bands.* On every triangle with $p \geq 2$ in the complete censuses (odd $N \leq 99$, $N = 105$, $N = 315$), and at 12 family points with $N = 1155$ (102 triangles in all), the triangle lies in exactly one strip $S_{q,m}$, $p \leq m$, and the p swapped-pair lengths are the p largest values of Proposition 6.2.
- *Theorem 6.4* is consistent with a direct trace for $4 \leq m \leq 30$ ($p = m$, no prong capped), and with [SPO, §9] ($N \leq 303$).
- *Theorem 6.5.* Thirteen exact decompositions at odd $N \leq 5\,475$ find free cylinders, all inside their family's certified interval; at all thirteen the class is completely periodic. At two of them the measured cylinder widths agree with the certificate's strip width to twelve digits.

Code availability. The certificates of Theorems 6.4 and 6.5, the symbolic identities of Proposition 6.1, the check of Proposition 6.2 and, of the computations above, the band census, the direct trace for Theorem 6.4 and the exact points of Theorem 6.5 are deposited at `https://github.com/josh-boorstein/billiards`, directory `triangles/`, whose manifest maps each one to the statement it supports. The version used here is release v1.2, archived at Zenodo as doi:10.5281/zenodo.24251628.

9. Open problems

1. **Conjecture 1.1 at composite** N, the class with every $\gcd(a_i, N) = 1$ first. It would give complete periodicity and property ($\perp$) on about 60% of triangles with odd N (with Theorem 4.3). Both halves stop at realisation (Remarks 7.1, 7.2).
2. **Prime powers.** Is $p = 0$ at every $N = \ell^k$ with all $\gcd(a_i, N) = 1$? Remark 7.3 gives the loops at two types when $\ell^{k-1} \mid a_v$.
3. **Which band points have** $p = m$**?** On the two lines of Theorem 6.4 the band prongs are all. A crossing band adds pairs, so a general statement must describe the crossings.
4. **Conjecture 2.2(b) at odd** N**.** By Theorem 5.3 a counterexample needs a minimal component bounded by swapped pairs with a signed relation among their lengths. Free cylinders of this kind exist (Theorem 6.5); is there a minimal one?

Appendix A. The development of the bands

We prove Propositions 6.1 and 6.2. Notation of §6.1; circumdiameter 1; frame $V_0 = 0$, $V_1 = \sin\Gamma$, $V_2 = \sin B\, e^{iA}$; $a = e^{iA}$, $b = e^{iB}$, $\zeta_q = e^{2\pi i/q} = a^d$. Write $\gamma_0 = (3n - 2m - 3)/2$; then $\pi/2 + B - \gamma_0 A = -\pi/2 + tA = \psi_0$, so $\psi_j - \pi/2 = B - (\gamma_0 + j)A \in (\pi/N)\mathbb{Z}$, and at odd N the direction ψ_j lies in the perpendicular class (the side directions being multiples of π/N). We give the case $j = 0$ in full and then the changes for general j. Every identity below marked *(exact)* is an identity of Laurent polynomials in $e^{i\beta/2}$, $e^{iA/2}$, $e^{itA/2}$, $e^{ik\beta/2}$ (and $e^{ijA/2}$), with the integers k, j free, verified by computer algebra; the code is deposited (§8).

A.1. **Development.** A rising run of length r acts, in the frame of the current image of V_0, as the rotation $z \mapsto a^r z$, a falling run of length f as $z \mapsto a^{-f} z$, and the letter 1 is the reflection $\sigma_1(z) = V_1 + b^{-2}(\bar z - V_1)$ in side 1. Call the fan of copies about the s-th image of V_0 along the word the s-th *star* ($s = 0, \dots, q-1$) and put $\varepsilon_s = r_s$ for $1 \le s \le h$ and $\varepsilon_s = -f_s$ for $h < s \le q-2$. Then $\varepsilon_s A \equiv 2s\beta + 2A \pmod{2\pi}$ for every s. Let G_s be the isometry carrying the base copy to the first copy of star s, so $G_1 = \sigma_0\sigma_1$ and $G_{s+1} = G_s \circ \rho_s \circ \sigma_1$ with $\rho_s(z) = a^{\varepsilon_s} z$. Writing $G_s = O_s + L_s$ (O_s the image of V_0, L_s linear or antilinear), induction gives

