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THE DEGREE PAIR $(88, 132)$ IN THE PLANE JACOBIAN PROBLEM: EXCLUSION OF THE SHAPE WITH STARTING CORNER $(11, 33)$ AND FINAL CORNER $(19/4, 8)$

DAVID NIEDBALA GIRAUDIN

Abstract. In the list of Guccione, Guccione, Horruitiner and Valqui of the possible shapes of a counterexample to the plane Jacobian conjecture with degrees at most 150, exactly one shape has maximal degree 132, up to the exchange of the two polynomials: the standard $(2, 3)$-pairs of degrees $(88, 132)$ with starting corner $(11, 33)$ and final corner $(19/4, 8)$. We show that no such pair exists, when it is normalised as in Corollary 5.21 of Guccione–Guccione–Valqui. The two sides of the Newton polygon are described by the results of Guccione–Guccione–Valqui. In suitable polynomial coordinates the pair is confined to an explicit finite system of polynomial equations. Its leading part has exactly two solutions, defined over $\mathbb{Q}(\sqrt{-3})$ and exchanged by its non-trivial automorphism; for each of them the system is solved order by order by linear algebra over that field, and the compatibility conditions of the first twelve orders, six polynomials in three unknowns, generate the unit ideal. The computation is exact and is reproduced by an ancillary script (SymPy, and Singular for three Gröbner bases). All statements are proved relative to the cited results. Nothing in this note has been checked by a third party.

1. Statement

Throughout, K is an algebraically closed field of characteristic zero, $L = K[x, y]$, $[P, Q] = P_x Q_y - P_y Q_x$ and $(m, n) = (2, 3)$. We use the notation of [1]: for a weight (ρ, σ), $v_{\rho,\sigma}$ is the weighted degree, $\ell_{\rho,\sigma}$ the leading form, $\mathrm{st}_{\rho,\sigma}$ and $\mathrm{en}_{\rho,\sigma}$ its end points; $L^{(l)} = K[x^{\pm 1/l}, y]$. *Standard (m, n)-pairs*, their *starting triple* and their *regular corners* are defined in [1, Definitions 4.3, 6.2, 5.5]. Put $t := xy^4$.

Standing data 1.1. (P, Q) is a standard $(2, 3)$-pair in L with $[P, Q] = J \in K^\times$, $\deg P = 88$, $\deg Q = 132$, starting triple $\big((11, 33), (3, 1), (4, -1)\big)$, and $(-1, 1) < \mathrm{Succ}_P(1, 0) < (-1, 0)$ as in [1, Corollary 5.21(4)]. Thus:

(D1) the support of P lies in $\{(i, j) : 4i - j \leq 22\}$, that of Q in $\{4i - j \leq 33\}$, and

$$\ell_{4,-1}(P) = \lambda_P \big(x^3 y\, h(t)\big)^2, \qquad \ell_{4,-1}(Q) = \lambda_Q \big(x^3 y\, h(t)\big)^3, \qquad \lambda_P, \lambda_Q \in K^\times,$$

with $h \in K[t]$ monic of degree 8, $h(0) \neq 0$ [1, Proposition 6.1, Theorem 7.6(2)].

Theorem 1.2. *There is no pair (P, Q) as in Standing data 1.1.*

This is the shape with corners $(11, 33)$, $(19/4, 8)$ and $(m, n) = (2, 3)$ of [2, Section 6]: by [2, Theorem 2.20, items (5), (8), (13)] the second corner lies on the line through $(11, 33)$ of direction $(1/4, 1)$ given by the direction of the starting triple, which is therefore $(4, -1)$, and Lemma 2.1 below shows that the middle entry of the triple is then necessarily $(3, 1)$. We found no exclusion of this shape in the literature we searched; its novelty is not otherwise verified. The four authors treat in [3] the shapes of degree below 125 and obtain the bound 108.

