hexagon:2611.00725v1

# Semiclassical Asymptotics of Wigner $3j$ and $6j$ Symbols at Growing Rank: Phase Horizons and Turning Connections

Antony Lewis

Department of Physics & Astronomy, University of Sussex, Brighton BN1 9QH, UK

**Abstract**

Wigner $3j$ and $6j$ symbols encode angular-momentum coupling and recoupling. Their large-spin asymptotics connect to tetrahedral geometry, three-dimensional quantum gravity and quantum topology. We study the intermediate regime in which the rank of a two-large/one-small $3j$, one-small $6j$, or one-small $q$-$6j$ symbol grows with the large scale $N$. In Weyl-midpoint coordinates $j+\frac{1}{2}$, their exact rank recurrences belong to a common class of even centered Jacobi chains. For this class we identify a generic phase-coherence horizon at rank $\asymp N^{2/3}$, where the leading quadratic phase-defect coefficient $\chi$ is nonzero. On compact turning-free corridors, a Hamilton–Jacobi effective-angle pullback approximates the physical solution to $O(N^{-2})$ in natural gauge, uniformly from fixed rank to fixed positive rank ratio; a real discrete-flux amplitude improves the bound to $O(N^{-3})$ without a new phase correction. Bulk zeros are located to $O(N^{-4})$, with an $O(N^{-3})$ Bohr–Sommerfeld law. At simple rank turns with lattice-quantized action we obtain a physically normalized Airy connection; under a forbidden-side bound, its turning-layer bracket is $\mathrm{Ai}(t)+N^{-4/3}\varpi\,\mathrm{Ai}'(t)+O(N^{-2})$. Exact hyperbolicity proves that bound for all three symbol families and extends the uniform Airy law through a fixed forbidden-side neighbourhood except for an $O(N^{-1}\log N)$ collar at the outer end of that neighbourhood. For $6j$ and $q$-$6j$ the rank momentum is a Euclidean or hyperbolic tetrahedral dihedral angle, and the $q$-$6j$ turning action obeys an exact Schläfli-type quantization identity.

## Contents

1 Introduction 2

2 Exact Weyl-midpoint recurrences 5

3 Even centered Wigner–Jacobi chains 6

4 Hahn/$3j$: verification and explicit data 26

5 Centered Racah/$6j$: verification and explicit data 31

6 The one-small quantum $6j$ chain 38

7 Comparison with exact symbols 41

8 Discussion 41

A General Weyl and oscillatory-transfer details 43

B Second-order closure for even centered chains 50

C Phase exactness of the second-order pullback 58

D Analytic continuation of bulk rank estimates and zero stability 61

E Rank-direction simple-turn Airy connection 63

F Hahn/$3j$ chain data 76

G Hahn Airy connection as a rank-direction turn 78

H Centered Racah chain data 80

I One-small quantum $6j$ chain: proofs 84

J Verification and a computable certificate 89

# 1 Introduction

Wigner $3j$ and $6j$ symbols describe how angular momenta couple and recouple. They appear throughout atomic, molecular and nuclear physics [8], in the harmonic analysis of fields on the sphere such as the cosmic microwave background [3], and, through the $6j$ symbol and its $U_q(\mathfrak{sl}_2)$ deformation, in spin-network models of three-dimensional quantum gravity [9, 40, 31], and in quantum topology, where the growth of quantum $6j$ symbols at roots of unity is governed by volumes of hyperbolic tetrahedra and feeds the volume conjecture for Turaev–Viro invariants [32, 33]. For large quantum numbers their behaviour is geometric: when all six spins of a $6j$ symbol are large it oscillates with the Regge action of a tetrahedron whose edge lengths are $j + \frac{1}{2}$ [9, 40], and when one angular momentum is small it reduces to a rotation matrix element. This paper is about the regime between these two limits, in which the small angular momentum grows with the others, from fixed values up to a fixed fraction of them.

In the Edmonds regime, where one angular momentum (the rank, $L$ for $3j$ and $f$ for $6j$) is small compared with the others, Wigner $3j$ and $6j$ symbols reduce to Wigner $d$-functions [27]. McFarlane improved these limits by averaging the large quantum numbers paired by the coupling, choosing the average by matching rank recurrences [4, 5], and Morehead derived the averaged $3j$ form from the Weyl correspondence with an $O(N^{-2})$ error [6]. Our earlier fixed-rank paper [2] made this exact at fixed rank: in midpoint variables built from the $SU(2)$ Weyl lengths $j + \frac{1}{2}$, where the Casimir $j(j+1) = (j+\frac{1}{2})^2 - \frac{1}{4}$ is invariant under the reflection $j + \frac{1}{2} \mapsto -(j + \frac{1}{2})$, the symbols admit all-order expansions in even inverse powers $N^{-2}$ away from caustics. Once the rank grows with $N$, however, the first surviving $O(N^{-2})$ local correction grows quadratically in the rank and accumulates an order-one phase, so every fixed-rank truncation eventually loses phase coherence. Describing what replaces it, and how far the replacement reaches, is the aim of this paper.

Two structural facts govern this transition. First, write $q$ for the rescaled rank, $p$ for its conjugate momentum and $h = N^{-1}$ for the lattice step. A centered nearest-neighbour recurrence is then an exact lattice Weyl quantization,

$$\mathrm{Op}_h^W(a(q)e^{ip})u(q) = a(q + h/2)u(q + h), \tag{1}$$

so centering eliminates the subprincipal terms linear in $h$ that would otherwise come from operator ordering. Second, for the Hahn, centered Racah, and centered $q$-Racah recurrences, reflected Casimir factors are exchanged by $h \mapsto -h$, which forces an even Weyl symbol. A coherent interaction-picture increment of size $k^s/N^r$ therefore accumulates a phase of size $K^{s+1}/N^r$ and breaks down at the coherence horizon $K \asymp N^{r/(s+1)}$; for $s = r = 2$ this gives the generic bulk $N^{2/3}$ law (where the coefficient $\chi(x)$ of the defect is nonzero). Beyond that horizon the natural object is a discrete Hamilton–Jacobi eikonal rather than a fixed-rank Taylor polynomial.

We prove the growing-rank theory once, for a class of recurrences that contains all three families. An *even centered Wigner–Jacobi chain* (Definition 3.2) is a three-term rank recurrence in Weyl-midpoint gauge whose coefficients are $h^2$-analytic deformations of the Wigner rank recurrence with integer or half-integer offsets $(a, b)$, together with an $h^2$-analytic starting value (seed). For every chain in this class we prove the following.

(i) *Automatic contact.* At every fixed rank the second-order construction satisfies the exact recurrence up to an $O_k(N^{-4})$ residual: the second-order fixed-rank contact identity holds identically across the class (Proposition 3.17), with no identity specific to the Hahn, Racah or $q$-$6j$ families needed.

(ii) *The $N^{2/3}$ coherence horizon.* With $\kappa = L + \frac{1}{2}$ and $x = \cos\theta$, the $N^{-2}$ Taylor series converges for $\kappa^3/N^2$ small (Theorem 3.7), and at $\kappa \asymp N^{2/3}$ the Wigner phase acquires the order-one shift

$-\chi(x)\kappa^3/(3N^2\sin\theta)$ (Theorem 3.9). This is the coherence horizon where $\chi(x) \neq 0$. Where $\chi$ vanishes (for Hahn at $x = 0$, and for Racah and $q$-6$j$ along a codimension-one strict-triangle locus when the two large pairs are unequal) the shift disappears.

(iii) *Phase-exact all-rank pullback.* The Darboux-renormalized second-order pullback $W_q^{[2]}\, d_{ab}^L(\Theta_q^{[2]})$, a Wigner function evaluated at a corrected Hamilton–Jacobi angle, approximates the physical solution to $O(N^{-2})$ in natural gauge, uniformly from fixed rank through fixed positive ratio $q = \kappa/N$ (Theorem 3.20). Its phase is correct to $O(N^{-3})$: up to that order the remainder is a real discrete-flux amplitude factor $1 + N^{-2}\alpha(q, x)$, with $\alpha = \delta_{\mathrm{fl}} - \mathsf{J}_2$ explicit and $\delta_{\mathrm{fl}} = 0$ for Hahn and Racah (Theorem 3.24).

(iv) *Bulk zeros.* Interior polynomial zeros lie within $O(N^{-4})$ of the pullback zeros and satisfy an $O(N^{-3})$ Bohr–Sommerfeld quantization law (Theorem 3.26).

(v) *Second-order Airy connection.* At a simple rank turn with lattice-quantized turning action, the physical solution is a normalized Airy function. When it grows at most polynomially in $N$ beyond the turn (the forbidden-side hypothesis (R)), its turning-layer bracket is $\mathrm{Ai}(t) + N^{-4/3}\varpi_\tau\,\mathrm{Ai}'(t) + O(N^{-2})$, with $\varpi_\tau$ given in closed jet and Olver forms. On the part $I'$ of a fixed turning neighbourhood $I$ consisting of the oscillatory side and an initial stretch of the forbidden side, the approximation has Olver coefficient $B_\tau(q)$ and an explicit $O(N^{-2})$ flux-matched amplitude, leaving an $O(N^{-3}\mathsf{M}/\mathsf{E})$ error (Theorem 3.27, Proposition 3.28).

(vi) *Forbidden-side decay.* For the Hahn, centered Racah, and one-small quantum 6$j$ chains, exact hyperbolicity beyond the turn proves recessivity without orthogonality inputs, and extends the uniform $O(N^{-3}\mathsf{M}/\mathsf{E})$ law to the end of $I$ except for a collar of width $O(N^{-1}\log N)$ (Propositions 3.31 and 3.34).

Sections 4 and 5 verify the class axioms for the fixed-offset Hahn/3$j$ and centered Racah/6$j$ chains (integer and half-integer offsets) and supply closed actions, $\mathsf{J}_2(0, x)$, closed Hahn and Racah $\varpi_\tau$, and exact turning-action integers; for Hahn the rank turn at a fixed node coincides with the spatial band edge. Section 5.3 identifies the Racah action with the Ponzano–Regge action of the principal Euclidean Weyl tetrahedron and with the bispectral generator of the dual $c$-lattice problem. Section 6 brings the one-small $U_q(\mathfrak{sl}_2)$ quantum 6$j$ symbol ($q_{\mathrm{def}} = e^{1/N}$) into the same framework via the $t$-dependent off-diagonal factor of (E1): its Weyl lengths form a hyperbolic triangle, its rank momentum is a dihedral angle of a hyperbolic tetrahedron, its turning action is quantized exactly by a Schläfli-type identity, and the full second-order, phase-exact (with flux defect $\delta_{\mathrm{fl}} = -\frac{1}{12}$ in the normalization of the quantum 6$j$ symbol used here), zero, and Airy theorems carry over.

Global fixed-ratio Hahn asymptotics are due to Lin and Wong [29], and varying-parameter Racah and difference-equation turning-point asymptotics are developed in [43, 47, 42, 44]. What is new here is not the existence of WKB or Airy approximations, but the following: (i) the exact physical Weyl-midpoint gauge and the coherent $N^{2/3}$ phase horizon; (ii) quantitative matching from fixed rank to fixed ratio via automatic second-order contact, yielding an all-rank $O(N^{-2})$ pullback that is phase exact to $O(N^{-3})$; and (iii) physically normalized turning connections with closed second-order Olver coefficients $\varpi_\tau$, $O(N^{-2})$ Airy amplitudes, and intrinsic forbidden-side decay. This recurrence-based analysis complements geometric and integral-representation treatments of Wigner symbols [9, 40, 15, 17, 16, 18, 10] and discrete orthogonal-polynomial methods [24, 11, 20].

For the $U_q(\mathfrak{sl}_2)$ 6$j$ symbol, all-large non-Euclidean tetrahedral asymptotics are due to Taylor and Woodward [31] (cf. [34] at roots of unity), and symmetric Jacobi formulations of the rank recurrence appear in [35, 39]. Section 6 treats the complementary one-small growing-rank regime without using [31]: we prove the midpoint recurrence from the Kirillov–Reshetikhin sum by a $q$-Zeilberger

certificate, evaluate the minimal-rank seed in closed form, establish exact Schläfli turning-action quantization, and derive the flux defect $-\frac{1}{12}$ of the quantum $6j$ normalization and the second-order Airy data.

The fixed-rank paper [2] treats fixed rank uniformly across the entire spatial lattice, including hard edges and corners, but its fixed-rank $N^{-2r}$ coefficients grow with the rank and break down as the rank grows. The present paper treats growing rank across the turning-free, critical, and simple-turning regimes; the two uniformities act in complementary variables. Only the following inputs from Ref. [2] are needed for the main results, all recalled explicitly below: the two exact recurrences and minimal-rank product structure in Section 2, the fixed-rank $N^{-2}$ constant fixing $\delta_{\mathrm{fl}} = 0$ for $3j$ and $6j$ (Corollaries 4.4 and 5.3), and, in a single geometric remark (Remark 5.9), the $c$-lattice Racah operator; the fourth-order comparison of Proposition F.3 also uses its $N^{-4}$ formula. Sections 3 and 6 are self-contained; Ref. [2] enters them only through the Hahn and Racah value $\delta_{\mathrm{fl}} = 0$ (Remark 3.35).

Throughout, asymptotic families run along admissible integer/half-integer sequences satisfying the triangle and parity constraints, and analyticity in $h^2$ refers to the branch of the exact finite-product or recurrence expressions through the physical point at fixed continuum parameters.

| Regime | Scale | Result |
|---|---|---|
| Mesoscopic / horizon | $L, f \asymp N^{2/3}$ | convergent $N^{-2}$ series below; first $O(1)$ phase shift at $N^{2/3}$ where $\chi(x) \neq 0$ |
| Turning-free bulk | $1 \le L, f \lesssim N$ ($q, \rho$ bounded) | all-rank pullback, $O(N^{-2})$ natural gauge, phase exact to $O(N^{-3})$; zeros to $O(N^{-4})$ |
| Simple turning window | $\zeta_\tau = O(N^{-2/3})$; $q \in I$ up to a log collar (general chain: $q \in I'$) | Airy bracket $\mathrm{Ai} + N^{-4/3}\varpi_\tau \mathrm{Ai}' + O(N^{-2})$; uniform $O(N^{-3}\mathsf{M}/\mathsf{E})$ law on $I$ up to the collar (general chain: on $I'$) |

Table 1: Distinct asymptotic regimes across Hahn/$3j$, centered Racah/$6j$, and quantum $6j$ chains. The $N^{2/3}$ rank coherence horizon (generic: where $\chi \neq 0$) is distinct from the $N^{-2/3}$ turning-point layer; $I' \subset I$ are the turning neighbourhoods of Theorem 3.27(ii) and Proposition 3.34.

### 1.1 Scope and conventions

**Error and claim conventions.** All-rank bulk theorems bound the *natural-gauge difference* $\sqrt{2k+1}\,|Y_k - U_k^{[2]}| \le CN^{-2}$, improving to $CN^{-3}$ with the real amplitude factor $1 + N^{-2}\alpha$ of Theorem 3.24 ($O(N^{-5/2})$ and $O(N^{-7/2})$ unscaled at fixed positive ratio); no relative bound is claimed at zeros of $d_{ab}^L$. Bulk zero estimates are coordinate bounds for simple zeros of the off-lattice polynomial continuation, and Airy bracket errors are additive inside the bracket (or weighted by $\mathsf{M}/\mathsf{E}$ in the uniform law). Open problems are collected in Remark 3.35 and Section 8.

**Scope.**

- The Racah and $q$-$6j$ turning theorems exclude $\tau = 0$ (limiting triangles right-angled at the vertex opposite $\alpha$ or $\beta$, resp. $\mathsf{X}$ or $\mathsf{Y}$, where the rank turn meets the hard edge) and the degenerate corner $A - B = O(1)$, $C = O(B^{1/2})$ (Remark 5.5).
- The proportional-index Hahn regime (where minimal rank and caustic move at order $N$), operator-level Egorov/Fourier-integral pullbacks, Christoffel–Darboux kernels, and all-order Ponzano–Regge remainders are not treated.

**Organization.** Section 2 recalls the exact $3j$ and $6j$ recurrences from Ref. [2]. Section 3 defines even centered Wigner–Jacobi chains and proves the general theorems. Sections 4 and 5 specialize to Hahn/$3j$ and centered Racah/$6j$, closing with the tetrahedral and bispectral geometry (Section 5.3). Section 6 treats the one-small quantum $6j$ chain, Section 7 compares with exact symbols, and Section 8 summarizes open problems and applications. Appendices A–E complete the general proofs, Appendices F–H and I contain the chain-specific proofs, and Appendix J records an a posteriori Jacobi bound.

## 2 Exact Weyl-midpoint recurrences

We recall from Ref. [2], without proof, the exact Weyl-midpoint recurrences and minimal-rank structure of the $3j$ and centered one-small $6j$ families; centering places every angular momentum $j$ at $j+\frac{1}{2}$ so that $h=N^{-1}$ enters the recurrence coefficients exclusively through $h^2$.

### 2.1 The $3j$ family

Let

$$j_\mp := \frac{N-1\mp\sigma}{2}, \qquad m_- := \mu - \frac{M}{2}, \qquad m_+ := -\mu - \frac{M}{2}, \qquad x := \frac{2\mu}{N}, \qquad h := N^{-1}, \quad (2)$$

so $(j_+ + \frac{1}{2}) + (j_- + \frac{1}{2}) = N$ and $(j_+ + \frac{1}{2}) - (j_- + \frac{1}{2}) = \sigma$, and define the phase-adjusted normalized coefficient [2, Eq. (17)]

$$\Phi_L^{M,\sigma}(x;h) := (-1)^{2L+j_++m_+}\sqrt{N}\begin{pmatrix} L & j_- & j_+ \\ M & m_- & m_+ \end{pmatrix}. \quad (3)$$

We use standard Condon–Shortley/Wigner conventions [25]. The rank $L$ and fixed offsets $M,\sigma$ are all integers or all half-integers (in the latter case $N\in\mathbb{Z}+\frac{1}{2}$); $2L$, $L\pm M$, $L\pm\sigma$, and $j_++m_+$ are always integers. The sign $(-1)^{j_++m_+}$ removes $(-1)^{2j_+}$ along fixed-$x$ sequences, while $(-1)^{2L}$ ensures the fixed-$L$ limit $+d^L_{M\sigma}(\arccos x)$ for half-integer $L$ as well. With $R_k := \sqrt{(k^2-M^2)(k^2-\sigma^2)}$, the exact rank recurrence is [2, Eq. (18)]

$$(2L+1)\big[L(L+1)x - M\sigma\big]\Phi_L = (L+1)R_L\sqrt{1-L^2h^2}\,\Phi_{L-1} + LR_{L+1}\sqrt{1-(L+1)^2h^2}\,\Phi_{L+1}, \quad (4)$$

and the magnetic recurrence gives the exact difference equation [2, Eq. (30)]

$$a_h(x+h)\Phi(x+2h)+a_h(x-h)\Phi(x-2h)-2(1-x^2)\Phi(x)+2h^2\big[2L(L+1)-M^2-\sigma^2+1\big]\Phi(x)=0, \quad (5)$$

with $a_h(y)=\sqrt{[(1-y)^2-h^2(\sigma-M)^2][(1+y)^2-h^2(\sigma+M)^2]}$. Every appearance of $N$ is through $h^2$. At minimal rank $L_{\min}=\max(|M|,|\sigma|)$, $\Phi_{L_{\min}}$ is, up to sign and constant, the square root of an explicit finite ratio of centered factors $u_\pm + jh$ ($u_\pm = \frac{1\pm x}{2}$) and $1-j^2h^2$ ($j\in\frac{1}{2}\mathbb{Z}$) [2, Eq. (27)], and at fixed $L$ and interior $x$, $\Phi_L^{M,\sigma}(x;h) = d^L_{M\sigma}(\arccos x) + f_2(x)h^2 + f_4(x)h^4 + \cdots$ with closed $f_2, f_4$ [2, Theorems 6.1 and 6.3].

### 2.2 The centered one-small $6j$ family

Consider

$$W_f(a,b,c;m,n) = \begin{Bmatrix} a & b & c \\ b+m & a+n & f \end{Bmatrix}, \quad (6)$$

with fixed admissible $m, n \in \frac{1}{2}\mathbb{Z}$, $m-n \in \mathbb{Z}$. The Weyl-midpoint lengths ($A$ and $B$ the averages of the Weyl lengths $a+\frac{1}{2}, a+n+\frac{1}{2}$ and $b+\frac{1}{2}, b+m+\frac{1}{2}$, and $C = c+\frac{1}{2}$) and their scaling are

$$A = a + \frac{n+1}{2} = N\alpha, \qquad B = b + \frac{m+1}{2} = N\beta, \qquad C = c + \frac{1}{2} = N\gamma, \tag{7}$$

with a strict limiting Euclidean triangle $(\alpha, \beta, \gamma)$. Set $x = (A^2 + B^2 - C^2)/(2AB) = \cos\theta$ and

$$\Psi_f(N) := (-1)^{\xi} 2\sqrt{AB}\, W_f, \qquad \xi := a+b+c+f+m \in \mathbb{Z}, \tag{8}$$

in the Edmonds/Condon–Shortley convention for which $\Psi_f(N) \to d^f_{mn}(\theta)$ at fixed $f$ [27] (since $d^f_{nm} = (-1)^{m-n} d^f_{mn}$, the kernel of Ref. [2] with exponent $a+b+c+f+n$ equals $(-1)^{m-n}\Psi_f$). With

$$R^{(6)}_k := \sqrt{(k^2-m^2)(k^2-n^2)}, \quad \eta_k := \sqrt{\Big(1-\frac{k^2}{4A^2}\Big)\Big(1-\frac{k^2}{4B^2}\Big)}, \quad \delta^{(6)}_f := \frac{2f(f+1)+1-m^2-n^2}{8AB}, \tag{9}$$

the exact centered Racah recurrence is [2, Lemma 7.1]

$$(2f+1)\{f(f+1)[x-\delta^{(6)}_f] - mn\}\Psi_f = fR^{(6)}_{f+1}\eta_{f+1}\Psi_{f+1} + (f+1)R^{(6)}_f \eta_f \Psi_{f-1}. \tag{10}$$

reducing at $N=\infty$ to the Wigner rank recurrence for $d^f_{mn}(\theta)$, and the minimal-rank value at $f_{\min} = \max(|m|,|n|)$ is a single-term finite product of factorial ratios [2, Lemma 8.1 and Eq. (147)].

# 3 Even centered Wigner–Jacobi chains

This section states the growing-rank theory once, for an abstract class of recurrences that contains both chains of Section 2. The class is defined by two properties: an exact Weyl-midpoint Jacobi form whose coefficients are analytic and even in $h = N^{-1}$ and reduce at $h=0$ to the Wigner rank recurrence, and an analytic stretched starting vector. The fixed-rank contact identity used by the second-order theorem then holds automatically (Proposition 3.17). Sections 4 and 5 verify these properties for the Hahn/$3j$ and centered Racah/$6j$ chains and supply their explicit data.

## 3.1 Exact Weyl structure and the class

**Proposition 3.1** (Exact even midpoint structure)**.** *For a finite Fourier symbol $c(q,p;h) = \sum_m c_m(q;h)e^{imp}$ define*

$$(\mathrm{Op}^W_h(c)u)(q) = \sum_m c_m(q+mh/2;h)u(q+mh). \tag{11}$$

*Hence*

$$\mathrm{Op}^W_h(a(q)e^{ip})u(q) = a(q+h/2)u(q+h). \tag{12}$$

*Any recurrence*

$$[x - b(q;h^2)]Y(q) = a(q+h/2;h^2)Y(q+h) + a(q-h/2;h^2)Y(q-h) \tag{13}$$

*is therefore the exact Weyl eigenvalue equation*

$$\mathrm{Op}^W_h\Big(b(q;h^2) + 2a(q;h^2)\cos p\Big)\, Y = xY. \tag{14}$$

*It is invariant under step reversal $h \mapsto -h$. Where $a(q \pm h/2; h^2) > 0$, its frozen transfer $T(q) = \left(\begin{smallmatrix} (x-b(q;h^2))/a(q+h/2;h^2) & -a(q-h/2;h^2)/a(q+h/2;h^2) \\ 1 & 0 \end{smallmatrix}\right)$ has determinant $a(q-h/2;h^2)/a(q+h/2;h^2)$, so after the scalar flux normalization (multiplying $T(q)$ by $\sqrt{a(q+h/2;h^2)/a(q-h/2;h^2)}$) it has determinant one. In the elliptic region $|x - b(q;h^2)| < 2\sqrt{a(q+h/2;h^2)a(q-h/2;h^2)}$, its eigenvalues are $e^{\pm i p_h}$ with*

$$\boxed{\cos p_h(q,x) = \frac{x - b(q;h^2)}{2\sqrt{a(q+h/2;h^2)a(q-h/2;h^2)}}.} \tag{15}$$

*If moreover $a(q;t)$ and $b(q;t)$ are analytic in $(q,t)$ near $t = 0$ (as (E1) of Definition 3.2 gives), then $p_h = p_0 + h^2 p_2 + h^4 p_4 + \cdots$, convergent on a compact elliptic set with $a_0 > 0$; evenness of the recurrence in $h$ alone does not give this (for $a(q;t) = 1 + \sqrt{t}$, $b = 0$, $x = 1$ one has $p_h = \pi/3 + |h|/\sqrt{3} + O(h^2)$). For a rank recurrence with lattice $q_k = (k + \frac{1}{2})h$, the half-rank coordinate is the unique affine coordinate compatible with the quadratic-lattice reflection $s \mapsto -s - 1$, since $s(s+1) = (s + \frac{1}{2})^2 - \frac{1}{4}$.*

*Proof.* The Weyl identity is immediate from the Fourier-mode definition. Step reversal exchanges the two half-step hoppings. The relation $k(k+1) = q_k^2 h^{-2} - 1/4$ places quadratic-lattice diagonal terms at the midpoint. □

We first fix the Wigner base chain. Let $a, b \in \frac{1}{2}\mathbb{Z}$ with $a - b \in \mathbb{Z}$, put $k_{\min} = \max(|a|, |b|)$ and $\kappa = k + \frac{1}{2}$; the ranks are $k \in k_{\min} + \mathbb{Z}_{\geq 0}$, integers or half-integers with $a, b$. In the natural gauge $\widehat{d}_k := \sqrt{2k+1}\, d^k_{ab}(\arccos x)$ the Wigner rank recurrence is the Jacobi equation

$$x\widehat{d}_k = A^{(0)}_{k+1}\widehat{d}_{k+1} + A^{(0)}_k \widehat{d}_{k-1} + B^{(0)}_k \widehat{d}_k, \qquad A^{(0)}_k = \frac{\sqrt{(k^2-a^2)(k^2-b^2)}}{k\sqrt{4k^2-1}}, \quad B^{(0)}_k = \frac{ab}{k(k+1)}, \tag{16}$$

with $A^{(0)}_{k_{\min}} := 0$ (the displayed quotient is $0/0$ when $|a| = |b| = \frac{1}{2}$) and $B^{(0)}_0 := 0$ when $a = b = 0$; in the exceptional channel $a = b = 0$ row $k = 0$ of (16) is $x\widehat{d}_0 = \frac{1}{\sqrt{3}}\widehat{d}_1$, whereas ungauged rank recurrences such as (4) and (10) vanish identically at $k = 0$ and iterate from $k \geq 1$. Since $2A^{(0)}_k = 1 + O(k^{-2})$ and $B^{(0)}_k = O(k^{-2})$, the Wigner chain is a summable perturbation of the constant elliptic chain. The ordered Condon–Shortley sign and the interior Darboux formula are

$$\epsilon_{ab} := (-1)^{\max(b-a,0)}, \qquad d^k_{ab}(\vartheta) = \epsilon_{ab}\sqrt{\frac{2}{\pi\kappa\sin\vartheta}}\left[\cos\left(\kappa\vartheta + \frac{p_{ab}\pi}{2} - \frac{\pi}{4}\right) + O(\kappa^{-1})\right], \tag{17}$$

with $p_{ab} := |a - b|$, uniformly for $\vartheta$ in compact subsets of $(0, \pi)$.

**Definition 3.2** (Even centered Wigner–Jacobi chain)**.** Let $\mathcal{P}$ be a compact parameter set and $X \Subset (-1, 1)$ compact. A family of real recurrences, indexed by $h = N^{-1}$ along an admissible sequence and by $(\pi, x) \in \mathcal{P} \times X$,

$$x\widehat{Y}_k = A_{k+1}(h)\widehat{Y}_{k+1} + A_k(h)\widehat{Y}_{k-1} + B_k(h)\widehat{Y}_k, \qquad k \geq k_{\min} \text{ if } (a,b) \neq (0,0), \quad k \geq 1 \text{ if } (a,b) = (0,0), \tag{18}$$

with $A_{k_{\min}}(h)\widehat{Y}_{k_{\min}-1} := 0$ (in the exceptional channel $a = b = 0$, row $k = 0$ is assumed only where stated in Theorems 3.24 and 3.26), together with a distinguished *physical solution* $\widehat{Y}_k = \sqrt{2k+1}\, Y_k$, is an *even centered Wigner–Jacobi chain* with offsets $(a, b)$ if:

(E1) *Even midpoint coefficients.* There are $s_1 > 0$, $t_0 > 0$ and functions $e_\pi(s,t)$, $v_\pi(s,t)$, independent of $x$ and $k$, continuous in $(\pi, s, t)$, analytic in $(s, t)$, and uniformly bounded on $\mathcal{P}$

times a fixed complex neighbourhood of $[0,s_1]\times\{|t|\le t_0\}$, real for real arguments, with $e_\pi(0,0)=1$, $v_\pi(0,0)=0$ and $e_\pi(s,0)\ge c>0$ on $[0,s_1]$, such that, with $\lambda_\pi(t):=e_\pi(0,t)$,

$$A_k(h)=A_k^{(0)}\,e_\pi(k^2h^2,h^2),\qquad B_k(h)=\lambda_\pi(h^2)\,B_k^{(0)}+v_\pi(q_k^2,h^2),\qquad q_k=\kappa h,\tag{19}$$

exactly, for all ranks with $q_k^2\le s_1$ (so also $k^2h^2\le s_1$).

(E2) *Analytic stretched seed.* There are a bounded open set $U\supset X$ with $\overline{U}\subset\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ and $t_0>0$ such that at the two lowest ranks $k_{\min},k_{\min}+1$ the physical solution is the value at $t=h^2$ of a function $\widehat{Y}_k(x;t)$ analytic and bounded in $(x,t)$ jointly on $U\times\{|t|\le t_0\}$, uniformly for $\pi\in\mathcal{P}$, real for real arguments, whose value at $t=0$ is the Wigner value $\widehat{d}_k(\arccos x)$ (which continues analytically to $U$, being a polynomial in $x$ times $(1-x)^\mu(1+x)^\nu$ with fixed exponents $\mu,\nu\in\frac12\mathbb{Z}$; when $k_{\min}>0$, or when $a=b=0$ and the row $k=0$ holds, joint analyticity at $k_{\min}$ alone gives $k_{\min}+1$ from (18) after shrinking $t_0$).

No further hypothesis is needed for the second-order theory: the fixed-rank contact property it uses follows from (E1) (Proposition 3.17).

*Remark* 3.3 (Analyticity of the seed in $x$). Joint analyticity in $(x,t)$ is used twice: it makes the fixed-rank datum $c(x)$ of Theorem 3.24, and with it $\alpha(q,x)$, analytic in $x$, and it supplies the continuation of the exact solution to a complex neighbourhood of $X$ in the proof of Theorem 3.26. Analyticity in $t$ alone, uniformly for real $x$, does not give either. For $e_\pi\equiv1$, $v_\pi\equiv0$ (the Wigner chain itself) the solution $\widehat{Y}_k=(1+tf(x))\,\widehat{d}_k$ with $f$ smooth and bounded but not analytic solves the recurrence and is analytic in $t$ at fixed real $x$, uniformly on $X$; here $\mathsf{J}_2\equiv0$ and $c=f$, so $\alpha(q,x)=f(x)$ in Theorem 3.24. Shrinking $U$ and $t_0$ if necessary, $\widehat{d}_{k_{\min}}(\arccos x)$ has no zero on $\overline{U}$ and $|\widehat{Y}_{k_{\min}}(x;t)-\widehat{d}_{k_{\min}}(\arccos x)|\le C|t|$ on $U\times\{|t|\le t_0\}$ by Cauchy's estimate, so for $N$ large the lowest-rank seed has no zeros on $U$. The Hahn, Racah and quantum $6j$ seeds are closed products and satisfy the joint form (Propositions 4.1, 5.1 and 6.1).

The Hahn and centered Racah chains have $e_\pi(s,t)=e_\pi(s)$ independent of $t$, so $\lambda_\pi\equiv1$; the one-small quantum $6j$ chain of Section 6 does not. The $t$-dependence is needed because the Wigner diagonal $B_k^{(0)}=ab/(k(k+1))$ is deformed there by the factor $\lambda_\pi(h^2)=e_\pi(0,h^2)$ (Remark 3.4).

Thus $Y_k(h)$ is analytic in $t=h^2$ at every fixed rank, with $Y_k(0)=d_{ab}^k(\arccos x)$, because the forward hopping $A_{k+1}(h)$ is analytic and nonzero near $t=0$. We write

$$\begin{gathered}e_\pi(s,t)=1-\varepsilon_1 s+\lambda_1 t+\cdots,\qquad v_\pi(s,t)=\beta_1 s+\beta_2 t+\cdots,\\ \chi(x):=x\varepsilon_1-\beta_1,\qquad \eta_\pi(s):=\partial_t\log e_\pi(s,t)\big|_{t=0},\end{gathered}\tag{20}$$

so $\eta_\pi(0)=\lambda_1$ (dots: terms of second order in $(s,t)$). We write $e_\pi(s):=e_\pi(s,0)$ for the *principal coefficient*. For Hahn and Racah $\lambda_1=0$ and $\eta_\pi\equiv0$.

The principal data of the chain are defined as follows. Put $a_0(q):=\frac12 e_\pi(q^2)=\frac12 e_\pi(q^2,0)$ and $b_0(q):=v_\pi(q^2,0)$. The principal symbol is $b_0(q)+2a_0(q)\cos p$, so the principal eikonal, discriminant, action, effective angle and transport are

$$\begin{gathered}\cos p(q,x)=\frac{x-b_0(q)}{e_\pi(q^2)},\qquad D(q,x):=e_\pi(q^2)^2-[x-b_0(q)]^2=4a_0^2\sin^2p,\\ S(q,x):=\int_0^q p(s,x)\,\mathrm{d}s,\qquad \Theta_q(x):=\frac{S(q,x)}{q},\qquad W_q(x):=\left[\frac{\sin^2\Theta_q(x)}{D(q,x)}\right]^{1/4},\end{gathered}\tag{21}$$

on the branch $p\to\theta=\arccos x$ as $q\downarrow0$. Since $p$ is analytic in $q^2$, $\Theta_q$ is even and analytic at $q=0$, with

$$\Theta_q=\theta-\frac{\chi(x)}{3\sin\theta}q^2+O(q^4),\qquad W_q=1+\frac{(\varepsilon_1-x\beta_1)/2-x\chi(x)/6}{1-x^2}\,q^2+O(q^4).\tag{22}$$

A *turning-free corridor* is a compact set $\mathcal{C}$ of points $(q, x, \pi)$ with $q^2 \le s_1 - \delta_{\mathcal{C}}$ for some $\delta_{\mathcal{C}} > 0$ (so that, for large $N$, the neighbouring ranks $k \pm 1$ of every corridor rank also satisfy (E1)) and $D > 0$ that contains $(r, x, \pi)$ for $0 \le r \le q$ whenever it contains $(q, x, \pi)$; on it $\sin p$, $e_\pi$ and $\sin\Theta_q$ are bounded below. The *leading pullback* at rank $k$ is

$$U_k := W_{q_k}(x)\, d^k_{ab}(\Theta_{q_k}(x)), \qquad \widehat{U}_k = \sqrt{2k+1}\, U_k. \tag{23}$$

Equation (22) follows from $\cos p = x + \chi q^2 + O(q^4)$, $\Theta_q = q^{-1}\int_0^q p$ and $\sin^2\Theta_q/D = (1 - x^2 - \frac{2}{3}x\chi q^2)/(1 - x^2 - 2(\varepsilon_1 - x\beta_1)q^2) + O(q^4)$.

*Remark* 3.4 (The $t$-dependence of $e_\pi$). Dividing the recurrence by $\lambda_\pi(h^2)$ gives a chain whose off-diagonal coefficient equals the Wigner one at $q = 0$, at the shifted spectral parameter $x/\lambda_\pi(h^2) = x - \lambda_1 x h^2 + \cdots$. Hence the $t$-dependence of $e_\pi$ has two effects, both of order $h^2$ and invisible at leading order: the shift $x \mapsto x - \lambda_1 x t$ at fixed rank, and the fixed-ratio term $\eta_\pi(q^2)$ in the order-$h^2$ coefficient of the off-diagonal symbol, (47). Everything that depends only on the principal data (the principal symbol, $\chi$, the critical phase law, the mesoscopic series, the first-order pullback, the turning-point geometry) is unchanged; the second-order data pick up $\eta_\pi$ and $\lambda_1$, and these terms are displayed where they occur. The $h^4/q^2$ coefficient of $B_k - B_k^{(0)}$ in the quantum $6j$ chain is $mn(m^2 + n^2 - 1)/6$, nonzero when $mn \neq 0$, so without the factor $\lambda_\pi$ that chain would not satisfy the analyticity of $v_\pi$.

*Remark* 3.5 (Scope of the class). The class is defined by the deformation of the Wigner chain, not by a special-function identity. Sections 4 and 5 show that the fixed-offset Hahn/$3j$ chain ($e = \sqrt{1-s}$, $v = 0$) and the centered Racah/$6j$ chain ($e$ a product of two square roots, $v$ affine) belong to it, and Section 6 that the one-small quantum $6j$ chain does, with $e_\pi(s, t)$ genuinely $t$-dependent. At principal level the Hahn rank symbol is a degeneration of the Racah one (Appendix H).

**Lemma 3.6** (Interior Wigner–Darboux expansion)**.** *Fix $a, b \in \frac{1}{2}\mathbb{Z}$ with $a - b \in \mathbb{Z}$ and $J \Subset (0, \pi)$, and put $\kappa = k + \frac{1}{2}$. Uniformly on $J$,*

$$\sqrt{2k+1}\, d^k_{ab}(\vartheta) = (\sin\vartheta)^{-1/2} \sum_{\pm} c^{\pm}_{ab} \exp\left[\pm i\left(\kappa\vartheta + \frac{\mathfrak{c}_{ab}(\vartheta)}{\kappa}\right)\right] + O_J(k^{-2}), \tag{24}$$

$$\mathfrak{c}_{ab}(\vartheta) := \frac{(a^2 + b^2 - 1/4)\cos\vartheta - 2ab}{2\sin\vartheta}. \tag{25}$$

*with $c^{\pm}_{ab} := \pi^{-1/2}\omega^{\pm 1}_{ab}$, $\omega_{ab} := \epsilon_{ab} e^{i(p_{ab}\pi/2 - \pi/4)}$ (so that the leading term is (17)). More generally, for every $R, s$ there are smooth branch amplitudes $A^\sigma_{r,ab}$, independent of $k$, such that*

$$\sqrt{2k+1}\, d^k_{ab}(\vartheta) = \sum_{\sigma=\pm 1} e^{i\sigma\kappa\vartheta}\Big[\sum_{r=0}^{R-1} \kappa^{-r} A^\sigma_{r,ab}(\vartheta) + \mathcal{E}^\sigma_{R,k}(\vartheta)\Big], \tag{26}$$

*where the remainders $\mathcal{E}^{\pm}_{R,k}$ are carrier-separated amplitude remainders: they stand inside the brackets, beside the carriers $e^{\pm i\kappa\vartheta}$, and*

$$\max_{0\le m\le s} \sup_{\vartheta\in J} |\partial^m_\vartheta \mathcal{E}^{\pm}_{R,k}(\vartheta)| \le C_{J,R,s}\, \kappa^{-R}, \tag{27}$$

*with $C_{J,R,s}$ independent of $k$. The derivatives fall on the amplitude remainders, not on the carriers; a derivative of a carrier costs a factor $\kappa$ and is taken explicitly.*

*Proof.* The expansion (24) is the standard interior Liouville–Green/Jacobi expansion; the first phase coefficient follows by integrating the $\kappa^{-1}$ correction in the Liouville normal form. The carrier-separated differentiated remainder (27) is proved in Appendix A: the undifferentiated Bessel-type (Frenzen–Wong/Hankel) expansion carries a remainder bounded only in the supremum norm, and the differential equation $w'' + (\kappa^2 - V)w = 0$ of $w = (\sin\vartheta)^{1/2}\sqrt{2k+1}\, d^k_{ab}$ upgrades it, at the price of one power of $\kappa$ per derivative, which an expansion $s$ orders deeper absorbs. □

### 3.2 Analytic transfer and the convergent mesoscopic series

The basic tool is the analytic oscillatory transfer bound of Lemma A.1 (Appendix A): if the transfer matrices are analytic in $t$ with order-$t^r$ coefficients $O(k^{2r})$ and the unperturbed fundamental matrix is naturally bounded, then in interaction form each order-$t^r$ insertion has size $O(k^{2r})$, and summing to rank $K$ produces the natural parameter $K^3 t$.

**Theorem 3.7** (Convergent mesoscopic even series)**.** *Let $Y_L$ be the physical solution of an even centered Wigner–Jacobi chain, and write $Y_L(t) = \sum_{r\ge 0} t^r y_{L,2r}(x)$ for its Taylor series at $t = h^2 = 0$, so that $y_{L,0} = d^L_{ab}(\arccos x)$. There are $\eta_* > 0$ and $C < \infty$, independent of $L$, such that the series converges uniformly on $\mathcal{P}\times X$ whenever $|t|\kappa^3 < \eta_*$, $\kappa = L + \frac{1}{2}$, and*

$$\sqrt{L+1}\,\sup_{\mathcal{P}\times X}|y_{L,2r}| \le C\eta_*^{-r}\kappa^{3r}. \tag{28}$$

*For $0 < \varrho < 1$ and $|t|\kappa^3 \le \varrho\eta_*$,*

$$\boxed{\sqrt{L+1}\,\sup_{\mathcal{P}\times X}\left|Y_L(t) - \sum_{r=0}^{R-1} t^r y_{L,2r}\right| \le \frac{C\varrho^R}{1-\varrho}.} \tag{29}$$

*In particular, for every $R$ there is $C_R$ with $\sqrt{L+1}\,|Y_L - \sum_{r<R} t^r y_{L,2r}| \le C_R(\kappa^3|t|)^R$ for $\kappa^3|t| \le \eta_*/2$, and $\sup|y_{L,2r}| \le C_r(L+1)^{3r-1/2}$.*

*Proof.* Take $\eta_* \le t_0/8$, so that $|t|\kappa^3 < \eta_*$ gives $|t| < t_0$ for every $\kappa \ge \frac{1}{2}$. For $L = k_{\min}$ and $L = k_{\min}+1$ the claim then holds directly by (E2). For $L > k_{\min}+1$, iterate $Z_{k+1} = T_k(t)Z_k$ for $k_{\min}+1 \le k < L$ from $Z_{k_{\min}+1} = (\widehat{Y}_{k_{\min}+1}, \widehat{Y}_{k_{\min}})^\top$, so $k \ge 1$ even when $a = b = 0$. By (E1), $T_k(t)$ depends on $t$ through $e_\pi(k^2t, t)$, $e_\pi((k+1)^2t, t)$, $\lambda_\pi(t)B^{(0)}_k$, and $v_\pi(\kappa_k^2 t, t)$ ($\kappa_k = k + \frac{1}{2}$), which are analytic with $|e_\pi(k^2t,t)-1| + |\lambda_\pi(t)-1|\,|B^{(0)}_k| + |v_\pi(\kappa_k^2 t, t)| \le C|t|k^2$ on $|t|k^2 \le c$ (using $k \ge 1$); hence $\|T_k(t) - T_k(0)\| \le C|t|k^2$ there, where $T_k(0)$ is the Wigner transfer of (16). On the compact set $X$, standard interior Jacobi/Wigner asymptotics give a naturally normalized Wigner fundamental matrix for $k \ge k_{\min}+1$ whose norm and inverse norm are uniformly bounded. For complex $t$ with $|t|\kappa^3 \le \eta_*$ one has $|k^2 t| \le \eta_*/\kappa$ and $|t| \le \eta_*$ for $k \le L$, so for fixed small $\eta_*$ all coefficients stay on their analytic branch, and the interaction factors $I + V_k(t)$ satisfy $\sum_{k=k_{\min}+1}^{L-1}\|V_k(t)\| \le C|t|\sum_{k<L}k^2 \le C\eta_*$. The ordered product and, by (E2), $Z_{k_{\min}+1}(t)$ are analytic and uniformly bounded on $|t| \le \eta_*/\kappa^3$. Cauchy's estimate gives (28), and the same bound on $|t|\kappa^3 \le \varrho\eta_*$ gives (29); the final assertions follow with $\varrho = \kappa^3|t|/\eta_*$ (cf. Lemma A.1). □

*Remark* 3.8. Theorem 3.7 is the rigorous analytic form of the backward-error picture in the small-$\kappa^3/N^2$ regime. It differs from the near-identity Hamiltonian interpolation theorem of [14]: convergence follows directly from analyticity of the finite transfer product on a disk of radius $\asymp \kappa^{-3}$. When $L \asymp N$ the scaled variable $\kappa^3/N^2$ grows like $N$, so this convergent local series is the wrong representation, and the closed action is the required nonperturbative resummation.

### 3.3 The $N^{2/3}$ critical phase law

The first coherent phase defect is quadratic in the rank, so, wherever its coefficient $\chi(x)$ is nonzero, the ordinary Wigner-$d$ phase first loses coherence at $L \asymp N^{2/3}$. The finite-$N$ law follows from the exact recurrence: the diagonal defect accumulates coherently, whereas the off-diagonal mode conversion oscillates and remains summable.

The finite-$N$ law rests on a critical transfer normal form, proved in Appendix A (Lemma A.2) as a finite-interval version of the Harris–Lutz asymptotic diagonalization method for difference systems (cf. Benzaid and Lutz [45]). If a real transfer system in the Wigner natural gauge satisfies, uniformly for $x \in K \Subset (-1,1)$ and $k \le L$,

$$T_k^{(N)} - T_k^{(0)} = 2\chi(x)\frac{k^2}{N^2}E_{11} + O\!\left(\frac{k}{N^2} + \frac{k^4}{N^4}\right), \tag{30}$$

where $T_k^{(0)}$ is the Wigner transfer and $E_{11}$ has a single nonzero upper-left entry, then for $L^3/N^2 \le \Lambda$ the ordered transfer from a fixed rank $k_0$ to $L$, written in the exact complex Wigner basis, is $\operatorname{diag}(e^{-\mathrm{i}\vartheta_{L,N}}, e^{\mathrm{i}\vartheta_{L,N}}) + O_{K,\Lambda}(L^2/N^2 + L^5/N^4)$ with

$$\vartheta_{L,N}(x) = \frac{\chi(x)}{\sin\theta}\sum_{k=k_0}^{L-1}\frac{k^2}{N^2} = \frac{\chi(x)}{\sin\theta}\left[\frac{L^3}{3N^2} + O\!\left(\frac{L^2}{N^2}\right)\right], \qquad x = \cos\theta. \tag{31}$$

The spectral-gap input is the bulk condition $\sin\theta \ge c_K > 0$; without such a dichotomy the off-diagonal cancellation need not hold.

**Theorem 3.9** (Critical phase law for even centered chains)**.** *Let $Y_L$ be the physical solution of an even centered Wigner–Jacobi chain and $\chi$ as in (20). Fix $\Lambda < \infty$. Uniformly on $\mathcal{P} \times X$, with $x = \cos\theta$, $\kappa = L + \frac{1}{2}$, $L \to \infty$ and $\kappa^3/N^2 \le \Lambda$,*

$$\sqrt{\frac{\pi\kappa\sin\theta}{2}}\,Y_L = \cos\!\left(\varphi_{L,ab}(\theta) - \frac{\chi(x)}{3\sin\theta}\frac{\kappa^3}{N^2}\right) + O_\Lambda\!\left(\frac{1}{\kappa} + \frac{\kappa^2}{N^2} + \frac{\kappa^5}{N^4}\right), \tag{32}$$

*where $\varphi_{L,ab}$ is the standard large-rank Wigner-d phase, $\sqrt{\pi\kappa\sin\theta/2}\,d^L_{ab}(\theta) = \cos\varphi_{L,ab}(\theta) + O(\kappa^{-1})$; by (17), $\varphi_{L,ab}(\theta) = \kappa\theta + \frac{p_{ab}\pi}{2} - \frac{\pi}{4}$, plus $\pi$ when $\epsilon_{ab} = -1$. In particular, if $\kappa^3/N^2 \to \lambda$, the additional phase tends to $-\lambda\chi(x)/(3\sin\theta)$, and $\sqrt{L+1}\,|Y_L - d^L_{ab}(\theta)| = O(\kappa^3/N^2)$ for $L = o(N^{2/3})$. The additional phase is of order one at $\kappa \asymp N^{2/3}$ exactly where $\chi(x) \ne 0$; at a point with $\chi(x) = 0$ (for Hahn, $x = 0$; for Racah, $x = 2\alpha\beta/(\alpha^2+\beta^2)$ when $\alpha \ne \beta$; for the quantum $6j$ chain, $x = 2\coth\mathsf{X}\coth\mathsf{Y}/(\coth^2\mathsf{X} + \coth^2\mathsf{Y})$ when $\mathsf{X} \ne \mathsf{Y}$) the leading shift disappears, (32) then says only that $Y_L$ agrees with the Wigner phase up to the stated error, and the $N^{2/3}$ scale is not a horizon there.*

*Proof.* In the natural gauge the transfer of (18) for $k \ge k_{\min} + 1$ is $T_k = \left(\begin{smallmatrix} (x-B_k)/A_{k+1} & -A_k/A_{k+1} \\ 1 & 0 \end{smallmatrix}\right)$. By (E1) and (20), $e_\pi((k+1)^2h^2, h^2)^{-1} = 1 + (\varepsilon_1 k^2 - \lambda_1)h^2 + O(kh^2 + k^4h^4)$, $\lambda_\pi(h^2)B_k^{(0)} = B_k^{(0)} + O(h^2k^{-2})$, $v_\pi(q_k^2, h^2) = \beta_1 k^2h^2 + O(kh^2 + h^2 + k^4h^4)$, and $(x - B_k^{(0)})/A_{k+1}^{(0)} = 2x + O(k^{-2})$; moreover $e_\pi(k^2h^2, h^2)/e_\pi((k+1)^2h^2, h^2) = 1 + O(kh^2)$ because $\partial_s \log e_\pi(s,t)$ is bounded near $(0,0)$ (the $\lambda_1 h^2$ term is $O(h^2)$ in $T_{11}$ and cancels to $O(kh^4)$ in $T_{12}$). Hence

$$T_k - T_k^{(0)} = 2(x\varepsilon_1 - \beta_1)\frac{k^2}{N^2}E_{11} + O\!\left(\frac{k}{N^2} + \frac{k^4}{N^4}\right), \tag{33}$$

which is (30) with $\chi = x\varepsilon_1 - \beta_1$. By (E2) the physical starting vector at $k_0 \geq k_{\min} + 1$ differs from the Wigner one by $O(N^{-2})$ in natural norm, and propagating it through the bounded critical transfer costs $O(N^{-2})$. The critical transfer normal form and $\sum_{k_0 \leq k < L} k^2 = L^3/3 + O(L^2)$ give the accumulated branch phase $-\chi \sum_{k<L} k^2/(N^2 \sin\theta) = -\chi L^3/(3N^2 \sin\theta) + O(L^2/N^2)$. The uniform Darboux formula $\sqrt{\pi\kappa \sin\theta/2}\, d^L_{ab} = \cos\varphi_{L,ab} + O(\kappa^{-1})$ and replacing $L^3$ by $\kappa^3$ at phase cost $O(\kappa^2/N^2)$ give (32). The subcritical statement is $R = 1$ of Theorem 3.7. □

*Remark* 3.10 (Even-midpoint coherence horizons). The mechanism of Theorem 3.9 is general: a coherent diagonal phase increment $\delta\varphi_k \approx \chi(k+1)^s/N^r$, $s > -1$, accompanied by an oscillatory off-diagonal interaction with bounded partial sums, accumulates the phase $\chi K^{s+1}/((s+1)N^r)$ and therefore has the coherence horizon

$$\boxed{K \asymp N^{r/(s+1)}}. \tag{34}$$

This is a heuristic organizing principle; the precise statement used in the proofs is the critical transfer normal form (30)–(31). For an even midpoint chain the first defect has $s = r = 2$, giving the $N^{2/3}$ horizon of Theorem 3.9 at every $x$ with $\chi(x) \neq 0$. After $R$ even principal phase terms have been resummed, an omitted increment $k^{2R+2}/N^{2R+2}$ gives

$$\boxed{K_R \asymp N^{(2R+2)/(2R+3)}} \tag{35}$$

provided its coefficient is nonzero. The Hamilton–Jacobi action below resums this entire hierarchy.

### 3.4 Hamilton–Jacobi action and the leading pullback

Beyond the critical scale the correct object is not a longer fixed-rank Taylor polynomial but the discrete eikonal (21). Its basic estimates are a turning-free adiabatic basis with fixed-start stability, and the first secular coefficient of the leading pullback (used for the $6j$ bound (328)); the all-rank error of the leading pullback follows from the second-order theory (Corollary 3.21).

**Lemma 3.11** (Turning-free adiabatic rank basis)**.** *Let $\mathcal{C}$ be a compact set of points $(q, x, \pi)$ with $q^2 < s_1$ and $D(q,x) > 0$, put $h = N^{-1}$, $q_k = (k + \frac{1}{2})h$, and*

$$z_k^{\pm} = D(q_k, x)^{-1/4} \exp\big(\pm \mathrm{i} N S(q_k, x)\big). \tag{36}$$

*If $K \to \infty$, $K/N \to 0$, and $(q_k, x, \pi) \in \mathcal{C}$ for $K \leq k \leq L$, then the exact Jacobi system (18) has a fundamental matrix $\mathcal{F}_k^{(N)}$ satisfying*

$$\mathcal{F}_k^{(N)} = P_k \left[ I + O_{\mathcal{C}}\Big(K^{-1} + L/N^2\Big) \right], \qquad P_k = \begin{pmatrix} z_k^+ & z_k^- \\ z_{k-1}^+ & z_{k-1}^- \end{pmatrix}, \tag{37}$$

*uniformly for $K \leq k \leq L$, in $(x, \pi)$, and in the final index $L$ as long as the path remains in $\mathcal{C}$.*

*Proof.* See Appendix A. □

**Corollary 3.12** (Fixed-start turning-free stability)**.** *On a turning-free corridor there is a fixed admissible rank $k_*$ such that, for every fixed $k_0 \geq k_*$ and all sufficiently large $N$, one may choose an exact naturally normalized fundamental matrix $G_k^{(N)}$ of (18) with*

$$\sup_{k_0 \leq k \leq L} \Big( \|G_k^{(N)}\| + \|(G_k^{(N)})^{-1}\| \Big) \leq C_{\mathcal{C},k_0} \tag{38}$$

*whenever the swept path stays in the corridor.*

*Proof.* See Appendix A. □

**Lemma 3.13** (First secular coefficient)**.** *With $y_{L,2}$ the t-coefficient of Theorem 3.7 and*

$$\mathcal{P}_2 := \frac{\chi(x)}{3}\,\partial_x + \frac{(\varepsilon_1 - x\beta_1)/2 - x\chi(x)/6}{1-x^2}, \tag{39}$$

*one has $\sqrt{L+1}\,|y_{L,2} - \kappa^2\mathcal{P}_2 d^L_{ab}(\arccos x)| \le C(L+1)$ uniformly on $\mathcal{P}\times X$. The operator $\kappa^2\mathcal{P}_2$ is the t-coefficient of the leading pullback $W_q d^L_{ab}(\Theta_q)$ at fixed rank, by (22).*

*Proof.* This follows from the fixed-rank contact of Proposition 3.17, whose proof (Appendix B) does not use the present lemma. At fixed rank the corrected pullback is $\widehat{U}^{[2]}_k = \widehat{d}_k + t\,\widehat{u}_k + O_k(t^2)$ with $\widehat{u}_k = \sqrt{2k+1}\,\mathcal{Q}_{2,k}d_k$, where

$$\mathcal{Q}_{2,k} - \kappa^2\mathcal{P}_2 = -\sqrt{1-x^2}\,\rho_2(0,x)\,\partial_x + \frac{x\,\rho_2(0,x)}{2\sqrt{1-x^2}},$$

an operator with coefficients bounded on $X$ and independent of $k$. Since $\widehat{Y}$ satisfies the recurrence exactly and $\mathcal{R}_k(\widehat{U}^{[2]})$ has no $t$-coefficient, differentiating both at $t=0$ shows that $\delta_k := \partial_t \widehat{Y}_k|_{t=0} - \widehat{u}_k$ solves the Wigner recurrence (16) for $k > k_{\min}$; its values at $k_{\min}, k_{\min}+1$ are bounded by (E2) and Proposition 3.16, and the Wigner propagator is uniformly bounded in natural gauge on $X$, so $|\delta_k| \le C$. As $\sqrt{2L+1}\,y_{L,2} = \widehat{u}_L + \delta_L$, and $\sqrt{2L+1}\,|d^L_{ab}| \le C$, $\sqrt{2L+1}\,|\partial_x d^L_{ab}| \le C\kappa$ on $X$ (Lemma 3.6), the claim follows. □

*Remark* 3.14 (Action-Jacobian zero transport). Let $S(r,x)$ be a smooth turning-free action, put $\Theta(r,x) = S(r,x)/r$ and $x_r(x) = \cos\Theta(r,x)$, and define

$$\varrho_r(x) := -\frac{1}{\pi r}\partial_x S(r,x), \qquad \rho_0(y) = \frac{1}{\pi\sqrt{1-y^2}}. \tag{40}$$

Then wherever $0 < \Theta < \pi$,

$$\boxed{\varrho_r(x)\,dx = \rho_0(x_r)\,dx_r.} \tag{41}$$

Indeed $\partial_x x_r = -\sin\Theta\, S_x/r$ and $\rho_0(x_r) = 1/(\pi\sin\Theta)$. Thus every effective coordinate obtained as the cosine of an action per unit rank transports the principal zero measure exactly; application-specific work enters only through the action and the finite-$N$ remainder.

### 3.5 Second Weyl eikonal and the second-order all-rank pullback

The leading pullback misses an order-$h$ accumulated phase at fixed ratio although the local symbol defect is $O(h^2)$. The correction is the order-$h^2$ Weyl eikonal, renormalized at $q=0$ by the finite-index Darboux phase.

**Proposition 3.15** (Second eikonal for an even midpoint Jacobi chain)**.** *Assume (13) with*

$$a(q;h^2) = a_0(q) + h^2 a_2(q) + O(h^4), \qquad b(q;h^2) = b_0(q) + h^2 b_2(q) + O(h^4), \tag{42}$$

*on a compact elliptic set with $a_0 > 0$. Let $p(q,x) \in (0,\pi)$ and $g(q,x)$ be defined by*

$$x - b_0(q) = 2a_0(q)\cos p(q,x), \qquad g(q,x) = -\frac{1}{2}\log(a_0(q)\sin p(q,x)). \tag{43}$$

*Then*

$$e^g \exp\{ih^{-1}(S + h^2 S_2)\}, \qquad S_q = p, \tag{44}$$

*solves the midpoint equation through order $h^2$ if and only if $S_{2,q} = \Sigma_2[a_0, a_2, b_2; p, g]$, where*

$$\Sigma_2[a_0, a_2, b_2; p, g] = \frac{1}{2a_0 \sin p}\Big[a_0(g_q^2 + g_{qq})\cos p - a_0 g_q p_q \sin p - \frac{a_0}{4}p_q^2 \cos p - \frac{a_0}{3}p_{qq}\sin p \tag{45}$$

$$+ 2a_2 \cos p + a_{0,q} g_q \cos p - \frac{a_{0,q}}{2}p_q \sin p + \frac{a_{0,qq}}{4}\cos p + b_2\Big]. \tag{46}$$

*Proof.* Insert the displayed ansatz into (13), Taylor-expand the two half-step coefficients and the phase, and use the principal eikonal and flux transport to cancel orders $h^0$ and $h^1$. The real order-$h^2$ scalar coefficient is exactly (46); the details are in Appendix B. □

For an even centered Wigner–Jacobi chain the exact midpoint coefficients are $a(s; h^2) = A_k$ at $s = kh$ and $b(q; h^2) = B_k$ at $q = q_k$. Since $A_k^{(0)} = \frac{1}{2}[(1 - a^2h^2/s^2)(1 - b^2h^2/s^2)/(1 - h^2/(4s^2))]^{1/2}$, $B_k^{(0)} = abh^2/(q^2 - h^2/4)$, $e_\pi(s^2, h^2) = e_\pi(s^2)\,(1 + h^2\eta_\pi(s^2) + O(h^4))$ and $\lambda_\pi(h^2) = 1 + O(h^2)$ (which changes $b$ only at order $h^4$), their order-$h^2$ coefficients at fixed positive $q$ are

$$a_2(q) = a_0(q)\Big[\frac{1}{q^2}\Big(\frac{1}{8} - \frac{a^2 + b^2}{2}\Big) + \eta_\pi(q^2)\Big], \qquad b_2(q) = \frac{ab}{q^2} + \partial_t v_\pi(q^2, 0). \tag{47}$$

The new term $\eta_\pi(q^2)$ is regular at $q = 0$ and is zero for the Hahn and Racah chains. It adds $\eta_\pi(q^2)\cot p$ to $\Sigma_2$ below. Put $g = -\frac{1}{2}\log(a_0 \sin p) = -\frac{1}{4}\log D + \text{const}$ and

$$\begin{aligned} &\Sigma_2 := \Sigma_2[a_0, a_2, b_2; p, g], \qquad \rho_2(q, x) := \Sigma_2(q, x) - \frac{d}{dq}\left(\frac{\mathfrak{c}_{ab}(\Theta_q(x))}{q}\right), \\ &\Gamma_2(q, x) := \int_0^q \rho_2(s, x)\,\mathrm{d}s, \qquad \Theta_q^{[2]} := \Theta_q + \frac{h^2}{q}\Gamma_2, \qquad W_q^{[2]} := \left[\frac{\sin^2\Theta_q^{[2]}}{D(q, x)}\right]^{1/4}, \end{aligned} \tag{48}$$

with $\mathfrak{c}_{ab}$ from (25). For the Hahn and centered Racah chains $\widetilde{\Gamma} = \Gamma_2 + \mathfrak{c}_{ab}(\Theta_q)/q$ has a three-term algebraic closed form in $\sqrt{D}$ (companion supplement [1]), so $\Theta_q^{[2]}$, and $B_\tau$ in Theorem 3.27, need no quadrature. The amplitude $W_q^{[2]}$ is not an extra fitted function: it keeps the principal complex-branch amplitude equal to $D^{-1/4}$ after the angle is moved. The *corrected pullback* is

$$U_k^{[2]} := W_{q_k}^{[2]}(x)\, d_{ab}^k(\Theta_{q_k}^{[2]}(x)), \qquad \widehat{U}_k^{[2]} = \sqrt{2k+1}\,U_k^{[2]}. \tag{49}$$

**Proposition 3.16** (Darboux renormalization of the second eikonal)**.** *On every compact interior $x$-set,*

$$\Sigma_2(q, x) = -\frac{\mathfrak{c}_{ab}(\theta)}{q^2} + O(1) \qquad (q \downarrow 0), \qquad \theta = \arccos x, \tag{50}$$

*and $\rho_2$ extends* analytically in $q^2$ *to $q = 0$. Consequently $\Gamma_2/q$, $\Theta_q^{[2]}$ and $W_q^{[2]}$ are analytic in $q^2$ at $q = 0$, and $U_k^{[2]}$ is analytic in $t = h^2$ at every fixed rank, with $U_k^{[2]}(0) = d_{ab}^k(\arccos x)$.*

*Proof.* By (E1), $a_0$, $b_0$, $p$, $g$, $\eta_\pi(q^2)$, $\partial_t v_\pi(q^2, 0)$ and $\Theta_q$ are analytic in $q^2$ near 0, with $a_0 = \frac{1}{2} + O(q^2)$, $p = \theta + O(q^2)$ and $g_q = O(q)$. In (46) only $2a_2\cos p + b_2$ has a pole (the term $\eta_\pi$ of $a_2$ is regular), so

$$\Sigma_2 = \frac{x(1/8 - (a^2 + b^2)/2) + ab}{q^2 \sin\theta} + O(1) = -\frac{\mathfrak{c}_{ab}(\theta)}{q^2} + O(1). \tag{51}$$

All terms of the density are even meromorphic in $q$ with at worst a $q^{-2}$ pole. Because $\Theta_q = \theta + O(q^2)$ is even analytic, $\frac{d}{dq}(\mathfrak{c}_{ab}(\Theta_q)/q) = -\mathfrak{c}_{ab}(\theta)q^{-2} + O(1)$ is even meromorphic with the same sole pole, so the difference is analytic in $q^2$. Then $\Gamma_2$ is odd analytic and $\Gamma_2/q$ is even analytic; at fixed rank $q^2 = \kappa^2 t$. Integrating the derivative term against $-\mathfrak{c}_{ab}(\theta)q^{-2}$ also writes $\Gamma_2(q, x) = (\mathfrak{c}_{ab}(\theta) - \mathfrak{c}_{ab}(\Theta_q(x)))/q + \int_0^q(\Sigma_2(s, x) + \mathfrak{c}_{ab}(\theta)s^{-2})\,\mathrm{d}s$ without nested integration of $\Theta_s$. □

**Proposition 3.17** (Automatic second-order fixed-rank contact)**.** *For every even centered Wigner–Jacobi chain and every fixed admissible rank $k$, the coefficient of $t$ in the Taylor expansion at $t = h^2 = 0$ of the natural-gauge residual*

$$\mathcal{R}_k(\widehat{U}^{[2]}) := x\widehat{U}_k^{[2]} - A_{k+1}\widehat{U}_{k+1}^{[2]} - A_k\widehat{U}_{k-1}^{[2]} - B_k\widehat{U}_k^{[2]} \tag{52}$$

*vanishes identically on $\mathcal{P} \times X$; hence $\mathcal{R}_k(\widehat{U}^{[2]}) = O_k(t^2)$. In the exceptional channel $a = b = 0$ this includes $k = 0$, with $B_0^{(0)} := 0$. Its $t^0$ coefficient vanishes because $U^{[2]}(0)$ is the Wigner solution. Moreover*

$$\begin{aligned}\rho_2(0,x) = \frac{1}{12(1-x^2)^{3/2}}\Big[&\varepsilon_1\big(9x - x^3 - 8(a^2+b^2)x + 4abx^2 + 12ab\big)\\ &+ \beta_1\big(8(a^2+b^2) - 16abx - 2x^2 - 6\big) + 12(\beta_2 + \lambda_1 x)(1-x^2)\Big].\end{aligned} \tag{53}$$

*The $\lambda_1$ term is the shift $\beta_2 \to \beta_2 + \lambda_1 x$ of Remark 3.4; it is zero for Hahn and Racah.*

*Proof.* The closed form of $\rho_2(0,x)$ follows by expanding (46) at $q = 0$; the contact identity reduces, through the Wigner rank recurrence and the ladder identities, to a rational identity in $(a, b, k, x)$. The computation is in Appendix B. □

**Theorem 3.18** (Weyl–Darboux midpoint engine)**.** *Let $h = N^{-1}$, $p \in \{3, 4\}$, $R := 7(p-1)/(5-p)+1$ ($R = 8$ for $p = 3$, $R = 22$ for $p = 4$), and let $U_k(h)$ be a corrected oscillatory pullback for a turning-free midpoint Jacobi chain. Assume:*

1. *for a fixed rank $k_0 \geq 1$ and all $k \geq k_0$ with $k^3h^2 \leq \eta_0$, the fixed-rank bound*

$$|\mathcal{R}_k(\widehat{U})| \leq Ch^4k^6; \tag{54}$$

2. *for $k \geq k_0$ in the turning-free corridor,*

$$|\mathcal{R}_k(\widehat{U})| \leq C(h^p + k^{-R}); \tag{55}$$

3. *the exact naturally gauged fundamental matrix and its inverse are uniformly bounded for $k_0 \leq k \leq L$.*

*Then, uniformly for every admissible terminal rank $L$ in the corridor with $k_0 \leq L = O(h^{-1})$,*

$$\boxed{\sup_{k_0 \leq k \leq L} |\widehat{Y}_k - \widehat{U}_k| \leq Ch^{p-1},} \tag{56}$$

*provided the two initial entries differ by $O(h^{p-1})$ at the fixed rank $k_0$. In particular the same bound holds simultaneously at fixed, mesoscopic, and fixed-positive-ratio rank as long as the swept corridor remains turning-free. On any subfamily for which the corrected comparison satisfies $|\widehat{U}_L| \geq c > 0$, the same estimate is equivalently a relative error $\widehat{Y}_L/\widehat{U}_L = 1 + O(h^{p-1})$.*

*Proof.* Let $K_* := \min\{L, \lfloor h^{-(5-p)/7} \rfloor\}$. Since $K_*^3h^2 \leq h^{(3p-1)/7} \to 0$, the fixed-rank estimate of hypothesis 1 applies up to $K_*$, and $\sum_{k=k_0}^{K_*} h^4k^6 = O(h^4K_*^7) = O(h^{p-1})$. If $L > K_*$, the high-rank estimate of hypothesis 2 gives $\sum_{k=K_*}^{L-1}(h^p + k^{-R}) = O(Lh^p + K_*^{-(R-1)}) = O(h^{p-1})$, since $K_* \asymp h^{-(5-p)/7}$, $(R-1)(5-p) = 7(p-1)$, and $L = O(h^{-1})$ (if $L \leq K_*$ the second sum is empty). Variation of constants with the uniformly bounded exact fundamental matrix of hypothesis 3

propagates the $O(h^{p-1})$ initial mismatch and these residual sums without amplification, which proves (56) for every terminal rank in the corridor.