$$G_s(z) = \begin{cases} O_s + b^2 a^{\tau_{s-1}} z & (s \text{ odd}), \\ O_s + a^{\tau_{s-1}} \bar z & (s \text{ even}), \end{cases} \qquad O_s = -2i \sin B \sin\Gamma \cdot b \sum_{l<s} (-1)^l a^{\tau_l},$$

with $\tau_{2l} A = -2l\beta$ and $\tau_{2l+1} A = 2(l+1)\beta + 2A$. Let $F(w) \coloneqq \Im\big((w - V_2)e^{-i\psi_0}\big)$, the signed distance from the line of the prong (positive on its left), and $F_s(Re^{i\phi}) \coloneqq F(G_s(Re^{i\phi}))$. Then (exact)

$$\begin{aligned} F_{2k}(Re^{i\phi}) &= F_{2k}(0) + R\cos(2k\beta + (2-t)A - \phi), \\ F_{2k+1}(Re^{i\phi}) &= F_{2k+1}(0) - R\cos(\phi - (2k+1)\beta - (2-t)A), \\ \cos(\beta/2)\, F_s(0) &= \sin B \,\big[\sin(\beta/2 - A)\sin(tA) \\ &\qquad + (-1)^{s+1}\cos(\beta/2 - tA)\cos(s\beta + A)\big] \end{aligned}$$

for $s \ge 1$, and $F_0(Re^{i\phi}) = R\cos(\phi - tA) - \sin B\cos((1-t)A)$. The vertices of star s are the G_s-images of the base fan's: images of V_1 at $\sin\Gamma\, e^{2iuA}$ and of V_2 at $\sin B\, e^{(2u+1)iA}$, $u \in \mathbb{Z}$. The side copies from the centre O_s to these vertices are the *rays* of the star (ray u in the frame of the base fan), and their far ends its *tips*.

A.2. **Closure and the star centres.** The word ends at the image $E = G_{q-1}(\sin B\, e^{-i(4m-1)A})$ of V_2, and $F(E) = 0$ is an identity (exact): the coefficients c_3, c_1, c_{-1} of b^3, b, b^{-1} in $E - V_2$, polynomials in a, satisfy

$$c_3(a) = a^{-2\gamma_0} c_{-1}(a^{-1}), \qquad c_1(a) = a^{-2\gamma_0} c_1(a^{-1})$$

modulo $\sum_{l<q} \zeta_q^l = 0$ and $a^n = -1$: two sums over all q-th roots of unity. So E lies on the line of the prong for every t.

From the formula for $F_s(0)$, the centre O_s of star s lies on the side $\operatorname{sign} F_s(0) = (-1)^{s+1}$ for $s \le h$ and $(-1)^s$ for $s > h$: for $s < h$, $\cos(s\beta+A) \ge \sin(3\beta/2 - A) > \sin(\beta/2 - A)$ and $\cos(\beta/2 - tA) > \sin tA$; for $s > h$, $s\beta + A \in [\pi/2 + \beta/2 + A, \pi - \beta + A]$, so $|\cos(s\beta + A)| \ge \min(\sin(\beta/2 + A), \cos(\beta - A)) > \sin(\beta/2 - A)$; at $s = h$, $\cos(h\beta + A) = \sin(\beta/2 - A)$ and $\sin tA < \cos(\beta/2 - tA)$.