2. The roots of h

Lemma 2.1. *Let (P, Q) be a standard $(2, 3)$-pair in L with a regular corner $((11, 33), (4, -1))$. Then $\frac{1}{2}\,\mathrm{st}_{4,-1}(P) = (3, 1)$. In Standing data 1.1, $h = (t - c)^8$ with $c \in K^\times$.*

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Proof. The point $A' := \frac{1}{2}\operatorname{st}_{4,-1}(P)$ lies in $\mathbb{Z} \times \mathbb{N}_0$, on the line through $(11, 33)$ of direction $(1/4, 1)$ and below $(11, 33)$ [1, Proposition 5.14 and its proof]: $A' = (11 - e, 33 - 4e)$ with $1 \leq e \leq 8$. The corner is of Case II in the sense of [1, Section 5, before Remark 5.9]: Case I would require $l - a/b = 1 - \frac{1}{3} > 1$ [1, Proposition 5.13], and Case III would require that $\rho = 4$ divides $l = 1$ [1, Proposition 5.17]. If $e \leq 7$ then $v_{1,-1}(A') < 0$, and A' is the first entry of a regular corner of (P, Q) [1, Remark 5.11], which requires $\gcd(11 - e, 33 - 4e) > 1$ [1, Proposition 5.19] (the other alternative of that proposition, $l - a/b > 1$, is impossible for $l = 1$); but $\gcd(11 - e, 33 - 4e) = \gcd(11 - e, 11) = 1$. Hence $e = 8$ and $A' = (3, 1)$.

In $\tilde{z} := x^{1/4}y$ one has $\ell_{4,-1}(P) = x^{11/2}p(\tilde{z})$ with $p(\tilde{z}) = \lambda_P \tilde{z}^2 h(\tilde{z}^4)^2$; the bracket of $\ell_{4,-1}(P)$ and $\ell_{4,-1}(Q)$ vanishes, both being powers of $x^3y\,h(t)$ up to constants, p has at least five distinct roots and $v_{1,-1}(\operatorname{st}_{4,-1}(P)) = v_{1,-1}(6, 2) > 0$. By [1, Proposition 5.16] some root $\lambda \neq 0$ of p has multiplicity at least $\frac{m}{l} \cdot \frac{a\rho+bl\sigma}{\rho+\sigma} = 2 \cdot \frac{44-33}{3} = \frac{22}{3}$ (the root can be taken different from 0, the multiplicity 2 of the root 0 being smaller than this bound). A root of h of multiplicity k gives roots of p of multiplicity $2k$; so some root c of h has multiplicity $k \geq 4$, and by [1, Proposition 5.18] some $(2, 3)$-pair in $L^{(4)}$ has a regular corner at $(\frac{11}{4}, 0) + k(\frac{1}{4}, 1) = (\frac{11+k}{4}, k)$. By [1, Proposition 5.19] this point would be the first entry of a regular corner of Case I or of Case II. Case II requires $\gcd(11+k, k) > 1$ [1, Proposition 5.14(4)], which fails for $4 \leq k \leq 8$. In Case I, with direction (ρ, σ), [1, Proposition 1.13, Corollary 5.7(2)] give $2(\frac{11+k}{4}\rho + k\sigma) = \frac{2}{5}(\rho+\sigma)$, which determines (ρ, σ), and [1, Proposition 5.13] requires a positive integer a' with $\frac{\rho}{4} \cdot 2a' = v(P)$ or a positive integer k' with $\frac{\rho}{4}k' \in \{v(P), v(Q)\}$, where $v(P) = \frac{2}{5}(\rho+\sigma)$ and $v(Q) = \frac{3}{5}(\rho+\sigma)$. For $k = 4, 5, 6, 7$ the directions are $(76, -71)$, $(24, -19)$, $(116, -81)$, $(68, -43)$ and no such integer exists (ancillary script). Hence $k = 8$. □