The split rank and the number of Darboux orders behind hypothesis 2 are set by the target global order, not by a special-function identity: $O(h^4k^6)$ sums to $O(h^4K^7) = O(h^{p-1})$ at $K \asymp h^{-(5-p)/7}$, where $O(k^{-R})$ sums to $O(K^{-(R-1)})$ and $R = 8$ ($p = 3$) or $R = 22$ ($p = 4$) is the smallest integer giving $O(h^{p-1})$. □

**Proposition 3.19** (The engine hypotheses for even centered chains)**.** *Let an even centered Wigner–Jacobi chain be given, and fix a closed turning-free corridor. For the corrected pullback (49) there exist $k_0, \eta_0, C > 0$ such that*

$$|\mathcal{R}_k(\widehat{U}^{[2]})| \le Ch^4\kappa^6, \qquad \kappa^3h^2 \le \eta_0, \tag{57}$$

$$|\mathcal{R}_k(\widehat{U}^{[2]})| \le C(h^3 + k^{-8}), \qquad k_0 \le k \le L-1. \tag{58}$$

*The exact naturally gauged fundamental matrix and its inverse are uniformly bounded on the same corridor, and the fixed starting entries of the exact and corrected physical branches differ by $O(h^2)$.*

*Proof.* The fixed-rank analytic bound (57) follows from Proposition 3.17 and Cauchy's estimate on the disk $|t| \le c\kappa^{-3}$, on which the residual is analytic and vanishes to second order at $t = 0$. It is also bounded there: $q^2 = \kappa^2 t = O(\kappa^{-1})$ and $\Theta_q^{[2]} - \theta = O(\kappa^{-1})$, so $\kappa|\Im\Theta_q^{[2]}| = O(1)$, and the interior Wigner–Darboux bounds, which continue to complex angles with $\kappa|\Im\vartheta| = O(1)$ (Lemma D.1), bound $\sqrt{2k+1}\, d_{ab}^k(\Theta_q^{[2]})$ uniformly. Appendix B proves the fixed-ratio $O(h^3)$ residual of the corrected pullback (Lemma B.1, a direct Taylor expansion of the Darboux branch that uses only (E1), Lemma 3.23 and Propositions 3.15 and 3.16) and the Darboux–Weyl corner matching that combines it with Proposition 3.17 to yield (58) (Corollary B.4). Turning-free stability is Corollary 3.12, and the statement about starting entries follows from (E2) and Proposition 3.16. □

**Theorem 3.20** (Second-order all-rank effective-angle pullback)**.** *Let an even centered Wigner–Jacobi chain be given, and fix a closed turning-free corridor $\mathcal{C}$. Then, for every admissible rank $L$ whose swept path stays in $\mathcal{C}$,*

$$\boxed{\sqrt{2L+1}\,|Y_L - W_q^{[2]}\, d_{ab}^L(\Theta_q^{[2]})| \le \frac{C_{\mathcal{C}}}{N^2}, \qquad q = \frac{L+\frac{1}{2}}{N}.} \tag{59}$$

*The constant includes the finitely many admissible ranks below the fixed starting rank of Theorem 3.18. At fixed positive ratio this is the absolute error $O(N^{-5/2})$.*

*Proof.* Proposition 3.19 verifies the $p = 3$ hypotheses of Theorem 3.18 for $U_k = U_k^{[2]}$; fixed ranks below its stability threshold are covered by the analytic seed (E2) and Proposition 3.16. Applying the engine and dividing by $\sqrt{2L+1}$ proves the estimate. □

**Corollary 3.21** (First-order all-rank effective-angle pullback)**.** *In the setting of Theorem 3.20, for every admissible rank $L$ whose swept path stays in $\mathcal{C}$,*

$$\boxed{\sqrt{2L+1}\,|Y_L - W_q\, d_{ab}^L(\Theta_q)| \le C_{\mathcal{C}}\,\frac{L+1}{N^2}, \qquad q = \frac{L+\frac{1}{2}}{N}.} \tag{60}$$

*At fixed positive ratio this is $O(N^{-1})$ in natural gauge: the leading pullback misses the accumulated phase $h\Gamma_2(q,x)$ (Remark 3.22).*

*Proof.* By Proposition 3.16, $\rho_2$ is continuous on the closed swept paths in $\mathcal{C}$, so $\Gamma_2/q$ is bounded on $\mathcal{C}$, so $\Theta_q^{[2]}-\Theta_q = h^2\Gamma_2/q = O(N^{-2})$ and $W_q^{[2]}-W_q = O(N^{-2})$, and for large $N$ both angles lie in a fixed compact $J \Subset (0,\pi)$. On $J$, $\sqrt{2L+1}\,|d_{ab}^L| \le C$ and $\sqrt{2L+1}\,|\partial_\vartheta d_{ab}^L| \le C\kappa$ (Lemma 3.6; the finitely many low ranks are bounded directly). Hence $\sqrt{2L+1}\,|W_q^{[2]} d_{ab}^L(\Theta_q^{[2]}) - W_q d_{ab}^L(\Theta_q)| \le C\kappa N^{-2}$, and Theorem 3.20 gives (60). □

*Remark* 3.22 (Transport curvature and the first odd fixed-ratio correction). The integral of the frozen shadow phase $p_2$ diverges and double-counts the finite-index Darboux phase. Subtracting the pole (Lemma A.4) leaves a bounded frozen density, but it tracks only the frozen characteristic phase. The additional terms in (46) are the Weyl transport-curvature contribution; using $\rho_2$ lowers the local residual by one full power of $h$. The exact centered operator is even in $h$, but a nonzero ratio $q = (L+\frac{1}{2})h$ couples the rank to $h^{-1}$, so the leading pullback misses the accumulated phase $(L+\frac{1}{2})(\Theta_q^{[2]} - \Theta_q) = h\,\Gamma_2(q,x)$. Thus the physical fixed-ratio expansion need not be even in $h$, although the operator and every fixed-rank coefficient are; including this odd correction improves the natural-gauge remainder from $O(h)$ to $O(h^2)$ uniformly from fixed rank through fixed ratio.

### 3.6 Phase exactness

The mechanism of phase exactness is parity: the exact operator is invariant under $h \mapsto -h$ at fixed $q$, and the Wigner–Darboux expansion has an even amplitude and an odd phase in $\kappa^{-1}$. Hence the first residual of the corrected pullback that survives at fixed ratio, at order $h^3$, is of transport type and is removed by a real amplitude factor; no order-$h^2$ phase is produced.

**Lemma 3.23** (Parity of the Wigner–Darboux expansion)**.** *Fix offsets $a, b$, both integers or both half-integers (so $a-b \in \mathbb{Z}$, as for Wigner $d$), and $J \Subset (0,\pi)$. The branch amplitudes of (26) can be written*

$$A_{r,ab}^{\pm}(\vartheta) = \omega_{ab}^{\pm 1}(\pm i)^r\, \mathfrak{a}_{r,ab}(\vartheta), \qquad \mathfrak{a}_{r,ab} \text{ real}, \qquad \mathfrak{a}_{0,ab} = (\pi \sin\vartheta)^{-1/2}, \tag{61}$$

*with $\omega_{ab}$ as in Lemma 3.6. Equivalently, for every $R$,*

$$\sqrt{\tfrac{\pi}{4}\sin\vartheta}\,\sqrt{2k+1}\,d_{ab}^k(\vartheta) = \epsilon_{ab}\,\mathfrak{A}_{ab}(\vartheta,\kappa)\cos\Big(\kappa\vartheta + \frac{p_{ab}\pi}{2} - \frac{\pi}{4} + \mathfrak{P}_{ab}(\vartheta,\kappa)\Big) + O_J(\kappa^{-R}), \tag{62}$$

*with $\epsilon_{ab}$ as in (17). Here $\mathfrak{A}_{ab} = 1 + \sum_{1\le j<R/2}\kappa^{-2j}\mathfrak{A}_{2j}(\vartheta)$ is even, and $\mathfrak{P}_{ab} = \mathfrak{c}_{ab}(\vartheta)/\kappa + \sum_{1\le j<R/2}\kappa^{-2j-1}\mathfrak{P}_{2j+1}(\vartheta)$ is odd in $\kappa^{-1}$, with smooth real coefficients. In particular the interior Darboux expansion has no $\kappa^{-1}$ amplitude term and no $\kappa^{-2}$ phase term.*

*Proof.* By the two-carrier uniqueness fact in the proof of Lemma B.3 (Appendix B) the coefficients $A_{r,ab}^{\pm}$ are determined by $d_{ab}^k$, and $A_{r,ab}^{-} = \overline{A_{r,ab}^{+}}$ because $d_{ab}^k$ is real; so it suffices to exhibit one expansion of the stated form. Since $J$ is compact, $J \subset (0, \pi-\delta]$ for some $\delta > 0$. Write $d_{ab}^k$ through the Jacobi polynomial $P_n^{(\alpha,\beta)}$ with $\alpha = p_{ab} = |a-b|$, $\beta = |a+b|$, $n = k - \max(|a|,|b|)$, so that $n+(\alpha+\beta+1)/2 = \kappa$. The Bessel-type uniform expansion of Frenzen and Wong, DLMF 18.15.6 [26, 49], gives

$$\begin{aligned}(\sin\tfrac{\vartheta}{2})^{\alpha+\frac12}(\cos\tfrac{\vartheta}{2})^{\beta+\frac12}P_n^{(\alpha,\beta)}(\cos\vartheta)& \\ = \frac{\Gamma(n+\alpha+1)}{2^{1/2}n!\,\kappa^{\alpha}}\Big[\vartheta^{1/2}\sum_m \frac{\mathsf{A}_m(\vartheta)J_\alpha(\kappa\vartheta)}{\kappa^{2m}} + \vartheta^{3/2}\sum_m \frac{\mathsf{B}_m(\vartheta)J_{\alpha+1}(\kappa\vartheta)}{\kappa^{2m+1}}\Big]&\end{aligned} \tag{63}$$

to any finite order, uniformly for $\vartheta \in [0, \pi-\delta]$, with real $\mathsf{A}_m, \mathsf{B}_m$ and remainder $O(\kappa^{-R})$ relative to the leading term on $J$. The Wigner normalization contributes $\sqrt{(n+\alpha)!(n+\alpha+\beta)!/(n!(n+\beta)!)}\cdot\kappa^{-\alpha}$

after combining with $\Gamma(n+\alpha+1)/n!$. Each of the factorial quotients $(n+\alpha+\beta)!/n!$ and $(n+\alpha)!/(n+\beta)!$ (or its reciprocal) is a product of linear factors $\kappa+c$ over a set of shifts $c$ symmetric about 0, hence of the parity of their number, $\alpha+\beta$ and $|\alpha-\beta|$ respectively; these numbers have equal parity (both odd for half-integer offsets), so the product of the two is an even rational function of $\kappa$ with leading term $\kappa^{2\alpha}$; its square root times $\kappa^{-\alpha}$ is a convergent series in $\kappa^{-2}$. Finally the Hankel expansion of $J_\nu(\kappa\vartheta)$ ([50, 10.17.3]) is $\sqrt{2/(\pi\kappa\vartheta)}[\mathcal{P}_\nu\cos\varpi_\nu-\mathcal{Q}_\nu\sin\varpi_\nu]$ with $\varpi_\nu=\kappa\vartheta-\nu\pi/2-\pi/4$, $\mathcal{P}_\nu$ even and $\mathcal{Q}_\nu$ odd in $\kappa^{-1}$. Since $\varpi_{\alpha+1}=\varpi_\alpha-\pi/2$, collecting terms gives $\sqrt{2k+1}\,d^k_{ab}(\sin\vartheta)^{1/2}=(2/\sqrt{\pi})\,[\mathcal{C}\cos\varpi_\alpha+\mathcal{S}\sin\varpi_\alpha]$ with $\mathcal{C}$ even and $\mathcal{S}$ odd in $\kappa^{-1}$, $\mathcal{C}=\pm1+O(\kappa^{-2})$; the factor $\sqrt{2\kappa}$ of the natural gauge cancels the $\kappa^{-1/2}$ of the Hankel expansion, so no fractional powers occur. Writing $\mathcal{C}\cos\varpi+\mathcal{S}\sin\varpi=\frac12\sum_\pm e^{\pm i\varpi}(\mathcal{C}\mp i\mathcal{S})$ gives (61), and $\mathfrak{A}=(\mathcal{C}^2+\mathcal{S}^2)^{1/2}$, $\mathfrak{P}=-\arctan(\mathcal{S}/\mathcal{C})$ give the amplitude–phase form; $\varpi_\alpha$ and $\kappa\vartheta+p_{ab}\pi/2-\pi/4$ differ by $p_{ab}\pi$, a sign, and comparison of the leading terms with (17) shows that the overall sign is $\epsilon_{ab}$. The coefficient of $\kappa^{-1}$ in $\mathfrak{P}$ is fixed by (24). □

On the continuum variable $q$, with $a(s;h^2)=A^{(0)}(s/h)e_\pi(s^2,h^2)$, $A^{(0)}(s/h)=\frac12[(1-a^2h^2/s^2)(1-b^2h^2/s^2)/(1-h^2/4s^2)]^{1/2}$ and $b(q;h^2)=\lambda_\pi(h^2)abh^2/(q^2-h^2/4)+v_\pi(q^2,h^2)$, the midpoint operator

$$(\mathcal{L}_hu)(q)=\big(x-b(q;h^2)\big)u(q)-a(q+\tfrac{h}{2};h^2)u(q+h)-a(q-\tfrac{h}{2};h^2)u(q-h) \tag{64}$$

reproduces the natural-gauge residual on the lattice $q=q_k$, and $\mathcal{L}_{-h}=\mathcal{L}_h$ exactly. By Lemma 3.23 the corrected pullback splits into two Darboux branches,

$$\widehat{U}^{[2],\pm}_k:=W^{[2]}_{q_k}(x)\,e^{\pm i\kappa\Theta^{[2]}_{q_k}}\sum_{r<R}\kappa^{-r}A^\pm_{r,ab}(\Theta^{[2]}_{q_k}),\qquad \widehat{U}^{[2]}=\widehat{U}^{[2],+}+\widehat{U}^{[2],-}+O(\kappa^{-R}). \tag{65}$$

**Theorem 3.24** (Phase exactness of the second-order pullback)**.** *Let an even centered Wigner–Jacobi chain be given; in the exceptional channel $a=b=0$ assume also that the physical solution satisfies the $k=0$ row $x\widehat{Y}_0=A_1\widehat{Y}_1+B_0\widehat{Y}_0$ with $B_0=v_\pi(h^2/4,h^2)$ (true for the Hahn, centered Racah, and $q$-6$j$ chains, by (91), (115), and Proposition 6.1(a)). Fix a closed turning-free corridor $\mathcal{C}$. There is a real function $\alpha(q,x)$, analytic and even in $q$ and analytic in $x$ on a complex neighbourhood of $\mathcal{C}$ (here the joint analyticity in (E2) is used, Remark 3.3), such that for every admissible rank $L$ whose swept path stays in $\mathcal{C}$,*

$$\boxed{\sqrt{2L+1}\,\Big|Y_L-(1+h^2\alpha(q,x))W^{[2]}_q(x)\,d^L_{ab}(\Theta^{[2]}_q(x))\Big|\le\frac{C_{\mathcal{C}}}{N^3},\qquad q=\frac{L+\frac12}{N}.} \tag{66}$$

*Explicitly,*

$$\alpha(q,x)=c(x)+\mathsf{J}_2(0,x)-\mathsf{J}_2(q,x),\qquad 2\mathsf{J}_2=\frac{\mathfrak{c}_{ab}(\Theta_q)\cot\Theta_q}{q^2}+\cot p\,\Big(\Sigma_2+\frac{p_{qq}}{24}\Big)-\frac{(\log D)_{qq}}{16}+\eta_\pi(q^2), \tag{67}$$

*with $p,D,\Theta_q$ from (21), $\Sigma_2$ from (48) (which contains $\eta_\pi\cot p$) and $\eta_\pi$ from (20); the last term is zero for Hahn and Racah. Here $c(x)$ is the fixed-rank datum: the coefficients of $t=h^2$ in $\widehat{Y}_k$ and $\widehat{U}^{[2]}_k$ differ by $c(x)\widehat{d}_k$ for every $k$. The function $\mathsf{J}_2$ is even and analytic in $q$, with*

$$\begin{aligned}\mathsf{J}_2(0,x)=-\frac{1}{24(1-x^2)^2}\Big[&\varepsilon_1\big((8(a^2+b^2)-5)x^2+4abx(x^2-5)-3\big)\\&+8\beta_1\big((1-a^2-b^2)x+ab(1+x^2)\big)-12\beta_2x(1-x^2)\Big]+\frac{\lambda_1}{2(1-x^2)},\end{aligned} \tag{68}$$

*in the notation (20).*

The theorem says that the $O(N^{-2})$ remainder of Theorem 3.20 is, up to $O(N^{-3})$, a real multiple of the corrected pullback itself: the effective angle $\Theta_q^{[2]}$ carries no phase error at order $N^{-2}$. The quantity $\frac{1}{2} + h^2\mathsf{J}_2$ is the flux $\pi a\,\Im(\overline{u(q-\frac{h}{2})}u(q+\frac{h}{2}))$ of the complex Darboux branch $u$ of the corrected pullback, and (67) says that the flux of $(1+h^2\alpha)u$ is constant to order $h^2$. The flux identity behind (67) is Lemma 3.25. The *flux defect* $\delta_{\mathrm{fl}}(x) := c(x) + \mathsf{J}_2(0,x)$ is thus the constant part of $\alpha + \mathsf{J}_2$, $\alpha = \delta_{\mathrm{fl}} - \mathsf{J}_2$. For the Hahn and centered Racah chains $\delta_{\mathrm{fl}} = 0$, i.e. $c(x) = -\mathsf{J}_2(0,x)$ and $\alpha = -\mathsf{J}_2$ (Corollaries 4.4 and 5.3); for the one-small quantum $6j$ chain $\delta_{\mathrm{fl}} = -\frac{1}{12}$ in units of the curvature scale squared (Corollary 6.3). For a general chain $c(x)$ depends on the normalization of the physical solution: multiplying the physical solution by $1 + h^2 f(x)$ with $f$ bounded and analytic on $U$ keeps (E1)–(E2) and replaces $\alpha$ by $\alpha + f$ (for $f$ not analytic, (E2) fails and $\alpha$ is no more regular than $f$, Remark 3.3; in the exceptional channel $a = b = 0$, omitting row $k = 0$ permits an $O(h^2)$ admixture of the second Legendre solution $\widehat{Q}_k$, shifting the phase at order $h^2$).

**Lemma 3.25** (Flux of the corrected pullback)**.** *On a closed turning-free corridor let $u = \widehat{U}^{[2],+}$ be the + Darboux branch (65) of the corrected pullback, regarded as a function of continuous $q > 0$ at fixed $h$ with $\kappa = q/h$, and put*

$$\mathsf{J}(s) := \pi\, a(s;h^2)\,\Im\big(\overline{u(s-\tfrac{h}{2})}\,u(s+\tfrac{h}{2})\big). \tag{69}$$

*(i)* $\pi\,\Im(\overline{u}\,\mathcal{L}_h u)(q) = -\big[\mathsf{J}(q+\frac{h}{2}) - \mathsf{J}(q-\frac{h}{2})\big]$ *exactly.*

*(ii)* $\mathsf{J}$ *is even in $h$, and at fixed $s > 0$,* $\mathsf{J}(s) = \frac{1}{2} + h^2\mathsf{J}_2(s,x) + O(h^4)$ *with* $\mathsf{J}_2$ *given by (67).*

*(iii)* $\mathsf{J}_2$ *extends to an even analytic function of $s$ at $s = 0$, with* $\mathsf{J}_2(0,x)$ *given by (68) in the notation (20).*

*(iv) With $C_3^+$, $B_0$ the leading coefficients of $\mathcal{L}_h u$ and $u$ at fixed ratio (Lemma C.1),* $\pi q^{-3}\Im(\overline{B_0}C_3^+) = -\partial_q \mathsf{J}_2(q,x)$.

*Proof.* (i) In $\overline{u}\,\mathcal{L}_h u$ the diagonal term $(x-b)|u|^2$ is real, and the two hopping terms are $-a(q\pm\frac{h}{2})\overline{u(q)}u(q\pm h)$; since $\Im(\overline{u(q)}u(q-h)) = -\Im(\overline{u(q-h)}u(q))$, their imaginary parts are $-\pi^{-1}\mathsf{J}(q+\frac{h}{2})$ and $+\pi^{-1}\mathsf{J}(q-\frac{h}{2})$.

(ii) Since $\Theta^{[2]}$ and $D$ depend on $h^2$ only, Lemma 3.23 gives $u(q;-h) = \omega_{ab}^2\overline{u(q;h)}$ (an identity of the explicit finite branch expressions, with $\kappa = q/h$ of either sign; see the proof of Lemma C.1), and $a$ is even in $h$; substituting gives $\mathsf{J}(s;-h) = \pi a\,\Im(u(s+\frac{h}{2};h)\overline{u(s-\frac{h}{2};h)}) = \mathsf{J}(s;h)$. By Lemma 3.23, up to $O(\kappa^{-R})$,

$$u = \omega_{ab}\pi^{-1/2}D^{-1/4}\mathfrak{A}_{ab}(\Theta_q^{[2]},\kappa)\exp\{i(\kappa\Theta_q^{[2]} + \mathfrak{P}_{ab}(\Theta_q^{[2]},\kappa))\}, \tag{70}$$

where $\kappa\Theta_q^{[2]} = S/h + h\Gamma_2$, $\mathfrak{P}_{ab} = h\,\mathfrak{c}_{ab}(\Theta_q)/q + O(h^3)$ and $\mathfrak{A}_{ab} = 1 + h^2\mathfrak{A}_2(\Theta_q)/q^2 + O(h^4)$. The phase increment is therefore $p(s) + h^2(p_{ss}/24 + \widetilde{\Gamma}_s) + O(h^4)$ with $\widetilde{\Gamma} = \Gamma_2 + \mathfrak{c}_{ab}(\Theta_q)/q$, and $\widetilde{\Gamma}_s = \Sigma_2$ by (48); the amplitude product is $\pi^{-1}D^{-1/2}[1 + h^2(2\mathfrak{A}_2(\Theta_s)/s^2 - (\log D)_{ss}/16)] + O(h^4)$; $a(s;h^2) = \frac{1}{2}e_\pi(s^2)(1 + h^2(c_{ab}/s^2 + \eta_\pi(s^2))) + O(h^4)$ with $c_{ab} = \frac{1}{8} - \frac{a^2+b^2}{2}$; and $D^{1/2} = e_\pi \sin p$. Multiplying out,

$$2\mathsf{J}_2 = \frac{c_{ab} + 2\mathfrak{A}_2(\Theta_s)}{s^2} + \cot p\,\Big(\Sigma_2 + \frac{p_{ss}}{24}\Big) - \frac{(\log D)_{ss}}{16} + \eta_\pi(s^2). \tag{71}$$

For the Wigner chain itself ($e_\pi = 1$, $v_\pi = 0$, so $\eta_\pi = 0$) one has $p = \Theta_s = \vartheta$, $D = \sin^2\vartheta$ and $\Sigma_2 = -\mathfrak{c}_{ab}(\vartheta)/s^2$, and $u$ is the + Darboux branch of $\widehat{d}$, a formal solution of the Wigner recurrence by the two-carrier uniqueness in the proof of Lemma B.3; by (i) its $\mathsf{J}$ is independent of $s$, which

forces $c_{ab} + 2\mathfrak{A}_2(\vartheta) = \mathfrak{c}_{ab}(\vartheta)\cot\vartheta$, i.e. $\mathfrak{A}_2 = -\frac{1}{2}\mathfrak{c}'_{ab}$ (by the identity $c_{ab} - \mathfrak{c}'_{ab} = \mathfrak{c}_{ab}\cot$). Substituting this gives (67).

(iii) By Proposition 3.16 every term of (67) is even meromorphic in $s$ with at most a double pole, and the double poles $\mathfrak{c}_{ab}(\theta)\cot\theta/s^2$ and $\cot\theta\cdot(-\mathfrak{c}_{ab}(\theta)/s^2)$ cancel. Expanding with (22), $\eta_\pi(0) = \lambda_1$ and $\cot p \to x/\sqrt{1-x^2}$ gives (68): the new terms are $\lambda_1 x\cdot x/(2(1-x^2))$ from the shift $\beta_2 \to \beta_2 + \lambda_1 x$ and $\lambda_1/2$ from $\eta_\pi$, which add to $\lambda_1/(2(1-x^2))$. As for $\rho_2(0,x)$, the higher jets of $e_\pi, v_\pi$ enter only at order $s^2$.

(iv) At fixed $q > 0$, $\mathcal{L}_h u = e^{i\Theta_q/z}[h^3 q^{-3} C_3^+ + O(h^4)]$ (with $z := h/q = \kappa^{-1}$) and $u = e^{i\Theta_q/z}[B_0 + O(h)]$, so the left side of (i) is $h^3\pi q^{-3}\Im(\overline{B_0}C_3^+) + O(h^4)$, while by (ii) the right side is $-h^3\partial_q \mathsf{J}_2 + O(h^5)$. □

*Proof of Theorem 3.24.* By Lemma C.1 (Appendix C) the order-$h^3$ fixed-ratio residual of $\widehat{U}^{[2],+}$ is $e^{i\Theta_q/z}h^3 q^{-3}C_3^+$ with $C_3^+/B_0 \in i\mathbb{R}$, which a real factor $1 + h^2\alpha$ cancels exactly when $\partial_q\alpha = \pi q^{-3}\Im(\overline{B_0}C_3^+)$; by Lemma 3.25(iv) this is $-\partial_q \mathsf{J}_2$. The initial value $\alpha(0,x) = c(x)$ matches the fixed-rank Taylor coefficient, an order-four corner bound controls the corrected residual from fixed rank to fixed ratio, and the engine summation with split rank $h^{-1/7}$ gives the $O(h^3)$ bound. The details are in Appendix C. □

### 3.7 Bulk zeros and rank-direction turning points

Bulk zero quantization needs no further hypothesis beyond Theorem 3.24; the Airy connection adds hypothesis (R) and a lattice quantization of the turning action.

**Theorem 3.26** (Bulk zero quantization)**.** *Let an even centered Wigner–Jacobi chain be given; in the exceptional channel $a = b = 0$ assume also the $k = 0$ row of Theorem 3.24. Fix a compact turning-free set of $(q,x,\pi)$ with $q$ bounded away from zero and swept paths in a turning-free corridor $\mathcal{C}$, and put $\widetilde{\Gamma}(q,x) := \Gamma_2(q,x) + \mathfrak{c}_{ab}(\Theta_q(x))/q$. After shrinking the complex neighbourhood $U$ of (E2) if necessary, for all sufficiently large admissible $N$ the lowest-rank seed $G_N(x) := Y_{k_{\min}}(x;h)$ is analytic and zero-free on $U$ (Remark 3.3) and $Y_L(x;h) = G_N(x)P_{L,N}(x)$ with $P_{L,N}$ a polynomial in $x$. Moreover, each zero $\widetilde{x}_j$ of $x \mapsto d^L_{ab}(\Theta_q^{[2]}(x))$, $q = (L+\frac{1}{2})/N$, in that set has exactly one zero $x_j$ of $P_{L,N}$ within $O(N^{-4})$ (all zeros of $P_{L,N}$ are real and simple); every zero of $P_{L,N}$ in the set lies within $O(N^{-4})$ of exactly one zero $\widetilde{x}_j$ (possibly in an $O(N^{-4})$ collar outside the set); and, with $p_{ab} = |a-b|$ and a consistent integer index $\nu_j$,*

$$\boxed{x_j = \widetilde{x}_j + O(N^{-4}), \qquad NS(q,x_j) + h\widetilde{\Gamma}(q,x_j) + \frac{p_{ab}\pi}{2} - \frac{\pi}{4} = \left(\nu_j + \frac{1}{2}\right)\pi + O(N^{-3}).} \tag{72}$$

*Consequently $x_j$ lies within $O(N^{-2})$ of the zero of $d^L_{ab}(\Theta_q(x))$ with the same index, and $NS(q,x_j) + p_{ab}\pi/2 - \pi/4 = (\nu_j+\frac{1}{2})\pi + O(N^{-1})$. Both parts of $h\widetilde{\Gamma}$ are needed at order $N^{-1}$: the Bohr–Sommerfeld condition with $h\Gamma_2$ but without the Darboux term $h\,\mathfrak{c}_{ab}(\Theta_q)/q = \mathfrak{c}_{ab}(\Theta_q)/\kappa$ is in error at order $N^{-1}$ wherever $\mathfrak{c}_{ab}(\Theta_q) \neq 0$. Theorem 3.20 alone, without Theorem 3.24, gives the same statements with $O(N^{-3})$ and $O(N^{-2})$ (assuming the $k = 0$ row when $a = b = 0$ for the polynomial factorization).*

*Proof.* On $\mathcal{C}$, $e_\pi(k^2h^2, h^2) > 0$ for $N$ large, so $A_k(h) > 0$ for $k_{\min} < k \le L$; the rank recurrence has off-diagonal coefficients independent of $x$ and diagonal coefficients affine in $x$ (including the $k = 0$ row when $a = b = 0$), so division by the nonvanishing seed gives a polynomial in $x$ by induction. Every real-axis estimate above continues to a strip $|\Im x| \le c/N$ with the same order, by a two-constants argument for the harmonic measure of a half-disc, because the exact solution and the pullbacks have exponential type $O(N)$ in $x$ (Lemmas D.2 and D.3). Cauchy's formula then

turns the $O(N^{-3})$ natural-gauge bound of Theorem 3.24 into an $O(N^{-2})$ bound on the $x$-derivative, while the comparison $(1+h^2\alpha)W_q^{[2]}d_{ab}^L(\Theta_q^{[2]})$ has the zeros $\widetilde{x}_j$ with slope of order $N$. This gives the zero correspondence; the quantization condition is the zero condition of the parity form of the Darboux expansion with $\kappa\Theta_q^{[2]} = NS + h\Gamma_2$. The details are in Appendix D. □

At a rank-direction turning point the leading pullback degenerates. Suppose that for fixed $(\pi, x)$ there is $q_*$ with $q_*^2 < s_1$,

$$D(q,x) > 0 \ (0 \le q < q_*), \qquad D(q_*,x) = 0, \qquad \partial_q D(q_*,x) < 0, \tag{73}$$

uniformly on a compact family (a *simple rank turn*). Then $u(q) := (x - b_0(q))/e_\pi(q^2)$ satisfies $u(q_*) = \tau \in \{+1,-1\}$, the *turn type*. For $\tau = +1$ define

$$\zeta_+(q) = \begin{cases} -\left[\dfrac{3}{2}\displaystyle\int_q^{q_*} p(s,x)\,\mathrm{d}s\right]^{2/3}, & q \le q_*, \\[2ex] \left[\dfrac{3}{2}\displaystyle\int_{q_*}^{q} \operatorname{arcosh} u(s)\,\mathrm{d}s\right]^{2/3}, & q \ge q_*, \end{cases} \tag{74}$$

and for $\tau = -1$ define $\zeta_-$ by the same formula with $p$ replaced by $\pi - p$ and $u$ by $-u$. Then $\zeta_\tau(q) = (-\partial_q D(q_*,x)/e_\pi(q_*^2)^2)^{1/3}(q-q_*) + O((q-q_*)^2)$, so $|\zeta_\tau/D|^{1/4}$ extends continuously and positively through the turn.

**Theorem 3.27** (Rank-direction simple-turn Airy connection)**.** *Let an even centered Wigner–Jacobi chain have a simple rank turn (73) of fixed type $\tau$ on a compact family, and assume the turning action is lattice-quantized: for every admissible $(L,N)$ in the family, with $\kappa = L + \frac{1}{2}$,*

$$K_+ := \frac{NS(q_*,x)}{\pi} + \frac{p_{ab}-1}{2} \in \mathbb{Z} \quad (\tau = +1), \qquad K_- := \kappa + \frac{NS(q_*,x)}{\pi} - Nq_* + \frac{p_{ab}}{2} \in \mathbb{Z} \quad (\tau = -1). \tag{75}$$

*(i) Uniformly for $|N^{2/3}\zeta_\tau(q)| \le A_0$ with $q = \kappa/N$,*

$$\boxed{Y_L = \epsilon_{ab}(-1)^{K_\tau}\sqrt{\frac{2}{q}}\,N^{-1/3}\left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4}\left[\mathrm{Ai}\Big(N^{2/3}\zeta_\tau(q)\Big) + O_{A_0}(N^{-1})\right].} \tag{76}$$

*(ii) Assume also that the physical solution is polynomially bounded on the forbidden side of the turn:*

*(R) there are $\delta, m, C > 0$, uniform on the family, with $|\widehat{Y}_k| \le CN^m$ whenever $q_* \le q_k \le q_* + \delta$.*

*Put $\widetilde{\Gamma} = \Gamma_2 + \mathfrak{c}_{ab}(\Theta_q)/q$, let $I = [q_* - \delta_0, q_* + \delta_0]$ be the fixed turning neighbourhood of the canonical Airy basis (Appendix E), shrunk if necessary so that $\delta_0 \le \delta$, and $I' = \{q \in I : \zeta_\tau(q) \le 4^{-2/3}\zeta_\tau(q_* + \delta_0)\}$. There is a real-analytic function $B_\tau(q) = B_\tau(q,x)$ on $I$ with*

$$B_\tau(q) = \frac{\tau\,\widetilde{\Gamma}(q,x)}{\sqrt{-\zeta_\tau(q)}} - \frac{5}{48\,\zeta_\tau(q)^2} \qquad (q < q_*), \tag{77}$$

*so that the limit*

$$\varpi_\tau(x) := B_\tau(q_*) = \lim_{q \uparrow q_*}\left[\frac{\tau\,\widetilde{\Gamma}(q,x)}{\sqrt{-\zeta_\tau(q)}} - \frac{5}{48\,\zeta_\tau(q)^2}\right] \tag{78}$$

*exists and equals the local expression*

$$\varpi_\tau(x) = \frac{2^{1/3}}{280\, q_*^2 u_1^{2/3}}\Big[35(4a^2+4b^2-1) - \frac{280\tau ab}{e_0} + q_*^2\Big(4u_1^2 - 8u_2 + \frac{60u_3}{u_1} - \frac{36u_2^2}{u_1^2} + \frac{35(2e_0e_2-e_1^2)}{e_0^2} - \frac{280\tau v_1}{e_0} - 280\,\eta_\pi(q_*^2)\Big)\Big] \tag{79}$$

*in the Taylor data of the class at the turn, $\tau u(q_*+\delta) = 1 + u_1\delta + u_2\delta^2 + u_3\delta^3 + O(\delta^4)$ (so $u_1 > 0$), $e_\pi((q_*+\delta)^2) = e_0 + e_1\delta + e_2\delta^2 + O(\delta^3)$ for the principal coefficient, $v_1 = \partial_t v_\pi(q_*^2, 0)$ and $\eta_\pi$ from (20) ($\eta_\pi \equiv 0$ for Hahn and Racah). Uniformly at admissible ranks with $q = \kappa/N \in I'$, on both sides of the turn,*

$$\boxed{\begin{aligned} Y_L &= \epsilon_{ab}(-1)^{K_\tau}\sqrt{\frac{2}{q}}\,N^{-1/3}\left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4} \\ &\quad\times\Big[(1+\mathfrak{a}_N(q))\,\mathrm{Ai}(t) + N^{-4/3}B_\tau(q)\,\mathrm{Ai}'(t) + \mathfrak{r}_N(q)\Big], \qquad t = N^{2/3}\zeta_\tau(q), \end{aligned}} \tag{80}$$

*where $\mathfrak{a}_N = N^{-2}\mathfrak{a}(q) + c_N$ with $\mathfrak{a}$ real-analytic on $I$ and constants $|c_N| \le CN^{-2}$, and $|\mathfrak{r}_N(q)| \le CN^{-3}\mathsf{M}(t)/\mathsf{E}(t)$, with Olver's modulus and weight functions $\mathsf{M}, \mathsf{E}$ of the Airy functions [46, Chapter 11]. Equivalently the bracket is $(1+O(N^{-2}))\,\mathrm{Ai}(N^{2/3}\widehat{\zeta}_\tau) + O(N^{-3})\mathsf{M}/\mathsf{E}$ with the second-order Langer variable $\widehat{\zeta}_\tau = \zeta_\tau + N^{-2}B_\tau$. In particular, uniformly for $|N^{2/3}\zeta_\tau(q)| \le A_0$, with the prefactor of (76),*

$$Y_L = \epsilon_{ab}(-1)^{K_\tau}\sqrt{\frac{2}{q}}\,N^{-1/3}\left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4}\Big[\mathrm{Ai}(t) + N^{-4/3}\varpi_\tau(x)\,\mathrm{Ai}'(t) + O_{A_0}(N^{-2})\Big]. \tag{81}$$

*The error terms in (76) and (81) are additive inside the brackets, and $\mathfrak{r}_N$ is measured against Olver's envelope; no relative-error assertion is made at zeros of* Ai.

*Proof.* (i) The exact recurrence satisfies the uniform simple-turning hypotheses of Geronimo–Bruno–Van Assche [42], which yield an exact canonical Ai / Bi basis in the coordinate $\zeta_\tau$ with the transport $|\zeta_\tau/D|^{1/4}$ and $O(N^{-1})$ envelope error; we use only the non-recessive envelope $(\mathrm{Ai}^2 + \mathrm{Bi}^2)^{1/2}$ of [48, Theorem 2.6]. The two connection coefficients are fixed by matching a two-component Jacobi vector at a fixed oscillatory rank ratio $q_0 < q_*$ to the fixed-ratio case of Corollary 3.21; (75) converts the bulk Wigner phase into $\pm$ the Airy phase (Appendix E).

(ii) Parity of the even midpoint operator gives a formal Airy parametrix $(g + h^2 p_2)\,\mathrm{Ai} + h^{4/3}q_1\,\mathrm{Ai}'$ with residual $O(h^4)\mathsf{M}/\mathsf{E}$ on $I$. A discrete Green function built from the canonical Bi solution and a recessive exact solution, constructed on the forbidden side from a final-value Volterra sum, turns it into an exact solution within $O(h^3)\mathsf{M}/\mathsf{E}$ of the parametrix on $I$. Condition (R) makes the Bi coefficient of the physical solution $O(N^{-\infty})$ relative to $\mathsf{M}/\mathsf{E}$ on $I'$, and its Ai scalar is fixed to relative $O(N^{-2})$ by amplitude matching with Theorem 3.20; this gives (80) with $B_\tau = q_1/g$. Phase matching at every oscillatory point gives (77); the transport equation for $q_1$ at $q_*$, where its leading coefficient vanishes, gives (79), and rewriting it in the jets of $F$ gives (82) (Appendix E.1, Lemma E.7). □

In the window of part (i) the physical solution is the normalized Ai branch, with no additional connection phase; a Bi component of relative size $O(N^{-1})$ is not excluded, and since it grows beyond the window, recessivity on the forbidden side needs (R) of part (ii) (Remarks 3.29(a) and 3.30).

The local expression (79) has two equivalent forms. With $F := \operatorname{arcosh}^2(\tau u) = \zeta_\tau \zeta_\tau'^2$ and $F(q_* + \delta) = F_1\delta + F_2\delta^2 + F_3\delta^3 + O(\delta^4)$,

$$\varpi_\tau = \frac{3(5F_1F_3 - 3F_2^2)}{35\,F_1^{8/3}} + \frac{1}{F_1^{2/3}}\Big[\frac{F_1^2}{240} + \frac{a^2+b^2-\frac14}{q_*^2} - 2\eta_\pi(q_*^2) - \frac{2(ab/q_*^2 + v_1)}{x - b_0(q_*)} + \frac14 (\log e_\pi(q^2))''\big|_{q_*}\Big], \quad (82)$$

whose first term is Olver's $\psi(0)$ for the comparison function $f = F$ (Remark 3.36); and in the order-$h^2$ coefficients $a_2, b_2$ of (47), it reads

$$\varpi_\tau = \frac{2^{1/3}}{u_1^{2/3}}\Big[\frac{4u_1^2 - 8u_2 + 60u_3/u_1 - 36u_2^2/u_1^2}{280} + \frac{2e_0e_2 - e_1^2}{8e_0^2} - \frac{a_2}{a_0}\Big|_{q_*} - \frac{\tau\, b_2(q_*)}{e_0}\Big]. \quad (83)$$

The local expressions are defined for every chain with a simple turn, but their identity with the limit (78) uses (R): without (R) the limit can diverge (Remark 3.29).

Thus, to this order, the bracket is Ai at the shifted argument $N^{2/3}(\zeta_\tau + N^{-2}\varpi_\tau)$: its origin sits at $\zeta_\tau = -N^{-2}\varpi_\tau$, while the turning point $q_*$, where $\zeta_\tau = 0$, does not move, and no further connection scalar is needed; under (R) the $O_{A_0}(N^{-1})$ bracket error of (i) improves to $O_{A_0}(N^{-4/3})$. The leading defect is $N^{-4/3}\varpi_\tau \operatorname{Ai}'(t)$, so where $\varpi_\tau(x) \neq 0$ the supremum of the error over the window $|t| \le A_0$ is exactly of order $N^{-4/3}$; at an individual point it is of that order only where $\operatorname{Ai}'(t) \neq 0$ as well. Off the layer the constant $\varpi_\tau$ does not suffice: at fixed $q \in I' \setminus \{q_*\}$ the relative error of $\operatorname{Ai} + N^{-4/3}\varpi_\tau \operatorname{Ai}'$ is of order $N^{-1}|\zeta_\tau|^{1/2}|B_\tau(q) - \varpi_\tau|$. With Theorem 3.20, (80) describes the physical solution, with additive error $O(N^{-2})$ in the natural gauge (measured against Olver's envelope in the Airy window), from fixed rank through the turn to a fixed distance into the forbidden side.

**Proposition 3.28** (Amplitude of the uniform Airy law)**.** *Let* $\mathfrak{j}$ *be the real-analytic function on* $I$ *of Lemma E.8 (the order-*$h^2$ *coefficient of the discrete flux of the Airy-form branch), and, assuming the* $k = 0$ *row of Theorem 3.24 when* $a = b = 0$*, let* $\delta_{\mathrm{fl}}(x) := c(x) + \mathsf{J}_2(0, x)$ *be the flux defect of the physical solution (Theorem 3.24). In Theorem 3.27(ii) one may take, with* $p_2$ *normalized by* $2p_2/g + \mathfrak{j} \equiv 0$*,*

$$\mathfrak{a}_N(q) = N^{-2}\Big(\delta_{\mathrm{fl}}(x) - \tfrac12 \mathfrak{j}(q)\Big) + O(N^{-3}), \quad (84)$$

*i.e.* $\mathfrak{a} = -\frac12\mathfrak{j}$ *and* $c_N = N^{-2}\delta_{\mathrm{fl}} + O(N^{-3})$ *(the* $O(N^{-3})$ *goes into* $\mathfrak{r}_N$*). For Hahn and centered Racah* $\delta_{\mathrm{fl}} \equiv 0$ *(Corollaries 4.4, 5.3), so the bracket of (80) is* $(1 - \frac{\mathfrak{j}(q)}{2N^2})\operatorname{Ai}(t) + N^{-4/3}B_\tau(q)\operatorname{Ai}'(t) + O(N^{-3})\mathsf{M}/\mathsf{E}$ *with no undetermined constant; for the one-small quantum* $6j$ *chain* $\delta_{\mathrm{fl}} = -\frac{1}{12}$ *(units of the curvature scale squared), so* $(1 - \frac{\mathfrak{j}(q)}{2N^2} - \frac{1}{12N^2})$ *replaces* $1 - \frac{\mathfrak{j}(q)}{2N^2}$ *(Corollary 6.3). At the turn* $\mathfrak{j}(q_*)$ *is (286); for Hahn with* $x \neq 0$ *and offsets* $(M, \sigma)$

$$\mathfrak{j}^{\mathrm{H}}(q_*) = \frac{1}{(1-x^2)^2}\Big[-\frac{2(M^2+\sigma^2)}{5} + \frac{(5x^2-1)M\sigma}{5x} + \frac{125x^4 - 35x^2 + 14}{900\,x^4}\Big]. \quad (85)$$

*Proof.* See Appendix E.2: the discrete flux of the Airy-form branch is equated with that of the Darboux branch at an oscillatory rank, and (286)–(85) are exact symbolic reductions. In Olver's notation $-\frac12 B_{\tau,\zeta}(0) = A_1(0)$ for $2\zeta B' + B = \psi_\tau$ ($A_1 = \frac12\zeta B^2 - \frac12 B'$), whereas $\delta_{\mathrm{fl}}$ has no ODE counterpart. □

*Remark* 3.29 (On hypothesis (R)). (a) Hypothesis (R) does not follow from (E1)–(E2), the simple turn and (75). The chain $e_\pi(s) = \sqrt{1-s}$, $v_\pi(s,t) = \beta_2 t$ with $\beta_2 \neq 0$, started from the Hahn/$3j$ seed at a physical node $x \in (0,1)$, satisfies (E1)–(E2) and has the principal data, simple turn and

turning quantization of the Hahn chain. Its solution, however, is up to a constant factor the Hahn solution at the off-lattice point $x - \beta_2 N^{-2}$, which is not recessive: its turning-layer bracket carries a Bi($t$) component of size about $-\pi\beta_2/(2N)$, so (R) fails and the bracket error is of order $N^{-1}$. Correspondingly $\Sigma_2$ gains $\beta_2 D^{-1/2}$, $\widetilde{\Gamma}$ gains $\beta_2 \arcsin(q/q_*) \to \pi\beta_2/2$, and the limit (78) diverges.

(b) A sufficient condition. Suppose the chain ends at a hard edge, $A_{k_{\max}+1}(h) = 0$, the physical solution satisfies the row $k = k_{\max}$, and the recurrence is exactly hyperbolic beyond the turn: $\tau(x - B_k(h)) \geq A_k(h) + A_{k+1}(h)$ for all $k \leq k_{\max}$ with $q_k \geq q_* + C_0 N^{-2}$. Then (R) holds with $m = \frac{1}{6}$. Indeed, in the gauge $\widetilde{Y}_k = \chi_\tau(k)\widehat{Y}_k$ the backward induction of Proposition 3.34(a) gives $|\widetilde{Y}_{k+1}| \leq |\widetilde{Y}_k|$ beyond the turn, so $|\widehat{Y}_k| \leq |\widehat{Y}_{k_1}| \leq C N^{1/6}$ for $k \geq k_1$, the first rank with $q_k \geq q_* + N^{-2/3}$ (the same $k_1$ as in Proposition 3.34, so $N^{2/3}\zeta_\tau(q_{k_1}) = O(1)$; Theorem 3.27(i), which does not use (R), bounds the window and $k_1$). For the Hahn, centered Racah, and one-small quantum $6j$ chains the hyperbolicity hypothesis holds (Proposition 3.31), so (R) holds without further input; it also follows directly from $3j$, $6j$, and $q$-$6j$ orthogonality (Sections 4, 5, and 6).

*Remark* 3.30 (Non-quantized turning action). The integrality (75) enters the proof of Theorem 3.27(i) only through the phase identities (265) and (268) (Appendix E). Let $K_\tau$ be real, for instance at a point $x$ of the family off the physical lattice, for a solution of the chain that has the fixed-ratio bulk asymptotics of Corollary 3.21 at the matching point (this is the input of the matching; at lattice nodes it is that corollary), and put $\mathcal{K}_+ := K_+$, $\mathcal{K}_- := K_-(L) - (L - k_{\min})$, writing $K_-(L)$ for $K_-$ of (75) to show its dependence on $L$ through $\kappa$; both are independent of $L$. The same two-component matching, with $\cos(\pi\mathcal{K}_+ + \frac{\pi}{4} - N\mathcal{A}) = \cos\pi\mathcal{K}_+ \cos(N\mathcal{A} - \frac{\pi}{4}) + \sin\pi\mathcal{K}_+ \sin(N\mathcal{A} - \frac{\pi}{4})$ for $\tau = +1$ (the sign of the second term is reversed for $\tau = -1$) and $\mathrm{Bi}(-z) \sim -\pi^{-1/2} z^{-1/4} \sin(\frac{2}{3}z^{3/2} - \frac{\pi}{4})$ [50, 9.7.11], gives uniformly for $|N^{2/3}\zeta_\tau(q)| \leq A_0$

$$Y_L = \epsilon_{ab}\chi_\tau(L)\sqrt{\frac{2}{q}}\, N^{-1/3} \left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4} \Big[\cos(\pi\mathcal{K}_\tau)\,\mathrm{Ai}(t) - \tau\sin(\pi\mathcal{K}_\tau)\,\mathrm{Bi}(t) + O_{A_0}(N^{-1})\Big], \tag{86}$$

with $\chi_+(L) = 1$, $\chi_-(L) = (-1)^{L-k_{\min}}$; for $\mathcal{K}_\tau \in \mathbb{Z}$ this is (76). A defect $\delta K := \mathcal{K}_\tau - \mathrm{round}(\mathcal{K}_\tau)$ thus gives a Bi component of relative size $\pi\delta K$ and, beyond the turn, violates (R). In Remark 3.29(a) the Hahn solution at $x - \beta_2 N^{-2}$ has $\partial_x S(q_*, x) = -\frac{\pi}{2}$, hence $\delta K = \beta_2/(2N)$ and the Bi coefficient $-\pi\beta_2/(2N)$ quoted there.

**Proposition 3.31** (Exact hyperbolicity beyond the rank turn)**.** *Let the chain be the Hahn/$3j$ chain with $x \neq 0$, the centered Racah/$6j$ chain, or the one-small quantum $6j$ chain, including its limit* $\coth\mathsf{Y} = 1$, $1/\sinh\mathsf{Y} = 0$ *in* (148) *(the $q$-Hahn chain, Remark 3.32), with a simple rank turn $(q_*, \tau)$ at a physical lattice node $x$ in a compact family in $(-1, 1)$, and $k_{\max}$ the hard edge, $A_{k_{\max}+1} = 0$. There are $C_0, N_0$, explicit in Appendix E.3, such that for $N \geq N_0$ and every $k \leq k_{\max}$ with $q_k \geq q_* + C_0 N^{-2}$,*

$$\tau\,(x - B_k) \;\geq\; A_k + A_{k+1}. \tag{87}$$

*(This coefficient inequality holds at any $x$ with a simple turn; at physical nodes the physical solution also satisfies row $k_{\max}$.) Hence, for the Hahn, centered Racah and one-small quantum $6j$ chains, (R) of Theorem 3.27 holds with $m = \frac{1}{6}$ (Remark 3.29(b)), without orthogonality inputs; for the $q$-Hahn limit only the coefficient inequality is claimed.*

*Proof.* See Appendix E.3; in the natural symbol variable $\sigma(q) = q^2$ (Hahn and Racah) or $\sigma(q) = \tanh^2 q$ (quantum $6j$), the secant-slope inequality uses only that the principal hopping is concave and decreasing in $\sigma$ and $b_0$ is linear in $\sigma$; the explicit constants use in addition elementary bounds on sinh, cosh and $\sigma''$ for the quantum chain. □

*Remark* 3.32 (The $q$-Hahn chain). Let $\mathsf{p} = e^{-2/N}$, $N_H \in \mathbb{Z}_{\geq 1}$, $a', b' \in \mathbb{Z}_{\geq 0}$, and let $J_H$ be the orthonormal Jacobi matrix of the $q$-Hahn polynomials $Q_{n_H}(\mathsf{p}^{-x_H}; \mathsf{p}^{a'}, \mathsf{p}^{b'}, N_H \,|\, \mathsf{p})$ [38, §14.6]: with the recurrence coefficients $A^H_{n_H}, C^H_{n_H}$ there (row $n_H$, $C^H_0 := 0$), diagonal $-(A^H_{n_H} + C^H_{n_H})$ and off-diagonal $\sqrt{A^H_{n_H-1} C^H_{n_H}}$, $0 \leq n_H \leq N_H$. Put $m = \frac{a'+b'}{2}$, $n = \frac{b'-a'}{2}$, $k_{\max} = N_H + m$, $\mathsf{X} = (k_{\max}+1)/N$, and let $J_6$ be the matrix of (148) on the ranks $k = n_H + m$ with $\coth \mathsf{Y} = 1$ and $1/\sinh \mathsf{Y} = 0$ (and the conventions of Proposition 6.1(a) at the lowest rank). Then, up to the gauge sign of the off-diagonal entries,

$$\begin{aligned} J_H &= \big(e^{(k_{\max}-a')/N} \cosh \mathsf{X} - 1\big) I - e^{(k_{\max}-a')/N} \sinh \mathsf{X}\, J_6, \\ \text{i.e.} \quad \mathsf{p}^{-x_H} &= e^{(k_{\max}-a')/N} \big(\cosh \mathsf{X} - x \sinh \mathsf{X}\big), \end{aligned} \tag{88}$$

an identity of rational functions of $e^{1/N}$, $e^{a'/(2N)}$, $e^{b'/(2N)}$, $e^{n_H/N}$ and $e^{N_H/N}$, verified by exact symbolic computation. Thus the $q$-Hahn chain is the one-small quantum $6j$ chain with an ideal vertex: $\sigma_2 = \tanh^2 \mathsf{Y} = 1$, $\beta_0 = \coth \mathsf{X} = (\sigma_1 \sigma_2)^{-1/2}$, the hard edge at $\sigma = \tanh^2 \mathsf{X}$, the analogue of the Hahn normalisation $\sigma_2^{-1} = 0$ ($\sigma_2 = \infty$) of Appendix E.3, with $\sigma_2$ now finite. Every constant in the proofs of the coefficient inequality of Proposition 3.31, of Proposition 3.34(a) and of the ratio bound in Proposition 3.34(b) stays finite in this limit ($q_m = \mathsf{X}$). The absolute bound $CN^{1/6} e^{-\xi}$ in (b), hence (R), uses $|\widehat{Y}_{k_1}| \leq CN^{1/6}$ from Theorem 3.27(i), and part (c) also uses Theorem 3.27(ii) and Proposition 3.28; their inputs ((E2), the turning quantization and $\delta_{\mathrm{fl}}$) are not established here for this chain. The identification needs the deformation tied to $N$: at fixed $\mathsf{p}$ the chain has no small parameter, and offsets growing with $N$ leave the class.

*Remark* 3.33. For the Hahn chain $q_* = \sin\theta$, that is $\sigma_* = q_*^2 = 1 - x^2$, while the hard edge is at $\sigma_{\max} = 1$. The condition $x \neq 0$ is therefore exactly $\sigma_* < \sigma_{\max}$, the condition that the forbidden zone beyond the turn is nonempty. Given a simple turn at some $|x| < 1$, the proof of the inequality uses nothing else about $x$.

**Proposition 3.34** (The physical solution on the forbidden side, up to the end of $I$)**.** *In the setting of Proposition 3.31, let* $\widetilde{Y}_k = \chi_\tau(k) \widehat{Y}_k$, $\rho_k = \widetilde{Y}_{k+1}/\widetilde{Y}_k$, *put* $\Upsilon(q) := \operatorname{arcosh}(\tau u(q)) = \zeta_\tau^{1/2} \zeta'_\tau$ *for* $q > q_*$ *and* $\Upsilon(q) := 0$ *for* $0 \leq q \leq q_*$, $\xi(q) = \frac{2}{3} N \zeta_\tau(q)^{3/2}$ *($q \geq q_*$), and* $I = [q_* - \delta_0, q_* + \delta_0]$ *with* $q_* + \delta_0$ *at a fixed distance from the hard edge.*

*(a)* $\widetilde{Y}_k$ *is strictly nonzero of one sign for* $q_k \geq q_* + C_0 N^{-2}$ *(up to the hard edge* $k_{\max}$*, where* $\widetilde{Y}_{k_{\max}+1} := 0$ *and so* $\rho_{k_{\max}} = 0$*), and* $0 < \rho_k \leq 1$ *there for* $k < k_{\max}$.

*(b) There is* $D_0 > 0$ *with* $\rho_{j-1} \leq \exp\big(-\Upsilon\big(\sqrt{(q_j^2 - D_0/N)_+}\big)\big)$ *for* $q_j \in I$ *with* $q_j \geq q_* + N^{-2/3}$ *and all sufficiently large* $N$*. Hence* $|\widehat{Y}_k| \leq CN^{1/6} e^{-\xi(q_k)}$ *there (at least the recessive Airy decay).*

*(c)* (80) *holds, with* $\mathfrak{a}_N$ *as in Proposition 3.28 and* $|\mathfrak{r}_N| \leq CN^{-3}\mathsf{M}/\mathsf{E}$*, at all lattice points* $q \in I$ *with* $q \leq q_* + \delta_0 - C_1 N^{-1} \log N$*; this replaces* $I'$ *of Theorem 3.27(ii) for these chains (for the* $q$*-Hahn limit of Remark 3.32 only (a) and the ratio bound in (b) are claimed).*

*Proof.* See Appendix E.3 (comparison with constant-ratio supersolutions; beyond $I$ only (a)–(b) is available, and the logarithmic collar is where the Bi component stops being below $N^{-3}\mathsf{M}/\mathsf{E}$). □

*Remark* 3.35 (Flux defect and the Gauss weight; open step). Since $\alpha = \delta_{\mathrm{fl}} - \mathsf{J}_2$, the identity $\alpha = -\mathsf{J}_2$ for Hahn and Racah is $\delta_{\mathrm{fl}} = 0$, proved from the exact fixed-rank $N^{-2}$ coefficients of Ref. [2]. For Racah it also follows without that input: the quantum $6j$ value of Corollary 6.3(iii) uses only the closed seed, the flat limit (vi) rescales it to $\delta_{\mathrm{fl}} = -\varsigma^2/12$, and as $\varsigma \to 0$ the seed and the (E1) data tend to those of the centered Racah chain. Since $\delta_{\mathrm{fl}}$ is determined at fixed rank, by the $t$-coefficients

of the seed and of the (E1) data (through $c(x)$), and these are analytic in $\varsigma$ at $\varsigma = 0$ ($q$-numbers $[n]$ are even analytic in $\varsigma$), the limit may be taken in $\delta_{\mathrm{fl}}$, so the Racah value is $\lim_{\varsigma\to 0}(-\varsigma^2/12) = 0$. An intrinsic route is the Gauss-weight identity $\lambda_j^2 w_j = 2\,(1 + O(h^4))$, where $\lambda_j^2 = \sum_k \widehat{Y}_k(x_j)^2$ is the natural-gauge column norm and $w_j$ is the node gap, equivalent to $\delta_{\mathrm{fl}} = 0$ once the constant physical-branch flux $\mathsf{J}_{\mathrm{phys}}(x_j) = \frac{1}{2} + h^2\delta_{\mathrm{fl}} + O(h^4)$ satisfies $\mathsf{J}_{\mathrm{phys}}(x_j) = \frac{1}{4}\lambda_j^2 w_j(1 + O(h^3))$. The weight identity holds exactly for $3j$ ($\lambda_j^2 = N$, $w_j = 2/N$) and $6j$ ($\lambda_j^2 = 4AB/(2C)$, $w_j = C/(AB)$); proving $\mathsf{J}_{\mathrm{phys}}(x_j) = \frac{1}{4}\lambda_j^2 w_j(1 + O(h^3))$ across the forbidden zone for a general chain in the class is *open*.