A.3. **The conditions.** The word is realised, with every crossing interior, if and only if every side copy it crosses has its endpoints strictly on opposite sides of the line. In star s a crossed ray needs its tip on the side opposite O_s, and the exit through side 1 needs the next tip on O_s's side. Within one vertex type the crossed tips form an arithmetic progression of angles spanning less than π, and $R\sin(\cdot)$ is concave where positive, so the two extreme tips of each type suffice; an image of V_1 between two crossed images of V_2 is implied, since $\sin\Gamma > \sin B \cos A$. The entry of star $s+1$ is the exit of star s. What remains is the following list of margins, each a positive multiple of the signed distance $\pm F$ of the vertex named, with the sign that makes the condition read *margin* > 0 (exact):

- J1(s) the V_1-vertex of the junction $s \mid s{+}1$ ($s = 0, \dots, q{-}2$): margin $P\,S_s + (-1)^s Q$, with $P = \sin((1-t)A)\cos(\beta/2 - tA)$, $Q = \sin(tA)\cos(\beta/2 + A - tA)\sin(\beta/2 - A)$, $S_s = \sin(s\beta + \beta/2 + A)$;
- J2(s) the V_2-vertex of that junction: margin a positive multiple of $\sin(tA)\cos(\beta/2 + A - tA)\cos((s+1)\beta/2)\sin(s\beta/2 + A)$ (s even), resp. $\dots \sin((s+1)\beta/2)\cos(s\beta/2 + A)$ (s odd);
- R2(s) the far V_2-extreme of a rising star ($1 \le s \le h$): $\cos(\beta/2 + A - tA)\,[\sin((1-t)A)\sin(s\beta + \beta/2) + \sin A \sin(s\beta - \beta/2 + tA) + (-1)^s \sin tA \sin(\beta/2 - A)]$;
- F2(s) the near V_2-extreme of a falling star ($h < s \le q - 2$): $\cos(\beta/2 + A - tA)\,[\sin((1-t)A)\sin(s\beta - \beta/2 + 2A) + \sin A \sin(s\beta + \beta/2 + (2-t)A) + (-1)^{s+1}\sin tA \sin(\beta/2 - A)]$, and in the last star $4\sin((1-t)A)\cos(\beta/2 + A - tA)\sin(\beta/2 - A)\cos(\beta/2 - A)$;
- L1, L2 the far extremes of the last star: $\sin A \sin(3\beta/2 - (2t+1)A)$ and $2\sin A \cos(\beta/2 + A - tA)\sin(\beta - (2+t)A)$.

Signs, $0 < t < 1$. J2: every factor is positive, as $(s+1)\beta/2 < \pi/2$ and $s\beta/2 + A \le \pi/2 - \beta + A$. R2: the first two terms are positive ($s\beta + \beta/2 \le \pi/2$ and $\beta/2 + tA \le s\beta - \beta/2 + tA \le \pi/2 - \beta + tA < \pi/2$ for $1 \le s \le h$), and $\sin A \sin(s\beta - \beta/2 + tA) \ge \sin A \sin(\beta/2 + tA) > \sin tA \sin(\beta/2 - A)$. F2: the first sine has argument in $[\pi/2 + 2A, \pi - 5\beta/2 + 2A] \subset (0, \pi)$; the second has argument in $[\pi/2 + \beta + (2-t)A, \pi - 3\beta/2 + (2-t)A] \subset (\pi/2, \pi)$, where sine decreases, so it is at least $\sin(3\beta/2 - (2-t)A) > \sin(\beta/2 - A)$ (both arguments in $(0, \pi/2]$, the first larger by $\beta - (1-t)A > 0$), which with $\sin A \ge \sin tA$ beats the third term; the last-star margin is a product of positive factors. L1: $3\beta/2 > 3A \ge (2t+1)A$. L2: $\beta \ge 3A > (2+t)A$. J1 for even s: $P, Q, S_s > 0$. J1 for odd s requires $PS_s > Q$; S_s is smallest at $s = q - 2$, where $S_{q-2} = \sin(3\beta/2 - A)$ (the arguments $s\beta + \beta/2 + A$,