3. The left side

Lemma 3.1. *There is $\gamma \in K$ such that*

$$\hat{P}(z, v) := P\big(zv^3 + \gamma v^2,\, v^{-1}\big), \qquad \hat{Q}(z, v) := Q\big(zv^3 + \gamma v^2,\, v^{-1}\big)$$

are polynomials in z, v. Then $\hat{P}$ has total degree 22 with top form $\lambda_P z^6 (z - cv)^{16}$, $\hat{Q}$ has total degree 33 with top form $\lambda_Q z^9 (z - cv)^{24}$, and

$$\hat{P}_z \hat{Q}_v - \hat{P}_v \hat{Q}_z = -J\, v.$$

Proof. The argument is the one of [3, Section 4, Case $(7, 21)$]; we give it in detail. Apply the flip ψ_1 of [1, Section 8], so that the weight $(4, -1)$ becomes $(\rho_0, \sigma_0) = (-1, 4)$, with $\operatorname{st}_{\rho_0,\sigma_0} = (66, 22)$ and $\operatorname{en}_{\rho_0,\sigma_0} = (2, 6)$ for the flipped P, and $\frac{1}{m}\operatorname{st} = \frac{1}{n}\operatorname{st} = (33, 11)$. Let F be the element of [1, Theorem 2.6] for (ρ_0, σ_0); then $v_{-1,4}(F) = 3$. If $\operatorname{st}(F) = (1, 1)$ then $\operatorname{en}(F) \neq (1, 1)$, since F is not a monomial [1, Remark 2.5], so $\operatorname{en}(F) \sim \operatorname{en}(P) \sim (1, 3)$ and $\operatorname{en}(F) = \frac{3}{11}(1, 3)$, which is impossible; hence $\operatorname{st}(F) \sim (33, 11)$, so $\operatorname{st}(F) = (9, 3) = \frac{3}{11}(33, 11)$ and $q = 11$ in [1, Corollary 7.4]. That corollary applies to the flipped pair (its hypotheses hold with $(a/l, b) = (33, 11)$, and $\rho_0 + \sigma_0 > 0$): for every weight between $(1, -3)$ and (ρ_0, σ_0) at which the pair has an edge, the edge form of P is a power R^{22}. In the original variables: for a weight $(-s, 1)$ with $s < 3$ carrying an edge with starting point $(22, 66)$, the form is $(xy^3 - \gamma y^{3-s})^{22}$ up to a constant, so $s \in \mathbb{Z}$, and $s \geq 2$ by the condition on $\operatorname{Succ}_P(1, 0)$; thus $s = 2$. Here $(22, 66) = \operatorname{en}_{4,-1}(P)$ is the starting point of the edge at $\operatorname{Succ}_P(1, 0)$: an edge of P with a direction strictly between $(4, -1)$ and $(1, 1)$ starting at $(22, 66)$ would end at a point with $v_{1,1} > 88 = \deg P$ [1, Remark 1.8], and by the condition on $\operatorname{Succ}_P(1, 0)$ there is no direction of P between $(1, 1)$ and $(-1, 1)$; then [1, Proposition 3.7] applies. The weights $(-s, 1)$ with $s < 3$ are in the range of the corollary, because $v_{-s,1}(22, 66) = 22(3 - s) > 0$. The translation $x \mapsto x + \gamma y^{-2}$ removes this edge [1, Proposition 3.9] (take $\gamma = 0$ if there is no such edge). By [1, Proposition 3.9] the leading forms of the translated pair at the weights between $(4, -1)$ and $(-2, 1)$ are unchanged, and its $(-2, 1)$-form is the monomial $x^{22}y^{66}$; so an edge with starting point $(22, 66)$ and $s < 3$ could only occur for $2 < s < 3$. The same corollary applies to the translated pair: it lies in $L^{(1)}$, the form ℓ_{ρ_0,σ_0} and the degrees $v_{1,1}$, $v_{0,1}$ of the flipped pair are unchanged, and so is F (the