For the one-small quantum $6j$ chain the deformed identity $\lambda_j^2 \bar{w}_j = 2h/\sinh h$ holds exactly, with $\lambda_j^2 = \sum_k \widehat{\Psi}_k(x_j)^2 = [2A][2B]/[2C]$ by $q$-orthogonality and $\bar{w}_j = |\partial_C x| = 2h \sinh \mathsf{Z}/(\sinh \mathsf{X} \sinh \mathsf{Y})$ the node density in $C$ (Section 6); the centred node gap $2 \sinh h \sinh \mathsf{Z}/(\sinh \mathsf{X} \sinh \mathsf{Y})$, the analogue of $w_j = C/(AB)$ above, equals $(\sinh h/h)\,\bar{w}_j$, and with it $\lambda_j^2 w_j = 2$ exactly. So the weight comparison does not single out a value for the $q$-chain: matching $\mathsf{J}_{\mathrm{phys}}(x_j)$ to $\frac{1}{4}\lambda_j^2 \bar{w}_j = \frac{1}{2}\frac{h}{\sinh h} = \frac{1}{2} - \frac{h^2}{12} + O(h^4)$ reproduces the value $\delta_{\mathrm{fl}} = -\frac{1}{12}$ proved by exact computer algebra from the closed seed, whereas the centred gap would give 0. The value $-\frac{1}{12}$ is relative to the normalization (147), which already contains the factors $\sqrt{[2k+1]}$ and $\sqrt{[2A][2B]}$: by the rule above, multiplying the physical solution by the constant $(\sinh h/h)^{1/2} = 1 + \frac{h^2}{12} + O(h^4)$ keeps (E1)–(E2) and replaces $\delta_{\mathrm{fl}}$ by 0. It is thus a property of the normalization (147) of the $q$-$6j$ symbol, not an obstruction to the class.

*Remark* 3.36 (Olver's form). For $w'' = (u^2 f + g)w$ with a simple turning point, Olver's uniform expansion is $\mathrm{Ai}(u^{2/3}\zeta)\sum_s A_s u^{-2s} + u^{-4/3}\,\mathrm{Ai}'(u^{2/3}\zeta)\sum_s B_s u^{-2s}$, $A_0 = 1$, where $B_0$ is the solution regular at $\zeta = 0$ of $2\zeta B_0' + B_0 = \psi$ [46, Chapter 11]. The bracket of (80) is the discrete analogue through $A_1$ and $B_0$, with $u = N$: $B_\tau$ solves the same transport equation with an explicit lattice function $\psi_\tau$ in place of $\psi$ (Lemma E.7), so $\varpi_\tau = B_\tau(q_*) = \psi_\tau(0)$. By (82), $\psi_\tau(0)$ is Olver's $\psi(0)$ for $f = F$ and $g(q_*) = F_1^2/240 + G_\tau$, where $G_\tau$ is the bracketed sum after $F_1^2/240$. The term $F_1^2/240$ is the $O(h^2)$ discreteness remnant of the lattice second difference at the turn, absent for a second-order ODE: for the model chain (constant $a$, linear $F$) with a symbol $T(p) = 1 + p^2/2 + t_2 p^4 + t_3 p^6 + \cdots$ in place of $\cosh p$ it equals $F_1^2(4t_2^2 - 2t_3)$, which is $F_1^2/240$ for cosh ($t_2 = \frac{1}{24}$, $t_3 = \frac{1}{720}$) and 0 for the ODE, and for the class the value $\frac{1}{240}$ is what the exact identity (82) = (79) gives. It arises because $F = \operatorname{arcosh}^2(\tau u)$ already carries the classical effect of the higher difference terms but not their action on the Airy parametrix at $\zeta = 0$, where $\mathrm{Ai}^{(6)}(0) = 4\,\mathrm{Ai}(0)$. It is a potential term, not a phase change: $\varpi_\tau$ is Olver's $\psi(0)$ for $w'' = (u^2 F + g_\tau)w$ with $g_\tau(q_*) = F_1^2/240 + G_\tau$, equivalently for the Langer variable $F + h^2 g_\tau$; omitting it would change $\varpi_\tau$ by exactly $-F_1^{4/3}/240$ (for the Hahn chain with $(M, \sigma) = (2, 1)$ and $x = \frac{3}{5}$ this is $-0.030$) and leave the bracket error of order $N^{-4/3}$. The coefficient blows up where the turn meets the zero of $e_\pi$ (Hahn: $x \to 0$; Racah: triangles right-angled opposite $\alpha$ or $\beta$, (137)); there a different comparison function is needed (Remark 5.5), as for a turning point coalescing with a singularity in Olver's theory.

# 4 Hahn/$3j$: verification and explicit data

This section verifies that the fixed-offset $3j$ rank recurrence is an even centered Wigner–Jacobi chain and records its explicit data and the resulting statements for $3j$ symbols. Put $\kappa = L + \frac{1}{2}$ and $q = \kappa/N$.

**Proposition 4.1** (The Hahn chain is an even centered chain)**.** *With $q_k = (k + \frac{1}{2})h$ and the natural gauge $\widehat{\Phi}_k = \sqrt{2k+1}\,\Phi_k$, the exact rank recurrence (4) is*

$$x\widehat{\Phi}_k = A_{k+1}^{(N)}\widehat{\Phi}_{k+1} + A_k^{(N)}\widehat{\Phi}_{k-1} + B_k\widehat{\Phi}_k, \qquad A_k^{(N)} = \frac{R_k}{k\sqrt{4k^2-1}}\sqrt{1-k^2h^2}, \quad B_k = \frac{M\sigma}{k(k+1)}. \tag{89}$$

*It is an even centered Wigner–Jacobi chain with offsets $(a,b)=(M,\sigma)$, on every compact $X \Subset (-1,1)$ and every $s_1<1$, with*

$$e(s)=\sqrt{1-s}, \qquad v\equiv 0, \qquad \varepsilon_1=\tfrac{1}{2}, \quad \beta_1=\beta_2=0, \quad \chi(x)=\frac{x}{2}. \tag{90}$$

*Its principal data are $\cos p = x/\sqrt{1-q^2}$ and $D(q,x)=1-x^2-q^2$.*

*Proof.* Dividing (4) by $k(k+1)(2k+1)$ and applying the square-root gauge gives the displayed coefficients, with $B_k=B_k^{(0)}$. For (E2), the lowest-rank value is, by [2, Eq. (27)], a fixed constant times the square root of $\mathfrak{C}_{L+\sigma}(u_+,h)\,\mathfrak{C}_{L-\sigma}(u_-,h)/\Pi_L(h^2)$ with $u_\pm=\frac{1\pm x}{2}$ (for general offsets after the symmetry reduction of Ref. [2], which changes it by a sign independent of $N$). Each $\mathfrak{C}_r(u,h)$ is a product of the factors $u+qh$ over a symmetric set of $q$, hence a polynomial in $(u,t)$, $t=h^2$, and $\Pi_L$ is a polynomial in $t$ with $\Pi_L(0)=1$. At $t=0$ the radicand is $u_+^{L+\sigma}u_-^{L-\sigma}>0$ for $x\in(-1,1)$ (in the chamber $M\ge|\sigma|$, $L_0=M$, to which the symmetry reduction brings every offset pair; there it is $\widehat{d}^{\,2}_{L_0}$ up to a constant), so on a complex neighbourhood $U$ of $X$ and a disk $|t|\le t_0$, uniform for $X\Subset(-1,1)$, the square root is analytic and bounded jointly in $(x,t)$, real for real arguments, with value $\widehat{d}_{L_0}(\arccos x)$ at $t=0$; when $L_0>0$ this also gives $L_0+1$ by (E2). In the exceptional sector $M=\sigma=0$, where the $k=0$ rank equation degenerates to $0=0$, one may start from

$$\Phi_0=1, \qquad \Phi_1=\frac{x}{\sqrt{1-t}}. \tag{91}$$

□

### 4.1 Closed Hahn action

Inserting $\Phi=\mathcal{A}(x)e^{\mathrm{i}S(x)/h}$ into the magnetic equation (5) gives at principal order $(1-x^2)\cos(2S_x)-(1-x^2)+2q^2=0$. On the branch reducing to the continuum Wigner phase,

$$S_x=-p_q(x), \qquad p_q(x)=\arcsin\!\left(\frac{q}{\sqrt{1-x^2}}\right), \tag{92}$$

with oscillatory band $1-x^2-q^2>0$ and turning points $x_\pm=\pm\sqrt{1-q^2}$. Direct integration of $F(q,x)=\int_0^x \arcsin(q/\sqrt{1-u^2})\,\mathrm{d}u$ gives

$$F(q,x)=q\arcsin\!\left(\frac{x}{\sqrt{1-q^2}}\right)+x\arcsin\!\left(\frac{q}{\sqrt{1-x^2}}\right)-\arctan\!\left(\frac{xq}{\sqrt{1-x^2-q^2}}\right), \tag{93}$$

and we put

$$S(q,x)=\frac{\pi q}{2}-F(q,x), \qquad \Theta_q(x)=\frac{S(q,x)}{q}, \qquad x_q:=\cos\Theta_q, \qquad W_q(x)=\left[\frac{1-x_q^2}{1-x^2-q^2}\right]^{1/4}. \tag{94}$$

Then $\Theta_q\to\arccos x$ as $q\to0$, $\partial_q S=\arccos(x/\sqrt{1-q^2})=p$ and $D=1-x^2-q^2$, so $S$, $\Theta_q$ and $W_q$ are the generic action data (21) of the Hahn chain, and the transport equation of (5) gives the same amplitude $(1-x^2-q^2)^{-1/4}$. For small $q$,

$$\begin{aligned}\Theta_q(x)&=\arccos x-\frac{q^2x}{6\sqrt{1-x^2}}-\frac{q^4x(3-2x^2)}{40(1-x^2)^{3/2}}+O_K(q^6),\\ W_q(x)&=1+\frac{q^2(3-x^2)}{12(1-x^2)}+\frac{q^4(41x^4-94x^2+225)}{1440(1-x^2)^2}+O_K(q^6).\end{aligned} \tag{95}$$

We now relate this to the secular growth of the fixed-rank coefficients. At order $N^{-2r}$ the maximal $L^{3r}$ growth of the fixed-rank coefficients comes from $r$ separate insertions of the first $k^2/N^2$ perturbation, so the leading terms are iterates of a phase displacement. For Hahn the exact fixed-rank coefficients of Ref. [2] satisfy $f_2 = \kappa^2 \mathcal{P}_2 f + O_K(L^{1/2})$ and $f_4 = \kappa^4 \mathcal{P}_4 f + O_K(L^{7/2})$, $f = d^L_{M\sigma}(\arccos x)$ (Proposition F.3), where $\mathcal{P}_2 = \frac{x}{6}\partial_x + \frac{3-x^2}{12(1-x^2)}$ is the operator (39) with (90) and

$$\mathcal{P}_4 = \frac{x^2}{72}\partial_x^2 + \frac{7x(3-2x^2)}{180(1-x^2)}\partial_x + \frac{41x^4 - 94x^2 + 225}{1440(1-x^2)^2}; \tag{96}$$

these are exactly the $q^2$ and $q^4$ terms of the pullback $W_q d^L_{M\sigma}(\Theta_q)$ by (95). The closed action resums the whole hierarchy (35) (Remark F.4).

**Proposition 4.2** (Hahn data)**.** *Fix $M, \sigma$ and a compact $K \Subset (-1,1)$, and let $\Phi_L = \Phi_L^{M,\sigma}$.*

*(a)* Mesoscopic and critical regimes. *Write $\Phi_L(x; t^{1/2}) = \sum_{r\geq 0} t^r f^L_{2r}(x)$ with $t = N^{-2}$ and $f^L_0 = d^L_{M\sigma}(\arccos x)$. The series converges and obeys the bounds (28)–(29) with $Y_L = \Phi_L$; in particular $\sqrt{L+1}\sup_K |\Phi_L - d^L_{M\sigma}(\arccos x)| = O(\kappa^3/N^2)$ for $L = o(N^{2/3})$. Uniformly for physical nodes $x = \cos\theta \in K$ and $\kappa^3/N^2 \leq \Lambda$,*

$$\sqrt{\frac{\pi\kappa\sin\theta}{2}}\,\Phi_L(x; N^{-1}) = \cos\left(\varphi_{L,M,\sigma}(\theta) - \frac{\kappa^3}{6N^2}\cot\theta\right) + O_{K,\Lambda}\left(\frac{1}{\kappa} + \frac{\kappa^2}{N^2} + \frac{\kappa^5}{N^4}\right), \tag{97}$$

*with $\varphi_{L,M,\sigma}$ the large-$L$ Wigner-d phase. If $\kappa^3/N^2 \to \lambda$ and $x_N \to \cos\theta_0$ with $|x_N - \cos\theta_0| = O(N^{-1})$, the added phase tends to $-\lambda\cot\theta_0/6$.*

*(b)* First-order pullback. *For every compact corridor $\mathcal{C} \Subset \{(q,x) : 0 \leq q < 1,\ D(q,x) > 0\}$ and every admissible $L$ whose swept path lies in $\mathcal{C}$, the bound (60) holds for $\Phi_L - W_q\, d^L_{M\sigma}(\arccos x_q)$ with the closed data (93)–(94). On compact fixed-ratio sets this is $O(N^{-1})$ in natural gauge and $O(N^{-3/2})$ absolutely; the remainder at this order is the omitted phase $h\Gamma_2(q,x)$ of Remark 3.22, and the pullback carries no further $O(1)$ $q$-dependent phase.*

*(c)* Second-order data. *The coefficients (47) are*

$$a_0 = \tfrac{1}{2}\sqrt{1-q^2}, \quad a_2 = \frac{a_0}{q^2}\Big(\frac{1}{8} - \frac{M^2+\sigma^2}{2}\Big), \quad b_2 = \frac{M\sigma}{q^2}, \tag{98}$$

*and for them the second-order data (48) define $\Sigma_2^{(3)}$, $\rho_2^{(3)}$, $\Gamma_2^{(3)}$, $\Theta_q^{[2]}$ and $W_q^{[2]}$, with $W_q^{[2]} = [\sin^2\Theta_q^{[2]}/(1-x^2-q^2)]^{1/4}$; $\rho_2^{(3)}$ is analytic in $q^2$ at $q=0$.*

*(d)* Amplitude and Airy data.

$$\mathsf{J}_2(0,x) = -\frac{(8(M^2+\sigma^2)-5)x^2 + 4M\sigma x(x^2-5) - 3}{48(1-x^2)^2}, \tag{99}$$

*and at the rank turn $q_* = \sqrt{1-x^2}$ of a fixed node $x \neq 0$, of type $\tau = \operatorname{sgn} x$,*

$$\varpi^{\mathrm{H}}(x) := \varpi_{\operatorname{sgn} x}(x) = \frac{2^{1/3}(140(M^2+\sigma^2)x^4 - 280M\sigma x^3 - 50x^4 + 15x^2 - 9)}{280\,|x|^{8/3}(1-x^2)^{4/3}}. \tag{100}$$

*Proof.* (a) is Theorems 3.7 and 3.9 with Proposition 4.1: the added phase is $-\chi\kappa^3/(3N^2\sin\theta) = -\kappa^3\cot\theta/(6N^2)$. For the limit, $\theta_N - \theta_0 = O(N^{-1})$, so the added phase at $\theta_N$ differs from its value at $\theta_0$ by $O(\kappa^3/N^3) = O(N^{-1})$ in the critical window $\kappa \asymp N^{2/3}$; no off-lattice continuation is needed. (b) is Corollary 3.21 with the identification of the action data above. (c) is Proposition 3.16 with $(a,b) = (M,\sigma)$. (d) is (68) with (90), and (79) with $\tau u(q) = |x|(1-q^2)^{-1/2}$, $e_0 = |x|$, $u_1 = q_*/x^2$, $v_1 = 0$ (exact computer algebra, Appendix G). □

*Remark* 4.3 (Sharpness of the $N^{2/3}$ coherence scale). For fixed interior $x = \cos\theta \neq 0$ the first pullback phase correction from (95) is $\kappa(\Theta_q - \theta) = -\kappa^3\cot\theta/(6N^2) + O(L^5/N^4)$. Thus $L = o(N^{2/3})$ is precisely the regime in which the uncorrected Wigner-$d$ phase perturbation vanishes uniformly on compact sets avoiding $x = 0$, and at $L \asymp N^{2/3}$ it has a finite nonzero limit. For $x \neq 0$ the principal part $\kappa^2\mathcal{P}_2 f$ of $f_2$ has exact order $L^{5/2}$ (its leading oscillating part is the phase term displayed above, which does not vanish there), so $\sqrt{L+1}\,N^{-2}f_2$ is of exact order $\kappa^3/N^2$ and the exponent 3 in (a) cannot be lowered.

**Corollary 4.4** (Phase-exact Hahn pullback)**.** *Fix $M, \sigma$ and a closed corridor $\mathcal{C} \Subset \{(q,x) : 0 \le q < 1,\ 1 - x^2 - q^2 > 0\}$. For every admissible rank $L$ whose swept path stays in $\mathcal{C}$,*

$$\boxed{\sqrt{2L+1}\left|\Phi_L^{M,\sigma}(x;N^{-1}) - \Big(1 - \frac{\mathsf{J}_2(q,x)}{N^2}\Big)W_q^{[2]}(x)\,d_{M\sigma}^L(\Theta_q^{[2]}(x))\right| \le \frac{C_{\mathcal{C}}}{N^3},} \tag{101}$$

*with $\mathsf{J}_2$ from (67) and (99); without the factor $1 - N^{-2}\mathsf{J}_2$ the bound is $C_{\mathcal{C}}N^{-2}$. At fixed positive ratio the absolute errors are $O(N^{-7/2})$ and $O(N^{-5/2})$.*

*Proof.* Theorems 3.20 and 3.24 with Proposition 4.1; the $k = 0$ row of the exceptional channel is (91). The fixed-rank datum is $c(x) = -\mathsf{J}_2(0,x)$ by the exact $N^{-2}$ coefficient [2, Theorem 6.1] (Appendix C). □

**Corollary 4.5** (Bulk Hahn zero quantization)**.** *Fix $M, \sigma$, $0 < q_0 < q_1 < 1$, and a compact $x$-interval on which $(q,x)$ stays separated from the two fixed-ratio turning curves for all $q \in [q_0, q_1]$. In a fixed Wigner/Hahn chamber extend the orthonormal Hahn expression (314) to the interior real node variable by the gamma-function continuation of its weight and norm; its zeros are those of the Hahn polynomial. Let $x_{j,N}^{\mathrm{H}}$ be such a zero in the compact bulk, $\widetilde{x}_{j,N}$ the zero with the same index of $x \mapsto d_{M\sigma}^L(\Theta_q^{[2]}(x))$, $q = (L+\frac{1}{2})/N$ (it may lie in an $O(N^{-4})$ collar outside the bulk), $p_+ = |M - \sigma|$ and $\widetilde{\Gamma}^{(3)} := \Gamma_2^{(3)} + \mathfrak{c}_{M\sigma}(\Theta_q)/q$. Then, with a consistent integer index $\nu_j$,*

$$\boxed{\begin{aligned} x_{j,N}^{\mathrm{H}} &= \widetilde{x}_{j,N} + O(N^{-4}),\\ NS(q, x_{j,N}^{\mathrm{H}}) + h\widetilde{\Gamma}^{(3)}(q, x_{j,N}^{\mathrm{H}}) + \frac{p_+\pi}{2} - \frac{\pi}{4} &= \Big(\nu_j + \frac{1}{2}\Big)\pi + O(N^{-3}). \end{aligned}} \tag{102}$$

*In particular, the zero with the same index of $d_{M\sigma}^L(\Theta_q(x))$ is within $O(N^{-2})$ of $x_{j,N}^{\mathrm{H}}$, and $NS(q, x_{j,N}^{\mathrm{H}}) + p_+\pi/2 - \pi/4 = (\nu_j + \frac{1}{2})\pi + O(N^{-1})$.*

*Proof.* The exact identity (314) writes $\Phi_L$ as a nonvanishing analytic prefactor times a Hahn polynomial on any compact interior band (for $N$ large), so the polynomial $P_{L,N}$ of Theorem 3.26 is a constant multiple of the Hahn polynomial in the node variable. The exceptional start (91) satisfies the $k = 0$ row with $B_0 = 0$. Theorem 3.26 with $p_{ab} = p_+$ gives the statements. □

Among rigorous Hahn results, Lin and Wong proved global fixed-ratio Hahn asymptotics for fixed parameters, with elementary oscillatory formulas in the band and Airy formulas near its edges [29]; see also DLMF §18.24 [26]. Through the exact Clebsch–Gordan/Hahn correspondence [28] (Appendix F.1) their band edges map to $x_{\mp}(q)$ up to $O(N^{-1})$, and their equilibrium density becomes $\rho_q = -S_x/(\pi q)$ (Section 5.3). Proposition 4.2(b) and Corollary 4.4 supply the turning-free normalization and phase matching directly from the midpoint recurrence, and Theorem 4.6 obtains the normalized Airy law at the band edges from the rank-direction theorem, without a parameter translation to their variables.

### 4.2 Normalized Airy connection at the fixed-ratio Hahn turning curve

Put $x_{\pm}(q) = \pm\sqrt{1-q^2}$ and $D(q,x) = 1 - x^2 - q^2$. For the upper turning point define the signed Airy coordinate

$$\xi_+(q,x) = \begin{cases} -\left[\dfrac{3}{2}\displaystyle\int_x^{x_+(q)} \left(\frac{\pi}{2} - p_q(u)\right) \mathrm{d}u\right]^{2/3}, & x \le x_+(q), \\[2ex] \left[\dfrac{3}{2}\displaystyle\int_{x_+(q)}^{x} \operatorname{arcosh}\!\left(\dfrac{q}{\sqrt{1-u^2}}\right) \mathrm{d}u\right]^{2/3}, & x \ge x_+(q), \end{cases} \tag{103}$$

and for the lower turning point put

$$\xi_-(q,x) := \xi_+(q,-x), \qquad \xi_+(q,x) = \left(\frac{2x_+}{q^2}\right)^{1/3}(x - x_+) + O\big((x-x_+)^2\big), \tag{104}$$

so $|\xi_{\pm}/D|^{1/4}$ extends continuously through the turning point, and each turning point is simple uniformly for $q$ in a compact subset of $(0,1)$.

With the Darboux sign $\epsilon_{M\sigma} = (-1)^{\max(\sigma - M, 0)}$ and $p_+ = |M - \sigma|$ of (17), write an upper physical lattice point as

$$x = 1 - \frac{p_+ + 1 + 2\nu_+}{N}, \qquad \nu_+ = \frac{N - 1 - p_+}{2} - \mu, \tag{105}$$

and a lower one, with $p_- = |M + \sigma|$, as

$$x = -1 + \frac{p_- + 1 + 2\nu_-}{N}, \qquad \nu_- = \mu + \frac{N - 1 - p_-}{2}. \tag{106}$$

The quantities $\nu_{\pm}$ are nonnegative integers on the corresponding physical grid.

**Theorem 4.6** (Weyl-normalized fixed-ratio Hahn Airy connection)**.** *Fix $M, \sigma$, $0 < q_0 < q_1 < 1$ and $A < \infty$, and let $q = (L + \frac{1}{2})/N \in [q_0, q_1]$. For all sufficiently large admissible $N$, uniformly at admissible pairs $(L, x)$, $x$ a physical lattice point, with $|N^{2/3}\xi_+(q,x)| \le A$,*

$$\boxed{\begin{aligned} \Phi_L^{M,\sigma}(x; N^{-1}) = \epsilon_{M\sigma}(-1)^{\nu_+ + p_+} \sqrt{\frac{2}{q}}\, N^{-1/3} \left|\frac{\xi_+(q,x)}{D(q,x)}\right|^{1/4} \\ \times \Big[\mathrm{Ai}(t) + N^{-4/3}\varpi^{\mathrm{H}}(x)\,\mathrm{Ai}'(t) + O_A(N^{-2})\Big], \qquad t = N^{2/3}\xi_+(q,x), \end{aligned}} \tag{107}$$

*and uniformly for $|N^{2/3}\xi_-(q,x)| \le A$,*

$$\boxed{\begin{aligned} \Phi_L^{M,\sigma}(x; N^{-1}) = \epsilon_{M\sigma}(-1)^{\nu_- + L + (p_+ - p_-)/2} \sqrt{\frac{2}{q}}\, N^{-1/3} \left|\frac{\xi_-(q,x)}{D(q,x)}\right|^{1/4} \\ \times \Big[\mathrm{Ai}(t) + N^{-4/3}\varpi^{\mathrm{H}}(x)\,\mathrm{Ai}'(t) + O_A(N^{-2})\Big], \qquad t = N^{2/3}\xi_-(q,x), \end{aligned}} \tag{108}$$

*with $\varpi^{\mathrm{H}}$ from (100). The exponent $\nu_- + L + (p_+ - p_-)/2$ is an integer for integer and half-integer channels. The error terms are additive inside the brackets; the error of* Ai *alone is $O_A(N^{-4/3})$, attained for the window supremum where $\varpi^{\mathrm{H}}(x) \neq 0$ (and pointwise where also $\mathrm{Ai}'(t) \neq 0$). Once the ordered Wigner-d sign $\epsilon_{M\sigma}$ is fixed, the midpoint normalization determines the Airy branch and the lattice parity, so no connection phase is left undetermined. The same window is the rank-direction turning layer at a fixed node: for fixed $x \neq 0$ the Hahn chain has the simple rank turn $q_* = \sqrt{1-x^2}$, and its Langer coordinate (74) is $\xi_{\operatorname{sgn} x}(q, x)$.*

*Proof.* Appendix G verifies the hypotheses of Theorem 3.27 at fixed nodes in a compact subset of $(0,1)$ or $(-1,0)$, identifies $\zeta_\pm(q;x) = \xi_\pm(q,x)$, and evaluates the lattice integers $K_+ = \nu_+ + p_+$ and $K_- = L - \nu_- + (p_+ - p_-)/2$. Condition (R) holds for every rank by $3j$ orthogonality. □

*Remark* 4.7 (Two distinct two-thirds scales). The phase horizon $L \asymp N^{2/3}$ keeps $L/N \to 0$ at fixed interior $x$; the Airy law keeps $q$ bounded away from zero and resolves the spatial layer $x - x_\pm(q) = O(N^{-2/3})$. The two share the exponent $2/3$ but are different limits, as are the centered-$6j$ critical horizon and the Racah turning layer. On the band side of the Airy layer, letting the negative Airy argument grow recovers the amplitude and phase of Proposition 4.2(b); Lin and Wong's parametrix [29] describes the same layer in uncentered variables, in which the band edge is displaced by $O(N^{-1})$.

## 5 Centered Racah/$6j$: verification and explicit data

This section verifies that the centered one-small $6j$ rank recurrence is an even centered Wigner–Jacobi chain and records its explicit data and the resulting statements for $6j$ symbols. Put $\kappa_f = f + \frac{1}{2}$ and $\rho = \kappa_f/N$. We restrict $(\alpha, \beta, \gamma)$ to a compact family of strict triangles, so $x$ lies in a compact $K \Subset (-1,1)$.

**Proposition 5.1** (The centered Racah chain is an even centered chain)**.** *In the natural gauge $\widehat{\Psi}_k = \sqrt{2k+1}\,\Psi_k$ the exact recurrence (10) is*

$$x\widehat{\Psi}_k = A_{k+1}^{(6,N)}\widehat{\Psi}_{k+1} + A_k^{(6,N)}\widehat{\Psi}_{k-1} + B_k^{(6,N)}\widehat{\Psi}_k, \tag{109}$$

*with*

$$A_k^{(6,N)} = \frac{R_k^{(6)}}{k\sqrt{4k^2-1}}\sqrt{\Big(1 - \frac{k^2h^2}{4\alpha^2}\Big)\Big(1 - \frac{k^2h^2}{4\beta^2}\Big)}, \tag{110}$$

$$B_k^{(6,N)} = \frac{h^2\{2k(k+1) + 1 - m^2 - n^2\}}{8\alpha\beta} + \frac{mn}{k(k+1)}. \tag{111}$$

*It is an even centered Wigner–Jacobi chain with offsets $(a,b) = (m,n)$, parameters $\pi = (\alpha,\beta)$, any $s_1 < 4\min(\alpha,\beta)^2$, and*

$$\begin{aligned} e(s) &= \sqrt{\Big(1 - \frac{s}{4\alpha^2}\Big)\Big(1 - \frac{s}{4\beta^2}\Big)}, \quad v(s,t) = \frac{s}{4\alpha\beta} + \frac{(\frac{1}{2} - m^2 - n^2)\,t}{8\alpha\beta}, \\ \varepsilon_1 &= \frac{\alpha^{-2} + \beta^{-2}}{8}, \quad \beta_1 = \frac{1}{4\alpha\beta}, \quad \chi_6(x) = \frac{x(\alpha^{-2} + \beta^{-2})}{8} - \frac{1}{4\alpha\beta}. \end{aligned} \tag{112}$$

*Its principal data are $p = \omega$, $D = D_6$ with*

$$\eta(\rho) = e(\rho^2), \qquad \cos\omega(\rho) = \frac{x - \rho^2/(4\alpha\beta)}{\eta(\rho)}, \tag{113}$$

$$D_6(\rho) = \eta(\rho)^2 - \Big[x - \frac{\rho^2}{4\alpha\beta}\Big]^2 = \sin^2\theta - \frac{\gamma^2\rho^2}{4\alpha^2\beta^2}, \tag{114}$$

*where the last form uses* $2\alpha\beta x = \alpha^2 + \beta^2 - \gamma^2$. *In particular* $D_6$ *decreases in* $\rho^2$, *so the swept path of any point a positive distance from the turning curve* $D_6 = 0$ *is turning-free.*

*Proof.* Dividing (10) by $k(k+1)(2k+1)$ and applying the square-root gauge gives the displayed coefficients, and $2k(k+1)+1 = 2q_k^2 h^{-2} + \frac{1}{2}$ gives $v$. For (E2), reducing to the canonical chamber $m \geq |n|$ ($f_{\min} = m$) by the column/row symmetries of the $6j$ symbol up to a fixed sign, the lowest-rank natural-gauge value is $\widehat{\Psi}_m = (-1)^{m-n}\sqrt{(2m+1)\,\mathcal{S}_m(A,B,C)}$ in terms of the one-term product $\mathcal{S}_m(A,B,C)$ of [2, Lemma 8.1] (the factor $(-1)^{m-n} = \operatorname{sgn} d^m_{mn}$ accounts for the phase $(-1)^{a+b+c+f+m}$ in (8)), whose factors are central-factorial products over symmetric sets of shifts of $A = N\alpha$, $B = N\beta$ and of the four combinations $A \pm B \pm C$, with $C = N\gamma(x)$, $\gamma(x) = (\alpha^2+\beta^2-2\alpha\beta x)^{1/2}$. With the powers of $N$ cancelled, as in Ref. [2], $(2m+1)\mathcal{S}_m$ is an analytic function of $(x,t)$, $t = N^{-2}$, near $X \times \{0\}$ ($\gamma(x)$ is analytic on a complex neighbourhood of $X$ because $\gamma > 0$ there), and it equals $\widehat{d}_m^{\,2} > 0$ at $t = 0$ because the limiting triangles are strict, so $\alpha \pm \beta \pm \gamma \neq 0$. Hence $\widehat{\Psi}_m$ is analytic and bounded jointly in $(x,t)$ on $U \times \{|t| \leq t_0\}$ for a complex neighbourhood $U$ of $X$, uniformly on the compact family, with value $\widehat{d}_m$ at $t = 0$; the second rank follows from the first by the recurrence at $f = f_{\min}$, where the backward coefficient vanishes and the forward one is analytic and nonzero. In the exceptional channel $m = n = 0$ one may take

$$\Psi_0 = 1, \qquad \Psi_1 = \frac{x - t/(8\alpha\beta)}{\sqrt{(1 - t/(4\alpha^2))(1 - t/(4\beta^2))}}. \tag{115}$$

Finally $\eta^2 - (x-b)^2 = 1 - x^2 - \rho^2(\alpha^2+\beta^2-2\alpha\beta x)/(4\alpha^2\beta^2)$. □

### 5.1 Closed Racah action

The rank action and the effective angle and transport

$$S_6(\rho) = \int_0^\rho \omega(s)\,\mathrm{d}s, \qquad \Theta_6(\rho) = \frac{S_6(\rho)}{\rho}, \qquad W_6(\rho) = \left[\frac{\sin^2\Theta_6(\rho)}{D_6(\rho)}\right]^{1/4} \tag{116}$$

are the generic action data (21) of the centered Racah chain. The action is elementary. Put $s_\theta := \sin\theta$ and

$$\rho_* := \frac{2\alpha\beta s_\theta}{\gamma}, \qquad \phi := \arcsin\frac{\rho}{\rho_*}, \qquad c_\beta := \frac{\alpha - \beta x}{\gamma}, \qquad c_\alpha := \frac{\beta - \alpha x}{\gamma}, \tag{117}$$

so that $c_\beta$ and $c_\alpha$ are the cosines of the angles of the triangle $(\alpha,\beta,\gamma)$ opposite $\beta$ and $\alpha$. Integration by parts, followed by $\rho^2 = \rho_*^2 \sin^2\phi$, gives for $0 \leq \rho < \rho_*$ the exact closed form

$$\boxed{S_6(\rho) = \rho\,\omega(\rho) + 2\gamma\phi - 2\alpha\,\operatorname{atan2}(c_\beta \sin\phi, \cos\phi) - 2\beta\,\operatorname{atan2}(c_\alpha \sin\phi, \cos\phi),} \tag{118}$$

since the remaining integrand reduces to $-2\gamma + 2\alpha c_\beta(1-(1-c_\beta^2)\sin^2\phi)^{-1} + 2\beta c_\alpha(1-(1-c_\alpha^2)\sin^2\phi)^{-1}$. The formula holds on every strict triangle and extends continuously to $\rho = \rho_*$ unless the triangle is right-angled at the vertex opposite $\alpha$ or $\beta$, where the turn meets an off-diagonal boundary.

**Proposition 5.2** (Racah data)**.** *Fix admissible* $m, n$ *and a compact family of strict limiting triangles.*

*(a)* Mesoscopic and critical regimes. *The Taylor series* $\Psi_f(t) = \sum_r t^r g_{2r}^{(f)}(x)$, $g_0^{(f)} = d^f_{mn}(\arccos x)$, *converges and obeys* (28)–(29)*; in particular* $\sqrt{f+1}\,|\Psi_f(N) - d^f_{mn}(\theta)| = O(\kappa_f^3/N^2)$ *for* $f = o(N^{2/3})$. *As* $\rho \to 0$,

$$\omega(\rho) = \theta - \frac{\chi_6(x)}{\sin\theta}\rho^2 + O(\rho^4), \tag{119}$$

$$NS_6(\rho) = \kappa_f\theta + \frac{\kappa_f^3}{N^2}C_6(\alpha,\beta,x) + O\Big(\frac{\kappa_f^5}{N^4}\Big), \qquad C_6 := \frac{1/(12\alpha\beta) - x(\alpha^{-2}+\beta^{-2})/24}{\sin\theta}, \tag{120}$$

*and uniformly when* $\kappa_f^3/N^2 \le \Lambda$,

$$\sqrt{\frac{\pi\kappa_f\sin\theta}{2}}\,\Psi_f(N) = \cos\left(\varphi_{f,mn}(\theta) + \frac{\kappa_f^3}{N^2}C_6(\alpha,\beta,x)\right) + O_\Lambda\left(\frac{1}{\kappa_f} + \frac{\kappa_f^2}{N^2} + \frac{\kappa_f^5}{N^4}\right). \tag{121}$$

*(b)* First-order pullback. *For a compact strict-triangle corridor whose swept family stays away from* $D_6 = 0$ *(hence from the off-diagonal boundaries* $\eta = 0$*, since* $\eta^2 \ge D_6$*), the bound* (60) *holds for* $\Psi_f - W_6(\rho)d^f_{mn}(\Theta_6(\rho))$*; at fixed ratio this is* $O(N^{-1})$ *in natural gauge and* $O(N^{-3/2})$ *absolutely, with no fixed-offset* $O(1)$ *phase beyond* $NS_6(\rho)$.

*(c)* Second-order data. *At fixed* $\rho > 0$,

$$a_{6,0} = \frac{\eta}{2}, \quad a_{6,2} = \frac{a_{6,0}}{\rho^2}\left(\frac{1}{8} - \frac{m^2+n^2}{2}\right), \quad b_{6,0} = \frac{\rho^2}{4\alpha\beta}, \quad b_{6,2} = \frac{1/2 - m^2 - n^2}{8\alpha\beta} + \frac{mn}{\rho^2}, \tag{122}$$

*and* $\Sigma_2^{(6)}$, $\rho_2^{(6)}$, $\Gamma_2^{(6)} = \int_0^\rho \rho_2^{(6)}$, $\Theta_6^{[2]} = \Theta_6 + h^2\Gamma_2^{(6)}/\rho$ *and* $W_6^{[2]} = [\sin^2\Theta_6^{[2]}/D_6]^{1/4}$ *are the data* (48)*;* $\rho_2^{(6)}$ *is analytic in* $\rho^2$ *at* $\rho = 0$.

*(d)* Amplitude and Airy data. $\mathsf{J}_2(0,x)$ *is* (68) *with* (112) *and* $\beta_2 = (\frac{1}{2} - m^2 - n^2)/(8\alpha\beta)$*; the Airy coefficient at the rank turn is* (137) *of Theorem 5.7.*

*Proof.* (a) is Theorems 3.7 and 3.9 with Proposition 5.1; the expansion of $\omega$ is $p = \theta - \chi_6 q^2/\sin\theta + O(q^4)$, and $-\chi_6/(3\sin\theta) = C_6$. (b) is Corollary 3.21; the swept family is turning-free by the monotonicity of $D_6$. (c) is Proposition 3.16 with $(a,b) = (m,n)$, and (d) is immediate. □

In the symmetric channel $m = n = 0$, Chen, Ismail and Simeonov proved the Legendre and phase-coherent cosine forms of the Ponzano–Regge asymptotics of the one-small $6j$ symbol $\{a\,b\,e;\ b\,a\,f\}$, with relative error $O(f^2/N)$ plus $O(f^{-3/2})$, for $f = o(N^{1/2})$ [30, Theorem 2], and they note that Ponzano and Regge's condition $f = O(N)$ is not sufficient for this. Part (a) of Proposition 5.2 extends the phase-coherent Wigner-$d$ window to $f = o(N^{2/3})$, for fixed nonzero offsets as well, for the absolute error in the natural gauge (a relative bound is not claimed near zeros). The coherence-horizon rule of Remark 3.10, itself heuristic, suggests that the next even term of (119) gives an $f \asymp N^{4/5}$ horizon once the first noncontinuum phase is resummed (Remark H.3); this is not proved here.

**Corollary 5.3** (Phase-exact centered-Racah pullback)**.** *Fix admissible* $m, n$ *and a closed strict-triangle corridor whose swept rank family stays away from* $D_6 = 0$ *(hence from the off-diagonal boundaries* $\eta = 0$*, since* $\eta^2 \ge D_6$*). For every admissible terminal rank* $f$ *in the corridor,*

$$\boxed{\sqrt{2f+1}\left|\Psi_f(N) - \Big(1 - \frac{\mathsf{J}_2(\rho,x)}{N^2}\Big)W_6^{[2]}(\rho)\,d^f_{mn}(\Theta_6^{[2]}(\rho))\right| \le \frac{C}{N^3},} \tag{123}$$

*with* $\mathsf{J}_2$ *from* (67)*; without the factor* $1 - N^{-2}\mathsf{J}_2$ *the bound is* $CN^{-2}$*. At fixed ratio the absolute errors are* $O(N^{-7/2})$ *and* $O(N^{-5/2})$.

*Proof.* Theorems 3.20 and 3.24 with Proposition 5.1; the $k = 0$ row of the exceptional channel is (115). The fixed-rank datum is $c(x) = -\mathsf{J}_2(0, x)$ by the exact degree $-2$ coefficient [2, Theorem 9.1] (Appendix C). □

Littlejohn and Yu proposed a uniform approximation of the $6j$ symbol by a rotation matrix, supported by numerical tests [18, 19]. For the centered one-small family, Corollary 5.3 is a rotation-matrix pullback with a proved error bound away from turning points.

**Corollary 5.4** (Bulk centered-Racah zero quantization)**.** *Fix $m, n$, $0 < \rho_0 < \rho_1$, and a compact turning-free fixed-ratio bulk set. For fixed $A, B, f, m, n$ regard the normalized $6j$ kernel through its Racah-polynomial continuation in the quadratic c-lattice, and let $x^{\mathrm{R}}_{j,N}$ be a zero in the compact set. Let $\widetilde{x}_{j,N}$ be the zero with the same index of $x \mapsto d^f_{mn}(\Theta_6^{[2]}(\rho, x))$, $\rho = (f + \frac{1}{2})/N$ (possibly in an $O(N^{-4})$ collar), $p_+ = |m - n|$ and $\widetilde{\Gamma}^{(6)} := \Gamma_2^{(6)} + \mathfrak{c}_{mn}(\Theta_6)/\rho$. Then, with a consistent integer index $\nu_j$,*

$$\boxed{x^{\mathrm{R}}_{j,N} = \widetilde{x}_{j,N} + O(N^{-4}), \qquad NS_6(\rho, x^{\mathrm{R}}_{j,N}) + h\widetilde{\Gamma}^{(6)}(\rho, x^{\mathrm{R}}_{j,N}) + \frac{p_+\pi}{2} - \frac{\pi}{4} = \left(\nu_j + \frac{1}{2}\right)\pi + O(N^{-3}).} \tag{124}$$

*In particular the zero of $d^f_{mn}(\Theta_6(\rho, x))$ with the same index lies within $O(N^{-2})$ of $x^{\mathrm{R}}_{j,N}$, and $NS_6(\rho, x^{\mathrm{R}}_{j,N}) + p_+\pi/2 - \pi/4 = (\nu_j + \frac{1}{2})\pi + O(N^{-1})$.*

*Proof.* By Proposition 5.1 and Theorem 3.26, the polynomial $P_{f,N}$ is a constant multiple of the Racah polynomial in the quadratic spectral variable, since $x$ is affine in $C^2 = c(c + 1) + \frac{1}{4}$. The exceptional start (115) satisfies the $k = 0$ row with $B_0 = h^2/(8\alpha\beta) = v(h^2/4, h^2)$. Theorem 3.26 with $p_{ab} = |m - n|$ gives the statements. □

## 5.2 Normalized turning connections for the centered Racah chain

Semiclassical (discrete WKB) analysis of angular-momentum recurrences goes back to Neville [7] and to Schulten and Gordon [41]; here the centered Jacobi gauge lets the modern difference-equation Airy theory be normalized directly against the Weyl bulk solution. Retain $\eta$, $D_6$ and $\omega$ from (114), and put

$$u_6(\rho) := \frac{x - \rho^2/(4\alpha\beta)}{\eta(\rho)}. \tag{125}$$

By the exact simplification (114), the unique rank-direction turning radius is

$$\boxed{\rho_* = \frac{2\alpha\beta \sin\theta}{\gamma}.} \tag{126}$$

We restrict to an interior simple turning point, equivalently to a compact strict-triangle family on which $\eta(\rho_*) \geq c > 0$. The characteristic roots collide at $+1$ or $-1$ according as

$$\tau := \operatorname{sgn} u_6(\rho_*) = \operatorname{sgn}\big((\alpha^2 - \beta^2)^2 - \gamma^4\big) \in \{+1, -1\}. \tag{127}$$

*Remark* 5.5 (Excluded boundary collision). The simple-turn results below exclude $\tau = 0$. By (127), $\tau = 0$ means $\alpha^2 - \beta^2 = \pm\gamma^2$, so the limiting triangle $(\alpha, \beta, \gamma)$ is right-angled at the vertex opposite $\alpha$ or $\beta$ (a right angle opposite $\gamma$ has $\tau = -1$ and is not excluded); then the formal turning radius meets the hard edge $\rho = 2\min(\alpha, \beta)$, where the off-diagonal hopping $\eta$ vanishes. They also exclude the degenerate corner $A - B = O(1)$, $C = O(B^{1/2})$, where the strict limiting-triangle hypothesis

fails. There the local comparison model is no longer Airy, and uniform two-column asymptotics through a fixed neighbourhood of $\tau = 0$, and through the degenerate corner, are open. Since $\varpi_\tau$ grows like $|\varepsilon|^{-8/3}$ as $\varepsilon \to 0$ ((137)), the correction $N^{-4/3}\varpi_\tau$ of the bracket is of size $(N\varepsilon^2)^{-4/3}$: the simple-turn expansion can remain uniform as the angle opposite $\alpha$ or $\beta$ approaches $\pi/2$ only while $N\varepsilon^2 \to \infty$.

For compact simple-turning families with $\tau \in \{+1, -1\}$ we use the following Airy coordinates. For $\tau = +1$, define the signed Airy coordinate

$$\zeta_+(\rho) = \begin{cases} -\left[\dfrac{3}{2}\displaystyle\int_\rho^{\rho_*} \omega(s)\,\mathrm{d}s\right]^{2/3}, & \rho \le \rho_*, \\ \left[\dfrac{3}{2}\displaystyle\int_{\rho_*}^{\rho} \operatorname{arcosh} u_6(s)\,\mathrm{d}s\right]^{2/3}, & \rho \ge \rho_*. \end{cases} \tag{128}$$

For $\tau = -1$, remove the alternating lattice phase: $\zeta_-$ is defined by (128) with $\omega$ replaced by $\pi - \omega$ and $u_6$ by $-u_6$,

$$\zeta_-(\rho) := \zeta_+(\rho)\big|_{\omega\to\pi-\omega,\ u_6\to-u_6}. \tag{129}$$

Write $\zeta_\tau = \zeta_+$ or $\zeta_-$ according to the turn type. With $d_* := \gamma^2\rho_*/(2\alpha^2\beta^2)$ and $\eta_* := \eta(\rho_*)$,

$$\zeta_\tau(\rho) = (d_*/\eta_*^2)^{1/3}(\rho - \rho_*) + O((\rho - \rho_*)^2), \tag{130}$$

so $|\zeta_\tau/D_6|^{1/4}$ extends continuously and positively through the turning point.

The remaining global phase can be evaluated without a Racah-polynomial parameter translation.

**Lemma 5.6** (Flat centered-Racah turning action)**.** *On the branch $0 < \omega < \pi$ with $S_6(0) = 0$,*

$$\boxed{\frac{S_6(\rho_*)}{\pi} = \begin{cases} \gamma - |\alpha - \beta|, & \tau = +1, \\ \rho_* - (\alpha + \beta - \gamma), & \tau = -1. \end{cases}} \tag{131}$$

*Proof.* Appendix H. □

To state the physical sign, let $p = |n - m|$. In the same ordered Wigner-$d$ convention, (17) gives the fixed sign $\epsilon^{(6)}_{mn} = (-1)^{\max(n-m,0)}$ and

$$d^f_{mn}(\vartheta) = \epsilon^{(6)}_{mn}\sqrt{\frac{2}{\pi F \sin\vartheta}}\left[\cos\left(F\vartheta + \frac{p\pi}{2} - \frac{\pi}{4}\right) + O(F^{-1})\right], \qquad F = f + \tfrac{1}{2}. \tag{132}$$

Define

$$K_+ := C - |A - B| + \frac{p-1}{2}, \qquad (\tau = +1), \tag{133}$$

$$K_- := F - A - B + C + \frac{p}{2}, \qquad (\tau = -1). \tag{134}$$

The triangle parities make $K_\pm$ integers, also for half-integer $m, n$. On the $-1$ branch, $K_-$ changes by one when $f$ changes by one, exactly encoding the alternating-lattice gauge.

**Theorem 5.7** (Weyl-normalized centered Racah Airy connection)**.** *Fix admissible $m, n$, $A_0 < \infty$, and a compact family of strict limiting triangles for which the turning point (126) is interior and simple and the turn type $\tau$ is fixed. Let $F = f + \frac{1}{2} = N\rho$. Uniformly at admissible ranks satisfying*

$$|N^{2/3}\zeta_\tau(\rho)| \le A_0, \tag{135}$$

*one has*

$$\boxed{\Psi_f(N) = \epsilon^{(6)}_{mn}(-1)^{K_\tau}\sqrt{\frac{2}{\rho}}\,N^{-1/3}\left|\frac{\zeta_\tau(\rho)}{D_6(\rho)}\right|^{1/4}\left[\mathrm{Ai}(t) + N^{-4/3}\varpi_\tau\,\mathrm{Ai}'(t) + O_{A_0}(N^{-2})\right],} \tag{136}$$

*where* $K_\tau = K_+$ *on the* $+1$ *branch and* $K_\tau = K_-$ *on the* $-1$ *branch,* $t = N^{2/3}\zeta_\tau(\rho)$*, and* $\varpi_\tau$*, the value at* $\rho_*$ *of* (79) *for the data* (112)*, is*

$$\boxed{\varpi_\tau = \frac{2^{1/3}\left[140\,\varepsilon^3\big((m^2+n^2)x - 2mn\big) + \mathcal{P}(x,s_*)\right]}{280\,(\rho_*\sin\theta)^{4/3}\,|\varepsilon|^{8/3}}, \qquad s_* := \frac{\rho_*^2}{4\alpha\beta}, \quad \varepsilon := x - s_*,} \tag{137}$$

$$\mathcal{P}(x,s_*) = -50x^4 + 15x^2 - 9 + 6s_*x(25x^2-3) + s_*^2(21 - 149x^2) + 36s_*^3x + 4s_*^4, \tag{138}$$

*with* $x = \cos\theta$ *and* $\tau = \operatorname{sgn}\varepsilon$*. Here* $s_* = b_0(\rho_*)$ *is the diagonal shift at the turn; for the Hahn chain* $b_0 = 0$*, and the same formula with* $\mathcal{P}(x,0)$*,* $\varepsilon = x$*,* $\rho_* \to q_* = \sin\theta$ *and offsets* $(M,\sigma)$ *in place of* $(m,n)$ *is* (100)*. In terms of the angles* $A_\alpha, A_\beta$ *of the limiting triangle opposite* $\alpha,\beta$*,* $s_* = b_0(\rho_*) = \sin A_\alpha \sin A_\beta$*,* $\varepsilon = \tau\eta(\rho_*) = -\cos A_\alpha \cos A_\beta$ *and* $\rho_*\sin\theta = 2\gamma\, s_*$*, so* $\tau = \operatorname{sgn}\varepsilon$ *is* (127)*, and* $\varpi_\tau$ *blows up like* $|\varepsilon|^{-8/3}$ *at the excluded triangles where* $A_\alpha$ *or* $A_\beta$ *is* $\pi/2$*. The uniform form* (80) *also holds on the neighbourhood* $I'$ *of* $\rho_*$ *of Theorem 3.27(ii), and, for this chain, to the end of* $I$ *except for a collar of width* $O(N^{-1}\log N)$ *(Proposition 3.34(c)). The* $O_{A_0}(N^{-2})$ *term is additive inside the square brackets, as in Theorem 4.6, so the error of* Ai *alone is* $O_{A_0}(N^{-4/3})$*, attained for the window supremum where* $\varpi_\tau \neq 0$ *(and pointwise where also* $\mathrm{Ai}'(t) \neq 0$*). Once the ordered Wigner-d sign* $\epsilon^{(6)}_{mn}$ *is chosen, the physical centered* $6j$ *solution continues as the normalized recessive* Ai *branch beyond either simple Racah turning point, with no additional connection phase.*

*Proof.* We verify the hypotheses of Theorem 3.27. By (114), $D_6 > 0$ on $0 \le \rho < \rho_*$ and $\partial_\rho D_6(\rho_*) = -\gamma^2\rho_*/(2\alpha^2\beta^2) < 0$, while $e(\rho_*^2) = \eta(\rho_*) \ge c$. Any $s_1 \in (\rho_*^2, 4\min(\alpha,\beta)^2)$, uniformly on the compact family, is admissible in Proposition 5.1, so Lemma E.1 gives the uniform coefficient bounds. The characteristic quotient is $u_6$, so (128)–(129) are the coordinates (74), and (17) with $(a,b) = (m,n)$ is (132). Since $A = N\alpha$, $B = N\beta$, $C = N\gamma$, Lemma 5.6 gives $NS_6(\rho_*) = \pi(C - |A-B|)$ for $\tau = +1$ and $NS_6(\rho_*) - \pi N\rho_* = -\pi(A+B-C)$ for $\tau = -1$, so the quantities (75) are exactly $K_\pm$ of (133)–(134), which are integers. Theorem 3.27 gives (136) with bracket error $O_{A_0}(N^{-1})$. Condition (R) of Theorem 3.27(ii) holds because $6j$ orthogonality, $\sum_f (2f+1)(2c+1)W_f^2 = 1$, gives $|\widehat{\Psi}_f| = 2\sqrt{(2f+1)AB}\,|W_f| \le 2\sqrt{AB/(2c+1)} = O(N^{1/2})$ for all $f$; that theorem gives the bracket $\mathrm{Ai}(t) + N^{-4/3}\varpi_\tau\,\mathrm{Ai}'(t) + O_{A_0}(N^{-2})$ and (80), with $\varpi_\tau$ from (79), which for the data (112) is (137) (Lemma H.7). □

For Racah polynomials, Wang and Wong developed large-degree Racah asymptotics with varying parameters, including outer and oscillatory regimes, and note that their uniform difference-equation technique supplies turning-point formulas without carrying out that specialization [44]. The present contribution is complementary and needs no translation to a printed Racah-polynomial parameter chamber: the Regge/Weyl centering selects the exact $N^{-2}$-even recurrence, and the evaluated turning action gives the canonical Airy basis the physical $6j$ normalization, branch and lattice parity directly.

### 5.3 Geometric interpretation

The results above use only the actions as integrals of the principal rank momentum. Their geometric meaning is recorded here; none of it is used in the proofs.

We begin with the bispectral meaning of the Hahn action. For fixed $M, \sigma$ the rank recurrence gives, at principal order, $\cos S_q = x/\sqrt{1-q^2}$, while the magnetic recurrence gives $S_x = -p_q$ with $p_q$ from (92), and $\partial_q S_x = -(1-x^2-q^2)^{-1/2} = \partial_x S_q$. Hence

$$\omega_0 = \arccos\left(\frac{x}{\sqrt{1-q^2}}\right) \mathrm{d}q - \arcsin\left(\frac{q}{\sqrt{1-x^2}}\right) \mathrm{d}x \tag{139}$$

is closed, its primitive is $S(q,x)$ of (94), and $\Lambda_S = \{(q, p_q; x, p_x) : p_q = S_q,\ p_x = -S_x\}$ is an exact Lagrangian graph on the turning-free bulk, an instance of the bispectral viewpoint of Duistermaat and Grünbaum [23]. The effective angle is the action per unit angular momentum, not an ad hoc coordinate correction. By Remark 3.14 the effective coordinates transport the zero measures exactly: with $\rho_q(x) = -S_x/(\pi q)$ and $\varrho_\rho^{(6)} := -\partial_x S_6/(\pi\rho)$, positive on the turning-free band because $\partial_x \omega = -1/(\eta \sin\omega)$,

$$\rho_q(x)\,\mathrm{d}x = \rho_0(x_q)\,\mathrm{d}x_q, \qquad \varrho_\rho^{(6)}(x)\,\mathrm{d}x = \rho_0(x_{6,\rho})\,\mathrm{d}x_{6,\rho}, \qquad x_{6,\rho} := \cos\Theta_6. \tag{140}$$

These are action-Jacobian identities, not asymptotic matchings of densities, and place the construction within the constrained-equilibrium picture for discrete orthogonal polynomials [21, 22, 52, 20].

The Racah action has a tetrahedral realization. Let $F = f + \frac{1}{2} = N\rho$ and consider the principal Weyl tetrahedron with edge array $(A, B, C; B, A, F)$. With the edge $F$ on an axis the transverse radii are $r_A = \sqrt{A^2 - F^2/4}$ and $r_B = \sqrt{B^2 - F^2/4}$, and if $\Theta_F$ is the dihedral angle at $F$, the opposite edge satisfies $C^2 = r_A^2 + r_B^2 - 2r_A r_B \cos\Theta_F$. So the principal Racah symbol is exactly $\omega(\rho) = \Theta_F$, on the branch with $\Theta_F \to \theta$ as $F \downarrow 0$. The Schläfli identity, with $\mathcal{S}_{\mathrm{PR}}$ written in terms of interior dihedral angles (with the exterior angles of the usual Ponzano–Regge form it reads $\mathrm{d}\mathcal{S}/\mathrm{d}F = \pi - \Theta_F$ and changes $\mathcal{S}_{\mathrm{PR}}$ by $\pi F$), gives $\mathrm{d}\mathcal{S}_{\mathrm{PR}}/\mathrm{d}F = \Theta_F$, so $\mathcal{S}_{\mathrm{PR}}(F) = NS_6(\rho) + \mathrm{const}$ when the other edges are fixed, and $\mathcal{V} = Fr_A r_B \sin\omega/6$, so the Ponzano–Regge amplitude, after the normalization $2\sqrt{AB}$ of (8), is

$$\mathcal{A}_6(\rho) = \sqrt{\frac{2}{\pi F\, \eta(\rho) \sin\omega(\rho)}}, \tag{141}$$

which in the channel $m = n = 0$ agrees with the amplitude of Proposition 5.2(b) because $\eta \sin\omega = \sqrt{D_6}$. Roberts proved the Ponzano–Regge formula [9] for uniformly dilated admissible labels [40]; that theorem does not literally cover the fixed-offset centered family, because $O(1)$ edge offsets contribute an $O(1)$ phase, which Proposition 5.2(b) supplies. The integral of $\rho_2^{(6)}$ is the first $h^2$ fixed-offset correction to this rank-direction Regge phase; its invariant tetrahedral form is open (Section 8).

**Lemma 5.8** (The Racah turning curve is the flat-tetrahedron locus)**.** *Let* CM *be the Cayley–Menger matrix of the principal Weyl tetrahedron and $T$ the area of the triangle with sides $(\alpha, \beta, \gamma)$. Then*

$$\boxed{\det \mathrm{CM} = 8N^6\rho^2\alpha^2\beta^2\, D_6(\rho) = 2N^6\rho^2(16T^2 - \gamma^2\rho^2).} \tag{142}$$

*Consequently the Racah turning curve $D_6 = 0$ is exactly the locus where the principal Weyl tetrahedron is flat; $D_6 > 0$ is the nondegenerate Euclidean tetrahedron, and for $D_6 < 0$ the six lengths are not those of any Euclidean tetrahedron.*

*Proof.* Appendix H. □

*Remark* 5.9 (Dual physical-$c$ problem). The fixed-rank theory constructs the exact self-adjoint Racah operator on the physical $c$-lattice, with eigenvalue $f(f+1)$ [2, Theorem 8.2]. Put $\Delta_C(\gamma) :=$

$[\gamma^2-(\alpha-\beta)^2][(\alpha+\beta)^2-\gamma^2]$. In the all-large scaling its principal symbol is $\mathcal{H}_C(\gamma,p_C)=\Delta_C(\gamma)(1-\cos p_C)/(2\gamma^2)$, so the dual Hamilton–Jacobi equation is

$$\rho^2=\frac{\Delta_C(\gamma)}{2\gamma^2}(1-\cos p_C),\qquad p_C=2\arcsin\!\left(\frac{\rho\gamma}{\sqrt{\Delta_C(\gamma)}}\right). \tag{143}$$

Since $\Delta_C=4\alpha^2\beta^2(1-x^2)$, the dual oscillatory condition $\Delta_C-\rho^2\gamma^2>0$ is exactly $D_6>0$, and the two momenta satisfy $\partial_\gamma\omega=\partial_\rho p_C=2\gamma/\sqrt{\Delta_C-\rho^2\gamma^2}$. Hence $S_6(\rho,\gamma)$ is a bispectral generating function,

$$\mathrm{d}S_6=\omega\,\mathrm{d}\rho+p_C\,\mathrm{d}\gamma, \tag{144}$$

and the rank and physical-edge problems share the principal caustic and the principal action (proof in Appendix H).

## 6 The one-small quantum $6j$ chain

The quantum $6j$ symbol of $U_q(\mathfrak{sl}_2)$ with deformation parameter $q_{\mathrm{def}}=e^{1/N}$ (distinct from the rank coordinate $q$) has the same one-small rank recurrence as the centered Racah chain, with $q$-numbers $[z]:=\sinh(z/N)/\sinh(1/N)$ in place of integers. This section shows that it is an even centered Wigner–Jacobi chain in the sense of Definition 3.2 (with $e_\pi$ genuinely $t$-dependent), so that Sections 3.4–3.7 apply, and gives the explicit data. The curvature scale is normalized to 1; a scale $\varsigma$ (that is, $q_{\mathrm{def}}=e^{\varsigma/N}$) is obtained by $N\to N/\varsigma$ in the $q$-numbers and the Weyl lengths, and multiplies the flux defect below by $\varsigma^2$. Proofs are in Appendix I; steps that rest on exact computer algebra (rational-function identities with symbolic labels) say so where they occur.

We first define the family. With the Kirillov–Reshetikhin $q$-$6j$ symbol [36] take, for fixed admissible $m,n\in\frac12\mathbb{Z}$ with $m-n\in\mathbb{Z}$,

$$W_f=\begin{Bmatrix} a & b & c\\ b+m & a+n & f\end{Bmatrix}_q,\qquad A=a+\tfrac{n+1}{2}=N\alpha,\quad B=b+\tfrac{m+1}{2}=N\beta,\quad C=c+\tfrac12=N\gamma, \tag{145}$$

with the same labels as in (6)–(7). Put $\mathsf{X}=2\alpha$, $\mathsf{Y}=2\beta$, $\mathsf{Z}=2\gamma$ and assume the strict triangle inequalities $|\mathsf{X}-\mathsf{Y}|<\mathsf{Z}<\mathsf{X}+\mathsf{Y}$. Then $(\mathsf{X},\mathsf{Y},\mathsf{Z})$ are the side lengths of a hyperbolic triangle of curvature $-1$, with angle $\theta$ opposite $\mathsf{Z}$,

$$\begin{aligned} x=\cos\theta&=\frac{\cosh\mathsf{X}\cosh\mathsf{Y}-\cosh\mathsf{Z}}{\sinh\mathsf{X}\sinh\mathsf{Y}},\qquad \mu:=\frac{\sinh\mathsf{Z}}{\sinh\mathsf{X}\sinh\mathsf{Y}},\\ \Delta_0&:=\coth^2\mathsf{X}+\coth^2\mathsf{Y}-2x\coth\mathsf{X}\coth\mathsf{Y}=\sin^2\theta+\mu^2.\end{aligned} \tag{146}$$

(The Weyl lengths $\alpha,\beta,\gamma$ play the role of the Racah triangle $(\alpha,\beta,\gamma)$; the flat limit is Corollary 6.3(vi).) The natural-gauge solution is

$$\widehat{\Psi}_k:=(-1)^{a+b+c+k+m}\sqrt{[2k+1][2A][2B]}\;W_k,\qquad \Psi_k=\widehat{\Psi}_k/\sqrt{2k+1}, \tag{147}$$

the $q$-analogue of (8), with $\Psi_k\to d^k_{mn}(\theta)$ at $h\to0$.