$1 \le s \le q-2$, lie in $[3\beta/2 + A, \pi - 3\beta/2 + A]$, where sine is concave, and $\sin(3\beta/2 + A) - \sin(3\beta/2 - A) = 2\cos(3\beta/2)\sin A \ge 0$), and the condition there is (6.1) with $>$. The ratio Q/P increases strictly on $(0,1)$ from 0 to ∞ ($P = \frac{1}{2}[\sin(\beta/2 + A - 2tA) + \sin(A - \beta/2)]$ decreases, Q increases), so (6.1) has one root t_{up} and every J1 holds exactly for $t < t_{\mathrm{up}}$. At $t = 2/3$, with $u = \beta/2$, the left side of (6.1) minus the right is (exact) $\frac{1}{2}\sin(A/3)\,[2\cos(3u - 4A/3)\sin(u - A/3) + \sin(5A/3) + \sin A] > 0$, so $t_{\mathrm{up}} > 2/3$.

Conclusion for $j = 0$. For $t \in (0, t_{\mathrm{up}})$ every condition holds, so the word is realised from V_2 to E with no vertex between; E is the image of V_2 on the base ray $-(4m-1)$. The direction $\psi_0 = -\pi/2 + tA$ lies in the open angle of directions $(A - \pi, -B)$ at V_2 ($\psi_0 - (A - \pi) = \pi/2 - (1-t)A > 0$, and $\psi_0 < -B$ because $(2t-1)A < \beta/2$), so (V_2, ψ_0) is a slot and the developed segment is its prong. $W_{q,m,0}$ is not a palindrome (first run 1, last $4m-1$), so the prong meets no side at a right angle (a prong that does retraces, and its word is a palindrome), and its end is a different slot at V_2.

General j. The runs shift every ε_s by $2j$, and the first run is $z \mapsto a^{-2j}\bar{z}$; so $G_s^{(j)}$ is a^{-2j} times the map above with the odd-index τ raised by $2j$, and $e^{-i\psi_j} = a^j e^{-i\psi_0}$. With $x \coloneqq (1+j)A$, $\cos(\beta/2)F_s^{(j)}(0) = \sin B\,[\sin(\beta/2 - x)\sin tA + (-1)^{s+1}\cos(\beta/2 - tA)\cos(s\beta + x)]$, the star-$s$ formulas acquire $+jA$ in the phase, and the closure and every condition are again exact identities. The margins are those above, with these changes:

J1(s) $S_s \to \sin(s\beta + \beta/2 + x)$ and $Q \to \sin tA\cos(\beta/2 + A - tA)\sin(\beta/2 - x)$;
J2(s) $\sin(s\beta/2 + A) \to \sin(s\beta/2 + x)$;
R2(s), F2(s) the arguments shifted by jA, and the last-star margin $\to \sin((1-t)A)\cos(\beta/2 + A - tA)\sin(\beta - (2+j)A)$;
L1 $\to \sin A\sin(3\beta/2 - (2t+1+j)A)$;
L2 $\to \sin A\cos(\beta/2 + A - tA)\sin(\beta - (2+t+j)A)$;
S1–S3 for $j \ge 1$ only, three conditions in the first star: $\sin A\sin(\beta/2 + (1 + j - 2t)A)$, $\sin A\cos(\beta/2 + A - tA)\sin((j-t)A)$ and $\cos(\beta/2 + A - tA)\sin((1-t)A)\sin jA$.

All are positive for $t \in (0,1)$: the arguments above go through with A replaced by $x \le mA < \beta/2$ where they used $A < \beta/2$, the side of every star centre is unchanged, and $(m+2)A \le \beta$ handles L1, L2 and the last ray. For F2 the second argument now lies in $(\pi/2, \pi - 3\beta/2 + (2 - t + j)A]$, so its sine is at least $\sin(3\beta/2 - (2 - t + j)A) > \sin(\beta/2 - x)$, the difference of the arguments being $\beta - (1-t)A > 0$; and the first argument, $s\beta - \beta/2 + (2+j)A$, stays in $(0, \pi)$ since $(2+j)A < 5\beta/2$. The exception is again J1 at $s = q-2$: $P\sin(3\beta/2 - x) > \sin tA\cos(\beta/2 + A - tA)\sin(\beta/2 - x)$; since $\sin(3u - x)/\sin(u - x)$ increases in $x \in [0, u)$, it follows from the case $j = 0$. The direction: $\psi_j - (A - \pi) = \pi/2 - (1 + j - t)A > 0$ and $\psi_j < \psi_0 < -B$. A reversed word $W_{q,m,j}$ starts with a run $4m - 1 - 2j \ge 2m + 1 > 2j' + 1$, so the $2m$ words and reverses are distinct, and the m pairs are distinct. This proves Proposition 6.1.