element G_0 of [1, Theorem 2.6] can be kept, and $\frac{3}{11}(1,3)$ is excluded in $L^{(1)}$ as well). Hence the form of such an edge would again be a power R^{22}, with R a $(-s,1)$-homogeneous element of $K[x,y^{\pm1}]$ whose only possible monomials are xy^3 and y^{3-s}, which forces $s\in\mathbb{Z}$. So no edge with $s<3$ and starting point $(22,66)$ remains. Since the weight $(-3,1)$ lies within a half-turn of $(4,-1)$, the point $(22,66)$ is then still leading for $(-3,1)$ [1, Remark 2.10, Proposition 3.7]; hence every monomial x^iy^j of the translated P satisfies $j\le 3i$, that is, $x^iy^j=z^iv^{3i-j}$ is a monomial in $z=xy^3$, $v=y^{-1}$. For every weight $(-s,1)$ with $s<3$ the values of P and Q are positive, since $(22,66)$ and $(33,99)$ remain in the supports and $v_{-s,1}(22,66)=22(3-s)$, $v_{-s,1}(33,99)=33(3-s)$. At the weight $(-2,1)$, if P has an edge there, [1, Proposition 7.3] with $T_0=Q$ gives $[\ell(Q),\ell(P)]=0$ and proportional values, so $\ell(Q)\sim R^{33}$ (the element $R=y(xy^2-\gamma)$, $\gamma\neq0$, is not a proper power) and the same translation applies. For the translated pair and $2\le s<3$, the leading forms commute by [1, Proposition 1.13] (the sum of the values exceeds $1-s$), so $\ell_{-s,1}(Q)$ is proportional to a power of the monomial $\ell_{-s,1}(P)=x^{22}y^{66}$ [1, Proposition 2.1(2)], hence a monomial at a point $(i,3i)$; by (D1) such a point of the support has $i\le33$, and $(33,99)$ is in the support, so $\ell_{-s,1}(Q)$ is the monomial at $(33,99)$ and $j-si\le33(3-s)$ on the support of the translated Q. Letting s tend to 3 gives $j\le3i$ there.

The translation replaces x^iy^j by monomials $x^{i-r}y^{j-2r}$ of $(4,-1)$-weight $4i-j-2r$, so (D1) holds for the translated pair with the same leading forms. Since $x^iy^j=z^iv^{3i-j}$ has $4i-j=i+(3i-j)$, the bound $4i-j\le22$ is the bound on the total degree, and for $x=zv^3$, $y=v^{-1}$ one has $t=zv^{-1}$ and, by Lemma 2.1, $(x^3y)^2(t-c)^{16}=z^6(z-cv)^{16}$, which gives the top form; likewise for Q. The Jacobian follows from $dx\wedge dy=-v\,dz\wedge dv$. □

4. The right side

Write $X:=x^{1/4}$, so that $L^{(4)}=K[X^{\pm1},y]$.

Lemma 4.1. *There are $\lambda\in K^\times$ with $\lambda^4=c$ and $\lambda_2\in K$ such that, with*

$$Y:=\eta+\lambda X^{-1}+\lambda_2X^{-2},\qquad P_1(X,\eta):=P\big(X^4,Y\big),\qquad Q_1(X,\eta):=Q\big(X^4,Y\big),$$

and with the weight $W(X^i\eta^j):=3i-7j$, every monomial of P_1 has $W\le2$ and every monomial of Q_1 has $W\le3$. The monomials of P_1 with $W=2$ lie between $X^3\eta$ and $X^{38}\eta^{16}$, both with non-zero coefficient; those of Q_1 with $W=3$ lie between X and $X^{57}\eta^{24}$, both with non-zero coefficient.