**Proposition 6.1** (The $q$-$6j$ chain is an even centered chain)**.** *(a)* Exact Jacobi form. *With* $\mathrm{s}_z:=\sinh(z/N)$,

$$\begin{aligned} x\widehat{\Psi}_k&=A_{k+1}\widehat{\Psi}_{k+1}+A_k\widehat{\Psi}_{k-1}+B_k\widehat{\Psi}_k,\\ A_k^2&=\frac{4\,\mathrm{s}_{k-n}\mathrm{s}_{k+n}\,\mathrm{s}_{k-m}\mathrm{s}_{k+m}\,(1-\mathrm{s}_k^2/\sinh^2\mathsf{X})(1-\mathrm{s}_k^2/\sinh^2\mathsf{Y})}{\mathrm{s}_{2k}^2\,\mathrm{s}_{2k+1}\mathrm{s}_{2k-1}},\\ B_k&=\coth\mathsf{X}\coth\mathsf{Y}\Big(1-\frac{2\cosh\frac{m}{N}\cosh\frac{n}{N}}{\cosh\frac{2k+1}{N}+\cosh\frac{1}{N}}\Big)+\frac{\mathrm{s}_m\mathrm{s}_n}{\mathrm{s}_k\,\mathrm{s}_{k+1}}.\end{aligned} \tag{148}$$

*with* $f_{\min} := \max(|m|,|n|)$ *and* $A_{f_{\min}} := 0$ *(the quotient for* $A_k^2$ *is* $0/0$ *when* $|m| = |n| = \tfrac12$*). For* $m = n = 0$ *the row* $k = 0$ *reads* $x\widehat{\Psi}_0 = A_1\widehat{\Psi}_1 + B_0\widehat{\Psi}_0$ *with* $B_0 = \coth\mathsf{X}\coth\mathsf{Y}(1 - 1/\cosh\frac{1}{N})$ *and* $\widehat{\Psi}_0 = 1$*.*

*(b)* (E1) with explicit data. *Put* $K = \sqrt{s}$*,* $H = \sqrt{t}$*,* $\Xi(u,w) := (\cosh 2u - \cosh 2w)/(2(u^2 - w^2))$ *(an analytic function of* $(u^2, w^2)$*,* $\Xi(0,0) = 1$*). Then* $A_k = A_k^{(0)} e(k^2h^2, h^2)$ *and* $B_k = \lambda(h^2)B_k^{(0)} + v(q_k^2, h^2)$ *exactly, with*

$$e(s,t)^2 = \frac{\Xi(K,Hn)\,\Xi(K,Hm)\,(1-\sinh^2 K/\sinh^2\mathsf{X})(1-\sinh^2 K/\sinh^2\mathsf{Y})}{\Xi(2K,0)\,\Xi(2K,H)},$$
$$\lambda(t) = e(0,t) = \frac{\operatorname{shc}(Hm)\operatorname{shc}(Hn)}{\operatorname{shc}(H)}, \qquad \operatorname{shc}(u) := \frac{\sinh u}{u},$$
$$v(s,t) = \coth\mathsf{X}\coth\mathsf{Y}\Big(1 - \frac{2\cosh Hm\cosh Hn}{\cosh 2K + \cosh H}\Big) + \frac{2\sinh Hm\sinh Hn}{\cosh 2K - \cosh H} - \lambda(t)\frac{mnH^2}{K^2 - H^2/4}, \tag{149}$$

*(the last term of* $v$ *is* $0$ *if* $mn = 0$*), jointly analytic in* $(s,t)$*, any* $s_1 < 4\min(\alpha,\beta)^2$ *being admissible. The class data are* $e(s) = e(s,0) = \sqrt{(1-\tanh^2 q\coth^2\mathsf{X})(1-\tanh^2 q\coth^2\mathsf{Y})}$ *and* $b_0(q) = \tanh^2 q\,\coth\mathsf{X}\coth\mathsf{Y}$ *(*$q^2 = s$*), and*

$$\varepsilon_1 = \tfrac12(\coth^2\mathsf{X} + \coth^2\mathsf{Y}), \quad \beta_1 = \coth\mathsf{X}\coth\mathsf{Y}, \quad \lambda_1 = \tfrac16(m^2+n^2-1),$$
$$\beta_2 = \tfrac12(\tfrac12 - m^2 - n^2)\coth\mathsf{X}\coth\mathsf{Y} - \tfrac13 mn, \quad \chi(x) = \tfrac{x}{2}(\coth^2\mathsf{X} + \coth^2\mathsf{Y}) - \coth\mathsf{X}\coth\mathsf{Y},$$
$$\eta(q^2) = \frac{1}{q^2}\Big[\frac{(m^2+n^2)(\sinh^2 q - q^2)}{2\sinh^2 q} + \frac12\Big(\frac{q^2}{\sinh^2 2q} - \frac14\Big)\Big], \tag{150}$$

*so that* $a_2/a_0 = \frac{1}{2\sinh^2 2q} - \frac{m^2+n^2}{2\sinh^2 q}$ *and* $\partial_t v(q^2,0) = -\coth\mathsf{X}\coth\mathsf{Y}(\frac{m^2+n^2}{2\cosh^2 q} - \frac{1}{4\cosh^4 q}) + mn(\frac{1}{\sinh^2 q} - \frac{1}{q^2})$*. The principal discriminant is*

$$D(q,x) = \frac{\sin^2\theta - \mu^2\sinh^2 q}{\cosh^2 q}, \qquad \sinh q_* = \frac{\sin\theta}{\mu}, \qquad \partial_q D(q_*) = -\frac{2\sin^2\theta}{\tanh q_*\cosh^2 q_*} < 0. \tag{151}$$

*(c)* (E2) and (R). *The two lowest ranks of* $\widehat{\Psi}$ *are the values at* $t = h^2$ *of functions analytic in* $(x,t)$ *on a complex neighbourhood of* $X$ *times a disk (even in* $h$*) that equal* $\widehat{d}_{f_{\min}}, \widehat{d}_{f_{\min}+1}$ *at* $h = 0$*. Moreover (R) holds both intrinsically at a simple turn by Proposition 3.31 and globally via* $\widehat{\Psi}_k^2 \le [2A][2B]/[2C] = O(N)$ *for all* $k$*.*

*(d)* Turn. *The turn* $q_*$ *of* (151) *is simple with* $q_* < \min(\mathsf{X},\mathsf{Y})$ *if and only if the triangle is not right-angled at the vertex opposite* $\mathsf{X}$ *or* $\mathsf{Y}$*. Its type is* $\tau = \operatorname{sgn}\varepsilon_*$*,*

$$\varepsilon_* := x - b_0(q_*) = \frac{(x\coth\mathsf{X} - \coth\mathsf{Y})(\coth\mathsf{X} - x\coth\mathsf{Y})}{\Delta_0},$$
$$\tau = -\operatorname{sgn}\big[(\cosh\mathsf{X} - \cosh\mathsf{Y}\cosh\mathsf{Z})(\cosh\mathsf{Y} - \cosh\mathsf{X}\cosh\mathsf{Z})\big], \tag{152}$$

*and the lattice quantization* (75) *holds exactly:* $K_+ = C - |A-B| + \frac{|m-n|-1}{2}$ *and* $K_- = \kappa - (A+B-C) + \frac{|m-n|}{2}$ *are integers.*

Part (a) is the three-term recurrence of the $q$-Racah polynomials in the degree variable (the $q$-analogue of the Schulten–Gordon recurrence [37, 38]) in the Weyl-midpoint gauge. It is proved

here from the Kirillov–Reshetikhin sum [36] by a $q$-Zeilberger certificate (after reducing general $(m,n)$ to the fundamental chamber $m \geq |n|$ by the tetrahedral symmetries of the sum): after the gauge change $\sqrt{[2f+1]}\,P(f)$ the recurrence of the sum is an identity of rational functions in the eight symbols $q, q^a, q^b, q^c, q^f, q^m, q^n, q^z$, verified by exact symbolic computation (Lemma I.1); only the definition (K1) is needed (with $q$-orthogonality (K2) supplying an independent global bound for (R) and the Gauss-weight identity of Remark 3.35). Everything else in the proposition is proved from (a) in Appendix I, including the closed seed, which is a one-term sum. The $q$-deformation also makes $e(0,t) = \lambda(t) \neq 1$, whereas $e(0,t) = 1$ for the Hahn and Racah chains: without the factor $\lambda$, the $h^4/q^2$ coefficient of $B_k - B_k^{(0)}$ is $mn(m^2+n^2-1)/6 \neq 0$ and $v$ would not be analytic at $q = 0$.

**Lemma 6.2** (Turning action of the $q$-6$j$ chain)**.** *With $p(q) = \arccos((x-b_0)/e)$ on the branch $p(0) = \theta$, $q_*$ as in (151),*

$$\boxed{\int_0^{q_*} p\,\mathrm{d}q = \begin{cases} \dfrac{\pi}{2}\,(\mathsf{Z} - |\mathsf{X}-\mathsf{Y}|), & \tau = +1, \\ \pi q_* - \dfrac{\pi}{2}\,(\mathsf{X}+\mathsf{Y}-\mathsf{Z}), & \tau = -1, \end{cases}} \tag{153}$$

*that is $NS(q_*)/\pi = C - |A-B|$ for $\tau = +1$ and $NS(q_*)/\pi - Nq_* = -(A+B-C)$ for $\tau = -1$: the Racah values of Lemma 5.6 with the Weyl labels of the quantum symbol in place of the flat ones.*

The geometry behind it is that $p(q)$ is the dihedral angle at the edge of length $2q$ of the hyperbolic tetrahedron with the edge array $(A,B,C;B,A,F)$ of Section 5.3, $F = f + \frac{1}{2}$, all lengths multiplied by $2/N$ (hyperbolic edges $2q; \mathsf{Z}; \mathsf{X}, \mathsf{X}; \mathsf{Y}, \mathsf{Y}$), by exact symbolic computation: $\cos^2 p = C_{34}^2/(C_{33}C_{44})$ for the cofactors of the matrix of cosh's of the edge lengths, the hyperbolic analogue of the principal Racah symbol being a dihedral angle. The proof of (153) avoids degenerate tetrahedra: it differentiates the action in $\mathsf{Z}$ at fixed $\mathsf{X}, \mathsf{Y}$, where $\partial_{\mathsf{Z}} p = \mu/\sqrt{D}$ turns the integral into $\arcsin(\mu \sinh q/\sin\theta)|_0^{q_*} = \pi/2$, and integrates from the degenerate triangle (Appendix I). This is a Schläfli-type identity for the one-parameter family of tetrahedra; the integer $K_\tau$ is the lattice quantization needed by Theorem 3.27.

**Corollary 6.3** (The growing-rank theory for the one-small $q$-6$j$)**.** *Let $(\alpha,\beta,\gamma)$ range over a compact family of strict triangles, with fixed admissible $m, n \in \frac{1}{2}\mathbb{Z}$, $m - n \in \mathbb{Z}$.*

*(i)* Horizon. *Theorem 3.7 and Theorem 3.9 hold with $\chi$ of (150): $\sqrt{\pi\kappa\sin\theta/2}\,\Psi_L = \cos(\varphi_{L,mn}(\theta) + C_q\kappa^3/N^2) + O(\kappa^{-1} + \kappa^2 N^{-2} + \kappa^5 N^{-4})$ with $C_q = -\chi(x)/(3\sin\theta)$. The coherence horizon is $L \asymp N^{2/3}$ where $\chi(x) \neq 0$, and $C_q \to C_6$ of (120) in the flat limit.*

*(ii)* Second-order pullback. *Theorem 3.20 holds for $\Psi_L$, with $\Theta_q^{[2]}$, $W_q^{[2]}$ built from (150) and (47), (48): $\sqrt{2L+1}\,|\Psi_L - W_q^{[2]} d_{mn}^L(\Theta_q^{[2]})| \leq CN^{-2}$ in any turning-free corridor.*

*(iii)* Phase exactness, with a flux defect. *Theorem 3.24 holds with $\mathsf{J}_2$ from (67)–(68) and*

$$\boxed{c(x) = -\mathsf{J}_2(0,x) - \frac{1}{12}, \qquad \delta_{\mathrm{fl}} = -\frac{1}{12}, \qquad \alpha(q,x) = -\mathsf{J}_2(q,x) - \frac{1}{12}.} \tag{154}$$

*This is proved from the closed seed by exact symbolic computation: in the fundamental chamber $m \geq |n|$ at $k = m$ (to which general $(m,n)$ reduce by the symmetries of (K1), which leave $\widehat{\Psi}_k/\widehat{U}_k^{[2]}$ and $\mathsf{J}_2$ invariant) both $[t]\log\widehat{\Psi}_m$ and $[t]\log\widehat{U}_m^{[2]}$ are explicit, and their difference plus $\mathsf{J}_2(0,x)$ is $-\frac{1}{12}$ identically in $(e^{\mathsf{X}}, e^{\mathsf{Y}}, e^{\mathsf{Z}}, m, n)$; the $k$-independence of $c$ is Proposition 3.17. The constant $-\frac{1}{12}$ is the $h^2$-coefficient of $\frac{1}{2}\,h/\sinh h$ in the exact Gauss-weight identity (Remark 3.35). Consequently $\sqrt{2L+1}\,|\Psi_L - (1 - N^{-2}(\mathsf{J}_2 + \frac{1}{12}))W_q^{[2]} d_{mn}^L(\Theta_q^{[2]})| \leq CN^{-3}$.*

*(iv)* Bulk zeros. *The polynomial $P_{L,N}$ is a constant multiple of the q-Racah polynomial in* $\cosh \mathsf{Z}$ *(x is affine in* $\cosh \mathsf{Z}$*), and Theorem 3.26 holds: its zeros are within $O(N^{-4})$ of those of $d^L_{mn}(\Theta^{[2]}_q)$ and satisfy the Bohr–Sommerfeld rule (72) with $p_{ab} = |m-n|$ and $S = \int_0^q p$.*

*(v)* Turning connection and forbidden-side decay. *If the triangle is not right-angled at the vertex opposite* $\mathsf{X}$ *or* $\mathsf{Y}$ *and the turn type $\tau$ is fixed on the compact family, Theorem 3.27 and Propositions 3.31–3.34 hold for $\Psi_L$ with $\epsilon_{mn} = (-1)^{\max(n-m,0)}$, $K_\pm$ and $\tau$ as in Proposition 6.1(d), $q_*, D$ from (151), $u_1 = \sin^2\theta/(\sinh q_* \cosh q_* \, \varepsilon_*^2)$, and $\varpi_\tau$ from (79)–(82) with the data (150), which contain the $\eta$-term. With Proposition 3.28 and $\delta_{\mathrm{fl}} = -\frac{1}{12}$ the uniform bracket (80) has the amplitude $1 + N^{-2}(-\frac{1}{12} - \frac{1}{2}\mathrm{j}(q))$ and holds on $I$ except for a collar of width $O(N^{-1}\log N)$ at its forbidden end.*

*(vi)* Flat limit. *Under $q_{\mathrm{def}} = e^{\varsigma/N}$ at step $h = N^{-1}$, put $\mathsf{X} = 2\varsigma\alpha$, $\mathsf{Y} = 2\varsigma\beta$, $\mathsf{Z} = 2\varsigma\gamma$ and $q = \varsigma\rho$ ($\rho = (f+\frac{1}{2})/N$), so the (E1) arguments rescale as $(s,t) \mapsto (\varsigma^2\rho^2, \varsigma^2 h^2)$, multiplying $\varepsilon_1, \beta_1, \beta_2, \chi, \lambda_1, \eta, \delta_{\mathrm{fl}}$ of (150) by $\varsigma^2$ and $\varpi_\tau$ by $\varsigma^{4/3}$. As $\varsigma \to 0$, $x \to (\alpha^2+\beta^2-\gamma^2)/(2\alpha\beta)$, $q_*/\varsigma \to \rho_*$, $e(\varsigma^2\rho^2) \to e^{(6)}(\rho^2)$, $b_0(\varsigma\rho) \to \rho^2/(4\alpha\beta)$ of (112), $\varsigma^2(\varepsilon_1, \beta_1, \beta_2, \chi)$ tend to their Racah values, $\varsigma^2(\lambda_1, \eta(\varsigma^2\rho^2), \delta_{\mathrm{fl}}) \to 0$, (151) becomes (114), (153) divided by $\varsigma$ has the form of Lemma 5.6 identically at every $\varsigma > 0$ (with the turning rank of the quantum chain), and $\varsigma^{4/3}\varpi_\tau \to \varpi_\tau$ of (137).*

## 7 Comparison with exact symbols

The proofs are analytic, and the comparisons in this section illustrate the rates; none is used in a proof. Table 2 compares the main statements with exact $3j$ and $6j$ symbols. Figure 1 uses the exact $3j$ rank recurrence, started from an exact minimal-rank value and run in 30-digit arithmetic, to compare the fixed-rank Wigner approximation with the leading and second-order pullbacks of Proposition 4.2 and the phase-exact pullback of Corollary 4.4.

For the quantum $6j$ symbols of Section 6, consider three families with curvature scale $\varsigma \in \{0.6, 0.7, 0.5\}$, $(m,n) = (2,1), (1,0), (3,-1)$, hyperbolic triangles $2\varsigma(1, 1.1, 1.5)$, $2\varsigma(1.5, 0.8, 1.1)$, $2\varsigma(1.2, 1, 1.7)$ and $N = 25$–$255$. Evaluated in 40-digit arithmetic from the closed seed and the Jacobi rows, the natural-gauge errors against the exact symbol scale as $N^{-1}$ (leading pullback), $N^{-2}$ (second order) and $N^{-3}$ (phase-exact with (154)); dropping $\eta$ restores an $N^{-1}$ error, and replacing $-\frac{1}{12}$ by 0 restores an $N^{-2}$ error. The Airy bracket $\mathrm{Ai} + N^{-4/3}\varpi_\tau \, \mathrm{Ai}'$ has an $O(N^{-2})$ error for both turn types at $(m,n) = (2,1)$, $\tau = -1$ and $(3,3)$, $\tau = +1$, $N \le 415$, while omitting the $\eta$-term of $\varpi_\tau$ leaves an $N^{-4/3}$ residual. The $O(\varsigma^2)$ rate of the flat limit of the rescaled class data, of the turn, and of $\varsigma^{4/3}\varpi_\tau$ (Corollary 6.3(vi)) is also observed.

Exact computer algebra reproduces the identities used in the proofs: the closed form (53) and the contact identity of Proposition 3.17, symbolically in $(a,b,k)$; the value (68) and the identity $c = -\mathsf{J}_2(0,\cdot)$ for the Hahn and Racah chains; the local formula (79) from the formal Airy parametrix, its Olver form (82), and its Hahn and Racah reductions (100) and (137); and the secular cancellation behind (96).

## 8 Discussion

Weyl-midpoint centering places the fixed-offset $3j$, one-small $6j$, and one-small quantum $6j$ rank recurrences in the single class of even centered Wigner–Jacobi chains. Across this class, the $k^2/N^2$ phase defect sets the generic $N^{2/3}$ coherence horizon (where $\chi \neq 0$), the Hamilton–Jacobi action

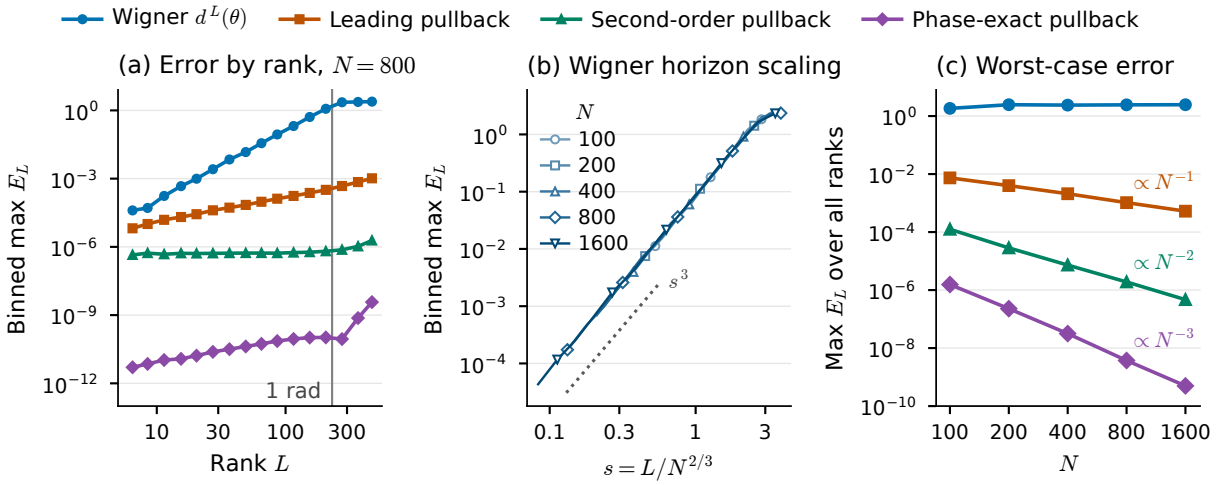



Figure 1: Natural-gauge error $E_L = \sqrt{2L+1}\,|\Phi_L - A_L|$ of an approximation $A_L$ to the $3j$ chain with $(M, \sigma) = (2, 1)$ and $x = \cos\theta = 0.3$, at every rank $3 \le L \le 0.65N$. Since $\Phi_L$ oscillates, $E_L$ passes near zero; panels (a) and (b) plot its maximum over logarithmic rank bins (eight per decade). **(a)** $N = 800$. The vertical line marks the rank at which the phase shift $\kappa^3 \cot\theta/(6N^2)$, $\kappa = L + \frac{1}{2}$, of (97) reaches one radian; there the fixed-rank approximation $d^L(\theta)$ has an order-one error. The three pullbacks stay accurate up to fixed ratio. **(b)** The error of $d^L(\theta)$ for $N = 100, \dots, 1600$ against $s = L/N^{2/3}$: the curves coincide, grow like $s^3$ (dotted) and level off at order one. **(c)** The maximum of $E_L$ over all ranks is of order one for $d^L(\theta)$ and decays like $N^{-1}$, $N^{-2}$ and $N^{-3}$ for the leading, second-order and phase-exact pullbacks.

resums the secular series, Darboux renormalization of the second Weyl eikonal produces an all-rank $O(N^{-2})$ pullback that $h$-parity makes phase exact to $O(N^{-3})$ via a real discrete-flux amplitude, bulk zeros follow to $O(N^{-4})$, and simple rank turns admit uniform second-order Airy asymptotics to $O(N^{-3}\mathsf{M}/\mathsf{E})$ with explicit Olver coefficients on a fixed turning neighbourhood $I'$ under the forbidden-side bound (R), and, for all three families, with intrinsic forbidden-side decay extending the law to the end of $I$ up to a logarithmic collar.

A separate technical supplement [1] addresses joint rank and hard-edge uniformity. It combines the physical-lattice edge factors of Ref. [2] with the present Hamilton–Jacobi construction and proves, for the fixed-offset Hahn/$3j$ chain, a joint error bound $O([N(1-|x|)]^{-2})$ in the natural gauge weighted by $D(q_L, x)^{1/4}$ for every physical position and all ranks below a fixed turning ratio. It also gives algebraic, fixed-rank and special-corner results for the centered Racah/$6j$ chain; the analogous global growing-rank Racah estimate remains conjectural. The supplement uses the results of Ref. [2] and of the present paper, but no theorem in the present paper depends on the supplement.

Several natural extensions remain open; none is used in the proofs above:

- *Boundary collisions.* Uniform two-column asymptotics where the Racah or $q$-$6j$ rank turn meets the hard edge ($\tau = 0$, limiting triangles right-angled opposite $\alpha$ or $\beta$, resp. $\mathsf{X}$ or $\mathsf{Y}$, where, for Racah, $\varpi_\tau$ diverges like $|\varepsilon|^{-8/3}$) and at the degenerate corner $A - B = O(1)$, $C = O(B^{1/2})$.
- *Dual Racah and Regge remainders.* Subprincipal two-direction transport for the dual $c$-lattice problem through the moving caustic, and an invariant tetrahedral formula for $\rho_2^{(6)}$ and higher Ponzano–Regge remainders.
- *Operator-level pullbacks and kernels.* Egorov or Fourier-integral operator formulations of the pullback and Christoffel–Darboux kernel asymptotics.

- *Global order $N^{-4}$.* In the proof of Theorem 3.18 a fixed-rank residual $O(h^{2m+2}\kappa^{3m+3})$, from contact through order $t^m$, sums to $O(h^{p-1})$ below a growing split rank $K_*$ ($1 \ll K_* \ll h^{-2/3}$, needed so the high-rank tail $K_*^{-(R-1)}$ is $O(h^{p-1})$) only when $p \le 2m+2$. Second-order contact ($m=1$) therefore stops at $p=4$, the $O(N^{-3})$ of Theorem 3.24; $O(N^{-4})$ would need a third-order pullback with fixed-rank contact through order $t^2$, a fixed-ratio residual $O(h^5)$ and an $O(h^4)$ start.
- *Proportional-index Hahn and general quantum $6j$.* Connection formulas when minimal rank and caustic move at order $N$; a direct semiclassical Gauss-weight proof of $\delta_{\mathrm{fl}} = 0$ (Hahn/Racah) and $\delta_{\mathrm{fl}} = -\frac{1}{12}$ ($q$-$6j$, Remark 3.35); and root-of-unity / Turaev–Viro regimes [34, 35].

The results bear on several applications. In full-sky cosmic microwave background bispectrum and lensing estimators, the two-large/one-small $3j$ symbol couples large multipoles $\ell \asymp N$ to a third multipole $L$, where the Regge-symmetric Wigner-$d$ approximation of Carron and Lewis [3] is used. Strongly squeezed configurations ($L$ fixed) are covered by the fixed-rank theory [2]; mildly squeezed configurations ($L$ in the hundreds for $\ell$ in the thousands) cross the $L \asymp N^{2/3}$ phase horizon, where Corollary 4.4 and Theorem 4.6 replace the fixed-$L$ formula to $O(N^{-3})$ on compact interior sets of $x$ and $O(N^{-2})$ in the band-edge Airy bracket. Likewise, growing-rank one-small $6j$ and $q$-$6j$ pullbacks bridge the Edmonds tensor-operator regime [8] and all-large spin-network asymptotics [9, 40, 31], evaluating a symbol at a given rank without stepping through all intermediate ranks [12, 13].

## Reproducibility

A companion GitHub repository, `https://github.com/cmbant/weyl-midpoint-semiclassics`, contains the technical supplement [1], the verification programs behind Section 7, their captured outputs, and the exact symbolic certificates used in the proofs (including the $q$-Zeilberger certificate of Lemma I.1 and the rational-function identity of Corollary 6.3(iii)). Symbolic checks certify the specified algebraic identities; finite numerical comparisons do not by themselves prove the uniform asymptotic bounds.

## Use of AI

All results, documents and scripts were generated by AI agents (Claude Opus and Sonnet 5.5, OpenAI's GPT 6.1 sol and Gemini 4 Hydrogen/Argon/Carbon) with human supervision and many cross-checks, including exact symbolic and high-precision numerical regression checks of the stated formulas. I thank DeepMind and Anthropic for access to their scientist programs.

## Acknowledgements

We acknowledge support by the UK STFC grant ST/T000473/1.

## A General Weyl and oscillatory-transfer details

This section and the next prove the general results of Section 3 for an arbitrary even centered Wigner–Jacobi chain. Only (E1)–(E2) of Definition 3.2 and the results of Section 3.5 are used; nothing specific to the Hahn or Racah coefficients enters.

## A.1 Analytic oscillatory transfer

**Lemma A.1** (Uniform analytic oscillatory transfer bound)**.** *Let $Z_{k+1} = T_k(t)Z_k$ and let $F_{k+1} = T_k(0)F_k$ be a naturally normalized oscillatory fundamental matrix with $F_k, F_k^{-1}$ uniformly bounded, $k \geq 1$. Suppose $T_k(t)$ is analytic in $t$ on $k^2|t| \leq c$ with*

$$T_k(t) = T_k(0) + \sum_{r=1}^{R-1} t^r E_{k,r} + \mathcal{R}_{k,R}(t), \tag{155}$$

*with*

$$\|E_{k,r}\| \leq C_r k^{2r}, \qquad \|\mathcal{R}_{k,R}(t)\| \leq C_R |t|^R k^{2R}, \tag{156}$$

*and assume the fixed starting vector is analytic in $t$ on $|t| \leq c$ with the analogous remainder (writing $Z_K(t) = \sum_{r\geq 0} t^r Z_K^{(r)}$). Then for $K^3|t| \leq \eta_0$,*

$$\left\| Z_K(t) - \sum_{r=0}^{R-1} t^r Z_K^{(r)} \right\|_{\mathrm{nat}} \leq C'_R (K^3|t|)^R, \tag{157}$$

$$\|Z_K^{(r)}\|_{\mathrm{nat}} \leq C'_r K^{3r}. \tag{158}$$

*Proof.* Put $Y_k = F_k^{-1} Z_k$. Since $F_{k+1} = T_k(0)F_k$, one has $Y_{k+1} = [I + V_k(t)]Y_k$ with $V_k(t) = F_{k+1}^{-1}[T_k(t) - T_k(0)]F_k$, which inherits the bounds (155)–(156). Taking $R = 2$ and shrinking the fixed analyticity disk if necessary gives $\|V_k(t)\| \leq C|t|k^2$ uniformly whenever $k^2|t| \leq c_1$. Hence on the disk $|t|K^3 \leq \eta$, with $\eta$ fixed sufficiently small, $\sum_{k<K} \|V_k(t)\| \leq C|t| \sum_{k<K} k^2 \leq C'|t|K^3$, and therefore the ordered product $\prod_{k<K}(I + V_k(t))$ and its inverse are analytic and bounded in norm by $\exp(C'|t|K^3)$ there. The same holds for the analytic starting vector, so $Y_K(t)$ is analytic and uniformly bounded on $|t| \leq \eta/K^3$.

Cauchy's estimate on that disk gives $\|Y_K^{(r)}(0)/r!\| \leq C_r K^{3r}$, and multiplication by the uniformly bounded $F_K$ gives (158). For the remainder at $t$ with $K^3|t| \leq \eta_0 < \eta/2$, applying Cauchy's integral formula on the circle of radius $\rho = \eta/(2K^3)$ and summing the geometric Taylor tail gives $\|Y_K(t) - \sum_{r=0}^{R-1} t^r Y_K^{(r)}\| \leq C_R(K^3|t|)^R$; the bounded change of basis $F_K$ yields (157). □

## A.2 The critical transfer normal form

**Lemma A.2** (Quantitative critical oscillatory transfer normal form)**.** *Fix $a, b \in \frac{1}{2}\mathbb{Z}$ with $a - b \in \mathbb{Z}$, a compact interval $K \Subset (-1, 1)$, and $\Lambda < \infty$. Let $T_k^{(0)}(x)$ be the transfer matrix of the rank recurrence for $d_{ab}^k(\arccos x)$ after the standard $N$-independent natural-amplitude gauge, and let $T_k^{(N)}$ be a second real transfer system in the same gauge. Suppose, uniformly for $x \in K$ and $k \leq L$, that $T_k^{(N)}(x) - T_k^{(0)}(x) = 2\chi(x)\frac{k^2}{N^2}E_{11} + \mathcal{R}_{k,N}(x)$ with $\|\mathcal{R}_{k,N}\| \leq C_K(k/N^2 + k^4/N^4)$, where $E_{11}$ has a single nonzero entry in its upper-left corner and $\chi$ is smooth on $K$. If*

$$\frac{L^3}{N^2} \leq \Lambda, \tag{159}$$

*then, for any fixed rank $k_0 \geq k_{\min} + 1$ and every $L \geq k_0$, the ordered transfer $\mathcal{U}_{L,N}(x) := F_L^{-1}(T_{L-1}^{(N)} \cdots T_{k_0}^{(N)})F_{k_0}$ in the exact complex Wigner basis has the form*

$$\mathcal{U}_{L,N}(x) = \begin{pmatrix} e^{-\mathrm{i}\vartheta_{L,N}(x)} & 0 \\ 0 & e^{\mathrm{i}\vartheta_{L,N}(x)} \end{pmatrix} + \mathcal{E}_{L,N}(x), \qquad \vartheta_{L,N}(x) = \frac{\chi(x)}{\sin\theta} \sum_{k=k_0}^{L-1} \frac{k^2}{N^2} \quad (x = \cos\theta), \tag{160}$$

*with*

$$\sup_{x\in K}\|\mathcal{E}_{L,N}(x)\| \le C_{K,\Lambda}\left(\frac{L^2}{N^2}+\frac{L^5}{N^4}\right), \qquad \sum_{k=k_0}^{L-1}\frac{k^2}{N^2}=\frac{L^3}{3N^2}+O\left(\frac{L^2}{N^2}\right). \tag{161}$$

The proof of Lemma A.2 uses the following matrix algebra. At leading Darboux order, write

$$F_k^{\mathrm{lead}}=\begin{pmatrix} e^{\mathrm{i}\varphi_k} & e^{-\mathrm{i}\varphi_k} \\ e^{\mathrm{i}(\varphi_k-\theta)} & e^{-\mathrm{i}(\varphi_k-\theta)}\end{pmatrix}, \qquad \varphi_{k+1}=\varphi_k+\theta. \tag{162}$$

Then direct inversion gives

$$(F_{k+1}^{\mathrm{lead}})^{-1}E_{11}F_k^{\mathrm{lead}}=\begin{pmatrix} -\dfrac{\mathrm{i}}{2\sin\theta} & \dfrac{e^{-\mathrm{i}(2\varphi_k-\theta)}}{e^{2\mathrm{i}\theta}-1} \\ \dfrac{e^{\mathrm{i}(2\varphi_k+\theta)}}{1-e^{2\mathrm{i}\theta}} & \dfrac{\mathrm{i}}{2\sin\theta}\end{pmatrix}. \tag{163}$$

The diagonal entries are the local eigenphase derivative, while the off-diagonal entries carry the mode-conversion phase $e^{\pm 2\mathrm{i}\varphi_k}$. The two-term Darboux expansion changes (163) by $O_K(k^{-1})$ with first difference $O_K(k^{-2})$, which is the regularity needed for the Abel summation of the mode-conversion terms in the next lemma.

**Lemma A.3** (Uniform Dirichlet bound for the Darboux phase)**.** *Let $z_k=e^{2\mathrm{i}\varphi_k}$ with $\varphi_{k+1}-\varphi_k=\theta+\delta_k$, $\theta\in[\theta_0,\pi-\theta_0]$, where $|\delta_k|\le\Lambda k^{-1}$ and $|\delta_{k+1}-\delta_k|\le\Lambda k^{-2}$. If $\sup_{a\le k\le b}|b_k|\le M$ and $\sum_{k=a}^{b-1}|b_{k+1}-b_k|\le V$, then $|\sum_{k=a}^{b}b_kz_k|\le C'(M+V)$ with $C'$ depending only on $\theta_0$ and $\Lambda$.*

*Proof.* Put $r_k=z_{k+1}/z_k=e^{2\mathrm{i}(\theta+\delta_k)}$. For $k\ge k_1(\theta_0,\Lambda)$ the spectral gap gives $|1-r_k|\ge c_{\theta_0}>0$, and the finitely many smaller ranks contribute at most $k_1M$. Moreover $|r_{k+1}-r_k|\le 2|\delta_{k+1}-\delta_k|\le 2\Lambda k^{-2}$. Thus $c_k=b_k/(1-r_k)$ satisfies $|c_k|\le M/c_{\theta_0}$ and $|c_{k+1}-c_k|\le|b_{k+1}-b_k|/c_{\theta_0}+2\Lambda Mk^{-2}/c_{\theta_0}^2$, so $\sum_k|c_{k+1}-c_k|\le C(V+M)$. Since $z_k=(z_k-z_{k+1})/(1-r_k)$, summation by parts gives $\sum_{k=a}^{b}b_kz_k=c_az_a-c_bz_{b+1}+\sum_{k=a+1}^{b}(c_k-c_{k-1})z_k$, which is bounded by $C'(M+V)$. □

## Proof of Lemma A.2: Quantitative critical oscillatory transfer normal form

*Proof.* In the natural amplitude gauge $\widehat{y}_k=\sqrt{2k+1}\,y_k$, the unperturbed Wigner recurrence is the Jacobi equation $x\widehat{y}_k=A_{k+1}\widehat{y}_{k+1}+A_k\widehat{y}_{k-1}+B_k\widehat{y}_k$ with $A_k=\frac12+O_K(k^{-2})$ and $B_k=O_K(k^{-2})$ for fixed $a,b$. Thus the unperturbed system is a summable perturbation of the constant elliptic transfer matrix with eigenvalues $e^{\pm\mathrm{i}\theta}$. The two exact complex solutions are built from the Darboux branches. By Lemma 3.6 at order $R\ge3$, $\sqrt{2k+1}\,d_{ab}^k(\theta)$ is a sum of two carrier-separated branches $e^{\pm\mathrm{i}(k+\frac12)\theta}\mathfrak{u}_k^{\pm}$ with $\mathfrak{u}_k^{\pm}$ a polynomial in $\kappa^{-1}$ with coefficients smooth on $K$, up to $O(k^{-R})$. Since the Wigner recurrence is exact for every $k$ and $e^{2\mathrm{i}\theta}\neq1$ on $K$, the residual of each branch separately is $O_K(k^{-R})$, so each branch is an approximate solution. Write $\mathfrak{u}_k^{\pm}:=e^{\pm\mathrm{i}\kappa\theta}\sum_{r<R}\kappa^{-r}A_{r,ab}^{\pm}(\theta)$ for the two branches, $\kappa=k+\frac12$, and $\mathfrak{F}_k$ for the matrix with columns $(\mathfrak{u}_k^{\pm},\mathfrak{u}_{k-1}^{\pm})^{\mathsf{T}}$. The Wigner transfer matrix is $T_k^{(0)}=\left(\begin{smallmatrix}(x-B_k)/A_{k+1} & -A_k/A_{k+1}\\ 1 & 0\end{smallmatrix}\right)$ with $A_k=\frac12+O_K(k^{-2})$ and $B_k=O_K(k^{-2})$, so $T_k^{(0)}$ is bounded and, for $k>k_{\min}$, invertible. Each branch alone has residual $O_K(k^{-R})$, so $\mathfrak{F}_{k+1}=T_k^{(0)}\mathfrak{F}_k+\mathcal{G}_k$ with $\|\mathcal{G}_k\|\le C_Kk^{-R}$ (only the first row of $\mathcal{G}_k$ is nonzero). Moreover $\mathfrak{F}_k$ is bounded and, since $c_{ab}^+c_{ab}^-=\pi^{-1}$, $\det\mathfrak{F}_k=\mathfrak{u}_k^+\mathfrak{u}_{k-1}^--\mathfrak{u}_k^-\mathfrak{u}_{k-1}^+=2\mathrm{i}/\pi+O_K(k^{-1})$, so $\mathfrak{F}_k^{-1}$ is bounded for $k\ge k_2$. Look for exact solutions $\mathfrak{F}_kC_k$ of $Y_{k+1}=T_k^{(0)}Y_k$. Then $C_{k+1}=(I+\eta_k)C_k$ with $\eta_k:=-\mathfrak{F}_{k+1}^{-1}\mathcal{G}_k$, $\|\eta_k\|\le Ck^{-R}$, which is summable since $R\ge3$. The ordered products $C_k^{\pm}:=\prod_{j\ge k}(I+\eta_j)^{-1}e_{\pm}$ converge and solve this recursion for $k\ge k_2$ (extend them to $k\ge k_0$ by the exact recurrence,

which is possible because $T_k^{(0)}$ is invertible); they satisfy $\|C_k^{\pm} - e_{\pm}\| \le C\sum_{j\ge k}\|\eta_j\| \le Ck^{1-R}$ and $\|C_{k+1}^{\pm} - C_k^{\pm}\| = \|\eta_k C_k^{\pm}\| \le Ck^{-R}$, uniformly on $K$. The first components $u_k^{\pm} := (\mathfrak{F}_k C_k^{\pm})_1$ are therefore exact solutions, with $u_k^{+} = \mathfrak{u}_k^{+}(1 + c_k^{(1)}) + \mathfrak{u}_k^{-} c_k^{(2)}$, where $c_k := C_k^{+} - e_{+}$ obeys $|c_k| \le Ck^{1-R}$ and $|c_{k+1} - c_k| \le Ck^{-R}$, and $u_k^{-} = \overline{u_k^{+}}$ because the recurrence is real. Dividing by the leading term $A_+ e^{\mathrm{i}\varphi_k}$ of $\mathfrak{u}_k^{+}$, the bracket is $1 + g_k$ with $g_k = \kappa^{-1}A_{1,ab}^{+}/A_{0,ab}^{+} + O_K(k^{-2})$ and $|g_{k+1} - g_k| \le C_K k^{-2}$: the finite branch sum is smooth in $\kappa^{-1}$ (its terms of order $r \ge 1$ have differences $O(k^{-r-1})$), and the correction $e^{-2\mathrm{i}\varphi_k}c_k^{(2)} + c_k^{(1)}$ is $O(k^{1-R})$ with difference $O(k^{1-R}) \le O(k^{-2})$, because $R \ge 3$. (The difference-system theorem of Benzaid and Lutz [45] by itself gives only $1 + o(1)$; the extra regularity comes from the explicit branch.) This yields two conjugate exact solutions $u_k^{\pm}$ and a fundamental matrix $F_k = \left(\begin{smallmatrix} u_k^{+} & u_k^{-} \\ u_{k-1}^{+} & u_{k-1}^{-} \end{smallmatrix}\right)$ such that

$$u_k^{\pm} = A_{\pm}(x)e^{\pm\mathrm{i}\varphi_k(x)}\left[1 + O_K(k^{-1})\right], \qquad \varphi_k = (k + \tfrac{1}{2})\theta + \varphi_0, \tag{164}$$

where $|A_{\pm}|$ is bounded above and below on $K$ and the discrete difference of the $O(k^{-1})$ coefficient is $O_K(k^{-2})$ (shown above). The ordinary Jacobi Darboux formula fixes the phase and normalization of the physical Wigner-$d$ combination uniformly in the interior [26]. Since $T_k^{(0)}F_k = F_{k+1}$, the perturbed coordinates $Y_k = F_k^{-1}Z_k$ satisfy

$$Y_{k+1} = \left[I + F_{k+1}^{-1}(T_k^{(N)} - T_k^{(0)})F_k\right]Y_k. \tag{165}$$

Inverting the leading Darboux matrices as in (163), with $x = \cos\theta$, one obtains

$$F_{k+1}^{-1}E_{11}F_k = \begin{pmatrix} -\dfrac{\mathrm{i}}{2\sin\theta} & \beta(x)e^{-2\mathrm{i}\varphi_k} \\ \overline{\beta(x)}e^{2\mathrm{i}\varphi_k} & \dfrac{\mathrm{i}}{2\sin\theta} \end{pmatrix} + O_K(k^{-1}), \tag{166}$$

where $\beta$ is bounded and smooth; the $O(k^{-1})$ term again has discrete difference $O_K(k^{-2})$. Hence (165) has the form

$$Y_{k+1} = [I + D_k + Q_k + S_k]Y_k, \qquad D_k = \frac{k^2}{N^2}\begin{pmatrix} -\mathrm{i}\chi/\sin\theta & 0 \\ 0 & \mathrm{i}\chi/\sin\theta \end{pmatrix}, \tag{167}$$

where $Q_k$ is off diagonal with entries $O_K(k^2/N^2)e^{\pm 2\mathrm{i}\varphi_k}$, and

$$\sum_{k\le L}\|S_k\| = O_K\left(\frac{L^2}{N^2} + \frac{L^5}{N^4}\right) = o(1) \tag{168}$$

under (159). Here $Q_k = 2\chi(x)\frac{k^2}{N^2}\left(\begin{smallmatrix} 0 & \beta e^{-2\mathrm{i}\varphi_k} \\ \overline{\beta}e^{2\mathrm{i}\varphi_k} & 0 \end{smallmatrix}\right)$ is the off-diagonal part of the leading term, and $S_k$ collects the $O_K(k^{-1})$ remainder of (166) times $2\chi k^2/N^2$ and the term $F_{k+1}^{-1}\mathcal{R}_{k,N}F_k = O(k/N^2 + k^4/N^4)$. We do not factor out the diagonal $D_k$ first: it is kept in the Harris–Lutz step below, and the diagonal product is extracted at the end.

It remains to remove $Q_k$. The geometric partial sums of $e^{\pm 2\mathrm{i}(k+1/2)\theta}$ are uniformly bounded on $K$; in the notation of Lemma A.3 the phase increment of the entries of $Q_k$ is exactly $\theta$ ($\delta_k = 0$). The amplitudes of $Q_k$ are $b_k = 2\chi\beta k^2/N^2$, so on $r \le k \le s \le L$ they have $M = O(L^2/N^2)$ and $V \le \sum_{k\le L} 2|\chi\beta|(2k+1)/N^2 = O(L^2/N^2)$. Abel summation (Lemma A.3) therefore gives

$$\sup_{r<s\le L}\left\|\sum_{k=r}^{s} Q_k\right\| \le C_{K,\Lambda}\frac{L^2}{N^2}, \tag{169}$$

while $\sum_{k\le L}\|D_k\| = O(L^3/N^2) = O(\Lambda)$, $\sum_{k\le L}\|Q_k\| = O(L^3/N^2) = O(\Lambda)$ and $\sum_{k\le L}\|D_k\|^2 = O(L^5/N^4)$. Put $H_k := -\sum_{j=k}^{L-1} Q_j$, so that $H_L = 0$, $H_{k+1} - H_k = Q_k$ and $\eta_H := \sup_k \|H_k\| \le C_{K,\Lambda} L^2/N^2$. The trivial bound first: $\|I + D_k + Q_k + S_k\| \le 1 + \|D_k\| + \|Q_k\| + \|S_k\|$, and these three sums are bounded by constants depending only on $K, \Lambda$, so $\|\mathcal{U}_{L,N}\|$ and the diagonal matrix below are bounded by $C_{K,\Lambda}$; if $\eta_H > \frac{1}{2}$ then $L^2/N^2$ is bounded below by a constant depending only on $K, \Lambda$ and (161) follows from these bounds, so we may assume $\eta_H \le \frac{1}{2}$.

Substitute $Y_k = (I+H_k)V_k$ in (167), keeping $D_k$. The matrix identity $(I+D_k+Q_k+S_k)(I+H_k) = (I + H_{k+1})(I + D_k) + \Pi_k$ holds with

$$\Pi_k := \big(D_k H_k - H_{k+1} D_k\big) + Q_k H_k + S_k(I + H_k), \tag{170}$$

because $I + H_k + Q_k = I + H_{k+1}$ cancels the terms linear in $Q_k$ and $H_k$. Therefore

$$V_{k+1} = [I + D_k + P_k]V_k, \qquad P_k := (I + H_{k+1})^{-1}\Pi_k, \qquad \|P_k\| \le 2\big[2\eta_H\|D_k\| + \eta_H\|Q_k\| + \tfrac{3}{2}\|S_k\|\big]. \tag{171}$$

The first term of $\Pi_k$ is the commutator-type contribution of the diagonal part; with $\sum\|D_k\| = O(\Lambda)$ it is the dominant one, and its total is $4\eta_H\sum_k\|D_k\| \le C_{K,\Lambda}\Lambda\, L^2/N^2$. With $\sum\|Q_k\| = O(\Lambda)$ and (168),

$$\sum_{k\le L}\|P_k\| \le C_{K,\Lambda}\Big(\eta_H\sum_k(\|D_k\| + \|Q_k\|) + \sum_k\|S_k\|\Big) \le C_{K,\Lambda}\Big(\frac{L^2}{N^2} + \frac{L^5}{N^4}\Big). \tag{172}$$

It remains to extract the diagonal product. The matrices $D_k$ are diagonal with purely imaginary entries $\mp\mathrm{i}\epsilon_k$, $\epsilon_k = \chi k^2/(N^2\sin\theta)$, so $\Delta_k := \prod_{j=k_0}^{k-1}(I + D_j)$ is diagonal, $\|\Delta_k^{-1}\| \le 1$ (each factor $1 \mp \mathrm{i}\epsilon_j$ has modulus $\ge 1$) and $\|\Delta_k\| \le \exp(\frac{1}{2}\sum\epsilon_j^2) \le C_{K,\Lambda}$. For $W_k := \Delta_k^{-1}V_k$ one has $W_{k+1} = W_k + \Delta_{k+1}^{-1}P_k\Delta_k W_k$, hence $\|W_L - W_{k_0}\| \le C\sum_k\|P_k\|\exp(C\sum_k\|P_k\|)$. Since $H_L = 0$, $Y_L = V_L = \Delta_L W_L$ and $Y_{k_0} = (I + H_{k_0})V_{k_0}$, so

$$\mathcal{U}_{L,N} = \Delta_L(I + H_{k_0})^{-1} + O\Big(\sum_k\|P_k\|\Big) = \Delta_L + O_{K,\Lambda}\Big(\frac{L^2}{N^2} + \frac{L^5}{N^4}\Big). \tag{173}$$

Finally $1 \mp \mathrm{i}\epsilon_j = \sqrt{1+\epsilon_j^2}\, e^{\mp\mathrm{i}\arctan\epsilon_j}$ and $\arctan\epsilon = \epsilon + O(\epsilon^3)$, so $\Delta_L = \mathrm{diag}(e^{-\mathrm{i}\vartheta_{L,N}}, e^{\mathrm{i}\vartheta_{L,N}}) + O(\sum_j\epsilon_j^2)$ with $\vartheta_{L,N} = \frac{\chi}{\sin\theta}\sum_{k=k_0}^{L-1}\frac{k^2}{N^2}$, an error $O(L^5/N^4)$. Thus the full ordered transfer is the diagonal phase product plus an error bounded by $C_{K,\Lambda}(L^2/N^2 + L^5/N^4)$ in the normalized Wigner basis (within the window (159), $L^5/N^4 \le \Lambda L^2/N^2$, so the second term is redundant; it is kept to match the statement). Any mismatch of the fixed starting vector is added separately in applications, and (161) is the elementary sum-of-squares formula. □

## A.3 Turning-free bulk: adiabatic basis and fixed-start stability

### Proof of Lemma 3.6: Differentiated interior Wigner–Darboux expansion

*Proof.* Fix $R, s$ and put $R' = R + s$. Enlarge $J$ to a compact interval $J_1 \Subset (0,\pi)$ and write $w = (\sin\vartheta)^{1/2}\sqrt{2k+1}\, d^k_{ab}$. It is real and solves exactly

$$w'' + (\kappa^2 - V)w = 0, \qquad V = \frac{a^2 + b^2 - \frac{1}{4} - 2ab\cos\vartheta}{\sin^2\vartheta}, \tag{174}$$

with $V$ analytic and bounded on $J_1$ (this is the Jacobi equation in Liouville normal form, as used in Lemma D.1).

*Step 1: undifferentiated carrier-separated expansion.* The Frenzen–Wong Bessel-type expansion of the Jacobi polynomial with the Hankel expansion of the Bessel functions (DLMF 18.15.6 and 10.17.3; this is the expansion written out in the proof of Lemma 3.23) gives smooth functions $\widetilde{A}_r^\sigma$ on $J_1$, independent of $k$, with

$$\sup_{J_1}\Big|w-\sum_{\sigma=\pm1}e^{i\sigma\kappa\vartheta}T^\sigma\Big|\le C\kappa^{-R'},\qquad T^\sigma:=\sum_{r<R'}\kappa^{-r}\widetilde{A}_r^\sigma(\vartheta). \tag{175}$$

Only the supremum norm of the remainder is available from these expansions; the remainder of the uniform Bessel-type expansion is not differentiated. Put $\widehat{w}:=\sum_\sigma e^{i\sigma\kappa\vartheta}T^\sigma$ and $\varrho:=w-\widehat{w}$, so $\sup_{J_1}|\varrho|\le C\kappa^{-R'}$.

*Step 2: the coefficients satisfy the Liouville–Green recursion.* Since $w$ is an exact solution, $\varrho''+(\kappa^2-V)\varrho=-\Omega$ with $\Omega:=\widehat{w}''+(\kappa^2-V)\widehat{w}$. Expanding the explicit sum, with $\widetilde{A}^\sigma_{R'}:=0$,

$$\Omega=\sum_\sigma e^{i\sigma\kappa\vartheta}\Big[2i\sigma\kappa\,(\widetilde{A}_0^\sigma)'+\sum_{j=0}^{R'-1}\kappa^{-j}\Omega_j^\sigma\Big],\qquad \Omega_j^\sigma:=(\widetilde{A}_j^\sigma)''-V\widetilde{A}_j^\sigma+2i\sigma(\widetilde{A}_{j+1}^\sigma)'. \tag{176}$$

Let $\chi\in C_c^\infty(\operatorname{int}J_1)$. Two integrations by parts give $\int\chi e^{-i\sigma\kappa\vartheta}(\varrho''+\kappa^2\varrho)\,\mathrm{d}\vartheta=\int\varrho\,e^{-i\sigma\kappa\vartheta}(\chi''-2i\sigma\kappa\chi')\,\mathrm{d}\vartheta=O(\kappa^{1-R'})$ and $\int\chi e^{-i\sigma\kappa\vartheta}V\varrho=O(\kappa^{-R'})$, so $\int\chi e^{-i\sigma\kappa\vartheta}\Omega\,\mathrm{d}\vartheta=O(\kappa^{1-R'})$. In this integral the terms with carrier $e^{-i\sigma\kappa\vartheta}$ are non-oscillatory, and the terms with the opposite carrier are $O(\kappa^{-n})$ for every $n$ (repeated integration by parts). Hence the polynomial in $\kappa^{-1}$

$$2i\sigma\kappa\int\chi(\widetilde{A}_0^\sigma)'+\sum_{j<R'}\kappa^{-j}\int\chi\,\Omega_j^\sigma \tag{177}$$

with constant coefficients is $O(\kappa^{1-R'})$ along $\kappa\in k_{\min}+\frac12+\mathbb{Z}_{\ge0}$, so its coefficients of $\kappa^1,\kappa^0,\dots,\kappa^{2-R'}$ vanish. As $\chi$ is arbitrary, $(\widetilde{A}_0^\sigma)'\equiv0$ and $\Omega_j^\sigma\equiv0$ for $j\le R'-2$. Therefore

$$\Omega=\kappa^{1-R'}\sum_\sigma e^{i\sigma\kappa\vartheta}[(\widetilde{A}^\sigma_{R'-1})''-V\widetilde{A}^\sigma_{R'-1}], \tag{178}$$

and $\sup_{J_1}|\partial^m\Omega|\le C_m\kappa^{m+1-R'}$.

*Step 3: derivative bounds with one power of $\kappa$ lost per derivative.* By the equation for $\varrho$, $|\varrho''|\le(\kappa^2+\sup|V|)|\varrho|+|\Omega|\le C\kappa^{2-R'}$ on $J_1$. The Landau–Kolmogorov inequality on $J_1$, $\sup|\varrho'|\le C(\sup|\varrho|^{1/2}\sup|\varrho''|^{1/2}+\sup|\varrho|)$, gives $\sup|\varrho'|\le C\kappa^{1-R'}$. Differentiating the equation, $\varrho^{(m+2)}=(V-\kappa^2)\varrho^{(m)}+\sum_{i\ge1}\binom{m}{i}V^{(i)}\varrho^{(m-i)}-\Omega^{(m)}$, and induction on $m$ gives $\sup_J|\varrho^{(m)}|\le C_m\kappa^{m-R'}$ for all $m\ge0$.

*Step 4: conclusion.* Put $A^\sigma_{r,ab}:=(\sin\vartheta)^{-1/2}\widetilde{A}_r^\sigma$ for $r<R$, and

$$\mathcal{E}^+_{R,k}:=(\sin\vartheta)^{-1/2}\Big[\sum_{R\le r<R'}\kappa^{-r}\widetilde{A}_r^+ +e^{-i\kappa\vartheta}\varrho\Big],\qquad \mathcal{E}^-_{R,k}:=(\sin\vartheta)^{-1/2}\sum_{R\le r<R'}\kappa^{-r}\widetilde{A}_r^-. \tag{179}$$

Then (26) holds exactly, and $\partial^m\mathcal{E}^\pm_{R,k}=O(\kappa^{-R}+\kappa^{m-R'})=O(\kappa^{-R})$ for $m\le s$, because $\partial^m(e^{-i\kappa\vartheta}\varrho)=e^{-i\kappa\vartheta}\sum_i\binom{m}{i}(-i\kappa)^{m-i}\varrho^{(i)}=O(\kappa^{m-R'})$. The coefficients $A^\sigma_{r,ab}$ do not depend on $R,s$, since coefficients of a carrier-separated expansion are unique (the uniqueness fact in the proof of Lemma B.3). □

## Proof of Lemma 3.11: Turning-free adiabatic rank basis

*Proof.* By (E1) the exact coefficients are the midpoint samples $A_k=a(kh;h^2)$ and $B_k=b(q_k;h^2)$ of

$$a(s;h^2)=A_k^{(0)}e_\pi(s^2,h^2),\qquad b(q;h^2)=\lambda_\pi(h^2)B_k^{(0)}+v_\pi(q^2,h^2), \tag{180}$$

with $A_k^{(0)}$ and $B_k^{(0)}$ from (16). Put $a_0(s) = \frac{1}{2}e_\pi(s^2)$ (principal coefficient) and $b_0(q) = v_\pi(q^2, 0)$ as in (21). Since $2A_k^{(0)} = 1 + O(k^{-2})$ and $B_k^{(0)} = O(k^{-2})$, with first differences one order smaller, $e_\pi(s^2, h^2) = e_\pi(s^2) + O(h^2)$, $\lambda_\pi(h^2) = 1 + O(h^2)$ and $v_\pi(q^2, h^2) = b_0(q) + O(h^2)$,

$$A_k = a_0(kh)[1 + O(N^{-2} + k^{-2})], \qquad B_k = b_0(q_k) + O(N^{-2} + k^{-2}). \tag{181}$$

At the midpoint $q = q_k$ the principal recurrence is

$$[x - b_0(q)]Y(q) = a_0(q + h/2)Y(q + h) + a_0(q - h/2)Y(q - h). \tag{182}$$

Write $Y(q) = A(q)e^{\pm \mathrm{i}S(q)/h}$. Taylor expansion of the two terms in (182) gives at orders $h^0$ and $h^1$

$$2a_0 \cos p = x - b_0, \qquad p = S_q, \tag{183}$$

$$2a_0 \sin p \, \frac{A_q}{A} + a_{0,q} \sin p + a_0 p_q \cos p = 0; \tag{184}$$

the diagonal term $b_0$ is not shifted and enters only at order $h^0$. On the branch continuous from $q = 0$, (183) is the principal eikonal of (21), so $S$ is the principal action. Equation (184) is equivalently

$$\frac{A_q}{A} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}q} \log(a_0 \sin p). \tag{185}$$

Since $a_0(q) \sin p(q, x) = \frac{1}{2}\sqrt{D(q, x)}$ by (21), we may take $A(q) = D(q, x)^{-1/4}$, which is the amplitude in (36).

Because $\mathcal{C}$ is compact and bounded away from the turning set, $a_0$, $D^{-1}$, $p$ and all derivatives needed in the preceding Taylor expansion are uniformly bounded, while $\sin p \geq c_{\mathcal{C}} > 0$. Taylor's formula with remainder therefore shows that the principal recurrence residual of each branch is bounded by $C_{\mathcal{C}} h^2 |z_k^\pm|$. Restoring the exact coefficients through (181) gives the explicit defect

$$r_k^\pm := x z_k^\pm - A_{k+1} z_{k+1}^\pm - A_k z_{k-1}^\pm - B_k z_k^\pm, \qquad |r_k^\pm| \leq C_{\mathcal{C}}(N^{-2} + k^{-2})|z_k^\pm|. \tag{186}$$

Now write the exact one-step transfer as

$$T_k = \begin{pmatrix} (x - B_k)/A_{k+1} & -A_k/A_{k+1} \\ 1 & 0 \end{pmatrix}. \tag{187}$$

The determinant of the branch matrix $P_k$ of (37) is

$$\det P_k = 2\mathrm{i}\, D(q_k, x)^{-1/4} D(q_{k-1}, x)^{-1/4} \sin(N[S(q_k, x) - S(q_{k-1}, x)]), \tag{188}$$

up to an irrelevant sign. Since

$$N[S(q_k, x) - S(q_{k-1}, x)] = p(q_k - h/2, x) + O_{\mathcal{C}}(h^2), \tag{189}$$

$P_k$ and $P_k^{-1}$ are uniformly bounded for all sufficiently large $N$. Equation (186) then gives

$$P_{k+1}^{-1} T_k P_k = I + E_k, \qquad \|E_k\| \leq C_{\mathcal{C}}(N^{-2} + k^{-2}). \tag{190}$$

Indeed $T_k P_k - P_{k+1}$ has zero second row and first row $(r_k^+/A_{k+1}, r_k^-/A_{k+1})$.

Initialize an exact fundamental matrix by $\mathcal{F}_K^{(N)} = P_K$ and write $\mathcal{F}_k^{(N)} = P_k C_k$. Then $C_{k+1} = (I + E_k)C_k$, $C_K = I$. Therefore

$$\sup_{K \leq k \leq L} \|C_k - I\| \leq \exp\left(\sum_{j=K}^{L-1} \|E_j\|\right) - 1 = O_{\mathcal{C}}(K^{-1} + L/N^2). \tag{191}$$

The term $L/N^2$ is retained so that the bound covers final ranks from mesoscopic to fixed ratio; on a fixed-ratio compact set it is simply $O(N^{-1})$. This proves (37). The argument is the finite-interval, turning-free discrete Liouville–Green construction; compare Geronimo–Smith [43]. □

### Proof of Corollary 3.12: Fixed-start turning-free stability

*Proof.* Choose an admissible intermediate rank $K = \min\{L, \lfloor N^{1/3} \rfloor + O(1)\}$ (if $L \le k_0$ there is nothing to prove). On the finite/mesoscopic interval $k_0 \le k \le K$, the exact transfer is, by (E1), an analytic $t = h^2$ perturbation of the Wigner transfer with order-$t^r$ coefficients $O(k^{2r})$. Apply the interaction representation of Lemma A.1 to the two columns of a bounded Wigner-$d$ fundamental matrix initialized at $k_0$. Since $K^3 h^2 = O(N^{-1})$, both the interaction product and its inverse are uniformly bounded, so the exact fundamental matrix and inverse are bounded through $K$. If $L > K$, on $K \le k \le L$ Lemma 3.11 applies because $K \to \infty$ and $K/N \to 0$; its comparison matrix $P_k$ and inverse are uniformly bounded by turning-free interiority. The two exact fundamental matrices differ at $K$ by a constant invertible change of basis whose norm and inverse norm are bounded by the preceding estimates. This propagates the bound through $L$. □

**Lemma A.4** (Shadow-phase pole)**.** *On a compact set of $(q, x, \pi)$ with $0 < q$, $q^2 < s_1$ and $D(q, x) > 0$, let $p_h$ be the exact frozen phase (15) of an even centered chain, with the exact midpoint coefficients*

$$a(s; h^2) = \frac{e_\pi(s^2, h^2)}{2} \left[ \frac{(1 - a^2 h^2/s^2)(1 - b^2 h^2/s^2)}{1 - h^2/(4s^2)} \right]^{1/2}, \qquad b(q; h^2) = \frac{\lambda_\pi(h^2)\, ab\, h^2}{q^2 - h^2/4} + v_\pi(q^2, h^2). \tag{192}$$

*Then, uniformly while $h/q$ is small,*

$$p_h = p + h^2 p_2 + O(h^4 q^{-4}), \tag{193}$$

*where*

$$p_2 = \frac{1}{\sin p} \left[ \frac{ab/q^2 + \partial_t v_\pi(q^2, 0)}{e_\pi(q^2)} + \cos p \left( \frac{(\log a_0)''(q)}{8} - \frac{a^2 + b^2 - \frac{1}{4}}{2q^2} + \eta_\pi(q^2) \right) \right], \tag{194}$$

*and*

$$p_2(q, x) = -\frac{\mathfrak{c}_{ab}(\theta)}{q^2} + O(1) \qquad (q \downarrow 0), \qquad \theta = \arccos x. \tag{195}$$

*In particular a leading-order adiabatic basis that uses only $p = S_q$ accumulates an $O(K^{-1})$ fixed-index shadow-phase error between $q \asymp K/N$ and a fixed positive ratio, since $N \int_{K/N}^{q_0} h^2 |p_2| \, \mathrm{d}q = O(K^{-1})$.*

*Proof.* At fixed $(q, x)$, average $\log a(s; h^2)$ over $s = q \pm h/2$: the factor $e_\pi(s^2, h^2)/2 = \frac{1}{2} e_\pi(s^2)(1 + h^2 \eta_\pi(s^2) + O(h^4))$ contributes $\log a_0(q) + h^2(\frac{1}{8}(\log a_0)''(q) + \eta_\pi(q^2)) + O(h^4)$, and the bracket contributes $-(a^2 + b^2 - \frac{1}{4})h^2/(2q^2) + O(h^4 q^{-4})$. Hence $2\sqrt{a(q + h/2)a(q - h/2)} = e_\pi(q^2)[1 + h^2(\frac{1}{8}(\log a_0)'' + \eta_\pi - (a^2 + b^2 - \frac{1}{4})/(2q^2)) + O(h^4 q^{-4})]$, while $x - b = x - b_0 - h^2(ab/q^2 + \partial_t v_\pi(q^2, 0)) + O(h^4 q^{-4})$. Expanding (15) and differentiating the analytic arccos branch gives (194). As $q \downarrow 0$, (E1) gives $e_\pi = 1 + O(q^2)$, $\cos p = x + O(q^2)$, $\sin p = \sin\theta + O(q^2)$, and $(\log a_0)''$, $\eta_\pi$ and $\partial_t v_\pi$ bounded, so the only pole is

$$\frac{ab - x(a^2 + b^2 - \frac{1}{4})/2}{q^2 \sin\theta} = -\frac{\mathfrak{c}_{ab}(\theta)}{q^2} \tag{196}$$

by (25). The accumulated estimate follows from $Nh^2 = h$ and $\int_{K/N}^{q_0} q^{-2} \mathrm{d}q \le N/K$. □

## B Second-order closure for even centered chains

### Proof of Proposition 3.15: Generic second midpoint eikonal

*Proof.* Write

$$Y(q) = \exp\{g(q) + i h^{-1}(S(q) + h^2 S_2(q))\}, \quad p = S_q, \quad \sigma_2 = S_{2,q}. \tag{197}$$

For either sign $\varepsilon \in \{+1,-1\}$, direct Taylor expansion gives

$$\begin{aligned} a(q+\varepsilon h/2;h^2) &= a_0 + \varepsilon\frac{h}{2}a_0' + h^2\left(a_2+\frac{1}{8}a_0''\right)+O(h^3),\\ \frac{Y(q+\varepsilon h)}{Y(q)} &= e^{i\varepsilon p}\left\{1+hA_\varepsilon + h^2\left(B_\varepsilon+\frac{1}{2}A_\varepsilon^2\right)+O(h^3)\right\},\\ A_\varepsilon &= \varepsilon g' + \frac{i}{2}p', \qquad B_\varepsilon = \frac{1}{2}g'' + i\varepsilon\left(\frac{1}{6}p''+\sigma_2\right). \end{aligned} \tag{198}$$

The $h^0$ coefficient yields $x - b_0 = 2a_0\cos p$. The coefficient of $ih$ is

$$2a_0 g'\sin p + a_0 p'\cos p + a_0'\sin p = 0, \tag{199}$$

equivalently $(a_0 e^{2g}\sin p)' = 0$; hence $g = -\frac{1}{2}\log(a_0\sin p)$ up to an additive constant. Using this value of $g$, the *real* $h^2$ equation is

$$\begin{aligned} 2a_0\sin p\,\sigma_2 = {}& a_0(g''+g'^2)\cos p - a_0 g' p'\sin p - \frac{a_0}{4}(p')^2\cos p - \frac{a_0}{3}p''\sin p\\ &+ 2a_2\cos p + a_0' g'\cos p - \frac{1}{2}a_0' p'\sin p + \frac{1}{4}a_0''\cos p + b_2. \end{aligned} \tag{200}$$

This is (46), including the curvature terms absent from a frozen dispersion expansion. All Taylor remainders and their finitely many $q$ derivatives are uniform on a fixed compact elliptic set where $a_0$ and $\sin p$ are bounded away from zero.