The endpoint. For the length law,

$$E_j = a^{-2j}\big[O_{q-1}^{(j)} + a^{\tau_{q-2}+2j}\sin B\, e^{i(2\beta-3A-2jA)}\big], \qquad \tau_{q-2}A = \pi-\beta+2A.$$

The last term is $-\sin B\, e^{i(\beta-A)}$, free of j. Summing two geometric series in $e^{2i\beta}$ (using $e^{2ih\beta} = -e^{-i\beta}$), $O_{q-1}^{(j)} = -2i\sin B\sin\Gamma\cdot b\cdot(-ie^{i\beta}e^{ix})\cos(\beta/2 - x)/\sin(\beta/2)$ with $x = (1+j)A$. Projecting with $e^{-i\psi_j} = ia^{j-t}$ and using $b\,e^{i\beta}e^{ix}a^{-j} = ie^{i(\beta/2+tA)}$,

$$(E_j - V_2)e^{-i\psi_j} = 2\sin B\cos(\beta/2-x)\Big[\frac{\sin\Gamma\, e^{i\beta/2}}{\sin(\beta/2)} - ie^{i(\beta/2-tA)}\Big].$$

Since $\sin\Gamma = \cos(\beta/2 - tA)$, the imaginary part vanishes and the real part is $\cos(\beta/2-tA)\cot(\beta/2)+\sin(\beta/2-tA) = \cos(tA)/\sin(\beta/2)$. With $\beta/2 - x = (2m-1-2j)A/2$ this is λ_j, proving Proposition 6.2.

Appendix B. The certificate for Theorem 6.4

Fix one of the two triangles and m. The $N = 18m+9$ slots lie at the three vertices; at each vertex we split them by residue modulo 3 into *strands* (slots three apart differ in direction by $3\pi/N$, the angle A at V_0) and index a slot by its position ν in its strand. Following a slot's prong produces a bounce word, written as runs in the letters $0, 2$ separated by letters 1, as in §6.1.

Word families. A *word family* is a set of slots in one strand, $\mathrm{lo}(m) \le \nu \le \mathrm{hi}(m)$, together with a word whose run lengths are affine functions of (m,ν) and a terminal event (a right-angle collision on a given letter, or arrival at a given vertex). For a family, a parity class of (m,ν) and a slot, every vertex image of the developed word is a sum of roots of unity $\zeta_{4N}^{am+b\nu+c}$ with fixed rational a, b, c, so each crossing condition (an extreme crossed ray of each type on the side opposite its star centre; the exit ray on the centre's side; the span of each run below π) is the sign of a finite cosine sum

$$f(m,\nu) = \sum_k c_k\cos\big((a_k m + b_k\nu + c_k')\pi/2N\big),$$

and the terminal event is the vanishing of another. A slot is *certified* when all its crossing conditions hold strictly and the terminal sum vanishes; then the developed line realises the word and the prong ends as stated. Since $N = 18m+9$, $m\pi/2N = \pi/36 - x/2$ with $x := \pi/2N$, and with $y := tx$ each f is a trigonometric polynomial in (x,y) with fixed frequencies. The perpendicular endings are identities: each terminal sum vanishes identically in (x,y), verified exactly in $\mathbb{Q}(\zeta_L)$ (for a family consisting of a single ν per m, after substituting $\nu = \mathrm{lo}(m)$).