Proof. By Lemma 2.1 and its proof, every root λ of $\tilde z^4=c$ has multiplicity $16\ge\frac{22}{3}$ in p, and by [1, Proposition 5.18] the automorphism $\varphi(y)=y+\lambda x^{-1/4}$ gives a $(2,3)$-pair $(\varphi P,\varphi Q)$ in $L^{(4)}$ with a regular corner at $A_1:=(\frac{11}{4},0)+8(\frac14,1)=(19/4,8)$, of direction $(\rho',\sigma')<(4,-1)$. We follow this corner as in the proof of [1, Proposition 5.19]. Case II is impossible, because [1, Proposition 5.14(4)] requires $\gcd(19,8)>1$; this holds at every step below, since it depends only on the point A_1. In Case III, [1, Proposition 5.17] gives $\rho'\mid4$, and the translation $y\mapsto y+\lambda' x^{\sigma'/\rho'}$ produces a $(2,3)$-pair in $L^{(4)}$ with a regular corner at the same point A_1 and a smaller direction. After finitely many steps Case I occurs, for a pair (P_1,Q_1) and a direction (ρ,σ). There $v_{\rho,\sigma}(P_1)+v_{\rho,\sigma}(Q_1)=\rho+\sigma$ [1, Proposition 1.13], while $\mathrm{en}(P_1)=2A_1=\frac23\,\mathrm{en}(Q_1)$ [1, Corollary 5.7(2)]. Hence $2(\frac{19}{4}\rho+8\sigma)=\frac25(\rho+\sigma)$, that is $(\rho,\sigma)=(12,-7)$, $v(P_1)=2$, $v(Q_1)=3$. By [1, Proposition 5.13] either $\frac12\,\mathrm{st}(P_1)=(a'/4,0)$ with $a'\in\mathbb{N}$, which would give $6a'=2$; or $\{\mathrm{st}(P_1),\mathrm{st}(Q_1)\}=\{(k'/4,0),(1-k'/4,1)\}$ with $k'\in\mathbb{N}$. If $\mathrm{st}(P_1)=(k'/4,0)$ then $3k'=2$; so $\mathrm{st}(Q_1)=(k'/4,0)$ with $3k'=3$, that is $\mathrm{st}(Q_1)=(1/4,0)$ and $\mathrm{st}(P_1)=(3/4,1)$. The directions of the Case III steps decrease strictly from $(4,-1)$ and stay larger than the direction $(12,-7)$ of the final Case I corner, which is determined by the point A_1; they have $\rho'\in\{1,2,4\}$. The only such direction is $(2,-1)$ (the direction $(4,-3)$ is smaller than $(12,-7)$), so there is at most one such step, a translation $y\mapsto y+\lambda_2x^{-1/2}$. Finally $v_{12,-7}(x^{i/4}y^j)=3i-7j$, $\mathrm{en}(P_1)=(38/4,16)$ and $\mathrm{en}(Q_1)=(57/4,24)$. □

5. Common coordinates

Let $\mathcal{A}$ be the set of formal sums $\sum c_{ij} X^i \eta^j$ ($i \in \mathbb{Z}$, $j \geq 0$) such that, for every r, only finitely many terms have $W = 3i - 7j \geq r$. It is a ring containing $K[X^{\pm 1}, \eta]$; for $f \neq 0$ let $W(f)$ be the largest weight of its terms and $\ell(f)$ the sum of the terms of that weight. Then $\ell(fg) = \ell(f)\ell(g)$ and $W(fg) = W(f) + W(g)$, and $1 + \varepsilon$ is invertible in $\mathcal{A}$ when $W(\varepsilon) < 0$.

Lemma 5.1. *Let γ, λ, λ_2 be as in Lemmas 3.1 and 4.1, and put*

$$\zeta := z - c\,v - \kappa, \qquad \kappa := 4\lambda^2 \lambda_2.$$

Write $\hat{P} = \sum p_{ab} \zeta^a v^b$ and $\hat{Q} = \sum q_{ab} \zeta^a v^b$ in $K[z, v] = K[\zeta, v]$. Then $p_{ab} \neq 0$ implies $3b - a \leq 2$ and $a + b \leq 22$; $q_{ab} \neq 0$ implies $3b - a \leq 3$ and $a + b \leq 33$. Moreover $p_{1,1}$, $p_{16,6}$, $q_{0,1}$, $q_{24,9}$ are non-zero, the part of $\hat{P}$ of degree 22 is $\lambda_P \zeta^{16} (\zeta + cv)^6$, and $\hat{P}_\zeta \hat{Q}_v - \hat{P}_v \hat{Q}_\zeta = -J\,v$.