Analyticity in $t = q^2$ at a matched endpoint does not follow from this formal expansion alone. For an even centered chain it is Proposition 3.16: the $t^{-1}$ part cancels against the derivative of the finite-rank Darboux phase. □

## Proof of Proposition 3.17: Automatic second-order fixed-rank contact

*Proof.* Write $s = 1-x^2$, $\rho_{20} := \rho_2(0,x)$ and $d_j = d_{ab}^j(\arccos x)$. Write $e_\pi(r,t) = 1-\varepsilon_1 r+\lambda_1 t+\varepsilon_2 r^2+\varepsilon_{11}rt+O(|r|^3+|t|^2)$ and $v_\pi(r,t) = \beta_1 r+\beta_2 t+\beta_{20}r^2+\beta_{11}rt+O(|r|^3+|t|^2)$. In (46) with (47), the coefficients $\varepsilon_2,\beta_{20}$ enter $a_0,b_0,p,g$ only at order $q^4$, hence their first two $q$-derivatives at order $q^2$; $\beta_{11}$ enters $b_2 = ab/q^2+\beta_2+\beta_{11}q^2+O(q^4)$ at order $q^2$, and $\varepsilon_{11}$ enters $\eta_\pi(q^2) = \lambda_1+(\varepsilon_{11}+\varepsilon_1\lambda_1)q^2+O(q^4)$ at order $q^2$; the only $q^{-2}$ terms are $2a_2\cos p+b_2$. So the $q^0$ coefficient of $\Sigma_2$, and with it $\rho_{20}$, depends only on $(\varepsilon_1,\beta_1,\beta_2,\lambda_1)$, linearly, and expanding with (22) and $\eta_\pi(0)\cot\theta = \lambda_1 x/\sqrt{s}$ gives (53).

At fixed rank $q^2 = \kappa^2 t$ and, by Proposition 3.16, $\Gamma_2/q = \rho_{20}+O(q^2)$. With (22) and $\partial_\theta = -\sqrt{s}\,\partial_x$,

$$\widehat{U}_k^{[2]} = \widehat{d}_k + t\,\widehat{u}_k + O_k(t^2), \qquad \widehat{u}_k = \sqrt{2k+1}\,\mathcal{Q}_{2,k}d_k, \tag{201}$$

$$\mathcal{Q}_{2,k} = \Big(\frac{\kappa^2\chi}{3} - \sqrt{s}\,\rho_{20}\Big)\partial_x + \frac{\kappa^2\big((\varepsilon_1 - x\beta_1)/2 - x\chi/6\big)}{s} + \frac{x\rho_{20}}{2\sqrt{s}}. \tag{202}$$

By (E1), $A_k(h) = A_k^{(0)}(1+(\lambda_1-\varepsilon_1 k^2)t)+O_k(t^2)$ and $B_k(h) = B_k^{(0)}(1+\lambda_1 t)+(\beta_1\kappa^2+\beta_2)t+O_k(t^2)$. Therefore the $t$ coefficient of $\mathcal{R}_k(\widehat{U}^{[2]})$ vanishes if and only if

$$\begin{aligned} &x\widehat{u}_k - A_{k+1}^{(0)}\widehat{u}_{k+1} - A_k^{(0)}\widehat{u}_{k-1} - B_k^{(0)}\widehat{u}_k\\ &\quad = -\varepsilon_1\big[(k+1)^2 A_{k+1}^{(0)}\widehat{d}_{k+1} + k^2 A_k^{(0)}\widehat{d}_{k-1}\big] + (\beta_1\kappa^2+\beta_2)\widehat{d}_k + \lambda_1 x\,\widehat{d}_k, \end{aligned} \tag{203}$$

where the last term collects the $\lambda_1$ contributions, $\lambda_1[A^{(0)}_{k+1}\widehat{d}_{k+1} + A^{(0)}_k\widehat{d}_{k-1} + B^{(0)}_k\widehat{d}_k] = \lambda_1 x\,\widehat{d}_k$ by the Wigner recurrence. Put $R_k = \sqrt{(k^2-a^2)(k^2-b^2)}$, so that $A^{(0)}_k = R_k/(k\sqrt{4k^2-1})$ and $B^{(0)}_k = ab/(k(k+1))$. The gauge factors cancel exactly,

$$A^{(0)}_{k+1}\sqrt{2k+3} = \frac{R_{k+1}}{(k+1)\sqrt{2k+1}}, \qquad A^{(0)}_k\sqrt{2k-1} = \frac{R_k}{k\sqrt{2k+1}}, \tag{204}$$

so dividing (203) by $\sqrt{2k+1}$ and writing $M_k := R_k d_{k-1}$, $P_k := R_{k+1}d_{k+1}$ (the operators $\mathcal{Q}_{2,j}$ being linear), it becomes the gauge-free identity

$$\begin{aligned}\Big(x - \frac{ab}{k(k+1)}\Big)\mathcal{Q}_{2,k}d_k - \frac{\mathcal{Q}_{2,k+1}P_k}{(k+1)(2k+1)} - \frac{\mathcal{Q}_{2,k-1}M_k}{k(2k+1)} \\ = -\varepsilon_1\,\frac{(k+1)P_k + kM_k}{2k+1} + (\beta_1\kappa^2 + \beta_2 + \lambda_1 x)d_k.\end{aligned} \tag{205}$$

The Wigner rank-shift (ladder) identities at ranks $k$ and $k+1$,

$$(1-x^2)d'_k = \Big(-kx + \frac{ab}{k}\Big)d_k + \frac{R_k}{k}d_{k-1}, \qquad (1-x^2)d'_{k-1} = \Big(kx - \frac{ab}{k}\Big)d_{k-1} - \frac{R_k}{k}d_k, \tag{206}$$

and the same pair with $k \to k+1$, express $M_k, M'_k, P_k, P'_k$ through $(d_k, d'_k)$ alone:

$$\begin{aligned} M_k &= k\,s\,d'_k + (k^2x - ab)d_k, \\ k\,s\,M'_k &= (k^2x-ab)M_k - (k^2-a^2)(k^2-b^2)\,d_k, \\ P_k &= ((k+1)^2x - ab)d_k - (k+1)\,s\,d'_k, \\ (k+1)\,s\,P'_k &= ((k+1)^2-a^2)((k+1)^2-b^2)d_k - ((k+1)^2x-ab)P_k, \end{aligned} \tag{207}$$

in which $R_k$ enters only through $R_k^2$. After this substitution the coefficients of $d_k$ and $d'_k$ in the difference of the two sides of (205) are rational functions of $(a,b,k,x)$, linear in $(\varepsilon_1,\beta_1,\beta_2,\lambda_1)$, and they vanish identically (a finite rational-function identity, verified by exact symbolic computation). No separate treatment of the edge row $k = k_{\min}$ is needed: $R_k$ multiplies only the $d_{k-1}$ terms, and $R_{k_{\min}} = 0$ gives $M_{k_{\min}} = 0$, which is what the displayed formulas for $M_k$ and $M'_k$ return there (the first ladder identity at $k_{\min}$ reads $(1-x^2)d'_k = (-kx + ab/k)d_k$). (The conventions $A^{(0)}_{k_{\min}} := 0$ and $B^{(0)}_0 := 0$ are needed only to define the coefficients; (205) has no $\sqrt{4k^2-1}$.) In the exceptional case $a = b = k = 0$, where $d_0 = 1$, $d_1 = x$, the identity is checked directly. The $\beta_2$ part is transparent: there $\chi = 0$, $\rho_{20} = s^{-1/2}$ and $\mathcal{Q}_{2,k} = -\partial_x + x/(2s)$ is independent of $k$, so it commutes with the $x$-independent Wigner hoppings, and $[x, -\partial_x] = 1$ produces the source $\beta_2\widehat{d}_k$; this is the shift $x \mapsto x - \beta_2 t$. The $\lambda_1$ part is the same mechanism with $\beta_2$ replaced by $\lambda_1 x$: it adds $\lambda_1 x/\sqrt{s}$ to $\rho_{20}$, hence $-\lambda_1 x\,\partial_x + \lambda_1 x^2/(2s)$ to $\mathcal{Q}_{2,k}$. The multiplier $\lambda_1 x^2/(2s)$ commutes with the $k$-hoppings and with $x - J^{(0)}$, where $J^{(0)}$ is the Wigner Jacobi operator of (16), so it drops out; and differentiating $x\widehat{d}_k = J^{(0)}\widehat{d}_k$ in $x$ gives $(x - J^{(0)})\partial_x\widehat{d}_k = -\widehat{d}_k$, so the term $-\lambda_1 x\,\partial_x$ produces exactly the source $\lambda_1 x\,\widehat{d}_k$ of (203). (This is the shift $x \mapsto x - \lambda_1 x t$ of Remark 3.4.) □

**Lemma B.1** (Fixed-ratio cubic residual of the corrected pullback)**.** *Fix $q_0 > 0$ and a closed turning-free corridor $\mathcal{C}_{q_0}$ of an even centered Wigner–Jacobi chain on which $q \geq q_0$. For the corrected pullback (49) the naturally gauged scalar Jacobi residual obeys*

$$|\mathcal{R}_k(\widehat{U}^{[2]})| \leq C_{\mathcal{C}_{q_0}} N^{-3}, \tag{208}$$

*uniformly whenever $(q_k, x, \pi) \in \mathcal{C}_{q_0}$. The proof uses only the Darboux expansion of Lemma 3.6 and Taylor expansion in $h$, so the bound holds for every sufficiently small $h > 0$ with $q = \kappa h$, not only at $h = N^{-1}$ for admissible $N$. More precisely, the coefficients $C_0^+, C_1^+, C_2^+$ of the expansion of $e^{-i\Theta_q/z}\mathcal{L}_h\widehat{U}^{[2],+}$ in $z = h/q$ (Lemma C.1) vanish.*

*Proof.* Fix $(x, \pi)$ and work at fixed $q \geq q_0$ with $h \to 0$, $\kappa = q/h$, $z = h/q$. The margin $\delta_{\mathcal{C}}$ of the corridor keeps $q \pm h$ in the region where (E1) holds, and every function of $(q, h)$ below is smooth there, with derivatives bounded uniformly on $\mathcal{C}_{q_0}$; $O(h^j)$ means a quantity bounded by $Ch^j$ uniformly. Write $\mathcal{L}_h$ for the midpoint operator (64), so that $\mathcal{R}_k(\widehat{U}^{[2]}) = (\mathcal{L}_h\widehat{U}^{[2]})(q_k)$ and $\widehat{U}^{[2]}(q)$ is regarded as a function of continuous $q$ with $\kappa = q/h$ (the ranks $k \pm 1$ are the points $q \pm h$).

*Step 1: one Darboux branch.* By (26) with $R = 3$, applied at the angles $\Theta_{q'}^{[2]}$, $q' \in \{q-h, q, q+h\}$ (which lie in a fixed compact subinterval of $(0, \pi)$, since $\Theta_{q'}^{[2]} - \Theta_{q'} = O(h^2)$), and with (65), $\widehat{U}^{[2]} = \widehat{U}^{[2],+} + \widehat{U}^{[2],-} + O(\kappa^{-3})$ at each of the three points, and $\widehat{U}^{[2],-} = \overline{\widehat{U}^{[2],+}}$. The coefficients of $\mathcal{L}_h$ are real and bounded, so $\mathcal{L}_h\widehat{U}^{[2],-} = \overline{\mathcal{L}_h\widehat{U}^{[2],+}}$ and

$$|\mathcal{R}_k(\widehat{U}^{[2]})| \leq 2\,|\mathcal{L}_h\widehat{U}^{[2],+}| + Ch^3. \tag{209}$$

By Lemma 3.23, as formal series in $\kappa^{-1}$ at fixed angle $\vartheta$,

$$\sum_r \kappa^{-r} A_{r,ab}^+(\vartheta) = \omega_{ab}(\pi \sin\vartheta)^{-1/2}\,\mathfrak{A}_{ab}(\vartheta, \kappa)\, e^{i\mathfrak{P}_{ab}(\vartheta,\kappa)}, \tag{210}$$

with $\mathfrak{A}_{ab} = 1 + \kappa^{-2}\mathfrak{A}_2 + O(\kappa^{-4})$ and $\mathfrak{P}_{ab} = \kappa^{-1}\mathfrak{c}_{ab} + O(\kappa^{-3})$: there is no $\kappa^{-1}$ amplitude term and no $\kappa^{-2}$ phase term. Hence the truncated sum $\sum_{r<3} \kappa^{-r} A_{r,ab}^+$ differs by $O(\kappa^{-3})$ from $\omega_{ab}(\pi \sin\vartheta)^{-1/2}(1 + \kappa^{-2}\mathfrak{A}_2)e^{i\mathfrak{c}_{ab}/\kappa}$, uniformly on compact angular sets. Replacing the former by the latter in (65) changes $\widehat{U}^{[2],+}$ by $O(\kappa^{-3})$ at each of the three points, hence changes $\mathcal{L}_h\widehat{U}^{[2],+}$ by $O(h^3)$, and does not change $C_r^+$ for $r \leq 2$. Since $W_q^{[2]}(\sin\Theta_q^{[2]})^{-1/2} = D^{-1/4}$ exactly (definition of $W^{[2]}$ in (48)), the branch to be studied is

$$u(q; h) := B_0(q)\Big(1 + \frac{\mathfrak{A}_2(\Theta_q^{[2]})}{\kappa^2}\Big)\exp\Big\{i\Big[\kappa\Theta_q^{[2]} + \frac{\mathfrak{c}_{ab}(\Theta_q^{[2]})}{\kappa}\Big]\Big\}, \qquad B_0 := \omega_{ab}\pi^{-1/2}D^{-1/4}. \tag{211}$$

The $h^2$ difference between $W_q^{[2]}$ and $W_q$ never appears separately: it is exactly what makes the principal amplitude $B_0$ independent of $h$. The only real $h^2$ amplitude correction is $\mathfrak{A}_2(\Theta_q^{[2]})/\kappa^2 = h^2\mathfrak{A}_2(\Theta_q^{[2]})/q^2$.

*Step 2: phase and amplitude.* Put $S := q\Theta_q$ (so $S_q = p$), $g := -\frac{1}{4}\log D$ (so $B_0(q')/B_0(q) = e^{g(q')-g(q)}$ and, since $D^{1/2} = 2a_0 \sin p$, $g = -\frac{1}{2}\log(a_0 \sin p) + \text{const}$ is the amplitude exponent of Proposition 3.15), and

$$S_2 := \Gamma_2 + \frac{\mathfrak{c}_{ab}(\Theta_q)}{q}, \qquad S_{2,q} = \rho_2 + \frac{d}{dq}\frac{\mathfrak{c}_{ab}(\Theta_q)}{q} = \Sigma_2 \tag{212}$$

by (48). This is where the two Darboux phases combine. Exactly, $\kappa\Theta_q^{[2]} = \kappa\Theta_q + \kappa\frac{h^2}{q}\Gamma_2 = S/h + h\Gamma_2$, while the Darboux phase of $d_{ab}^k(\Theta_q^{[2]})$ is $\mathfrak{c}_{ab}(\Theta_q^{[2]})/\kappa = h\,\mathfrak{c}_{ab}(\Theta_q^{[2]})/q$, and $\mathfrak{c}_{ab}(\Theta_q^{[2]}) - \mathfrak{c}_{ab}(\Theta_q) = O(h^2)$. So the Darboux phase $\mathfrak{c}_{ab}(\Theta_q)/\kappa$ restores, up to $O(h^3)$, precisely the term subtracted in $\Gamma_2$:

$$\kappa\Theta_q^{[2]} + \frac{\mathfrak{c}_{ab}(\Theta_q^{[2]})}{\kappa} = \frac{S}{h} + hS_2 + \psi, \qquad \psi := \frac{h}{q}\big[\mathfrak{c}_{ab}(\Theta_q^{[2]}) - \mathfrak{c}_{ab}(\Theta_q)\big] = O(h^3). \tag{213}$$

Therefore

$$u = e^{iS/h}\,V\,E, \qquad V := B_0\,e^{ihS_2}, \qquad E := \Big(1 + \frac{h^2\mathfrak{A}_2(\Theta_q^{[2]})}{q^2}\Big)e^{i\psi} = 1 + h^2 m(q;h), \tag{214}$$

with $m$ smooth and bounded together with its $q$-derivatives ($\psi$ is smooth and divisible by $h^3$). Thus $E-1 = O(h^2)$: it contains the real amplitude correction and the $O(h^3)$ phase correction $\psi$, and no term of order $h$.

*Step 3: splitting the residual.* Put $a_\varepsilon := a(q+\varepsilon h/2; h^2)$ and $\delta_\varepsilon := [S(q+\varepsilon h) - S(q)]/h$ for $\varepsilon = \pm 1$. Then

$$e^{-iS(q)/h}(\mathcal{L}_h u)(q) = E(q)\,T(q) - \sum_{\varepsilon=\pm1} a_\varepsilon e^{i\delta_\varepsilon} V(q+\varepsilon h)\big[E(q+\varepsilon h) - E(q)\big], \tag{215}$$

where $T(q) := (x - b(q;h^2))V(q) - \sum_\varepsilon a_\varepsilon e^{i\delta_\varepsilon}V(q+\varepsilon h)$ is the residual of the pure ansatz $e^g\exp\{ih^{-1}(S + h^2S_2)\}$ of Proposition 3.15 (times the constant $\omega_{ab}\pi^{-1/2}$ and $e^{\text{const}}$). Since $E - 1 = h^2 m$ with $m$ smooth, $E(q+\varepsilon h) - E(q) = h^2[m(q+\varepsilon h) - m(q)] = O(h^3)$; the coefficients $a_\varepsilon$ and $V$ are bounded, so the sum in (215) is $O(h^3)$. The amplitude corrections are therefore invisible at orders $h^0, h^1, h^2$ for two reasons: the slowly varying factor $E = 1 + O(h^2)$ multiplies $T$, whose principal coefficients vanish (Step 4), and its variation across a step is $O(h^3)$.

*Step 4: the orders $h^0, h^1, h^2$ of $T$.* Taylor expansion with $S_q = p$, $S_{2,q} = \Sigma_2$ gives

$$\begin{aligned} \delta_\varepsilon &= \varepsilon p + \tfrac{h}{2}p_q + \varepsilon\tfrac{h^2}{6}p_{qq} + O(h^3), \\ g(q+\varepsilon h) - g(q) &= \varepsilon h g_q + \tfrac{h^2}{2}g_{qq} + O(h^3), \qquad ih\big[S_2(q+\varepsilon h) - S_2(q)\big] = i\varepsilon h^2\Sigma_2 + O(h^3), \end{aligned} \tag{216}$$

so that $V(q+\varepsilon h)/V(q) = \exp X_\varepsilon$ with $X_\varepsilon = i\varepsilon p + h\alpha_\varepsilon + h^2\beta_\varepsilon + O(h^3)$, $\alpha_\varepsilon = \varepsilon g_q + \frac{i}{2}p_q$, $\beta_\varepsilon = \frac{1}{2}g_{qq} + i\varepsilon(\frac{1}{6}p_{qq} + \Sigma_2)$. With $a_\varepsilon = a_0 + \varepsilon\frac{h}{2}a_{0,q} + h^2(a_2 + \frac{1}{8}a_{0,qq}) + O(h^3)$ and $b = b_0 + h^2 b_2 + O(h^4)$ (from (47)), and the elementary sums $\sum_\varepsilon e^{i\varepsilon p} = 2\cos p$, $\sum_\varepsilon \varepsilon e^{i\varepsilon p} = 2i\sin p$, one finds

$$\frac{T}{V} = c_0 + hc_1 + h^2 c_2 + O(h^3), \tag{217}$$

$$c_0 = x - b_0 - 2a_0\cos p, \tag{218}$$

$$c_1 = -i\big[2a_0 g_q \sin p + a_0 p_q\cos p + a_{0,q}\sin p\big], \tag{219}$$

$$\begin{aligned} c_2 = -b_2 - \Big\{ & a_0(g_{qq} + g_q^2)\cos p - a_0 g_q p_q \sin p - \tfrac{a_0}{4}p_q^2\cos p - \tfrac{a_0}{3}p_{qq}\sin p \\ & + 2a_2\cos p + a_{0,q}g_q\cos p - \tfrac{1}{2}a_{0,q}p_q\sin p + \tfrac{1}{4}a_{0,qq}\cos p - 2a_0\Sigma_2\sin p\Big\}. \end{aligned} \tag{220}$$

(The coefficient of $h^2$ in $\sum_\varepsilon a_\varepsilon e^{X_\varepsilon}$ is $\sum_\varepsilon e^{i\varepsilon p}\big[a_0(\beta_\varepsilon + \frac{1}{2}\alpha_\varepsilon^2) + \frac{1}{2}\varepsilon a_{0,q}\alpha_\varepsilon + a_2 + \frac{1}{8}a_{0,qq}\big]$, where $\beta_\varepsilon + \frac{1}{2}\alpha_\varepsilon^2 = \frac{1}{2}(g_{qq} + g_q^2) - \frac{1}{8}p_q^2 + i\varepsilon(\frac{1}{6}p_{qq} + \Sigma_2 + \frac{1}{2}g_q p_q)$; with $\sum_\varepsilon e^{i\varepsilon p}(R + i\varepsilon I) = 2R\cos p - 2I\sin p$ this is the brace in $c_2$.) Now $c_0 = 0$ is the principal eikonal $x - b_0 = 2a_0\cos p$; $c_1 = 0$ is the transport equation, because $g = -\frac{1}{2}\log(a_0\sin p) + \text{const}$ gives $2a_0 g_q\sin p = -a_{0,q}\sin p - a_0 p_q\cos p$; and $c_2 = 2a_0\sin p\,(\Sigma_2 - \Sigma_2[a_0, a_2, b_2; p, g])$ with $\Sigma_2[\cdot]$ the density (46), which is 0 because $S_{2,q} = \Sigma_2$ by (212). The same three equations are the orders $h^0, h^1, h^2$ in the proof of Proposition 3.15; $c_1$ is purely imaginary and $c_2$ real, so a real phase correction $S_2$ alone closes the order $h^2$. Hence $T = V\cdot O(h^3) = O(h^3)$.

*Step 5: conclusion.* By (215) and Step 4, $e^{-i\Theta_q/z}\mathcal{L}_h u = O(h^3) = O(z^3)$ at fixed $q$, with $S/h = \Theta_q/z$; that is, $C_0^+ = C_1^+ = C_2^+ = 0$, and $|\mathcal{L}_h u| \le Ch^3$. With Step 1, $|\mathcal{R}_k(\widehat{U}^{[2]})| \le Ch^3$ uniformly on $\mathcal{C}_{q_0}$, which is (208). The first coefficient that can survive is $C_3^+$, to which both the $h^3$ terms of $T$ and the second term of (215) contribute; Lemma C.1 shows that $C_3^+/B_0$ is imaginary. □

The next two lemmas give the corner argument behind the large-rank bound (58). The first is the model argument for a scalar remainder. The second, which is the one used here, repeats it carrier by carrier for the two-carrier Wigner–Darboux residual, where the carriers must first be separated.

**Lemma B.2** (Two-scale corner divisibility)**.** *Let $p, s, R$ be positive integers with $R > p$. Suppose*

$$A(q,z) = \sum_{r=0}^{R-1} z^r C_r(q) + E_R(q,z), \qquad 0 \le q \le q_*, \quad 0 < z \le z_*, \tag{221}$$

*where $C_r \in C^s([0,q_*])$ and*

$$\max_{0\le m<s} \sup_{0\le q\le q_*} |\partial_q^m E_R(q,z)| \le C z^R. \tag{222}$$

*Assume that*

$$A(q,z) = O_q(z^p) \quad (z \to 0) \tag{223}$$

*for every fixed $q > 0$, and that for every fixed admissible discrete $z$,*

$$A(q,z) = O_z(q^s) \quad (q \to 0), \tag{224}$$

*where the admissible $z$-set contains a sequence tending to zero. Then*

$$\boxed{|A(q,z)| \le C\{q^s z^p + z^R\}} \tag{225}$$

*uniformly on the closed corner.*

*Proof.* The fixed-positive-$q$ estimate first gives $C_0 = \cdots = C_{p-1} = 0$ on $(0, q_*]$, hence on $[0, q_*]$ by continuity. For $0 \le m < s$, differentiate the remaining expansion at $q = 0$. The fixed-$z$ hypothesis gives

$$\sum_{r=p}^{R-1} z^r C_r^{(m)}(0) = O(z^R) \tag{226}$$

along a sequence $z \to 0$. Divide successively by $z^p, z^{p+1}, \dots, z^{R-1}$ to obtain $C_r^{(m)}(0) = 0$ for $p \le r < R$ and $0 \le m < s$. Taylor's theorem gives $C_r(q) = q^s \widetilde{C}_r(q)$ with uniformly bounded $\widetilde{C}_r$. Summing the finite series proves (225). □

**Lemma B.3** (Darboux–Weyl corner matching)**.** *Let $p \in \{3, 4\}$, let $R > p$ be an integer, put $\kappa = k + \frac{1}{2}$ and $z = \kappa^{-1}$, and treat $(q, z)$ as independent continuum variables with $h := qz$ (so $q = \kappa h$ at the lattice points). Consider*

$$\widehat{U}_k(h) = \sqrt{2k+1}\, W(q, h^2) d_{ab}^k\Big(\Theta(q, h^2)\Big). \tag{227}$$

*Assume on $0 \le q \le q_* + \delta$ (for some $\delta > 0$; the conclusion concerns $0 \le q \le q_*$) that the exact Jacobi coefficients, $W$, and $\Theta$ are smooth in the blown-up variables $(q, z)$ after $h = qz$, with at least four $q$ derivatives and enough uniform inverse-rank derivatives for a finite Darboux expansion of order $R + 3$ with three derivatives of its carrier-separated remainders (Lemma 3.6 with order $R + 3$ and $s = 3$). Assume also the* even-$h$ midpoint dependence

$$\Theta(q, h^2) = \Theta_0(q) + h^2 H(q, h^2) \tag{228}$$

*with smooth bounded $H$ on this closed blown-up corner, and that $\Theta_0(q) := \Theta(q,0)$ stays in a compact subinterval of $(0,\pi)$. Put*

$$S_0(q) := q\Theta_0(q), \qquad p_0(q) := S_0'(q). \tag{229}$$

*If (i) at fixed $q > 0$ the scalar residual is $O(h^p)$ and (ii) at every fixed admissible rank $k$ the residual is analytic in $h^2$ with vanishing $h^0$ and $h^2$ coefficients, then*

$$\boxed{|\mathcal{R}_k(\widehat{U})| \le C(h^p + \kappa^{-R})} \tag{230}$$

*uniformly on the closed corner for all sufficiently large $k$.*

*Proof.* The two complex Darboux branches must be separated explicitly: cancellation for the *real* Wigner solution does not by itself give cancellation in each branch.

*The independent variables.* The residual is regarded as a function of two independent continuum variables $(q,z)$ on the closed corner $0 \le q \le q_*$, $0 < z \le z_0$; the step $h = qz$ and the rank parameter $\kappa = z^{-1}$ are derived from them, so the relation $q = \kappa h$ of the lattice is the definition of $h$ and is not a constraint. A point $(q,z)$ is physical when $z = 1/\kappa$ for an admissible rank and $h = qz = N^{-1}$. Fixed ratio is the curve $q = q_0$, $z \to 0$; fixed rank is the curve $z = z_k$, $q \downarrow 0$; both lie in the corner. The exact coefficients are functions of $(q,z)$ through (E1): $k = z^{-1} - \frac{1}{2}$, $e_\pi(k^2h^2, h^2) = e_\pi(q^2(1-\frac{z}{2})^2, q^2z^2)$, and so on (Corollary B.4). Put $\widetilde{\Theta}(q,z) := \Theta(q, q^2z^2)$, so that by the even-$h$ hypothesis

$$\widetilde{\Theta}(q,z) = \Theta_0(q) + z^2q^2H(q,z), \tag{231}$$

where $H$ and its required derivatives are bounded on the corner, and write $\widetilde{W}(q,z) := W(q, q^2z^2)$. The hypotheses are assumed on $0 \le q \le q_* + \delta$ for some $\delta > 0$, so that the neighbouring ranks below stay in the domain (this is the margin $\delta_{\mathcal{C}}$ in the definition of a turning-free corridor); the conclusion concerns $q \le q_*$.

*Carrier-separated residual.* Insert the differentiated expansion of Lemma 3.6 of order $R+3$ with $s = 3$ at the angle $\vartheta = \widetilde{\Theta}(q,z)$, which stays in a compact subinterval of $(0,\pi)$:

$$\widehat{U}_k = \sum_\sigma e^{i\sigma\Theta_0(q)/z}\,\mathfrak{U}^\sigma(q,z), \qquad \mathfrak{U}^\sigma = \widetilde{W}e^{i\sigma zq^2H}\Big[\sum_{r\le R+2} z^r A^\sigma_{r,ab}(\widetilde{\Theta}) + \mathcal{E}^\sigma_{R+3,k}(\widetilde{\Theta})\Big], \tag{232}$$

using $\kappa\widetilde{\Theta} = \Theta_0/z + zq^2H$. The prefactor $\widetilde{W}e^{i\sigma zq^2H}$ and the finite sum are smooth functions of $(q,z)$ on the corner (including $z = 0$). The remainder part is $\widetilde{W}e^{i\sigma zq^2H}\mathcal{E}^\sigma_{R+3,k}(\widetilde{\Theta}(q,z))$ at *fixed* $k$, i.e. fixed $z$; by the chain rule (Faà di Bruno, with $\partial_q^j\widetilde{\Theta}$ bounded) and (27) its $q$-derivatives of order $m \le 3$ are $O(z^{R+3})$. The derivatives that fall on the carrier $e^{i\sigma\Theta_0/z}$ are not included here: they are taken explicitly and cost $z^{-1}$ each.

*Neighbouring ranks.* At the same $h$ the ranks $k \pm 1$ correspond to the continuum points

$$(q,z) \longmapsto (q_\pm, z_\pm) = \left(q(1\pm z), \frac{z}{1\pm z}\right), \tag{233}$$

since $\kappa \pm 1 = 1/z_\pm$ and $(\kappa \pm 1)h = q_\pm$; note $q_\pm z_\pm = qz = h$, so $\Theta(q_\pm, h^2) = \widetilde{\Theta}(q_\pm, z_\pm)$ and the same functions $\widetilde{\Theta}, \widetilde{W}, H$ apply at the shifted points. Hence

$$\widehat{U}_{k\pm1} = \sum_\sigma e^{i\sigma\Theta_0(q_\pm)/z_\pm}\, e^{i\sigma z_\pm q_\pm^2 H(q_\pm,z_\pm)}\, \widetilde{W}(q_\pm, z_\pm)\,[\cdots], \tag{234}$$

with the same bracket evaluated at $(q_\pm, z_\pm)$, and its remainder part is the above remainder composed with $(q,z) \mapsto (q_\pm, z_\pm)$. This composition preserves the bounds: the map is smooth,

$\partial_q[q(1\pm z)] = 1\pm z$, the fixed rank is now $k\pm 1$ (so the same lemma applies with $\kappa\pm 1 \asymp \kappa$), and $z_\pm \asymp z$, so the $q$-derivatives of order $m \le 3$ of the shifted remainder are $O(z^{R+3})$. The carrier quotients have smooth expansions: with $\Theta_0(q(1\pm z)) = \Theta_0(q) \pm zq\Theta_0'(q) + O(z^2)$,

$$\frac{\Theta_0(q_\pm)}{z_\pm} - \frac{\Theta_0(q)}{z} = \pm(\Theta_0 + q\Theta_0') + zg_\pm^{(0)}(q,z) = \pm p_0(q) + zg_\pm^{(0)}(q,z), \tag{235}$$

with $g_\pm^{(0)}$ smooth on the corner. Thus $e^{i\sigma\Theta_0(q_\pm)/z_\pm} = e^{i\sigma\Theta_0(q)/z}\, e^{\pm i\sigma p_0(q) + i\sigma z g_\pm^{(0)}}$, and the extra factor is smooth on the corner. The exact coefficients $A_{k\pm1}, A_k, B_k$ and the neighbouring-rank shifts are smooth in $(q,z)$, so

$$\mathcal{R}_k(q,z) = e^{i\Theta_0(q)/z}\left[\sum_{r=0}^{R+2} z^r C_r^+(q) + E_+(q,z)\right] + e^{-i\Theta_0(q)/z}\left[\sum_{r=0}^{R+2} z^r C_r^-(q) + E_-(q,z)\right], \tag{236}$$

where $\sum_r z^r C_r^\pm(q)$ is the Taylor polynomial in $z$ at fixed $q$ of the smooth finite parts (so $C_r^\pm \in C^4([0,q_*])$), and $E_\pm$ collects the Taylor remainder, of size $O(z^{R+3})$ with $q$-derivatives, and the three composed remainders of the carrier-separated expansions, so that $\|\partial_q^m E_\pm(\cdot,z)\|_\infty \le C z^{R+3}$ for $0 \le m \le 3$. The carriers stand outside the brackets, so each $q$-derivative that falls on a carrier costs one factor $z^{-1}$; the expansion is therefore carried three orders beyond the order $z^R$ needed in (230). It is a finite-order statement and does not assert convergence of the Darboux series. All coefficient bounds are uniform because the angles stay in an interior compact set and the hopping and its reciprocal are smooth in the blown-up variables. Thus the one-step recurrence produces no uncontrolled factors $e^{i\,\mathrm{const}/z}$ beyond the two carriers in (236).

We use the following elementary uniqueness fact. If $0 < \vartheta < \pi$ and

$$e^{i(k+1/2)\vartheta}\sum_{r=0}^{M-1} a_r (k+1/2)^{-r} + e^{-i(k+1/2)\vartheta}\sum_{r=0}^{M-1} b_r (k+1/2)^{-r} = O(k^{-M}) \tag{237}$$

for every sufficiently large rank $k$ of an admissible consecutive sequence $k \in k_{\min} + \mathbb{Z}_{\ge 0}$ ($k_{\min}$ integer or half-integer), then $a_r = b_r = 0$ for $0 \le r < M$. At the first nonzero order, multiply by $e^{i(k+1/2)\vartheta}$ and compare $k$ with $k+1$: the coefficient of $e^{2i(k+1/2)\vartheta}$ must vanish because $e^{2i\vartheta} \ne 1$; the constant coefficient then vanishes. Induction on the inverse-rank order proves the assertion. The same argument works on a fixed compact angular interval with uniform separation from its endpoints.

At each fixed $q > 0$ the assumed $O(h^p) = O(q^p z^p)$ scalar residual, together with this uniqueness argument applied to (236), forces $C_0^\pm = \cdots = C_{p-1}^\pm = 0$ (first for fixed $q$ and then at $q = 0$ by continuity).

Next fix $k$, hence $z$, and let $q \downarrow 0$, with $h = qz$. Even analyticity in $h$ and the vanishing $h^0, h^2$ contact coefficients imply $\partial_q^m \mathcal{R}_k(0,z) = 0$ for $0 \le m < 4$. We show by induction on $m \in \{0,1,2,3\}$ that

$$\partial_q^m C_r^\pm(0) = 0 \qquad (p \le r \le R+2-m). \tag{238}$$

Differentiate (236) $m$ times at $q = 0$. The remainder contributes $\partial_q^m(e^{\pm i\Theta_0/z}E_\pm)(0,z) = O(z^{R+3-m})$, since each derivative on the carrier costs at most $z^{-1}$. In the Leibniz expansion of $\partial_q^m(e^{\pm i\Theta_0/z} z^r C_r^\pm)$ at $q = 0$, a term with $j \ge 1$ derivatives on the carrier is $O(z^{r-j})\,\partial_q^{m-j} C_r^\pm(0)$; it vanishes by the induction hypothesis when $r \le R+2-(m-j)$, and otherwise $r - j \ge R+3-m$, so it is $O(z^{R+3-m})$. (For $m = 0$ there are no such terms.) What remains is

$$e^{i\Theta_0(0)/z}\sum_{r=p}^{R+2-m} z^r \partial_q^m C_r^+(0) + e^{-i\Theta_0(0)/z}\sum_{r=p}^{R+2-m} z^r \partial_q^m C_r^-(0) = O(z^{R+3-m}), \tag{239}$$

and the uniqueness fact with $\vartheta = \Theta_0(0)$ and $M = R+3-m$ gives $\partial_q^m C_r^{\pm}(0) = 0$ for $p \le r \le R+2-m$.

In particular $\partial_q^m C_r^{\pm}(0) = 0$ for $p \le r < R$ and $0 \le m \le 3$, so Taylor's theorem gives $|C_r^{\pm}(q)| \le Cq^4$ for $p \le r < R$, while $\sum_{r=R}^{R+2} z^r |C_r^{\pm}(q)| \le Cz^R$ because the $C_r^{\pm}$ are bounded on $[0, q_*]$. Hence

$$|\mathcal{R}_k| \le C(q^4 z^p + z^R) \le C'(h^p + \kappa^{-R}), \tag{240}$$

because $h = qz$, $p \in \{3, 4\}$, and $q \le q_*$. This proves the bound. The fixed-ratio hypothesis (i) gives $C_0^{\pm} = \cdots = C_{p-1}^{\pm} = 0$, and the fixed-rank hypothesis (ii) gives the vanishing of the $q$-derivatives at $q = 0$. □

**Corollary B.4** (Matched remainder for the corrected pullback)**.** *On every closed turning-free corridor of an even centered chain reaching $q = 0$, there are $k_0$ and $C_{\mathcal{C}}$ such that*

$$|\mathcal{R}_k(\widehat{U}^{[2]})| \le C_{\mathcal{C}}(h^3 + k^{-8}), \qquad k_0 \le k \le L - 1. \tag{241}$$

*This is* (58)*.*

*Proof.* Apply Lemma B.3 with $(p, R) = (3, 8)$. In the blown-up variables $q = \kappa h$, $z = \kappa^{-1}$ the exact coefficients are analytic: $k^{-1} = z/(1 - z/2)$, so $A_k^{(0)}$ and $B_k^{(0)}$ of (16) are analytic in $z$ near 0, while by (E1) $e_\pi(k^2h^2, h^2) = e_\pi(q^2(1 - z/2)^2, q^2z^2)$, $\lambda_\pi(h^2) = \lambda_\pi(q^2z^2)$, and $v_\pi(q_k^2, h^2) = v_\pi(q^2, q^2z^2)$ are analytic in $(q, z)$ on the closed corridor. Moreover $\Theta_q$, $\Gamma_2(q)/q$ and $W_q^{[2]}$ are analytic in $q^2$ at $q = 0$ by Proposition 3.16, so $\Theta^{[2]}$ has the even-$h$ form required there, and the angle remains in a fixed interior interval.

Hypothesis (i) of the lemma is Lemma B.1. Hypothesis (ii) is the fixed-rank analyticity of $U^{[2]}$ (Proposition 3.16) and of the exact coefficients, the vanishing of the order-$h^0$ residual because $U^{[2]}(0)$ is the Wigner solution, and the vanishing of the order-$h^2$ residual coefficient, which is Proposition 3.17. Finally $\kappa^{-8} \asymp k^{-8}$ for $k \ge k_0$. □

# C Phase exactness of the second-order pullback

This section completes the proof of Theorem 3.24. The parity of the Wigner–Darboux expansion (Lemma 3.23) and the flux identity (Lemma 3.25) are in Section 3.6.

**Lemma C.1** (The cubic fixed-ratio residual is of transport type)**.** *On a closed turning-free corridor reaching $q = 0$, split the corrected pullback into its two Darboux branches* (65)*, with differentiated remainder. In the blown-up variables $z = \kappa^{-1}$, $h = qz$ write*

$$\mathcal{R}_k(\widehat{U}^{[2],+}) = e^{i\Theta_q/z}\Big[\sum_{r<R} z^r C_r^+(q) + O(z^R)\Big], \qquad \widehat{U}_k^{[2],+} = e^{i\Theta_q/z}\Big[\sum_{r<R} z^r B_r(q) + O(z^R)\Big]. \tag{242}$$

*Then the $C_r^+$ and $B_r$ are analytic and even in $q$, and analytic in $x$ (at fixed $\pi$), on the complex neighbourhood $\mathcal{V}$ of the corridor described in the proof, $C_0^+ = C_1^+ = C_2^+ = 0$, $C_3^+ = O(q^4)$, $B_0 = \omega_{ab}\pi^{-1/2}D^{-1/4}$, and*

$$\frac{C_3^+(q)}{B_0(q)} \in i\mathbb{R} \qquad (\textit{real } q, x). \tag{243}$$

*Proof. Domain of analyticity.* Fix $\pi$ and the corridor $\mathcal{C}$. On $\mathcal{C}$ one has $D > 0$ and $e_\pi$, $\sin p$, $\sin\Theta_q$ bounded below, and $\mathcal{C}$ contains the whole segment $\{(r, x, \pi) : 0 \le r \le q\}$ with each of its points. By compactness and continuity there is a complex neighbourhood $\mathcal{V}$ of $\mathcal{C}$ in the variables $(q^2, x)$, with $(\sigma q^2, x) \in \mathcal{V}$ whenever $(q^2, x) \in \mathcal{V}$ and $0 \le \sigma \le 1$, and with the following properties for

all $(q^2,x)\in\mathcal{V}$: the data $e_\pi(r)$, $v_\pi(r,0)$ of (E1) are analytic and $w:=(x-b_0(r))/e_\pi(r)$ stays in a fixed compact neighbourhood of a subset of $(-1,1)$ disjoint from the cuts $(-\infty,-1]\cup[1,\infty)$ of arccos, equivalently $D(r,x)=e_\pi(r)^2(1-w^2)\neq 0$ (here $r=q^2$ and all $r=\sigma q^2$); and $\Theta_q$ lies in a fixed complex neighbourhood of a compact subinterval of $(0,\pi)$, so that $\sin\Theta_q\neq 0$. On $\mathcal{V}$, $p=\arccos w$ and $D$ are analytic in $(q^2,x)$, hence so are $\Theta_q=\int_0^1 p(\sigma^2q^2,x)\,\mathrm{d}\sigma$ (the substitution $s=\sigma q$ in $S=\int_0^q p(s^2,x)\,\mathrm{d}s=q\Theta_q$), $S$, and $D^{-1/4}$ (on a small neighbourhood $D$ stays in a disc not containing 0); $\Sigma_2$ is a fixed algebraic expression in $a_0,a_2,b_2,p,g$ and their $q$-derivatives, hence analytic in $(q^2,x)$ for $q\neq 0$, and by Proposition 3.16 $q^2\rho_2$ is analytic in $(q^2,x)$ and vanishes at $q^2=0$ identically in $x$, so $\rho_2$, $\Gamma_2/q$, $\Theta_q^{[2]}$, $W_q^{[2]}$ are analytic in $(q^2,x,h^2)$ on $\mathcal{V}$. The Darboux data $\mathfrak{c}_{ab}$ and the amplitudes $\mathfrak{a}_{r,ab}$ of (61) are analytic functions of $\vartheta\in(0,\pi)$, being obtained from the analytic potential $V$ of the proof of Lemma 3.6 by the Liouville–Green recursion there (each step integrates an analytic function; the constants of integration are numbers), so their compositions with $\Theta_q$ and $\Theta_q^{[2]}$ are analytic on $\mathcal{V}$. Finally $a$, $b$ of (64) do not involve $x$ at all, and $x$ enters $\mathcal{L}_h$ only through the diagonal term $x-b$. Each $C_r^+$ and $B_r$ is a fixed polynomial in these functions and their $q$-derivatives (the formal expansion described at the end of this proof), so it is analytic in $(q^2,x)$ on $\mathcal{V}$. The reason is thus not any special dependence of the recurrence on $x$: $p$, $D$, $\Theta_q$, $\Sigma_2$ and the Darboux amplitudes are analytic in $(q^2,x)$ wherever $D\neq 0$ and $\sin\Theta_q\neq 0$.

*Evenness and vanishing.* At fixed $k$ every factor of $\widehat{U}_k^{[2],+}$ is analytic and even in $q$ ($h^2=q^2z^2$; $\Theta^{[2]}$, $D$ are even analytic by Proposition 3.16), and so are the exact coefficients; the carriers of the neighbouring ranks are $e^{i\Theta_q/z}$ times functions analytic in $(q,z)$, as in the proof of Lemma B.3. This gives the even coefficients. The vanishing of $C_0^+,C_1^+,C_2^+$ is the direct expansion in the proof of Lemma B.1 (Steps 3–5 there); the bound $C_3^+=O(q^4)$ is proved in the proof of Lemma B.3 applied to $\widehat{U}^{[2]}$ (Corollary B.4), whose fixed-ratio hypothesis (i) is Lemma B.1; and $B_0=W_q^{[2]}A_{0,ab}^+(\Theta_q^{[2]})=\omega_{ab}\pi^{-1/2}D^{-1/4}$ by (61), since $W_q^{[2]}=(\sin\Theta_q^{[2]})^{1/2}D^{-1/4}$.

For (243) regard $q>0$ as fixed and $h$ as a real parameter of either sign, with $\kappa=q/h$. This is a statement about *formal finite branch expansions*, not about Wigner functions of negative rank: $\widehat{U}^{[2],\pm}$ denotes the explicit expression (65), a carrier $e^{\pm i\kappa\Theta_q^{[2]}}$ times a finite sum $W_q^{[2]}\sum_{r<R}\kappa^{-r}A_{r,ab}^{\pm}(\Theta_q^{[2]})$ with the amplitudes of Lemma 3.23, evaluated at $(q,h)$ through $\kappa=q/h$ (no $d_{ab}^k$ with $k<0$ enters, and the Darboux series is not summed). Likewise $C_r^+(q)$ are, by definition, the coefficients of the formal expansion in $z=h/q$ of $e^{-i\Theta_q/z}\mathcal{L}_h\widehat{U}^{[2],+}$, obtained by Taylor-expanding the shifted carriers and amplitudes at $q\pm h$ and the coefficients $a,b$ of (64); each $C_r^+$ is a fixed algebraic expression in the coefficients of these expansions. The operator $\mathcal{L}_h$ of (64) satisfies $\mathcal{L}_{-h}=\mathcal{L}_h$ exactly, as an operator on functions of $q$ (the two links are exchanged and $a,b$ are even in $h$). Since $\Theta^{[2]}$ and $D$ depend on $h^2$ only, and $A_{r,ab}^{\pm}=\omega_{ab}^{\pm1}(\pm i)^r\mathfrak{a}_{r,ab}$ with $\mathfrak{a}_{r,ab}$ real (61), the substitution $h\mapsto -h$ ($\kappa\mapsto-\kappa$, $z\mapsto -z$) maps the carrier $e^{i\kappa\Theta_q^{[2]}}$ to $e^{-i\kappa\Theta_q^{[2]}}$ and $\kappa^{-r}A_{r,ab}^+$ to $(-1)^r\kappa^{-r}\omega_{ab}(i)^r\mathfrak{a}_{r,ab}=\omega_{ab}^2\,\overline{\kappa^{-r}A_{r,ab}^+}$, term by term; this is the exact algebraic identity $\widehat{U}^{[2],+}(q;-h)=\omega_{ab}^2\,\widehat{U}^{[2],-}(q;h)$ between the finite expressions, with $\widehat{U}^{[2],-}=\overline{\widehat{U}^{[2],+}}$ for real data. Applying $\mathcal{L}_{-h}=\mathcal{L}_h$ gives $\varrho^+(q;-h)=\omega_{ab}^2\overline{\varrho^+(q;h)}$ for $\varrho^+(q;h):=\mathcal{L}_h\widehat{U}^{[2],+}$, again as an identity of finite expressions, and so, comparing the coefficients of the formal series in $z$ one at a time, $(-1)^rC_r^+=\omega_{ab}^2\overline{C_r^+}$ and $B_0=\omega_{ab}^2\overline{B_0}$. For $r=3$ this is $\overline{C_3^+/B_0}=-C_3^+/B_0$. □

## Proof of Theorem 3.24: Phase exactness of the second-order pullback

*Proof. The initial value of $\alpha$.* Write $\widehat{Y}_k=\widehat{d}_k+t\,y_k+O_k(t^2)$ and $\widehat{U}_k^{[2]}=\widehat{d}_k+t\,u_k+O_k(t^2)$ at fixed rank. Differentiating the exact recurrence and the residual identity of Proposition 3.17 at $t=0$ shows that $\delta_k:=y_k-u_k$ solves the Wigner recurrence (16) for all $k\geq k_{\min}$, including the

boundary row $k = k_{\min}$ (where $A^{(0)}_{k_{\min}} = 0$; in the exceptional channel this is the $k = 0$ row of the hypothesis, with $B^{(0)}_0 = 0$ as in Proposition 3.17). The solutions of the Wigner recurrence satisfying the boundary row form the line spanned by $\widehat{d}$, and $\widehat{d}_{k_{\min}}(\arccos x) \neq 0$ on $X$. Hence $\delta_k = c(x)\widehat{d}_k$ for all $k$, with $c(x) = \delta_{k_{\min}}/\widehat{d}_{k_{\min}}$. Here $y_k = \partial_t \widehat{Y}_k(x;0)$ is analytic in $x$ on $U$ by the joint analyticity in (E2) (Cauchy's formula in $t$), $u_k$ is analytic in $x$ on a complex neighbourhood of $X$, and $\widehat{d}_{k_{\min}}$ has no zero there, so $c$ is analytic in $x$ on a complex neighbourhood of $X$.

*Construction of $\alpha$.* With $C_3^+, B_0$ from Lemma C.1, put

$$\alpha(q,x) := c(x) + \int_0^q \frac{C_3^+(s,x)}{i\, s^3\, e_\pi(s^2)\sin p(s,x)\, B_0(s,x)}\,\mathrm{d}s. \tag{244}$$

By (243) the integrand is real, and since $C_3^+ = q^4\times$(even analytic) it is odd analytic in $s$; so $\alpha$ is real and even analytic in $q$. It is analytic in $x$ on the complex neighbourhood $\mathcal{V}$ of Lemma C.1 (at fixed $\pi$): $c$ is, by the first step (this is where the joint analyticity in (E2) enters), and the integrand is built from $e_\pi$, $p$, $B_0$ and $C_3^+$, which are analytic in $(s^2,x)$ on $\mathcal{V}$ because $p$, $D$, $\Theta_s$, $\Sigma_2$ and the Darboux amplitudes are, wherever $D \neq 0$ and $\sin\Theta_s \neq 0$ (proof of that lemma); $\mathcal{V}$ contains $(s^2,x)$ for $0 \le s \le q$ with $(q^2,x)$, so the integral (244) stays in the domain of analyticity. Since $C_3^+/B_0 \in i\mathbb{R}$ and $|B_0|^2 = (\pi e_\pi \sin p)^{-1}$, the integrand equals $\pi s^{-3}\Im(\overline{B_0}C_3^+) = -\partial_s \mathsf{J}_2(s,x)$ by Lemma 3.25(iv), so $\alpha(q,x) = c(x) + \mathsf{J}_2(0,x) - \mathsf{J}_2(q,x)$, which is (67). Set $\widehat{U}^{[3]}_k := (1+h^2\alpha(q_k,x))\widehat{U}^{[2]}_k$.

*Fixed-ratio residual.* At fixed $q > 0$, Lemma C.1 gives $\mathcal{L}_h\widehat{U}^{[2],+} = e^{i\Theta_q/z}[h^3 q^{-3}C_3^+ + O(h^4)]$. For a smooth $\alpha$,

$$\mathcal{L}_h(h^2\alpha u) = h^2\alpha\,\mathcal{L}_h u - h^2\sum_{\pm} a(q\pm\tfrac{h}{2})(\alpha(q\pm h)-\alpha(q))u(q\pm h), \tag{245}$$

and with $u = \widehat{U}^{[2],+}$ one has $u(q\pm h) = e^{\pm ip}u(q)\,(1+O(h))$ and $a = \frac{1}{2}e_\pi + O(h^2)$, so the sum is $ih\,\alpha_q\, e_\pi \sin p\, B_0 e^{i\Theta_q/z} + O(h^2)$. Thus the $h^3$ coefficient of $\mathcal{L}_h\widehat{U}^{[3],+}$, with $\widehat{U}^{[3],+} = (1+h^2\alpha)\widehat{U}^{[2],+}$, is

$$q^{-3}C_3^+ - i\,\alpha_q\, e_\pi \sin p\, B_0 = 0 \tag{246}$$

by (244). The same holds for the conjugate branch because $\alpha$ is real, and the Darboux remainder contributes $O(\kappa^{-R})$. Hence $|\mathcal{R}_k(\widehat{U}^{[3]})| \le Ch^4$ at fixed positive ratio.

*Fixed-rank residual, corner, and summation.* At fixed $k$, $h^2\alpha(q_k,x)$ is analytic in $t$ (since $\alpha(q,x)$ is even analytic in $q$ and $q_k^2 = \kappa^2 t$), so $\widehat{U}^{[3]}$ is analytic at fixed rank; the $t$ coefficient of its residual is that of $\widehat{U}^{[2]}$, which vanishes by Proposition 3.17, plus $\alpha(0,x)$ times the residual of $\widehat{d}$, which is zero. Cauchy's estimate on $|t| \le c\kappa^{-3}$ gives $|\mathcal{R}_k(\widehat{U}^{[3]})| \le Ch^4k^6$ for $k^3h^2 \le \eta_0$, as in Proposition 3.19, and Lemma B.3 with $(p,R) = (4,22)$ (its blown-up hypotheses hold as in Corollary B.4, since $1+q^2z^2\alpha$ is analytic in $(q,z)$) gives $|\mathcal{R}_k(\widehat{U}^{[3]})| \le C(h^4 + k^{-22})$ for $k \ge k_0$. At the fixed start rank $k_0$, $\widehat{Y}_{k_0} - \widehat{U}^{[3]}_{k_0} = t(\delta_{k_0} - \alpha(0,x)\widehat{d}_{k_0}) + O(t^2) = O(h^4)$ by the choice $\alpha(0,x) = c(x)$, at both start entries. Applying Theorem 3.18 with $p = 4$ (split rank $K_* = \min\{L, \lfloor h^{-1/7}\rfloor\}$) and Corollary 3.12 gives $\sup_{k_0\le k\le L}|\widehat{Y}_k - \widehat{U}^{[3]}_k| = O(h^3)$; finitely many ranks below $k_0$ are covered by fixed-rank analyticity. □

## Proof of Corollaries 4.4 and 5.3: the fixed-rank datum

*Proof.* It remains to show $c(x) = -\mathsf{J}_2(0,x)$. By the proof of Proposition 3.17, the coefficient of $t$ in $U^{[2]}_k$ is $\mathcal{Q}_{2,k}d_k$, a first-order operator in $x$ applied to $d_k = d^k_{ab}(\arccos x)$, so $c(x)d_k$ is the coefficient of $t$ in $Y_k$ minus $\mathcal{Q}_{2,k}d_k$, at any one rank $k$. For the Hahn chain [2, Theorem 6.1] gives that coefficient as $(A_2d_k' + B_2d_k)/(12(x^2-1))$ with $A_2 = 2J(x^3-x) - 4x(M^2+\sigma^2) + 2M\sigma(x^2+3) + 4x$, $B_2 = J(x^2-3) + 2M\sigma x$, $J = k(k+1)$. With (90) and (53) the $d_k'$ coefficients of the two agree identically in $(k,x)$,

and the $d_k$ coefficients differ by $((8(M^2+\sigma^2)-5)x^2+4M\sigma x(x^2-5)-3)/(48(1-x^2)^2) = -\mathsf{J}_2(0,x)$. For the centered Racah chain [2, Theorem 9.1] gives the coefficient as $P_k d'_k + Q_k d_k$ for its kernel $(-1)^{m-n}\Psi_k$, which has the same ratio to the Wigner function; with $A, B$ replaced by $\alpha, \beta$ and (112), the $d'_k$ coefficients again agree and the $d_k$ coefficients differ by $-\mathsf{J}_2(0,x)$ from (68), identically in $(k, m, n, \alpha, \beta, x)$. Both identities are rational-function identities, checked by exact computer algebra. □

# D Analytic continuation of bulk rank estimates and zero stability

**Lemma D.1** (Polynomial continuation and complex-angle Darboux bounds)**.** *For an even centered Wigner–Jacobi chain on a turning-free corridor $\mathcal{C}$ (satisfying the $k=0$ row of Theorem 3.24 when $a=b=0$), for all sufficiently large admissible $N$ the exact solution continues in the spectral coordinate $x$ as*

$$Y_k(x;h) = G_N(x)P_{k,N}(x) \qquad (k_{\min} \le k \le L), \tag{247}$$

*where $G_N$ is analytic and zero-free on a fixed complex neighbourhood $U$ of the real compact set and $P_{k,N}$ is a polynomial in $x$ of degree $k-k_{\min}$ whose zeros are all real and simple; for the Hahn, centered Racah, and quantum $6j$ chains its zeros are those of the Hahn, Racah, or $q$-Racah polynomial. Moreover, for $\Re\vartheta$ in a compact subinterval of $(0,\pi)$ and $\kappa|\Im\vartheta| \le C$, one has $\sqrt{2k+1}\,|d^k_{ab}(\vartheta)| \le C'$.*

*Proof.* The minimal admissible rank recurrence has one vanishing backward hopping, $A_{k_{\min}} = 0$, while on $\mathcal{C}$, $e_\pi(k^2h^2, h^2) > 0$ and hence $A_k(h) > 0$ for $k_{\min} < k \le L$ and $N$ large. By (E2) (Remark 3.3), after shrinking $U$ if necessary, the lowest-rank seed $G_N := Y_{k_{\min}}(\cdot;h)$ is analytic and zero-free on $U$ for all sufficiently large $N$; in the exceptional channel $a=b=0$ the $k=0$ row gives $Y_1/Y_0 = (x-B_0)/(\sqrt{3}\,A_1)$, which is affine in $x$ since $A_1, B_0$ are independent of $x$ by (E1). All subsequent rank steps have off-diagonal coefficients independent of $x$ and diagonal coefficients affine in $x$; division by $G_N$ therefore gives a polynomial in $x$ by induction. Explicitly, $\widehat{P}_k := \sqrt{2k+1}\,P_{k,N}$ satisfies $A_{k+1}\widehat{P}_{k+1} = (x-B_k)\widehat{P}_k - A_k\widehat{P}_{k-1}$ with $\widehat{P}_{k_{\min}-1} = 0$ and $\widehat{P}_{k_{\min}}$ a nonzero constant, so $\widehat{P}_k$ is a nonzero constant times $\det(x-J_k)$, where $J_k$ is the real symmetric tridiagonal matrix with diagonal $B_{k_{\min}}, \dots, B_{k-1}$ and off-diagonal $A_{k_{\min}+1}, \dots, A_{k-1} > 0$; its eigenvalues, the zeros of $P_{k,N}$, are real and simple. In the Racah and quantum $6j$ cases $x$ is affine in $C^2 = c(c+1) + \frac{1}{4}$ or $\cosh \mathsf{Z}$, so these are precisely zeros in the standard spectral variable.

For the complex-angle bound, the function $w = (\sin\vartheta)^{1/2} d^k_{ab}(\vartheta)$ solves $w'' + (\kappa^2 - V)w = 0$ with $V = (a^2+b^2-\frac{1}{4}-2ab\cos\vartheta)/\sin^2\vartheta$ analytic and bounded on the stated set, so the energy $|w'|^2 + \kappa^2|w|^2$ grows at most by the factor $e^{(2\kappa + C/\kappa)|\Im\vartheta|}$ along vertical segments, and the real interior bounds $\sqrt{2k+1}\,|d^k_{ab}| \le C$, $\sqrt{2k+1}\,|\partial_\vartheta d^k_{ab}| \le C\kappa$ (Lemma 3.6) give $\sqrt{2k+1}\,|d^k_{ab}(\vartheta)| \le C'$. □

**Lemma D.2** (Real-axis bounds continue to an $h$-strip)**.** *Let $F$ be analytic on a neighbourhood of the closed disc $|x-x_0| \le r$, with $|F| \le \varepsilon$ on $[x_0-r, x_0+r]$ and $|F(x)| \le Me^{\Lambda|\Im x|}$ on the disc, $M \ge \varepsilon > 0$. Then for real $|y| < r$*

$$|F(x_0+iy)| \le \varepsilon \exp\Big(\frac{4|y|}{\pi r}\big(\log(M/\varepsilon) + \Lambda r\big)\Big). \tag{248}$$

*In particular, if $\Lambda \le C_0 N$, $M \le C_0 N^m$ and $\varepsilon \ge N^{-m'}$, then $|F(x_0+iy)| \le C\varepsilon$ for $|y| \le c/N$, and Cauchy's estimate gives $|F'(x_0)| \le C\varepsilon N$ and $|F''(x_0)| \le C\varepsilon N^2$, with $C$ depending only on $C_0, m, m', r, c$.*

*Proof.* On the upper half-disc the harmonic measure of the semicircle is $\omega(x_0+\xi+iy) = \frac{2}{\pi}\arg\frac{r+\xi+iy}{r-\xi-iy}$, so $\omega(x_0+iy) = \frac{4}{\pi}\arctan(y/r) \le \frac{4y}{\pi r}$. The subharmonic function $\log|F|$ is at most $\log\varepsilon$ on the

diameter and $\log M+\Lambda r$ on the semicircle, hence at most $(1-\omega)\log\varepsilon+\omega(\log M+\Lambda r)$ inside; the lower half-disc is the same. For the corollary, $|y|\le c/N$ makes the exponent $O(N^{-1}(\log N+N))=O(1)$; Cauchy's formula on the circle of radius $c/(2N)$ about $x_0$, with the first part applied at the centres $x_0+\Re\xi$ and $r/2$ in place of $r$ (for $c/(2N)\le r/2$ these discs lie in $|x-x_0|\le r$), gives the derivative bounds. □

**Lemma D.3** (Exponential-type bounds off the real axis)**.** *For an even centered chain and a compact turning-free fixed-ratio family with swept paths in a corridor, there are $r, C>0$ such that, for $\Re x$ in a slightly enlarged real compact set, $|\Im x|\le r$ and every rank $L$ of the family,*

$$|\widehat{Y}_L(x)|\le Ce^{CN|\Im x|},\qquad |\widehat{U}_L^{[2]}(x)|+|\widehat{U}_L^{[3]}(x)|\le CN^{1/2}e^{CN|\Im x|},\tag{249}$$

*where $\widehat{U}^{[3]}=(1+h^2\alpha)\widehat{U}^{[2]}$ is the comparison of Theorem 3.24.*

*Proof.* For real $\vartheta$, $d_{ab}^k(\vartheta)=\langle k,a|e^{-i\vartheta J_y}|k,b\rangle=\sum_\mu\langle k,a|v_\mu\rangle e^{-i\vartheta\mu}\langle v_\mu|k,b\rangle$ over an orthonormal eigenbasis of $J_y$ with $|\mu|\le k$; this formula continues to complex $\vartheta$, and Cauchy–Schwarz gives $|d_{ab}^k(\vartheta)|\le e^{k|\Im\vartheta|}$. The functions $\Theta_q^{[2]}(x)$, $W_q^{[2]}(x)$ and $\alpha(q,x)$ are analytic in $x$ on a fixed complex neighbourhood and real on the real axis, so $|\Im\Theta_q^{[2]}(x)|\le C|\Im x|$; this gives the pullback bounds. For the exact solution let $G_k(x_r)$, $x_r=\Re x$, be the bounded real fundamental matrix of Corollary 3.12. The exact transfer at $x$ differs from that at $x_r$ only by $i\Im x\,A_{k+1}^{-1}E_{11}$, so in the interaction representation each step is $I+V_k$ with $\|V_k\|\le C|\Im x|$, and the product over $L=O(N)$ ranks is at most $e^{CN|\Im x|}$. The finitely many ranks below the start rank are analytic and bounded on a fixed complex neighbourhood by (E2) (the real set $\{|x-X|\le r\}$ lies in $U$ for $r$ small, so the chain satisfies (E1)–(E2) on this slightly enlarged real compact set). □

## Proof of Theorem 3.26: Bulk zero quantization

*Proof.* The polynomial structure, and the reality and simplicity of the zeros of $P_{L,N}$, are proved in Lemma D.1, so it suffices to count real zeros. Put $F=\widehat{Y}_L-\widehat{U}_L^{[3]}$. By Theorem 3.24, applied on a slightly larger turning-free set, $|F|\le Ch^3$ on the real axis; by Lemma D.3 the growth hypothesis of Lemma D.2 holds with $\Lambda=CN$ and $M=CN^{1/2}$. Hence $|F|\le Ch^3$, $|F'|\le Ch^2$ and $|F''|\le Ch$ on the real compact set. The zeros of $\widehat{U}_L^{[3]}$ are those of $d_{ab}^L(\Theta_q^{[2]}(x))$, because $1+h^2\alpha>0$. By Lemma 3.23 (with differentiated remainder, Lemma 3.6), $\widehat{U}_L^{[3]}=\epsilon_{ab}(2/\sqrt{\pi})(1+h^2\alpha)D^{-1/4}[\mathfrak{A}\cos\Phi+O(h^R)]$ with $\Phi=\kappa\Theta_q^{[2]}+p_{ab}\pi/2-\pi/4+\mathfrak{P}$ and $\partial_x\Phi=N\partial_xS+O(1)$ (each $x$-derivative on the carrier-separated remainder costs at most $O(N)$: since $\mathfrak{A}$ and $\mathfrak{P}$ are smooth in $(\vartheta,\kappa^{-1})$, the $O(\kappa^{-R})$ remainder of Lemma 3.23 inherits the $C^s$ bounds of Lemma 3.6, with a factor $\kappa$ per derivative); on a turning-free set $\partial_xS=-\int_0^q\mathrm{d}s/(e_\pi\sin p)\le -cq$. So the comparison zeros $\widetilde{x}_j$ are simple, $|\partial_x\widehat{U}_L^{[3]}(\widetilde{x}_j)|\ge cN$, $|\partial_x^2\widehat{U}_L^{[3]}|\le CN^2$, consecutive zeros are $\asymp N^{-1}$ apart, and $|\widehat{U}_L^{[3]}(x)|\ge cN\operatorname{dist}(x,\{\widetilde{x}_j\})$ within distance $cN^{-1}$ of the zero set and $\ge c$ further away. Now count zeros. At $\widetilde{x}_j\pm C_1h^4$ Taylor's theorem gives $\widehat{U}_L^{[3]}$ opposite signs and size at least $cC_1h^3-CN^2h^8\ge\frac{c}{2}C_1h^3$, while $|F|\le Ch^3$; for $C_1$ large, $\widehat{Y}_L$ has a zero in between. Within distance $c'N^{-1}$ of $\widetilde{x}_j$, with $c'$ small, $|\partial_x\widehat{U}_L^{[3]}|\ge cN/2$ and $|F'|\le Ch^2$, so $\widehat{Y}_L$ is strictly monotone there and that zero is the only one. At distance between $C_1h^4$ and $c'N^{-1}$ from the zero set, $|\widehat{U}_L^{[3]}|\ge cC_1h^3>|F|$, and further away $|\widehat{U}_L^{[3]}|\ge c>|F|$; so $\widehat{Y}_L$ has no other zeros. Running this on a slightly larger compact turning-free set, every zero of $\widehat{Y}_L$ in the original set lies within $C_1h^4$ of exactly one comparison zero, which may lie in an $O(h^4)$ collar outside it.