There are 28 (for Δ_1) and 24 (for Δ_2) families ending at a right angle, plus the band prongs. Their ranges $[\mathrm{lo}(m), \mathrm{hi}(m)]$ are bounded by affine functions of m (floors of affine functions), except at four boundaries per triangle, which are *transcendental*: the boundary between families i and $i+1$ is the sign change of one cosine sum f_i.

Proposition B.1. *Each of the eight transcendental boundary functions has the form* $\sum_{k=1}^{4} A_k \cos((6\nu + o_k)\pi/2N)$ *with* $A_k \in \{2, 2\sqrt{3}\}$*, at most one* $A_k = 2\sqrt{3}$*, and* o_k *affine in* m*, and does not vanish at any integer point* (m, ν)*,* $m \geq 1$*.*

Proof. The four-term form is an exact identity. Write $f = \frac{1}{2}\sum_k (Z_k + \bar{Z}_k)$ with $Z_k = A_k \xi^{6\nu+o_k}$, $\xi = e^{i\pi/2N}$; a term with $A_k = 2$ is 2 times one root of unity, a term with $A_k = 2\sqrt{3}$ is 2 times two roots of unity of ratio $e^{i\pi/3}$. So f is $\frac{1}{2} \cdot 2 = 1$ times a sum of at most ten roots of unity with coefficient 1. By Mann's theorem [Man65] a minimal vanishing sub-sum of at most ten roots, after rotation, consists of M-th roots of unity with M squarefree and divisible only by primes at most 7, so of 210-th roots of unity. The ratio of two roots from different terms on the same side (both from the Z_k, or both from the $\bar{Z}_k$) is ν-free, of the form $\xi^{\alpha m+\beta'}$, and outside an explicit finite set $\mathcal{E}$ of values of m none lies in μ_{210}. So off $\mathcal{E}$ a minimal vanishing sub-sum contains roots from at most one term on each side, and it cannot lie on one side only. The single $2\sqrt{3}$-term cannot be split between sub-sums, nor matched with a term with $A_k = 2$: either would need a same-side ratio $e^{\pm i\pi/3}$ between distinct terms, excluded off $\mathcal{E}$; so it can only be matched with its own conjugate. Hence $f = 0$ forces a perfect matching $Z_i = -\bar{Z}_j$ between terms of equal weight, each a congruence $12\nu + o_i + o_j \equiv 2N \pmod{4N}$; the differences of two such congruences are ν-free and leave finitely many m, each checked over ν modulo $4N$: no matching is solvable. For all eight functions $\mathcal{E} = \{2, 3, 17\}$, and there interval arithmetic over a full period in ν gives $|f| \geq 7.6 \cdot 10^{-4}$. □

The case $m \geq 100$**.** The slots of each family lie in the region of the (x, y)-plane with $0 < x \leq \pi/3618$ between the lines $y = \mathrm{lo}(m)x$ and $y = \mathrm{hi}(m)x$, an affine bound $\nu = am+b$ becoming the line $y = a\pi/36+(b-a/2)x$. Every crossing condition of every family (2 084 for Δ_1, 1 936 for Δ_2) is certified to have the required strict sign on its whole region. The certificate encloses f/x^k by Taylor models, the vanishing order $k \in \{0, 1\}$ at $x = 0$ being decided exactly; where the leading coefficient vanishes at a rational corner it changes chart, to $y = y_c + sx$ and then to $x = \rho u$ with the factor x^k removed, splitting at a double zero. Conditions that apply only when a run has length at least 2 are certified on the corresponding half-plane, which is lattice-tight because a run's parity is fixed on a parity class. A family with a single slot per m is certified on its line, in the parity class fixed by that of m. Every run length is certified ≥ 1 (or identically 0) on its family's range, by its affine formula. At each transcendental boundary $i \mid i+1$, with f_i vanishing on a curve through $(0, y_c)$, every condition of families i and $i+1$ that is identically $\pm f_i$ is certified outside the band $|y - y_c| \leq 0.002$, and every other condition across it; each such condition requires the sign of f_i that f_i itself has in family i, and the opposite sign in family $i+1$. So at a lattice point in the band, where $f_i \neq 0$ by Proposition B.1, the sign of f_i selects a family all of whose conditions hold.