Proof. The substitution $z = \zeta + cv + \kappa$ does not increase the total degree and turns the top form of Lemma 3.1 into $\lambda_P \zeta^{16} (\zeta + cv)^6$; it preserves the bracket. In $\mathcal{A}$ the element Y of Lemma 4.1 is $\lambda X^{-1}(1 + \varepsilon)$ with $W(\varepsilon) < 0$; it is invertible. The assignment $v \mapsto Y^{-1}$, $z \mapsto X^4 Y^3 - \gamma Y$ defines a ring morphism $K[z, v] \to \mathcal{A}$ which sends $\hat{P}$ to P_1 and $\hat{Q}$ to Q_1, because $P(x, y) = \hat{P}\big((x - \gamma y^{-2}) y^3, y^{-1}\big)$ in $K[x, y^{\pm 1}]$. Expanding (ancillary script), the image of v has $W = 3$ and leading term X/λ, and the image of ζ has $W = -1$ and leading term $4\lambda^2 \eta X^2$: the terms of weights 3 and 0 cancel by the choice of $c = \lambda^4$ and κ. Hence the image of $\zeta^a v^b$ has leading term a non-zero multiple of $\eta^a X^{2a+b}$, of weight $3b - a$, and these monomials are distinct for distinct (a, b). Therefore, for a non-zero polynomial, the value of W on the image of $\sum c_{ab} \zeta^a v^b$ is $\max\{3b - a : c_{ab} \neq 0\}$, and the leading part is the image of the corresponding terms. Lemma 4.1 gives the values 2 and 3 for the images of $\hat{P}$ and $\hat{Q}$; the monomials $X^3 \eta$, $X^{38} \eta^{16}$, X, $X^{57} \eta^{24}$ correspond to ζv, $\zeta^{16} v^6$, v, $\zeta^{24} v^9$. □

6. The finite system

By Lemmas 2.1–5.1, a pair as in the standing data yields a solution of the following system.

(S1) $\hat{P} = \sum p_{ab} \zeta^a v^b$ with $a, b \geq 0$, $a + b \leq 22$, $3b - a \leq 2$; $\hat{Q} = \sum q_{ab} \zeta^a v^b$ with $a, b \geq 0$, $a + b \leq 33$, $3b - a \leq 3$; and $p_{1,1}\, p_{16,6}\, q_{0,1} \neq 0$.

(S2) $\hat{P}_\zeta \hat{Q}_v - \hat{P}_v \hat{Q}_\zeta = -J\,v$ with $J \in K^\times$.

(S3) The part of $\hat{P}$ on the line $a + b = 22$ is $\lambda_P\, \zeta^{16} (\zeta + c\,v)^6$, with $\lambda_P, c \in K^\times$.

In (S1), $\hat{P}$ has 89 possible monomials and $\hat{Q}$ has 187. Call the terms with $3b - a = 2 - k$ the *layer* k of $\hat{P}$ and those with $3b - a = 3 - k$ the layer k of $\hat{Q}$, and write $\hat{P} = \sum_{k \geq 0} P_k$, $\hat{Q} = \sum_{k \geq 0} Q_k$. The bracket of a term of layer i of $\hat{P}$ with a term of layer j of $\hat{Q}$ is a monomial $\zeta^a v^b$ with $3b - a = 3 - (i + j)$, and v has $3b - a = 3$. So (S2) is equivalent to the *equations of order k*:

$$\sum_{i+j=k} \{P_i, Q_j\} = \begin{cases} -J\,v, & k = 0, \\ 0, & k \geq 1, \end{cases} \qquad \{f, g\} := f_\zeta g_v - f_v g_\zeta. \tag{1}$$

With $s := \zeta^3 v$, the layers of order 0 are

$$P_0 = p_{1,1}\, \zeta v\, (1 + a_1 s + \cdots + a_5 s^5), \qquad Q_0 = q_{0,1}\, v\, (1 + h_1 s + \cdots + h_8 s^8),$$

with $a_5 \neq 0$ by (S1).