For the quantization condition, at $\widetilde{x}_j$ one has $\cos\Phi=O(h^R)$ and $\mathfrak{A}\ge\frac12$, so $\Phi=(\nu_j+\frac12)\pi+O(h^R)$. Here $\kappa\Theta_q^{[2]}=NS+h\Gamma_2$ because $\kappa h=q$, and $\mathfrak{P}=\mathfrak{c}_{ab}(\Theta_q^{[2]})/\kappa+O(\kappa^{-3})=h\,\mathfrak{c}_{ab}(\Theta_q)/q+O(h^3)$. Moving

from $\widetilde{x}_j$ to $x_j$ costs $O(N \cdot h^4) = O(h^3)$ in $NS + h\widetilde{\Gamma}$. This proves (72). Finally $\Theta_q^{[2]} - \Theta_q = h^2\Gamma_2/q$ moves the angle by $O(h^2)$, hence the zero of $d_{ab}^L(\Theta_q(x))$ with the same index by $O(h^2)$, and dropping $h\widetilde{\Gamma}$, or only its Darboux part $h\,\mathfrak{c}_{ab}(\Theta_q)/q$, changes the left side of the quantization condition by a quantity of exact order $h$ wherever that term is nonzero. □

# E Rank-direction simple-turn Airy connection

This section proves Theorem 3.27(i): it verifies the simple-turning hypotheses for an even centered chain, constructs the canonical Airy basis, and identifies the physical combination. Throughout, the chain has a simple rank turn (73) of fixed type $\tau$ on a compact family, $u(q) = (x - b_0(q))/e_\pi(q^2)$, and $\zeta_\tau$ is the coordinate (74).

**Lemma E.1** (Simple-turning hypotheses for even centered chains)**.** *On a fixed neighbourhood of $q_*$ the natural-gauge recurrence (18) has off-diagonal coefficient bounded away from zero and coefficient functions admitting uniform smooth asymptotic expansions in powers of $N^{-1}$. Its principal characteristic quotient is $u$, and*

$$1 - u(q)^2 = \frac{D(q,x)}{e_\pi(q^2)^2}, \qquad \partial_q(1-u^2)(q_*) = \frac{\partial_q D(q_*,x)}{e_\pi(q_*^2)^2} < 0. \tag{250}$$

*Consequently the two principal characteristic roots coalesce simply at $\lambda = \tau$. When $\tau = -1$, the exact gauge $Y_k = (-1)^{k-k_{\min}}\widetilde{Y}_k$ converts the collision at $\lambda = -1$ into the same simple $+1$ turning-point normal form.*

*Proof.* Since $q_*^2 < s_1$ and $e_\pi(q_*^2) > 0$, (E1) gives a fixed neighbourhood of $q_*$ on which $e_\pi$ is bounded below and $e_\pi$, $v_\pi$ are analytic; there $k \asymp N$, so $A_k^{(0)}$ and $B_k^{(0)}$ are analytic functions of $k^{-1} = O(N^{-1})$ with $2A_k^{(0)} = 1 + O(N^{-2})$. This gives the positive lower bound and the uniform expansions. Freezing the principal coefficients gives $\lambda + \lambda^{-1} = 2u(q)$. The first identity in (250) is the definition $D = e_\pi^2 - (x - b_0)^2$, and the second follows from $D(q_*, x) = 0$ and (73). Hence $u^2 - 1$ has a simple zero at $q_*$ and $u(q_*) = \tau$. The alternating gauge changes the signs of both characteristic roots and therefore reduces the $-1$ collision to the $+1$ case without changing its order. □

**Theorem E.2** (Canonical Airy basis at a simple rank turn)**.** *In the natural gauge $\widehat{U}_L = \sqrt{2L+1}\,U_L$, there is a real canonical Airy pair $\widehat{U}_{\mathrm{Ai}}^{(N)}, \widehat{U}_{\mathrm{Bi}}^{(N)}$ of exact solutions of (18) on a fixed neighbourhood of $q_*$ such that, with $t = N^{2/3}\zeta_\tau(q)$ and $q = (L + \frac{1}{2})/N$,*

$$\widehat{U}_{\mathrm{Ai}}^{(N)}(L) = 2\chi_\tau(L)N^{1/6}\left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4}\left[\mathrm{Ai}(t) + O_{A_0}(N^{-1})\right], \tag{251}$$

$$\widehat{U}_{\mathrm{Bi}}^{(N)}(L) = 2\chi_\tau(L)N^{1/6}\left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4}\left[\mathrm{Bi}(t) + O_{A_0}(N^{-1})\right], \tag{252}$$

*for $|t| \le A_0$. Here $\chi_+(L) = 1$, while $\chi_-(L) = (-1)^{L-k_{\min}}$ is the exact alternating gauge, real for integer and half-integer ranks. The displayed remainders are additive inside the square brackets; no relative-error claim is intended at zeros of* Ai *or* Bi*. On the entire fixed turning neighbourhood, before restricting to $|t| \le A_0$, the same exact pair satisfies the uniform estimate of Geronimo–Bruno–Van Assche Theorem 3.8 in the form of Geronimo's Theorem 2.6: each bracket error is at most $CN^{-1}(\mathrm{Ai}^2 + \mathrm{Bi}^2)^{1/2}(t)$. On the forbidden side $t > 0$ this weight has the size of* Bi*, so the* Ai *member is not asserted to be recessive there; a recessive exact solution is constructed in Lemma E.5.*

*Undoing the natural gauge gives the physical scale*

$$U_{\mathrm{Ai}}^{(N)}(L) = \chi_\tau(L)\sqrt{\frac{2}{q}}\,N^{-1/3}\left|\frac{\zeta_\tau(q)}{D(q,x)}\right|^{1/4}\left[\mathrm{Ai}(t) + O_{A_0}(N^{-1})\right], \tag{253}$$

*with the analogous formula for* Bi. *In an oscillatory overlap* $N^{-2/3} \ll |q - q_*| \ll 1$, *the* Ai *member is the canonical WKB cosine branch and the* Bi *member is its quadrature branch.*

*Proof.* Set $\varepsilon = N^{-1}$ and $s = k\varepsilon$; the midpoint variable is $q = s + \varepsilon/2$. In the natural gauge the exact equation (18) is a smoothly varying Jacobi recurrence

$$xY_k = A_{k+1}Y_{k+1} + B_kY_k + A_kY_{k-1}. \tag{254}$$

On a fixed turning neighbourhood, (E1) and $k \asymp N$ give

$$A_k = a_0(s) + O(\varepsilon^2), \qquad B_k = b_0(s + \varepsilon/2) + O(\varepsilon^2), \quad a_0(s) = \tfrac{1}{2}e_\pi(s^2), \quad b_0(q) = v_\pi(q^2, 0), \tag{255}$$

with the corresponding discrete differences one power of $\varepsilon$ smaller. The lower bound on $e_\pi$ keeps $a_0$ uniformly positive.

To match Geronimo–Bruno–Van Assche Eq. (2.6) exactly, distinguish their smooth comparison coefficients from the coefficients of the actual recurrence. Set

$$\varepsilon^{-1} = N, \qquad y = x, \qquad a(s,\varepsilon) = a_0(s), \qquad b(s,\varepsilon) = b_0(s + \varepsilon/2), \tag{256}$$

and use (E1) to regard

$$a_1(k\varepsilon,\varepsilon) = A_k, \qquad b_1(k\varepsilon,\varepsilon) = B_k \tag{257}$$

as samples of smooth actual-coefficient functions on the fixed neighbourhood. Then (255) is literally their perturbation condition (3.52),

$$\sup|a - a_1| + \sup|b - b_1| = O(\varepsilon^2). \tag{258}$$

Moreover their characteristic quotient, Eq. (2.7), is exactly

$$\frac{y - b(s,\varepsilon)}{2a(s + \varepsilon/2,\varepsilon)} = \frac{x - b_0(q)}{e_\pi(q^2)} = u(q), \qquad q = s + \frac{\varepsilon}{2}. \tag{259}$$

By (250), $u^2 - 1$ has a simple zero with positive derivative in the outward $q$ direction. After shrinking the fixed neighbourhood if necessary, the squared characteristic exponent $k^2$ of Geronimo–Bruno–Van Assche is therefore increasing in $s$ with one zero, as their Theorems 3.3 and 3.8 require: their hyperbolic side $t \geq t_0$ (Eqs. (2.7)–(2.8)) is the forbidden side $q \geq q_*$ here.

Because $e_\pi$ and $v_\pi$ are analytic on a fixed complex neighbourhood and $A_k^{(0)}, B_k^{(0)}$ are analytic in $k^{-1}$, the $N^{-1}$ expansions of the coefficients and the finitely many scaled derivatives required by Geronimo–Bruno–Van Assche are uniform in $N$ and in the parameter family. Thus the compact family satisfies their uniform coefficient hypotheses, not merely the pointwise turning geometry.

In the $\tau = -1$ case the exact transformation $Y_k = (-1)^{k-k_{\min}}\widetilde{Y}_k$ replaces the characteristic quotient by $-u$, changing the colliding root from $-1$ to $+1$ while leaving the coefficient bounds and $O(\varepsilon^2)$ comparison estimates unchanged.

With the uniform hypotheses verified in Lemma E.1, Geronimo–Bruno–Van Assche Theorem 3.3, using their explicit Airy functions (2.10)–(2.11), gives an $O(\varepsilon^2)$ residual with Airy envelopes, and their Theorem 3.8 converts those parametrices to exact solutions with $O(\varepsilon)$ uniform envelope error under Eq. (3.52) [42].

In particular this control holds on the whole fixed turning neighbourhood, not only when the scaled Airy argument is bounded. In the form restated by Geronimo [48, Theorem 2.6, Eqs. (2.22), (2.25)], the exact solutions $f_i$ and the parametrices $\psi_1 = g\,\mathrm{Ai}(\varepsilon^{-2/3}\varrho)$, $\psi_2 = g\,\mathrm{Bi}(\varepsilon^{-2/3}\varrho)$ satisfy $|f_i - \psi_i| \le c\varepsilon\, g\,|\tilde{u}_i(\varepsilon^{-2/3}\varrho)|$ at every lattice point of the neighbourhood (Eq. (2.26) there after the alternating gauge), with the Hankel envelope $\tilde{u}_1(x) = (-x/3)^{1/2}e^{i\pi/6}H^{(1)}_{1/3}(\frac{2}{3}(-x)^{3/2})$ and $\tilde{u}_2(x) = (-x/3)^{1/2}e^{-i\pi/6}H^{(2)}_{1/3}(\frac{2}{3}(-x)^{3/2})$. For $x<0$, $\tilde{u}_1 = \mathrm{Ai} - i\,\mathrm{Bi}$ and $\tilde{u}_2 = \overline{\tilde{u}_1}$, so $|\tilde{u}_i| = (\mathrm{Ai}^2+\mathrm{Bi}^2)^{1/2}$; for $x>0$, evaluating $(-x)^{1/2} = i\sqrt{x}$ with the principal branches of $H^{(1,2)}_{1/3}$ at $-i\frac{2}{3}x^{3/2}$ gives $\tilde{u}_1 = \mathrm{Ai} + i\,\mathrm{Bi}$ and $\tilde{u}_2 = -2\,\mathrm{Ai}$. Only the bound $|\tilde{u}_i(x)| \le 2(\mathrm{Ai}(x)^2+\mathrm{Bi}(x)^2)^{1/2}$ for real $x$ is used here, that is, the weight $(\mathrm{Ai}^2+\mathrm{Bi}^2)^{1/2}$ up to a factor 2. We use only this non-recessive envelope; the recessive solution is constructed separately (Lemma E.5). The Wang–Wong theory provides an equivalent general simple-turning framework [47].

For the comparison equation the Langer coordinate is exactly the coordinate (74) at the lattice points. The Langer variable $\varrho$ of Geronimo–Bruno–Van Assche integrates, between the turning point $t_0$ and the lattice point $t$, the function arcosh on the hyperbolic side $t \ge t_0$ and arccos on the oscillatory side $t \le t_0$ of $(y - b(r,\varepsilon))/(2a(r+\varepsilon/2,\varepsilon))$ over $r$ [42, Eqs. (2.7)–(2.8), (2.13)]. (Geronimo [48, Eqs. (2.1)–(2.2)] places the hyperbolic side at $t \le t_p$; the reflection $n \mapsto -n$ of the lattice maps one convention to the other and leaves the quotient, with its midpoint $a(t+\varepsilon/2)$, unchanged.) With the comparison coefficients above this quotient is $u(r+\varepsilon/2)$, so the oscillatory integrand is $p$ (after the alternating gauge, $\pi - p$ on the $-1$ branch). The substitution $q' = r + \varepsilon/2$ maps the lattice point $t = k\varepsilon$ to $q_k$ and $t_0$ to $q_*$, and turns these integrals into the defining actions in (74), including their real hyperbolic continuations. Hence $\varepsilon^{-2/3}\varrho(k\varepsilon) = N^{2/3}\zeta_\tau(q_k)$ exactly, with no $O(\varepsilon)$ shift of the turning point. For half-integer ranks one indexes the lattice by the integer $n = k - k_{\min}$ and shifts both comparison coefficients by $k_{\min}\varepsilon$, which changes none of these identities. Since $(\mathrm{Ai}^2+\mathrm{Bi}^2)^{1/2}$ is bounded above and below on $|t| \le A_0$, the envelope estimate gives the bounded-window displays (251)–(252) with additive $O_{A_0}(N^{-1})$ bracket error.

The transport factor is equally explicit. Geronimo–Bruno–Van Assche Eq. (2.9) (Eq. (2.3) of [48]) gives, on the oscillatory side,

$$g = \left|\frac{\zeta_\tau}{a(s+\varepsilon/2,\varepsilon)^2\sin^2 p}\right|^{1/4}. \tag{260}$$

Since $a(s+\varepsilon/2,\varepsilon) = e_\pi(q^2)/2$ and $D = e_\pi^2\sin^2 p$ by (21), this simplifies exactly to

$$g = \sqrt{2}\left|\frac{\zeta_\tau}{D}\right|^{1/4}. \tag{261}$$

Multiplying this canonical pair by $\sqrt{2}\,N^{1/6}$ makes its oscillatory Airy expansion have natural-gauge WKB amplitude $2\pi^{-1/2}D^{-1/4}$, exactly the normalization used in the fixed-ratio basis. Finally,

$$\frac{2N^{1/6}}{\sqrt{2L+1}} = \sqrt{\frac{2}{q}}\,N^{-1/3}, \qquad 2L+1 = 2Nq, \tag{262}$$

which gives (253). The same Airy asymptotics identify the Ai member with the cosine WKB branch and the Bi member with its quadrature branch in the overlap $N^{-2/3} \ll |q - q_*| \ll 1$. □

## Proof of Theorem 3.27(i)

*Proof.* Theorem E.2 constructs a canonically normalized local Ai / Bi pair in the coordinate $\zeta_\tau$, including the alternating gauge on the $-1$ branch and the physical transport normalization. Thus

the only remaining task is to identify the physical linear combination and its lattice sign. Choose a fixed oscillatory matching point $q_0 < q_*$ inside the fixed turning neighbourhood but a positive distance from $q_*$; since $D > 0$ on $[0, q_*)$, the swept path of $q_0$ is a turning-free corridor. This point lies outside the bounded-$A_0$ display as $N \to \infty$, so the matching uses the fixed-neighbourhood Airy-envelope estimate of Geronimo–Bruno–Van Assche Theorem 3.8, verified in Theorem E.2, rather than only the bounded-scaled-argument display. At that point, the fixed-ratio case of Corollary 3.21, $W_q = (\sin\Theta_q)^{1/2} D^{-1/4}$, $\kappa\Theta_q = NS$ and (17) give

$$Y_L = \epsilon_{ab}\sqrt{\frac{2}{\pi N q}}\, D(q,x)^{-1/4} \cos\!\left(NS(q,x) + \frac{p_{ab}\pi}{2} - \frac{\pi}{4}\right) + O(N^{-3/2}), \tag{263}$$

For fixed negative $\zeta$,

$$\mathrm{Ai}(N^{2/3}\zeta) = \frac{N^{-1/6}|\zeta|^{-1/4}}{\sqrt{\pi}} \left[\cos\!\left(N\mathcal{A} - \frac{\pi}{4}\right) + O(N^{-1})\right], \qquad \mathcal{A} = \frac{2}{3}(-\zeta)^{3/2}. \tag{264}$$

Using the exact transport factor of the comparison parametrix, the canonical Ai member therefore has physical oscillatory amplitude $\sqrt{2/(\pi N q)}\, D^{-1/4}$ at the matching point, and this matches (263) exactly.

For $\tau = +1$, $\mathcal{A} = S(q_*, x) - S(q, x)$, and the quantization (75), $NS(q_*, x) = \pi K_+ - \pi(p_{ab} - 1)/2$, gives

$$NS(q,x) + \frac{p_{ab}\pi}{2} - \frac{\pi}{4} = \pi K_+ + \frac{\pi}{4} - N\mathcal{A}, \tag{265}$$

so the bulk cosine is $(-1)^{K_+}\cos(N\mathcal{A} - \pi/4)$. For $\tau = -1$,

$$\mathcal{A} = \int_q^{q_*} [\pi - p(s,x)]\,\mathrm{d}s = \pi(q_* - q) - S(q_*, x) + S(q, x), \tag{266}$$

and $Nq = \kappa$ together with (75) yields the exact physical-lattice identity

$$NS(q,x) = \pi\Big[K_- - \frac{p_{ab}}{2}\Big] + N\mathcal{A}. \tag{267}$$

Hence

$$NS(q,x) + \frac{p_{ab}\pi}{2} - \frac{\pi}{4} = \pi K_- + N\mathcal{A} - \frac{\pi}{4}. \tag{268}$$

To identify the two coefficients of the *exact* Airy basis one must match a two-component Jacobi vector, not a single value. Choose consecutive ranks $L_0, L_0 - 1$ at the fixed oscillatory matching point $q_0 < q_*$. Write $\widehat{\mathcal{A}}, \widehat{\mathcal{B}}$ for the exact natural-gauge canonical Ai/Bi solutions of Theorem E.2, and set

$$\mathcal{F}_{L_0} = \begin{pmatrix} \widehat{\mathcal{A}}_{L_0} & \widehat{\mathcal{B}}_{L_0} \\ \widehat{\mathcal{A}}_{L_0-1} & \widehat{\mathcal{B}}_{L_0-1} \end{pmatrix}. \tag{269}$$

The fixed-neighbourhood Airy-envelope estimates and the oscillatory asymptotics of both Ai and Bi give, after the constant factor $\chi_\tau(L_0)$ is extracted,

$$\mathcal{F}_{L_0} = \frac{2D(q_0,x)^{-1/4}}{\sqrt{\pi}} \begin{pmatrix} \cos\phi_0 & \pm\sin\phi_0 \\ \cos(\phi_0 + \tau p_0) & \pm\sin(\phi_0 + \tau p_0) \end{pmatrix} + O(N^{-1}), \tag{270}$$

where the sign in the second column is fixed by the Bi convention, $p_0 = p(q_0, x)$, and $\phi_0$ is the Airy cosine phase. The determinant of the leading matrix is, up to sign,

$$\frac{4}{\pi} D(q_0,x)^{-1/2} \sin p_0, \tag{271}$$

which is bounded away from zero on the fixed matching compact set. Consequently $\mathcal{F}_{L_0}^{-1} = O(1)$, uniformly in $N$.

The leading fixed-ratio theorem gives the *same two consecutive values* of the exact physical natural-gauge solution, with an $O(N^{-1})$ error in each entry. The phase identities (265) and (268) then give the *rank-independent* parity factor

$$\sigma_\tau := \frac{(-1)^{K_\tau(L)}}{\chi_\tau(L)} \in \{+1, -1\}, \tag{272}$$

since $K_+$ is independent of $L$, whereas both $K_-$ and the exponent in $\chi_-(L) = (-1)^{L-k_{\min}}$ increase by one as $L$ increases. The product $\chi_\tau(L)\sigma_\tau = (-1)^{K_\tau(L)}$ is independent of the reference rank $k_{\min}$ in $\chi_-$. Matching both physical entries therefore gives

$$\begin{pmatrix} \widehat{Y}_{L_0} \\ \widehat{Y}_{L_0-1} \end{pmatrix} = \epsilon_{ab}\sigma_\tau \begin{pmatrix} \widehat{\mathcal{A}}_{L_0} \\ \widehat{\mathcal{A}}_{L_0-1} \end{pmatrix} + O(N^{-1}). \tag{273}$$

Inverting the matrix gives the *constant* Airy-basis connection coefficients

$$c_{\mathrm{Ai}} = \epsilon_{ab}\sigma_\tau + O(N^{-1}), \qquad c_{\mathrm{Bi}} = O(N^{-1}). \tag{274}$$

Because both canonical columns are exact recurrence solutions, these coefficients are unchanged by propagation to the turning layer. There $|\mathrm{Ai}(t)| + |\mathrm{Bi}(t)| = O_{A_0}(1)$ for $|t| \le A_0$, so $c_{\mathrm{Ai}}\widehat{\mathcal{A}}_L + c_{\mathrm{Bi}}\widehat{\mathcal{B}}_L$ differs from $\epsilon_{ab}\sigma_\tau\widehat{\mathcal{A}}_L$ by $O_{A_0}(N^{-1})$ inside the bracket of (253), whose own error is $O_{A_0}(N^{-1})$. In particular no order-one Bi component survives, and (76) follows with the asserted parity and the additive $O_{A_0}(N^{-1})$ bracket error. □

## E.1 Second-order Airy connection

This subsection proves Theorem 3.27(ii). The construction is the even-midpoint version of the Fedotov–Klopp induction for difference Airy equations [51], with the Airy shift identity of Geronimo–Bruno–Van Assche; the canonical Bi solution of Theorem E.2, with the recessive solution of Lemma E.5, is used only as an exact basis for a discrete Green function.

Work at fixed $(x, \pi)$ on the fixed turning neighbourhood $I = [q_* - \delta_0, q_* + \delta_0]$ of Theorem E.2 (with analyticity domains and constants uniform on the compact family), after the alternating gauge when $\tau = -1$, and write $\varepsilon = h$. With $a(s; h^2)$, $b(q; h^2)$ as in Lemma C.1 put

$$(L\psi)(q) = a(q+\tfrac{h}{2}; h^2)\psi(q+h) + a(q-\tfrac{h}{2}; h^2)\psi(q-h) - 2a(q; h^2)z_h(q)\psi(q), \qquad z_h := \tau\,\frac{x - b(q; h^2)}{2a(q; h^2)}, \tag{275}$$

so that $L$ is minus the natural-gauge residual, $a, b$ are even analytic in $h$, and $z_0 = \tau u$. Let $\zeta = \zeta_\tau$ be the coordinate (74); it is analytic on $I$ with $\zeta' > 0$ and satisfies the eikonal $\cosh(\zeta^{1/2}\zeta') = z_0$ (with $\zeta^{1/2} \in i\mathbb{R}_+$ on the oscillatory side). Let $C(s; \zeta, h)$, $S(s; \zeta, h)$ solve $Y'' = (\zeta + hs)Y$ with $C = 1$, $C' = 0$, $S = 0$, $S' = 1$ at $s = 0$, and put $s_\pm(q, h) = (\zeta(q \pm h) - \zeta(q))/h$.

**Lemma E.3** (Airy shift, parity and first-order operators)**.** *(i) For every solution $\chi$ of $\chi'' = X\chi$, $X = h^{-2/3}\zeta(q)$, one has $\chi(X + h^{1/3}s) = \chi(X)C(s) + h^{1/3}\chi'(X)S(s)$ and $\chi'(X + h^{1/3}s) = h^{-1/3}\chi(X)C'(s) + \chi'(X)S'(s)$. Consequently, for $\psi = P\chi(h^{-2/3}\zeta) + h^{1/3}Q\chi'(h^{-2/3}\zeta)$,*

$$L\psi = \chi\,\mathfrak{A}[P, Q] + h^{1/3}\chi'\,\mathfrak{B}[P, Q], \tag{276}$$

$$\mathfrak{A} = \sum_{\pm} a(q \pm \tfrac{h}{2})\big[P_{\pm}C(s_{\pm}) + Q_{\pm}C'(s_{\pm})\big] - 2az_hP,$$
$$\mathfrak{B} = \sum_{\pm} a(q \pm \tfrac{h}{2})\big[P_{\pm}S(s_{\pm}) + Q_{\pm}S'(s_{\pm})\big] - 2az_hQ, \tag{277}$$

*$P_{\pm} = P(q \pm h)$, independently of $\chi$. (ii) If $P$ is even and $Q$ odd in $h$, then $\mathfrak{A}$ is even and $\mathfrak{B}$ is odd in $h$; if $P$ is odd and $Q$ even, then $\mathfrak{A}$ is odd and $\mathfrak{B}$ is even. (iii) For $h$-independent $p, q$: $\mathfrak{A}[p,0] = O(h^2)$, $\mathfrak{B}[0,q] = O(h^2)$, and*

$$\mathfrak{B}[p,0] = h\,\frac{(Vp^2)'}{p} + O(h^3), \qquad \mathfrak{A}[0,q] = h\,\frac{(Wq^2)'}{q} + O(h^3), \tag{278}$$

*with $V = a_0S_0(\zeta')$, $W = \zeta V$, $S_0(s) = \sinh(\zeta^{1/2}s)/\zeta^{1/2}$. $V$ has no zero on $I$, $W$ has exactly one zero, simple, at $q_*$, and $g := |V|^{-1/2} = \sqrt{2}\,|\zeta/D|^{1/4}$ satisfies $(Vg^2)' = 0$.*

*Proof.* (i) $Y(s) = \chi(X + h^{1/3}s)$ solves $Y'' = (\zeta + hs)Y$ with $Y(0) = \chi(X)$, $Y'(0) = h^{1/3}\chi'(X)$; and $h^{-2/3}\zeta(q \pm h) = X + h^{1/3}s_{\pm}$. (ii) Under $h \mapsto -h$ the two links exchange, $s_{\pm} \mapsto -s_{\mp}$, and uniqueness for the ODE gives $C(s;\zeta,-h) = C(-s;\zeta,h)$, $S(s;\zeta,-h) = -S(-s;\zeta,h)$; substitution gives $\mathfrak{A}[P,Q](-h) = \mathfrak{A}[\check{P},-\check{Q}](h)$ and $\mathfrak{B}[P,Q](-h) = -\mathfrak{B}[\check{P},-\check{Q}](h)$, $\check{P}(h) = P(-h)$. (iii) At $h = 0$, $C_0 = \cosh(\zeta^{1/2}s)$ is even and $S_0$ odd, $s_{\pm} = \pm\zeta'$, and the eikonal gives $[h^0]\mathfrak{A}[p,0] = 2a_0p(\cosh(\zeta^{1/2}\zeta') - z_0) = 0$ and likewise $[h^0]\mathfrak{B}[0,q] = 0$; the $h^1$ coefficients of these two vanish by (ii). For the $h^1$ coefficients of the other two use $s_{\pm} = \pm\zeta' + \frac{h}{2}\zeta'' + O(h^2)$ and $S_1 = \frac{s}{2}\partial_\zeta S_0$, $C_1' = \frac{s}{2}\partial_\zeta C_0'$ (both from the variational equation of $Y'' = (\zeta + hs)Y$); then $[h^1]\mathfrak{B}[p,0] = (a_0'p + 2a_0p')S_0 + a_0p\frac{\mathrm{d}}{\mathrm{d}q}S_0(\zeta'(q);\zeta(q)) = (Vp^2)'/p$, and the same computation with $C_0' = \zeta S_0$ gives $(Wq^2)'/q$. Finally $|V| = a_0\sqrt{|z_0^2 - 1|}/|\zeta|^{1/2} = \sqrt{D}/(2|\zeta|^{1/2})$ by (250), which is bounded and nonzero through the turn, and $W = \zeta V$. □

**Theorem E.4** (Formal Airy parametrix through relative order $h^2$)**.** *There are functions $q_1$, $p_2$, real-analytic on $I$, with analyticity domains and bounds uniform on the compact family, such that*

$$\psi^{[2]} := (g + h^2p_2)\,\mathrm{Ai}(h^{-2/3}\zeta) + h^{4/3}q_1\,\mathrm{Ai}'(h^{-2/3}\zeta) \tag{279}$$

*has $\mathfrak{A}[g + h^2p_2, hq_1] = O(h^4)$ and $\mathfrak{B}[g + h^2p_2, hq_1] = O(h^5)$ uniformly on $I$; hence $|(L\psi^{[2]})(q_k)| \le Ch^4\,\mathsf{M}(X_k)/\mathsf{E}(X_k)$ at the lattice points, with Olver's modulus $\mathsf{M}$ and weight $\mathsf{E}$ for the Airy functions. Here $q_1$ is the unique solution analytic at $q_*$ of $(Wq_1^2)'/q_1 = -[h^2]\mathfrak{A}[g,0]$.*

*Proof.* By Lemma E.3, $\mathfrak{A}[g,0]$ is even with vanishing $h^0$ term and $\mathfrak{B}[g,0]$ is odd with vanishing $h^1$ term (since $(Vg^2)' = 0$). Adding $Q = hq_1$ changes $[h^2]\mathfrak{A}$ by $(Wq_1^2)'/q_1$ and changes $\mathfrak{B}$ only at order $h^3$. The equation $2Wq_1' + W'q_1 = f$ with $W = (q - q_*)\omega$, $\omega$ analytic and nonzero, has the analytic solution $q_1(q) = \omega(q)^{-1/2}\int_0^1 f(q_* + (q - q_*)v^2)\,\omega(q_* + (q - q_*)v^2)^{-1/2}\mathrm{d}v$, while the homogeneous solution $|W|^{-1/2}$ is singular; so $q_1$ is unique. Now $\mathfrak{A}$ is $O(h^4)$ by parity, and $\mathfrak{B} = h^3B_3 + O(h^5)$. Adding $h^2p_2$ changes $[h^3]\mathfrak{B}$ by $(Vp_2^2)'/p_2$ and $\mathfrak{A}$ only at order $h^4$; the regular equation $(\sqrt{|V|}p_2)' = -\operatorname{sgn}(V)B_3/(2\sqrt{|V|})$ fixes $p_2$ up to a multiple of $g$. Taylor remainders of functions analytic on a compact set are uniform. No differentiability or analyticity in the parameter $\pi$ is asserted, since (E1) assumes only continuity in $\pi$. For the envelope bound use $|\,\mathrm{Ai}\,| \le \mathsf{M}/\mathsf{E}$, $|\,\mathrm{Ai}'\,| \le \mathsf{N}/\mathsf{E}$ and $h^{1/3}\mathsf{N}(X)/\mathsf{M}(X) \le C(h^{2/3} + |\zeta|)^{1/2}$. □

**Lemma E.5** (Recessive solution)**.** *Put $\psi_1 = g\,\mathrm{Ai}(h^{-2/3}\zeta)$, $\psi_2 = g\,\mathrm{Bi}(h^{-2/3}\zeta)$ and $X_k = h^{-2/3}\zeta(q_k)$ at the lattice points $q_k$ of $I$, let $f_1$ be the exact solution of $Lf = 0$ on the lattice of $I$ with $f_1 = \psi_1$ at the last two lattice points, and let $J$ be the set of lattice points with $X_k \ge -1$. Then*

*(i)* $|f_{1,k}-\psi_{1,k}|\le Ch\,\psi_{1,k}$ *for* $k\in J$*;*

*(ii)* $|f_1|\le C\mathsf{M}/\mathsf{E}$ *on the lattice of* $I$*; the exact canonical* Bi *solution* $f_2$ *of Theorem E.2, normalized to* $\psi_2$*, satisfies* $|f_2|\le C\mathsf{ME}$ *there; and the Casoratian* $\mathcal{W}=a(q_k+\frac{h}{2})(f_{1,k+1}f_{2,k}-f_{1,k}f_{2,k+1})$ *satisfies* $|\mathcal{W}|\ge ch^{1/3}$*.*

*Proof.* (i) Since $\mathrm{Ai}>0$ on $[-1,\infty)$, $\psi_1>0$ on $J$, and $J$ is a final segment of the lattice because $\zeta'>0$. Write $f_1=\psi_1(1+\sigma)$ on $J$ and $D_k=a(q_k+\frac{h}{2})\psi_{1,k}\psi_{1,k+1}(\sigma_{k+1}-\sigma_k)$. Then $Lf_1=0$ reads $D_k-D_{k-1}=-\psi_{1,k}(L\psi_1)_k(1+\sigma_k)$, and $\sigma$ vanishes at the last two lattice points, so summing twice gives the final-value Volterra equation

$$\sigma_k=-\sum_{i>k}K_{ki}(1+\sigma_i),\qquad K_{ki}=(L\psi_1)_i\sum_{j=k}^{i-1}\frac{\psi_{1,i}}{a(q_j+\frac{h}{2})\psi_{1,j}\psi_{1,j+1}},\tag{280}$$

both sums running over $J$ up to the second-last lattice point. By Lemma E.3 with $P=g$, $Q=0$, $\mathfrak{A}[g,0]=O(h^2)$ and $\mathfrak{B}[g,0]=O(h^3)$; with (276) and $|\mathrm{Ai}'(X)|\le C(2+X)^{1/2}\,\mathrm{Ai}(X)$ on $[-1,\infty)$, $h^{1/3}(2+X)^{1/2}\le C$ on $I$, this gives $|(L\psi_1)_i|\le Ch^2\psi_{1,i}$ on $J$. Next $\mathrm{Ai}(X)\asymp(2+X)^{-1/4}e^{-\xi(X)}$ on $[-1,\infty)$ with $\xi=\frac{2}{3}X_+^{3/2}$, and $g\asymp1$, so $\psi_{1,i}\le Ce^{-(\xi_i-\xi_j)}\psi_{1,j}$ for $j\le i$ in $J$; and $X_{j+1}-X_j\asymp h^{1/3}$ because $\zeta'\asymp1$. Hence $|K_{ki}|\le Ch^2\sum_{j<i}e^{-2(\xi_i-\xi_{j+1})}$, and the last sum is at most $Ch^{-1/3}(1+X_i^+)^{-1/2}$: if $X_i\le4$ it has at most $Ch^{-1/3}$ terms; if $X_i>4$, the terms with $X_{j+1}\ge X_i/4$ satisfy $\xi_i-\xi_{j+1}\ge\frac{1}{2}X_i^{1/2}(X_i-X_{j+1})$ and form a geometric sum $\le1+Ch^{-1/3}X_i^{-1/2}$, and the at most $Ch^{-1/3}(1+X_i)$ others are $\le e^{-X_i^{3/2}}$ each; recall $X_i\le Ch^{-2/3}$. So $|K_{ki}|\le Ch^{5/3}(1+X_i^+)^{-1/2}$, and $\sum_{i\in J}(1+X_i^+)^{-1/2}\le Ch^{-1/3}\int_{-1}^{Ch^{-2/3}}(1+X_+)^{-1/2}\mathrm{d}X\le Ch^{-2/3}$. Thus $\|K\|_\infty:=\max_k\sum_i|K_{ki}|\le Ch$, and $\max_J|\sigma|\le\|K\|_\infty(1+\max_J|\sigma|)$ gives $\max_J|\sigma|\le2\|K\|_\infty$ for $h$ small.

(ii) For $X\ge c_0$ ($c_0\approx-0.37$, where Ai = Bi) Olver's functions give $\mathsf{M}/\mathsf{E}=\sqrt{2}\,\mathrm{Ai}$ and $\mathsf{ME}=\sqrt{2}\,\mathrm{Bi}\ge(\mathrm{Ai}^2+\mathrm{Bi}^2)^{1/2}$, while $\mathsf{E}=1$, $\mathsf{M}=(\mathrm{Ai}^2+\mathrm{Bi}^2)^{1/2}$ for $X\le c_0$; on $[-1,c_0]$ also $\mathrm{Ai}\asymp\mathsf{M}$. So (i) gives $|f_1|\le C\mathsf{M}/\mathsf{E}$ on $J$, and the envelope estimate of Theorem E.2, with weight $g(\mathrm{Ai}^2+\mathrm{Bi}^2)^{1/2}\le Cg\mathsf{ME}$, gives $|f_2|\le C\mathsf{ME}$ and, for the canonical Ai solution $F_1$, $|F_1-\psi_1|\le Ch\,\mathsf{ME}$. Take consecutive lattice points $k_0,k_0+1$ with $X\in[-1,1]$. There $f_1-\psi_1$, $F_1-\psi_1$ and $f_2-\psi_2$ are $O(h)$ while $\psi_1,\psi_2=O(1)$, so the constant Casoratians satisfy $\mathcal{W}(f_1,f_2),\mathcal{W}(F_1,f_2)=\mathcal{W}_{k_0}(\psi_1,\psi_2)+O(h)$ and $\mathcal{W}(F_1,f_1)=O(h)$. By Lemma E.3(i), $\mathcal{W}_{k_0}(\psi_1,\psi_2)=a(q_{k_0}+\frac{h}{2})g_{k_0}g_{k_0+1}(\mathrm{Ai}(X+h^{1/3}s_+)\,\mathrm{Bi}(X)-\mathrm{Ai}(X)\,\mathrm{Bi}(X+h^{1/3}s_+))=-a\,g_{k_0}g_{k_0+1}h^{1/3}S(s_+)/\pi$ with $X=X_{k_0}$, and $|S(s_+)|\ge c$ on $I$; so $|\mathcal{W}(f_1,f_2)|\ge ch^{1/3}$, and $f_1=\alpha F_1+\beta f_2$ with $\alpha=\mathcal{W}(f_1,f_2)/\mathcal{W}(F_1,f_2)=1+O(h^{2/3})$, $\beta=\mathcal{W}(F_1,f_1)/\mathcal{W}(F_1,f_2)=O(h^{2/3})$. At lattice points with $X_k\le0$, $\mathsf{E}\le\mathsf{E}(0)$ and $|F_1|+|f_2|\le C\mathsf{M}$, so $|f_1|\le C\mathsf{M}\le C\mathsf{M}/\mathsf{E}$ there. □

**Proposition E.6** (Exact solution near the parametrix)**.** *Let* $f_1$ *be the recessive solution and* $f_2$ *the canonical* Bi *solution of Lemma E.5, so that* $|f_1|\le C\mathsf{M}/\mathsf{E}$ *and* $|f_2|\le C\mathsf{ME}$ *on the lattice of* $I$*. If a lattice function* $\psi$ *on* $I$ *has* $|(L\psi)_k|\le\eta\,\mathsf{M}(X_k)/\mathsf{E}(X_k)$*, the recurrence has an exact solution* $f$ *with* $f=\psi$ *at the last two lattice points of* $I$ *and* $|f_k-\psi_k|\le C\eta h^{-1}\mathsf{M}(X_k)/\mathsf{E}(X_k)$*. In particular there is an exact solution* $f_h$ *with* $|f_h-\psi^{[2]}|\le Ch^3\mathsf{M}/\mathsf{E}$ *on* $I$*.*

*Proof.* The Casoratian $\mathcal{W}=a(q_k+\frac{h}{2})(f_{1,k+1}f_{2,k}-f_{1,k}f_{2,k+1})$ is constant, and $|\mathcal{W}|\ge ch^{1/3}$ by Lemma E.5(ii). Let $n$ be the last lattice point of $I$. The final-value Green function $e_k=\sum_{k<i<n}(f_{2,k}f_{1,i}-f_{1,k}f_{2,i})\,r_i/\mathcal{W}$, with the sum running up to the second-last lattice point as in Lemma E.5(i), satisfies $e_{n-1}=e_n=0$ and $(Le)_i=r_i$ at every lattice point $i<n$ of $I$ (the kernel solves the homogeneous recurrence in $k$, vanishes at $k=i$, and its value at $k=i-1$ times $a(q_i-\frac{h}{2})$ is 1); take $r=-L\psi$ and $f=\psi+e$. For $i>k$, $\mathsf{E}(X_k)\le\mathsf{E}(X_i)$, so both terms $f_{2,k}f_{1,i}r_i$

and $f_{1,k}f_{2,i}r_i$ of the summand are bounded by $C\eta\,(\mathsf{M}_k/\mathsf{E}_k)\mathsf{M}_i^2$, and $\sum_i \mathsf{M}(X_i)^2 \le Ch^{-2/3}$ because $\mathsf{M}(X)^2 \le C(1+|X|)^{-1/2}$ and $X_{i+1}-X_i \asymp h^{1/3}$. Hence $|e_k| \le C\eta h^{-1}\mathsf{M}_k/\mathsf{E}_k$. With $\eta = Ch^4$ from Theorem E.4 this gives $f_h$. □

**Lemma E.7** (Olver form of the Airy coefficient)**.** *Put $B_\tau := q_1/g$ on $I$ and $\psi_\tau := -g\,[h^2]\mathfrak{A}[g,0]/\zeta'$. (i) $\psi_\tau$ is real-analytic on $I$, and $B_\tau$, as a function of $\zeta=\zeta_\tau(q)$, is the solution analytic at $\zeta=0$ of Olver's transport equation*

$$2\zeta\frac{\mathrm{d}B}{\mathrm{d}\zeta}+B=\psi_\tau,\qquad B_\tau=\frac{1}{2\zeta^{1/2}}\int_0^\zeta \psi_\tau(v)\,v^{-1/2}\,\mathrm{d}v=\int_0^1\psi_\tau(\zeta y^2)\,\mathrm{d}y; \tag{281}$$

*in particular $B_\tau(q_*)=\psi_\tau(q_*)$. (ii) $\psi_\tau(q_*)$ equals (79) and (82).*

*Proof.* (i) $[h^2]\mathfrak{A}[g,0]$ is a Taylor coefficient of a function analytic in $(q,h)$, and $\zeta'>0$. By Lemma E.3(iii) $V>0$ on $I$ (for $\zeta<0$, $\sqrt{-\zeta}\,\zeta'\in(0,\pi)$ is $p$ or $\pi-p$), so $Vg^2=1$, and $W=\zeta V$. For $q_1=gB$, using $(Vg^2)'=g(2Vg'+V'g)=0$,

$$\frac{(Wq_1^2)'}{q_1}=2\zeta VgB'+\zeta' VgB+\zeta B(2Vg'+V'g)=\frac{\zeta'}{g}\Big(2\zeta\frac{\mathrm{d}B}{\mathrm{d}\zeta}+B\Big), \tag{282}$$

so the transport equation of Theorem E.4 is (281); the homogeneous solution $\zeta^{-1/2}$ is singular, which leaves the displayed integral. (ii) The first identity is the evaluation at the end of the proof of Theorem 3.27(ii) below. The eikonal $\cosh(\zeta^{1/2}\zeta')=\tau u$ gives $F=\operatorname{arcosh}^2(\tau u)=\zeta\zeta'^2$, with $F_1=2u_1$, $F_2=2u_2-\frac{1}{3}u_1^2$, $F_3=2u_3-\frac{2}{3}u_1u_2+\frac{4}{45}u_1^3$; substituting these, $x-b_0(q_*)=\tau e_0$ and $(2e_0e_2-e_1^2)/e_0^2=(\log e_\pi(q^2))''(q_*)$ turns (79) into (82), an identity checked by exact computer algebra. □

## Proof of Theorem 3.27(ii)

*Proof.* Work in the gauge of the neighbourhood and write the physical solution as $\widehat{Y}=\kappa_h f_h+\mu_h f_2$, with $f_1,f_2$ as in Lemma E.5; this is possible because $\mathcal{W}(f_h,f_2)=\mathcal{W}(f_1,f_2)+O(h)$ (evaluate at the points $k_0,k_0+1$ of that proof, where $f_h-f_1=O(h)$) and $|\mathcal{W}(f_1,f_2)|\ge ch^{1/3}$.

*Recessive selection.* $\mu_h=\mathcal{W}(f_h,\widehat{Y})/\mathcal{W}(f_h,f_2)$. At the last lattice points of $I$, with $\xi_{\mathrm{end}}=\frac{2}{3}N\zeta(q_*+\delta_0)^{3/2}$, $|f_h|\le C\mathsf{M}/\mathsf{E}\le Ce^{-\xi_{\mathrm{end}}}$ (Proposition E.6; at these two points $\xi_{\mathrm{end}}-\xi=O(1)$, since $\xi'=N\Upsilon$ and $\xi$ increases by at most $\max_I\Upsilon$ per rank) and by (R) $|\widehat{Y}|\le CN^m$, so $|\mu_h|\le CN^{m+1/3}e^{-\xi_{\mathrm{end}}}$. On $I'=\{\zeta\le 4^{-2/3}\zeta(q_*+\delta_0)\}$, $\mathsf{E}(X)^2\le Ce^{\frac{4}{3}X_+^{3/2}}\le Ce^{\xi_{\mathrm{end}}/2}$, hence $|\mu_h f_2|\le CN^{m+1/3}e^{-\xi_{\mathrm{end}}}\mathsf{ME}\le CN^{m+1/3}e^{-\xi_{\mathrm{end}}/2}\,\mathsf{M}/\mathsf{E}=O(N^{-\infty})\,\mathsf{M}/\mathsf{E}$ (for the Hahn, centered Racah, and quantum $6j$ chains the larger range of Proposition 3.34(c) holds).

*The scalar $\kappa_h$.* Fix $q_0\in(q_*-\delta_0,q_*)$; its swept path is a turning-free corridor, so Theorem 3.20, Lemma 3.23 and $W^{[2]}(\sin\Theta^{[2]})^{-1/2}=D^{-1/4}$ give, at the lattice points $k$ near $q_0N$ (in the physical frame: for $\tau=-1$ the alternating gauge is restored, which adds $\pi(k-k_{\min})$ to the phases $\phi_k^f$ below),

$$\widehat{Y}_k=\frac{2}{\sqrt{\pi}}\epsilon_{ab}D(q_k)^{-1/4}\big[(1+O(h^2))\cos\phi_k^Y+O(h^2)\big],\qquad \phi_k^Y-\phi_{k-1}^Y=p(q_k-\tfrac{h}{2})+O(h^2), \tag{283}$$

with $\phi_k^Y=NS(q_k)+h\widetilde{\Gamma}(q_k)+p_{ab}\pi/2-\pi/4+O(h^3)$, $\widetilde{\Gamma}:=\Gamma_2+\mathfrak{c}_{ab}(\Theta_q)/q$, where the amplitude is even in $h$ by Lemma 3.23. On the other side, $X=h^{-2/3}\zeta(q_0)\to-\infty$, and the large-argument expansions $\mathrm{Ai}(-y)=\pi^{-1/2}y^{-1/4}[\cos(\xi-\frac{\pi}{4})(1+O(\xi^{-2}))+\frac{5}{72\xi}\sin(\xi-\frac{\pi}{4})+O(\xi^{-3})]$, $\mathrm{Ai}'(-y)=$

$\pi^{-1/2}y^{1/4}[\sin(\xi-\frac{\pi}{4})+O(\xi^{-1})]$, $\xi=\frac{2}{3}y^{3/2}$ ([50, 9.7.9–9.7.10]), together with Proposition E.6, give

$$\begin{aligned} f_{h,k} &= \sqrt{\tfrac{2}{\pi}}\,h^{1/6}D(q_k)^{-1/4}\big[(1+O(h^2))\cos\phi_k^f+O(h^2)\big],\\ \phi_k^f &= N\mathcal{A}(q_k)-\frac{\pi}{4}-h\Big[\frac{5}{72\mathcal{A}}+\sqrt{-\zeta}\,\frac{q_1}{g}\Big](q_k)+O(h^2), \end{aligned} \tag{284}$$

with $\mathcal{A}=\frac{2}{3}(-\zeta)^{3/2}$; the $h^{4/3}q_1\,\mathrm{Ai}'$ term is of size $h$ and in quadrature, so it shifts the phase and changes the amplitude only at order $h^2$. With the alternating gauge restored, the increments of $\phi^f$ are $-\tau p(q_k-\frac{h}{2})+O(h^2)$, since $N\mathcal{A}$ decreases by $p$ per rank for $\tau=+1$ and by $\pi-p$ for $\tau=-1$. For a phase sequence with increments $p_k+O(h^2)$ the vector $v_k=(\cos\phi_k,\cos\phi_{k-1})$ satisfies $\mathcal{Q}_k(v_k)=1+O(h^2)$, where $\mathcal{Q}_k(v)=(v_1^2+v_2^2-2v_1v_2\cos p_k)/\sin^2 p_k$. Applying this to the exact identity $\widehat{Y}=\kappa_h f_h+\mu_h f_2$ at two consecutive ranks, after dividing each row $j$ by the amplitude factor $D(q_j,x)^{-1/4}$ common to $\widehat{Y}_j$ and $f_{h,j}$ (these differ by $1+O(h)$ between the rows, so they must be removed row by row), gives $|\kappa_h|\,h^{1/6}/\sqrt{2}=1+O(h^2)$, and then, since the map $\phi\mapsto(\cos\phi,\cos(\phi-p))$ is an embedding of the circle with derivative bounded below, $\phi_k^Y\equiv-\tau\phi_k^f \pmod\pi$ up to $O(h^2)$ (the pair $(\cos\phi_k^f,\cos\phi_{k-1}^f)$ is $(\cos\psi,\cos(\psi-p_k))$ with $\psi=-\tau\phi_k^f$). The sign of $\kappa_h$ is that of Theorem 3.27.

*The bracket.* By the scalar step, $\kappa_h=\pm\sqrt{2}N^{1/6}(1+c_h)$ with $c_h$ real and $|c_h|\le Ch^2$, and by Proposition E.6, $f_h=\psi^{[2]}+O(h^3)\mathsf{M}/\mathsf{E}$ on $I$. Since $g\ge c$ on $I$, dividing $\widehat{Y}=\kappa_h f_h+\mu_h f_2$ by $\pm\sqrt{2}N^{1/6}g$ gives on $I'$

$$(1+c_h)\Big[\Big(1+h^2\frac{p_2}{g}\Big)\,\mathrm{Ai}(X)+h^{4/3}\frac{q_1}{g}\,\mathrm{Ai}'(X)\Big]+O(h^3)\frac{\mathsf{M}}{\mathsf{E}}(X). \tag{285}$$

Here $c_h h^{4/3}(q_1/g)\,\mathrm{Ai}'=O(h^{10/3})\mathsf{N}/\mathsf{E}=O(h^3)\mathsf{M}/\mathsf{E}$, because $|\,\mathrm{Ai}'\,|\le\mathsf{N}/\mathsf{E}$, $\mathsf{N}\le C(1+|X|)^{1/2}\mathsf{M}$ and $|X|\le Ch^{-2/3}$ on $I$; likewise $c_h h^2(p_2/g)\,\mathrm{Ai}=O(h^4)\mathsf{M}/\mathsf{E}$. With $\mathfrak{a}_N=h^2p_2/g+c_h$, $B_\tau=q_1/g$, $\sqrt{2}g=2|\zeta/D|^{1/4}$ and $\sqrt{2L+1}=\sqrt{2Nq}$ this is (80). For the shifted form put $s=h^{4/3}B_\tau$: $\mathrm{Ai}(X+s)=\mathrm{Ai}+s\,\mathrm{Ai}'+\frac{1}{2}s^2X\,\mathrm{Ai}+\frac{1}{6}s^3\,\mathrm{Ai}'''(\tilde{X})$, where $\frac{1}{2}s^2X=\frac{1}{2}h^2\zeta B_\tau^2=O(h^2)$ joins the amplitude, and $\mathrm{Ai}'''=\mathrm{Ai}+X\,\mathrm{Ai}'$ bounds the last term by $Ch^4(1+|X|)^{3/2}\mathsf{M}/\mathsf{E}\le Ch^3\mathsf{M}/\mathsf{E}$ (on the forbidden side $(\mathsf{M}/\mathsf{E})(\tilde{X})\le e^{C|s|X^{1/2}}(\mathsf{M}/\mathsf{E})(X)$ and $|s|X^{1/2}\le Ch$). On $|t|\le A_0$, $\mathsf{M}/\mathsf{E}\le C$ and $q-q_*=O(h^{2/3})$, so $B_\tau(q)=\varpi_\tau+O(h^{2/3})$ with $\varpi_\tau:=B_\tau(q_*)$, which gives (81).

*The coefficient.* On the oscillatory side (265) and (268) give $NS+p_{ab}\pi/2-\pi/4\equiv-\tau(N\mathcal{A}-\pi/4)$ $(\mathrm{mod}\ \pi)$ for both turn types, so $\phi^Y\equiv-\tau(N\mathcal{A}-\pi/4-\tau h\widetilde{\Gamma})\ (\mathrm{mod}\ \pi)$. Comparing with $-\tau\phi^f$ at every $q_0\in(q_*-\delta_0,q_*)$, the coefficients of $h$ must agree: $\tau\widetilde{\Gamma}=5/(72\mathcal{A})+\sqrt{-\zeta}\,q_1/g$. This is (77), since $5/(72\mathcal{A}\sqrt{-\zeta})=5/(48\zeta^2)$; as $q_1/g$ is analytic on $I$, $B_\tau$ is the analytic continuation of its right side, and (78) follows.

For (79), evaluate the regular equation $2Wq_1'+W'q_1=-[h^2]\mathfrak{A}[g,0]$ of Theorem E.4 at $q_*$, where $W(q_*)=0$ and $W'(q_*)=\zeta'(q_*)V(q_*)=\frac{1}{2}e_0\zeta'(q_*)^2\neq0$: so $q_1(q_*)=-[h^2]\mathfrak{A}[g,0](q_*)/W'(q_*)$ and $\varpi_\tau=q_1(q_*)/g(q_*)$, with $g(q_*)=(2/(e_0\zeta'(q_*)))^{1/2}$. At $q_*$ one has $\zeta=0$, so $C(s;0,h)=1+\frac{1}{6}hs^3+\frac{1}{180}h^2s^6+O(h^3)$ and $s_\pm=\zeta(q_*\pm h)/h$; the eikonal $\cosh(\zeta^{1/2}\zeta')=\tau u$ determines the jets of $\zeta$ at $q_*$ from those of $\tau u$ (in particular $\zeta'(q_*)=(2u_1)^{1/3}$), and $V=\frac{1}{2}e_\pi S_0(\zeta')$ gives those of $g$. Expanding $\mathfrak{A}[g,0](q_*)=\sum_\pm a(q_*\pm\frac{h}{2};h^2)g(q_*\pm h)C(s_\pm)-\tau(x-b(q_*;h^2))g(q_*)$ with $a(s;h^2)=\frac{1}{2}e_\pi(s^2)(1+h^2((\frac{1}{8}-\frac{a^2+b^2}{2})/s^2+\eta_\pi(s^2)))+O(h^4)$ and $\tau(x-b(q_*;h^2))=e_0-\tau h^2(ab/q_*^2+v_1)+O(h^4)$, the coefficients of $h^0$ and $h^1$ vanish and the coefficient of $h^2$ yields (79), an identity checked by exact computer algebra. The Olver form is Lemma E.7(ii). □

## E.2 Flux of the Airy-form branch and the amplitude

Work in the setting of Lemma E.3 and Theorem E.4. Put $\chi = \mathrm{Ai} + i\,\mathrm{Bi}$, $X = h^{-2/3}\zeta(q)$, $P = g + h^2 p_2$, $Q = hq_1 = hgB_\tau$ (real, analytic), $\psi^{\mathrm{c}} := P\chi(X) + h^{1/3}Q\chi'(X)$, so $\Re\psi^{\mathrm{c}} = \psi^{[2]}$. With $\hat{s}_\epsilon = (\zeta(q+\epsilon\frac{h}{2}) - \zeta(q))/h$, $\hat{P}_\epsilon = P(q+\epsilon\frac{h}{2})$, $\hat{Q}_\epsilon = Q(q+\epsilon\frac{h}{2})$ ($\epsilon = \pm$) and $C, S$ as in Lemma E.3 (parameter $\zeta(q)$), put $\mathcal{A}_\epsilon = \hat{P}_\epsilon C(\hat{s}_\epsilon) + \hat{Q}_\epsilon C'(\hat{s}_\epsilon)$, $\mathcal{B}_\epsilon = \hat{P}_\epsilon S(\hat{s}_\epsilon) + \hat{Q}_\epsilon S'(\hat{s}_\epsilon)$ and $\mathfrak{F}(q;h) := a(q;h^2)(\mathcal{A}_-\mathcal{B}_+ - \mathcal{B}_-\mathcal{A}_+)$.

**Lemma E.8** (Flux of the Airy-form branch)**.** *Let* $J^{\mathrm{c}}(s) := \pi a(s;h^2)\Im(\overline{\psi^{\mathrm{c}}(s-\frac{h}{2})}\psi^{\mathrm{c}}(s+\frac{h}{2}))$.

*(i)* $J^{\mathrm{c}} = h^{1/3}\mathfrak{F}$ *and* $\pi\Im(\overline{\psi^{\mathrm{c}}}L\psi^{\mathrm{c}})(q) = J^{\mathrm{c}}(q+\frac{h}{2}) - J^{\mathrm{c}}(q-\frac{h}{2})$.

*(ii)* $\mathfrak{F}$ *is even in* $h$, $\mathfrak{F} = 1 + h^2(2p_2/g + \mathfrak{j}(q)) + O(h^4)$ *uniformly on* $I$, *with* $\mathfrak{j} := [h^2]\mathfrak{F}|_{p_2=0}$ *real-analytic* ($p_2$*-independent).*

*(iii) For* $p_2$ *of Theorem E.4,* $2p_2/g + \mathfrak{j}$ *is constant on* $I$.

*(iv) With* $\zeta^{(k)}$ *the derivatives of* $\zeta_\tau$ *at* $q_*$, $\varpi_\tau = B_\tau(q_*)$, $B_{\tau,\zeta}(0) = \psi'_\tau(\zeta{=}0)/3$ *(by (281)) and* $a_0 = \frac{1}{2}e_\pi(q^2)$,

$$\mathfrak{j}(q_*) = B_{\tau,\zeta}(0) + \frac{\zeta'^2}{6}\varpi_\tau + \frac{13\,\zeta'^6}{10080} - \frac{\zeta'^2\zeta''}{12} - \frac{\zeta'''}{12\zeta'} + \frac{\zeta''^2}{8\zeta'^2} + \frac{\frac{1}{8} - \frac{a^2+b^2}{2}}{q_*^2} + \eta_\pi(q_*^2) - \frac{1}{8}\big(\log e_\pi(q^2)\big)''\Big|_{q_*}. \tag{286}$$

*(v) For the Hahn chain with* $x \neq 0$, $\mathfrak{j}(q_*)$ *is the rational function (85).*

*Proof.* (i) By Lemma E.3(i) at the links $q \pm \frac{h}{2}$, $\psi^{\mathrm{c}}(q+\epsilon\frac{h}{2}) = \chi\mathcal{A}_\epsilon + h^{1/3}\chi'\mathcal{B}_\epsilon$ with real $\mathcal{A}_\epsilon, \mathcal{B}_\epsilon$, so $\Im(\overline{\psi^{\mathrm{c}}(q-\frac{h}{2})}\psi^{\mathrm{c}}(q+\frac{h}{2})) = h^{1/3}(\mathcal{A}_-\mathcal{B}_+ - \mathcal{B}_-\mathcal{A}_+)\Im(\bar{\chi}\chi')$, $\Im(\bar{\chi}\chi') = \mathrm{Ai}\,\mathrm{Bi}' - \mathrm{Bi}\,\mathrm{Ai}' = \frac{1}{\pi}$. As $L$ has real coefficients, $\Im(\bar{\psi}L\psi)(q) = a(q+\frac{h}{2})\Im(\bar{\psi}(q)\psi(q+h)) - a(q-\frac{h}{2})\Im(\bar{\psi}(q-h)\psi(q))$.

(ii) By Lemma E.3(ii), $\mathcal{A}_\epsilon(-h) = \mathcal{A}_{-\epsilon}(h)$, $\mathcal{B}_\epsilon(-h) = -\mathcal{B}_{-\epsilon}(h)$ ($\hat{Q}_\epsilon$ is odd in $h$), so $\mathfrak{F}$ is even. At $h = 0$, $\mathcal{A}_\pm = g\cosh(\zeta'^{1/2}\zeta'/2)$, $\mathcal{B}_\pm = \pm gS_0(\zeta'/2)$, so $\mathfrak{F}|_{h=0} = a_0 g^2 S_0(\zeta') = Vg^2 = 1$. $p_2$ enters through $\hat{P}_\pm$ at order $h^2$, linearly in each of $\hat{P}_\pm$, giving $2(p_2/g)$ times the leading flux.

(iii) $L\psi^{\mathrm{c}} = \chi\mathfrak{A} + h^{1/3}\chi'\mathfrak{B}$ with the real $\mathfrak{A} = O(h^4)$, $\mathfrak{B} = O(h^5)$ of Theorem E.4, so $\pi\Im(\overline{\psi^{\mathrm{c}}}L\psi^{\mathrm{c}}) = h^{1/3}(P\mathfrak{B} - Q\mathfrak{A}) = O(h^{16/3})$. By (i)–(ii) it is $h^{10/3}\partial_q(2p_2/g + \mathfrak{j}) + O(h^{16/3})$, so $\partial_q(2p_2/g + \mathfrak{j}) = 0$.

(iv) At $q_*$, $\zeta = 0$, $C(s;0,h) = 1 + \frac{h}{6}s^3 + \frac{h^2}{180}s^6 + O(h^3)$, $S(s;0,h) = s + \frac{h}{12}s^4 + \frac{h^2}{504}s^7 + O(h^3)$, and $g$ has the jets of $(a_0 S_0(\zeta'))^{-1/2}$. Expanding $\mathfrak{F}(q_*;h)$ in the generic jets $(\zeta', \zeta'', \zeta''', a_0, a_0', a_0'', a_2, B_\tau, B'_\tau)$ is a finite polynomial computation, done by exact computer algebra: $1 + 0\cdot h + h^2(\cdots)$ with $(\cdots)$ the right side of (286) ($B'_\tau(q_*) = \zeta' B_{\tau,\zeta}(0)$, $a_2/a_0 = (\frac{1}{8} - \frac{a^2+b^2}{2})/q_*^2 + \eta_\pi(q_*^2)$).