Tiling. For every $m \geq 7$, exactly: in each strand the first family starts at $\nu = 0$, each family's range ends one below the next one's start, the last ends at the strand's last slot, and every range is non-empty. The band prongs are exactly m slots, as Proposition 6.1 requires.

The case $7 \leq m \leq 100$. Every one of the 81 310 slots per triangle outside the band prongs is assigned to its family by walking its strand (transcendental boundaries by the sign of f_i), and each of its conditions (787 290 for Δ_1, 737 279 for Δ_2) evaluated in double precision with the argument reduced exactly modulo $8N$ and an a priori error bound $\sum_k |c_k| \cdot 10^{-13}$; the check would fall back to interval arithmetic below this bound, which never happens. All hold; run lengths and spans are checked at every slot.

The case $m \leq 6$. Each prong's word is certified at the point, with interval signs and exact endings in $\mathbb{Q}(\zeta_{4N})$.

Counting. So all $16m + 9$ slots outside the band prongs end at a right angle, and at most $2m$ slots end at a vertex. A prong ending at a vertex meets no side at a right angle, so it is not its own reverse and its two end slots are distinct. Proposition 6.1 gives m swapped pairs ($t = 2/3, 1/3 < t_{\mathrm{up}}$), distinct by Proposition 6.2: $2m$ distinct slots. So they are all, and $p = m$. This proves Theorem 6.4.

Controls. The certificate can fail: each of the 19 families whose affine upper bound is not a strand end loses a certificate when extended one slot further, and at $m = 3, 5$ every perpendicular word with one run changed by ± 1 (1 159 words) fails.

Appendix C. The certificate for Theorem 6.5

For each row, c is an explicit cyclic word of length $|c|$ in the letters $0, 1, 2$ (deposited), with no cyclically adjacent repeat. On the line $A = \pi/n$, writing $y = e^{iB}$:

(i) $|c|$ is even and the linear part of the composite reflection along c is the identity, identically in y (exact, on direction monomials);

(ii) its translation part is $U\,\zeta_{2n}^{e} y^{s}$ with U real on the line (exact in $\mathbb{Q}(\zeta_{2n})[y^{\pm 1}]$), so c closes by a translation in direction $\psi = \pi/2 + sB + eA$, which lies in the perpendicular class at every rational point of the line; $|U| > 0$ on J;

(iii) the width $W(B)$ of the strip of lines in direction ψ with itinerary c — the minimum, over the crossed side copies, of the transverse extent, each a trigonometric polynomial in B with exactly computed coefficients — is positive on J, by a Lipschitz bound on adaptive cells ($\min W(\text{ends}) - \mathrm{Lip} \cdot h/2 > 0$) evaluated in ball arithmetic [Joh17] at 200 bits with a rigorous Lipschitz constant; J is a float estimate rounded inward;

(iv) c is not a cyclic rotation of its reverse.

By (i)–(iii) the lines with itinerary c fill an open strip of closed leaves avoiding every marked point, so a cylinder Z of the perpendicular class. ι reverses the

leaves and preserves side labels, so ιZ has itinerary c reversed. A maximal cylinder contains no marked point (every vertex image is marked), so its leaves share one cyclic itinerary, and by (iv) $Z \neq \iota Z$: the two maximal cylinders are free. The certificate can fail: $W < 0$ just outside each J (by $2 \cdot 10^{-5}$ to $1.6 \cdot 10^{-4}$), and the true zeros of W lie within $6 \cdot 10^{-7}$ of J.

Use of AI

This paper — its text, its proofs and the code it deposits (Section 8, *Code availability*) — was drafted by an AI system (Claude, Anthropic) working under the author's direction, and checked by independent AI review passes and by the exact computations described in Section 8. The author has read parts of it in detail and takes responsibility for its content.

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