Lemma 6.1. *If (S1)–(S3) has a solution, it has one with $p_{1,1} = q_{0,1} = a_1 = c = 1$ and $p_{0,0} = q_{0,0} = q_{16,6} = 0$. For such a solution $J = -1$, $\lambda_P = a_5$, and $(a_2, \ldots, a_5, h_1, \ldots, h_8)$ is one of exactly two tuples, which have their entries in $\mathbb{Q}(\sqrt{-3})$ and are exchanged by its non-trivial automorphism; in both, $T := 1/a_5$ is a root of*

$$150994944\, T^2 - 314828451840\, T + 1291756832812537 = 0,$$

whose discriminant is $-3\cdot 476472471552^2$, *and*

$$a_2=\tfrac{33207559840-1179648T}{87282533261},\quad a_3=\tfrac{9142619992576-948436992T}{111808925107341},\quad a_4=\tfrac{35168631085056-8204058624T}{4222117981434353},$$
$$h_1=4,\quad h_2=\tfrac{132959026424-1179648T}{37406799969},\quad h_3=\tfrac{17130979691392-721944576T}{10164447737031},$$
$$h_4=\tfrac{2319025500854272 0-2265018531840T}{47742411020834607},\quad h_5=\tfrac{65536T+154611184}{1175013\,T},\quad h_6=\tfrac{6291456T+42564276664}{1839678687\,T},$$
$$h_7=\tfrac{132830239360-4718592T}{69469771371\,T},\quad h_8=\tfrac{25761882159104-5173673984T}{484505342131811\,T}.$$

Proof. The equation of order 0 in (1) reads, for the coefficient of v, $p_{1,1}q_{0,1}=-J$, and for the coefficient of vs^k, $1\le k\le 12$,

$$\sum_i (5i-2k+1)\,a_i\,h_{k-i}=0 \qquad (a_0=h_0=1,\ 0\le i\le 5,\ 0\le k-i\le 8), \tag{2}$$

because $\{\zeta^{3i+1}v^{i+1},\zeta^{3j}v^{j+1}\}=(3i-2j+1)\,\zeta^{3(i+j)}v^{i+j+1}$. The system (2) has no solution with $a_1=0$ and $a_5\neq 0$ (Gröbner basis, ancillary script). The system (S1)–(S3) is preserved by $(\hat P,\hat Q)\mapsto(\gamma'\hat P(\alpha\zeta,\beta v),\delta\hat Q(\alpha\zeta,\beta v))$, which multiplies $p_{1,1}$ by $\gamma'\alpha\beta$, $q_{0,1}$ by $\delta\beta$, each a_i by $(\alpha^3\beta)^i$ and c by β/α. Since $a_1\neq 0$ we may assume $p_{1,1}=q_{0,1}=a_1=c=1$; then $J=-1$, and comparing the coefficients of $\zeta^{16}v^6$ in P_0 and in (S3) gives $\lambda_P=a_5$. Adding constants to $\hat P$ and $\hat Q$, and replacing $\hat Q$ by $\hat Q-(q_{16,6}/a_5)\hat P$, preserves (S1)–(S3) and the layers of order 0, since $\hat P$ has no term in layer 0 of $\hat Q$. Finally, with $a_1=1$ and $a_5\neq 0$ the system (2) has exactly two solutions, in which $T=1/a_5$ satisfies the quadratic equation stated (Gröbner basis, ancillary script), and the displayed expressions satisfy (2) for each root T (ancillary script). □

Proposition 6.2. *The system* (S1)–(S3), *normalised as in Lemma 6.1, has no solution.*