(v) By (281), $B_\tau(q_*) = \psi_\tau(q_*)$, $B'_\tau(q_*) = \frac{1}{3}\psi'_\tau(q_*)$, $\psi_\tau = -g[h^2]\mathfrak{A}[g,0]/\zeta'$. The truncated-series computation of (iv), one order further in $q - q_*$, gives $[h^2]\mathfrak{A}[g,0](q_* + \delta)$ to first order in $\delta$ in the jets of $\zeta$ (to order 4), $a_0$ (order 3), $a_2$, $\tau b_2$ (order 1); at $\delta = 0$ it reproduces (79). For Hahn, $e_\pi = \sqrt{1-q^2}$, $\tau u = |x|(1-q^2)^{-1/2}$, $a_2 = a_0(\frac{1}{8} - \frac{M^2+\sigma^2}{2})/q^2$, $b_2 = M\sigma/q^2$, and with $x = 2w/(1+w^2)$, $q_* = (1-w^2)/(1+w^2)$ all jets (those of $\zeta$ from $\cosh(\zeta'^{1/2}\zeta') = \tau u$, $\zeta'^3 = 2u_1$) are rational in $w$; (286) reduces modulo $\zeta'^3 = 2u_1$ to a cube-root-free rational function of $w$ equal to (85), ($\tau = -1$: via $\tau M\sigma$, $|x|$). □

*Proof of Proposition 3.28.* Fix an oscillatory $q_0 \in (q_* - \delta_0, q_*)$. As in the proof of Theorem 3.27(ii), $\widehat{Y} = \kappa_h f_h + O(N^{-\infty})\mathsf{M}/\mathsf{E}$, $\kappa_h = \pm\sqrt{2}N^{1/6}(1 + c_h)$, $f_h = \psi^{[2]} + O(h^3)\mathsf{M}/\mathsf{E}$; so near $q_0$, $\widehat{Y} = 2\Re u_A + O(h^3)$ with $u_A = \frac{1}{2}\kappa_h\psi^{\mathrm{c}}$ ($\tau = +1$), $\frac{1}{2}\kappa_h(-1)^{k-k_{\min}}\overline{\psi^{\mathrm{c}}}$ ($\tau = -1$, alternating gauge restored), and also $\widehat{Y} = 2\Re u_D + O(h^3)$, $u_D = (1 + h^2\alpha)\widehat{U}^{[2],+}$ (Theorem 3.24). Both are + carriers: $\chi(-y) \sim \pi^{-1/2}y^{-1/4}e^{-i(\xi - \pi/4)}$ has phase increasing with $q$ ($\xi = N\mathcal{A}(q)$ decreases), and for $\tau = -1$ the factor $(-1)^k(\cdot)$ makes it increasing in the physical frame. Put $\eta_k := u_A(q_k)/u_D(q_k) - 1$ at the lattice

points of a compact interval $J_0 \subset (q_* - \delta_0, q_*)$ around $q_0$. Then $\Re(u_D\eta) = O(h^3)$ at every lattice point of $J_0$, but $\eta_k$ is a priori only a sequence: two consecutive real equations $\Re(u_{D,k}\eta_k) = O(h^3)$, $\Re(u_{D,k+1}\eta_{k+1}) = O(h^3)$ involve two different complex numbers, and smoothness of $\eta$ does not bound $\eta_{k+1} - \eta_k$ (the sequence could alternate in sign). What is needed, and proved next, is the slow variation

$$\eta_{k+1} - \eta_k = O(h^3) \qquad (q_k, q_{k+1} \in J_0). \tag{287}$$

*Proof of (287): carrier-amplitude expansions of both representations.* All functions of $q$ below are smooth on $J_0$ and independent of $h$.

(D) By Lemma 3.23 and the proof of Lemma 3.25(ii), with $\kappa\Theta_q^{[2]} = NS + h\Gamma_2$, $\mathfrak{P}_{ab} = h\,\mathfrak{c}_{ab}(\Theta_q)/q + O(h^3)$ and $\mathfrak{A}_{ab} = 1 + h^2\mathfrak{A}_2(\Theta_q)/q^2 + O(h^4)$,

$$u_D(q_k) = e^{iNS(q_k)}\beta(q_k)\big[1 + ih\widetilde{\Gamma}(q_k) + h^2 d_2(q_k)\big] + O(h^3), \qquad \beta := \omega_{ab}\pi^{-1/2}D^{-1/4}, \tag{288}$$

with $\widetilde{\Gamma} = \Gamma_2 + \mathfrak{c}_{ab}(\Theta_q)/q$ and $d_2 = \alpha + \mathfrak{A}_2/q^2 - \frac{1}{2}\widetilde{\Gamma}^2$ (expand the exponential; $1 + h^2\alpha$ is real). There is no order-$h$ amplitude term because $\mathfrak{A}_{ab}$ is even.

(A) At $q \in J_0$ the Airy argument is $X = -y$, $y = h^{-2/3}(-\zeta(q)) \gg 1$, and $\xi = \frac{2}{3}y^{3/2} = N\mathcal{A}(q)$, $\xi^{-1} = h/\mathcal{A}$. The large-argument expansions of Ai and Bi (DLMF 9.7.9 and 9.7.11) combine to $\chi(-y) = \mathrm{Ai}(-y) + i\,\mathrm{Bi}(-y) = \pi^{-1/2}y^{-1/4}e^{-i(\xi-\pi/4)}\big[1 + iu_1\xi^{-1} - u_2\xi^{-2} + O(\xi^{-3})\big]$, $u_1 = \frac{5}{72}$, and $\chi'(-y) = i\pi^{-1/2}y^{1/4}e^{-i(\xi-\pi/4)}[1 + O(\xi^{-1})]$, so in $\psi^{\mathrm{c}} = (g + h^2 p_2)\chi(X) + h^{1/3}hq_1\chi'(X)$ the $\chi'$ term has relative size $h^{4/3}y^{1/2} = h\sqrt{-\zeta}$ and is imaginary at leading order. Using $\frac{1}{2}\kappa_h\, g\, y^{-1/4} = \pm(1 + c_h)D^{-1/4}$ (from $\kappa_h = \pm\sqrt{2}N^{1/6}(1 + c_h)$ and $g = \sqrt{2}|\zeta/D|^{1/4}$),

$$\tfrac{1}{2}\kappa_h\psi^{\mathrm{c}}(q_k) = \pm(1 + c_h)\, e^{i\pi/4}e^{-iN\mathcal{A}(q_k)}\pi^{-1/2}D^{-1/4}\big[1 + ihA_1 + h^2A_2\big](q_k) + O(h^3), \tag{289}$$

with $A_1 = u_1/\mathcal{A} + \sqrt{-\zeta}\,B_\tau$ real and $A_2$ smooth. There is no order-$h$ amplitude term here either. For $\tau = +1$ this is $u_A$. For $\tau = -1$, $u_A = (-1)^{k-k_{\min}}\overline{\frac{1}{2}\kappa_h\psi^{\mathrm{c}}}$ has the bracket $1 - ihA_1 + h^2\overline{A_2}$. By the exact phase identities (265), (268), and the rank-independence of $\sigma_\tau = (-1)^{K_\tau(L)}/\chi_\tau(L)$, the product of the carrier of $u_A$ and the lattice factor is a constant times $e^{iNS(q_k)}$: $e^{-iN\mathcal{A}(q_k)} = e^{-iNS(q_*)}e^{iNS(q_k)}$ for $\tau = +1$ ($\mathcal{A} = S(q_*) - S$), and $(-1)^{k-k_{\min}}e^{iN\mathcal{A}(q_k)} = \mathrm{const}\cdot e^{iNS(q_k)}$ for $\tau = -1$. Hence

$$u_A(q_k) = \Lambda_N\, e^{iNS(q_k)}\pi^{-1/2}D^{-1/4}(q_k)\big[1 + ih\gamma_1 + h^2\gamma_2\big](q_k) + O(h^3), \tag{290}$$

with a constant $\Lambda_N$ independent of $k$ ($|\Lambda_N| \le C$), $\gamma_1 = \tau A_1$ and $\gamma_2$ smooth. Finally $\tau\widetilde{\Gamma} = A_1$ is exactly the order-$h$ phase matching (77) (note $5/(48\zeta^2)\sqrt{-\zeta} = u_1/\mathcal{A}$), so $\gamma_1 = \widetilde{\Gamma}$.

Dividing (A) by (D), the carriers $e^{iNS}$ and the amplitudes $D^{-1/4}$ cancel, and the order-$h$ terms cancel because $\gamma_1 = \widetilde{\Gamma}$:

$$1 + \eta_k = \Lambda'_N\big[1 + h^2\rho(q_k)\big] + O(h^3), \qquad \rho := \gamma_2 - d_2, \tag{291}$$

with $\Lambda'_N = \Lambda_N/\omega_{ab}$ independent of $k$ and bounded. Since $\rho$ is Lipschitz on $J_0$ and $q_{k+1} - q_k = h$, $\eta_{k+1} - \eta_k = \Lambda'_N h^2(\rho(q_{k+1}) - \rho(q_k)) + O(h^3) = O(h^3)$, which is (287). $\square$

*Conclusion.* By (287), $\Re(u_{D,k}\eta_k) = O(h^3)$ and $\Re(u_{D,k+1}\eta_k) = \Re(u_{D,k+1}\eta_{k+1}) + O(h^3) = O(h^3)$: the real vector $(\Re\eta_k, \Im\eta_k)$ is annihilated, up to $O(h^3)$, by the matrix with rows $(\Re u_{D,k}, -\Im u_{D,k})$ and $(\Re u_{D,k+1}, -\Im u_{D,k+1})$, whose determinant is $\pm\Im(\overline{u_{D,k}}u_{D,k+1}) = \pm(1 + h^2\alpha_k)(1 + h^2\alpha_{k+1})\,\mathsf{J}(q_{k+1/2})/(\pi a(q_{k+1/2})) = \pm(1 + O(h^2))/(2\pi a)$ by Lemma 3.25(ii), bounded away from zero; here and below $q_{k+1/2} := q_k + \frac{h}{2}$ is the link midpoint and $\alpha_j := \alpha(q_j)$. As $|u_D| \le C$ on $J_0$, this gives $\eta_k = O(h^3)$, and the fluxes of $u_A$ and $u_D$ agree to relative $O(h^3)$.

The flux of $u_D$ across the link $(q_k, q_{k+1})$ is $(1 + h^2\alpha_k)(1 + h^2\alpha_{k+1})\mathsf{J}(q_{k+1/2})$. Since $\alpha$ is smooth, $(1 + h^2\alpha_k)(1 + h^2\alpha_{k+1}) = (1 + h^2\alpha(q_{k+1/2}))^2 + O(h^4)$, so with Lemma 3.25(ii) the flux is $(1 +$

$h^2\alpha)^2(\frac12+h^2\mathsf{J}_2)\big|_{q_{k+1/2}}+O(h^4)=\frac12+h^2\delta_{\mathrm{fl}}+O(h^4)$, all evaluated at the midpoint, where $\alpha+\mathsf{J}_2(q)=c+\mathsf{J}_2(0,x)$ is constant; that of $u_A$ is $\frac14\kappa_h^2h^{1/3}\mathfrak{F}=\frac12(1+c_h)^2\mathfrak{F}$ (Lemma E.8(i); for $\tau=-1$, $\Im(\overline{(-1)^k\bar\psi_k}(-1)^{k+1}\bar\psi_{k+1})=\Im(\bar\psi_k\psi_{k+1})$). Equating, with Lemma E.8(ii)–(iii) and $2p_2/g+\mathfrak{j}\equiv0$,

$$c_h=h^2\delta_{\mathrm{fl}}+O(h^3),\qquad \mathfrak{a}_N=h^2p_2/g+c_h=h^2\big(\delta_{\mathrm{fl}}-\tfrac12\mathfrak{j}\big)+O(h^3),\tag{292}$$

with the $O(h^3)$ absorbed in $\mathfrak{r}_N$; $\delta_{\mathrm{fl}}=0$ for Hahn and Racah since $c=-\mathsf{J}_2(0,x)$, and $\delta_{\mathrm{fl}}=-\frac{1}{12}$ for the quantum $6j$ chain (Corollary 6.3). □

### E.3 Hyperbolicity and the physical solution on the forbidden side

Proofs of Propositions 3.31 and 3.34. Write $\sigma(q)=q^2$ for Hahn and Racah and $\sigma(q)=\tanh^2q$ for one-small quantum $6j$, with $\sigma_k=\sigma(kh)$, $\bar\sigma_k=\frac12(\sigma_k+\sigma_{k+1})$, $q_k=(k+\frac12)h$, and offsets $(a,b)$. By Propositions 4.1, 5.1 and 6.1, with $\mathrm{s}_z=\sinh(zh)$ and $\mathrm{c}_z=\cosh(zh)$,

$$A_k=\tfrac12r_k\widetilde e(\sigma_k),\qquad \widetilde e(\sigma)=\sqrt{(1-\sigma/\sigma_1)(1-\sigma/\sigma_2)},\tag{293}$$

where $(\beta_0,\sigma_1,\sigma_2^{-1})=(0,1,0)$ for Hahn, $(\frac{1}{4\alpha\beta},4\alpha^2,(4\beta^2)^{-1})$ for Racah, and $(\coth\mathsf{X}\coth\mathsf{Y},\tanh^2\mathsf{X},\coth^2\mathsf{Y})$ for quantum $6j$, so $\beta_0=(\sigma_1\sigma_2)^{-1/2}$ throughout (the $q$-Hahn limit $\coth\mathsf{Y}=1$, $1/\sinh\mathsf{Y}=0$ of Remark 3.32 is included: $\sigma_2=1$, and all constants below stay finite); here

$$r_k^2=\frac{(1-a^2/k^2)(1-b^2/k^2)}{1-1/(4k^2)}\ \text{(Hahn, Racah)},\qquad r_k^2=\frac{(1-\mathrm{s}_a^2/\mathrm{s}_k^2)(1-\mathrm{s}_b^2/\mathrm{s}_k^2)}{1-\mathrm{s}_1^2/\mathrm{s}_{2k}^2}\ \text{(quantum }6j\text{)},\tag{294}$$

using $\mathrm{s}_{2k}^2\mathrm{s}_{2k+1}\mathrm{s}_{2k-1}=16\mathrm{s}_k^4\mathrm{c}_k^4(1-\mathrm{s}_1^2/\mathrm{s}_{2k}^2)$ and $(1-\mathrm{s}_k^2/\sinh^2\mathsf{X})/\mathrm{c}_k^2=1-\sigma_k/\sigma_1$, while

$$B_k=\beta_0\bar\sigma_k-\frac{\beta_0h^2}{2}(a^2+b^2)+\frac{ab}{k(k+1)}\ \text{(Hahn, Racah)},\quad B_k=\beta_0\Big(1-\frac{\mathrm{c}_a\mathrm{c}_b}{\mathrm{c}_k\mathrm{c}_{k+1}}\Big)+\frac{\mathrm{s}_a\mathrm{s}_b}{\mathrm{s}_k\mathrm{s}_{k+1}}\ \text{(quantum }6j\text{)}.\tag{295}$$

Put $\sigma_{\max}=\min(\sigma_1,\sigma_2)$ (1 for Hahn) and $q_m=\sigma^{-1}(\sigma_{\max})$ ($\sqrt{\sigma_{\max}}$ for Hahn and Racah, $\min(\mathsf{X},\mathsf{Y})$ for quantum $6j$); the hard edge is $\sigma_{k_{\max}+1}=\sigma_{\max}$, $\widetilde e(\sigma_{\max})=0$. Put $\sigma_*=\sigma(q_*)$, $m_*=(1-\tau x)/\sigma_*>0$, and $\Phi(\sigma)=\tau(x-\beta_0\sigma)-\widetilde e(\sigma)$, which is $\tau(x-\Psi_\tau(\sigma))$ for $\Psi_\tau=\beta_0\sigma+\tau\widetilde e$; a simple turn means $\Phi(\sigma_*)=0$.

(F1) $\widetilde e$ is concave and decreasing, $0\le\widetilde e\le1$ on $[0,\sigma_{\max}]$: $\widetilde e''=-(\sigma_1^{-1}-\sigma_2^{-1})^2/(4\widetilde e^3)\le0$. (F2) $\Phi$ is convex on $[0,\sigma_{\max})$ for either sign of $\tau$, being an affine function minus the concave $\widetilde e$. Since $\widetilde e(0)=1$, $\Phi(0)=\tau x-1$, so the secant slope of $\Phi$ on $[0,\sigma_*]$ is $-\Phi(0)/\sigma_*=m_*$. The secant slopes of a convex function are nondecreasing, so beyond $\sigma_*$ both the difference quotient and the derivative are at least $m_*$:

$$\Phi'(\sigma)\ge m_*,\qquad \Phi(\sigma)\ge m_*(\sigma-\sigma_*)\qquad(\sigma_*\le\sigma<\sigma_{\max}).\tag{296}$$

(F3) Set $N_0:=\max(2\max(|a|,|b|,1)/q_*,\max(|a|,|b|,1))$. Since $\sinh u/u$ increases on $(0,\infty)$, for $k\ge q_*N/2$ and $N\ge N_0$ all three chains satisfy $1-\frac{a^2+b^2}{k^2}\le r_k^2\le1+\frac{1}{3k^2}$, so $|r_k-1|\le\rho_1h^2$ with $\rho_1:=4(a^2+b^2+\frac13)/q_*^2$. For Hahn and Racah $\bar\sigma_k=q_k^2+\frac{h^2}{4}$ gives $|B_k-\beta_0q_k^2|\le\rho_2h^2$ with $\rho_2:=\beta_0(1+2(a^2+b^2))/4+4|ab|/q_*^2$; for quantum $6j$, since $1-(\mathrm{c}_k\mathrm{c}_{k+1})^{-1}-\bar\sigma_k=\frac12(\mathrm{c}_k^{-1}-\mathrm{c}_{k+1}^{-1})^2\in[0,\frac18h^2]$, $0\le\mathrm{c}_a\mathrm{c}_b-1\le(a^2+b^2)h^2$ ($|a|h,|b|h\le1$), and $|\mathrm{s}_a\mathrm{s}_b/(\mathrm{s}_k\mathrm{s}_{k+1})|\le|ab|/(k(k+1))$, one has $|B_k-\beta_0\bar\sigma_k|\le\rho_2h^2$ with $\rho_2:=\beta_0(\frac18+a^2+b^2)+4|ab|/q_*^2$. (F4) For $\sigma>\sigma_*$, $\tilde u-1=\Phi/\widetilde e$ with $\tilde u(\sigma):=\tau(x-\beta_0\sigma)/\widetilde e(\sigma)$ is positive and increasing, so $\Upsilon(q)=\operatorname{arcosh}\tilde u(\sigma(q))$ is positive and increasing on $(q_*,q_m)$, while $\Upsilon(q):=0$ for $0\le q\le q_*$.

*Proof of Proposition 3.31.* Let $q_k \ge q_*$, $c_k = \tau(x - B_k)$. By (F3) and concavity of $\widetilde{e}$, $A_k + A_{k+1} \le \frac{1}{2}(1+\rho_1 h^2)(\widetilde{e}(\sigma_k) + \widetilde{e}(\sigma_{k+1})) \le (1+\rho_1 h^2)\,\widetilde{e}(\bar{\sigma}_k) \le \widetilde{e}(\bar{\sigma}_k) + \rho_1 h^2$ ($k_{\max}$ included): the middle step is concavity of $\widetilde{e}$ and the last is $\widetilde{e} \le 1$ from (F1). For Hahn and Racah $\widetilde{e}(\bar{\sigma}_k) \le \widetilde{e}(q_k^2)$ and $q_k^2 - q_*^2 \ge 2q_*(q_k - q_*)$, so (296) gives

$$c_k - A_k - A_{k+1} \ge \Phi(q_k^2) - (\rho_1+\rho_2)h^2 \ge 2m_* q_*(q_k - q_*) - (\rho_1+\rho_2)h^2 \ge 0 \tag{297}$$

for $q_k - q_* \ge C_0 h^2$, $C_0 := (\rho_1+\rho_2)/(2m_*q_*)$. For quantum $6j$, $\sigma'(q) = 2\tanh q\,\mathrm{sech}^2 q$ is unimodal on $[0,\infty)$ with $\sigma''(q) = 2\,\mathrm{sech}^2 q(1 - 3\tanh^2 q) \ge -\frac{2}{3}$, so $c_\sigma := \min_{[q_*,q_m]}\sigma' = \min(\sigma'(q_*), \sigma'(q_m)) > 0$ and $\bar{\sigma}_k \ge \sigma(q_k) - \frac{1}{12}h^2 \ge \sigma_* + c_\sigma(q_k - q_*) - \frac{1}{12}h^2$; hence $c_k - A_k - A_{k+1} \ge \Phi(\bar{\sigma}_k) - (\rho_1+\rho_2)h^2 \ge 0$ for $q_k - q_* \ge C_0 h^2$ with $C_0 := (\frac{1}{12}m_* + \rho_1 + \rho_2)/(m_* c_\sigma)$. □

*Proof of Proposition 3.34.* Let $c_k = \tau(x - B_k)$, $k_c$ the first rank with $q_k \ge q_* + C_0 N^{-2}$, $k_1$ the first rank with $q_k \ge q_* + N^{-2/3}$ (so that $N^{2/3}\zeta_\tau(q_{k_1}) = O(1)$, as in Remark 3.29(b)), and $k_e$ the last rank in $I$. At a physical node the physical solution satisfies row $k_{\max}$ ($A_{k_{\max}+1} = 0$): if $\widetilde{Y}_{k_{\max}}$ were 0, then since $A_k > 0$ for $k_{\min} < k \le k_{\max}$, downward induction on $A_k\widetilde{Y}_{k-1} = c_k\widetilde{Y}_k - A_{k+1}\widetilde{Y}_{k+1}$ would force $\widetilde{Y}_k = 0$ down to $k_{\min}$ (and to $k = 0$ when $a = b = 0$ since $A_1 > 0$), contradicting (E2). Thus $\widetilde{Y}_{k_{\max}} \ne 0$.

*Comparison with constant ratios.* Let $y > 0$ and $j \ge k_c$, and suppose

$$F_i(y) := A_{i+1}y + A_i/y \le c_i \qquad (j \le i \le k_{\max}). \tag{298}$$

Then $\widetilde{Y}_i \ne 0$, of the sign of $\widetilde{Y}_{k_{\max}}$, for $j-1 \le i \le k_{\max}$, and $0 < \rho_i \le y$ for $j-1 \le i < k_{\max}$. Indeed, with $\rho_{k_{\max}} = 0$ (the convention $\widetilde{Y}_{k_{\max}+1} = 0$), suppose $0 \le \rho_i \le y$ for some $i \ge j$. Then (298) gives $c_i - A_{i+1}\rho_i \ge c_i - A_{i+1}y \ge A_i/y > 0$, so $\widetilde{Y}_{i-1} \ne 0$ has the sign of $\widetilde{Y}_i$ and

$$\rho_{i-1} = \frac{A_i}{c_i - A_{i+1}\rho_i} \in (0, y], \tag{299}$$

because $\rho \mapsto A_i/(c_i - A_{i+1}\rho)$ is increasing and $A_i/(c_i - A_{i+1}y) \le y$. Backward induction from $i = k_{\max}$ gives the claim.

*(a)* is the case $y = 1$: $F_i(1) = A_i + A_{i+1} \le c_i$ for $i \ge k_c$ is Proposition 3.31.

*(b)* The inequality (298) is verified for explicit $y$ as follows.

Let $y = e^{-\vartheta}$, $\cosh\vartheta \le \Lambda := \tilde{u}(\sigma(q_* + \delta_0))$. By (F3) and Jensen's inequality for $\widetilde{e}$ (weights $\lambda = 1/(1+y^2)$, $1-\lambda$), $F_i(y) \le (1+\rho_1 h^2)\cosh\vartheta\,\widetilde{e}(\tilde{\sigma}_i) + \rho_1\Lambda h^2$ (using $\widetilde{e} \le 1$ and $\cosh\vartheta \le \Lambda$) with $\tilde{\sigma}_i = \lambda\sigma_i + (1-\lambda)\sigma_{i+1} = \bar{\sigma}_i - \frac{1}{2}(\sigma_{i+1} - \sigma_i)\tanh\vartheta$. Put $\Phi_\vartheta(\sigma) := \tau(x - \beta_0\sigma) - \cosh\vartheta\,\widetilde{e}(\sigma)$; with $\sigma_\vartheta > \sigma_*$ defined by $\tilde{u}(\sigma_\vartheta) = \cosh\vartheta$ we have $\Phi_\vartheta(\sigma_\vartheta) = 0$, and $\Phi'_\vartheta = \Phi' - (\cosh\vartheta - 1)\widetilde{e}' \ge \Phi' \ge m_*$ on $[\sigma_\vartheta, \sigma_{\max})$ by (296), since $\cosh\vartheta \ge 1$ and $\widetilde{e}' \le 0$. For Hahn and Racah ($\sigma_{i+1} - \sigma_i = 2q_i h$), $\tilde{\sigma}_i = q_i^2 - hq_i\tanh\vartheta + \frac{h^2}{4} \ge q_i^2 - q_m h$ and $c_i \ge \tau(x - \beta_0\tilde{\sigma}_i) - (\beta_0 q_m + \frac{\beta_0}{4} + \rho_2)h$ ($h \le 1$ since $N \ge N_0$), so $c_i - F_i(y) \ge \Phi_\vartheta(\tilde{\sigma}_i) - D_3 h$ with $D_3 := \beta_0 q_m + \frac{\beta_0}{4} + \rho_2 + \rho_1\Lambda$; setting $D_0 := q_m + D_3/m_*$ and $\sigma_\vartheta = q_j^2 - D_0 h$ whenever $q_j^2 - D_0 h > q_*^2$ gives $\tilde{\sigma}_i \ge \sigma_\vartheta + (D_0 - q_m)h$ for $i \ge j$, hence $c_i - F_i(y) \ge 0$ and (298) for $y = \exp(-\Upsilon(\sqrt{q_j^2 - D_0 h}))$. For quantum $6j$, $0 \le \sigma' \le 1$ gives $\sigma_{i+1} - \sigma_i \le h$, $\tilde{\sigma}_i \ge \sigma_i \ge \sigma(q_i) - \frac{1}{2}h$ and $|\bar{\sigma}_i - \tilde{\sigma}_i| \le \frac{1}{2}h$, so $c_i - F_i(y) \ge \Phi_\vartheta(\tilde{\sigma}_i) - D_3 h$ with $D_3 := \frac{1}{2}\beta_0 + \rho_2 + \rho_1\Lambda$. Put $D_\sigma := \frac{1}{2} + D_3/m_*$, $c'_\sigma := \min(\sigma'(q_*), \sigma'(q_* + \delta_0)) > 0$, and $D_0 := \frac{2(q_*+\delta_0)}{c'_\sigma}D_\sigma$: whenever $q_j \in [q_*, q_* + \delta_0]$ has $\tilde{q}_j := \sqrt{q_j^2 - D_0 h} > q_*$, the mean value theorem on $[\tilde{q}_j, q_j]$ gives $\sigma(q_j) - \sigma(\tilde{q}_j) \ge c'_\sigma(q_j^2 - \tilde{q}_j^2)/(2(q_* + \delta_0)) = D_\sigma h$, so $\sigma_\vartheta := \sigma(q_j) - D_\sigma h \ge \sigma(\tilde{q}_j) > \sigma_*$ satisfies $\tilde{\sigma}_i \ge \sigma_\vartheta + (D_\sigma - \frac{1}{2})h$ for $i \ge j$, giving $c_i - F_i(y) \ge 0$ and $\rho_{j-1} \le \exp(-\operatorname{arcosh}\tilde{u}(\sigma_\vartheta)) \le \exp(-\Upsilon(\tilde{q}_j))$.

When $q_j^2 - D_0 h \le q_*^2$, $\Upsilon(\sqrt{(q_j^2 - D_0 h)_+}) = 0$ and $\rho_{j-1} \le 1$ by (a); thus in all three chains, for every $j \ge k_1$ ($j \ge k_c$ for large $N$) with $q_j \le q_* + \delta_0$,

$$\rho_{j-1} \le \exp(-\Upsilon(\sqrt{(q_j^2 - D_0 h)_+})). \tag{300}$$

*The product.* Put $m := \lceil D_0/(2q_*) \rceil$ and $\Upsilon_{\max} := \Upsilon(q_* + \delta_0)$. Since adjacent $q_j^2$ are spaced by at least $2q_* h$, for $j \ge k_1 + m + 1$ one has $q_j^2 - D_0 h \ge q_{j-m}^2 \ge q_{k_1+1}^2 > q_*^2$, so $\Upsilon(\sqrt{(q_j^2 - D_0 h)_+}) \ge \Upsilon(q_{j-m})$; for $k_1 < j \le \min(k, k_1 + m)$ use $\Upsilon(\sqrt{(q_j^2 - D_0 h)_+}) \ge 0 = \Upsilon(q_j) - \Upsilon(q_j) \ge \Upsilon(q_j) - \Upsilon_{\max}$. Since $\Upsilon$ is nondecreasing and $\xi' = N\zeta_\tau^{1/2}\zeta_\tau' = N\Upsilon$ on $[q_*, q_* + \delta_0]$, for any $k_1 \le k \le k_e$,

$$\begin{aligned} \sum_{j=k_1+1}^{k} \Upsilon(\sqrt{(q_j^2 - D_0 h)_+}) &\ge \sum_{i=k_1+1}^{k-m} \Upsilon(q_i) \ge \sum_{j=k_1+1}^{k} \Upsilon(q_j) - m\Upsilon_{\max} \\ &\ge N \int_{q_{k_1}}^{q_k} \Upsilon - m\Upsilon_{\max} = \xi(q_k) - \xi(q_{k_1}) - m\Upsilon_{\max}. \end{aligned} \tag{301}$$

Hence $|\widehat{Y}_k|/|\widehat{Y}_{k_1}| = \prod_{j=k_1+1}^{k} \rho_{j-1} \le e^{m\Upsilon_{\max}} e^{-(\xi(q_k) - \xi(q_{k_1}))}$, while $|\widehat{Y}_{k_1}| \le CN^{1/6}$ by Theorem 3.27(i) and $\xi(q_{k_1}) = O(1)$.

(c) Write $\widehat{Y} = \kappa_h f_h + \mu_h f_2$ as in the proof of Theorem 3.27(ii); $\mu_h = \mathcal{W}(f_h, \widehat{Y})/\mathcal{W}(f_h, f_2)$, $|\mathcal{W}(f_h, f_2)| \ge ch^{1/3}$. The exact solution $f_h$ of Proposition E.6 is defined on the lattice of $I$ and agrees with the parametrix $\psi^{[2]}$ at its last two lattice points $k_e - 1, k_e$ only, so the constant Casoratian $\mathcal{W}(f_h, \widehat{Y})$ is evaluated at $(k_e - 1, k_e)$, both in $I$. There $f_h = \psi^{[2]}$, $|f_h| \le Ce^{-\xi}$ ($\mathsf{M}/\mathsf{E} \le Ce^{-\xi}$ for $X \ge 0$), and $|\widehat{Y}_k| \le CN^{1/6} e^{-\xi(q_k)}$ for $k = k_e - 1, k_e$ by (a)–(b). Since $\xi$ is nondecreasing and $\xi(q_{k_e}) - \xi(q_{k_e - 1}) = N \int_{q_{k_e-1}}^{q_{k_e}} \Upsilon \le \Upsilon_{\max}$, each product $|f_{h,k} \widehat{Y}_{k'}|$ with $k, k' \in \{k_e - 1, k_e\}$ is at most $CN^{1/6} e^{-2\xi(q_{k_e-1})} \le Ce^{2\Upsilon_{\max}} N^{1/6} e^{-2\xi(q_{k_e})}$, so the decay bound changes only by the uniform factor $e^{2\Upsilon_{\max}}$ against the evaluation at $(k_e, k_e + 1)$. Thus $|\mu_h| \le CN^{1/2} e^{-2\xi(q_{k_e})}$ and, with $|f_2| \le C\mathsf{ME}$, $\mathsf{E}^2 \le Ce^{2\xi}$, at $q \in I$

$$|\mu_h f_2| \le CN^{1/2} e^{-2(\xi(q_{k_e}) - \xi(q))} \mathsf{M}/\mathsf{E} \le CN^{-3} \mathsf{M}/\mathsf{E} \quad \text{for } q \le q_* + \delta_0 - C_1 N^{-1} \log N, \tag{302}$$

with $c := \Upsilon(q_* + \frac{\delta_0}{2}) > 0$ and $C_1 := 7/(4c)$. Since $q_* + \delta_0 - q_{k_e} < h$, every such $q$ has $q_{k_e} - q > (C_1 \log N - 1)h$. For $q \ge q_* + \frac{\delta_0}{2}$ one has $\Upsilon \ge c$ on $[q, q_{k_e}]$, so $\xi(q_{k_e}) - \xi(q) = N \int_q^{q_{k_e}} \Upsilon \ge cN(q_{k_e} - q) \ge \frac{7}{4} \log N - c$, and the factor $e^{2c}$ lost against $N^{-7/2}$ is absorbed in $C$; while for $q \le q_* + \frac{\delta_0}{2}$ monotonicity of $\xi$ ($\xi = 0$ for $q \le q_*$) gives $\xi(q_{k_e}) - \xi(q) \ge \xi(q_{k_e}) - \xi(q_* + \frac{\delta_0}{2}) \ge cN(q_{k_e} - q_* - \frac{\delta_0}{2}) \ge \frac{c\delta_0}{4} N$ for large $N$. The rest of the proof of Theorem 3.27(ii) is unchanged ($\kappa_h$ is fixed at an oscillatory $q_0$; the other remainders were estimated on all of $I$); the amplitude is Proposition 3.28. □

# F Hahn/$3j$ chain data

This section records the Hahn specialization of the fixed-rank contact identity, the explicit shadow phase, the principal secular terms, and the explicit higher phase horizons of Section 4.

**Corollary F.1** (Fixed-rank second-order contact for Hahn)**.** *For the Hahn chain* $(\varepsilon_1, \beta_1, \beta_2) = (\frac{1}{2}, 0, 0)$*, so Proposition 3.17 gives, with* $s = 1 - x^2$*,*

$$\rho_{20}^{(3)}(x) = \frac{9x - x^3 - 8x(M^2 + \sigma^2) + 4M\sigma x^2 + 12M\sigma}{24 s^{3/2}}, \tag{303}$$

*the contact source* $-\frac{1}{2}[(k+1)^2 A_{k+1}^{(0)} \widehat{d}_{k+1} + k^2 A_k^{(0)} \widehat{d}_{k-1}]$ *in* (203)*, and*

$$\mathcal{R}_k(\widehat{U}^{[2]}) = O_k(h^4) \qquad \text{at every fixed admissible rank } k. \tag{304}$$

**Proposition F.2** (Hahn shadow phase)**.** *For the Hahn chain, on a compact subset of* $0 < q < 1$ *and* $D(q,x) > 0$*,*

$$p_h(q,x) = p(q,x) + h^2 p_2^{(3)}(q,x) + O(h^4), \tag{305}$$

*where*

$$p_2^{(3)}(q,x) = -\frac{4(1-q^2)^2[(M^2+\sigma^2)x - 2M\sigma] + x(3q^2-1)}{8q^2(1-q^2)^2\sqrt{1-x^2-q^2}}. \tag{306}$$

*In particular*

$$p_2^{(3)}(q,x) = -\frac{\mathfrak{c}_{M\sigma}(\theta)}{q^2} + O(1), \qquad q \downarrow 0. \tag{307}$$

*Proof.* This is (194) with $(a,b) = (M,\sigma)$, $e_\pi(s) = \sqrt{1-s}$, $v_\pi = 0$ and $\sin p = \sqrt{1-x^2-q^2}/\sqrt{1-q^2}$: here $(\log a_0)'' = -(1+q^2)/(1-q^2)^2$, and clearing the denominator $8q^2(1-q^2)^2$ gives the displayed numerator, since $(1-q^2)^2 - q^2(1+q^2) = 1-3q^2$. The pole is (195). □

**Proposition F.3** (Principal secular terms at orders $N^{-2}$ and $N^{-4}$)**.** *For fixed* $M, \sigma$ *and* $x$ *in a compact set* $K \Subset (-1,1)$*, let* $f = d_{M\sigma}^L(\arccos x)$ *and* $\kappa = L + \frac{1}{2}$*, and let* $f_2, f_4$ *be the exact fixed-rank coefficients of* $N^{-2}, N^{-4}$ *in* $\Phi_L^{M,\sigma}$*. Then*

$$f_2 = \kappa^2 \mathcal{P}_2 f + O_K(L^{1/2}), \qquad \mathcal{P}_2 = \frac{x}{6}\partial_x + \frac{3-x^2}{12(1-x^2)}, \tag{308}$$

*and the exact fourth-order coefficient* $f_4 = \mathcal{D}_4 f$ *satisfies*

$$\boxed{f_4 = \kappa^4 \mathcal{P}_4 f + O_K(L^{7/2}),} \tag{309}$$

*with* $\mathcal{P}_4$ *from* (96)*.*

*Proof.* The first statement is the large-$L$ part of the exact fixed-rank $f_2$ formula of Ref. [2, Theorem 6.1]. For the second, the same reference gives $f_4 = \mathcal{D}_4 f$ with $\mathcal{D}_4 = A_4(x)\partial_x + B_4(x)$ and $A_4, B_4$ rational in $(x, J, \Sigma, \tau_\star)$, $J = L(L+1)$, $\Sigma = M^2+\sigma^2$, $\tau_\star = M\sigma$ [2, Theorem 6.3 and Eqs. (323)–(324)]. Reducing $f''$ in $\kappa^4\mathcal{P}_4 f$ by the Wigner equation $(1-x^2)^2 f'' = 2x(1-x^2)f' + [-J(1-x^2) + \Sigma - 2\tau_\star x]f$ gives a first-order form $\widetilde{A}_4 f' + \widetilde{B}_4 f$, and exact rational algebra gives $\deg_J(A_4 - \widetilde{A}_4) = 1$ and $\deg_J(B_4 - \widetilde{B}_4) = 2$: the $J^2 f'$ and $J^3 f$ terms cancel. Since $f = O_K(L^{-1/2})$ and $f' = O_K(L^{1/2})$ on compact bulk sets, $(\mathcal{D}_4 - \kappa^4\mathcal{P}_4)f = O_K(L^{5/2} + L^{7/2})$. □

## F.1 Exact Hahn representation

In the fundamental Hahn chamber $\sigma \geq |M|$, the standard $3j$–Clebsch–Gordan relation and Koornwinder's Hahn identity, Eq. (4.5) of [28], give the exact parameter dictionary

$$n_H = L - \sigma, \qquad N_H = N - 1 - \sigma, \qquad \alpha_H = \sigma - M, \qquad \beta_H = \sigma + M, \tag{310}$$

$$k_H = \frac{N - 1 - \sigma - 2\mu + M}{2}. \tag{311}$$

Writing

$$w_H(k) = \frac{(\alpha_H+1)_k}{k!}\frac{(\beta_H+1)_{N_H-k}}{(N_H-k)!}, \tag{312}$$

$$h_H(n) = \frac{(-1)^n(n+\alpha_H+\beta_H+1)_{N_H+1}(\beta_H+1)_n n!}{(2n+\alpha_H+\beta_H+1)(\alpha_H+1)_n(-N_H)_n N_H!}, \tag{313}$$

Koornwinder's exact identity and the standard $3j$–Clebsch–Gordan phase can be written in the orthonormal form

$$\widehat{\Phi}_L^{M,\sigma} = \sqrt{2L+1}\,\Phi_L^{M,\sigma} = \varsigma_H\sqrt{\frac{N w_H(k_H)}{h_H(n_H)}}\,Q_{n_H}(k_H;\alpha_H,\beta_H,N_H), \tag{314}$$

where $\varsigma_H = +1$ in the fundamental chamber $\sigma \geq |M|$ ($k_H = 0$ gives $m_- = j_-$, $Q_{n_H}(0) = 1$, and a single Racah term of sign $(-1)^{4L} = +1$); it is not an undetermined connection scalar. All other fixed $(M,\sigma)$ chambers are finite exact Wigner/Regge symmetry images of the fundamental chamber, up to a fixed sign $\varsigma_H \in \{\pm 1\}$.

*Remark* F.4 (Higher Hahn phase horizons). Expanding the arcsine in (92), which converges on every compact turning-free set,

$$NS(q,x) = \kappa\arccos x - \sum_{r\geq 1}\frac{\binom{2r}{r}}{4^r(2r+1)}\frac{\kappa^{2r+1}}{N^{2r}}\int_0^x \frac{\mathrm{d}u}{(1-u^2)^{r+1/2}}, \tag{315}$$

so every coefficient of the hierarchy (35) is explicit and nonzero for $x \neq 0$, and the closed action resums it. The first correction displaces a bulk zero by $O(L^2/N^2)$ in $\theta$, one zero spacing (of order $1/\kappa$) when $L^3/N^2 \asymp 1$ and $x \neq 0$.

# G Hahn Airy connection as a rank-direction turn

This section proves Theorem 4.6 from Theorem 3.27. At a fixed node $x \neq 0$ the Hahn chain turns in the rank direction exactly where the node meets the band edge, and the two Langer coordinates coincide.

**Lemma G.1** (Hahn rank turn and Langer coordinate)**.** *Let $X$ be a compact subset of $(0,1)$ or of $(-1,0)$, and take the Hahn chain of Proposition 4.1 with any $s_1 \in (\max_X(1-x^2),1)$. For $x \in X$ put $q_*(x) = \sqrt{1-x^2}$. Then $q_*^2 < s_1$, $D(q,x) = 1-x^2-q^2 > 0$ for $0 \leq q < q_*$, $D(q_*,x) = 0$ and $\partial_q D(q_*,x) = -2q_* < 0$, so (73) holds uniformly on $X$, with turn type $\tau = \operatorname{sgn} x$. The Langer coordinate (74) is the spatial one,*

$$\zeta_\tau(q;x) = \xi_\tau(q,x) \qquad (0 < q < 1), \tag{316}$$

*and the turning action is*

$$S(q_*,x) = \frac{\pi}{2}(1-x) \quad (x>0), \qquad S(q_*,x) = \pi q_* - \frac{\pi}{2}(1+x) \quad (x<0). \tag{317}$$

*Proof.* The turning statements are immediate from $D = 1-x^2-q^2$, and $u(q) = x/\sqrt{1-q^2}$ gives $u(q_*) = \operatorname{sgn} x$.

Take $x > 0$. On the oscillatory side $q \le q_*$, equivalently $x \le x_+(q)$, write $p(s,x) = \arccos(x/\sqrt{1-s^2}) = \int_x^{\sqrt{1-s^2}} (1-s^2-u^2)^{-1/2}\,\mathrm{d}u$. Fubini's theorem on $\{(s,u) : s \ge q,\ u \ge x,\ s^2+u^2 \le 1\}$ gives

$$\int_q^{q_*} p(s,x)\,\mathrm{d}s = \int_x^{x_+(q)} \Big[\arcsin \frac{s}{\sqrt{1-u^2}}\Big]_{s=q}^{s=\sqrt{1-u^2}} \mathrm{d}u = \int_x^{x_+(q)} \Big(\frac{\pi}{2} - p_q(u)\Big)\mathrm{d}u, \tag{318}$$

which is (316) there. On the forbidden side $q \ge q_*$, equivalently $x \ge x_+(q)$, the rank action $I_{\mathrm{r}}(q) = \int_{q_*}^{q} \operatorname{arcosh}(x/\sqrt{1-s^2})\,\mathrm{d}s$ and the spatial action $I_{\mathrm{s}}(q) = \int_{x_+(q)}^{x} \operatorname{arcosh}(q/\sqrt{1-u^2})\,\mathrm{d}u$ both vanish at $q = q_*$, and, since the integrand of $I_{\mathrm{s}}$ vanishes at $u = x_+(q)$,

$$\partial_q I_{\mathrm{s}} = \int_{x_+(q)}^{x} \frac{\mathrm{d}u}{\sqrt{u^2-(1-q^2)}} = \operatorname{arcosh} \frac{x}{\sqrt{1-q^2}} = \partial_q I_{\mathrm{r}}. \tag{319}$$

Hence $I_{\mathrm{r}} = I_{\mathrm{s}}$. For $x < 0$, $p(s,-x) = \pi - p(s,x)$ and $u \mapsto -u$, so $\zeta_-(q;x) = \zeta_+(q;-x)$, while $\xi_-(q,x) = \xi_+(q,-x)$ by definition. Finally, (93)–(94) at $x = x_\pm(q)$ give $F = \frac{\pi}{2}(q+x-1)$ for $x > 0$ and $F = \frac{\pi}{2}(x+1-q)$ for $x < 0$ (the arctangent tends to $\pm\pi/2$), and $S = \pi q/2 - F$ is (317) with $q = q_*(x)$. □

## Proof of Theorem 4.6: Weyl-normalized fixed-ratio Hahn Airy connection

*Proof.* Upper turn. By (104), $\xi_+(q,\cdot)$ is continuous and increasing with derivative bounded below near $x_+(q)$, uniformly for $q \in [q_0,q_1]$. Hence for $N \ge N_0(q_0,q_1,A)$ every pair with $q \in [q_0,q_1]$ and $|N^{2/3}\xi_+(q,x)| \le A$ has $x$ in the compact set $X = [\frac12\sqrt{1-q_1^2}, \frac12(1+\sqrt{1-q_0^2})] \subset (0,1)$. Lemma G.1 gives a simple rank turn of type $+1$, uniformly on $X$. At a physical node $x = 1-(p_+ + 1 + 2\nu_+)/N$ of (105), (317) gives $NS(q_*,x)/\pi = (p_+ + 1 + 2\nu_+)/2$, so

$$K_+ = \frac{NS(q_*,x)}{\pi} + \frac{p_+ - 1}{2} = \nu_+ + p_+ \in \mathbb{Z}, \tag{320}$$

with $p_{ab} = p_+ = |M-\sigma|$. Theorem 3.27 with $(a,b) = (M,\sigma)$, $Y_L = \Phi_L^{M,\sigma}$ and $\epsilon_{ab} = \epsilon_{M\sigma}$ applies on the compact family $X$; by (316) its window $|N^{2/3}\zeta_+| \le A$ is the stated one, and it gives (107) with $(-1)^{K_+} = (-1)^{\nu_+ + p_+}$ and additive $O_A(N^{-1})$ bracket error. Condition (R) of Theorem 3.27(ii) holds for every rank: by $3j$ orthogonality the $3j$ symbol $w_L$ of (3) satisfies $\sum_L (2L+1)w_L^2 = 1$, so $|\widehat{\Phi}_L| = \sqrt{(2L+1)N}\,|w_L| \le N^{1/2}$. Theorem 3.27(ii) therefore gives the bracket $\mathrm{Ai}(t) + N^{-4/3}\varpi_+(x)\,\mathrm{Ai}'(t) + O_A(N^{-2})$. For $e_\pi = \sqrt{1-s}$, $v_\pi = 0$ and $q_* = \sqrt{1-x^2}$ one has $\tau u(q) = |x|(1-q^2)^{-1/2}$, $e_0 = |x|$, $u_1 = q_*/x^2$ and $v_1 = 0$, and (79) reduces to (100) (exact computer algebra). In the window $x - x_\pm(q) = O(N^{-2/3})$, so $\varpi^{\mathrm{H}}$ may equally be evaluated at the band edge, at cost $O(N^{-2})$.

Lower turn. The same argument applies on $-X$, with type $\tau = -1$. At a node $x = -1 + (p_- + 1 + 2\nu_-)/N$ of (106), (317) gives

$$K_- = \kappa + \frac{NS(q_*,x)}{\pi} - Nq_* + \frac{p_+}{2} = \kappa - \frac{N(1+x)}{2} + \frac{p_+}{2} = L - \nu_- + \frac{p_+ - p_-}{2}. \tag{321}$$

This is an integer: $(p_+ - p_-)/2 = -\operatorname{sgn}(M\sigma)\min(|M|,|\sigma|)$ and $L \pm M$, $L \pm \sigma$ are integers. Since $K_-$ and $\nu_- + L + (p_+ - p_-)/2$ differ by the even integer $2\nu_-$, Theorem 3.27 gives (108). Equivalently, the lower turn follows from the upper one by the exact reflection $\Phi_L^{M,\sigma}(-x;N^{-1}) = (-1)^{L-\sigma}\Phi_L^{-M,\sigma}(x;N^{-1})$ (a $3j$ symmetry), which exchanges $p_+ \leftrightarrow p_-$ and $\nu_+ \leftrightarrow \nu_-$, takes $\xi_+(q,-x)$ to $\xi_-(q,x)$, and leaves $D$ and $\varpi^{\mathrm{H}}$ unchanged ($\varpi^{\mathrm{H}}$ depends on $x$ only through $x^2$ and $M\sigma x$); the signs agree because $\epsilon_{M\sigma}\epsilon_{-M,\sigma} = (-1)^{\sigma+\max(|M|,|\sigma|)}$. □

The proof uses no parameter translation to a printed Hahn-polynomial normalization: the exact representation (314) is needed only for the bulk zeros (Corollary 4.5).

# H Centered Racah chain data

This section records the Racah specialization of the fixed-rank contact identity, the first pullback coefficient, the Hahn degeneration and second secular phase of the principal symbol, and the proofs of the dual, tetrahedral and flat-turn statements of Section 5.

**Corollary H.1** (Fixed-rank second-order contact for centered Racah)**.** *Put* $s = 1 - x^2$, $\mathfrak{e}_6 = \alpha^{-2} + \beta^{-2}$, $\mathfrak{p}_6 = (\alpha\beta)^{-1}$ *and* $\mathfrak{q}_{mn} = \frac{1}{8} - \frac{m^2+n^2}{2}$. *For the centered Racah chain* $(\varepsilon_1, \beta_1, \beta_2) = (\mathfrak{e}_6/8,\ \mathfrak{p}_6/4,\ (\frac{1}{2} - m^2 - n^2)\mathfrak{p}_6/8)$, *so Proposition 3.17 gives*

$$\begin{aligned}\rho_{20}^{(6)} = \frac{1}{96 s^{3/2}}\Big[&16\mathfrak{e}_6\mathfrak{q}_{mn}x + 4\mathfrak{e}_6 mnx^2 + 12\mathfrak{e}_6 mn - \mathfrak{e}_6 x^3 + 7\mathfrak{e}_6 x \\ &- 24\mathfrak{p}_6\mathfrak{q}_{mn}x^2 - 8\mathfrak{p}_6\mathfrak{q}_{mn} - 32\mathfrak{p}_6 mnx - 7\mathfrak{p}_6 x^2 - 5\mathfrak{p}_6\Big],\end{aligned} \tag{322}$$

*the contact source* $-\frac{\mathfrak{e}_6}{8}\big[(k+1)^2 A_{k+1}^{(0)}\widehat{d}_{k+1} + k^2 A_k^{(0)}\widehat{d}_{k-1}\big] + \delta_k^{(6)}\widehat{d}_k$ *in* (203), *with* $\delta_k^{(6)} = \beta_1\kappa^2 + \beta_2 = (2k(k+1) + 1 - m^2 - n^2)/(8\alpha\beta)$, *and*

$$\mathcal{R}_k(\widehat{U}^{[2]}) = O_k(h^4) \qquad \text{at every fixed admissible rank } k, \tag{323}$$

*including* $k = 0$ *in the channel* $m = n = 0$, *whose exact two-start seed is* (115).

**Proposition H.2** (First centered-Racah pullback coefficient)**.** *Put*

$$\mathfrak{a}_6 := \alpha^{-2} + \beta^{-2}, \qquad \chi_6(x) := \frac{x\mathfrak{a}_6}{8} - \frac{1}{4\alpha\beta}. \tag{324}$$

*Then, as* $\rho \to 0$,

$$\Theta_6(\rho) = \theta - \frac{\chi_6(x)}{3\sin\theta}\rho^2 + O(\rho^4), \tag{325}$$

$$W_6(\rho) = 1 + \rho^2\frac{(3-x^2)\mathfrak{a}_6/48 - x/(12\alpha\beta)}{1-x^2} + O(\rho^4). \tag{326}$$

*Hence the first pullback operator is*

$$\boxed{\mathcal{P}_2^{(6)} = \left(\frac{x\mathfrak{a}_6}{24} - \frac{1}{12\alpha\beta}\right)\partial_x + \frac{(3-x^2)\mathfrak{a}_6/48 - x/(12\alpha\beta)}{1-x^2}.} \tag{327}$$

*If* $g_2^{(f)}$ *denotes the coefficient of* $t = h^2 = N^{-2}$ *in the centered Racah fixed-rank series of* $\Psi_f$, *then on compact strict-triangle families*

$$\sqrt{f+1}\,\Big|g_2^{(f)} - \kappa_f^2\mathcal{P}_2^{(6)}d_{mn}^f(\theta)\Big| \le C(f+1). \tag{328}$$

*Proof.* The expansions are (22) with (112), and the bound (328) is Lemma 3.13. □

Hahn also appears as a principal-symbol degeneration of Racah, a useful unification at the Hamilton–Jacobi level. Set $\alpha = \frac{1}{2}$ and let $\beta \to \infty$ while $x$ is held fixed through the law of cosines, so $\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta x$. Then (113) tends locally uniformly before the turning point to

$$\cos\omega_{\mathrm{H}}(\rho) = \frac{x}{\sqrt{1-\rho^2}}, \tag{329}$$

which is exactly the rank-direction Hahn symbol associated with the action (94). In particular

$$\frac{x}{\sqrt{1-\rho^2}} = x \sum_{m\geq 0} \frac{\binom{2m}{m}}{4^m} \rho^{2m}, \tag{330}$$

so every even Hahn principal-symbol coefficient is the boundary value

$$\chi^{(\mathrm{H})}_{2m} = x\frac{\binom{2m}{m}}{4^m}. \tag{331}$$

Moreover the Racah turning radius tends to $\sqrt{1-x^2}$, while $\partial_\rho S_6 = \omega$ tends to

$$\partial_q S = \arccos\left(\frac{x}{\sqrt{1-q^2}}\right). \tag{332}$$

Thus the Hahn and centered-Racah principal actions are two members of the same two-parameter symbol family. This is a principal-symbol degeneration only; no uniform $\beta \to \infty$ theorem for the full offset-dependent recurrence is asserted.

*Remark* H.3 (Second centered-Racah secular phase and the $N^{4/5}$ horizon). With $E_6 = \alpha^{-2} + \beta^{-2}$, $P_6 = (\alpha\beta)^{-1}$, $\chi^{(6)}_2 = xE_6/8 - P_6/4$ and $\chi^{(6)}_4 = x(3E_6^2/128 - P_6^2/32) - E_6P_6/32$, the principal symbol is $\cos\omega = x + \chi^{(6)}_2\rho^2 + \chi^{(6)}_4\rho^4 + O(\rho^6)$, so with $s_\theta = \sin\theta$

$$NS_6(\rho) = \kappa_f\theta - \frac{\chi^{(6)}_2}{3s_\theta}\frac{\kappa_f^3}{N^2} - \frac{1}{5}\Big[\frac{\chi^{(6)}_4}{s_\theta} + \frac{x(\chi^{(6)}_2)^2}{2s_\theta^3}\Big]\frac{\kappa_f^5}{N^4} + O\Big(\frac{\kappa_f^7}{N^6}\Big). \tag{333}$$

Once the first term is resummed, the heuristic rule of Remark 3.10 gives $f \asymp N^{4/5}$ as the next generic horizon (not proved here); on the locus $\chi^{(6)}_2 = 0$ it is the first, unless the bracket also vanishes.

## Proof of the statements of Remark 5.9: dual principal symbol and bispectral action

*Proof.* For the exact self-adjoint $c$-lattice operator of Ref. [2, Theorem 8.2], let $a_c^{\pm}$ be the two hopping coefficients and $b_c$ the diagonal term. At $C = c + \frac{1}{2} = N\gamma$ with fixed $m, n$, expanding its four-factor edge coefficient gives $N^{-2}a_c^{\pm} = a_C^{(0)} + O(N^{-1})$ with $a_C^{(0)} = \Delta_C(\gamma)/(4\gamma^2)$ and $N^{-2}b_c = b_C^{(0)} + O(N^{-1})$ with $b_C^{(0)} = \Delta_C(\gamma)/(2\gamma^2)$. Since the exact eigenvalue is $f(f+1) = N^2\rho^2 + O(1)$, the principal WKB ansatz in the $c$ direction gives $b_C^{(0)} - 2a_C^{(0)}\cos p_C = \rho^2$, proving (143). The law of cosines $2\alpha\beta x = \alpha^2 + \beta^2 - \gamma^2$ gives $\Delta_C = 4\alpha^2\beta^2(1-x^2)$, and hence the common caustic by (114). Since $\eta(\rho)$ is independent of $\gamma$, differentiating (113) at fixed $\rho$ gives $\partial_\gamma\omega = \gamma/(\alpha\beta\eta\sin\omega) = 2\gamma/\sqrt{\Delta_C - \rho^2\gamma^2}$, because $4\alpha^2\beta^2\eta^2\sin^2\omega = \Delta_C - \rho^2\gamma^2$ by (114); differentiating (143) in $\rho$ gives the same expression. Since $S_6(\rho,\gamma) = \int_0^\rho \omega(s,\gamma)\,\mathrm{d}s$ and $p_C(0,\gamma) = 0$, $\partial_\gamma S_6 = \int_0^\rho \partial_s p_C(s,\gamma)\,\mathrm{d}s = p_C(\rho,\gamma)$, proving (144). □

### Proof of Lemma 5.8: The Racah turning curve is the flat-tetrahedron locus

*Proof.* Order the vertices $1,2,3,4$ with $d_{12}=d_{14}=A$, $d_{23}=d_{34}=B$, $d_{13}=C$, $d_{24}=F$, and write $a=A^2$, $b=B^2$, $c=C^2$, $f=F^2$. The Cayley–Menger matrix, bordered by the vector of ones and indexed $0,1,2,3,4$, is

$$\mathrm{CM}=\begin{pmatrix} 0 & 1 & 1 & 1 & 1\\ 1 & 0 & a & c & a\\ 1 & a & 0 & b & f\\ 1 & c & b & 0 & b\\ 1 & a & f & b & 0 \end{pmatrix}. \tag{334}$$

The following computation is a polynomial identity valid for all real $N,\alpha,\beta,\gamma,\rho$: only the squares $a,b,c,f$ enter, and no tetrahedron is assumed to exist.

*Direct expansion.* Regard CM as a linear map on $\mathbb{R}^5$ with standard basis $e_0,\dots,e_4$. Exchanging the vertices 2 and 4 leaves it invariant, so it preserves $v=e_2-e_4$ and the span of $e_0,e_1,e_3,u$ with $u=e_2+e_4$. Indeed $\mathrm{CM}\,v=-f\,v$, and in the basis $(e_0,e_1,e_3,u)$ of the complement the images $\mathrm{CM}\,e_0=e_1+e_3+u$, $\mathrm{CM}\,e_1=e_0+c\,e_3+a\,u$, $\mathrm{CM}\,e_3=e_0+c\,e_1+b\,u$, $\mathrm{CM}\,u=2e_0+2a\,e_1+2b\,e_3+f\,u$ give

$$\det\mathrm{CM}=-f\det\begin{pmatrix} 0 & 1 & 1 & 2\\ 1 & 0 & c & 2a\\ 1 & c & 0 & 2b\\ 1 & a & b & f \end{pmatrix}. \tag{335}$$

Subtracting the second row from the third and fourth and expanding along the first column, the $4\times4$ determinant in (335) equals

$$-\det\begin{pmatrix} 1 & 1 & 2\\ c & -c & 2(b-a)\\ a & b-c & f-2a \end{pmatrix}=-2\big[4ab-(a+b-c)^2-cf\big]. \tag{336}$$

With $a=N^2\alpha^2$, $b=N^2\beta^2$, $c=N^2\gamma^2$, $f=N^2\rho^2$, (335) and (336) give

$$\det\mathrm{CM}=2N^6\rho^2\big[4\alpha^2\beta^2-(\alpha^2+\beta^2-\gamma^2)^2-\gamma^2\rho^2\big]. \tag{337}$$

*The two forms of* (142). Heron's formula gives $4\alpha^2\beta^2-(\alpha^2+\beta^2-\gamma^2)^2=16T^2$, which with (337) is the second equality. For $\alpha,\beta\neq0$, $2\alpha\beta x=\alpha^2+\beta^2-\gamma^2$ and (114) give $4\alpha^2\beta^2D_6=4\alpha^2\beta^2-(\alpha^2+\beta^2-\gamma^2)^2-\gamma^2\rho^2$, which is the first.

*Sign trichotomy.* The prefactor $8N^6\rho^2\alpha^2\beta^2$ is strictly positive for $\rho>0$, so $\det\mathrm{CM}$ has the sign of $D_6$. A tetrahedron with prescribed edge lengths exists in $\mathbb{R}^3$ if and only if $\det\mathrm{CM}\geq0$ and every face satisfies the triangle inequalities. The four faces are $(\alpha,\beta,\gamma)$ twice, $(A,A,F)$ and $(B,B,F)$, and $D_6>0$ already makes all of them admissible. Indeed $D_6>0$ gives $\gamma^2\rho^2<16T^2$, so $T^2>0$ and the $(\alpha,\beta,\gamma)$ faces are nondegenerate triangles; and

$$4\alpha^2\gamma^2-16T^2=(\alpha^2-\beta^2+\gamma^2)^2\geq0 \tag{338}$$

gives $\gamma^2\rho^2<16T^2\leq4\alpha^2\gamma^2$, that is $\rho<2\alpha$, the triangle inequality for $(A,A,F)$; exchanging $\alpha$ and $\beta$ gives $\rho<2\beta$ for $(B,B,F)$. So $D_6>0$ is the nondegenerate Euclidean tetrahedron, $D_6=0$ the flat one, and for $D_6<0$ the determinant is negative and the six lengths are not those of any Euclidean tetrahedron. □

*Remark* H.4 (Geometric check for genuine tetrahedra). For an actual tetrahedron $\det \mathrm{CM} = 288\mathcal{V}^2$, and $\mathcal{V} = Fr_A r_B \sin\omega/6$ with $r_A r_B = N^2\alpha\beta\,\eta$ and $\sin\omega = \sqrt{D_6}/\eta$ by (113) and (114). Hence $288\mathcal{V}^2 = 8N^6\rho^2\alpha^2\beta^2 D_6$, which is the first equality of (142) again. This route covers only $D_6 \geq 0$, where the tetrahedron exists; the expansion above is the statement for all real parameters.

*Remark* H.5 (Relation to the full six-edge wall). The lemma identifies the one-parameter turning curve of this paper with a slice of the full six-edge Cayley–Menger wall $\{\det \mathrm{CM} = 0\}$ in the space of squared edge lengths $x_{ij}$. At a point of the wall where the bordered distance matrix has corank one, with kernel vector $(v_0, \lambda)$, its adjugate is $c\,(v_0,\lambda)(v_0,\lambda)^{\mathsf{T}}$ for a scalar $c \neq 0$, and differentiating the determinant gives $\partial \det \mathrm{CM}/\partial x_{ij} = 2c\lambda_i\lambda_j$. The Airy turn treated here is therefore the pullback of the full wall along the symmetric family $(A, B, C; B, A, F)$; this gradient supplies the normal and the transversality of the wall in directions the one-parameter recurrence does not see. Conversely, the normalized Airy connection of Theorem 5.7 is the analytic local model on this slice.

*Remark* H.6 (Volume and the equilateral family). First, $D_6$ decreases monotonically in $\rho^2$, but the tetrahedral volume does *not*: (142) gives

$$\mathcal{V}^2 = \frac{N^6}{144}\rho^2(16T^2 - \gamma^2\rho^2) = \frac{N^6\gamma^2}{144}\rho^2(\rho_*^2 - \rho^2). \tag{339}$$

Thus, on $0 \leq \rho \leq \rho_*$ where the tetrahedron exists, $\mathcal{V}$ grows from zero at $\rho = 0$, reaches its unique maximum at $\rho = \rho_*/\sqrt{2}$, and then decreases to zero at the positive flattening radius $\rho_* = 4T/\gamma$. This distinguishes the monotonicity of the *characteristic discriminant* from that of the geometric volume. Second, for the equilateral family $\alpha = \beta = \gamma = L$ one gets $\det \mathrm{CM} = 2N^6L^2\rho^2(3L^2 - \rho^2)$ and $\rho_* = \sqrt{3}\,L$, recovering the classical five-equal-edge caustic ratio. Since $\rho_*/L = \sqrt{3}$ is irrational whereas the ratio of two nonzero admissible Weyl edge lengths is rational, no exactly equilateral admissible half-integer label set lies on that caustic. The turning curve is met only through the $\rho - \rho_* = O(N^{-2/3})$ layer of Theorem 5.7, equivalently a layer of width $O(N^{1/3})$ in the edge length $F$, consistent with the simple-turn hypothesis used there.