Proof (exact computation). Fix one of the two tuples of Lemma 6.1; its entries lie in $E:=\mathbb{Q}(\sqrt{-3})$, and (S3) fixes the coefficients of $\hat P$ on the line $a+b=22$: $p_{16+j,6-j}=a_5\binom{6}{j}$. For $k\ge 1$ the equation of order k in (1) is $\{P_k,Q_0\}+\{P_0,Q_k\}=-\sum_{0<i<k}\{P_i,Q_{k-i}\}$: a linear system $A_kx_k=b_k$ in the vector x_k of the coefficients of layer k not fixed by (S3) or by the normalisation, with a matrix A_k with entries in E and a right-hand side depending on the earlier layers. Reducing A_k to echelon form expresses the pivot coordinates of x_k through b_k and the coordinates which remain free, and the rows of the echelon form without pivot give *conditions*: polynomials over E in the free coordinates of the layers $\le k$, which vanish on every solution. Carrying this out for $k=1,\dots,12$ (ancillary script): exactly three coordinates remain free, namely $q_{23,8}$ (layer 2), $q_{24,8}$ (layer 3) and $q_{22,7}$ (layer 4), and there are six conditions, one for each of $k=8,9,10,11$ and two for $k=12$. These six polynomials generate the unit ideal of $E[q_{23,8},q_{24,8},q_{22,7}]$ (Gröbner basis, Singular [4]; the script writes the input file). Hence there is no solution with this tuple. The other tuple is obtained by the non-trivial automorphism of E, which transforms the whole computation, so the conclusion is the same. □

As a check of the procedure, the script repeats the computation without condition (S3): it then finds four free coordinates and four conditions for $k\le 12$, all vanishing when the free coordinates are 0, in agreement with the fact that (P_0,Q_0) alone satisfies (S1) and (S2).

Proof of Theorem 1.2. A pair as in the standing data gives a solution of (S1)–(S3) by Lemmas 2.1–5.1, hence a normalised one by Lemma 6.1, contradicting Proposition 6.2. □

7. Scope

- **Proved here, relative to the cited results:** the reduction of Theorem 1.2 to Proposition 6.2 (Lemmas 2.1–6.1). The inputs are (D1), which rests on [1, Proposition 6.1, Theorem 7.6(2), Remark 3.1], and [1, Definition 6.2, Theorem 2.6, Remarks 2.5, 2.10, 5.11, Corollaries 5.7, 7.4, Propositions 1.13, 2.1, 3.7, 3.9, 5.13, 5.14, 5.16, 5.17, 5.18, 5.19, 7.3], as indicated. The condition on $\mathrm{Succ}_P(1,0)$ in the standing data is the one of [1, Corollary 5.21(4)], which is established there for the standard pairs attached to a minimal counterexample; the theorem is stated under this condition.

- **Verified by exact computation:** Proposition 6.2; the statements on the system (2) in Lemma 6.1; the arithmetic of Lemmas 2.1 and 4.1; the leading terms in Lemma 5.1 (ancillary file `verify_88_132.py` in the folder `anc`; SymPy for the linear algebra over $\mathbb{Q}(\sqrt{-3})$, and Singular for three Gröbner bases, for which the script writes the input files). The result depends on the correctness of these computations.
- **Not claimed:** a statement on the plane Jacobian conjecture itself; a new bound on the degree of a counterexample (see [3] for the one obtained by the four authors). The same shape with P and Q exchanged is covered by exchanging them, provided the condition on Succ holds for the polynomial of degree 88; both conditions are part of [1, Corollary 5.21(4)]. The completeness of the list of [2] is not re-examined.
- [1, Proposition 7.3 and Corollary 7.4], on which Lemma 3.1 rests, are stated in [1] with the indication that their proofs mimic those of Proposition 7.1 and Corollary 7.2; they are applied here to a Laurent translate, as in [1, Section 8] and [3]. The references [2] and [3] are arXiv preprints.
- The argument has been prepared by the author with AI assistance (Claude, Anthropic). Nothing in this note has been checked by a third party; it has not been validated by professionals.

References

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[2] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Some algorithms related to the Jacobian Conjecture*, arXiv:1708.07936v1.

[3] J. A. Guccione, J. J. Guccione, R. Horruitiner, C. Valqui, *Increasing the degree of a possible counterexample to the Jacobian Conjecture from 100 to 108*, arXiv:2204.14178.

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Independent researcher, France