**Lemma H.7** (Closed Racah Airy coefficient)**.** *For the data* (112) *at a simple turn,* (79) *equals* (137).

*Proof.* Write $z = \rho^2$, $E(z) = e(z)^2$ and $K = \gamma^2/(4\alpha^2\beta^2)$, so $D_6 = \sin^2\theta - Kz$ by (114) and $K\rho_*^2 = \sin^2\theta$. At the turn $E(\rho_*^2) = (x - s_*)^2 = \varepsilon^2$, so $e_0 = |\varepsilon|$, $\tau = \operatorname{sgn}\varepsilon$ and $\tau/e_0 = 1/\varepsilon$. Since $\tau u_6 = (1 + w)^{1/2}$ with $w = K(z - \rho_*^2)/E(z)$ and $z - \rho_*^2 = 2\rho_*\delta + \delta^2$, the jets $u_j$ are rational in $(\alpha, \beta, x, \rho_*)$, and $u_1 = K\rho_*/\varepsilon^2 = \sin^2\theta/(\rho_*\varepsilon^2)$; hence $\rho_*^2 u_1^{2/3} = (\rho_* \sin\theta)^{4/3}|\varepsilon|^{-4/3}$. With $v_1 = (\frac{1}{2} - m^2 - n^2)/(8\alpha\beta)$ and $(2e_0e_2 - e_1^2)/e_0^2 = \frac{1}{2}(\log E(\rho^2))''(\rho_*)$, the bracket of (79) reduces modulo $\rho_*^2 = 4\alpha^2\beta^2\sin^2\theta/\gamma^2$ to $[140\varepsilon^3((m^2 + n^2)x - 2mn) + \mathcal{P}(x, s_*)]/\varepsilon^4$; the offset terms combine through $1/\rho_*^2 + 1/(4\alpha\beta\varepsilon) = x/(\rho_*^2\varepsilon)$. This is an identity of rational functions of $(\alpha, \beta, x)$, checked by exact computer algebra. □

## Proof of Lemma 5.6: Flat centered-Racah turning action

*Proof.* At the turning radius $R = \rho_*$ one has $\phi = \pi/2$ in (118), so each atan2 term equals $\frac{\pi}{2}\operatorname{sgn} c$, and

$$S_6(R) = R\,\omega(R) + \pi\gamma - \pi\alpha\operatorname{sgn} c_\beta - \pi\beta\operatorname{sgn} c_\alpha. \tag{340}$$

By the law of cosines, $x - R^2/(4\alpha\beta) = -c_\alpha c_\beta$, and $\eta(R) > 0$, so $\omega(R) = 0$ when $c_\alpha c_\beta < 0$ and $\omega(R) = \pi$ when $c_\alpha c_\beta > 0$. If $\tau = +1$ the characteristic roots meet at $+1$, so $\omega(R) = 0$: one base angle is obtuse, necessarily the one opposite the longer of $\alpha, \beta$, and $S_6(R) = \pi(\gamma - |\alpha - \beta|)$. If $\tau = -1$ they meet at $-1$, so $\omega(R) = \pi$, both base angles are acute, and $S_6(R) = \pi[R - (\alpha + \beta - \gamma)]$. This is (131). □

# I One-small quantum $6j$ chain: proofs

Two facts about the quantum $6j$ symbol are cited, both known. (K1) The Kirillov–Reshetikhin formula [36], which we take as the definition of the symbol,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q = \prod_{4\ \text{triads}} \Delta(\cdot) \sum_z \frac{(-1)^z [z+1]!}{\prod_i [z-\alpha_i]!\ \prod_i [\beta_i - z]!}, \tag{341}$$

$$\Delta(a,b,c)^2 = \frac{[a+b-c]!\,[a-b+c]!\,[-a+b+c]!}{[a+b+c+1]!}, \tag{342}$$

with $[z]! = \prod_{j=1}^{z} [j]$, the four triads $(j_1, j_2, j_{12}), (j_3, j, j_{12}), (j_1, j, j_{23}), (j_3, j_2, j_{23})$ with sums $\alpha_i$, the three quadrilateral sums $\beta_1 = j_1{+}j_2{+}j_3{+}j$, $\beta_2 = j_2{+}j_{12}{+}j{+}j_{23}$, $\beta_3 = j_{12}{+}j_1{+}j_{23}{+}j_3$, and $z$ running over $\max \alpha_i \le z \le \min \beta_i$ in steps of 1 (with the convention $1/[k]! = 0$ for $k < 0$ the sum may be taken over all of $\mathbb{Z}$). (K2) $q$-orthogonality: $\sum_f [2f+1][2c+1] W_f^2 = 1$ for the family $W_f$ of Section 6; it supplies an independent global bound for (R) (alongside Proposition 3.31) and the Gauss-weight identity of Remark 3.35. The three-term recurrence (a) is *not* cited: it is proved from (K1) in Lemma I.1 below.

**Lemma I.1** (Three-term recurrence of the Kirillov–Reshetikhin sum)**.** *For the family $W_f$ in the fundamental chamber $m \ge |n|$, put $\alpha_1 = a{+}b{+}c$, $\alpha_2 = \alpha_1{+}m{+}n$, $\alpha_3 = 2a{+}n{+}f$, $\alpha_4 = 2b{+}m{+}f$, $\beta_1 = 2a{+}2b{+}m{+}n$, $\beta_2 = a{+}b{+}c{+}n{+}f$, $\beta_3 = a{+}b{+}c{+}m{+}f$,*

$$t(z,f) = \frac{(-1)^z [z+1]!}{\prod_{i=1}^{4} [z-\alpha_i]! \prod_{i=1}^{3} [\beta_i - z]!}, \qquad S_f = \sum_{z \in \mathbb{Z}} t(z,f), \tag{343}$$

*so that $W_f = P_0\, P(f)\, S_f$ with $P_0 = \Delta(a,b,c)\Delta(b{+}m, a{+}n, c)$ independent of $f$ and $P(f) = \Delta(a, a{+}n, f)\Delta(b{+}m, b, f)$. Then $S_f = 0$ unless $m \le f \le f_{\max} := \min(2a{+}n, 2b{+}m)$, and for every $f \ge \frac{1}{2}$ with $f - m \in \mathbb{Z}$*

$$(x - B_f)\, S_f = \rho_+(f)\, S_{f+1} + \rho_-(f)\, S_{f-1}, \tag{344}$$

*with, for $\mathrm{s}_z = \sinh(z/N)$ and $2A = 2a{+}n{+}1$, $2B = 2b{+}m{+}1$,*

$$\rho_+(f) = -\frac{2\,\mathrm{s}_{f+1-n}\mathrm{s}_{f+1+n}\mathrm{s}_{f+1-m}\mathrm{s}_{f+1+m}}{\mathrm{s}_{2A}\mathrm{s}_{2B}\,\mathrm{s}_{2f+2}\mathrm{s}_{2f+1}}, \qquad \rho_-(f) = -\frac{2\,\mathrm{s}_{2A-f}\mathrm{s}_{2A+f}\mathrm{s}_{2B-f}\mathrm{s}_{2B+f}}{\mathrm{s}_{2A}\mathrm{s}_{2B}\,\mathrm{s}_{2f}\mathrm{s}_{2f+1}}, \tag{345}$$

*and $B_f$ of (148). For $m = n = 0$ the row $f = 0$ reads $(x - B_0)S_0 = \rho_+(0)S_1$ with $B_0 = \coth \mathsf{X} \coth \mathsf{Y}(1 - 1/\cosh \frac{1}{N})$.*

*Proof. Support.* For $f < m$ one has $\beta_2 < \alpha_2$, and for $f > f_{\max}$ one of $\alpha_3, \alpha_4$ exceeds $\beta_1$; the sum is empty.

*Termwise relations.* In this appendix $q$ inside $q$-numbers, powers $q^z$ and the Kirillov–Reshetikhin sum denotes the deformation parameter $q_{\mathrm{def}} = e^{1/N}$ of Section 6, while $q_k$, $q_*$ and the argument of $p(q)$ remain the rank coordinate. Put $q = e^{1/N}$ and write each $q$-number as $[k] = (u - u^{-1})/(q - q^{-1})$ with $u = q^k$. Let

$$r(z) = \frac{(-1)^z [z+1]!}{[z{-}\alpha_1]![z{-}\alpha_2]![z{-}\alpha_3{+}1]![z{-}\alpha_4{+}1]![\beta_1{-}z]![\beta_2{+}1{-}z]![\beta_3{+}1{-}z]!} \tag{346}$$

(with the same $f$). Since $[k]/[k]! = 1/[k-1]!$ for every integer $k$ (both sides vanish for $k \le 0$),

$$\begin{aligned} t(z, f{+}1) &= r(z)\,[z{-}\alpha_3{+}1][z{-}\alpha_3][z{-}\alpha_4{+}1][z{-}\alpha_4], \\ t(z,f) &= r(z)\,[\beta_2{+}1{-}z][\beta_3{+}1{-}z][z{-}\alpha_3{+}1][z{-}\alpha_4{+}1], \\ t(z, f{-}1) &= r(z)\,[\beta_2{+}1{-}z][\beta_2{-}z][\beta_3{+}1{-}z][\beta_3{-}z]. \end{aligned} \tag{347}$$

Let $G(z) = r(z)\,[z{-}\alpha_1][z{-}\alpha_2][z{-}\alpha_3{+}1][z{-}\alpha_4{+}1]\,\mathrm{Pol}(q^z)$, where $\mathrm{Pol}(w) = p_{-2}w^{-2} + p_0 + p_2 w^2$ has coefficients to be fixed below. $G$ has finite support, and by the same rule $G(z{+}1) = -r(z)\,[z{+}2][\beta_1{-}z][\beta_2{+}1{-}z][\beta_3{+}1{-}z]\,\mathrm{Pol}(q^{z+1})$. Hence the telescoping identity

$$\rho_- t(z, f{-}1) + \rho_0\, t(z, f) + \rho_+ t(z, f{+}1) = G(z{+}1) - G(z), \qquad \rho_0 := -(x - B_f), \tag{348}$$

is, after division by $r(z)$ (it is polynomial in the displayed $q$-numbers, so holds also where $r(z) = 0$), the identity

$$\begin{aligned}\rho_-[\beta_2{+}1{-}z][\beta_2{-}z][\beta_3{+}1{-}z][\beta_3{-}z] + \rho_0[\beta_2{+}1{-}z][\beta_3{+}1{-}z][z{-}\alpha_3{+}1][z{-}\alpha_4{+}1] \\ + \rho_+[z{-}\alpha_3{+}1][z{-}\alpha_3][z{-}\alpha_4{+}1][z{-}\alpha_4] \\ = -[z{+}2][\beta_1{-}z][\beta_2{+}1{-}z][\beta_3{+}1{-}z]\mathrm{Pol}(q^{z+1}) - [z{-}\alpha_1][z{-}\alpha_2][z{-}\alpha_3{+}1][z{-}\alpha_4{+}1]\mathrm{Pol}(q^z).\end{aligned} \tag{349}$$

Both sides are Laurent polynomials in $w = q^z$ whose coefficients are rational functions of $q$ and of $u_a = q^a,\ u_b = q^b,\ u_c = q^c,\ u_f = q^f,\ u_m = q^m,\ u_n = q^n$ ($\alpha_i, \beta_i$ are monomials in these, $x$ and $B_f$ rational in them). Linear algebra over $\mathbb{Q}(q, u_a, u_b, u_c, u_f, u_m, u_n)$ on the coefficients of $w^{-4}, \dots, w^4$ determines $\rho_0$ (equal to $-(x - B_f)$, a check) and $p_{-2}, p_0, p_2$ (explicit rational functions; their denominators are products of $\mathrm{s}_f, \mathrm{s}_{f+1}, \mathrm{s}_{2f+1}, \cosh\frac{f}{N}, \cosh\frac{f+1}{N}, \sinh\mathsf{X}, \sinh\mathsf{Y}$ and monomials, all strictly nonzero for $f > 0$, so they are finite for every $f \geq \frac{1}{2}$, with $S_{-1/2} = 0$ by support when $m = f = \frac{1}{2}$), and then (348) holds as an identity of rational functions in all eight symbols, verified by exact symbolic computation; it holds for every integer $z$ whenever the denominators do not vanish. Summing (348) over $z \in \mathbb{Z}$, the right side telescopes to 0 because $G$ has finite support, which is (344) (with the sign convention $\rho_0 = -(x - B_f)$).

*The row $f = 0$ for $m = n = 0$.* Here $\mathrm{s}_f = 0$ is a pole of the certificate, and the row is checked directly: $S_0 = t(a{+}b{+}c, 0)$ has one term, and $S_1 = t(z_0, 1) + t(z_0{+}1, 1)$, $z_0 = a{+}b{+}c$, with $S_1/S_0 = [b{+}c{-}a][a{+}c{-}b] - [a{+}b{+}c{+}2][a{+}b{-}c]$ (the factors $[b{+}c{-}a]$, $[a{+}b{-}c]$ vanish exactly when the corresponding term is outside the range). The identity $(x - B_0) = \rho_+(0)\, S_1/S_0$ is an identity of rational functions in $(q, q^a, q^b, q^c)$, verified by exact symbolic computation. □

**Lemma I.2** (Closed seed)**.** *For $m \geq |n|$ and $f = m$ the sum (K1) has exactly one term, and with $\Sigma_L := \{-\frac{L-1}{2}, \dots, \frac{L-1}{2}\}$ (a symmetric multiset of $L$ shifts),*

$$\begin{aligned}W_m^2 = \frac{[2m]!}{[m+n]!\,[m-n]!} \prod_{j=-m}^{m} \frac{1}{[2A+j]\,[2B+j]} \\ \times \prod_{\sigma\in\Sigma_{m+n}} [A{+}B{-}C{+}\sigma]\,[A{+}B{+}C{+}\sigma] \prod_{\sigma\in\Sigma_{m-n}} [B{+}C{-}A{+}\sigma]\,[A{-}B{+}C{+}\sigma],\end{aligned} \tag{350}$$

*where $j$ runs over $-m, -m+1, \dots, m$ (the $2m+1$ values).*

*Proof.* For $f = m$ the triad sums are $\alpha = (a{+}b{+}c,\ a{+}b{+}c{+}m{+}n,\ 2a{+}n{+}m,\ 2b{+}2m)$ and the quadrilateral sums $\beta = (2a{+}2b{+}m{+}n,\ a{+}b{+}c{+}m{+}n,\ a{+}c{+}b{+}2m)$. So $\alpha_2 = \beta_2 =: z_0$ and $\max\alpha \leq z \leq \min\beta$ forces $z = z_0$. Write the squared one-term expression as a product of $q$-factorials. Collecting the exponents of the four $\Delta^2$ and of the square of the summand, the factorials $[a{+}b{-}c]!, [b{+}c{-}a]!, [a{-}b{+}c{+}n{-}m]!$ occur to the power $-1$, and $[a{-}b{+}c]!$, $[a{+}b{+}m{+}n{-}c]!$, $[b{-}a{+}m{-}n{+}c]!$, $[z_0{+}1]!$ to the power $+1$, $[a{+}b{+}c{+}1]!$ to the power $-1$; the triads containing $m$ contribute $[2m]!\,[m{-}n]!\,[m{+}n]!\,[2a{+}n{-}m]!\,[2b]!/([2a{+}n{+}m{+}1]!\,[2b{+}2m{+}1]!)$ and the square contributes $[m{+}n]!^{-2}[m{-}n]!^{-2}$. With $2A = 2a + n + 1$, $2B = 2b + m + 1$,

$$\frac{[2a{+}n{-}m]!}{[2a{+}n{+}m{+}1]!} = \prod_{j=-m}^{m} [2A+j]^{-1}, \qquad \frac{[2b]!}{[2b{+}2m{+}1]!} = \prod_{j=-m}^{m} [2B+j]^{-1}, \tag{351}$$

$$\frac{[z_0{+}1]!}{[a{+}b{+}c{+}1]!}=\prod_{\sigma\in\Sigma_{m+n}}[A{+}B{+}C{+}\sigma],\quad \frac{[a{+}b{+}m{+}n{-}c]!}{[a{+}b{-}c]!}=\prod_{\sigma\in\Sigma_{m+n}}[A{+}B{-}C{+}\sigma],\tag{352}$$

$$\frac{[b{-}a{+}m{-}n{+}c]!}{[b{+}c{-}a]!}=\prod_{\sigma\in\Sigma_{m-n}}[B{+}C{-}A{+}\sigma],\quad \frac{[a{-}b{+}c]!}{[a{-}b{+}c{+}n{-}m]!}=\prod_{\sigma\in\Sigma_{m-n}}[A{-}B{+}C{+}\sigma],\tag{353}$$

where the shifts are matched by

$$\begin{aligned} A{+}B{-}C &= a{+}b{-}c+\tfrac{m+n+1}{2}, & A{+}B{+}C &= a{+}b{+}c+\tfrac{m+n+3}{2},\\ B{+}C{-}A &= b{+}c{-}a+\tfrac{m-n+1}{2}, & A{-}B{+}C &= a{-}b{+}c-\tfrac{m-n-1}{2}. \end{aligned}\tag{354}$$

The remaining prefactor is $[2m]!/([m{+}n]!\,[m{-}n]!)$. (The identity is also verified in exact rational arithmetic for integer and half-integer families.) □

**Proof of Proposition 6.1.** *Symmetry reduction to $m\geq|n|$.* In (K1), swapping $(j_1,j_3)\leftrightarrow(j_2,j)$ leaves the triads and quadrilateral sums invariant and sends $(a,b,m,n)\mapsto(b,a,n,m)$ with $(A,B,\mathsf{X},\mathsf{Y})\leftrightarrow(B,A,\mathsf{Y},\mathsf{X})$; composing this with the row flip $(j_1,j_2)\leftrightarrow(j_3,j)$ sends $(a,b,m,n)\mapsto(a+n,b+m,-m,-n)$ at fixed $(A,B,C)$. These two involutions, together with their composition (the row flip $(j_1,j_2)\leftrightarrow(j_3,j)$ sending $(a,b,m,n)\mapsto(b+m,a+n,-n,-m)$ with $(A,B,\mathsf{X},\mathsf{Y})\leftrightarrow(B,A,\mathsf{Y},\mathsf{X})$), map every admissible pair $(m,n)$ into the fundamental chamber $m\geq|n|$, leave $x,\mu,\Delta_0$, $f_{\min}=\max(|m|,|n|)$, $f_{\max}=\min(2A-1,2B-1)$, and all coefficients and class data of (a)–(d) invariant, and multiply both $\widehat{\Psi}_k$ of (147) and $\widehat{d}_k(\theta)=\sqrt{2k+1}\,d^k_{mn}(\theta)$ by the same $k$-independent sign ($+1$ for the row flip and $(-1)^{m-n}$ for the two generators, using $d^k_{mn}=(-1)^{m-n}d^k_{nm}=(-1)^{m-n}d^k_{-m,-n}=d^k_{-n,-m}$). It therefore suffices to prove (a)–(d) in the fundamental chamber $m\geq|n|$, where $f_{\min}=m$.

*(a)* By (147),

$$\widehat{\Psi}_f=K_0(-1)^{f-m}\sqrt{[2f+1]}\,P(f)\,S_f,\qquad K_0=(-1)^{a+b+c+2m}\sqrt{[2A][2B]}\,P_0\ \text{independent of } f.\tag{355}$$

Multiply (344) by $K_0(-1)^{f-m}\sqrt{[2f+1]}P(f)$. For $m\leq f<f_{\max}$ the first term on the right is $A_{f+1}\widehat{\Psi}_{f+1}$ provided

$$-\rho_+(f)=A_{f+1}\frac{\sqrt{[2f+3]}\,P(f+1)}{\sqrt{[2f+1]}\,P(f)},\tag{356}$$

and for $m<f\leq f_{\max}$ the second is $A_f\widehat{\Psi}_{f-1}$ provided

$$-\rho_-(f)=A_f\frac{\sqrt{[2f-1]}\,P(f-1)}{\sqrt{[2f+1]}\,P(f)}.\tag{357}$$

On these respective index ranges both sides of each relation are strictly positive ($[2f-1]>0$ for $f>m\geq0$; the $q$-factorials in $P$ have non-negative arguments on $[m,f_{\max}]$; $\mathrm{s}_{2A\pm k},\mathrm{s}_{2B\pm k}>0$ for $k\leq f_{\max}$ except at $k=f_{\max}+1$, which is not needed; and $\mathrm{s}_{k\pm m},\mathrm{s}_{k\pm n}>0$ for $k>m$), so it suffices that their squares agree. From $P(f)^2=[2a{+}n{-}f]![2b{+}m{-}f]![f{-}n]![f{+}n]![f{-}m]![f{+}m]!/([2a{+}n{+}f{+}1]![2b{+}m{+}f{+}1]!)$,

$$\begin{aligned} \frac{P(f{+}1)^2}{P(f)^2}&=\frac{[f{+}1{-}n][f{+}1{+}n][f{+}1{-}m][f{+}1{+}m]}{[2A{-}f{-}1][2B{-}f{-}1][2A{+}f{+}1][2B{+}f{+}1]},\\ \frac{P(f{-}1)^2}{P(f)^2}&=\frac{[2A{-}f][2B{-}f][2A{+}f][2B{+}f]}{[f{-}n][f{+}n][f{-}m][f{+}m]}, \end{aligned}\tag{358}$$

and $1-\mathrm{s}_k^2/\sinh^2\mathsf{X}=\mathrm{s}_{2A-k}\mathrm{s}_{2A+k}/\mathrm{s}_{2A}^2$ (similarly for $\mathsf{Y}$), so the factors $\mathrm{s}_{2A\mp k}$ cancel against $P(f\pm1)^2/P(f)^2$ and $\rho_+(f)^2=A_{f+1}^2\frac{[2f+3]}{[2f+1]}\frac{P(f+1)^2}{P(f)^2}$, $\rho_-(f)^2=A_f^2\frac{[2f-1]}{[2f+1]}\frac{P(f-1)^2}{P(f)^2}$, an identity of rational functions (verified symbolically). At $f=m\geq\frac12$ the term $A_m\widehat{\Psi}_{m-1}$ is absent ($A_m=0$ and $S_{m-1}=0$ by Lemma I.1), and at $f=f_{\max}$ the term $A_{f_{\max}+1}\widehat{\Psi}_{f_{\max}+1}$ is absent ($A_{f_{\max}+1}=0$ and $S_{f_{\max}+1}=0$); $W_f=0$ outside $[m,f_{\max}]$ by the triangle conditions. This gives (148). For $m=n=0$ the row $k=0$ is row $f=0$ of Lemma I.1, with $\widehat{\Psi}_0^2=[1][2A][2B]W_0^2=1$ by (350).

*(b) Evenness and analyticity of $e$.* With $K=hk$, $H=h$: $\sinh h(k-n)\sinh h(k+n)=\Xi(K,Hn)(K^2-H^2n^2)$, $\sinh^2 2hk=4K^2\,\Xi(2K,0)$ and $\mathrm{s}_{2k+1}\mathrm{s}_{2k-1}=\Xi(2K,H)(4K^2-H^2)$, so that $A_k^2=\frac{(k^2-m^2)(k^2-n^2)}{k^2(4k^2-1)}\,e(K^2,H^2)^2=A_k^{(0)2}e^2$ with the stated $\Xi$-quotient $e^2$. $\Xi(u,w)$ is analytic in $(u^2,w^2)$ with $\Xi(0,0)=1$, so $e^2$ is jointly analytic, equals 1 at $(0,0)$, and is positive on $[0,s_1]\times\{0\}$ iff $\sinh K<\sinh\mathsf{X},\sinh\mathsf{Y}$, that is $s_1<4\min(\alpha,\beta)^2$. At $t=0$, $\Xi(K,0)=\sinh^2K/K^2$ and $\Xi(2K,0)=\sinh^2K\cosh^2K/K^2$, so $e(s)^2=(1-\sinh^2K/\sinh^2\mathsf{X})(1-\sinh^2K/\sinh^2\mathsf{Y})/\cosh^4K=(1-\tanh^2K\coth^2\mathsf{X})(1-\tanh^2K\coth^2\mathsf{Y})$. At $K=0$, $\Xi(0,w)=\sinh^2w/w^2$ gives $e(0,t)=\lambda(t)$. Differentiating $\log\Xi(K,Hn)=\log((\sinh^2K-n^2t)/(K^2-n^2t))+O(t^2)$ and $\log\Xi(2K,H)=\log((\sinh^2 2K-t)/(4K^2-t))+O(t^2)$ in $t$ at $t=0$ gives

$$\eta(K^2)=\tfrac12\Big[(m^2+n^2)\Big(\frac{1}{K^2}-\frac{1}{\sinh^2K}\Big)+\frac{1}{\sinh^2 2K}-\frac{1}{4K^2}\Big],\tag{359}$$

which is (150), and $\eta(0)=\frac12[\frac{m^2+n^2}{3}-\frac13]=\lambda_1$.

*The split of $B$.* With $U=h(k+\frac12)$, $B_k=\coth\mathsf{X}\coth\mathsf{Y}(1-\frac{2\cosh Hm\cosh Hn}{\cosh 2U+\cosh H})+\frac{2\sinh Hm\sinh Hn}{\cosh 2U-\cosh H}$. The first term is analytic in $(U^2,H^2)$. In the second, $\cosh 2U-\cosh H=w\varphi$ with $w=U^2-H^2/4$ and $\varphi$ analytic, $\varphi(0,0)=2$, $\varphi|_{w=0}=2\sinh H/H$; therefore $2/(w\varphi)-(H/\sinh H)/w=(2-(H/\sinh H)\varphi)/(w\varphi)$ has a numerator divisible by $w$ and is analytic. Since $B_k^{(0)}=mnH^2/w$ and $\lambda(t)=\sinh Hm\sinh Hn/(mnH\sinh H)$, $B_k-\lambda(h^2)B_k^{(0)}=v(q_k^2,h^2)$ is analytic, jointly with $e$. Expanding $v(0,t)$ to first order gives $\beta_2$, and $\partial_s v(0,0)=\beta_1$, $v(s,0)=\coth\mathsf{X}\coth\mathsf{Y}\tanh^2\sqrt{s}$ gives $b_0$. Without $\lambda$, the coefficient of $h^4/q^2$ in $B_k-B_k^{(0)}$ is $mn(m^2+n^2-1)/6$, nonzero when $mn\neq0$. The identities (151): with $\Delta_0=\sin^2\theta+\mu^2$ (hyperbolic identity $\coth^2\mathsf{X}+\coth^2\mathsf{Y}-2x\coth\mathsf{X}\coth\mathsf{Y}=1-x^2+\mu^2$), $e^2-(x-b_0)^2=\sin^2\theta-\tanh^2q\,\Delta_0=(\sin^2\theta-\mu^2\sinh^2q)/\cosh^2q$. These are identities between rational functions of $e^{\mathsf{X}},e^{\mathsf{Y}},e^{\mathsf{Z}},e^q$, verified by exact symbolic computation.

*(c) Seed (E2).* Put $\ell_\pm:=\frac12(\mathsf{X}+\mathsf{Y})\pm\frac12\mathsf{Z}$ and $\ell'_\pm:=\frac12(\mathsf{Z}\pm(\mathsf{Y}-\mathsf{X}))$, and note $[A{+}B{\mp}C{+}\sigma]=\sinh(\ell_\mp+h\sigma)/\sinh h$, $[B{+}C{-}A{+}\sigma]=\sinh(\ell'_++h\sigma)/\sinh h$, $[A{-}B{+}C{+}\sigma]=\sinh(\ell'_-+h\sigma)/\sinh h$ and $[2A+j]=\sinh(\mathsf{X}+hj)/\sinh h$. In (350) the powers of $\sinh h$ cancel against $[2A][2B]$, so

$$\begin{aligned}\widehat{\Psi}_m^2=[2m+1]\,\frac{[2m]!}{[m{+}n]![m{-}n]!}\,\frac{\sinh\mathsf{X}\sinh\mathsf{Y}}{\prod_{j=-m}^m\sinh(\mathsf{X}{+}hj)\sinh(\mathsf{Y}{+}hj)}\\ \times\prod_{\Sigma_{m+n}}\sinh(\ell_-{+}h\sigma)\sinh(\ell_+{+}h\sigma)\prod_{\Sigma_{m-n}}\sinh(\ell'_-{+}h\sigma)\sinh(\ell'_+{+}h\sigma),\end{aligned}\tag{360}$$

Each product over a symmetric multiset is even in $h$, the $q$-binomial and $[2m+1]$ are even analytic functions of $h$ with values $\binom{2m}{m+n}$ and $2m+1$ at $h=0$, and by the symmetry of $\Sigma_L$ the factors pair into $\sinh(\ell_-{+}h\sigma)\sinh(\ell_+{+}h\sigma)=\frac12[\cosh(\mathsf{X}+\mathsf{Y}+2h\sigma)-\cosh\mathsf{Z}]$ and (re-pairing $\sigma$ with $-\sigma$) $\sinh(\frac{\mathsf{Z}+\mathsf{Y}-\mathsf{X}}{2}+h\sigma)\sinh(\frac{\mathsf{Z}+\mathsf{X}-\mathsf{Y}}{2}-h\sigma)=\frac12[\cosh\mathsf{Z}-\cosh(\mathsf{X}-\mathsf{Y}+2h\sigma)]$. Thus $\widehat{\Psi}_m^2=G(x;h^2)$, where $G$ is a polynomial of degree $2m$ in $\cosh\mathsf{Z}$, hence in $x=(\cosh\mathsf{X}\cosh\mathsf{Y}-\cosh\mathsf{Z})/(\sinh\mathsf{X}\sinh\mathsf{Y})$, with coefficients analytic in $t=h^2$ and real zeros at $x=-1+O(h)$ (from $\Sigma_{m+n}$) and $x=1+O(h)$ (from $\Sigma_{m-n}$); in particular $G$ is jointly analytic in $(x,t)$. At $h=0$ the hyperbolic half-angle formulas $\cos^2\frac{\theta}{2}=\sinh s\sinh(s-\mathsf{Z})/(\sinh\mathsf{X}\sinh\mathsf{Y})$, $\sin^2\frac{\theta}{2}=\sinh(s-\mathsf{X})\sinh(s-\mathsf{Y})/(\sinh\mathsf{X}\sinh\mathsf{Y})$,

$s=\frac12(\mathsf{X}+\mathsf{Y}+\mathsf{Z})$, give $G(x;0)=(2m+1)\binom{2m}{m+n}\cos^{2(m+n)}\frac{\theta}{2}\sin^{2(m-n)}\frac{\theta}{2}=\widehat{d}_m(\theta)^2$. Hence $\widehat{\Psi}_m=(-1)^{m-n}\sqrt{G(x;h^2)}$ (the sign being $(-1)^{a+b+c+2m}(-1)^{z_0}=(-1)^{m-n}$, from (147) and $\operatorname{sgn}W_m=(-1)^{z_0}$; it is the sign of $d^m_{mn}$) is analytic, bounded, and nonvanishing in $(x,t)$ on $U\times\{|t|\le t_0\}$, for a complex neighbourhood $U$ of $X$ (because $G>0$ on $X\times\{0\}$, where it equals $\widehat{d}_m^{\,2}$), with value $\widehat{d}_m$ at $t=0$. By (a) at $k=m$, where $A_m=0$ ($\mathrm{s}_{k-m}=0$), $\widehat{\Psi}_{m+1}=(x-B_m)\widehat{\Psi}_m/A_{m+1}$, jointly analytic because $A_{m+1}=A^{(0)}_{m+1}e((m+1)^2t,t)$ is analytic and nonzero, with value $\widehat{d}_{m+1}$ at $t=0$ by the Wigner recurrence; the exceptional case $m=n=0$ is the row $k=0$, with $\widehat{\Psi}_0=1$ and $\widehat{\Psi}_1=(x-B_0)/A_1$ affine in $x$. This gives (E2) (and polynomiality in $x$ for Theorem 3.26).

*(R).* At a simple turn (R) holds intrinsically by Proposition 3.31; independently, by (K2), $\widehat{\Psi}_k^2=[2k+1][2A][2B]W_k^2\le[2A][2B]/[2c+1]=\sinh\mathsf{X}\sinh\mathsf{Y}/(\sinh\frac1N\sinh\mathsf{Z})=O(N)$ for all $k$.

*(d)* Using $\tanh^2q_*=\sin^2\theta/\Delta_0$ (from $D(q_*)=0$) in $\varepsilon_*=x-\tanh^2q_*\coth\mathsf{X}\coth\mathsf{Y}$ gives $\varepsilon_*=[x\Delta_0-(1-x^2)\coth\mathsf{X}\coth\mathsf{Y}]/\Delta_0=(x\coth\mathsf{X}-\coth\mathsf{Y})(\coth\mathsf{X}-x\coth\mathsf{Y})/\Delta_0$ with $\Delta_0>0$, and $x\coth\mathsf{X}-\coth\mathsf{Y}=(\cosh\mathsf{Y}-\cosh\mathsf{X}\cosh\mathsf{Z})/(\sinh^2\mathsf{X}\sinh\mathsf{Y})$, $\coth\mathsf{X}-x\coth\mathsf{Y}=(\cosh\mathsf{Y}\cosh\mathsf{Z}-\cosh\mathsf{X})/(\sinh^2\mathsf{Y}\sinh\mathsf{X})$. By the hyperbolic sine rule $\sinh q_*=\sinh\mathsf{Y}\sin A_{\mathsf{X}}=\sinh\mathsf{X}\sin A_{\mathsf{Y}}$ ($A_{\mathsf{X}}$ the angle opposite $\mathsf{X}$), so $q_*<\min(\mathsf{X},\mathsf{Y})$ unless $A_{\mathsf{X}}$ or $A_{\mathsf{Y}}=\pi/2$, i.e. $\cosh\mathsf{X}=\cosh\mathsf{Y}\cosh\mathsf{Z}$ or $\cosh\mathsf{Y}=\cosh\mathsf{X}\cosh\mathsf{Z}$ (hyperbolic Pythagoras), which is $\varepsilon_*=0$. Otherwise $e(q_*^2)>0$, $u(q_*)=\varepsilon_*/e=\tau$, and (151) gives the simple turn. The integers: $K_+=C-|A-B|+\frac{|m-n|-1}{2}$ equals $c+b-a+\max(m-n,0)$ if $A\ge B$ and $c+a-b+\max(n-m,0)$ if $A<B$, and $K_-=\kappa-(A+B-C)+\frac{|m-n|}{2}=f+c-a-b-\min(m,n)$, all integers by the triad conditions; the identification of $NS(q_*)/\pi$ is Lemma 6.2. □

**Proof of Lemma 6.2.** *Geometry*, by exact symbolic computation. Let $M_{ij}=\cosh d_{ij}$ be the matrix of hyperbolic edge lengths $d_{12}=2q$, $d_{34}=\mathsf{Z}$, $d_{13}=d_{23}=\mathsf{X}$, $d_{14}=d_{24}=\mathsf{Y}$, with cofactors $C_{ij}$. The dihedral angle $p_{\mathrm{tet}}$ at the edge 12 satisfies $\cos p_{\mathrm{tet}}=-C_{34}/\sqrt{C_{33}C_{44}}$, and $\cos^2p_{\mathrm{tet}}=C_{34}^2/(C_{33}C_{44})$ equals $(x-b_0)^2/e^2$ as an identity of rational functions of $e^{\mathsf{X}},e^{\mathsf{Y}},e^{\mathsf{Z}},e^q$, with the sign checked on the branch $p(0)=\theta$: this identifies $p=p_{\mathrm{tet}}$, as in the flat case of Section 5.3. This identification explains (153) through Schläfli's formula but is not used in the proof.

*Direct proof.* Fix $\mathsf{X},\mathsf{Y}$ and vary $\mathsf{Z}$ in an interval on which the type $\tau$ and $q_*<\min(\mathsf{X},\mathsf{Y})$ are constant. By (151) $D\cosh^2q=\sin^2\theta-\mu^2\sinh^2q$; the substitution $\sin\phi=\mu\sinh q/\sin\theta$, $\phi\in[0,\frac{\pi}{2}]$, turns $S_*(\mathsf{Z}):=\int_0^{q_*}p\,\mathrm{d}q$ into the integral over a fixed $\phi$-interval of a function smooth in $(\phi,\mathsf{Z})$ (since $e\ge c>0$ on $[0,q_*]$ and $e\sin p=\sqrt{D}$), so it may be differentiated under the integral sign, with the endpoint term $p(q_*)\partial_{\mathsf{Z}}q_*$. In $\cos p=(x-b_0)/e$ only $x$ depends on $\mathsf{Z}$, with $\partial_{\mathsf{Z}}x=-\mu$, so $\partial_{\mathsf{Z}}p=\mu/(e\sin p)=\mu/\sqrt{D}$ and

$$\int_0^{q_*}\partial_{\mathsf{Z}}p\,\mathrm{d}q=\int_0^{q_*}\frac{\mu\cosh q\,\mathrm{d}q}{\sqrt{\sin^2\theta-\mu^2\sinh^2q}}=\arcsin\frac{\mu\sinh q}{\sin\theta}\Big|_0^{q_*}=\frac{\pi}{2}.\tag{361}$$

At $q_*$, $\cos p=\varepsilon_*/e=\tau$, so $p(q_*)=0$ for $\tau=+1$ and $\pi$ for $\tau=-1$. Hence $\partial_{\mathsf{Z}}S_*=\pi/2$ ($\tau=+1$) and $\partial_{\mathsf{Z}}(S_*-\pi q_*)=\pi/2$ ($\tau=-1$). By (152): for $\mathsf{X}>\mathsf{Y}$ the second factor is negative, so $\tau=+1$ exactly for $\cosh\mathsf{Z}<\cosh\mathsf{X}/\cosh\mathsf{Y}=:\cosh\mathsf{Z}_c$, i.e. on $(\mathsf{X}-\mathsf{Y},\mathsf{Z}_c)$ (note $\mathsf{Z}_c>\mathsf{X}-\mathsf{Y}$), and $\tau=-1$ on $(\mathsf{Z}_c,\mathsf{X}+\mathsf{Y})$; for $\mathsf{X}=\mathsf{Y}$, $\tau=-1$ throughout; $\mathsf{X}<\mathsf{Y}$ is symmetric. As $\mathsf{Z}\downarrow|\mathsf{X}-\mathsf{Y}|$ or $\mathsf{Z}\uparrow\mathsf{X}+\mathsf{Y}$ the triangle degenerates, $\sin\theta\to0$, and $\sinh q_*=\sinh\mathsf{Y}\sin A_{\mathsf{X}}\to0$ while $p$ stays bounded, so $S_*\to0$ and $q_*\to0$. Integrating $\partial_{\mathsf{Z}}$ from the degenerate end of each interval gives $S_*=\frac{\pi}{2}(\mathsf{Z}-|\mathsf{X}-\mathsf{Y}|)$ on the first and $S_*-\pi q_*=-\frac{\pi}{2}(\mathsf{X}+\mathsf{Y}-\mathsf{Z})$ on the second. With $\mathsf{X}=2hA$ etc. and $N=1/h$ this is $NS(q_*)/\pi=C-|A-B|$, resp. $NS(q_*)/\pi-Nq_*=-(A+B-C)$. □

**Proof of Corollary 6.3.** (i), (ii), (iv), (v) are Theorems 3.7, 3.9, 3.20, 3.26, 3.27 and Propositions 3.31–3.34 applied to the chain of Proposition 6.1; $\widehat{\Psi}_k\to\widehat{d}_k$ at $h\to0$ because $\widehat{\Psi}_{f_{\min}}$ has

the sign of $d_{mn}^{f_{\min}}$, so Theorem 3.27 carries the Darboux sign $\epsilon_{mn} = (-1)^{\max(n-m,0)}$; (R) is Proposition 6.1(c); the Airy amplitude is Proposition 3.28 with (iii). In (v), $\partial_q D(q_*) = -2e_*^2 u_1$ gives $u_1$ from (151). For (vi), under $N \to N/\varsigma$ and $q = \varsigma\rho$ at step $h = N^{-1}$, the (E1) functions in the $\rho$-variable are $e(\varsigma^2 s, \varsigma^2 t)$ and $v(\varsigma^2 s, \varsigma^2 t)$, which multiplies $\varepsilon_1, \beta_1, \beta_2, \chi, \lambda_1, \eta, \delta_{\mathrm{fl}}$ by $\varsigma^2$ and $\varpi_\tau$ by $\varsigma^{4/3}$ (since $(\varsigma/N)^{4/3}\varpi_\tau = N^{-4/3}(\varsigma^{4/3}\varpi_\tau)$ while $(\varsigma/N)^{-2/3}\zeta_\tau(\varsigma\rho) = N^{2/3}\zeta_{\tau,\rho}(\rho)$); expanding $\coth(2\varsigma\alpha) = (2\varsigma\alpha)^{-1}(1 + O(\varsigma^2))$, $\sinh(\varsigma\rho) = \varsigma\rho(1 + O(\varsigma^2))$, and $\mu \sinh(\varsigma\rho) = \gamma\rho/(2\alpha\beta) + O(\varsigma^2)$ gives the Racah limits.

(iii) *The flux defect, from the closed seed*, by exact symbolic computation. By the symmetry reduction in Proposition 6.1 it suffices to take $m \geq |n|$, since $\widehat{\Psi}_k / \widehat{U}_k^{[2]}$ and $\mathsf{J}_2(0, x)$ are invariant under both involutions. By Theorem 3.24 and the contact identity, the coefficients of $t$ in $\widehat{\Psi}_k$ and $\widehat{U}_k^{[2]}$ differ by $c(x)\widehat{d}_k$ for every $k$, so at $k = m$ (both start from $\widehat{d}_m$ at $t = 0$)

$$c(x) = [t] \log \widehat{\Psi}_m - [t] \log \widehat{U}_m^{[2]} = \tfrac{1}{2}\mathsf{Q} - \nu_m. \tag{362}$$

*Seed side.* From (360), with $\log[z] = \log z + \frac{(z^2-1)t}{6} + O(t^2)$ and, for a symmetric multiset, $\sum_{\sigma \in \Sigma_L} \log\sinh(\ell + h\sigma) = L \log\sinh \ell - \frac{t}{2}\frac{L(L^2-1)}{12}\frac{1}{\sinh^2 \ell} + O(t^2)$,

$$\begin{aligned} \mathsf{Q} = [t] \log \widehat{\Psi}_m^2 = {} & \mathsf{f}(2m{+}1) - \mathsf{f}(m{+}n) - \mathsf{f}(m{-}n) \\ & - \tfrac{1}{2}\Big[\mathsf{T}(m{+}n)\big(\tfrac{1}{\sinh^2 \ell_-} + \tfrac{1}{\sinh^2 \ell_+}\big) + \mathsf{T}(m{-}n)\big(\tfrac{1}{\sinh^2 \ell'_-} + \tfrac{1}{\sinh^2 \ell'_+}\big) \\ & \qquad - \mathsf{T}(2m{+}1)\big(\tfrac{1}{\sinh^2 \mathsf{X}} + \tfrac{1}{\sinh^2 \mathsf{Y}}\big)\Big], \end{aligned} \tag{363}$$

$\mathsf{f}(N) = \frac{N(N+1)(2N+1)}{36} - \frac{N}{6} = \sum_{j \leq N} \frac{j^2-1}{6}$, $\mathsf{T}(N) = \frac{N(N^2-1)}{12}$, *Pullback side.* By the proof of Proposition 3.17, $\widehat{U}_m^{[2]} = \widehat{d}_m + t\,\widehat{u}_m + O(t^2)$ with $\widehat{u}_m = \sqrt{2m+1}\,\mathcal{Q}_{2,m} d_m$, so that, with $\kappa = m + \frac{1}{2}$, $s = 1 - x^2$, $\partial_x \log d_m = -(mx - n)/s$ ($d_m \propto \cos^{m+n}\frac{\theta}{2} \sin^{m-n}\frac{\theta}{2}$) and the data (150), (53),

$$\nu_m = \Big(\frac{\kappa^2 \chi}{3} - \sqrt{s}\,\rho_2(0, x)\Big)\partial_x \log d_m + \frac{\kappa^2\big((\varepsilon_1 - x\beta_1)/2 - x\chi/6\big)}{s} + \frac{x\,\rho_2(0, x)}{2\sqrt{s}}. \tag{364}$$

*The identity.* In the variables $u = e^{\mathsf{X}/2}$, $v = e^{\mathsf{Y}/2}$, $w = e^{\mathsf{Z}/2}$, every quantity above is a rational function of $(u, v, w)$ with coefficients polynomial in $(m, n)$ ($x$, $\coth \mathsf{X}$, $\coth \mathsf{Y}$, $\sinh \ell_\pm$, $\sinh \ell'_\pm$ and $\sqrt{s}\,\rho_2(0, x)$ are rational, the square root cancelling in $\sqrt{s}\rho_2$ and in $x\rho_2/\sqrt{s}$), and

$$\tfrac{1}{2}\mathsf{Q} - \nu_m + \mathsf{J}_2(0, x) + \tfrac{1}{12} = 0 \tag{365}$$

is an identity of rational functions of $(u, v, w)$ with symbolic $(m, n)$: its numerator, a polynomial in $(u, v, w, m, n)$, expands to zero in exact arithmetic. This gives $c(x) = -\mathsf{J}_2(0, x) - \frac{1}{12}$, i.e. (154) (in the scalar channel $m = n = 0$, where $\widehat{\Psi}_0 = 1$ and $\mathsf{Q} = 0$, all $\varepsilon_1, \beta_1, \beta_2$ terms in $-\nu_0 + \mathsf{J}_2(0, x)$ cancel directly to $\frac{1}{2}\lambda_1 = -\frac{1}{12}$). The relation to the Gauss weight is Remark 3.35. □

# J Verification and a computable certificate

The proofs are analytic; none rests on numerical fitting. Exact computer-algebra checks reproduce the identities used in the proofs, and high-precision numerical checks illustrate the stated rates against exact $3j$ and $6j$ symbols (Section 7); the latter enter no proof. For a self-adjoint Jacobi recurrence $a_{k+1}Y_{k+1} + a_k Y_{k-1} = b_k Y_k$ the Casoratian $a_k(Y_k W_{k-1} - Y_{k-1} W_k)$ of two solutions is

constant, so a fundamental matrix $F_k$ has $\det F_k = \mathcal{C}/a_k$ and $\|F_k^{-1}\|_2 = a_k\|F_k\|_2/|\mathcal{C}|$. Variation of constants then bounds the error of any approximation $\widetilde{Z}_{k+1} = T_k\widetilde{Z}_k + \rho_k$ by

$$\|Z_K - \widetilde{Z}_K\|_2 \le \frac{\|F_K\|_2}{|\mathcal{C}|}\Big[a_{k_0}\|F_{k_0}\|_2\|Z_{k_0} - \widetilde{Z}_{k_0}\|_2 + \sum_{k=k_0}^{K-1} a_{k+1}\|F_{k+1}\|_2\|\rho_k\|_2\Big], \tag{366}$$

a bound in finite exact-recurrence quantities that can be evaluated in interval arithmetic; it is not used in the proofs.

# References


[1] A. Lewis, *Joint Rank–Position Uniformity for Weyl–Midpoint Wigner $3j$ Symbols: A Technical Supplement with Racah $6j$ Extensions*, companion technical supplement to the present paper, with its verification programs in the companion GitHub repository https://github.com/cmbant/weyl-midpoint-semiclassics (2026).

[2] A. Lewis, *Even Asymptotic Expansions and Uniform Hard-Edge Approximations for Wigner $3j$ and $6j$ Symbols at Fixed Rank*, Hexagon 2611.00533v1 (2026); https://hexagonmath.org/2611.00533v1.

[3] J. Carron and A. Lewis, *Spherical bispectrum expansion and quadratic estimators*, JCAP **07** (2024) 067. https://doi.org/10.1088/1475-7516/2024/07/067.

[4] S. C. McFarlane, *Angular momentum and Heisenberg's correspondence principle*, J. Phys. B **25** (1992), 4045–4057.

[5] S. C. McFarlane, *Angular momentum and Heisenberg's correspondence principle. II. More complex matrix elements*, J. Phys. B **26** (1993), 1871–1884.

[6] J. J. Morehead, *Semiclassical integrable matrix elements*, Phys. Rev. A **53** (1996), 1285–1294.

[7] D. Neville, *A technique for solving recurrence relations approximately and its application to the 3-J and 6-J symbols*, J. Math. Phys. **12** (1971), 2438–2453.

[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press (1957).

[9] G. Ponzano and T. Regge, *Semiclassical limit of Racah coefficients*, in *Spectroscopic and Group Theoretical Methods in Physics*, ed. F. Bloch, North-Holland, Amsterdam (1968).

[10] V. Bonzom and P. Fleury, *Asymptotics of Wigner $3nj$-symbols with small and large angular momenta: an elementary method*, J. Phys. A: Math. Theor. **45** (2012) 075202; arXiv:1108.1569.

[11] P. A. Braun, *Discrete semiclassical methods in the theory of Rydberg atoms in external fields*, Rev. Mod. Phys. **65** (1993) 115–161. https://doi.org/10.1103/RevModPhys.65.115.

[12] K. Schulten and R. G. Gordon, *Exact recursive evaluation of $3j$- and $6j$-coefficients for quantum-mechanical coupling of angular momenta*, J. Math. Phys. **16** (1975), 1961–1970.

[13] H. T. Johansson and C. Forssén, *Fast and accurate evaluation of Wigner $3j$, $6j$, and $9j$ symbols using prime factorization and multiword integer arithmetic*, SIAM J. Sci. Comput. **38** (2016) A376–A384. https://doi.org/10.1137/15M1021908.

[14] G. Benettin and A. Giorgilli, *On the Hamiltonian interpolation of near-to-the-identity symplectic mappings with application to symplectic integration algorithms*, J. Stat. Phys. **74** (1994), 1117–1143. https://doi.org/10.1007/BF02188219.

[15] M. W. Reinsch and J. J. Morehead, *Asymptotics of Clebsch–Gordan coefficients*, J. Math. Phys. **40** (1999), 4782–4806.

[16] M. Dupuis and E. R. Livine, *Pushing further the asymptotics of the $6j$-symbol*, Phys. Rev. D **80** (2009), 024035.

[17] V. Aquilanti, H. M. Haggard, R. G. Littlejohn, and L. Yu, *Semiclassical analysis of Wigner $3j$-symbol*, J. Phys. A: Math. Theor. **40** (2007), 5637–5674. https://doi.org/10.1088/1751-8113/40/21/013.

[18] R. G. Littlejohn and L. Yu, *Uniform semiclassical approximation for the Wigner $6j$-symbol in terms of rotation matrices*, J. Phys. Chem. A **113** (2009), 14904–14922. https://doi.org/10.1021/jp905056y.

[19] L. Yu, *Uniform approximation from symbol calculus on a spherical phase space*, J. Phys. A: Math. Theor. **44** (2011), 505202; arXiv:1105.4220. https://doi.org/10.1088/1751-8113/44/50/505202.

[20] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin, and P. D. Miller, *Discrete Orthogonal Polynomials: Asymptotics and Applications*, Annals of Mathematics Studies **164**, Princeton University Press, 2007.

[21] E. A. Rakhmanov, *Equilibrium measure and the distribution of zeros of the extremal polynomials of a discrete variable*, Sb. Math. **187** (1996), 1213–1228. https://doi.org/10.1070/SM1996v187n08ABEH000153.

[22] P. D. Dragnev and E. B. Saff, *Constrained energy problems with applications to orthogonal polynomials of a discrete variable*, J. Anal. Math. **72** (1997), 223–259. https://doi.org/10.1007/BF02843160.

[23] J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Comm. Math. Phys. **103** (1986), 177–240.

[24] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer Series in Computational Physics, Springer, 1991. https://doi.org/10.1007/978-3-642-74748-9.

[25] NIST Digital Library of Mathematical Functions, *Chapter 34: $3j$, $6j$, $9j$ Symbols*, especially Secs. 34.3–34.5. https://dlmf.nist.gov/34.

[26] NIST Digital Library of Mathematical Functions, *Chapter 18: Orthogonal Polynomials*, especially Secs. 18.15, 18.24, and 18.38. https://dlmf.nist.gov/18.

[27] J. P. M. Flude, *The Edmonds asymptotic formulae for the $3j$ and $6j$ symbols*, J. Math. Phys. **39** (1998), 3906–3915; arXiv:physics/9710037.

[28] T. H. Koornwinder, *Clebsch–Gordan coefficients for $SU(2)$ and Hahn polynomials*, Nieuw Arch. Wisk. (3) **29** (1981), 140–155.

[29] Y. Lin and R. Wong, *Global asymptotics of the Hahn polynomials*, Analysis and Applications **11** (2013), 1350018. https://doi.org/10.1142/S0219530513500188.

[30] L.-C. Chen, M. E. H. Ismail, and P. Simeonov, *Asymptotics of Racah coefficients and polynomials*, J. Phys. A: Math. Gen. **32** (1999), 537–553. https://doi.org/10.1088/0305-4470/32/3/009.

[31] Y. U. Taylor and C. T. Woodward, *$6j$ symbols for $U_q(\mathfrak{sl}_2)$ and non-Euclidean tetrahedra*, Selecta Math. (N.S.) **11** (2006), 539–571. https://doi.org/10.1007/s00029-005-0014-9.

[32] F. Costantino, *$6j$ symbols, hyperbolic structures and the volume conjecture*, Geom. Topol. **11** (2007), 1831–1854. https://doi.org/10.2140/gt.2007.11.1831.

[33] Q. Chen and T. Yang, *Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants*, Quantum Topol. **9** (2018), 419–460. https://doi.org/10.4171/QT/111.

[34] S. Mizoguchi and T. Tada, *Three-dimensional gravity from the Turaev–Viro invariant*, Phys. Rev. Lett. **68** (1992), 1795–1798. https://doi.org/10.1103/PhysRevLett.68.1795.

[35] V. Bonzom, M. Dupuis, and F. Girelli, *Towards the Turaev–Viro amplitudes from a Hamiltonian constraint*, Phys. Rev. D **90** (2014), 104038. https://doi.org/10.1103/PhysRevD.90.104038.

[36] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of the algebra $U_q(\mathfrak{sl}(2))$, $q$-orthogonal polynomials and invariants of links*, in *Infinite-dimensional Lie algebras and groups (Luminy–Marseille, 1988)*, Adv. Ser. Math. Phys. **7**, World Scientific, Teaneck, NJ, 1989, 285–339.

[37] R. Askey and J. Wilson, *A set of orthogonal polynomials that generalize the Racah coefficients or $6$-$j$ symbols*, SIAM J. Math. Anal. **10** (1979), 1008–1016. https://doi.org/10.1137/0510092.

[38] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics, Springer, Berlin, 2010. https://doi.org/10.1007/978-3-642-05014-5.

[39] I. Khavkine, *Recurrence relation for the $6j$-symbol of $su_q(2)$ as a symmetric eigenvalue problem*, Int. J. Geom. Methods Mod. Phys. **12** (2015), 1550117. https://doi.org/10.1142/S0219887815501170.

[40] J. Roberts, *Classical $6j$-symbols and the tetrahedron*, Geometry & Topology **3** (1999), 21–66. https://doi.org/10.2140/gt.1999.3.21.

[41] K. Schulten and R. G. Gordon, *Semiclassical approximations to $3j$- and $6j$-coefficients for quantum-mechanical coupling of angular momenta*, J. Math. Phys. **16** (1975), 1971–1988.

[42] J. S. Geronimo, O. Bruno, and W. Van Assche, *WKB and turning point theory for second-order difference equations*, in *Spectral Methods for Operators of Mathematical Physics*, Operator Theory: Advances and Applications **154** (2004), 101–138. https://doi.org/10.1007/978-3-0348-7947-7_7.

[43] J. S. Geronimo and D. T. Smith, *WKB (Liouville–Green) analysis of second order difference equations and applications*, J. Approx. Theory **69** (1992), 269–301. https://doi.org/10.1016/0021-9045(92)90003-7.

[44] X.-S. Wang and R. Wong, *Asymptotics of Racah polynomials with varying parameters*, J. Math. Anal. Appl. **436** (2016), 1149–1164. https://doi.org/10.1016/j.jmaa.2015.12.035.

[45] Z. Benzaid and D. A. Lutz, *Asymptotic representation of solutions of perturbed systems of linear difference equations*, Stud. Appl. Math. **77** (1987), 195–221. https://doi.org/10.1002/sapm1987773195.

[46] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974; reprinted A K Peters, Wellesley, MA, 1997.

[47] Z. Wang and R. Wong, *Asymptotic expansions for second-order linear difference equations with a turning point*, Numer. Math. **94** (2003), 147–194. https://doi.org/10.1007/s00211-002-0416-y.

[48] J. S. Geronimo, *WKB and turning point theory for second order difference equations: external fields and strong asymptotics for orthogonal polynomials*, arXiv:0905.1684 (2009).

[49] C. L. Frenzen and R. Wong, *A uniform asymptotic expansion of the Jacobi polynomials with error bounds*, Canad. J. Math. **37** (1985), 979–1007. https://doi.org/10.4153/CJM-1985-053-1.

[50] NIST Digital Library of Mathematical Functions, *Chapters 9 and 10: Airy and Related Functions; Bessel Functions*, especially Secs. 9.7–9.8 and 10.17. https://dlmf.nist.gov/9, https://dlmf.nist.gov/10.

[51] A. Fedotov and F. Klopp, *The complex WKB method for difference equations and Airy functions*, SIAM J. Math. Anal. **51** (2019), 4413–4447. https://doi.org/10.1137/18M1228694.

[52] A. B. J. Kuijlaars and W. Van Assche, *The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory **99** (1999), 167–197. https://doi.org/10.1006/jath.1999.3316.

| case | $N = 160$ | | $N = 640$ | |
|---|---|---|---|---|
| | without | with | without | with |
| *(a) Fixed-ratio pullback,* $\max\sqrt{2L+1}\,\lvert Y_L - U_L\rvert$*;* $U = U^{[2]}$ *without,* $(1 - N^{-2}\mathsf{J}_2)U^{[2]}$ *with* | | | | |
| $3j$, $(M,\sigma) = (2,1)$, $x = 0.3$, $q \approx 0.45$ | $2.3 \cdot 10^{-5}$ | $3.8 \cdot 10^{-8}$ | $1.3 \cdot 10^{-6}$ | $4.3 \cdot 10^{-10}$ |
| $3j$, $(\frac{3}{2}, -\frac{1}{2})$, $x \approx 0.2$, $q \approx 0.5$ † | $2.5 \cdot 10^{-6}$ | $4.1 \cdot 10^{-8}$ | $1.5 \cdot 10^{-7}$ | $6.6 \cdot 10^{-10}$ |
| $6j$, $(m,n) = (2,1)$, $(\alpha,\beta,\gamma) = (1, \frac{6}{5}, \frac{3}{2})$, $\rho \approx 0.9$ | $4.7 \cdot 10^{-6}$ | $1.3 \cdot 10^{-7}$ | $3.0 \cdot 10^{-7}$ | $2.0 \cdot 10^{-9}$ |
| | $N = 100$ | | $N = 800$ | |
| *(b) Bulk zero,* $\lvert x_j - \widetilde{x}_j\rvert$ *against* $d^L(\Theta_q)$ *(without) and* $d^L(\Theta_q^{[2]})$ *(with),* $q = 0.4$ | | | | |
| $3j$, $(2,1)$, $x \approx 0.5$ | $6.4 \cdot 10^{-5}$ | $1.5 \cdot 10^{-8}$ | $1.0 \cdot 10^{-6}$ | $3.6 \cdot 10^{-12}$ |
| $6j$, $(1,0)$, $(\alpha,\beta) = (1, \frac{5}{4})$, $x \approx 0.3$ | $1.3 \cdot 10^{-6}$ | $3.7 \cdot 10^{-11}$ | $2.1 \cdot 10^{-8}$ | $9.4 \cdot 10^{-15}$ |
| | $N = 500$ | | $N = 4000$ | |
| *(c) Airy bracket,* $\max_{\lvert t\rvert\le 2}\lvert$*bracket* $-\,\mathrm{Ai}(t) -$ *term*$\rvert$*;* *term* $= 0$ *without,* $N^{-4/3}\varpi\,\mathrm{Ai}'(t)$ *with* | | | | |
| $3j$, $(2,1)$, upper turn, $x = 0.6$ | $1.9 \cdot 10^{-4}$ | $3.6 \cdot 10^{-6}$ | $1.2 \cdot 10^{-5}$ | $5.5 \cdot 10^{-8}$ |
| $3j$, $(\frac{3}{2}, -\frac{1}{2})$, lower turn, $x \approx -0.5$ † | $7.7 \cdot 10^{-5}$ | $3.9 \cdot 10^{-6}$ | $5.0 \cdot 10^{-6}$ | $6.6 \cdot 10^{-8}$ |
| | $N = 40$ | | $N = 320$ | |
| $6j$, $(2,1)$, $(1, \frac{6}{5}, \frac{3}{2})$, $\tau = -1$ | $3.2 \cdot 10^{-3}$ | $2.9 \cdot 10^{-4}$ | $2.2 \cdot 10^{-4}$ | $4.9 \cdot 10^{-6}$ |
| $6j$, $(1,0)$, $(1,2,\frac{6}{5})$, $\tau = +1$ | $2.3 \cdot 10^{-3}$ | $4.1 \cdot 10^{-4}$ | $1.4 \cdot 10^{-4}$ | $6.7 \cdot 10^{-6}$ |
| | $N \approx 40$ | | $N \approx 320$ | |
| *(d) Uniform law,* $\max\lvert$*bracket* $-\,\mathrm{Ai} - N^{-4/3}B\,\mathrm{Ai}'\rvert/(\mathsf{M}/\mathsf{E})$*;* $B = \varpi_\tau$ *without,* $B_\tau(\rho)$ *with* | | | | |
| $6j$, $(2,1)$, $(1, \frac{6}{5}, \frac{3}{2})$, $\tau = -1$ | $2.2 \cdot 10^{-3}$ | $2.1 \cdot 10^{-4}$ | $3.8 \cdot 10^{-4}$ | $3.3 \cdot 10^{-6}$ |
| $6j$, $(1,0)$, $(1,2,\frac{6}{5})$, $\tau = +1$ | $1.3 \cdot 10^{-3}$ | $1.8 \cdot 10^{-4}$ | $1.6 \cdot 10^{-4}$ | $3.4 \cdot 10^{-6}$ |
| $6j$, $(\frac{1}{2}, -\frac{1}{2})$, $(1, \frac{5}{3}, 2)$, $\tau = -1$ ‡ | $9.9 \cdot 10^{-4}$ | $1.1 \cdot 10^{-5}$ | $1.2 \cdot 10^{-4}$ | $1.9 \cdot 10^{-7}$ |
| $q$-$6j$, $(2,1)$, $(1, \frac{11}{10}, \frac{3}{2})$, $\varsigma = 0.6$, $\tau = -1$ § | $1.1 \cdot 10^{-3}$ | $2.3 \cdot 10^{-5}$ | $5.2 \cdot 10^{-5}$ | $2.2 \cdot 10^{-8}$ |
| $q$-$6j$, $(3,3)$, $(2,1,\frac{13}{10})$, $\varsigma = 0.5$, $\tau = +1$ § | $1.1 \cdot 10^{-3}$ | $2.5 \cdot 10^{-5}$ | $2.9 \cdot 10^{-5}$ | $9.7 \cdot 10^{-9}$ |

Table 2: Errors against exact symbols. (a) Corollaries 4.4 and 5.3: over seven ranks around the stated ratio, natural gauge; a fourfold increase of $N$ lowers the error by about $4^2$ without and $4^3$ with the amplitude factor. (b) Corollaries 4.5 and 5.4: one zero tracked in $N$; rates $N^{-2}$ and $N^{-4}$. (c) Theorems 4.6 and 5.7: rates $N^{-4/3}$ and $N^{-2}$. (d) the Airy bracket of (80) (for the Racah rows without its $N^{-2}$ amplitude): maximum over the ranks in $\rho \in [\rho_* - \frac{1}{4}, \rho_* + \frac{1}{5}]$ (every rank of this window, both sides of the turn), $N = 40$ and 320; observed rates $N^{-1}$ (the "without" column, still pre-asymptotic at these $N$ in some rows) and $N^{-2}$. ‡ Half-integer $6j$ family, $N \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$: $N = \frac{159}{4}$ and $\frac{1281}{4}$. § Quantum $6j$ at curvature scale $\varsigma$ in the rescaled variable $\rho = (f + \frac{1}{2})/N$ of Corollary 6.3(vi) (Section 6): eight equally spaced ranks of the window $[\rho_* - \delta_-, \rho_* + \delta_+]$ with $\delta_- \le 0.10$ and $\delta_+ = \min(0.08,\, 0.2\, d_{\mathrm{edge}})$, $d_{\mathrm{edge}}$ the distance of $\rho_*$ from the hard edge (so the window stays away from the edge collar), $N = 35$ and 315; the column with $B_\tau$ includes the amplitude $1 + N^{-2}(\delta_{\mathrm{fl}} - \frac{1}{2}\mathsf{j})$ with rescaled flux defect $\delta_{\mathrm{fl}} = -\varsigma^2/12$ and decays like $N^{-3}$ (log-log slopes $-3.0$ and $-3.5$ over $N = 35, 125, 315, 795$; at $N = 795$ the entries are $1.9 \cdot 10^{-9}$ and $5.1 \cdot 10^{-10}$); omitting $\delta_{\mathrm{fl}}$ leaves slopes $-2.1$ and $-2.2$. The $\tau = +1$ turn of the $(3,3)$ row lies at $d_{\mathrm{edge}} \approx 0.105$ (at $\varsigma = 0.5$ every $\tau = +1$ turn lies within about 0.13 of the hard edge), so its window top is about 3 ranks from the edge at $N = 35$ and about 67 at $N = 795$; the small-$N$ entries of that row are pre-asymptotic. For $3j$ in half-integer channels (†) $N$ is the nearest half-integer above the stated value.